

Modeling the Price Dynamics of Catastrophe Bonds

Submitted by: D. Scott Taylor

Undergraduate Economics Program
Tepper School of Business
Carnegie Mellon University

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Advisor: Benoit Morel

Benoit Morel
Associate Teaching Professor
College of Humanities and Social Sciences
Carnegie Mellon University

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D. Scott Taylor

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1 Introduction

Catastrophe bonds and other classes of insurance linked securities, whose payoffs are tied to the occurrence of property catastrophes, provide primary insurers and reinsurers a mechanism by which they can securitize and transfer some of their systematic risk to the market. At the same time, catastrophe bonds provide capital markets investors the unique opportunity to enhance the optimality of their portfolio by purchasing instruments with lucrative returns and a low degree of covariability with financial markets.

While these instruments are theoretically very attractive from the perspectives of both insurance companies and investors, securitized insurance products have remained marginal compared to more traditional forms of risk financing, i.e. reinsurance and retrocession. A possible explanation for the marginality and exoticness of insurance linked securities is that there has been limited public discussion about the pricing and valuation of these instruments relative to other structured financial products, e.g. mortgage backed securities and collateralized debt obligations.

While specialist hedge funds and broker-dealers, with expertise in both structured finance and reinsurance, are able to create sophisticated pricing models for catastrophe bonds, we propose that a lack of public discussion about the valuation of insurance linked securities has created an educational barrier to entry for non-specialist investors, e.g. pension funds and mutual funds. With

the hope of making catastrophe bonds more accessible to a non-specialist audience, this paper will extend well-known methods from credit and default risk analysis to build a tractable pricing model for catastrophe bonds.

The paper is organized as follows. Section 2 discussed methods of risk financing used in insurance markets. In this section we juxtapose traditional risk financing using reinsurance and alternative risk financing using securitization. Section 3 outlines the proposed pricing methodology for catastrophe bonds. This sections has been broken into various sections to highlight the various features of the model. The discussion of pricing begins by providing a theoretical introduction to interest rate modeling, and the pricing of defaultable zero-coupon and coupon bonds in a reduced-form setting. We then specify the models definition of default to provide a more economically intuitive notion of default in the context of a catastrophe bond. We subsequently speak to the filtration of information available to investors over time and potential uncertainty associated with recoverables given default. Lastly, we extend our model to allow for principal-dependent coupon payments to investors.

2 Risk Financing in Insurance Markets

The traditional mechanism transferring and managing risks in the insurance industry is reinsurance. That being said, it is important to note that in recent

years securitized financial products like insurance linked bonds, options, and swaps have been introduced to the market as a substitute for traditional reinsurance. While the focus of this paper is on the pricing of insurance linked securities, specifically catastrophe bonds; this section will provide an overview of both the advantages and disadvantages of traditional reinsurance and insurance securitization.

2.1 Traditional Risk Financing with Reinsurance

The traditional and predominant method of risk transfer and management in the insurance industry is the risk warehouse, i.e. primary insurers and reinsurers underwrite risks from the economy and silo them in their portfolio of assumed risks. To specify the concept of risk warehousing further, we note that while most insurance companies provide both risk diversification and risk management services to their clients, they do not pass their assumed risks to capital markets, rather they hold them internally on balance sheet.

Before discussing alternative forms of risk financing, e.g. securitization, it is important to describe traditional risk transfer methods used in the insurance industry in detail. In an economy, businesses and individuals exposed to insurable risks may choose to hedge their exposures to these risks by transferring them to a primary insurance company. These hedgers can pay premiums to a primary insurer in exchange for the promise that they will be indem-

nified, i.e. reimbursed, for their potential losses incurred by pre-determined insurance events.

A primary insurer provides a risk warehousing function in the economy since these firms often choose to retain the majority of their assumed insurance risks on their balance sheet. Since primary insurance companies provide coverage to many businesses and individuals in the economy, whose risks can be assumed to be mostly independent, primary insurers are able to reduce their exposures substantially through diversification.¹ This result follows as a result of the weak law of large numbers which is given by,

$$\lim_{n \rightarrow \infty} \Pr \left(\left| \bar{X}_n - \mu \right| < \epsilon \right) = 1, \quad (1)$$

where \lim denotes the limit operator, \Pr denotes the probability operator, \bar{X}_n denotes the sample average, μ denotes the true expected value, and ϵ denotes an arbitrary positive number.

