Supplement to Singular Invariant Markov Equilibrium in Stochastic Overlapping Generations Models

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Abstract

In this supplementary material, we provide the proofs of propositions, lemmas, and corollaries, background on iterated function system, and a numerical algorithm for computing equilibria in the models for the paper "Singular Invariant Markov Equilibrium in Stochastic Overlapping Generations Models" written by E. Kim and S. E. Spear.

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1 Proofs

1.1 Proof of Proposition 1:

The first Euler equation combined with the second Euler equation reduces to:

$$\frac{\pi^h}{m_m\left(m_{y,t},h\right)}+\frac{\pi^l}{m_m\left(m_{y,t},l\right)}=\frac{1}{q\left(m_{y,t-1},s_t\right)\omega_y-m_y\left(m_{y,t-1},s_t\right)} \tag{1}$$

where $q=\frac{1}{\bar{p}}$.

Then, the second Euler equation reduces to:

$$m_m\left(m_{y,t},s_{t+1}\right)=\frac{m_y\left(m_{y,t-1},s_t\right)+q\left(m_{y,t},s_{t+1}\right)\omega_m}{2} \tag{2}$$

Putting the money market clearing conditions into Eq. (1) and (2) rearranging them, we get:

$$M^{s_{t+1}}-m_y\left(m_{y,t},s_{t+1}\right)=\frac{m_y\left(m_{y,t-1},s_t\right)+q\left(m_{y,t},s_{t+1}\right)\omega_m}{2} \tag{3}$$

and

$$\frac{\pi^h\left(M^l-m_y\left(m_{y,t},l\right)\right)+\pi^l\left(M^h-m_y\left(m_{y,t},h\right)\right)}{\left(M^h-m_y\left(m_{y,t},h\right)\right)\left(M^l-m_y\left(m_{y,t},l\right)\right)}=\frac{1}{\left(q\left(m_{y,t-1},s_t\right)\omega_y-m_y\left(m_{y,t-1},s_t\right)\right)} \tag{4}$$

We consider the following linear forecast functions for the money holding of the young and the price of the single good in terms of the fiat money. $m_y\left(m,s_{t+1}\right)=\bar{m}^{s_{t+1}}-\gamma^{s_{t+1}}\left(m-\bar{m}^{s_{t+1}}\right)$ and $q\left(m,s_{t+1}\right)=\bar{q}^{s_{t+1}}+\rho^{s_{t+1}}\left(m-\bar{m}^{s_{t+1}}\right)$ where $m=m_{y,t}$. Plugging these linear forecast functions into Eq. (3) and (4) and rearranging them, we obtain:[1]

$$2\gamma^{s_t}m+2\left(M^{s_t}-\left(1+\gamma^{s_t}\right)\bar{m}^{s_t}\right)=\left(1+\rho^{s_t}\omega_m\right)m+\left(\bar{q}^{s_t}-\rho^{s_t}\bar{m}^{s_t}\right)\omega_m \tag{5}$$

and

$$\begin{aligned}
&\left(m-\bar{m}^{s_t}\right)\Big[\left(\rho^{s_t}\omega_y+\gamma^{s_t}\right)\left(\pi^h\gamma^l+\pi^l\gamma^h\right)\gamma^{s_t}\left(m-\bar{m}^{s_t}\right)\\
&-\left(\rho^{s_t}\omega_y+\gamma^{s_t}\right)\left(\pi^h\left(M^l-\bar{m}^l+\gamma^l\left(\bar{m}^{s_t}-\bar{m}^l\right)\right)+\pi^l\left(M^h-\bar{m}^h+\gamma^h\left(\bar{m}^{s_t}-\bar{m}^h\right)\right)\right)\\
&+\left(\pi^h\gamma^l+\pi^l\gamma^h\right)\gamma^{s_t}\left(\bar{q}^{s_t}\omega_y-\bar{m}^{s_t}\right)\Big]\\
&-\left(\bar{q}^{s_t}\omega_y-\bar{m}^{s_t}\right)\left(\pi^h\left(M^l-\bar{m}^l+\gamma^l\left(\bar{m}^{s_t}-\bar{m}^l\right)\right)+\pi^l\left(M^h-\bar{m}^h+\gamma^h\left(\bar{m}^{s_t}-\bar{m}^h\right)\right)\right)\\
=&\left(m-\bar{m}^{s_t}\right)\Big[-\gamma^h\gamma^l\left(\gamma^{s_t}\right)^2\left(m-\bar{m}^{s_t}\right)\\
&+\left(\gamma^h\left(M^l-\left(\bar{m}^l-\gamma^l\left(\bar{m}^{s_t}-\bar{m}^l\right)\right)\right)+\gamma^l\left(M^h-\left(\bar{m}^h-\gamma^h\left(\bar{m}^{s_t}-\bar{m}^h\right)\right)\right)\right)\gamma^{s_t}\Big]\\
&-\left(M^l-\bar{m}^l+\gamma^l\left(\bar{m}^{s_t}-\bar{m}^l\right)\right)\left(M^h-\bar{m}^h+\gamma^h\left(\bar{m}^{s_t}-\bar{m}^h\right)\right)
\end{aligned} \tag{6}$$

where $m=m_{t-1}$.

[1] Since this equation should hold for all periods, we use the time subscript t instead of $t+1$.

From Eq. (5) and (6), we can derive the following ten equations:

$$2\gamma^h = \left(1 + \omega_m \rho^h\right) \tag{7}$$

$$2\gamma^l = \left(1 + \omega_m \rho^l\right) \tag{8}$$

$$2\left(M^h - \left(1 + \gamma^h\right)\bar{m}^h\right) = \left(\bar{q}^h - \bar{m}^h \rho^h\right)\omega_m \tag{9}$$

$$2\left(M^l - \left(1 + \gamma^l\right)\bar{m}^l\right) = \left(\bar{q}^l - \bar{m}^l \rho^l\right)\omega_m \tag{10}$$

$$-\left(\rho^h \omega_y + \gamma^h\right)\left(\pi^h \gamma^l + \pi^l \gamma^h\right) = \gamma^l \left(\gamma^h\right)^2 \tag{11}$$

$$-\left(\rho^l \omega_y + \gamma^l\right)\left(\pi^h \gamma^l + \pi^l \gamma^h\right) = \gamma^h \left(\gamma^l\right)^2 \tag{12}$$

$$\begin{aligned}
&\left(\rho^h \omega_y + \gamma^h\right)\left(\pi^h \left(M^l - \bar{m}^l + \gamma^l \left(\bar{m}^h - \bar{m}^l\right)\right) + \pi^l \left(M^h - \bar{m}^h + \gamma^h \left(\bar{m}^h - \bar{m}^h\right)\right)\right)\\
&- \left(\pi^h \gamma^l + \pi^l \gamma^h\right)\gamma^h \left(\bar{q}^h \omega_y - \bar{m}^h\right)\\
&= -\left(\gamma^h \left(M^l - \left(\bar{m}^l - \gamma^l \left(\bar{m}^h - \bar{m}^l\right)\right)\right) + \gamma^l \left(M^h - \left(\bar{m}^h - \gamma^h \left(\bar{m}^h - \bar{m}^h\right)\right)\right)\right)\gamma^h
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\left(\rho^l \omega_y + \gamma^l\right)\left(\pi^h \left(M^l - \bar{m}^l + \gamma^l \left(\bar{m}^l - \bar{m}^l\right)\right) + \pi^l \left(M^h - \bar{m}^h + \gamma^h \left(\bar{m}^l - \bar{m}^h\right)\right)\right)\\
&- \left(\pi^h \gamma^l + \pi^l \gamma^h\right)\gamma^l \left(\bar{q}^l \omega_y - \bar{m}^l\right)\\
&= -\left(\gamma^h \left(M^l - \left(\bar{m}^l - \gamma^l \left(\bar{m}^l - \bar{m}^l\right)\right)\right) + \gamma^l \left(M^h - \left(\bar{m}^h - \gamma^h \left(\bar{m}^l - \bar{m}^h\right)\right)\right)\right)\gamma^l
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\left(\bar{q}^h \omega_y - \bar{m}^h\right)\left(\pi^h \left(M^l - \bar{m}^l + \gamma^l \left(\bar{m}^h - \bar{m}^l\right)\right) + \pi^l \left(M^h - \bar{m}^h + \gamma^h \left(\bar{m}^h - \bar{m}^h\right)\right)\right)\\
&= \left(M^l - \bar{m}^l + \gamma^l \left(\bar{m}^h - \bar{m}^l\right)\right)\left(M^h - \bar{m}^h + \gamma^h \left(\bar{m}^h - \bar{m}^h\right)\right)
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\left(\bar{q}^l \omega_y - \bar{m}^l\right)\left(\pi^h \left(M^l - \bar{m}^l + \gamma^l \left(\bar{m}^l - \bar{m}^l\right)\right) + \pi^l \left(M^h - \bar{m}^h + \gamma^h \left(\bar{m}^l - \bar{m}^h\right)\right)\right)\\
&= \left(M^l - \bar{m}^l + \gamma^l \left(\bar{m}^l - \bar{m}^l\right)\right)\left(M^h - \bar{m}^h + \gamma^h \left(\bar{m}^l - \bar{m}^h\right)\right)
\end{aligned} \tag{16}$$

There are eight variables, $\{\bar{m}^h, \bar{m}^l, \bar{q}^h, \bar{q}^l, \gamma^h, \gamma^l, \rho^h, \rho^l\}$. By solving the first eight equations, (7) – (14), we get values for $\{\bar{m}^h, \bar{m}^l, \bar{q}^h, \bar{q}^l, \gamma^h, \gamma^l, \rho^h, \rho^l\}$. These values satisfy the last two equations, (15) and (16), for sufficiently small shocks.

From Eq. (7) and (8):

$$\rho^s = \frac{2\gamma^s - 1}{\omega_m} \tag{17}$$

From Eq. (9) and (10) combined with Eq. (17):

$$\bar{m}^s = \frac{2M^s - \omega_m \bar{q}^s}{3} \tag{18}$$

So far, there are four variables, $\{\bar{q}^h, \bar{q}^l, \gamma^h, \gamma^l\}$, and six equations left, (11) – (16).

From Eq. (11) and (12):

$$\begin{aligned}
(19)\qquad & \frac{\left(\rho^h\omega_y+\gamma^h\right)}{\left(\rho^l\omega_y+\gamma^l\right)}=\frac{\gamma^h}{\gamma^l} \\
\iff & \left(\rho^h\omega_y+\gamma^h\right)\gamma^l=\left(\rho^l\omega_y+\gamma^l\right)\gamma^h \\
\iff & \frac{\rho^h}{\gamma^h}=\frac{\rho^l}{\gamma^l} \\
\iff & \frac{2\gamma^h-1}{\gamma^h\omega_m}=\frac{2\gamma^l-1}{\gamma^l\omega_m} \\
\iff & \gamma^h=\gamma^l=\gamma
\end{aligned}$$

From this result considering Eq. (17):

$$(20)\qquad \rho^h=\rho^l=\rho$$

From Eq. (11) or (12):

$$\begin{aligned}
(21)\qquad & -\left(\rho\omega^y+\gamma\right)\left(\pi^h\gamma+\pi^l\gamma\right)=\gamma^3 \\
\iff & \gamma^2+\gamma+\rho\omega^y=0 \\
\iff & \gamma^2+\gamma+\frac{2\gamma-1}{\omega^m}\omega^y=0 \\
\iff & \gamma=-\frac{1}{2}-\frac{\omega^y}{\omega^m}+\sqrt{\left(\frac{1}{2}+\frac{\omega^y}{\omega^m}\right)^2+\frac{\omega^y}{\omega^m}}
\end{aligned}$$

where we focus on the positive solution of this quadratic equation because a true monetary equilibrium in which young agents hold fiat money cannot exist with the negative value of γ.

Hence, the linear forecast functions are parallel over states. So far, there are two variables, $\{\bar{q}^h,\bar{q}^l\}$, and four equations left, (13) – (16).

From Eq. (13):

$$\begin{aligned}
(22)\qquad & \left(\rho\omega_y+\gamma\right)\left(\pi^h\left(M^l-\bar{m}^l+\gamma\left(\bar{m}^h-\bar{m}^l\right)\right)+\pi^l\left(M^h-\bar{m}^h\right)\right)-\gamma^2\left(\bar{q}^h\omega_y-\bar{m}^h\right) \\
& =-\left(M^l-\left(\bar{m}^l-\gamma\left(\bar{m}^h-\bar{m}^l\right)\right)+M^h-\bar{m}^h\right)\gamma^2 \\
\iff & \left(\pi^h\left(M^l-\bar{m}^l+\gamma\left(\bar{m}^h-\bar{m}^l\right)\right)+\pi^l\left(M^h-\bar{m}^h\right)\right)+\left(\bar{q}^h\omega_y-\bar{m}^h\right) \\
& =\left(M^l-\left(\bar{m}^l-\gamma\left(\bar{m}^h-\bar{m}^l\right)\right)+M^h-\bar{m}^h\right) \\
\iff & -\pi^lM^l-\pi^hM^h+\pi^l\bar{m}^l+\pi^h\bar{m}^h-\pi^l\gamma\left(\bar{m}^h-\bar{m}^l\right)+\left(\bar{q}^h\omega_y-\bar{m}^h\right)=0
\end{aligned}$$

From Eq. (14):

$$
\begin{aligned}
(23) \qquad & (\rho\omega_y + \gamma)\left(\pi^h\left(M^l - \bar{m}^l\right) + \pi^l\left(M^h - \bar{m}^h + \gamma\left(\bar{m}^l - \bar{m}^h\right)\right)\right) - \gamma^2\left(\bar{q}^l\omega_y - \bar{m}^l\right) \\
& = -\left(M^l - \bar{m}^l + M^h - \left(\bar{m}^h - \gamma\left(\bar{m}^l - \bar{m}^h\right)\right)\right)\gamma^2 \\
\Longleftrightarrow & \left(\pi^h\left(M^l - \bar{m}^l\right) + \pi^l\left(M^h - \bar{m}^h + \gamma\left(\bar{m}^l - \bar{m}^h\right)\right)\right) + \left(\bar{q}^l\omega_y - \bar{m}^l\right) \\
& = \left(M^l - \bar{m}^l + M^h - \left(\bar{m}^h - \gamma\left(\bar{m}^l - \bar{m}^h\right)\right)\right) \\
\Longleftrightarrow & -\pi^l M^l - \pi^h M^h + \pi^l\bar{m}^l + \pi^h\bar{m}^h - \pi^h\gamma\left(\bar{m}^l - \bar{m}^h\right) + \left(\bar{q}^l\omega_y - \bar{m}^l\right) = 0
\end{aligned}
$$

Combining Eq. (22) and (23) to get $\bar{q}^h - \bar{q}^l$:

$$
\begin{aligned}
(24) \qquad & -\pi^l\gamma\left(\bar{m}^h - \bar{m}^l\right) + \left(\bar{q}^h\omega_y - \bar{m}^h\right) = -\pi^h\gamma\left(\bar{m}^l - \bar{m}^h\right) + \left(\bar{q}^l\omega_y - \bar{m}^l\right) \\
\Longleftrightarrow & \; \bar{q}^h - \bar{q}^l = \frac{2\left(1+\gamma\right)\left(M^h - M^l\right)}{3\omega_y + \left(1+\gamma\right)\omega_m}
\end{aligned}
$$

After plugging Eq. (24) into Eq. (22) and rearranging it:

$$
(25) \qquad \bar{q}^h = -\frac{A + C\left(1+\gamma\right)\omega_m\pi^l}{3\omega_y}
$$

where $A = -\left(2 + \pi^h + 2\pi^l\gamma\right)M^h + \left(-\pi^l + 2\pi^l\gamma\right)M^l$ and $C = \frac{2(1+\gamma)\left(M^h - M^l\right)}{3\omega_y + (1+\gamma)\omega_m}$.

After plugging Eq. (24) into Eq. (23) and rearranging it:

$$
(26) \qquad \bar{q}^l = \frac{-\tilde{A} + C\left(1+\gamma\right)\omega_m\pi^h}{3\omega_y}
$$

where $\tilde{A} = \left(-\pi^h + 2\pi^h\gamma\right)M^h - \left(2 + \pi^l + 2\pi^h\gamma\right)M^l$.

So far, there are no variables, but two equations left, (15) and (16). The values for the eight variables given above satisfy Eq. (15) and (16) for sufficiently small shocks which we numerically check by calculating the errors for different sizes of shocks in Table 1. This is consistent with Proposition 2 that there are ME generated by a LIFS for small enough shocks in any three-period SOLG models with a single asset.

1.2 Proof of Lemma 1:

The market clearing condition in time t is:

$$
(27) \qquad e_y\left(p_t, p_{t+1}, p_{t+2}\right) + e_m\left(p_{t-1}, p_t, p_{t+1}\right) = a
$$

where p_t is the price of equity in time t and a is the total asset quantity in the deterministic economy.

The asset demand function of an young agent born in time $t-1$ can be denoted by:

$$
(28) \qquad e_{y,t-1} = e_y\left(p_{t-1}, p_t, p_{t+1}\right)
$$

Table 1: Errors in Eq. (15) and (16)

Size of shock (M^l, M^h)	No shock $(1,1)$	1% $(0.995, 1.005)$	5% $(0.975, 1.025)$	10% $(0.950, 1.050)$
Error in Eq. (15)	0%	0.0008%	0.0204%	0.0812%
Error in Eq. (16)	0%	0.0008%	0.0206%	0.0830%

- Errors in Table 1 are relative errors calculated by dividing the absolute values of errors with the values of the left hand side in Eq. (15) and (16).
- The size of shock measures differences in the total money suppply between high and low states compared to the money supply base, 1.

