LOW FREQUENCY EXPANSIONS FOR TWO–DIMENSIONAL INTERFACE SCATTERING PROBLEMS

by

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1 Introduction

The purpose of this paper is to make some remarks on a model problem in the scattering by obstacles. We have a bounded region in which a non-dissipative hyperbolic equation holds. In the (infinite) exterior a different non-dissipative hyperbolic equation holds. The effect of the infinite exterior is to introduce a dissipation effect and it is this aspect we wish to pursue.

The results are different in two or three dimensions. We treat a two dimensional case since two dimensionality produces serious technical difficulties and we want to explore these. At the end of the section we indicate what happens in three dimensions. The specific problem considered is the following. Let Ω be a bounded region in the plane, with boundary Γ and let $\Omega^+ = \bar{\Omega}^c$. Let ρ and μ be positive functions in Ω and τ a positive function on Γ. We put $Lu : div(\mu \operatorname{grad} u)$ and consider the problem;

$$\rho u_{tt} = Lu + h \text{ in } \Omega, u_{tt} = \Delta u \text{ in } \Omega^+, \text{for } t > 0$$

$$u(x,0) = u_t(x,0) = 0 \text{ in } \mathbf{R}^2 \qquad\qquad P(f,g,h)$$

$$u^- = u^+ + f \ , \ \mu u_n^- = \tau(\mu_n^+ + g) \ \text{ on } \Gamma \ t > 0.$$

Here f, g and h are given, n is the normal to Γ and the plus and minus signs denote limits from Ω^+ and Ω.

For simplicity we will assume throughout that τ is a constant.

The above problem arises in the scattering of anti-plane elastic waves, in an infinite homogeneous medium, by an inhomogeneous cylinder [(2)].

It also arises in the scattering of tranverse magnetic waves by dielectric cylinders ([9]).

Our goal is to investigate what happens to the solution when f, g and h tend to steady states $f_\infty, g_\infty, h_\infty$ as t tends to infinity. We would call the system **dissipative** if u then tends to a steady state u_∞. For purposes of comparison let us consider the interior Dirichlet and Neumann problems:

$$\rho u_{tt} = Lu + h \ \text{ in } \Omega \qquad u(x,0) = u_t(x,0) = 0 \text{ in } \Omega$$

$$u^- = f \ \text{ or } \ \mu u_n^- = \tau g \text{ on } \Gamma \tag{1.1}$$

If $f, g, h \to f_\infty, g_\infty, h_\infty$, and there were a steady state, u_∞, one would expect it to solve the problems,

$$Lu_\infty + h_\infty = 0 \text{ in } \Omega \ , \ u_\infty^- = f_\infty \text{ or } \mu u_{\infty,n}^- = \tau g_\infty \text{ on } \Gamma \tag{1.2}$$

The (Dirichlet) problem (1.2) will always have a solution but it is not true (in general) that it will be the limit of the solution of the (Dirichlet) problem (1.1). We observe that the (Neumann) problem (1.2) will have a solution if and only if the quantity,

$$Z(g_\infty, h_\infty) = \tau \int_\Gamma g_\infty + \int_\Omega h_\infty \tag{1.3}$$

is zero. Moreover even if one has a solution it is not unique. The methods of [6] can be used to show that (formally) the solution of the (Neumann) problem (1.1) satisfies an estimate of the form $u(x,t) \to cZ(g_\infty, h_\infty)t + 0(1)$ as $t \to \infty$. Thus the problems (1.1) are certainly not dissipative.

The standard approach to asymptotic stability questions is to Laplace transform the problem. If U, F, G and H are the transforms of u, f, g and h we obtain the reduced problem:

$$p^2 \rho U = LU + H \text{ in } \Omega, \ p^2 U = \triangle U \text{ in } \Omega^+$$

$$\hat{P}_p(F, G, H).$$

$$U^- = U^+ + F \ , \ \mu U_n^- = \tau(U_n^+ + G) \text{ on } \Gamma$$

Here p is complex, with Re $p \geq 0$, and $U \to 0$ as $|x| \to \infty$. Assume that $f(x,t) = f_\infty(x) + \tilde{f}(x,t)$ where $f(x,\cdot) \in L_1(0,\infty)$. Then $F(x,p) = f_\infty(x)p^{-1} + \hat{F}(x,p)$ with $\tilde{F}$ continuous in Re $p \geq 0$. We suppose similar decompositions hold for g and h. Then one seeks the solution of $\hat{P}_p$ in the

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form $U(x,p) = U_\infty(x)p^{-1} + \bar{U}$ and expects U_∞ to satisfy,

$$LU_\infty + U_\infty = 0 \text{ in } \Omega, \triangle u_\infty = 0 \text{ in } \Omega^+,$$

$$U_\infty^- = U_\infty^+ + f_\infty \ , \ \mu U_{\infty,n}^- = \tau(U_{n,\infty}^+ + g_\infty) \text{ on } \tau \qquad P_0(f_\infty, g_\infty, h_\infty).$$

Then one expects the solution of $P(f,g,h)$ to tend to $U_\infty(x)$ as $t \to \infty$. We are warned that there may be trouble with the above idea by the fact that P_0 lacks uniqueness. Indeed any constant is a solution of the homogeneous problem.