While primary insurers are able to substantially reduce their risk via diversification, it is important to note that as a result of non-zero covariability between assumed risks, there will be some residual, i.e. systematic, risk that remains on a primary insurer's balance sheet. This residual risk can often be attributed to natural disasters, which can cause sizable losses to primary insurer's capital base. To limit a primary insurer's insolvency risk, the firm

¹Cummins, J. David and Trainar, Philippe, Securitization, Insurance, and Reinsurance. *Journal of Risk and Insurance*, Vol. 76, Issue 3, pp. 467, September 2009.

may choose to transfer some of its residual risk to a reinsurer in return for a percentage of its received premiums.

A traditional reinsurer may warehouse residual risks of a primary insurer and then can diversify their portfolio of exposures by issuing policies to many primary insurers from various different geographic regions. We note that reinsurers provide many types of reinsurance coverage to primary insurers, which in the context of reinsurance terminology are called ceding companies or cedants. Reinsurers can diversify their exposures further by providing both proportional and non-proportional reinsurance coverages and by underwriting multiple lines of insurance risks, in addition to diversifying their exposures geographically.

Traditional reinsurance provides an important second layer of risk warehousing for the economy and helps keep the premiums of individual hedgers low and the insolvency risk of primary insurers within pre-calculated margins. We propose that reinsurance is an intuitive mechanism to transfer the residual risk of primary insurers since the insurance policies of individual hedgers are not sufficiently liquid or transparent to be transferred directly to capital markets. Additionally, we note that it would be prohibitively expensive to transfer the policies of individual hedgers to capital markets due to transactional and other frictional costs.

While it is likely to be infeasible to transfer the policies of individual hedgers, as a result of technological advances in structured finance, e.g. the establish-

ment of collateralized debt obligations and credit default swaps as mainstays in the financial landscape, primary insurers and reinsurers are now able to securitize and transfer some of their residual risks to capital markets. That being said, before discussing insurance securitization as an alternative form of risk financing in detail, we propose it is a good idea to summarize some of the advantages and disadvantages of traditional reinsurance.

Primary insurance and reinsurance entails the pooling of insurance risks which are subsequently warehoused, managed, and diversified. We note that the typical association between insurance companies and capital markets is via the issuance of common stock or debt to investors. By reaping the benefits of the weak law of large numbers, risk diversification and pooling enables reinsurance companies to realize a substantial degree of risk reduction.

While the risk warehousing function of traditional reinsurance generates many important market efficiencies, it is important to note that this risk transfer mechanism also has many associated disadvantages. Reinsurance treaties held on balance sheet are often opaque to capital markets, making it difficult for investors to appropriately assess the associated risks of a reinsurer's assumed risks. This opacity creates an informational asymmetry between a reinsurance company's management and their investors, raising the cost of capital.²

²Cummins, J. David and Trainar, Philippe, *Securitization, Insurance, and Reinsurance*. Journal of Risk and Insurance, Vol. 76, Issue 3, pp. 473, September 2009.

Fluctuations in capital costs and global reinsurance capacity may provide a partial explanation of the reinsurance underwriting cycle, which is a source of noted inefficiency in the insurance and reinsurance market. Additionally, the effectiveness of risk warehousing and diversification diminishes when dealing with covarying risks or risks linked to natural disasters, which can create capacity impairing shocks to a reinsurance industry's capital base. While risk warehousing is effective means of risk management for primary insurers, it is unclear whether or not it is an appropriate response when dealing with covarying risks or natural disasters. For this reason, we now will introduce insurance securitization as a means of helping reinsurers manage their insolvency risk.

2.2 Risk Financing with Securitization

As noted above, the effectiveness of traditional reinsurance as a risk warehouse diminishes when insurance risks are covarying or linked to natural catastrophes. Additionally since retained risks are often opaque to investors, informational asymmetries may arise which can raise a reinsurer's cost of capital. In the aftermath of a large insurance event, the capacity of the global reinsurance market may be substantially impaired, potentially making the price of reinsurance prohibitively high. This paper proposes that insurance securitization can help resolve some of these market inefficiencies in several ways.