At the steady state, Eq. (27) and (28) can be reduced into:

$$e_y(\bar{p}, \bar{p}, \bar{p}) + e_m(\bar{p}, \bar{p}, \bar{p}) = a \tag{29}$$

and

$$\bar{e} = e_y(\bar{p}, \bar{p}, \bar{p}) \tag{30}$$

We first show the existence of the steady state price, $\bar{p}$. Let's assume that $W(\bar{p}) = e_y(\bar{p}, \bar{p}, \bar{p}) + e_m(\bar{p}, \bar{p}, \bar{p})$. e_y and e_m are continuous in the given prices to the agents because they are derived from applying the IFT to the agent's optimization problem under strict concave preferences. Then, the function W should be continuous in $\bar{p}$ as well. As the constant price $\bar{p}$ goes to zero, the equity holdings e_y and e_m diverge to infinity because agents can receive extremely high returns by buying the asset. The asset holdings will be close to zero as the constant asset price goes to infinity because it requires a small number of asset quantities to transfer endowments to the next period. By the intermediate value theorem, there exists at least one $\bar{p}$ such that $W(\bar{p}) = a$ for $\forall a \in \mathbb{R}_{++}$.

We now introduce two linear forecast functions: $e_y = \bar{e} - \gamma(e_{y,-1} - \bar{e}) = G(e_{y,-1})$ and $p = \bar{p} + \rho(e_{y,-1} - \bar{e}) = H(e_{y,-1})$. Hence, the market clearing condition and the optimal young agent's asset holdings can be rewritten as:

$$\begin{aligned} & G \circ G(e) + e_m(H(e), H \circ G(e), H \circ G \circ G(e)) = a \\ \Longleftrightarrow\ & \bar{e} + \gamma^2(e - \bar{e}) + e_m\left(\bar{p} + \rho(e - \bar{e}), \bar{p} - \rho\gamma(e - \bar{e}), \bar{p} + \rho\gamma^2(e - \bar{e})\right) = a \end{aligned} \tag{31}$$

and

$$\begin{aligned} & G(e) = e_y(H(e), H \circ G(e), H \circ G \circ G(e)) \\ \Longleftrightarrow\ & \bar{e} - \gamma(e - \bar{e}) = e_y\left(\bar{p} + \rho(e - \bar{e}), \bar{p} - \rho\gamma(e - \bar{e}), \bar{p} + \rho\gamma^2(e - \bar{e})\right) \end{aligned} \tag{32}$$

where $e = e_{y,-2}$.

Under $e = \bar{e}$, Eq. (31) and (32) always hold because

$$\begin{aligned} & \bar{e} + \gamma^2(e - \bar{e}) + e_m\left(\bar{p} + \rho(e - \bar{e}), \bar{p} - \rho\gamma(e - \bar{e}), \bar{p} + \rho\gamma^2(e - \bar{e})\right) = a \\ \Longleftrightarrow\ & \bar{e} + e_m(\bar{p}, \bar{p}, \bar{p}) = a \end{aligned} \tag{33}$$

and

$$\bar{e} - \gamma\left(e - \bar{e}\right) = e_y\left(\bar{p} + \rho\left(e - \bar{e}\right), \bar{p} - \rho\gamma\left(e - \bar{e}\right), \bar{p} + \rho\gamma^2\left(e - \bar{e}\right)\right) \tag{34}$$
$$\Longleftrightarrow \bar{e} = e_y\left(\bar{p}, \bar{p}, \bar{p}\right)$$

Therefore, $\bar{e}$ and $\bar{p}$ can be the solution to the system at $e_{y,-2} = \bar{e}$ in a three-period deterministic OLG model with a single long-lived asset.

1.3 Proof of Proposition 2:

With the linear forecast functions in (22), the optimality condition for the household problem is given by:

$$G\left(e, s_{t-1}\right) = e_y\begin{pmatrix} H\left(e, s_{t-1}\right), H\left(G\left(e, s_{t-1}\right), h\right), H\left(G\left(e, s_{t-1}\right), l\right), \\ H\left(G\left(G\left(e, s_{t-1}\right), h\right), h\right), H\left(G\left(G\left(e, s_{t-1}\right), l\right), h\right), \\ H\left(G\left(G\left(e, s_{t-1}\right), h\right), l\right), H\left(G\left(G\left(e, s_{t-1}\right), l\right), l\right) \end{pmatrix} \tag{35}$$

for $s_{t-1} \in \{h, l\}$ where $e = e_{y,t-2}$.

The market-clearing condition becomes:

$$G\left(G\left(e, s_{t-1}\right), s_t\right) + e_m\begin{pmatrix} H\left(e, s_{t-1}\right), H\left(G\left(e, s_{t-1}\right), h\right), H\left(G\left(e, s_{t-1}\right), l\right), \\ H\left(G\left(G\left(e, s_{t-1}\right), h\right), h\right), H\left(G\left(G\left(e, s_{t-1}\right), l\right), h\right), \\ H\left(G\left(G\left(e, s_{t-1}\right), h\right), l\right), H\left(G\left(G\left(e, s_{t-1}\right), l\right), l\right) \end{pmatrix} = 1 \tag{36}$$

for $\left(s_{t-1}, s_t\right) \in \{h, l\}^2$ where $e = e_{y,t-2}$.

The market-clearing conditions and the equations from the equity holding for the young, (35) and (36), are now a system of six equations in the six variables, $\left\{\bar{e}^h, \bar{e}^l, \bar{p}^h, \bar{p}^l, \gamma, \rho\right\}$. There is, however, a functional dependency between γ and ρ, since

$$\frac{e_y^{s_t} - \bar{e}^{s_t}}{p_t^{s_t} - \bar{p}^{s_t}} = -\frac{\gamma\left(e - \bar{e}^{s_t}\right)}{\rho\left(e - \bar{e}^{s_t}\right)} = -\frac{\gamma}{\rho} \tag{37}$$
$$\Longleftrightarrow \gamma = -\rho\left(\frac{e_y^{s_t} - \bar{e}^{s_t}}{p_t^{s_t} - \bar{p}^{s_t}}\right)$$

Thus, we have at most five independent equations in five variables. If we denote the system of the equilibrium conditions by $Z : \mathbb{R}^5_{++} \to \mathbb{R}^5_{++}$, then we want to show that $Z \pitchfork 0$. Since we know $\bar{e}$ and $\bar{p}$ can be the solution to the system at the steady state when ω and δ don't depend on s by Lemma 1, the IFT will then guarantee the existence of a LIFS in a neighborhood of the deterministic steady state.

The assumed linear forecast functions generate the following relationships:

$$G\left(G\left(e, s_{t-1}\right), s_t\right) = \bar{e}^{s_t} - \gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^{s_t}\right) + \gamma^2\left(e - \bar{e}^{s_{t-1}}\right) \tag{38}$$
$$H\left(G\left(e, s_{t-1}\right), s_t\right) = \bar{p}^{s_t} + \rho\left(\bar{e}^{s_{t-1}} - \bar{e}^{s_t}\right) - \rho\gamma\left(e - \bar{e}^{s_{t-1}}\right) \tag{39}$$
$$H\left(G\left(G\left(e, s_{t-1}\right), s_t\right), s_{t+1}\right) = \bar{p}^{s_{t+1}} + \rho\left(\bar{e}^{s_t} - \bar{e}^{s_{t+1}}\right) - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^{s_t}\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right) \tag{40}$$

It will simplify the calculations to show the transversality result if we allow the asset quantities to vary over each state, which we denote by $\bar{a}^h$ and $\bar{a}^l$, respectively. We will include these variables in the rank calculation of the Jacobian matrix, and use the transversal density theorem to infer that for almost all asset quantities, $Z \pitchfork 0$, i.e. the IFT applies. With the relationships above and the inclusion of the asset quantities as variables, we can rewrite the market-clearing conditions as:

$$\bar{a}^{s_t} = \bar{e}^{s_t} - \gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^{s_t}\right) + \gamma^2\left(e - \bar{e}^{s_{t-1}}\right)$$

$$+e_m \begin{pmatrix} \bar{p}^{s_{t-1}} + \rho\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^h + \rho\left(\bar{e}^{s_{t-1}} - \bar{e}^h\right) - \rho\gamma\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^l + \rho\left(\bar{e}^{s_{t-1}} - \bar{e}^l\right) - \rho\gamma\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^h - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^h\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^h + \rho\left(\bar{e}^l - \bar{e}^h\right) - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^l\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^l + \rho\left(\bar{e}^h - \bar{e}^l\right) - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^h\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^l - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^l\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right) \end{pmatrix} \tag{41}$$

Also, we can transform the definition of the equity holding for the young into:

$$0 = \bar{e}^{s_{t-1}} - \gamma\left(e - \bar{e}^{s_{t-1}}\right)$$

$$-e_y \begin{pmatrix} \bar{p}^{s_{t-1}} + \rho\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^h + \rho\left(\bar{e}^{s_{t-1}} - \bar{e}^h\right) - \rho\gamma\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^l + \rho\left(\bar{e}^{s_{t-1}} - \bar{e}^l\right) - \rho\gamma\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^h - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^h\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^h + \rho\left(\bar{e}^l - \bar{e}^h\right) - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^l\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^l + \rho\left(\bar{e}^h - \bar{e}^l\right) - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^h\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right), \\ \bar{p}^l - \rho\gamma\left(\bar{e}^{s_{t-1}} - \bar{e}^l\right) + \rho\gamma^2\left(e - \bar{e}^{s_{t-1}}\right) \end{pmatrix} \tag{42}$$

For the rank calculation of the Jacobian matrix, let:

$$\begin{aligned} A_{hh} =& 1 - \gamma^2 - e_{m1}\rho + e_{m2h}\rho\gamma + e_{m2l}\left(\rho + \rho\gamma\right) - e_{m3hh}\rho\gamma^2 \\ & - e_{m3lh}\left(\rho + \rho\gamma + \rho\gamma^2\right) + e_{m3hl}\left(\rho - \rho\gamma^2\right) - e_{m3ll}\left(\rho\gamma + \rho\gamma^2\right) \\ A_{hl} =& - \gamma - \gamma^2 - e_{m1}\rho + e_{m2h}\rho\gamma + e_{m2l}\left(\rho + \rho\gamma\right) - e_{m3hh}\rho\gamma^2 \\ & - e_{m3lh}\left(\rho + \rho\gamma + \rho\gamma^2\right) + e_{m3hl}\left(\rho - \rho\gamma^2\right) - e_{m3ll}\left(\rho\gamma + \rho\gamma^2\right) \\ A_{lh} =& 1 + \gamma - e_{m2h}\rho + e_{m3hh}\rho\gamma - e_{m3lh}\rho + e_{m3hl}\left(\rho + \rho\gamma\right) \\ A_{ll} =& - e_{m2h}\rho + e_{m3hh}\rho\gamma - e_{m3lh}\rho + e_{m3hl}\left(\rho + \rho\gamma\right) \\ B_{hh} =& - e_{m2l}\rho + e_{m3lh}\left(\rho + \rho\gamma\right) - e_{m3hl}\rho + e_{m3ll}\rho\gamma \\ B_{hl} =& 1 + \gamma - e_{m2l}\rho + e_{m3lh}\left(\rho + \rho\gamma\right) - e_{m3hl}\rho + e_{m3ll}\rho\gamma \\ B_{lh} =& - \gamma - \gamma^2 - e_{m1}\rho + e_{m2h}\left(\rho + \rho\gamma\right) + e_{m2l}\rho\gamma - e_{m3hh}\left(\rho\gamma + \rho\gamma^2\right) \\ & + e_{m3lh}\left(\rho - \rho\gamma^2\right) - e_{m3hl}\left(\rho + \rho\gamma + \rho\gamma^2\right) - e_{m3ll}\rho\gamma^2 \\ B_{ll} =& 1 - \gamma^2 - e_{m1}\rho + e_{m2h}\left(\rho + \rho\gamma\right) + e_{m2l}\rho\gamma - e_{m3hh}\left(\rho\gamma + \rho\gamma^2\right) \\ & + e_{m3lh}\left(\rho - \rho\gamma^2\right) - e_{m3hl}\left(\rho + \rho\gamma + \rho\gamma^2\right) - e_{m3ll}\rho\gamma^2 \end{aligned} \tag{43}$$

$$
\begin{aligned}
F_h =& e_{m1} + e_{m2h} + e_{m3hh} + e_{m3lh} \\
F_l =& e_{m2h} + e_{m3hh} + e_{m3lh} \\
G_h =& e_{m2l} + e_{m3hl} + e_{m3ll} \\
G_l =& e_{m1} + e_{m2l} + e_{m3hl} + e_{m3ll} \\
L_h =& e_{m1}\left(e-\bar{e}^h\right) - e_{m2h}\gamma\left(e-\bar{e}^h\right) + e_{m2l}\left(\bar{e}^h-\bar{e}^l-\gamma\left(e-\bar{e}^h\right)\right) \\
&+ e_{m3hh}\gamma^2\left(e-\bar{e}^h\right) + e_{m3lh}\left(\bar{e}^l-\bar{e}^h-\gamma\left(\bar{e}^h-\bar{e}^l\right)+\gamma^2\left(e-\bar{e}^h\right)\right) \\
&+ e_{m3hl}\left(\bar{e}^h-\bar{e}^l+\gamma^2\left(e-\bar{e}^h\right)\right) + e_{m3ll}\left(-\gamma\left(\bar{e}^h-\bar{e}^l\right)+\gamma^2\left(e-\bar{e}^h\right)\right) \\
L_l =& e_{m1}\left(e-\bar{e}^l\right) + e_{m2h}\left(\bar{e}^l-\bar{e}^h-\gamma\left(e-\bar{e}^l\right)\right) - e_{m2l}\gamma\left(e-\bar{e}^l\right) \\
&+ e_{m3hh}\left(-\gamma\left(\bar{e}^l-\bar{e}^h\right)+\gamma^2\left(e-\bar{e}^l\right)\right) + e_{m3lh}\left(\bar{e}^l-\bar{e}^h+\gamma^2\left(e-\bar{e}^l\right)\right) \\
&+ e_{m3hl}\left(\bar{e}^h-\bar{e}^l-\gamma\left(\bar{e}^l-\bar{e}^h\right)+\gamma^2\left(e-\bar{e}^l\right)\right) + e_{m3ll}\gamma^2\left(e-\bar{e}^l\right) \\
M_{hh} =& 2\gamma\left(e-\bar{e}^h\right) - e_{m2h}\rho\left(e-\bar{e}^h\right) - e_{m2l}\rho\left(e-\bar{e}^h\right) + 2e_{m3hh}\rho\gamma\left(e-\bar{e}^h\right) \\
&+ e_{m3lh}\left(-\rho\left(\bar{e}^h-\bar{e}^l\right)+2\rho\gamma\left(e-\bar{e}^h\right)\right) + 2e_{m3hl}\rho\gamma\left(e-\bar{e}^h\right) \\
&+ e_{m3ll}\left(-\rho\left(\bar{e}^h-\bar{e}^l\right)+2\rho\gamma\left(e-\bar{e}^h\right)\right) \\
M_{hl} =& \bar{e}^l-\bar{e}^h+2\gamma\left(e-\bar{e}^h\right) - e_{m2h}\rho\left(e-\bar{e}^h\right) - e_{m2l}\rho\left(e-\bar{e}^h\right) \\
&+ 2e_{m3hh}\rho\gamma\left(e-\bar{e}^h\right) + e_{m3lh}\left(-\rho\left(\bar{e}^h-\bar{e}^l\right)+2\rho\gamma\left(e-\bar{e}^h\right)\right) \\
&+ 2e_{m3hl}\rho\gamma\left(e-\bar{e}^h\right) + e_{m3ll}\left(-\rho\left(\bar{e}^h-\bar{e}^l\right)+2\rho\gamma\left(e-\bar{e}^h\right)\right) \\
M_{lh} =& \bar{e}^h-\bar{e}^l+2\gamma\left(e-\bar{e}^l\right) - e_{m2h}\rho\left(e-\bar{e}^l\right) - e_{m2l}\rho\left(e-\bar{e}^l\right) \\
&+ e_{m3hh}\left(-\rho\left(\bar{e}^l-\bar{e}^h\right)+2\rho\gamma\left(e-\bar{e}^l\right)\right) + 2e_{m3lh}\rho\gamma\left(e-\bar{e}^l\right) \\
&+ e_{m3hl}\left(-\rho\left(\bar{e}^l-\bar{e}^h\right)+2\rho\gamma\left(e-\bar{e}^l\right)\right) + 2e_{m3ll}\rho\gamma\left(e-\bar{e}^l\right) \\
M_{ll} =& 2\gamma\left(e-\bar{e}^l\right) - e_{m2h}\rho\left(e-\bar{e}^l\right) - e_{m2l}\rho\left(e-\bar{e}^l\right) \\
&+ e_{m3hh}\left(-\rho\left(\bar{e}^l-\bar{e}^h\right)+2\rho\gamma\left(e-\bar{e}^l\right)\right) + 2e_{m3lh}\rho\gamma\left(e-\bar{e}^l\right) \\
&+ e_{m3hl}\rho\gamma\left(-\rho\left(\bar{e}^l-\bar{e}^h\right)+2\rho\gamma\left(e-\bar{e}^l\right)\right) + 2e_{m3ll}\rho\gamma\left(e-\bar{e}^l\right) \\
H_h =& 1+\gamma+e_{y1}\rho - e_{y2h}\rho\gamma - e_{y2l}\left(\rho+\rho\gamma\right) + e_{y3hh}\rho\gamma^2 \\
&+ e_{y3lh}\left(\rho+\rho\gamma+\rho\gamma^2\right) - e_{y3hl}\left(\rho-\rho\gamma^2\right) + e_{y3ll}\left(\rho\gamma+\rho\gamma^2\right) \\
H_l =& e_{y2h}\rho + e_{y3hh}\rho\gamma + e_{y3lh}\rho - e_{y3hl}\left(\rho+\rho\gamma\right) \\
I_h =& e_{y2l}\rho - e_{y3lh}\left(\rho+\rho\gamma\right) + e_{y3hl}\rho - e_{y3ll}\rho\gamma \\
I_l =& 1+\gamma+e_{y1}\rho - e_{y2h}\left(\rho+\rho\gamma\right) - e_{y2l}\rho\gamma + e_{y3hh}\left(\rho\gamma+\rho\gamma^2\right) \\
&- e_{y3lh}\left(\rho-\rho\gamma^2\right) + e_{y3hl}\left(\rho+\rho\gamma+\rho\gamma^2\right) + e_{y3ll}\rho\gamma^2
\end{aligned}
$$