The plan of the paper is as follows. We will treat $\hat{P}_p$ completely, within the framework of generalized solutions. In section two we review a variational procedure from [2] and in section three we use it to obtain an estimate for the solution U_p of $\hat{P}_p$ for p small, under the assumption that F, G and H are independent of p. The result is that,

$$U_p = Z(G,H)(\log p + \gamma) + U_0(F,G,H) + 0(p^2(\log p)^2) \qquad (1.4)$$

Here Z is as in (1.3), γ is a constant and $U_0(F,G,H)$ is a special solution of $P_0(F,G,H)$. In section four we give a formal analysis of what we believe are the implications of (1.4) for time decay. We assume that

$$f(x,t) = f_\infty(x) + \tilde{f}(x,t), g(x,t) = g_\infty(x) + \tilde{g}(x,t), h(x,t) = h_\infty(x) + \tilde{h}(x,t)$$

where $\tilde{f}, \tilde{g}$ and $\tilde{h}$ are in $L_1(0,\infty)$ with respect to t. Our formal asymptotic results are that the solution u_p of $P(f,g,h)$ satisfies,

$$u_p(x,t) = Z(g_\infty, h_\infty)\log 2t + U_0(f_\infty, g_\infty, h_\infty) + Z(\bar{G}, \bar{H})\chi(t) + 0(t^{-2}\log t) \text{ as } t \to \infty \qquad (1.5)$$

Here $\chi(t) = 0$ for $t < \frac{1}{2}, \chi(t) = t^{-1}$ for $t > \frac{1}{2}$ and $\bar{G}, \bar{H}$ are given by

$$\bar{G}(x) = \int_0^\infty \tilde{g}(x,t)dt, \quad \bar{H}(x) = \int_0^\infty \tilde{h}(x,t)dt \qquad (1.6)$$

Let us comment on the meaning of (1.5). Observe first that there is a steady state limit if and only if $Z(g_\infty, h_\infty) = 0$. If this quantity is not zero the effect of the exterior is not quite strong enough to produce dissipation but it does reduce the order of growth from $0(t)$ in the interior problem to $0(\log t)$ for the interface problem. If $Z(g_\infty, h_\infty)$ is zero then the interface solution will tend to a (special) solution of $P_0(f_\infty, g_\infty, h_\infty)$ and we have dissipation. The term $Z(\bar{G}, \bar{H})\chi(t)$ is a first measure of the rate of decay in the dissipative situation. It depends on the time dependent part of the data but only on the special quantities (1.6).

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Remark 1.1 *The special solution U_0 in (1.4) and (1.5) is a solution of Laplace's equation in Ω^+ and it will be characterized by the estimate*

$$U_0(x) = A \log|x| + 0(|x|)^{-1} \text{ as } |x| \to \infty \qquad (1.7)$$

and we will see that $A = 0$ if and only if $Z(G, H) = 0$.

Remark 1.2 *In the physical problems of [2] and [9] the solution u of $P(f, g, h)$ represents a scattered wave in Ω^+. In pure scattering problems one has an incident field $u^0(x, t)$ which is, typically, a solution of $u_{tt} = \triangle u$ in all space. In this case $f = u^0$ and $g = u^0$ on τ and $h \equiv 0$. If $u^0 \to u^{0,\infty}(x)$ as $t \to \infty$ then $\triangle u^{0,\infty} = 0$ in all space. In this case it will automatically be true that $Z(g_\infty, h_\infty)$ is zero.*

The study of low frequency (small p) expansions for interior and exterior problems has a long history. See, for instance, [3], [4], [6], [7], [8] and [12]. The application to the study of long time asymptotics for exterior problems is treated in [1], [10] and [11]. The latter results, in contrast to ours, are rigorously proved. What is needed to make our results rigorous is a high frequency (large p) theory. We do not have this, as yet, since we do not know how to handle the interior region.

Remark 1.3 *The presence of $\log p$ in (1.4) and $\log t$ in (1.5) is characteristic of two dimensional problems in exterior regions as indicated in the references. Since two-dimensional problems serve as numerical examples it seems important to understand this phenomenon. This motivated our consideration of two-dimensional problems.*

Remark 1.4 *Let us describe, without proof, the results for three dimensions. The comments concerning the interior problems (1.1) and (1.2) remain in force. Now, however, it will be true that the solution of $\hat{P}_p$ in three dimensions will tend to the unique function U_∞ which satisfies P_0 and vanishes as $|x| \to \infty$. Formally, this means that the solution of $P(f, g, h)$ tends to U_∞ as $t \to \infty$. Thus in three dimensions the effect of the infinite exterior is enough to produce dissipation.*

2   The reduced interface problem

We follow the ideas of [2]. Let $K(z)$ denote the singular Bessel function of second kind and order zero. $K(z)$ behaves like $z^{-\frac{1}{2}}e^{-z}$ for large z and for

small z,

$$K(z) = \log z + \gamma + \sum_1^\infty a_n z^{2n} \log z + \sum_1^\infty b_n z^{2n}, \gamma = -\log 2 - \int_0^\infty e^{-\eta} \log \eta \, d\eta$$

(2.1)

We form the simple layer

$$\mathcal{S}_p[\varphi](x) = (2\pi)^{-1} \int_\Gamma \varphi(y) K(p\,|x-y|) d\tau_y$$

(2.2)

It satisfies $\Delta v = p^2 v$ in $\Omega \cup \Omega^+$ and tends to zero as $|x| \to \infty$. It is continuous in all space with the limit value.

$$\mathcal{S}_p[\varphi](x) = (2\pi)^{-1} \int_\Gamma \varphi(y) K(p\,|x-y|)\,|_{x\in\Gamma}\, d\tau_y \quad \text{on } \Gamma$$

(2.3)

$\mathcal{S}_p$ is an integral operator with logarithmic kernel. $\mathcal{S}_p$ satisfies the jump relations,

$$\left(\frac{\partial}{\partial n} \mathcal{S}_p[\varphi]\right)^\pm = \pm\frac{1}{2}\varphi + N_p[\varphi] \quad \text{on } \Gamma$$

(2.4)

Here n is the normal to Γ and N_p is an integral operator with continuous kernel if Γ is a smooth curve.