First, we propose that residual or systematic risks within insurance markets are orthogonal, i.e. uncorrelated, with other risks in financial markets. While the risk of natural catastrophes like hurricanes or earthquakes can create risk covariability within the reinsurance industry, these risks are mostly unassociated with the economic forces that move the securities market. It follows that if these risks can be transferred to the securities market, it may be possible to reduce the covariability loading in insurance premiums used to hedge a reinsurers insolvency risk.³ Also, the low covariability with the securities market makes securitized insurance products attractive to investors since they can serve as a portfolio diversifier.

While there are many different types of securitized insurance products, this paper will focus its efforts on insurance linked bonds, specifically those bonds tied to property catastrophe risk, which have been among the most successful securitization structures to date. A catastrophe bond transaction is typically initiated by a sponsoring primary insurer or reinsurer, who establishes a special purpose vehicle in a tax-favorable domicile to help them transfer some of their residual risk to the capital markets.

A special purpose vehicle is a free-standing financial entity that is purposefully isolated from the balance sheet of the sponsoring primary insurer or reinsurer. The special purpose vehicle raises capital by issuing debt to private investors; these funds are then held in trust on behalf of the sponsor

³Cummins, J. David and Trainar, Philippe, *Securitization, Insurance, and Reinsurance*. Journal of Risk and Insurance, Vol. 76, Issue 3, pp. 473, September 2009.

and invested in stable securities like U.S. Treasury bonds or a money market account. The special purpose vehicle then enters a reinsurance transaction with the sponsor, which entails the agreement to release the collateral held in the trust to the sponsor on the occurrence of a pre-defined insurance event, typically a natural disaster.

In return for bearing the risk of possibly losing part or all of the catastrophe bond's principal, in the event of large losses to the sponsor, investors are compensated with a spread or risk premium which can range anywhere from 5 to 22 percent.⁴ It follows that if the principal held in trust by the special purpose vehicle is not released during the bonds risk period, the total return to investors is equal to the return of the collateral in the trust plus the accrued spread over the risk period. We note that the spread is paid to investors by the sponsor in return for reinsurance coverage. It is important to highlight the differences between the model of the risk warehouse, which internally diversifies risk, and insurance securitization, which externalizes and transfers residual risks directly to capital markets.

The comparative pricing of catastrophe bonds and property catastrophe reinsurance treaties is challenging for many reasons. First, we note that catastrophe bonds have multi-year risk periods, in contrast to traditional reinsurance which are typically for 1 year. Second, catastrophe bonds are fully collateralized financial instruments and resultantly have a lower degree of

⁴Lane, Morton N., and Roger Beckwith. "Annual Review for the Four Quarters, Q2 2009 to Q1 2010." Trade Notes (2009): 8. Web. 12 May 2010.

counterparty risk than most reinsurance transactions where indemnification is promised on a best efforts basis. Third, most reinsurance transactions include reinstatement provisions, whereas catastrophe bonds do not. Having provided an introduction to both traditional risk financing with reinsurance and alternative risk financing with securitization, we will turn our focus to the pricing of and modeling of catastrophe bonds in a modified reduced-form setting.

3 Pricing Methodology

As mentioned above, the comparative pricing of catastrophe bonds and excess-of-loss reinsurance treaties is difficult for many reasons. That being said, we propose it is possible to utilize existing methods, borrowed from reduced-form credit risk analysis, to model the price dynamics of catastrophe bonds over time. With the hope of making our paper as accessible as possible to the reader, we introduce the features of our pricing model in various subsections. First, we provide a general overview of interest rate mathematics for the purposes of discounting. Second, we introduce the central building block of our pricing model, the defaultable zero-coupon bond. In the subsequent subsections, we extend our framework to treat defaultable coupon bonds, defaultable coupon bonds with uncertain recovery, and defaultable coupon bonds with principal-dependent coupons. Additionally, we specify the nota-

tion of our pricing model to provide an economically intuitive interpretation of default in the context of catastrophe bonds. The objective of this section is to provide an overview of some of the methods used in reduced-form credit risk modeling and to show how these techniques can be extended to model the price dynamics of catastrophe bonds and other insurance linked securities.