$$
\begin{aligned}
J_h &= -e_{y1} - e_{y2h} - e_{y3hh} - e_{y3lh} \\
J_l &= -e_{y2h} - e_{y3hh} - e_{y3lh} \\
K_h &= -e_{y2l} - e_{y3hl} - e_{y3ll} \\
K_l &= -e_{y1} - e_{y2l} - e_{y3hl} - e_{y3ll} \\
N_h &= -e_{y1}\left(e - \bar{e}^h\right) + e_{y2h}\gamma\left(e - \bar{e}^h\right) - e_{y2l}\left(\bar{e}^h - \bar{e}^l - \gamma\left(e - \bar{e}^h\right)\right) \\
&\quad - e_{y3hh}\gamma^2\left(e - \bar{e}^h\right) - e_{y3lh}\left(\bar{e}^l - \bar{e}^h - \gamma\left(\bar{e}^h - \bar{e}^l\right) + \gamma^2\left(e - \bar{e}^h\right)\right) \\
&\quad - e_{y3hl}\left(\bar{e}^h - \bar{e}^l + \gamma^2\left(e - \bar{e}^h\right)\right) - e_{y3ll}\left(-\gamma\left(\bar{e}^h - \bar{e}^l\right) + \gamma^2\left(e - \bar{e}^h\right)\right) \\
N_l &= -e_{y1}\left(e - \bar{e}^l\right) - e_{y2h}\left(\bar{e}^l - \bar{e}^h - \gamma\left(e - \bar{e}^l\right)\right) + e_{y2l}\gamma\left(e - \bar{e}^l\right) \\
&\quad - e_{y3hh}\left(-\gamma\left(\bar{e}^l - \bar{e}^h\right) + \gamma^2\left(e - \bar{e}^l\right)\right) - e_{y3lh}\left(\bar{e}^l - \bar{e}^h + \gamma^2\left(e - \bar{e}^l\right)\right) \\
&\quad - e_{y3hl}\left(\bar{e}^h - \bar{e}^l - \gamma\left(\bar{e}^l - \bar{e}^h\right) + \gamma^2\left(e - \bar{e}^l\right)\right) - e_{y3ll}\gamma^2\left(e - \bar{e}^l\right) \\
O_h &= -\left(e - \bar{e}^h\right) + e_{y2h}\rho\left(e - \bar{e}^h\right) + e_{y2l}\rho\left(e - \bar{e}^h\right) \\
&\quad - 2e_{y3hh}\rho\gamma\left(e - \bar{e}^h\right) - e_{y3lh}\left(-\rho\left(\bar{e}^h - \bar{e}^l\right) + 2\rho\gamma\left(e - \bar{e}^h\right)\right) \\
&\quad - 2e_{y3hl}\rho\gamma\left(e - \bar{e}^h\right) - e_{y3ll}\left(-\rho\left(\bar{e}^h - \bar{e}^l\right) + 2\rho\gamma\left(e - \bar{e}^h\right)\right) \\
O_l &= -\left(e - \bar{e}^l\right) + e_{y2h}\rho\left(e - \bar{e}^l\right) + e_{y2l}\rho\left(e - \bar{e}^l\right) \\
&\quad - e_{y3hh}\left(-\rho\left(\bar{e}^l - \bar{e}^h\right) + 2\rho\gamma\left(e - \bar{e}^l\right)\right) - 2e_{y3lh}\rho\gamma\left(e - \bar{e}^l\right) \\
&\quad - e_{y3hl}\left(-\rho\left(\bar{e}^l - \bar{e}^h\right) + 2\rho\gamma\left(e - \bar{e}^l\right)\right) - 2e_{y3ll}\rho\gamma\left(e - \bar{e}^l\right)
\end{aligned}
$$

where $A_{hh} - A_{hl} = A_{lh} - A_{ll} = B_{hl} - B_{hh} = B_{ll} - B_{lh} = 1 + \gamma$, $A_{hl} - A_{ll} = B_{ll} - B_{hl}$, $F_h - F_l = G_l - G_h = e_{m1}$, $H_h - H_l = I_l - I_h$ and $J_l - J_h = K_h - K_l = e_{y1}$.

Evaluating these variables at the steady state, $e = \bar{e}^h = \bar{e}^l = \bar{e}$, and $\bar{p}^h = \bar{p}^l = \bar{p}$, yields:

$$
\begin{aligned}
L_h &= 0 \\
L_l &= 0 \\
M_{hh} &= 0 \\
M_{hl} &= 0 \\
M_{lh} &= 0 \\
M_{ll} &= 0 \\
N_h &= 0 \\
N_l &= 0 \\
O_h &= 0 \\
O_l &= 0
\end{aligned}
\tag{44}
$$

Then, the Jacobian matrix with respect to the parameters in the price and equity linear forecast functions and the asset quantity parameters takes the form:

$$
(45) \qquad DZ = \begin{array}{c}
\begin{array}{cccccccc} \partial\bar{e}^h & \partial\bar{e}^l & \partial\bar{p}^h & \partial\bar{p}^l & \partial\rho & \partial\gamma & \partial\bar{a}^h & \partial\bar{a}^l \end{array} \\
\begin{bmatrix}
A_{hh} & B_{hh} & F_h & G_h & L_h & M_{hh} & -1 & 0 \\
A_{hl} & B_{hl} & F_h & G_h & L_h & M_{hl} & 0 & -1 \\
A_{lh} & B_{lh} & F_l & G_l & L_l & M_{lh} & -1 & 0 \\
A_{ll} & B_{ll} & F_l & G_l & L_l & M_{ll} & 0 & -1 \\
H_h & I_h & J_h & K_h & N_h & O_h & 0 & 0 \\
H_l & I_l & J_l & K_l & N_l & O_l & 0 & 0
\end{bmatrix}
\end{array}
$$

Evaluating this matrix at the deterministic steady state yields:

$$
(46) \qquad DZ = \begin{bmatrix}
A_{hh} & B_{hh} & F_h & G_h & 0 & 0 & -1 & 0 \\
A_{hl} & B_{hl} & F_h & G_h & 0 & 0 & 0 & -1 \\
A_{lh} & B_{lh} & F_l & G_l & 0 & 0 & -1 & 0 \\
A_{ll} & B_{ll} & F_l & G_l & 0 & 0 & 0 & -1 \\
H_h & I_h & J_h & K_h & 0 & 0 & 0 & 0 \\
H_l & I_l & J_l & K_l & 0 & 0 & 0 & 0
\end{bmatrix}
$$

We reduce this matrix via row and column operations. Subtract the second and fourth rows from the first and third rows, respectively. Then, subtract the first row with the third row and clear the third row to get:

$$
(47) \qquad DZ = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{hl} & B_{hl} & F_h & G_h & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
A_{ll} & B_{ll} & F_l & G_l & 0 & 0 & 0 & -1 \\
H_h & I_h & J_h & K_h & 0 & 0 & 0 & 0 \\
H_l & I_l & J_l & K_l & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Subtract the fourth row from the second row and the sixth row from the fifth row and then, clear the fourth row to obtain:

$$
(48) \qquad DZ = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{hl} - A_{ll} & B_{hl} - B_{ll} & e_{m1} & -e_{m1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
H_h - H_l & I_h - I_l & -e_{y1} & e_{y1} & 0 & 0 & 0 & 0 \\
H_l & I_l & J_l & K_l & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Add the second and fourth columns to the first and third columns, respectively. Then, clear the last row to derive:

$$
(49) \qquad DZ = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B_{hl} - B_{ll} & 0 & -e_{m1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & I_h - I_l & 0 & e_{y1} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

From this reduction, the following submatrix has rank 2:

$$\begin{bmatrix} B_{hl} - B_{ll} & -e_{m1} \\ I_h - I_l & e_{y1} \end{bmatrix} \tag{50}$$

Therefore, the reduced Jacobian matrix is given by:

$$DZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

which has rank 5.

From the rank calculation, the IFT applies. Hence, we can infer that there will be ME generated by a LIFS around the deterministic steady states for sufficiently small shocks.

1.4 Proof of Proposition 3:

The first-order forecast function should be consistent with the full price dynamics system:

$$f \circ f \circ f(p) = z\left(f \circ f\left(p\right), f\left(p\right), p\right) \tag{52}$$

After taking the derivative of the first-order vector system for Eq. (52) and evaluating it at the steady state, $p_{t-2} = \bar{p}$, we have:

$$f' \begin{bmatrix} f'^2 \\ f' \\ 1 \end{bmatrix} = Z \begin{bmatrix} f'^2 \\ f' \\ 1 \end{bmatrix} \tag{53}$$

Eq. (53) implies that f' takes one of the eigenvalues of Z and the vector in Eq. (53) is its corresponding eigenvector. One can construct a forward-stable first-order forecast function by associating it with the eigenvalue of Z inside the unit circle (see Kim and Spear 2017).

We show there exists a functional relationship between the equity holdings of the young in period t and the young in period $t-1$ from Proposition 2 on a neighborhood of the deterministic steady-state. Hence, we can write:

$$\hat{e}_y\left(p_{t+1}\right) = G\left(\hat{e}_y\left(p_t\right)\right) \tag{54}$$

For the function G to be consistent with the underlying full price dynamics of the model, it requires that:

$$\hat{e}_y\left(f\left(p\right)\right) = G\left(\hat{e}_y\left(p\right)\right) \tag{55}$$

which imposes the condition on the derivatives at the steady state that:

$$D_p\hat{e}_y f' = G'\left(\bar{e}_y\right) D_p\hat{e}_y \tag{56}$$

where $D_p\hat{e}_y = \hat{e}'_y = \frac{\partial e_y}{\partial p_t} + \frac{\partial e_y}{\partial p_{t+1}} f' + \frac{\partial e_y}{\partial p_{t+2}} f'^2 \,|_{p_t=\bar{p}}$.

It follows, then, that $-\gamma = G'\left(\bar{e}_y\right) = f'\left(\bar{p}\right)$. Thus, the slope parameter of the LIFS can coincide with the stable eigenvalue of Z.

To associate the parameter ϱ with γ, we write the relationship between p_{t+1} and $\hat{e}_y\left(p_t\right)$ as:

$$p_{t+1} = H\left(\hat{e}_y\left(p_t\right)\right) \tag{57}$$

Model consistency then requires that:

$$f\left(p_t\right) = H\left(\hat{e}_y\left(p_t\right)\right) \tag{58}$$

Eq. (58) indicates that the derivative of this equation will be:

$$f' = H'\hat{e}'_y \tag{59}$$

Since $H' = \varrho$, we have:

$$\varrho = \frac{f'}{\hat{e}'_y} = \frac{-\gamma}{\hat{e}'_y}. \tag{60}$$

1.5 Proof of Lemma 2:

The market clearing condition in time t is:

$$\sum_{i=1}^{L-1}\sum_{j=1}^{M} e_{i,j}\left(p_{t+1-i}, \ldots, p_{t+L-i}\right) = a \tag{61}$$

The $((L-1)M-1)$ dimensional vector of the asset demands functions in time t can be denoted by:

$$\xi_t = \begin{bmatrix} \left\{e_{(L-1),j}\left(p_{t-L+2}, \ldots, p_{t+1}\right)\right\}_{j=1,\ldots,M-1} \\ \left\{e_{(L-2),j}\left(p_{t-L+3}, \ldots, p_{t+2}\right)\right\}_{j=1,\ldots,M} \\ \vdots \\ \left\{e_{1,j}\left(p_t, \ldots, p_{t+L-1}\right)\right\}_{j=1,\ldots,M} \end{bmatrix} \tag{62}$$

where the demand functions for all but the type-M of the second oldest generation are included. At the steady state, Eq. (61) and (62) can be reduced into:

$$\sum_{i=1}^{L-1}\sum_{j=1}^{M} e_{i,j}\left(\bar{p}, \ldots, \bar{p}\right) = a \tag{63}$$

and

$$\bar{\xi} = \begin{bmatrix} \left\{e_{(L-1),j}\left(\bar{p}, \ldots, \bar{p}\right)\right\}_{j=1,\ldots,M-1} \\ \left\{e_{(L-2),j}\left(\bar{p}, \ldots, \bar{p}\right)\right\}_{j=1,\ldots,M} \\ \vdots \\ \left\{e_{1,j}\left(\bar{p}, \ldots, \bar{p}\right)\right\}_{j=1,\ldots,M} \end{bmatrix} \tag{64}$$

As in the deterministic three-period model, the asset demand functions are also continuous in the prices. As the constant price goes to zero, the equity holdings diverge to infinity. The asset holdings will be close to zero as the constant asset price goes to infinity. By appealing to the intermediate value theorem, there also exists at least one steady-state price satisfying the market clearing conditions.

We now introduce two linear forecast functions:$\xi_t = \bar{\xi} + \Gamma\left(\xi_{t-1} - \bar{\xi}\right) = G\left(\xi_{t-1}\right)$ and $p_t = \bar{p} + \Lambda^T\left(\xi_{t-1} - \bar{\xi}\right) = H\left(\xi_{t-1}\right)$. Hence, the market clearing condition and the individual optimality conditions can be rewritten as:

$$
\begin{aligned}
& a = \iota^T G^{L-1}(\xi) + e_{(L-1),M}\left(H(\xi), H\circ G(\xi), \ldots, H\circ G^{L-1}(\xi)\right) \\
\Longleftrightarrow & a = \iota^T\left(\left(\sum_{i=1}^{L-1}\Gamma^{i-1}\right)(I-\Gamma)\bar{\xi} + \Gamma^{L-1}\xi\right) \\
& + e_{(L-1),M}\left(\begin{array}{c} \bar{p} + \Lambda^T\left(\xi - \bar{\xi}\right), \bar{p} + \Lambda^T\left((I-\Gamma)\bar{\xi} + \Gamma\xi - \bar{\xi}\right), \ldots, \\ \bar{p} + \Lambda^T\left(\left(\sum_{i=1}^{L-1}\Gamma^{i-1}\right)(I-\Gamma)\bar{\xi} + \Gamma^{L-1}\xi - \bar{\xi}\right) \end{array}\right)
\end{aligned}
\tag{65}
$$

and

$$
\begin{aligned}
& G^{L-1}(\xi) = \\
& \left[\begin{array}{c}
\left\{e_{(L-1),j}\left(H(\xi), H\circ G(\xi), \ldots, H\circ G^{L-1}(\xi)\right)\right\}_{j=1,\ldots,M-1} \\
\left\{e_{(L-2),j}\left(H\circ G(\xi), H\circ G^2(\xi), \ldots, H\circ G^{L}(\xi)\right)\right\}_{j=1,\ldots,M} \\
\vdots \\
\left\{e_{1,j}\left(H\circ G^{L-2}(\xi), H\circ G^{L-1}(\xi), \ldots, H\circ G^{2L-3}(\xi)\right)\right\}_{j=1,\ldots,M}
\end{array}\right] \\
\Longleftrightarrow & \left(\sum_{i=1}^{L-1}\Gamma^{i-1}\right)(I-\Gamma)\bar{\xi} + \Gamma^{L-1}\xi = \\
& \left[\begin{array}{c}
\left\{e_{(L-1),j}\left(\begin{array}{c} \bar{p} + \Lambda^T\left(\xi - \bar{\xi}\right), \ldots, \\ \bar{p} + \Lambda^T\left(\left(\sum_{i=1}^{L-1}\Gamma^{i-1}\right)(I-\Gamma)\bar{\xi} + \Gamma^{L-1}\xi - \bar{\xi}\right) \end{array}\right)\right\}_{j=1,\ldots,M-1} \\
\vdots \\
\left\{e_{1,j}\left(\begin{array}{c} \bar{p} + \Lambda^T\left(\left(\sum_{i=1}^{L-2}\Gamma^{i-1}\right)(I-\Gamma)\bar{\xi} + \Gamma^{L-2}\xi - \bar{\xi}\right), \ldots, \\ \bar{p} + \Lambda^T\left(\left(\sum_{i=1}^{2L-3}\Gamma^{i-1}\right)(I-\Gamma)\bar{\xi} + \Gamma^{2L-3}\xi - \bar{\xi}\right) \end{array}\right)\right\}_{j=1,\ldots,M}
\end{array}\right]
\end{aligned}
\tag{66}
$$

where $\xi = \xi_{t-L+1}$, G^N is the composition of N number of G functions and we let $\Gamma^0 = I$.