We seek to represent the solution U of $P(F,G,H)$ in Ω^+ as,

$$U = \mathcal{S}_p[\varphi]$$

(2.5)

From (2.5), (2.3), (2.4) and the interface conditions we are led to the problem

$$p^2 \rho U = LU + H \quad \text{in } \Omega$$

$$\hat{P}'_p(F,G,H)$$

$$U^- = S_p[\varphi] + F \, , \ \mu U_n^- = \tau \left(\tfrac{1}{2}\varphi + N_p[\varphi] + G\right) \quad \text{on } \Gamma,$$

for the pair (U, φ). It is easy to verify the following result.

Theorem 2.1 *If (U,φ) is a solution of $\hat{P}'_p(F,G,H)$ and U is defined in Ω^+ by (2.5) then U is a solution of $\hat{P}_p(F,G,H)$.*

We reformulate $\hat{P}'_p$ as a variational problem. Let $H_1, H_{-1/2}$ and $H_{1/2}$ denote the complexifications of $H_1(\Omega), H_{1/2}(\Gamma)$ and $H_{1/2}(\Gamma)$ and let $\mathcal{H} =$

$H_1 \times H_{-1/2}$. We write $\mathcal{U} = (U, \varphi)$ and $\mathcal{W} = (W, \psi)$ for elements of $\mathcal{H}$. Note that $\mathcal{U} \in \mathcal{H}$ implies $U^- \in H_{1/2}$. We define the bilinear form,

$$\mathcal{A}_p(\mathcal{U}, \mathcal{W}) = A_p(\mathcal{U}, W) + B_p(\mathcal{U}, \psi),$$

$$A_p(\mathcal{U}, W) = \int_\Omega (\mu \nabla U \cdot \nabla \bar{W} + p^2 \rho U \bar{W}) - \tau \int_\Gamma \left(\frac{1}{2} \varphi + N_p[\varphi]) \bar{W} \right)$$

$$B_p(\mathcal{U}, \psi) = \int_\Gamma U^- \bar{\psi} - \int_\Gamma S_p[\varphi] \bar{\psi},$$

$$(2.6)$$

and the functional,

$$\mathcal{F}(\mathcal{W}) = \mathcal{F}_1(W) + \mathcal{F}_2(\psi),$$

$$(2.7)$$

$$\mathcal{F}_1(W) = \int_\Omega H \bar{W} + \tau \int_\Gamma G \bar{W}^-, \mathcal{F}_2(\psi) = \int_\Gamma F \bar{\psi}$$

Remark 2.1 S_p *extends to an operator from* $H_{-1/2}$ *to* $H_{1/2}$ *and* N_p *to operator from* $H_{-1/2}$ *to* $H_{3/2}$ [1], *the complexification of* $H_{3/2}(\tau)$ *(see [5]). The integrals over* Γ *in (2.6) and (2.7) should, accordingly, be interpreted as duality pairings but we will continue to use the integral notation.*

Definition 2.1 U_p *is a generalized solution of* $\hat{P}_p(F, G, H)$ *if* $\mathcal{U}_p = (U_p, \varphi_p)$ *is a solution of*

$$\mathcal{A}_p(\mathcal{U}_p, \mathcal{W}) = \mathcal{F}(\mathcal{W}) \ \forall \ \mathcal{W} \in \mathcal{H} \quad VP_p(F, G, H)$$

and $U_p = S_p[\varphi]$ *in* Ω^+.

Remark 2.2 *The variational formulations* VP_p *and* VP_0 *in the next section are amenable to implementation with finite elements. This is described in [7] with proofs of optimal convergence.*

The following result is easy to prove and is given in [2].

Theorem 2.2 *If* U_p *is a generalized solution of* $\hat{P}_p(F, G, H)$ *and is sufficiently smooth, then* U_p *is a solution of* $\hat{P}_p(F, G, H)$.

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[1] If Γ is a $C^{(2)}$ curve

(The required smoothness can be achieved by appropriate regularity assumptions on the data).

There is a technical difficulty with VP_p which bears on the results of this paper. VP_p is treated in [2] as a compact perturbation of a coercive problem. This reduces existence to uniqueness and that can fail. The result is this.

Theorem 2.3 *Let* $-\lambda_n^2$ *be the eigen-values of* $\triangle$ *in* Ω *with Neumann boundary conditions. Let* $F, G,$ *and* H *be in* $H_{1/2}, H_{-1/2}$ *and* H_1. *Then for any* p *with* $\mathrm{Re}\, p \geq 0$, $p \neq i\lambda_n$, $VP_p(F, G, H)$ *has a unique solution.*

Remark 2.3 *Notice that* $p = 0$ *is one of the excluded* $p's$. *We will see that, in general, this exclusion is necessary. For* $p = \pm i\lambda_n, X_n \neq 0$, *however, the exclusion is a consequence only of the ansatz not of the problem. In [2] an alternate variational procedure is given which yields existence for those values.*

3 Low frequency asymptotics

In this section we study the behavior of the generalized solution of $\hat{P}_p(F, G, H)$ under the assumption that $F, G,$ and H are independent of p.

Let us begin with some analysis of the formal limit problem

$$Lv + H = 0 \text{ in } \Omega , \quad \triangle v = 0 \text{ in } \Omega^+ ,$$

$$v^- = v^+ + F , \quad \mu v_n^- = \tau(v^+ + G) \text{ on } \Gamma \tag{3.1}$$

A first question is how solutions should behave for large $|x|$. Note that

$$\tau \left(\int_\Gamma v_n^+ + G \right) = \int_\Gamma \mu v_n^- = - \int_\Gamma LV = - \int_\Omega H$$

so that

$$\int_\Gamma v_n^+ = -\frac{1}{\tau} \int_\Omega H - \int_\Gamma G = \frac{-Z(G, H)}{\tau} , \tag{3.2}$$

Z as in (1.3). Thus we expect that,

$$v = A \log |x| + 0(1) \text{ as } |x| \to \infty \tag{3.2}$$

with A then determined by $A = -(2\pi\tau)^{-1} Z(G, H)$.