3.1 Interest Rate Modeling

Speaking to the basics of interest rate modeling, we firstly consider a loan promising to pay P dollars at the maturity time T . If the issuer of the loan is default-free, then pricing can proceed using a conventional default-free term-structure model. We note that such term-structure models are usually based on the notion of some short-rate process r_t , whose stochastic behavior is modeled under risk-neutral probability assessments. To be concise, by risk-neutral probabilities we are speaking about probability assessments under which the market value of a security is calculated as the expectation of the discounted present value of its cash flows, using the compounded short interest rate for discounting purposes.⁵

We will highlight the basics of interest rate models with an example. If the short interest rate process changes only at discrete time intervals of length

⁵Duffie, Darrell, and Kenneth J. Singleton. *Credit Risk: Pricing, Measurement, and Management*. 2. Princeton, NJ: Princeton University Press, 2003. 101. Print.

1, the value of a default-free zero-coupon bond maturing at time T , with promised payoff of P at maturity, has a price at time t given by,

$$\begin{aligned}\delta_1(t, T) &= E_t^* \left[P \cdot e^{-r_t} \cdot e^{-r_{(t+1)}} \dots e^{-r_{(T-1)}} \right] \\ &= E_t^* \left[P \cdot e^{-[r_t + r_{(t+1)} \dots r_{(T-1)}]} \right],\end{aligned}\tag{2}$$

where E_t^* denotes risk-neutral expectation assessed at time t . We note that there are computational advantages, particularly when working with default times, to modeling in a continuous-time setting. In continuous-time, the analogue to the equation (2) is

$$\delta_2(t, T) = E_t^* \left[P \cdot e^{-\int_t^T r_s ds} \right],\tag{3}$$

where $e^{-\int_t^T r_s ds}$ is the short interest rate process, and P is the bond's principal.

Referencing Duffie and Singleton (1999), it is important to point out that risk-neutral probability assessments exist under extremely weak no-arbitrage conditions.⁶ When pricing default-free securities, these probabilities are often specified so as to make the computation of the expectation in equation (3) more manageable. When markets are not financially complete, like the case of catastrophe bonds or other credit linked instruments, it is possible that

⁶Duffie, Darrell, and Kenneth J. Singleton. Credit Risk: Pricing, Measurement, and Management. 2. Princeton, NJ: Princeton University Press, 2003. 101. Print.

the risk-neutral probability assessments associating with the valuation of a security may be non-unique. For this reason, there may be many sets of risk-neutral probabilities which are consistent with the prevailing prices of securities in the market.⁷

Duffie and Singleton write that whether one is pricing in a complete or an incomplete market setting, the knowledge of risk-neutral probabilities is usually not enough information to fit a credit risk model to historical data.⁸ This follows because the behavior of a security's observed prices over time reflect the actual, i.e. real world, likelihoods that create a price history. Thus, when one hopes to both price and model the risks associated with a security, it is typically necessary to specify both the risk-neutral and actual probabilities.

3.2 Zero-Coupon Defaultable Bonds

If the security of interest may default prior to the maturity time T , then in addition to the risk of changes in the short interest rate process r_t , both the magnitude and timing of the security's payoff to investors may be uncertain. Being mindful of this additional uncertainty, we can structure a defaultable zero-coupon bond as a portfolio of two securities:

⁷Duffie, Darrell, and Kenneth J. Singleton. *Credit Risk: Pricing, Measurement, and Management*. 2. Princeton, NJ: Princeton University Press, 2003. 102. Print.

⁸Duffie, Darrell, and Kenneth J. Singleton. *Credit Risk: Pricing, Measurement, and Management*. 2. Princeton, NJ: Princeton University Press, 2003. 102. Print.

- A security that pays P at time T if and only if the bond survives to maturity.
- A security that pays $P \cdot W$ at time T if and only if the bond defaults prior to maturity.

To clarify our notation, W denotes the bond's random recovery rate given default which is bounded between 0 and 1, i.e. $W \in [0, 1]$. Additionally, we define $1_{\{\tau > t\}}$ as the indicator of the event $\tau > t$, which has an outcome of 1 if the issuer has not defaulted prior to time t and an outcome of 0 otherwise.