Under $\xi = \bar{\xi}$, Eq. (65) and (66) always hold because they degenerate to:

$$
\iota^T\bar{\xi} + e_{(L-1),M}\left(\bar{p}, \ldots, \bar{p}\right) = a \tag{67}
$$

and

$$
\bar{\xi} = \left[\begin{array}{c}
\left\{e_{(L-1),j}\left(\bar{p}, \ldots, \bar{p}\right)\right\}_{j=1,\ldots,M-1} \\
\left\{e_{(L-2),j}\left(\bar{p}, \ldots, \bar{p}\right)\right\}_{j=1,\ldots,M} \\
\vdots \\
\left\{e_{1,j}\left(\bar{p}, \ldots, \bar{p}\right)\right\}_{j=1,\ldots,M}
\end{array}\right] \tag{68}
$$

Therefore, $\bar{\bar{\xi}}$ and $\bar{P}$ can be the solution to the system at $\xi_{t-L+1} = \bar{\bar{\xi}}$ in a more realistic version of the deterministic OLG model with a single long-lived asset.

1.6 Proof of Proposition 4:

We calculate the number of variables in the linear forecast functions. $\bar{p}^{s_t}$ has S values depending on s_t. Λ^{s_t} is a $((L-1)M-1)$ dimensional vector for $s_t \in \{z_1, \ldots, z_S\}$ and thus there are $((L-1)M-1)S$ variables for Λ^{s_t}. $\bar{\bar{\xi}}^{s_t}$ is also a $((L-1)M-1)$ dimensional vector varing over states. Hence, it has $((L-1)M-1)S$ variables. Γ^{s_t} is a $((L-1)M-1) \times ((L-1)M-1)$ matrix. The affine matrix has $((L-1)M-1)^2$ variables for every state $s_t \in \{z_1, \ldots, z_S\}$. The total number of variables is $((L-1)M)^2 S$ if we let Λ^{s_t} and Γ^{s_t} vary over states while it is $(L-1)M((L-1)M+S-1)$ if we restrict such coefficient matrices to be uniform over states i.e. $\Lambda^{s_t} = \Lambda$ and $\Gamma^{s_t} = \Gamma$ for $\forall s_t \in \{z_1, \ldots, z_S\}$.

With the linear forecast functions in (38) and (39), the market clearing conditions can be rewritten as:

$$
\begin{aligned}
(69)\qquad \bar{a}^{s_t} &= \iota^T G^{L-1}\left(\xi; S_{t-L+2}^{t}\right) \\
&+ e_{(L-1),M}\left(H\left(\xi, s_{t-L+2}\right), \left\{H\left(G\left(\xi; S_{t-L+2}^{t-L+3}\right)\right)\right\}, \ldots, \left\{H\left(G^{L-1}\left(\xi; S_{t-L+2}^{t+1}\right)\right)\right\}; S_{t-L+3}^{t} \right) \\
\Longleftrightarrow \bar{a}^{s_t} &= \iota^T\left(\sum_{i=1}^{L-1}\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}}\right)\left(I-\Gamma^{s_{t-i+1}}\right)\bar{\bar{\xi}}^{s_{t-i+1}} + \left(\prod_{k=1}^{L-1}\Gamma^{s_{t-k+1}}\right)\xi\right) \\
&+ e_{(L-1),M}\left(\begin{array}{c}
\bar{p}^{s_{t-L+2}} + \left(\Lambda^{s_{t-L+2}}\right)^T\left(\xi - \bar{\bar{\xi}}^{s_{t-L+2}}\right), \\
\left\{\bar{p}^{s_{t-L+3}} + \left(\Lambda^{s_{t-L+3}}\right)^T\left(\begin{array}{c}\left(I-\Gamma^{s_{t-L+2}}\right)\bar{\bar{\xi}}^{s_{t-L+2}} \\ +\Gamma^{s_{t-L+2}}\xi - \bar{\bar{\xi}}^{s_{t-L+3}}\end{array}\right)\right\}, \ldots, \\
\left\{\bar{p}^{s_{t+1}} + \left(\Lambda^{s_{t+1}}\right)^T\left(\begin{array}{c}\sum_{i=1}^{L-1}\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}}\right)\left(I-\Gamma^{s_{t-i+1}}\right)\bar{\bar{\xi}}^{s_{t-i+1}} \\ +\left(\prod_{k=1}^{L-1}\Gamma^{s_{t-k+1}}\right)\xi - \bar{\bar{\xi}}^{s_{t+1}}\end{array}\right)\right\}; S_{t-L+3}^{t}
\end{array}\right)
\end{aligned}
$$

where $\xi = \xi_{t-L+1}$.

We denote the set of the equity prices in time τ that an agent born in time t and node S^t should expect as: $\left(p\left(S^t, S_{t+1}^{\tau}\right)\right)_{S_{t+1}^{\tau}} = \left\{H\left(G^{\tau-t}\left(\xi; S_t^{\tau}\right)\right)\right\}$, where $\xi = \xi_{t-1}$. Instead of inserting the shocks into every forecast function, we write the history of shocks only in the outermost forecast function via an obvious abuse of notation. For example, $H\left(G\left(\xi; S_t^{t+1}\right)\right) = H\left(G\left(\xi, s_t\right), s_{t+1}\right)$ or $G^2\left(\xi; s_t^{t+1}\right) = G\left(G\left(\xi, s_t\right), s_{t+1}\right)$. Note that the vector of the asset demand functions in time t, $G^{L-1}\left(\xi; S_{t-L+2}^{t}\right)$, is evaluated at a realized history of shocks from time $t-L+2$ to time t. This is in contrast to the set of the equity prices in the sense that they are evaluated over all possible paths of shocks because agents need to expect the asset prices in the future. We let $\prod_{k=1}^{0}\Gamma^{s_{t-k+1}} = I$. Since $\Gamma^{s_{t-k+1}}$ is a matrix, one needs to be careful about the order of multiplication in the product. For example, $\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}} = \Gamma^{s_t}\Gamma^{s_{t-1}}\ldots\Gamma^{s_{t-i+2}}$.

We can rewrite the optimality condition for the household problem as:

(70)
$$G^{L-1}\left(\xi; S_{t-L+2}^{t}\right) =$$
$$\left[\begin{array}{c}
\left\{e_{(L-1),j}\left(H\left(\xi, s_{t-L+2}\right), \ldots, \left\{H\left(G^{L-1}\left(\xi; S_{t-L+2}^{t+1}\right)\right)\right\}; S_{t-L+3}^{t}\right)\right\}_{j=1,\ldots,M-1} \\
\left\{e_{(L-2),j}\left(H\left(G\left(\xi; S_{t-L+2}^{t-L+3}\right)\right), \ldots, \left\{H\left(G^{L}\left(\xi; S_{t-L+2}^{t+2}\right)\right)\right\}; S_{t-L+4}^{t}\right)\right\}_{j=1,\ldots,M} \\
\vdots \\
\left\{e_{1,j}\left(\ H\left(G^{L-2}\left(\xi; S_{t-L+2}^{t}\right)\right), \ldots, \left\{H\left(G^{2L-3}\left(\xi; S_{t-L+2}^{t+L-1}\right)\right)\right\}\ \right)\right\}_{j=1,\ldots,M}
\end{array}\right]$$
$$\iff \sum_{i=1}^{L-1}\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}}\right)\left(I-\Gamma^{s_{t-i+1}}\right)\bar{\xi}^{s_{t-i+1}} + \left(\prod_{k=1}^{L-1}\Gamma^{s_{t-k+1}}\right)\xi =$$
$$\left[\begin{array}{c}
\left\{e_{(L-1),j}\left(\begin{array}{c}
\bar{p}^{s_{t-L+2}} + \left(\Lambda^{s_{t-L+2}}\right)^{T}\left(\xi - \bar{\xi}^{s_{t-L+2}}\right), \ldots, \\
\left\{\bar{p}^{s_{t+1}} + \left(\Lambda^{s_{t+1}}\right)^{T}\left(\begin{array}{c}
\sum_{i=1}^{L-1}\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}}\right)\left(I-\Gamma^{s_{t-i+1}}\right)\bar{\xi}^{s_{t-i+1}} \\
+\left(\prod_{k=1}^{L-1}\Gamma^{s_{t-k+1}}\right)\xi - \bar{\xi}^{s_{t+1}}
\end{array}\right)\right\}; S_{t-L+3}^{t}
\end{array}\right)\right\}_{j} \\
\vdots \\
\left\{e_{1,j}\left(\begin{array}{c}
\bar{p}^{s_{t}} + \left(\Lambda^{s_{t}}\right)^{T}\left(\begin{array}{c}
\sum_{i=1}^{L-2}\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k}}\right)\left(I-\Gamma^{s_{t-i}}\right)\bar{\xi}^{s_{t-i}} \\
+\left(\prod_{k=1}^{L-2}\Gamma^{s_{t-k}}\right)\xi - \bar{\xi}^{s_{t}}
\end{array}\right), \ldots, \\
\left\{\bar{p}^{s_{t+L-1}} + \left(\Lambda^{s_{t+L-1}}\right)^{T}\left(\begin{array}{c}
\sum_{i=1}^{2L-3}\left(\prod_{k=1}^{i-1}\Gamma^{s_{t+L-k-1}}\right)\left(I-\Gamma^{s_{t+L-i-1}}\right)\bar{\xi}^{s_{t+L-i-1}} \\
+\left(\prod_{k=1}^{2L-3}\Gamma^{s_{t+L-k-1}}\right)\xi - \bar{\xi}^{s_{t+L-1}}
\end{array}\right)\right\}
\end{array}\right)\right\}_{j}
\end{array}\right]$$

The market clearing condition has S^{L-1} equations depending on the history of shocks from time $t-L+2$ to t. The expression for the asset demand functions has $((L-1)M-1)S^{L-1}$ equations if there are at least two agents in each period. This is because the vector of asset demand functions is a $((L-1)M-1)$ dimensional vector and there are S^{L-1} types of the history of shocks from time $t-L+2$ to t. Hence, the total number of equations in the system is $(L-1)MS^{L-1}$.

If there is a representative agent in each period, the number of equations for the asset demand functions will be $(L-2)S^{L-2}$ since the asset holdings of the second oldest generation is removed in the vector of the endogenous state variables, and thus there are S^{L-2} types of the history of shocks from time $t-L+3$ to t. In this case, the total number of equations of the system is $(L-2+S)S^{L-2}$.

With an indicator function, we can denote the total number of equations in the system by $((L-1)M+(S-1)\mathbf{1}(M=1))S^{L-1-\mathbf{1}(M=1)}$ where $\mathbf{1}(M=1)$ is one if there is a representative agent in each period and zero otherwise.

For the rank calculation of the Jacobian matrix, let

$$
\begin{aligned}
(71)\quad A_{z_s} =\, & \iota^T\left(\sum_{i=1}^{L-1}\mathbf{1}\left(s_{t-i+1}=z_s\right)\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}}\right)\left(I-\Gamma^{s_{t-i+1}}\right)\right)\\
& -\mathbf{1}\left(s_{t-L+2}=z_s\right)e_{(L-1),M,p_{t-L+2}}\left(\Lambda^{s_{t-L+2}}\right)^T\\
& +\sum_{\bar{s}_{t-L+3}^{t-L+3}} e_{(L-1),M,p_{t-L+3}}\left(\Lambda^{\bar{s}_{t-L+3}}\right)^T\begin{pmatrix}\mathbf{1}\left(s_{t-L+2}=z_s\right)\left(I-\Gamma^{s_{t-L+2}}\right)\\ -\mathbf{1}\left(\bar{s}_{t-L+3}=z_s\right)I\end{pmatrix}+\dots\\
& +\sum_{\bar{s}_{t-L+3}^{t+1}} e_{(L-1),M,p_{t+1}}\left(\Lambda^{\bar{s}_{t+1}}\right)^T\begin{pmatrix}\sum_{i=1}^{L-1}\mathbf{1}\left(\bar{s}_{t-i+1}=z_s\right)\left(\prod_{k=1}^{i-1}\Gamma^{\bar{s}_{t-k+1}}\right)\left(I-\Gamma^{\bar{s}_{t-i+1}}\right)\\ -\mathbf{1}\left(\bar{s}_{t+1}=z_s\right)I\end{pmatrix}\\
F_{z_s} =\, & \mathbf{1}\left(s_{t-L+2}=z_s\right)e_{(L-1),M,p_{t-L+2}}+\sum_{\bar{s}_{t-L+3}^{t-L+3}}\mathbf{1}\left(\bar{s}_{t-L+3}=z_s\right)e_{(L-1),M,p_{t-L+3}}+\dots\\
& +\sum_{\bar{s}_{t-L+3}^{t+1}}\mathbf{1}\left(\bar{s}_{t+1}=z_s\right)e_{(L-1),M,p_{t+1}}\\
L_{z_s} =\, & \mathbf{1}\left(s_{t-L+2}=z_s\right)e_{(L-1),M,p_{t-L+2}}\left(\xi-\bar{\xi}^{s_{t-L+2}}\right)^T\\
& +\sum_{\bar{s}_{t-L+3}^{t-L+3}}\mathbf{1}\left(\bar{s}_{t-L+3}=z_s\right)e_{(L-1),M,p_{t-L+3}}\left(\left(I-\Gamma^{s_{t-L+2}}\right)\bar{\xi}^{s_{t-L+2}}+\Gamma^{s_{t-L+2}}\xi-\bar{\xi}^{\bar{s}_{t-L+3}}\right)^T+\dots\\
& +\sum_{\bar{s}_{t-L+3}^{t+1}}\mathbf{1}\left(\bar{s}_{t+1}=z_s\right)e_{(L-1),M,p_{t+1}}\begin{pmatrix}\sum_{i=1}^{L-1}\left(\prod_{k=1}^{i-1}\Gamma^{\bar{s}_{t-k+1}}\right)\left(I-\Gamma^{\bar{s}_{t-i+1}}\right)\bar{\xi}^{\bar{s}_{t-i+1}}\\ +\left(\prod_{k=1}^{L-1}\Gamma^{\bar{s}_{t-k+1}}\right)\xi-\bar{\xi}^{\bar{s}_{t+1}}\end{pmatrix}^T\\
M_{z_s}^i =\, & \sum_{j=1}^{L-2}\left(\frac{\partial\iota^T\left(\prod_{k=1}^{j}\Gamma^{s_{t-k+1}}\right)\left(\bar{\xi}^{s_{t-j}}-\bar{\xi}^{s_{t-j+1}}\right)}{\partial\Gamma_i^{z_s}}\right)+\frac{\partial\iota^T\left(\prod_{k=1}^{L-1}\Gamma^{s_{t-k+1}}\right)\left(\xi-\bar{\xi}^{s_{t-L+2}}\right)}{\partial\Gamma_i^{z_s}}\\
& +e_{(L-1),M,p_{t-L+3}}\frac{\partial\left(\Lambda^{\bar{s}_{t-L+3}}\right)^T\Gamma^{s_{t-L+2}}\left(\xi-\bar{\xi}^{s_{t-L+2}}\right)}{\partial\Gamma_i^{z_s}}+\dots\\
& +\sum_{\bar{s}_{t-L+3}^{t+1}} e_{(L-1),M,p_{t+1}}\begin{pmatrix}\sum_{j=1}^{L-2}\left(\frac{\partial\left(\Lambda^{\bar{s}_{t+1}}\right)^T\left(\prod_{k=1}^{j}\Gamma^{\bar{s}_{t-k+1}}\right)\left(\bar{\xi}^{\bar{s}_{t-j}}-\bar{\xi}^{\bar{s}_{t-j+1}}\right)}{\partial\Gamma_i^{z_s}}\right)\\ +\frac{\partial\left(\Lambda^{\bar{s}_{t+1}}\right)^T\left(\prod_{k=1}^{L-1}\Gamma^{\bar{s}_{t-k+1}}\right)\left(\xi-\bar{\xi}^{s_{t-L+2}}\right)}{\partial\Gamma_i^{z_s}}\end{pmatrix}\\
H_{z_s} =\, & \sum_{i=1}^{L-1}\mathbf{1}\left(s_{t-i+1}=z_s\right)\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k+1}}\right)\left(I-\Gamma^{s_{t-i+1}}\right)\\
& -\begin{bmatrix}\left\{\begin{matrix}-\mathbf{1}\left(s_{t-L+2}=z_s\right)e_{(L-1),j,p_{t-L+2}}\left(\Lambda^{s_{t-L+2}}\right)^T+\dots\\ +\sum_{\bar{s}_{t-L+3}^{t+1}} e_{(L-1),j,p_{t+1}}\left(\Lambda^{\bar{s}_{t+1}}\right)^T\begin{pmatrix}\sum_{i=1}^{L-1}\mathbf{1}\left(\bar{s}_{t-i+1}=z_s\right)\left(\prod_{k=1}^{i-1}\Gamma^{\bar{s}_{t-k+1}}\right)\\ \cdot\left(I-\Gamma^{\bar{s}_{t-i+1}}\right)-\mathbf{1}\left(\bar{s}_{t+1}=z_s\right)I\end{pmatrix}\end{matrix}\right\}_j\\ \vdots\\ \left\{\begin{matrix}e_{1,j,p_t}\left(\Lambda^{s_t}\right)^T\begin{pmatrix}\sum_{i=1}^{L-2}\mathbf{1}\left(s_{t-i}=z_s\right)\left(\prod_{k=1}^{i-1}\Gamma^{s_{t-k}}\right)\left(I-\Gamma^{s_{t-i}}\right)\\ -\mathbf{1}\left(s_t=z_s\right)I\end{pmatrix}+\dots\\ +\sum_{\bar{s}_{t+1}^{t+L-1}} e_{1,j,p_{t+L-1}}\left(\Lambda^{\bar{s}_{t+L-1}}\right)^T\begin{pmatrix}\sum_{i=1}^{2L-3}\mathbf{1}\left(\bar{s}_{t+L-i-1}=z_s\right)\left(\prod_{k=1}^{i-1}\Gamma^{\bar{s}_{t+L-k-1}}\right)\\ \cdot\left(I-\Gamma^{\bar{s}_{t+L-i-1}}\right)-\mathbf{1}\left(\bar{s}_{t+L-1}=z_s\right)I\end{pmatrix}\end{matrix}\right\}_j\end{bmatrix}
\end{aligned}
$$