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We note that the problem (3.1), (3.3) lacks uniqueness, any constant is a solution of the homogenous problem. We can eliminate this by strengthening (3.2) and we arrive at the following problem

$$Lv + H = 0 \text{ in } \Omega , \qquad \triangle v = 0 \text{ in } \Omega^+$$

$$v = A\log|x| + 0(|x|^{-1}) \text{ as } |x| \to \infty \qquad\qquad P_0(F, G, H)$$

$$v^- = v^+ + F , \qquad \mu v_n^- = \tau(v_n^+ + g) \text{ on } \Gamma$$

Theorem 3.1 *There exists at most one solution of $P_0(F, G, H)$*

Proof : Suppose v is a solution of the homogenous problem. The argument preceding shows that A must be 0 so that $v = 0(|x|^{-1})$ as $|x| \to \infty$. Then Green's theorem arguments yield,

$$\int_\Omega |\nabla w|^2 + \int_{\Omega+} |\nabla v|^2 = 0$$

and, since $v \to 0$ as $|x| \to \infty$ $v \equiv 0$.

Problem $P_0(F, G, H)$ admits of a formulation like $\hat{P}_p$. We represent v in Ω^+ by $v = \mathcal{S}_0[\varphi]$ where $\mathcal{S}_0$ is obtained by replacing $K(p|x - y|)$ by $\log|x - y|$. This leads to an analog of P'_p and to a variational formulation. Define, for $\mathcal{U} = (U, \varphi), \mathcal{W} = (W, \psi)$

$$\mathcal{A}_0(\mathcal{U}, \mathcal{W}) = A_0(\mathcal{U}, W) + B_0(\mathcal{U}, \psi),$$

$$A_0(\mathcal{U}, W) = \int_\Omega \mu\nabla U \cdot \nabla\bar{W} - \tau\int_\Gamma \left(\frac{1}{2}\varphi + N_0[\varphi]\right)\bar{W}^- \qquad (3.3)$$

$$B_0(\mathcal{U}, \psi) = \int_\Gamma U^-\bar{\psi} - \int_\Gamma \mathcal{S}_0[\varphi]\bar{\psi} .$$

The appropriate variational problem to consider is

$$\mathcal{A}_0(\mathcal{U}, \mathcal{W}) = \mathcal{F}(\mathcal{W}) \text{ for any } \mathcal{W} \in \mathcal{H} \qquad VP_0(F, G, H)$$

Definition 3.1 *v is a generalized solution of $P_0(F, G, H)$ if $\mathcal{U} = (U, \varphi)$ is a solution of $VP_0(F, G, H)$ and $U = \mathcal{S}_0[\varphi]$ in Ω^+.*

One has the analog of Theorem 2.2.

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Theorem 3.2 *If v is a generalized solution of $P_0(F, G, H)$ and is sufficiently smooth then is a solution of $P_0(F, G, H)$*

We set,

$$m[\varphi] = (2\pi)^{-1} \int_\Gamma \varphi d\sigma \ , \ \delta = \log p + \gamma \tag{3.4}$$

where γ is the constant in (2.1). We can now give our main results.

Theorem 3.3 *i) For any $F \in H_{1/2}, G \in H_{-1/2}$ and $H \in H_1$ there exists a unique generalized solution U_0 of $P_0(F, G, H)$.*
ii) If $\mathcal{U}_0 = (U_0, \varphi_0)$ for the generalized solution we have,

$$m[\varphi_0] = -(2\pi\tau)^{-1} Z(G, H) \tag{3.5}$$

Theorem 3.4 *There exists a $p_0 > 0$ such that for all $p, Re \ p \geq 0, 0 < |p| < p_0$ the generalized solution U_p of $\hat{P}_p(F, G, H)$ satisfies,*

$$U_p = -(2\pi\tau)^{-1} Z(G, H)\delta + U_0 + 0(p^2(\log p)^2) \tag{3.6}$$

where U_0 is as in Theorem 3.3.

Note that U_p has a limit at $p = 0$ if and only if $Z(G, H) = 0$.

We begin the proofs with expansions of the various bilinear functionals. On the basis of (2.1) we have,

$$\begin{aligned} \frac{\mathcal{S}_p[\varphi]}{S_p[\varphi]} &= \frac{\mathcal{S}_0[\varphi]}{S_0[\varphi]} + \delta m[\varphi] + \frac{\tilde{\mathcal{S}}_p[varphi]}{\tilde{S}_p[\varphi]} \\ N_p[\varphi] &= N_0[\varphi] + \tilde{N}_p[\varphi] \end{aligned} \tag{3.7}$$

Here the subscript zero denotes the integral operators with the logarithm as kernel and the tildas denote operators which are of order $p^2 \log p$. These yield, with $\mathcal{U} = (U, \varphi), \mathcal{W} = (W, \psi)$,

$$\mathcal{A}_p(\mathcal{U}, \mathcal{W}) = A_p(\mathcal{U}, W) + B_p(\mathcal{U}, \psi) - 2\pi m[\varphi]\overline{m[\psi]}\delta$$

$$A_p(\mathcal{U}, \mathcal{W}) = A_0(\mathcal{U}, W) + \tilde{A}_p(\mathcal{U}, W) \tag{3.8}$$

$$B_p(\mathcal{U}, \psi) = B_0(\mathcal{U}, \psi) + \tilde{B}_p(\mathcal{U}, \psi)$$

where A_0 and B_0 are as in (3.3) and $\bar{A}_p$ and $\bar{B}_p$ are of order $p^2 \log p$.