Expressed concisely,

$$1_{\{\tau > t\}} = \begin{cases} 1 & \text{if } \tau > t, \\ 0 & \text{if } \tau \leq t \end{cases}. \quad (4)$$

It follows that if the security of interest has not defaulted by time t , the associated risk-neutral probability of survival is given by,

$$E_t^* [1_{\{\tau > t\}}] = p^*(t, T). \quad (5)$$

We note that as probabilities assessments must sum to 1, the security's associated risk-neutral conditional probability of default is given by,

$$E_t^* [1_{\{\tau \leq T\}}] = 1 - p^*(t, T). \quad (6)$$

Having specified this paper's notation, the price of a defaultable zero-coupon bond, assessed at time t , can be expressed as,

$$\begin{aligned} b_0(t, T) = & E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot 1_{\{\tau > t\}} \right] \\ & + E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot W \cdot 1_{\{\tau \leq T\}} \right]. \end{aligned} \quad (7)$$

where $e^{-\int_t^T r_s ds}$ is the continuously compounded short interest rate process, P is the bond's principal paid to investors the maturity time T , and W is the random recovery rate given default.⁹

3.3 Defaultable Coupon Bonds

In return for bearing the risk of possibly losing part or all of the bond's principal in the event of default, investors are compensated with coupon payments. For this reason, it is natural to extend equation (7) to include the coupons paid to investors. At this time, we will make these assumptions about the nature of the coupon payments:

- Coupons are paid to investors continuously between the bond's inception time, $t = 0$, and maturity time, $t = T$.
- Coupon payments are paid to investors until the maturity time T , irrespective of whether or not the security has defaulted.

⁹Duffie, Darrell, and Kenneth J. Singleton. *Credit Risk: Pricing, Measurement, and Management*. 2. Princeton, NJ: Princeton University Press, 2003. 102. Print.

- Coupon payments are paid to investors with a fixed interest rate, expressed as a percentage of par.

Having clarified this paper's assumptions surrounding coupon payments, we can structure a defaultable coupon bond as a portfolio of three securities:

- A security that pays P at time T if and only if the bond survives to maturity.
- A security that pays $P \cdot W$ at time T if and only if the bond defaults prior to maturity.
- A security that continuously pays coupons to investors between the security's inception $t = 0$ and maturity $t = T$.

As in the case of the zero-coupon bond, we set $1_{\{\tau > t\}}$ be the indicator of the event $\tau > t$, which has outcome 1 if the issuer has not defaulted prior to time t and zero otherwise. Subsequently, the price of a defaultable coupon bond, assessed at time t , is given by,

$$\begin{aligned}
 b_1(t, T) &= E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot 1_{\{\tau > t\}} \right] \\
 &+ E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot W \cdot 1_{\{\tau \leq T\}} \right] \\
 &+ E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \int_t^T C du \right], \tag{8}
 \end{aligned}$$

where $e^{-\int_t^T \tau_s ds}$ is the continuously compounded short interest rate process, P is the bond's principal paid to investors the maturity time T , W is the random recovery rate given default (expressed as a percentage of par), and C is the coupon rate (expressed as a percentage of par).

3.4 Default in the Context of Catastrophe Bonds

The prior two subsections provide a brief introduction to the techniques and notation used in reduced-form credit risk modeling. Interested readers are referred to Duffie and Singleton (1999) and Schönbucher (2003) for a more comprehensive and technically rigorous discussion of reduced-form models.

It is important to note that equations (7) and (8) are to be used for the pricing of defaultable bonds in a theoretical setting. Thus far, the default time τ has been presented in this paper as an abstract probabilistic process. To improve our pricing model for catastrophe bonds, we will specify the default time τ in greater detail with the hope of providing a more economically intuitive interpretation of default in the context of catastrophe bonds and reinsurance transactions.

Before discussing our approach in detail, we note that there are many sophisticated techniques used in reduced-form credit risk analysis which can be used to model a security's default time. When pricing credit-linked instruments in a reduced-form setting, the distribution of default probabilities

and timings is typically poorly defined. For this reason, many default timing models are based on the notion of the arrival intensity of default. The simplest version of such a model defines a security's default time as the first arrival time τ of a Poisson process with some constant mean arrival rate, called intensity, which is often denoted by λ . In this specific class of default timing model:

- The security's probability of survival to time t is $p^*(t, T) = e^{-\lambda t}$, i.e. the time to default is an exponentially distributed variable.
- The expected time to default is $\frac{1}{\lambda}$.