$$J_{z_s} = -\begin{bmatrix} \left\{ \begin{array}{c} \mathbf{1}\left(s_{t-L+2}=z_s\right) e_{(L-1),j,p_{t-L+2}} + \dots \\ + \sum_{\bar{s}_{t-L+3}^{t+1}} \mathbf{1}\left(\bar{s}_{t+1}=z_s\right) e_{(L-1),j,p_{t+1}} \end{array} \right\}_{j=1,\dots,M-1} \\ \vdots \\ \left\{ \begin{array}{c} \mathbf{1}\left(s_t = z_s\right) e_{1,j,p_t} + \dots \\ + \sum_{\bar{s}_{t+1}^{t+L-1}} \mathbf{1}\left(\bar{s}_{t+L-1}=z_s\right) e_{1,j,p_{t+L-1}} \end{array}\right\}_{j=1,\dots,M} \end{bmatrix} \tag{72}$$

$$N_{z_s} = -\begin{bmatrix} \left\{ \begin{array}{c} \mathbf{1}\left(s_{t-L+2}=z_s\right) e_{(L-1),j,p_{t-L+2}} \left(\xi - \bar{\xi}^{s_{t-L+2}}\right)^T + \dots + \sum_{\bar{s}_{t-L+3}^{t+1}} \mathbf{1}\left(\bar{s}_{t+1}=z_s\right) \\ \cdot e_{(L-1),j,p_{t+1}} \left(\begin{array}{c} \sum_{i=1}^{L-1} \left(\prod_{k=1}^{i-1} \Gamma^{\bar{s}_{t-k+1}}\right) \left(I - \Gamma^{\bar{s}_{t-i+1}}\right) \bar{\xi}^{\bar{s}_{t-i+1}} \\ + \left(\prod_{k=1}^{L-1} \Gamma^{\bar{s}_{t-k+1}}\right) \xi - \bar{\xi}^{\bar{s}_{t+1}} \end{array} \right)^T \end{array} \right\}_j \\ \vdots \\ \left\{ \begin{array}{c} \mathbf{1}\left(s_t = z_s\right) e_{1,j,p_t} \left(\begin{array}{c} \sum_{i=1}^{L-2} \left(\prod_{k=1}^{i-1} \Gamma^{s_{t-k}}\right) \left(I - \Gamma^{s_{t-i}}\right) \bar{\xi}^{s_{t-i}} \\ + \left(\prod_{k=1}^{L-2} \Gamma^{s_{t-k}}\right) \xi - \bar{\xi}^{s_t} \end{array} \right)^T + \dots \\ + \sum_{\bar{s}_{t+1}^{t+L-1}} \mathbf{1}\left(\bar{s}_{t+L-1}=z_s\right) e_{1,j,p_{t+L-1}} \left(\begin{array}{c} \sum_{i=1}^{2L-3} \left(\prod_{k=1}^{i-1} \Gamma^{\bar{s}_{t+L-k-1}}\right) \left(I - \Gamma^{\bar{s}_{t+L-i-1}}\right) \\ \cdot \bar{\xi}^{\bar{s}_{t+L-i-1}} + \left(\prod_{k=1}^{2L-3} \Gamma^{\bar{s}_{t+L-k-1}}\right) \xi - \bar{\xi}^{\bar{s}_{t+L-1}} \end{array} \right)^T \end{array} \right\}_j \end{bmatrix}$$

$$O^i_{z_s} = \sum_{j=1}^{L-2} \left(\frac{\partial \left(\prod_{k=1}^{j} \Gamma^{s_{t-k+1}}\right) \left(\bar{\xi}^{s_{t-j}} - \bar{\xi}^{s_{t-j+1}}\right)}{\partial \Gamma_i^{z_s}} \right) + \frac{\partial \left(\prod_{k=1}^{L-1} \Gamma^{s_{t-k+1}}\right) \left(\xi - \bar{\xi}^{s_{t-L+2}}\right)}{\partial \Gamma_i^{z_s}}$$

$$- \begin{bmatrix} \left\{ \begin{array}{c} e_{(L-1),j,p_{t-L+3}} \frac{\partial \left(\Lambda^{\bar{s}_{t-L+3}}\right)^T \Gamma^{s_{t-L+2}} \left(\xi - \bar{\xi}^{s_{t-L+2}}\right)}{\partial \Gamma_i^{z_s}} + \dots \\ + \sum_{\bar{s}_{t-L+3}^{t+1}} e_{(L-1),j,p_{t+1}} \left(\begin{array}{c} \sum_{j=1}^{L-2} \left(\frac{\partial \left(\Lambda^{\bar{s}_{t+1}}\right)^T \left(\prod_{k=1}^{j} \Gamma^{\bar{s}_{t-k+1}}\right) \left(\bar{\xi}^{\bar{s}_{t-j}} - \bar{\xi}^{\bar{s}_{t-j+1}}\right)}{\partial \Gamma_i^{z_s}} \right) \\ + \frac{\partial \left(\Lambda^{\bar{s}_{t+1}}\right)^T \left(\prod_{k=1}^{L-1} \Gamma^{\bar{s}_{t-k+1}}\right) \left(\xi - \bar{\xi}^{s_{t-L+2}}\right)}{\partial \Gamma_i^{z_s}} \end{array} \right) \end{array} \right\}_j \\ \vdots \\ \left\{ \begin{array}{c} e_{1,j,p_t} \left(\begin{array}{c} \sum_{j=1}^{L-3} \left(\frac{\partial \left(\Lambda^{s_t}\right)^T \left(\prod_{k=1}^{j} \Gamma^{s_{t-k}}\right) \left(\bar{\xi}^{s_{t-j-1}} - \bar{\xi}^{s_{t-j}}\right)}{\partial \Gamma_i^{z_s}} \right) \\ + \frac{\partial \left(\Lambda^{s_t}\right)^T \left(\prod_{k=1}^{L-2} \Gamma^{s_{t-k}}\right) \left(\xi - \bar{\xi}^{s_{t-L+2}}\right)}{\partial \Gamma_i^{z_s}} \end{array} \right) + \dots \\ + \sum_{\bar{s}_{t+1}^{t+L-1}} e_{1,j,p_{t+L-1}} \left(\begin{array}{c} \sum_{j=1}^{2L-4} \left(\frac{\partial \left(\Lambda^{\bar{s}_{t+L-1}}\right)^T \left(\prod_{k=1}^{j} \Gamma^{\bar{s}_{t+L-k-1}}\right) \left(\bar{\xi}^{\bar{s}_{t+L-j-2}} - \bar{\xi}^{\bar{s}_{t+L-j-1}}\right)}{\partial \Gamma_i^{z_s}} \right) \\ + \left(\frac{\partial \left(\Lambda^{\bar{s}_{t+L-1}}\right)^T \left(\prod_{k=1}^{2L-3} \Gamma^{\bar{s}_{t+L-k-1}}\right) \left(\xi - \bar{\xi}^{s_{t-L+2}}\right)}{\partial \Gamma_i^{z_s}} \right) \end{array} \right) \end{array} \right\}_j \end{bmatrix}$$

where s denotes the history of shocks through which the economy evolves but $\bar{s}$ represents the possible histories of shocks in the future that agents need to expect. The state at time $t - L + 2$ is given and thus, $\bar{s}_{t-L+2} = s_{t-L+2}$. $\mathbf{1}\left(s_t = z_s\right)$ is one if s_t is z_s and zero otherwise. e_{i,j,p_τ} is the derivative of the asset holding of age-i and type-j agent with respect to p_τ. The derivatives of the system with respect to $\Gamma_i^{z_s}$ – the i-th row of the matrix Γ^{z_s}, $M^i_{z_s}$ and $O^i_{z_s}$ will be zero at the deterministic steady state. This is because the terms in the small parentheses cancel out since the transition vectors in the LIFS are the same at the steady state: $\xi = \bar{\xi}^{z_s} = \bar{\xi}$ for $\forall s \in \{1, \dots, S\}$.

Let these variables at the steady state, $\Gamma^{z_s} = \Gamma$, $\Lambda^{z_s} = \Lambda$, and $\bar{p}^{z_s} = \bar{p}$ for $\forall s \in \{1, \ldots, S\}$:

$$
\begin{aligned}
A_{z_s} &= \iota^T \left(\sum_{i=1}^{L-1} \mathbf{1}\left(s_{t-i+1} = z_s\right) \Gamma^{i-1} \left(I - \Gamma\right) \right) - \mathbf{1}\left(s_{t-L+2} = z_s\right) e_{(L-1),M,p_{t-L+2}} \Lambda^T \\
&\quad + \sum_{\bar{s}_{t-L+3}^{t-L+3}} e_{(L-1),M,p_{t-L+3}} \Lambda^T \left[\begin{array}{c} \mathbf{1}\left(s_{t-L+2} = z_s\right) \left(I - \Gamma\right) \\ -\mathbf{1}\left(\bar{s}_{t-L+3} = z_s\right) I \end{array} \right] + \ldots \\
&\quad + \sum_{\bar{s}_{t-L+3}^{t+1}} e_{(L-1),M,p_{t+1}} \Lambda^T \left[\begin{array}{c} \sum_{i=1}^{L-1} \mathbf{1}\left(\bar{s}_{t-i+1} = z_s\right) \Gamma^{i-1} \left(I - \Gamma\right) \\ -\mathbf{1}\left(\bar{s}_{t+1} = z_s\right) I \end{array} \right] \\
&= A^1_{z_s,n} + \mathbf{1}\left(s_{t-L+2} = z_s\right) A^2 + A^3 \\
F_{z_s} &= \mathbf{1}\left(s_{t-L+2} = z_s\right) e_{(L-1),M,p_{t-L+2}} + \sum_{\bar{s}_{t-L+3}^{t-L+3}} \mathbf{1}\left(\bar{s}_{t-L+3} = z_s\right) e_{(L-1),M,p_{t-L+3}} + \ldots \\
&\quad + \sum_{\bar{s}_{t-L+3}^{t+1}} \mathbf{1}\left(\bar{s}_{t+1} = z_s\right) e_{(L-1),M,p_{t+1}} \\
&\quad = \mathbf{1}\left(s_{t-L+2} = z_s\right) F^1 + F^2 \\
L_{z_s} &= \mathbf{0} \\
M^i_{z_s} &= \mathbf{0} \\
H_{z_s} &= \sum_{i=1}^{L-1} \mathbf{1}\left(s_{t-i+1} = z_s\right) \Gamma^{i-1} \left(I - \Gamma\right) \\
&\quad - \left[\begin{array}{c} \left(\begin{array}{c} -\mathbf{1}\left(s_{t-L+2} = z_s\right) e_{(L-1),j,p_{t-L+2}} \Lambda^T + \ldots \\ + \sum_{\bar{s}_{t-L+3}^{t+1}} e_{(L-1),j,p_{t+1}} \Lambda^T \left[\begin{array}{c} \sum_{i=1}^{L-1} \mathbf{1}\left(\bar{s}_{t-i+1} = z_s\right) \Gamma^{i-1} \left(I - \Gamma\right) \\ -\mathbf{1}\left(\bar{s}_{t+1} = z_s\right) I \end{array} \right] \end{array} \right)_j \\ \vdots \\ \left(\begin{array}{c} e_{1,j,p_t} \Lambda^T \left[\begin{array}{c} \sum_{i=1}^{L-2} \mathbf{1}\left(s_{t-i} = z_s\right) \Gamma^{i-1} \left(I - \Gamma\right) \\ -\mathbf{1}\left(s_t = z_s\right) I \end{array} \right] + \ldots \\ + \sum_{\bar{s}_{t+1}^{t+L-1}} e_{1,j,p_{t+L-1}} \Lambda^T \left[\begin{array}{c} \sum_{i=1}^{2L-3} \mathbf{1}\left(\bar{s}_{t+L-i-1} = z_s\right) \Gamma^{i-1} \left(I - \Gamma\right) \\ -\mathbf{1}\left(\bar{s}_{t+L-1} = z_s\right) I \end{array} \right] \end{array} \right)_j \end{array} \right] \\
&= H^1_{z_s,n} + H^2_{z_s,n} + H^3 \\
J_{z_s} &= - \left[\begin{array}{c} \left(\mathbf{1}\left(s_{t-L+2} = z_s\right) \ e_{(L-1),j,p_{t-L+2}} + \ldots + \sum_{\bar{s}_{t-L+3}^{t+1}} \mathbf{1}\left(\bar{s}_{t+1} = z_s\right) e_{(L-1),j,p_{t+1}} \right)_j \\ \vdots \\ \left(\mathbf{1}\left(s_t = z_s\right) e_{1,j,p_t} + \ldots + \sum_{\bar{s}_{t+1}^{t+L-1}} \mathbf{1}\left(\bar{s}_{t+L-1} = z_s\right) e_{1,j,p_{t+L-1}} \right)_j \end{array} \right] \\
&= J^1_{z_s,n} + J^2 \\
N_{z_s} &= \mathbf{0} \\
O^i_{z_s} &= \mathbf{0}
\end{aligned}
\tag{73}
$$

where $A^1_{z_s,n}$ denotes the first bracket in A_{z_s}. With an abuse of notation, we let n be either an index for the n-th branch in a date-event tree from time $t - L + 2$ to time t or the sequence of shocks itself. For example, $n = 1$ designates the first branch in which all realized states are z_1. $n = 2$ is an index for a history of shocks having z_1 for all periods but time t when $s_t = z_2$. Terms in $A^1_{z_s,n}$ are evaluated at the n-th history of shocks. A^2 points out all elements multiplied by $\mathbf{1}\left(s_{t-L+2} = z_s\right)$, and it remains the same no matter what a history of shocks is at the deterministic steady state. Thus, there are no history specific indices. A^3 indicates the rest part of A_{z_s} and this term stays the same no matter what z_s and a history of shock are since it is only affected by the projected histories of shocks forward after the date of birth and such projected histories of shocks are symmetric in the sense that each state of nature has the same cases of shock histories. We can define F^1 and F^2 similar to A^2 and A^3. $H^1_{z_s,n}$ and H^3 are defined as $A^1_{z_s,n}$ and A^3. $H^2_{z_s,n}$ extracts terms in the big bracket of H_{z_s} corresponding to the n-th realized history of shocks. $J^1_{z_s,n}$ and J^2 are terms characterized similar to $H^2_{z_s,n}$ and H^3. Note that $\sum_{s=1}^{S} A^1_{z_s,n} = \sum_{s=1}^{S} A^1_{z_s,n'}$, $\sum_{s=1}^{S} H^1_{z_s,n} = \sum_{s=1}^{S} H^1_{z_s,n'}$, $\sum_{s=1}^{S} H^2_{z_s,n} = \sum_{s=1}^{S} H^2_{z_s,n'}$ and $\sum_{s=1}^{S} J^1_{z_s,n} = \sum_{s=1}^{S} J^1_{z_s,n'}$ for all $(n,n') \in \{z_1,\ldots,z_S\}^{L-1} \times \{z_1,\ldots,z_S\}^{L-1}$.

Then, the Jacobian matrix with respect to the unknowns in a LIFS including the asset quantity parameters, $\left\{\bar{\xi}^{z_s}, \bar{P}^{z_s}, \Lambda^{z_s}, \Gamma^{z_s}, \bar{a}^{z_s}\right\}_{s=1}^{S}$, takes the form:

(74)
$$DZ = \begin{array}{c} \begin{array}{ccccccc} \partial\left(\bar{\xi}^{z_s}\right)^T & \partial\bar{P}^{z_s} & \partial\left(\Lambda^{z_s}\right)^T & \partial\Gamma_1^{z_s} & \ldots & \partial\Gamma^{z_s}_{((L-1)M-1)} & \partial\bar{a}^{z_s} \end{array} \\ \left[\begin{array}{ccccccc} \mathbf{A}_{S^t_{t-L+2}} & \mathbf{F}_{S^t_{t-L+2}} & \mathbf{L}_{S^t_{t-L+2}} & \mathbf{M}^1_{S^t_{t-L+2}} & \ldots & \mathbf{M}^{((L-1)M-1)}_{S^t_{t-L+2}} & -\mathbf{1}\left(s_t = z_s\right)_{S^t_{t-L+2}} \\ \mathbf{H}_{S^t_{t-L+2}} & \mathbf{J}_{S^t_{t-L+2}} & \mathbf{N}_{S^t_{t-L+2}} & \mathbf{O}^1_{S^t_{t-L+2}} & \ldots & \mathbf{O}^{((L-1)M-1)}_{S^t_{t-L+2}} & \mathbf{0} \end{array}\right] \end{array}$$

where the partial derivatives should be taken for $\forall s \in \{1,\ldots,S\}$, but we write them only for an unspecified state z_s for every variable for notational simplicity. This Jacobian matrix is a $\left(\left((L-1)M + (S-1)\mathbf{1}(M=1)\right) S^{L-1-\mathbf{1}(M=1)}\right) \times \left(\left(\left((L-1)M\right)^2 + 1\right) S\right)$ matrix. The submatrices with the subscript, S^t_{t-L+2}, in the Jacobian matrix are matrices created by stacking up rowwise the derivative matrices with respect to the price and equity parameters evaluated over all the histories of shocks from time $t-L+2$ to time t.

The Jacobian matrix evaluated at the deterministic steady state yields the following matrix. Note that the submatrices in the Jacobian matrix corresponding to Λ^{z_s} and $\Gamma_i^{z_s}$ become a zero matrix if evaluated at the steady state, so we remove them in the Jacobian matrix since zero matrices do not affect the rank calculation.