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We rewrite $VP_p(F, G, H)$ in the form,

$$A_0(\mathcal{U}, \mathcal{W}) = \mathcal{F}_1(W) - \bar{A}_p(\mathcal{U}, W)$$

$$B_0(\mathcal{U}, \psi) - 2\pi m[\varphi]\overline{m[\psi]}\delta = \mathcal{F}_2(\psi) - \bar{B}_p(\mathcal{U}, \psi) \tag{3.9}$$

The idea is to solve (3.9) by successive approximations for small p. The key result is the following.

Lemma 3.1 *For any $K_1 \in H_1'$ and $K_2 \in H_{-1/2}'$ the problem*

$$A_0(\mathcal{U}, W) = K_1(W) \quad \forall W \; ; \; B_0(\mathcal{U}, \psi) = K_2(\psi) \quad \forall \psi \tag{3.10}$$

has a unique solution $\mathcal{U} = (U, \varphi)$ with

$$m[\varphi] = -(2\pi\tau)^{-1} K_1(1) \tag{3.11}$$

The proof of this result is a little technical and we postpone it until the end of the section. We observe that it yields immediately the proof of Theorem 3.3. We observe also that we will have for the solution (U, φ) of (3.10),

$$U = \mathcal{T}_U[K] \, , \; \varphi = \mathcal{T}_\varphi[K] \tag{3.12}$$

where $\mathcal{T}_U$ and $\mathcal{T}_\varphi$ are bounded linear operators.

Lemma 3.2 *For any $K_1 \in H_1'$ and $K_2 \in H_{-1/2}'$ the problem,*

$$A_0(\mathcal{U}', W) = K_1(W) \quad \forall W$$

$$B_0(\mathcal{U}', W) - 2\pi m[\varphi]\overline{m[\psi]}\delta = K_2(\psi) \quad \forall \psi \tag{3.13}$$

has a unique solution $\mathcal{U}' = (\mathcal{U}', \varphi)$ with

$$\mathcal{U}' = -(2\pi\tau)^{-1}K_1(1) + \mathcal{U} \tag{3.14}$$

where $(\mathcal{U}, \varphi)$ is the solution of (3.10)

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Proof : This is an immediate consequence of Lemma (3.1) (with the formula (3.11)).

We are now ready to prove Theorem 3.4. Let U_0 be the generalized solution of $P_0(F, G, H)$. Thus (U_0, φ_0) is a solution of $VP_0(F, G, H)$ and $U_0 = S_0[\varphi_0]$ in Ω^+. According to (3.5) we will have $m[\varphi_0] = -(2\pi\tau)^{-1}Z(G, H)$. If we set $V_0 = -(2\pi\tau)^{-1}Z(G, H) + U_0$ and put $\mathcal{V}_0 = (V_0, \varphi_0)$ then by Lemma 3.2 we have,

$$A_0(\mathcal{V}_0, W) = \mathcal{F}_1(W)$$

$$(3.15)$$

$$B_0(\mathcal{V}_0, \psi) - 2\pi m[\varphi_0]\overline{m[\psi]} = \mathcal{F}_2(\psi)$$

Let U_p denote the generalized solution of $\hat{P}_p(F, G, H)$. Then $\mathcal{U}_p = (U_p, \varphi_p)$ will be a solution of (3.9) and we will have $U_p = S_p[\varphi_p]$ in Ω^+. Put $U_p = V_0 + \tilde{U}_p$ in Ω and $\varphi_p = \varphi_0 + \tilde{\varphi}_p$ and set $\tilde{\mathcal{U}}_p = (\tilde{U}_p, \tilde{\varphi}_p)$. Then if we substract (3.15) from (3.9) we obtain,

$$A_0(\tilde{\mathcal{U}}_p, W) = J_1(W) - \tilde{A}_p(\mathcal{V}_p, W))$$

$$(3.16)$$

$$B_0(\tilde{\mathcal{U}}_p, \psi) - 2\pi m[\tilde{\varphi}_p]\overline{m[\psi]}\delta = J_2(\psi) - \tilde{B}_p(\mathcal{V}_p, \psi),$$

where

$$J_1(W) = -\tilde{A}_p(\mathcal{V}_0, W) = -p^2 \int_\Omega U_0 \bar{W} + \tau \int_\Gamma \varphi_0 \tilde{N}_p$$

$$(3.17)$$

$$J_2(\psi) = -\tilde{B}_p(\mathcal{V}_0, \psi) = -\int_\Gamma \varphi_0 \tilde{N}_p$$

Let $\mathcal{V}' = (V', \varphi')$ be the solution of

$$A_0(\mathcal{V}', W) = J_1(W)$$

$$B_0(\mathcal{V}', \psi) - 2\pi m[\varphi']\overline{m[\psi]} = J_2(\psi)$$

This exists by Lemma (3.1). Recall that V_0 will be a constant times δ plus a term independent of p and φ_0 will be independent of $p \cdot \tilde{N}_p$ and $\tilde{S}_p$ are both of order $p^2 \log p$. It follows that both J_1 and J_2 will be of order $p^2 \log p$. We can then invoke the estimates (3.14) and (3.12) from Lemmas 3.1 and 3.2 to conclude that for $\mathcal{V}'$,

$$\mathcal{V}' = 0(p^2(\log p)^2) , \quad \varphi' = 0(p^2 \log p)$$

$$(3.18)$$

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We can now apply successive approximations to (3.16) for $0 < |p| \leq p_0$ for some p_0 to conclude that $\mathcal{V}_p = (\tilde{U}_p, \tilde{\varphi}_p)$ satisfies the same estimates (3.18). This yields the estimate 3.6 in Ω. Finally we have in Ω^+