Relaxing the assumption of a constant mean arrival rate, we are led to the doubly-stochastic model of default. It follows that conditional on the information given by the path of intensity $\lambda(t)$, the security's default time is modeled by a Poisson process with a time-varying intensity. In this setting, given the security has survived to time t , the conditional survival probability is given by,

$$\begin{aligned} p^*(t, T) &= E_t^* \left[1_{\{\tau > t\}} \right] \\ &= E_t^* \left[e^{-\int_t^T \lambda(s) ds} \right]. \end{aligned} \tag{9}$$

Reader's interested in learning more about the pricing of catastrophe bonds

using intensity-based models or default are referred to Burnecki, Kukla, and Taylor (2005). We note that there is no relation between the aforementioned Taylor and the author of this paper.

While the use of intensity-based models of default are appropriate for the pricing of most credit-linked instruments, we propose they are inappropriate for the pricing of catastrophe bonds as they do not speak to the default triggering mechanism of catastrophe bonds. Referencing subsection 2.2, we know that a catastrophe bond's principal can only be released from a special purpose vehicle on the occurrence of a pre-defined catastrophic event. Specifically, the losses sustained by the sponsor during the life of the bond must be greater than or equal to the attachment point of the reinsurance treaty. With this knowledge, we can then express the catastrophe bond's default time as follows,

$$\tau = \inf \{t \in [0, T] : A \leq L_t\}, \quad (10)$$

where τ is the bond's time of default, L_t are the losses sustained by the sponsor at time t , $t \in [0, T]$ is the risk period of the bond, A is the point of attachment for the reinsurance treaty, and \inf denotes the infimum operator.

Equation (10) is an important result as it provides an economically intuitive interpretation of default in the context of catastrophe bonds. Additionally, equation (10) facilitates a focal shift from the simulation of the default time

τ , to the modeling of the probability that the losses to the sponsor will exceed the catastrophe bond's point of attachment, i.e. $\Pr(L_t \geq A)$. For the sake of clarity, we will define the sponsor's exceedance curve as follows,

$$S(X) = \Pr(L_t \geq X) \quad (11)$$

where S is the survivorship function, X is an arbitrary loss threshold, L_t is the loss sustained by the sponsor at time t , and \Pr denotes the probability operator. Based on the payout structure of a catastrophe bond, described in (??), we know that the security will have survived to time t if and only if $L_t < A$. Conversely, the catastrophe bond will have defaulted if the losses sustained by the sponsor are greater than or equal to the point of attachment, i.e. $A \leq L_t$.

3.5 Recovery Modeling

In equations (7) and (8), W indicates the random recovery rate received by investors at default time τ . Thus far, we have made few explicit assumptions about the characteristics of a catastrophe bond's recoverables given default. To add a layer of sophistication our pricing model, we can specify the catastrophe bond's recovery regime in greater detail to account for and the filtration of relevant information available to investors and for potential uncertainty surrounding the recovery amount.

In reduced-form credit risk analysis, it is possible that the random recovery rate W , may only be revealed at the default time τ . For the purposes of pricing catastrophe bonds prior to default, one can replace W with $E_t^*[W | \mathcal{F}_{\tau-}]$ where $\mathcal{F}_{\tau-}$ denotes all of the information known by investors in the market up to, but not including the default time τ .¹⁰ Referencing Duffie and Singleton (1998) and Schönbucher (1998), it can be shown that there is a process ω , which is the conditional risk-neutral expected recovery in the event of default at time t , based on all of the information available up to but not including time t .¹¹ Under certain technical conditions, we can then simplify the bond pricing calculation to replace W with ω_t . The important distinction here is that ω_t is known based on the information available to investors at time t .

It is important to acknowledge the potential uncertainty associated with the conditional risk-neutral expected recovery ω_t . As in the case of the random recovery W , the conditional risk-neutral expected recovery rate ω_t , is a fraction of the par value of the bond and is bounded between 0 and 1, i.e. $\omega_t \in [0, 1]$. For this reason, it is intuitive to model a catastrophe bond's conditional risk-neutral recovery density using the beta family of distributions, which are conveniently defined on the interval $(0, 1)$ and parameterized by two positive shape parameters typically denoted by α and β .¹² If we assume

¹⁰Duffie, Darrell, and Kenneth J. Singleton. Credit Risk: Pricing, Measurement, and Management. 2. Princeton, NJ: Princeton University Press, 2003. 131. Print.