The upper submatrix above the horizontal line denotes the partial derivatives for the market clearing condition. This submatrix has S^{L-1} rows. We order the rows by the history of shocks from time $t-L+2$ to t. In the first row, all states across time $t-L+2$ to t are z_1. In the second row, states are z_1 for all times but time t when the state is z_2. States are z_S for all periods from time $t-L+2$ to time t in the last row.

The lower submatrix denotes the partial derivatives for the asset demand functions. It has $((L-1)M-1)S^{L-1}$ rows under heterogeneity. We also order the rows in the lower submatrix by the history of shocks, but we note that variables regarding the asset demand functions in each history of shocks have $(L-1)M-1$ rows. Thus, in the lower submatrix, we use identity and zero matrices of which row dimension is $(L-1)M-1$ as well.

$$
DZ = \left[\begin{array}{cccccccccccc}
A^1_{z_1,1}+A^2+A^3 & A^1_{z_2,1}+A^3 & \cdots & A^1_{z_S,1}+A^3 & F^1+F^2 & F^2 & \cdots & F^2 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,2}+A^2+A^3 & A^1_{z_2,2}+A^3 & \cdots & A^1_{z_S,2}+A^3 & F^1+F^2 & F^2 & \cdots & F^2 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S}+A^2+A^3 & A^1_{z_2,S}+A^3 & \cdots & A^1_{z_S,S}+A^3 & F^1+F^2 & F^2 & \cdots & F^2 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,S^{L-2}+1}+A^3 & A^1_{z_2,S^{L-2}+1}+A^2+A^3 & \cdots & A^1_{z_S,S^{L-2}+1}+A^3 & F^2 & F^1+F^2 & \cdots & F^2 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,S^{L-2}+2}+A^3 & A^1_{z_2,S^{L-2}+2}+A^2+A^3 & \cdots & A^1_{z_S,S^{L-2}+2}+A^3 & F^2 & F^1+F^2 & \cdots & F^2 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S^{L-2}+S}+A^3 & A^1_{z_2,S^{L-2}+S}+A^2+A^3 & \cdots & A^1_{z_S,S^{L-2}+S}+A^3 & F^2 & F^1+F^2 & \cdots & F^2 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1(S-1)S^{L-2}+1}+A^3 & A^1_{z_2,(S-1)S^{L-2}+1}+A^3 & \cdots & \begin{matrix}A^1_{z_S,(S-1)S^{L-2}+1}+\\ A^2+A^3\end{matrix} & F^2 & F^2 & \cdots & F^1+F^2 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,(S-1)S^{L-2}+2}+A^3 & A^1_{z_2,(S-1)S^{L-2}+2}+A^3 & \cdots & \begin{matrix}A^1_{z_S,(S-1)S^{L-2}+2}+\\ A^2+A^3\end{matrix} & F^2 & F^2 & \cdots & F^1+F^2 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,(S-1)S^{L-2}+S}+A^3 & A^1_{z_2,(S-1)S^{L-2}+S}+A^3 & \cdots & \begin{matrix}A^1_{z_S,(S-1)S^{L-2}+S}+\\ A^2+A^3\end{matrix} & F^2 & F^2 & \cdots & F^1+F^2 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
H^1_{z_1,1}+H^2_{z_1,1}+H^3 & H^1_{z_2,1}+H^2_{z_2,1}+H^3 & \cdots & H^1_{z_S,1}+H^2_{z_S,1}+H^3 & J^1_{z_1,1}+J^2 & J^1_{z_2,1}+J^2 & \cdots & J^1_{z_S,1}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,2}+H^2_{z_1,2}+H^3 & H^1_{z_2,2}+H^2_{z_2,2}+H^3 & \cdots & H^1_{z_S,2}+H^2_{z_S,2}+H^3 & J^1_{z_1,2}+J^2 & J^1_{z_2,2}+J^2 & \cdots & J^1_{z_S,2}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H^1_{z_1,S}+H^2_{z_1,S}+H^3 & H^1_{z_2,S}+H^2_{z_2,S}+H^3 & \cdots & H^1_{z_S,S}+H^2_{z_S,S}+H^3 & J^1_{z_1,S}+J^2 & J^1_{z_2,S}+J^2 & \cdots & J^1_{z_S,S}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{matrix}H^1_{z_1,(S-1)S^{L-2}+1}+\\ H^2_{z_1,(S-1)S^{L-2}+1}+H^3\end{matrix} & \begin{matrix}H^1_{z_2,(S-1)S^{L-2}+1}+\\ H^2_{z_2,(S-1)S^{L-2}+1}+H^3\end{matrix} & \cdots & \begin{matrix}H^1_{z_3,(S-1)S^{L-2}+1}+\\ H^2_{z_S,(S-1)S^{L-2}+1}+H^3\end{matrix} & \begin{matrix}J^1_{z_1,(S-1)S^{L-2}+1}\\ +J^2\end{matrix} & \begin{matrix}J^1_{z_2,(S-1)S^{L-2}+1}\\ +J^2\end{matrix} & \cdots & \begin{matrix}J^1_{z_S,(S-1)S^{L-2}+1}\\ +J^2\end{matrix} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\begin{matrix}H^1_{z_1,(S-1)S^{L-2}+2}+\\ H^2_{z_1,(S-1)S^{L-2}+2}+H^3\end{matrix} & \begin{matrix}H^1_{z_2,(S-1)S^{L-2}+2}+\\ H^2_{z_2,(S-1)S^{L-2}+2}+H^3\end{matrix} & \cdots & \begin{matrix}H^1_{z_S,(S-1)S^{L-2}+2}+\\ H^2_{z_S,(S-1)S^{L-2}+2}+H^3\end{matrix} & \begin{matrix}J^1_{z_1,(S-1)S^{L-2}+2}\\ +J^2\end{matrix} & \begin{matrix}J^1_{z_2,(S-1)S^{L-2}+2}\\ +J^2\end{matrix} & \cdots & \begin{matrix}J^1_{z_S,(S-1)S^{L-2}+2}\\ +J^2\end{matrix} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\begin{matrix}H^1_{z_1,(S-1)S^{L-2}+S}+\\ H^2_{z_1,(S-1)S^{L-2}+S}+H^3\end{matrix} & \begin{matrix}H^1_{z_2,(S-1)S^{L-2}+S}+\\ H^2_{z_2,(S-1)S^{L-2}+S}+H^3\end{matrix} & \cdots & \begin{matrix}H^1_{z_S,(S-1)S^{L-2}+S}+\\ H^2_{z_S,(S-1)S^{L-2}+S}+H^3\end{matrix} & \begin{matrix}J^1_{z_1,(S-1)S^{L-2}+S}\\ +J^2\end{matrix} & \begin{matrix}J^1_{z_2,(S-1)S^{L-2}+S}\\ +J^2\end{matrix} & \cdots & \begin{matrix}J^1_{z_S,(S-1)S^{L-2}+S}\\ +J^2\end{matrix} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right] \tag{75}
$$

We reduce this Jacobian matrix via row and column operations. We remove all A^3 and F^2 with identity matrices in the rightmost columns and obtain the following matrix.

$$
DZ = \left[\begin{array}{cccccccccccc}
A^1_{z_1,1}+A^2 & A^1_{z_2,1} & \cdots & A^1_{z_S,1} & F^1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,2}+A^2 & A^1_{z_2,2} & \cdots & A^1_{z_S,2} & F^1 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S}+A^2 & A^1_{z_2,S} & \cdots & A^1_{z_S,S} & F^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,S^{L-2}+1} & A^1_{z_2,S^{L-2}+1}+A^2 & \cdots & A^1_{z_S,S^{L-2}+1} & 0 & F^1 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,S^{L-2}+2} & A^1_{z_2,S^{L-2}+2}+A^2 & \cdots & A^1_{z_S,S^{L-2}+2} & 0 & F^1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S^{L-2}+S} & A^1_{z_2,S^{L-2}+S}+A^2 & \cdots & A^1_{z_S,S^{L-2}+S} & 0 & F^1 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1(S-1)S^{L-2}+1} & A^1_{z_2,(S-1)S^{L-2}+1} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1}+A^2 & 0 & 0 & \cdots & F^1 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,(S-1)S^{L-2}+2} & A^1_{z_2,(S-1)S^{L-2}+2} & \cdots & A^1_{z_S,(S-1)S^{L-2}+2}+A^2 & 0 & 0 & \cdots & F^1 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,(S-1)S^{L-2}+S} & A^1_{z_2,(S-1)S^{L-2}+S} & \cdots & A^1_{z_S,(S-1)S^{L-2}+S}+A^2 & 0 & 0 & \cdots & F^1 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
H^1_{z_1,1}+H^2_{z_1,1}+H^3 & H^1_{z_2,1}+H^2_{z_2,1}+H^3 & \cdots & H^1_{z_S,1}+H^2_{z_S,1}+H^3 & J^1_{z_1,1}+J^2 & J^1_{z_2,1}+J^2 & \cdots & J^1_{z_S,1}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,2}+H^2_{z_1,2}+H^3 & H^1_{z_2,2}+H^2_{z_2,2}+H^3 & \cdots & H^1_{z_S,2}+H^2_{z_S,2}+H^3 & J^1_{z_1,2}+J^2 & J^1_{z_2,2}+J^2 & \cdots & J^1_{z_S,2}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H^1_{z_1,S}+H^2_{z_1,S}+H^3 & H^1_{z_2,S}+H^2_{z_2,S}+H^3 & \cdots & H^1_{z_S,S}+H^2_{z_S,S}+H^3 & J^1_{z_1,S}+J^2 & J^1_{z_2,S}+J^2 & \cdots & J^1_{z_S,S}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{array}{c} H^1_{z_1,(S-1)S^{L-2}+1}+ \\ H^2_{z_1,(S-1)S^{L-2}+1}+H^3 \end{array} & \begin{array}{c} H^1_{z_2,(S-1)S^{L-2}+1}+ \\ H^2_{z_2,(S-1)S^{L-2}+1}+H^3 \end{array} & \cdots & \begin{array}{c} H^1_{z_S,(S-1)S^{L-2}+1}+ \\ H^2_{z_S,(S-1)S^{L-2}+1}+H^3 \end{array} & \begin{array}{c} J^1_{z_1,(S-1)S^{L-2}+1} \\ +J^2 \end{array} & \begin{array}{c} J^1_{z_2,(S-1)S^{L-2}+1} \\ +J^2 \end{array} & \cdots & \begin{array}{c} J^1_{z_S,(S-1)S^{L-2}+1} \\ +J^2 \end{array} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\begin{array}{c} H^1_{z_1,(S-1)S^{L-2}+2}+ \\ H^2_{z_1,(S-1)S^{L-2}+2}+H^3 \end{array} & \begin{array}{c} H^1_{z_2,(S-1)S^{L-2}+2}+ \\ H^2_{z_2,(S-1)S^{L-2}+2}+H^3 \end{array} & \cdots & \begin{array}{c} H^1_{z_S,(S-1)S^{L-2}+2}+ \\ H^2_{z_S,(S-1)S^{L-2}+2}+H^3 \end{array} & \begin{array}{c} J^1_{z_1,(S-1)S^{L-2}+2} \\ +J^2 \end{array} & \begin{array}{c} J^1_{z_2,(S-1)S^{L-2}+2} \\ +J^2 \end{array} & \cdots & \begin{array}{c} J^1_{z_S,(S-1)S^{L-2}+2} \\ +J^2 \end{array} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\begin{array}{c} H^1_{z_1,(S-1)S^{L-2}+S}+ \\ H^2_{z_1,(S-1)S^{L-2}+S}+H^3 \end{array} & \begin{array}{c} H^1_{z_2,(S-1)S^{L-2}+S}+ \\ H^2_{z_2,(S-1)S^{L-2}+S}+H^3 \end{array} & \cdots & \begin{array}{c} H^1_{z_S,(S-1)S^{L-2}+S}+ \\ H^2_{z_S,(S-1)S^{L-2}+S}+H^3 \end{array} & \begin{array}{c} J^1_{z_1,(S-1)S^{L-2}+S} \\ +J^2 \end{array} & \begin{array}{c} J^1_{z_2,(S-1)S^{L-2}+S} \\ +J^2 \end{array} & \cdots & \begin{array}{c} J^1_{z_S,(S-1)S^{L-2}+S} \\ +J^2 \end{array} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right] \tag{76}
$$

Subtract the first S^{L-2} rows corresponding to the histories of shocks where $s_{t-L+2} = z_1$ from the next every S^{L-2} rows within the upper submatrix. Likewise, subtract the first $((L-1)\,M-1)\,S^{L-2}$ rows in the lower submatrix corresponding to the histories of shocks in which $s_{t-L+2} = z_1$ from the next every $((L-1)\,M-1)\,S^{L-2}$ rows. Then, we derive the following matrix.

$$DZ = \left[\begin{array}{cccccccccccc}
A^1_{z_1,1}+A^2 & A^1_{z_2,1} & \cdots & A^1_{z_S,1} & F^1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,2}+A^2 & A^1_{z_2,2} & \cdots & A^1_{z_S,2} & F^1 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S}+A^2 & A^1_{z_2,S} & \cdots & A^1_{z_S,S} & F^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,S^{L-2}+1,1}-A^2 & A^1_{z_2,S^{L-2}+1,1}+A^2 & \cdots & A^1_{z_S,S^{L-2}+1,1} & -F^1 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
A^1_{z_1,S^{L-2}+2,2}-A^2 & A^1_{z_2,S^{L-2}+2,2}+A^2 & \cdots & A^1_{z_S,S^{L-2}+2,2} & -F^1 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S^{L-2}+S,S}-A^2 & A^1_{z_2,S^{L-2}+S,S}+A^2 & \cdots & A^1_{z_S,S^{L-2}+S,S} & -F^1 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,(S-1)S^{L-2}+1,1}-A^2 & A^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1,1}+A^2 & -F^1 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
A^1_{z_1,(S-1)S^{L-2}+2,2}-A^2 & A^1_{z_2,(S-1)S^{L-2}+2,2} & \cdots & A^1_{z_S,(S-1)S^{L-2}+2,2}+A^2 & -F^1 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,(S-1)S^{L-2}+S,S}-A^2 & A^1_{z_2,(S-1)S^{L-2}+S,S} & \cdots & A^1_{z_S,(S-1)S^{L-2}+S,S}+A^2 & -F^1 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
H^1_{z_1,1}+H^2_{z_1,1}+H^3 & H^1_{z_2,1}+H^2_{z_2,1}+H^3 & \cdots & H^1_{z_S,1}+H^2_{z_S,1}+H^3 & J^1_{z_1,1}+J^2 & J^1_{z_2,1}+J^2 & \cdots & J^1_{z_S,1}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,2}+H^2_{z_1,2}+H^3 & H^1_{z_2,2}+H^2_{z_2,2}+H^3 & \cdots & H^1_{z_S,2}+H^2_{z_S,2}+H^3 & J^1_{z_1,2}+J^2 & J^1_{z_2,2}+J^2 & \cdots & J^1_{z_S,2}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H^1_{z_1,S}+H^2_{z_1,S}+H^3 & H^1_{z_2,S}+H^2_{z_2,S}+H^3 & \cdots & H^1_{z_S,S}+H^2_{z_S,S}+H^3 & J^1_{z_1,S}+J^2 & J^1_{z_2,S}+J^2 & \cdots & J^1_{z_S,S}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^1_{z_1,(S-1)S^{L-2}+1,1}+H^2_{z_1,(S-1)S^{L-2}+1,1} & H^1_{z_2,(S-1)S^{L-2}+1,1}+H^2_{z_2,(S-1)S^{L-2}+1,1} & \cdots & H^1_{z_S,(S-1)S^{L-2}+1,1}+H^2_{z_S,(S-1)S^{L-2}+1,1} & J^1_{z_1,(S-1)S^{L-2}+1,1} & J^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,(S-1)S^{L-2}+2,2}+H^2_{z_1,(S-1)S^{L-2}+2,2} & H^1_{z_2,(S-1)S^{L-2}+2,2}+H^2_{z_2,(S-1)S^{L-2}+2,2} & \cdots & H^1_{z_S,(S-1)S^{L-2}+2,2}+H^2_{z_S,(S-1)S^{L-2}+2,2} & J^1_{z_1,(S-1)S^{L-2}+2,2} & J^1_{z_2,(S-1)S^{L-2}+2,2} & \cdots & J^1_{z_S,(S-1)S^{L-2}+2,2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H^1_{z_1,(S-1)S^{L-2}+S,S}+H^2_{z_1,(S-1)S^{L-2}+S,S} & H^1_{z_2,(S-1)S^{L-2}+S,S}+H^2_{z_2,(S-1)S^{L-2}+S,S} & \cdots & H^1_{z_S,(S-1)S^{L-2}+S,S}+H^2_{z_S,(S-1)S^{L-2}+S,S} & J^1_{z_1,(S-1)S^{L-2}+S,S} & J^1_{z_2,(S-1)S^{L-2}+S,S} & \cdots & J^1_{z_S,(S-1)S^{L-2}+S,S} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right] \tag{77}$$