$$U_p + (2\pi\tau)^{-1}Z(G,H)\delta - V_0 = \mathcal{S}_0[\varphi_p] + m[\varphi_p]\delta + (2\pi\tau)^{-1}Z(G,H)\delta$$

$$+ \, \bar{\mathcal{S}}_p\,[\varphi_p] - \mathcal{S}_0[\varphi_o] = \mathcal{S}_0[\tilde{\varphi}_p] + m[\varphi_0]\delta + (2\pi\tau)^{-1}Z(G,H)\delta$$

$$+m[\tilde{\varphi}_p]\delta + \bar{\mathcal{S}}_p\,[\varphi_p] = \mathcal{S}_0[\tilde{\varphi}_p] + m[\hat{\varphi}_p]\delta + \mathcal{T}_p[\varphi_p] = 0(p^2\log p)^2)\,,$$

since $m[\varphi_o] + (2\pi\tau)^{-1}Z(G,H) = 0$. This completes the proof of Theorem 3.4.

Remark 3.1 *For our work in the next section we want a sharpening of (3.6). If one inserts the expansions (2.1) and carries out successive approximations with a little more details one finds*

$$U_p = -(2\pi\tau)^{-1}Z(G,H)\delta + U_0(\mathcal{F}) + p^2(\log p)^2 M_{22} + p^2\log p M_{21} + p^2 M_{20} + 0(p^4(\log p)^2)$$
$$(3.19)$$

Here $U_0(\mathcal{F})$ is the solution in Theorem 3.3. The quantities M_{ij} are bounded linear operators over the data $\mathcal{F}$. They could, in principle, be determined but we will not use their explicit form.

Proof of Lemma 3.1 We need two preliminary results.

Proposition 3.1 *For any Γ there exists constants $k_0 > 0$ and $k_1 > 0$ such that*

$$\int_\Gamma \mathcal{S}_0[\varphi]\bar{\varphi} \geq k_0\|\varphi\|^2_{H_{-1/2}} - k_1 m[\varphi]|^2 \quad \forall\varphi \in H_{-1/2} \qquad (3.20)$$

Proof : It is shown in [5] that if Γ is sufficiently small (mapping radius less than one) there is a $k > 0$ such that $\int_\Gamma \mathcal{S}_0[\varphi]\varphi \geq k\|\varphi\|^2_{H_{-1/2}}$. The result (3.20) follows by a scaling argument.

Proposition 3.2 $\int_\Gamma \dfrac{1}{2}\varphi + N_0[\varphi] = \int_\Gamma \varphi.$

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Proof : Put $u = \mathcal{S}_0[\varphi]$. We have $\triangle u = 0$ in Ω^+ with $u = m[\varphi]\log|x| + 0|x|^{-1})$ as $|x| \to \infty$. We have then

$$\int_\Gamma u_n^+ = \lim_{R \to \infty} \int_{\Gamma_R} u_n = 2\pi m[\varphi]$$

The jump relation (3.4) holds for $\mathcal{S}_0[\varphi]$ also and the result follows.

We are going to treat the problem (3.10) as a compact perturbation of a coercive form. We set, for $\mathcal{U} = (U, \varphi), \mathcal{W} = (W, \psi)$,

$$A_0'(\mathcal{U}, \mathcal{W}) = \int_\Omega \mu \nabla U \cdot \nabla \bar{W} + U\bar{V} - \frac{\tau}{2} \int_\Gamma \varphi \bar{V}^-$$

$$B_0'(\mathcal{U}, \psi) = \int_\Gamma U^- \bar{\psi} - \int_\Gamma \mathcal{S}_0[\varphi]\bar{\psi} + k_1 m[\varphi]\overline{m[\psi]} \tag{3.21}$$

with k_1 as in (3.20). Then we consider the problem,

$$A_0'(\mathcal{U}, \mathcal{W}) = K_1(\mathcal{W}) \, , \quad \frac{\tau}{2} B_0'(\mathcal{U}, \psi) = \frac{\tau}{2} K_2(\psi) \tag{3.22}$$

From (3.21) and Proposition 3.1 we have,

$$Re(A_0'(\mathcal{U}, U) + B_0'(\mathcal{U}, \varphi)) = \int_\Omega \mu \nabla U \cdot \nabla \bar{U} + |U|^2 - Re \int_\Gamma \mathcal{S}_0[\varphi]\bar{\varphi} + k_1|m(\varphi)|^2$$

$$\geq K \left(\|U\|_{H_1}^2 + \|\varphi\|_{H_{-1/2}}^2 \right)$$

for some $K > 0$. It follows that (3.22) has a unique solution.

Next we consider the problem

$$A_0(\mathcal{U}, \mathcal{W}) = K_1(\mathcal{W}) \, , \quad \frac{\tau}{2} B_0'(\mathcal{U}, \psi) = \frac{\tau}{2} K_2(\psi) \tag{3.23}$$

We see that (3.23) and (3.22) differ only by the terms

$$-\int_\Omega U^- \bar{V}^- + \tau \int_\Gamma N_0[\varphi]\bar{V}^-$$

Since N_0 maps $H_{-1/2}$ into $H_{3/2}$ we see that this term represents a compact perturbation. Thus existence for (3.23) is reduced to uniqueness and this we establish now.