¹¹Duffie, Darrell, and Kenneth J. Singleton. Credit Risk: Pricing, Measurement, and Management. 2. Princeton, NJ: Princeton University Press, 2003. 131. Print.

¹²Schönbucher, Phillip J. Credit Derivatives Pricing Models: Models, Pricing, Implementation. 1. London: Wiley Finance, 2003. Print.

that a catastrophe bond's recovery density is a beta distributed variable, the recovery density is given by,

$$f(\omega_t; \alpha, \beta) \sim \frac{(\omega_t)^{\alpha-1} (1 - \omega_t)^{\beta-1}}{B(\alpha, \beta)} \quad (12)$$

where the beta function, $B(\cdot)$, denotes the normalization constant to insure that the probability distribution integrates to unity. We can then re-write equation (11) to account for the filtration of information available to investors \mathcal{F}_t and for potential uncertainty surrounding the recovery amount, as follows,

$$\begin{aligned} b_3(t, T) &= E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot 1_{\{\tau > t\}} \mid \mathcal{F}_t \right] \\ &+ E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \omega_t \cdot 1_{\{\tau \leq T\}} \mid \mathcal{F}_t \right] \\ &+ E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \int_t^T C du \mid \mathcal{F}_t \right] \end{aligned} \quad (13)$$

where $e^{-\int_t^T r_s ds}$ is the short interest rate process integrated over time, P is the bond's principal paid to investors the maturity time T , ω_t is the conditional risk-neutral expected recovery assessed at time t (expressed as a percent of par), and C is the coupon rate (expressed as a percent of par).

3.6 Principal-Dependent Coupons

Thus far, we have explicitly assumed the coupon of catastrophe bonds are paid to investors are continuously accrued with a fixed interest rate, expressed as a percentage of par. While this is a valid theoretical assumption, in practice, we note that the coupons of catastrophe bonds are paid to investors as a fixed percentage of the bond's outstanding principal in the special purpose vehicle. That is to say, in the event of a default event or partial write down, coupons are paid to investors as a fixed percentage of the recoverables. We can modify our model in equation (13) to allow for the principal-dependent coupons as follows,

$$\begin{aligned}
 b_4(t, T) = & E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot 1_{\{\tau > t\}} \mid \mathcal{F}_t \right] \\
 & + E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \int_t^T C du \cdot 1_{\{\tau > t\}} \mid \mathcal{F}_t \right] \\
 & + E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \omega_t \cdot 1_{\{\tau \leq T\}} \mid \mathcal{F}_t \right] \\
 & + E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \omega_t \cdot \int_t^T C du \cdot 1_{\{\tau > t\}} \mid \mathcal{F}_t \right]. \quad (14)
 \end{aligned}$$

We note that equation (14) is a bit cumbersome and can be condensed as follows,

$$\begin{aligned}
b_5(t, T) &= E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \left(1 + \int_t^T C du \right) \cdot 1_{\{\tau > t\}} \mid \mathcal{F}_t \right] \\
&+ E_t^* \left[e^{-\int_t^T r_s ds} \cdot P \cdot \omega_t \cdot \left(1 + \int_t^T C du \right) \cdot 1_{\{\tau \leq T\}} \mid \mathcal{F}_t \right]. \quad (15)
\end{aligned}$$

4 Conclusion

Catastrophe bonds, whose payoffs to investors are tied to the occurrence of natural disasters, provide primary insurers and reinsurers a mechanism by which they can securitize and transfer some of their residual risk to capital markets. We note that catastrophe bonds can be very attractive to investors as they provide the unique opportunity to improve a portfolio managers allocation by offering them an instrument that has both an attractive return and low covariability with financial markets.

While catastrophe bonds are theoretically attractive, they have remained marginal relative to traditional reinsurance. This paper proposes that a possibly explanation for this marginality can be attributed to the lack of discussion about the pricing and valuation of these instruments in a public forum. We hope that this paper, which draws parallels between the modeling of defaultable bonds in a reduced-form setting and the pricing of catastrophe bonds, can serve as a springboard for future research in the insurance linked securities space.

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