In the matrix above, $A^1_{z_s,n,n'} = A^1_{z_s,n} - A^1_{z_s,n'}$ for $\forall s \in \{1,\dots,S\}$ and $(n,n') \in \{z_1,\dots,z_S\}^{L-1} \times \{z_1,\dots,z_S\}^{L-1}$. We define $H^1_{z_s,n,n'}, H^2_{z_s,n,n'}$ and $J^1_{z_s,n,n'}$ similarly. Note that $A^1_{z_s,s'\cdot S^{L-2}+1,1} = A^1_{z_s,s'\cdot S^{L-2}+2,2} = \dots = A^1_{z_s,s'\cdot S^{L-2}+S^{L-2},S^{L-2}}$ for $\forall\,(s,s') \in \{1,\dots,S\} \times \{1,\dots,S-1\}$. In other words, rows corresponding to the same state in time $t-L+2$, s_{t-L+2}, are identical in the upper submatrix of (77). This result also applies to H^1, H^2, and J^1. Therefore, we can clear every block of S^{L-2} rows using the first row within each block. Similarly, clear every block of $((L-1)\,M-1)\,S^{L-2}$ rows using the first $(L-1)\,M-1$ rows within each block in the lower submatrix. Via this procedure, we obtain the following matrix.

$$DZ = \left[\begin{array}{cccccccccccc}
A^1_{z_1,1}+A^2 & A^1_{z_2,1} & \cdots & A^1_{z_S,1} & F^1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
A^1_{z_1,2}+A^2 & A^1_{z_2,2} & \cdots & A^1_{z_S,2} & F^1 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A^1_{z_1,S}+A^2 & A^1_{z_2,S} & \cdots & A^1_{z_S,S} & F^1 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,S^{L-2}+1,1}-A^2 & A^1_{z_2,S^{L-2}+1,1}+A^2 & \cdots & A^1_{z_S,S^{L-2}+1,1} & -F^1 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,(S-1)S^{L-2}+1,1}-A^2 & A^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1,1}+A^2 & -F^1 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
H^1_{z_1,1}+H^2_{z_1,1}+H^3 & H^1_{z_2,1}+H^2_{z_2,1}+H^3 & \cdots & H^1_{z_S,1}+H^2_{z_S,1}+H^3 & J^1_{z_1,1}+J^2 & J^1_{z_2,1}+J^2 & \cdots & J^1_{z_S,1}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,2}+H^2_{z_1,2}+H^3 & H^1_{z_2,2}+H^2_{z_2,2}+H^3 & \cdots & H^1_{z_S,2}+H^2_{z_S,2}+H^3 & J^1_{z_1,2}+J^2 & J^1_{z_2,2}+J^2 & \cdots & J^1_{z_S,2}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H^1_{z_1,S}+H^2_{z_1,S}+H^3 & H^1_{z_2,S}+H^2_{z_2,S}+H^3 & \cdots & H^1_{z_S,S}+H^2_{z_S,S}+H^3 & J^1_{z_1,S}+J^2 & J^1_{z_2,S}+J^2 & \cdots & J^1_{z_S,S}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{array}{c} H^1_{z_1,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_1,(S-1)S^{L-2}+1,1} \end{array} & \begin{array}{c} H^1_{z_2,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_2,(S-1)S^{L-2}+1,1} \end{array} & \cdots & \begin{array}{c} H^1_{z_S,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_S,(S-1)S^{L-2}+1,1} \end{array} & J^1_{z_1,(S-1)S^{L-2}+1,1} & J^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right] \tag{78}$$

To simplify the first S^{L-2} rows in the upper submatrix and the first $((L-1)M-1)S^{L-2}$ rows in the lower submatrix, we repeat the same procedures as above. Subtract the first S^{L-3} rows corresponding to the histories of shocks where $(s_{t-L+2}, s_{t-L+3}) = (z_1, z_1)$ from the next every S^{L-3} rows within the upper submatrix. Likewise, subtract the first $((L-1)M-1)S^{L-3}$ rows in the lower submatrix corresponding to the histories of shocks in which $(s_{t-L+2}, s_{t-L+3}) = (z_1, z_1)$ from the next every $((L-1)M-1)S^{L-3}$ rows. Then, rows corresponding to the same states in time $t-L+2$ and $t-L+3$, are identical in both submatrices. Clear every block of S^{L-3} rows using the first row within each block. Similarly, clear every block of $((L-1)M-1)S^{L-3}$ rows using the first $(L-1)M-1$ rows within each block. Repeat this process with the first S^{L-4} rows, the first S^{L-5} rows and up to the first S rows for the upper submatrix. Likewise, iterate the same methods of reducing rows with the first $((L-1)M-1)S^{L-4}$ rows, the first $((L-1)M-1)S^{L-5}$ rows and up to the first $(L-1)M-1$ rows for the lower submatrix. Lastly, we can clear the first S rows using an identity matrix in the rightmost columns. Then, we derive the following upper and lower submatrices respectively.

For the upper submatrix,

$$
DZ^u = \left[\begin{array}{cccccccccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
A^1_{z_1,S+1,1} & A^1_{z_2,S+1,1} & \cdots & A^1_{z_S,S+1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,S^{L-3}+1,1} & A^1_{z_2,S^{L-3}+1,1} & \cdots & A^1_{z_S,S^{L-3}+1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,S^{L-2}+1,1} - A^2 & A^1_{z_2,S^{L-2}+1,1} + A^2 & \cdots & A^1_{z_S,S^{L-2}+1,1} & -F^1 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A^1_{z_1,(S-1)S^{L-2}+1,1} - A^2 & A^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1,1} + A^2 & -F^1 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}\right] \tag{79}
$$

For the lower submatrix,

$$
DZ^L = \begin{bmatrix}
H^1_{z_1,1}+H^2_{z_1,1}+H^3 & H^1_{z_2,1}+H^2_{z_2,1}+H^3 & \cdots & H^1_{z_S,1}+H^2_{z_S,1}+H^3 & J^1_{z_1,1}+J^2 & J^1_{z_2,1}+J^2 & \cdots & J^1_{z_S,1}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,2}+H^2_{z_1,2}+H^3 & H^1_{z_2,2}+H^2_{z_2,2}+H^3 & \cdots & H^1_{z_S,2}+H^2_{z_S,2}+H^3 & J^1_{z_1,2}+J^2 & J^1_{z_2,2}+J^2 & \cdots & J^1_{z_S,2}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H^1_{z_1,S}+H^2_{z_1,S}+H^3 & H^1_{z_2,S}+H^2_{z_2,S}+H^3 & \cdots & H^1_{z_S,S}+H^2_{z_S,S}+H^3 & J^1_{z_1,S}+J^2 & J^1_{z_2,S}+J^2 & \cdots & J^1_{z_S,S}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
H^1_{z_1,S+1,1}+H^2_{z_1,S+1,1} & H^1_{z_2,S+1,1}+H^2_{z_2,S+1,1} & \cdots & H^1_{z_S,S+1,1}+H^2_{z_S,S+1,1} & J^1_{z_1,S+1,1} & J^1_{z_2,S+1,1} & \cdots & J^1_{z_S,S+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{matrix} H^1_{z_1,S^{L-3}+1,1}+ \\ H^2_{z_1,S^{L-3}+1,1} \end{matrix} & \begin{matrix} H^1_{z_2,S^{L-3}+1,1}+ \\ H^2_{z_2,S^{L-3}+1,1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-3}+1,1}+ \\ H^2_{z_S,S^{L-3}+1,1} \end{matrix} & J^1_{z_1,S^{L-3}+1,1} & J^1_{z_2,S^{L-3}+1,1} & \cdots & J^1_{z_S,S^{L-3}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{matrix} H^1_{z_1,S^{L-2}+1,1}+ \\ H^2_{z_1,S^{L-2}+1,1} \end{matrix} & \begin{matrix} H^1_{z_2,S^{L-2}+1,1}+ \\ H^2_{z_2,S^{L-2}+1,1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-2}+1,1}+ \\ H^2_{z_S,S^{L-2}+1,1} \end{matrix} & J^1_{z_1,S^{L-2}+1,1} & J^1_{z_2,S^{L-2}+1,1} & \cdots & J^1_{z_S,S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\begin{matrix} H^1_{z_1,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_1,(S-1)S^{L-2}+1,1} \end{matrix} & \begin{matrix} H^1_{z_2,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_2,(S-1)S^{L-2}+1,1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_S,(S-1)S^{L-2}+1,1} \end{matrix} & J^1_{z_1,(S-1)S^{L-2}+1,1} & J^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{80}
$$

Adds the second to the $(S-1)$-th block of columns corresponding to the derivatives with respect to $\left\{\left(\bar{\xi}^{z_s}\right)^T\right\}_{s=2}^{S-1}$ to the first block of columns denoting the derivatives with respect to $\left(\bar{\xi}^{z_1}\right)^T$. Similarly, adds columns corresponding to the derivatives with respect to $\{\bar{P}^{z_s}\}_{s=2}^{S-1}$ to a column with regards to the derivative with respect to $\bar{P}^{z_1}$. Then, we obtain the following submatrices due to the property that $\sum_{s=1}^{S} A^1_{z_s,n} = \sum_{s=1}^{S} A^1_{z_s,n'}$, $\sum_{s=1}^{S} H^1_{z_s,n} = \sum_{s=1}^{S} H^1_{z_s,n'}$, $\sum_{s=1}^{S} H^2_{z_s,n} = \sum_{s=1}^{S} H^2_{z_s,n'}$ and $\sum_{s=1}^{S} J^1_{z_s,n} = \sum_{s=1}^{S} J^1_{z_s,n'}$ for all $(n,n') \in \{z_1,\ldots,z_S\}^{L-1} \times \{z_1,\ldots,z_S\}^{L-1}$.

For the upper submatrix,

$$DZ^U = \left[\begin{array}{ccccccccccccc}
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\mathbf{0} & A^1_{z_2,S+1,1} & \cdots & A^1_{z_S,S+1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,S^{L-3}+1,1} & \cdots & A^1_{z_S,S^{L-3}+1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,S^{L-2}+1,1} + A^2 & \cdots & A^1_{z_S,S^{L-2}+1,1} & 0 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1,1} + A^2 & 0 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}\right. \tag{81}$$

For the lower submatrix,

$$
DZ^{L}=\begin{bmatrix}
H & H^1_{z_2,1}+H^2_{z_2,1}+H^3 & \cdots & H^1_{z_S,1}+H^2_{z_S,1}+H^3 & J & J^1_{z_2,1}+J^2 & \cdots & J^1_{z_S,1}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
H & H^1_{z_2,2}+H^2_{z_2,2}+H^3 & \cdots & H^1_{z_S,2}+H^2_{z_S,2}+H^3 & J & J^1_{z_2,2}+J^2 & \cdots & J^1_{z_S,2}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
H & H^1_{z_2,S}+H^2_{z_2,S}+H^3 & \cdots & H^1_{z_S,S}+H^2_{z_S,S}+H^3 & J & J^1_{z_2,S}+J^2 & \cdots & J^1_{z_S,S}+J^2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\mathbf{0} & H^1_{z_2,S+1,1}+H^2_{z_2,S+1,1} & \cdots & H^1_{z_S,S+1,1}+H^2_{z_S,S+1,1} & \mathbf{0} & J^1_{z_2,S+1,1} & \cdots & J^1_{z_S,S+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\mathbf{0} & \begin{matrix}H^1_{z_2,S^{L-3}+1,1}+\\ H^2_{z_2,S^{L-3}+1,1}\end{matrix} & \cdots & \begin{matrix}H^1_{z_S,S^{L-3}+1,1}+\\ H^2_{z_S,S^{L-3}+1,1}\end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-3}+1,1} & \cdots & J^1_{z_S,S^{L-3}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\mathbf{0} & \begin{matrix}H^1_{z_2,S^{L-2}+1,1}+\\ H^2_{z_2,S^{L-2}+1,1}\end{matrix} & \cdots & \begin{matrix}H^1_{z_S,S^{L-2}+1,1}+\\ H^2_{z_S,S^{L-2}+1,1}\end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-2}+1,1} & \cdots & J^1_{z_S,S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\mathbf{0} & \begin{matrix}H^1_{z_2,(S-1)S^{L-2}+1,1}+\\ H^2_{z_2,(S-1)S^{L-2}+1,1}\end{matrix} & \cdots & \begin{matrix}H^1_{z_S,(S-1)S^{L-2}+1,1}+\\ H^2_{z_S,(S-1)S^{L-2}+1,1}\end{matrix} & \mathbf{0} & J^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{82}
$$

In this lower submatrix, $H=\sum_{s=1}^{S} H^1_{z_s,n}+\sum_{s=1}^{S} H^2_{z_s,n}+3H^3$ and $J=\sum_{s=1}^{S} J^1_{z_s,n}+3J^2$ are placed in rows corresponding to the first n histories of shocks.

Subtract the first block of $(L-1)\,M-1$ rows from the second to S-th block of $(L-1)\,M-1$ rows. Next, transform the first block into a $((L-1)\,M-1) \times ((L-1)\,M-1)$ identity matrix via column operations and clear the first block of $(L-1)\,M-1$ rows using the identity matrix. Then, we can draw a lower submatrix as below.

$$
DZ^L = \begin{bmatrix}
\boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,2,1}+H^2_{z_2,2,1} & \cdots & H^1_{z_S,2,1}+H^2_{z_S,2,1} & \mathbf{0} & J^1_{z_2,2,1} & \cdots & J^1_{z_S,2,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & H^1_{z_2,S,1}+H^2_{z_2,S,1} & \cdots & H^1_{z_S,S,1}+H^2_{z_S,S,1} & \mathbf{0} & J^1_{z_2,S,1} & \cdots & J^1_{z_S,S,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,S+1,1}+H^2_{z_2,S+1,1} & \cdots & H^1_{z_S,S+1,1}+H^2_{z_S,S+1,1} & \mathbf{0} & J^1_{z_2,S+1,1} & \cdots & J^1_{z_S,S+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-3}+1,1}+ \\ H^2_{z_2,S^{L-3}+1,1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-3}+1,1}+ \\ H^2_{z_S,S^{L-3}+1,1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-3}+1,1} & \cdots & J^1_{z_S,S^{L-3}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-2}+1,1}+ \\ H^2_{z_2,S^{L-2}+1,1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-2}+1,1}+ \\ H^2_{z_S,S^{L-2}+1,1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-2}+1,1} & \cdots & J^1_{z_S,S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_2,(S-1)S^{L-2}+1,1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,(S-1)S^{L-2}+1,1}+ \\ H^2_{z_S,(S-1)S^{L-2}+1,1} \end{matrix} & \mathbf{0} & J^1_{z_2,(S-1)S^{L-2}+1,1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1,1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{83}
$$

For both submatrices, $A^1_{z_{s'},n,1} = A^1_{z_{s'},n}$, $H^1_{z_{s'},n,1} = H^1_{z_{s'},n}$, $H^2_{z_{s'},n,1} = H^2_{z_{s'},n}$ and $J^1_{z_{s'},n,1} = J^1_{z_{s'},n}$ for $s' \neq 1$ because the first history of shocks consists of full z_1 from time $t - L + 2$ to t so that $A^1_{z_{s'},1} = 0$, $H^1_{z_{s'},1} = H^2_{z_{s'},1} = \mathbf{0}$ and $J^1_{z_{s'},1} = 0$ if $s' \neq 1$. Thus, both submatrices are given as next.

$$
DZ^U = \left[\begin{array}{cccccccccccc}
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\mathbf{0} & A^1_{z_2,S+1} & \cdots & A^1_{z_S,S+1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,S^{L-3}+1} & \cdots & A^1_{z_S,S^{L-3}+1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,S^{L-2}+1} + A^2 & \cdots & A^1_{z_S,S^{L-2}+1} & 0 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,(S-1)S^{L-2}+1} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1} + A^2 & 0 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}\right] \tag{84}
$$

$$
DZ^L = \begin{bmatrix}
\boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,2} + H^2_{z_2,2} & \cdots & H^1_{z_S,2} + H^2_{z_S,2} & \mathbf{0} & J^1_{z_2,2} & \cdots & J^1_{z_S,2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & H^1_{z_2,S} + H^2_{z_2,S} & \cdots & H^1_{z_S,S} + H^2_{z_S,S} & \mathbf{0} & J^1_{z_2,S} & \cdots & J^1_{z_S,S} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,S+1} + H^2_{z_2,S+1} & \cdots & H^1_{z_S,S+1} + H^2_{z_S,S+1} & \mathbf{0} & J^1_{z_2,S+1} & \cdots & J^1_{z_S,S+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-3}+1} + \\ H^2_{z_2,S^{L-3}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-3}+1} + \\ H^2_{z_S,S^{L-3}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-3}+1} & \cdots & J^1_{z_S,S^{L-3}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-2}+1} + \\ H^2_{z_2,S^{L-2}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-2}+1} + \\ H^2_{z_S,S^{L-2}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-2}+1} & \cdots & J^1_{z_S,S^{L-2}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,(S-1)S^{L-2}+1} + \\ H^2_{z_2,(S-1)S^{L-2}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,(S-1)S^{L-2}+1} + \\ H^2_{z_S,(S-1)S^{L-2}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,(S-1)S^{L-2}+1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{85}
$$

For the upper submatrix, $A^1_{z_s, s' \cdot S^{L-2}+1} = 0$ if $s \neq s'+1$ for $\forall\,(s, s') \in \{2, \ldots, S\} \times \{1, \ldots, S-1\}$ because $s_{t-L+2} = z_{s'+1}$ and $s_\tau = z_1$ for $\forall \tau \in \{t-L+3, \ldots, t\}$ in the $\left(s' \cdot S^{L-2}+1\right)$-th history of shocks. For the lower submatrix, $H^1_{z_s, s'} = H^2_{z_s, s'} = \mathbf{0}$ and $J^1_{z_s, s'} = 0$ if $s \neq s'$ for $\forall\,(s, s') \in \{2, \ldots, S\}^2$ because $s_\tau = z_1$ for $\forall \tau \in \{t-L+2, \ldots, t-1\}$ and $s_t = z_{s'}$ in the s'-th history of shocks. Therfore, we can derive the following submatrices.