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Let $\mathcal{U} = (U, \varphi)$ be a solution of (3.23) for K_1 and K_2 equal to zero. If we put $V = 1, \psi = 0$ on the first equation we find

$$\tau \int_\Gamma \frac{1}{2}\varphi + N_0[\varphi] = K_1(1) = 0 \ ,$$

or, by Proposition (3.2), $m[\varphi] = 0$. Now define U in Ω^+ by $U = \mathcal{S}_0[\varphi]$. We have $\triangle U = 0$ in Ω^+ and $U = 0(|x|^{-1})$ as $|x| \to \infty$. Further we have $U^+ = \mathcal{S}_0[\varphi], U_n^+ = \frac{1}{2}\varphi + N_0[\varphi]$. The second equation in (3.23), with $K_2 = 0$ and $m[\varphi] = 0$ gives $U^- = \mathcal{S}_0[\varphi] = U^+$. Then returning to the first equation we have, with $K_1 = 0$,

$$\int_\Omega \mu \nabla U \cdot \nabla \bar{U} - \tau \int_\Gamma \left(\frac{1}{2}\varphi + N_0[\varphi]\right)\bar{U}^- = 0$$

Elliptic theory yields $LU = 0$ in Ω, $\mu U_n^- = \tau\left(\frac{1}{2}\varphi + N_0[\varphi]\right) = \tau U_n^+$. Thus U is a solution of $P_0(0,0,0)$ and by Theorem 3.1 $U \equiv 0$ in $\mathbf{R}^2$. But $U = \mathcal{S}_0[\varphi]$ in Ω^+ hence we must have $\mathcal{S}_0[\varphi] \equiv 0$ in Ω^+. We assert this implies $\varphi \equiv 0$. Indeed if we consider $\mathcal{S}_0[\varphi]$ in Ω it must be identically zero since $\mathcal{S}_0[\varphi]^- = \mathcal{S}_0[\varphi]^+ = 0$. Then the jump relation (2.4) implies $\varphi = 0$. Thus we have uniqueness and can infer existence for (3.23).

Finally we consider the problem,

$$A_0(\mathcal{U}, W) = K_1(W)$$

$$B_0'(\mathcal{U}, \psi) = K_2(\psi) - k_1(2\pi\tau)^{-1}K_1(1)\overline{m(\psi)}$$

(3.24)

This problem has a solution. Moreover if we put $V = 1$ in the first equation we find $m[\varphi] = -(2\pi\tau)^{-1}K_1(1)$ so that the second term on the right of (3.24) cancels the extra term in B_0' and (3.24) is equivalent to (3.10) and we see also that (3.11) holds.

4   Remarks on long time asymptotics

The analysis in this section is very formal for the reason indicated in the introduction. We believe however, that the results are correct.

We want to study approach to steady state and we first make precise our assumptions on the data. We assume,

$$f(x,t) = f_\infty(x) + \tilde{f}(x,t), g(x,t) = g_\infty(x) + \tilde{g}(x,t), h(x,t) = h_\infty(x) + \tilde{h}(x,t)$$

(4.1)

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$\tilde{f}$ is to be such that the maps $t \to t^k \tilde{f}(\cdot, t)$ are in $L_1(0, \infty : H_{1/2}(\tau))$ for $k = 0, 1, 2$. It will follow that f will have a Laplace transform F which satisfies,

$$F = f_\infty p^{-1} + F_0 + F_1 p + 0(p^2) \text{ as } p \to 0. \tag{4.2}$$

Here,

$$F_0(x) = \int_0^\infty \tilde{f}(x, t)dt \,, \quad F_1(x) = \int_0^\infty t\hat{f}(x, t)dt \tag{4.3}$$

Analogous results are to hold for $\tilde{g}$ and $\tilde{h}$. It follows that the functional $\mathcal{F}$ in (2.7) has the form,

$$\mathcal{F}(p) = \mathcal{F}_{-1}p^{-1} + \mathcal{F}_0 + \mathcal{F}_1 p + O(p^2) \text{ as } p \to 0. \tag{4.4}$$

From (4.3), and its analogs for $\tilde{g}$ and $\tilde{h}$, together with (2.7), we have

$$\mathcal{F}_{-1}(\mathcal{W}) = \mathcal{F}^1_{-1}(W) + \mathcal{F}^2_{-1}(\psi) \,; \quad \mathcal{F}^2_{-1}(\psi) = \int_\Gamma f_\infty$$

$$\mathcal{F}^1_{-1}(\mathcal{W}) = \int_\Omega h_\infty \bar{W} + \tau \int_\Gamma g_\infty W^-$$

$$\mathcal{F}_0(\mathcal{W}) = \mathcal{F}^1_0(W) + \mathcal{F}^2_0(\psi) \,, \quad \mathcal{F}^2_0(\psi) = \int_\Gamma \left(\int_0^\infty \tilde{f}(x, t)dt \right) \bar{\psi}(x)d\sigma$$

$$\mathcal{F}^1_0(W) = \int_\Omega \left(\int_0^\infty \tilde{h}(x, t)dt \right) \bar{W}(x)dx + \tau \int_\Gamma \left(\int_0^\infty \tilde{g}(x, t)dt \right) W(x)d\sigma \tag{4.5}$$

We substitute the estimates (4.2) into (3.19) . We decompose the result as

$$U_p = U_p^1 + U_p^2 \,,$$

$$U_p^1 = -(2\pi\tau)Z(g_\infty, h_\infty)p^{-1}\delta + U_0(\mathcal{F}_{-1})p^{-1} - (2\pi\tau)^{-1}Z(G_0, H_0)\delta$$

$$U_p^2 = R_{12}p(\log p)^2 + R_{11}p\log p + R_{10}p + R_{20}p^2 + 0(p^3(\log p)^2) \tag{4.5}$$

Theorem 4.1 *The function U_p^1 is the Laplace transform of,*

$$u^1 = (2\pi\tau)Z(g_\infty, h_\infty)\log 2t + U_0(\mathcal{F}_{-1}) + (2\pi\tau)^{-1}Z(G_0, H_0)\chi(t) \tag{4.6}$$

where $\chi(t) = 0$ for $t < \frac{1}{2}, \chi(t) = t^{-1}$ for $t > \frac{1}{2}$.

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Proof : We have only to identify the quantities $p^{-1}\delta, p^{-1}$ and δ. p^{-1} is the transform of 1 which identifies the second term in (4.6). For the first we have,

$$\int_0^\infty e^{-pt} \log 2t \, dt = p^{-1} \int_0^\infty e^{-\eta} (\log 2\eta - \log p) d\eta = -\delta p^{-1}$$

For the third we have,

$$\int_{1/2}^\infty e^{-pt} t^{-1} dt = e^{-pt} \log 2t \Big|_{1/2}^\infty + p \int_0^\infty e^{-pt} \log 2t \, dt = -\delta$$

So far our results are rigorous. Now, however, we wish to invoke the complex inversion formula for the Laplace transform. Thus we assume the solution u_p of $P(f, g, h)$ can be recovered from U_p by

$$u_p(x, t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{pt} U_p(x, p) dp \qquad (4.7)$$

It is here and in the calculations below that we would need estimates for U_p for large p. Let us proceed formally.

We insert (4.5) into (4.7). We can identify the term involving U_p^1 as u^1. U_p^2 has a limit at Re $p = 0$ so we take $c = 0$ in (4.7) for that term. Thus our formal calculation gives $u_p(x, t) = u^1(x, t) + u^2(x, t)$ where,

$$u^2(x, t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{i\eta t} U_p^2(x, i\eta) d\eta \qquad (4.8)$$

We show, formally, that $u^2(x, t) = O\left(\frac{\log t}{t^2}\right)$ as $t \to \infty$. As a first step we deal with the terms $p(\log p)^2$ and $p \log p$ in (4.5). We have,

$$\begin{aligned}
\int_1^\infty e^{-pt} t^{-2} dt &= -t^{-1} e^{-pt}\Big|_1^\infty - p \int_1^\infty e^{-pt} t^{-1} dt \\
&= e^{-p} - pe^{-pt} \log t\Big|_1^\infty - p^2 \int_1^\infty e^{-pt} \log t \, dt \\
&= e^{-p} - p^2 \int_0^\infty e^{-pt} \log t \, dt + p^2 \int_0^p e^{-pt} \log t \, dt \\
&= -p \log p + \alpha + \beta p + \gamma p^2 + 0(p^3) \text{ as } p \to 0 ,
\end{aligned} \qquad (4.8)$$

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for some α, β, γ. Also

$$
\begin{aligned}
\int_1^\infty e^{-pt}\left\{t^{-2} - t^{-2}\log t\right\} &= \int_1^\infty e^{-pt}(t^{-1}\log t)'dt = p\int_1^\infty e^{-pt}(t^{-1}\log t) \\
&= pe^{-pt}\frac{(\log t)^2}{2}\Big|_1^\infty + \frac{p^2}{2}\int_1^\infty e^{-pt}(\log t)^2 dt \\
&= \frac{p^2}{2}\int_0^\infty e^{-pt}(\log t)^2 dt - \frac{p^2}{2}\int_0^1 e^{-pt}(\log t)^2 dt \\
&= -\frac{p}{2}\log p\int_0^\infty e^{-\eta}\log\eta\, d\eta + \frac{p}{2}(\log p)^2 \\
&\quad + \alpha' + \beta'p + \gamma'p^2 +)(p^3) \text{ as } p \to 0.
\end{aligned}
$$

(4.9)

The conclusion from (4.8) and (4.9) is this. If we set

$$
\gamma_1(t) = \begin{matrix} 0 & t < 1 \\ \\ -t^{-2} & t > 1 \end{matrix} \; ; \; \gamma_2(t) = \begin{matrix} 0 \\ \\ t^{-2}\log t - t^{-2} \end{matrix} + \frac{1}{2}\left(\int_0^\infty e^{-\eta}\log\eta\, d\eta\right)\gamma_1(t) \begin{matrix} t < 1 \\ \\ t > 1 \end{matrix}
$$

then the transforms $\Gamma_1(p)$ and $\Gamma_2(p)$ of γ_1 and γ_2 satisfy,

$$
\Gamma_1(p) = p\log p + \tilde\Gamma_1(p), \Gamma_2(p) = p(\log p)^2 + \tilde\Gamma_2(p)
$$

where $\tilde\Gamma_1$ and $\tilde\Gamma_2$ have continuous derivatives of order less than or equal to two in Re $p \geq 0$.

Thus we can further decompose U_p^2 into $U_{p,1}^2 + U_{p,2}^2$ where $U_{p,1}^2 = a\Gamma_1(p) + b\Gamma_2(p)$, for some constants a and b and $U_{p,2}^2$ is twice continuously differentiable in $Rep \geq 0$. We have then, formally,

$$
u^2(x,t) = a\gamma_1(t) + b\gamma_2(t) + (2\pi)^{-1}\int_{-\infty}^{+\infty} e^{i\eta t}U_{p,2}^2(x,i\eta)d\eta \qquad (4.9)
$$

We assume that we can integrate by parts twice and obtain,

$$
(2\pi)^{-1}\int_{-\infty}^{+\infty} e^{i\eta t}U_{p,2}^2(x,i\eta)d\eta = (-2\pi)^{-1}t^{-2}\int_{-\infty}^{+\infty} e^{i\eta t}\frac{\partial^2}{\partial p^2}U_{p,2}^2(x,i\eta)d\eta.
$$

This is of order t^{-2} and we obtain the result in (1.5). Once again we emphasize this result would require some detailed information about the behavior of U_p for large p.

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