$$DZ^U = \left[\begin{array}{cccccccccccc}
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\mathbf{0} & A^1_{z_2,S+1} & \cdots & A^1_{z_S,S+1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,S^{L-3}+1} & \cdots & A^1_{z_S,S^{L-3}+1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & A^1_{z_2,S^{L-2}+1}+A^2 & \cdots & \mathbf{0} & 0 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & A^1_{z_S,(S-1)S^{L-2}+1}+A^2 & 0 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}\right. \tag{86}$$

(87)

$$
DZ^L = \begin{bmatrix}
\boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,2} + H^2_{z_2,2} & \cdots & \mathbf{0} & \mathbf{0} & J^1_{z_2,2} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & H^1_{z_S,S} + H^2_{z_S,S} & \mathbf{0} & \mathbf{0} & \cdots & J^1_{z_S,S} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,S+1} + H^2_{z_2,S+1} & \cdots & H^1_{z_S,S+1} + H^2_{z_S,S+1} & \mathbf{0} & J^1_{z_2,S+1} & \cdots & J^1_{z_S,S+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-3}+1} + \\ H^2_{z_2,S^{L-3}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-3}+1} + \\ H^2_{z_S,S^{L-3}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-3}+1} & \cdots & J^1_{z_S,S^{L-3}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-2}+1} + \\ H^2_{z_2,S^{L-2}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-2}+1} + \\ H^2_{z_S,S^{L-2}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-2}+1} & \cdots & J^1_{z_S,S^{L-2}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,(S-1)S^{L-2}+1} + \\ H^2_{z_2,(S-1)S^{L-2}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,(S-1)S^{L-2}+1} + \\ H^2_{z_S,(S-1)S^{L-2}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,(S-1)S^{L-2}+1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

Now, we clear $\{J^1_{z_s,s}\}_{s=2}^{S}$ elements in the lower submatrix by using rows having F^1 elements in the upper submatrix appropriately. This subtraction will affect $\{\ H^1_{z_s,s} + H^2_{z_s,s}\ \}_{s=2}^{S}$ submatrices. However, rows having these submatrices do not have any other non-zero elements so that we can apply row operations to transform such matrices to be identity matrices. Hence, the lower submatrix is given as follows.

$$
DZ^L = \begin{bmatrix}
\boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \boldsymbol{I} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & H^1_{z_2,S+1} + H^2_{z_2,S+1} & \cdots & H^1_{z_S,S+1} + H^2_{z_S,S+1} & \mathbf{0} & J^1_{z_2,S+1} & \cdots & J^1_{z_S,S+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-3}+1} + \\ H^2_{z_2,S^{L-3}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-3}+1} + \\ H^2_{z_S,S^{L-3}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-3}+1} & \cdots & J^1_{z_S,S^{L-3}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,S^{L-2}+1} + \\ H^2_{z_2,S^{L-2}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,S^{L-2}+1} + \\ H^2_{z_S,S^{L-2}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,S^{L-2}+1} & \cdots & J^1_{z_S,S^{L-2}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \begin{matrix} H^1_{z_2,(S-1)S^{L-2}+1} + \\ H^2_{z_2,(S-1)S^{L-2}+1} \end{matrix} & \cdots & \begin{matrix} H^1_{z_S,(S-1)S^{L-2}+1} + \\ H^2_{z_S,(S-1)S^{L-2}+1} \end{matrix} & \mathbf{0} & J^1_{z_2,(S-1)S^{L-2}+1} & \cdots & J^1_{z_S,(S-1)S^{L-2}+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{88}
$$

Lastly, we can clear all terms in columns corresponding to the derivatives with respect to $\left\{ \left(\bar{\xi}^{z_s} \right)^T \right\}_{s=2}^{S-1}$ using a big diagonal matrix consisting of identity matrices in the upper-left part of the lower submatrix. Then, clear all elements in columns corresponding to the derivatives with respect to $\{\bar{P}^{z_s}\}_{s=2}^{S-1}$ using rows having F^1 in the upper submatrix properly. In the end, we can derive the following reduced Jacobian matrix.

$$
DZ = \left[\begin{array}{cccccccccccc}
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & F^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & \cdots & F^1 & 0 & 0 & \cdots & 0 \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}
\end{array}\right] \tag{89}
$$

The rank of the reduced Jacobian matrix is $((L-1)\,M+1)\,S-1$ no matter what M is because the lower submatrix has $((L-1)\,M-1)\,S$ independent rows after the row and column operations even in the no cohort heterogeneity case. The number of variables is $((L-1)\,M)^2\,S$ for the heterogeneous LIFS where $\Lambda^{z_s} \neq \Lambda^{z_{s'}}$ and $\Gamma^{z_s} \neq \Gamma^{z_{s'}}$ if $z_s \neq z_{s'}$.

Since we deal with the multi-period SOLG models where agents live more than two periods, the number of variables is always greater than the rank of the Jacobian matrix, i.e. $((L-1)\,M)^2\,S \geq ((L-1)\,M+1)\,S-1$ for $\forall L \geq 3$ and $\forall M \geq 1$. Therefore, the IFT applies from the rank calculation for the heterogeneous LIFS. Hence, we can infer that there will be ME generated by a LIFS in a neighborhood of the deterministic steady state for sufficiently small shocks in these general SOLG models.

1.7 Proof of Corollary 2:

For the homogeneous LIFS where $\Lambda^{z_s} = \Lambda$ and $\Gamma^{z_s} = \Gamma$, the number of variables is $(L-1)\,MS + (L-1)\,M\,((L-1)\,M-1)$. Thus, the IFT applies for the homogeneous LIFS as long as S is smaller or equal to $(L-1)\,M\,((L-1)\,M-1)+1$ from the proof of Proposition 4.

1.8 Proof of Proposition 5:

Kim and Spear (2017) shows that one can construct a forward-stable general forecast function so that F can take a subset of the stable eigenvalues of Z as its own from the fact that $ZK = KF$. Therefore, it is enough to show here that Γ can take the eigenvalues of F as its own.

Let the asset holding dynamics in time $t+1$ be:

$$\xi\left(p_{t+L},\ldots,p_{t+1},p_t,\ldots,p_{t-L+4}\right) = G\left(\xi\left(p_{t+L-1},\ldots,p_{t+1},p_t,\ldots,p_{t-L+3}\right)\right) \tag{90}$$

If taking a derivative of Eq. (90) with respect to the predetermined price vector q_t^T, we obtain the following equation when evaluated at the steady state:

$$\hat{\Xi}F = \Gamma\hat{\Xi} \tag{91}$$

because

$$\frac{\partial\hat{\xi}\left(q_{t+1}\right)}{\partial q_t^T} = \frac{\partial\hat{\xi}\left(q_{t+1}\right)}{\partial q_{t+1}^T}\frac{\partial q_{t+1}}{\partial q_t^T} = \hat{\Xi}F \tag{92}$$

and

$$\frac{\partial G\left(\hat{\xi}\left(q_t\right)\right)}{\partial q_t^T} = \frac{\partial G\left(\hat{\xi}\left(q_t\right)\right)}{\partial\hat{\xi}\left(q_t\right)^T}\frac{\partial\hat{\xi}\left(q_t\right)}{\partial q_t^T} = \Gamma\hat{\Xi} \tag{93}$$

By the Jordan decomposition of F, $F = M\Omega M^{-1}$, we can re-write Eq. (91) as:

$$\hat{\Xi}M\Omega = \Gamma\hat{\Xi}M \tag{94}$$

where Ω is a Jordan matrix with the eigenvalues of F on its main diagonal.

Eq. (94) implies that Γ takes the eigenvalues of F as its own and column vectors in $\hat{\Xi}M$ are the associated eigenvectors in models both with and without heterogeneity.

To find the relationship between Γ and Λ, let the law of motion for the price be given by:

$$p_{t+1} = f\left(p_t,\ldots,p_{t-L+3}\right) = H\left(\xi\left(p_{t+L-1},\ldots,p_{t+1},p_t,\ldots,p_{t-L+3}\right)\right) \tag{95}$$

Taking the derivative of Eq. (95) with respect to q_t^T yields the following equation:

$$Df = \Lambda^T\hat{\Xi} \tag{96}$$

We first consider the no heterogeneity case. Assuming the $(L-2)$ dimensional square matrix $\hat{\Xi}$ is invertible, we can re-written Eq. (96) as:

$$\Lambda^T = Df\hat{\Xi}^{-1} \tag{97}$$

From Eq. (91), we can derive an expression for $\hat{\Xi}^{-1}$ as follows:

$$\hat{\Xi}^{-1} = F^{-1}\hat{\Xi}^{-1}\Gamma \tag{98}$$

By plugging Eq. (98) into Eq. (97), we can obtain the relationship between Γ and Λ:

$$\Lambda^T = DfF^{-1}\hat{\Xi}^{-1}\Gamma \tag{99}$$

For the heterogeneity case, $\hat{\Xi}$ is not a square matrix and thus not invertible. Thus, one can relate Λ^T and Γ implicitly as follows. Multiply F to both sides of Eq. (96) and then replace $\hat{\Xi}F$ with $\Gamma\hat{\Xi}$ from Eq. (91) to obtain;

$$DfF = \Lambda^T\hat{\Xi}F = \Lambda^T\Gamma\hat{\Xi} \tag{100}$$

1.9 Proof of Proposition 6:

The dimension of an invariant set generated by a LIFS with S number of individual functions can be at most $(S-1)$ because a $(S-1)$-dimensional object has at least S number of vertexes. The invariant set of a LIFS without gaps is similar to the figure formed by connecting the transition vectors, $\left\{\bar{\xi}^s\right\}_s$. Therefore, a LIFS consisting of S individual mappings can yield a $(S-1)$-dimensional invariant set without gaps. Any k-dimensional objects have Lebesgue measure zero relative to $\mathbb{R}^n$ if $k < n$. Therefore, the invariant measure of a LIFS will be singular if S is smaller than or equal to $((L-1)\,M-1)$ for both homogeneous and heterogeneous LIFSs.

When S is greater than $((L-1)\,M-1)$, whether the invariant set has Lebesgue measure zero is determined by the eigenvalues of the affine matrices as seen in the one-dimensional case with two states. The system in (52) transformed from the iterated mappings, $\{G_s\}_s$, is a collection of $((L-1)\,M-1)$-number of one-dimensional LIFS. There are two-well known results in the theory of IFS which are helpful to find the weak sufficient condition for the singularity in this case. One is that the no-overlap property is satisfied if the maximum of the affine coefficients is less than $\frac{1}{S}$ for a one-dimensional LIFS with S states assuming that S-number of transition coefficients are distinct. The condition that $\gamma < 1/2$ for a one-dimensional LIFS with two states is a special case. The other one is that a cartesian product of n number of one-dimensional invariant sets has Lebesgue measure zero relative to $\mathbb{R}^n$ if there exists at least one invariant set with Lebesgue measure zero relative to $\mathbb{R}$.

Combining these two facts, we can conclude that the invariant set of a LIFS with S states has Lebesgue measure zero, i.e. a singular invariant measure if there exists i such that $\max_s \left\{\lambda_i^s\right\} < \frac{1}{S_i}$ where S_i is the number of distinct i-th elements in $\left\{\bar{\boldsymbol{\theta}}^s\right\}_s$. For the homogeneous LIFS, $\lambda_i^s = \lambda_i$ for $\forall s$. Therefore, the weak sufficient condition for this system to have a singular invariant measure requires to replace $\max_s \left\{\lambda_i^s\right\}$ with λ_i in the one for the heterogeneous LIFS above.

2 Background on Iterated Function System

We begin this section by defining the iterated function system. The IFS is a finite set of contraction mappings and their corresponding probabilities, denoted by (f_i, p_i) where $i \in \{1, 2, \ldots, N\}$. Each function f_i has the same domain and range called X, which is a closed subset of $\mathbb{R}^n$ and endowed with a metric d, such that (X, d) is a complete metric space. The sum of the probabilities assigned to the functions should be 1. Thus, the dynamic system represented by the IFS operates through selecting a function in $\{f_i\}_{i=1}^N$ corresponding to an exogenous shock $s_t \in \{1, 2, \ldots, N\}$. For example, if the shock s_t is at state i, the dynamic system is evaluated at f_i. The probability of selecting f_i is p_i.

We can summarize the definition of the IFS as follows:

$$\left\{ (f_i, p_i) \mid f_i : X \to X \text{ with } p_i, i = 1, 2, \ldots, N \text{ and } \sum_{i=1}^{N} p_i = 1 \right\} \tag{101}$$

In the context of the three-period SOLG model, $\{f_i\}$ can be thought as the policy functions of the young's equity holdings and $\{p_i\}$ is the set of the probability weights for an aggregate shock. Since there are two states – high and low – in the model, i can be either 1 and 2. (X, d) are the set of the lagged endogenous state variables and the Euclidean norm defined on X, respectively.

We define two well-known concepts from dynamic system theory in the context of the IFS. One is the invariant set and the other is the invariant measure. An invariant set of the IFS is a subset, $J \subset X$ which is non-empty, compact and satisfies:

$$\mathbf{H}(J) = \bigcup_{i=1}^{N} f_i(J) = J \tag{102}$$

where the operator $\mathbf{H}$ on X is called Hutchinson or Barnsley operator and this operator simply means the union of the images of each function, f_i, in the IFS.

Let $\mathbf{B}(X)$ be the σ-algebra of X. $P(X)$ is the space of probability measures on $\mathbf{B}(\boldsymbol{X})$. An invariant measure of the IFS, μ_f, is a probability measure on the $\mathbf{B}(X)$, which satisfies:

$$\mathbf{M}(\mu_f) = \mu_f \tag{103}$$

where the operator $\mathbf{M}$ on $\mathbf{P}(X)$ is called Markov operator, which determines the evolution of probabilities and is given by:

$$\mathbf{M}(\mu)(B) = \sum_{i=1}^{N} p_i \mu \left(f_i^{-1}(B) \right) \tag{104}$$

for all $\mu \in \mathbf{P}(X)$ and $B \in \mathbf{B}(X)$.

Hence, an invariant set is the fixed point of the Hutchinson operator and an invariant measure is the fixed point of the Markov operator.

An attractor of the IFS is the support of the unique invariant measure $\bar{\mu}_f$ which is an invariant set. One well-known result in the theory of the IFS is that there exists the unique invariant measure $\bar{\mu}_f$ and the unique invariant set $\bar{J}$ by the contraction mapping principle as shown by

Hutchinson (1981). Hence, the support of the unique invariant measure should be the unique invariant set, and the unique invariant measure must be ergodic. As with all contraction mappings, the Hutchinson operator is globally contracting: starting from any point $x \in X$, iterations of $\mathbf{H}$ will converge to J.

We give the definitions of a singular and absolutely continuous measure as follows. Two positive measures μ and ν defined on a measurable space $(X, \mathbf{B}(X))$ are called singular to each other if there exist two disjoint sets B and C in $\mathbf{B}(X)$ whose union is X such that μ is zero on all measurable subsets of B while ν is zero on all measurable subsets of C. For two positive measures μ and ν defined on a measurable space $(X, \mathbf{B}(X))$, μ is called absolutely continuous with respect to ν if for every measurable set B in $\mathbf{B}(X)$, $v(B) = 0$ implies $\mu(B) = 0$.

Σ is the space of the history of the exogenous shocks from minus infinite in time to present, i.e. $S_{-\infty}^{t} = \{\ldots, s_0, \ldots, s_t\} \in \Sigma$. We can define $\pi : \Sigma \rightarrow J$ as a coding map that associates with each infinite sequence of shock realizations a limit point in the attractor of the IFS:

$$\pi\left(S_{-\infty}^{t}\right) = \lim_{k\rightarrow\infty} f_{S_1} \circ f_{S_2} \circ \cdots \circ f_{S_k}(X) \tag{105}$$

where the f_{S_i} is the function in the IFS corresponding to the state of nature S_i. S_i is the i-th shock of $S_{-\infty}^{t}$ from today to backwards. For example, $S_1 = s_t$ and $S_2 = s_{t-1}$.

The *fiber* of a point x in the attractor of the IFS is the set $\{S \in \Sigma | \pi(S) = x\}$, i.e. the histories of shocks that lead to x. The coding map is point fibered if its fiber is a singleton for any $x \in J$.

Finally, the no-overlap property of the IFS means the images of any two different individual functions in the IFS have an empty intersection set, i.e.

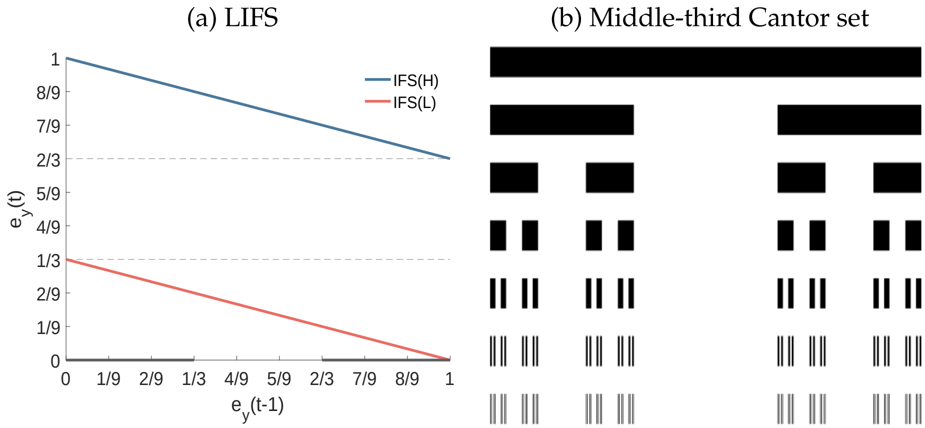
$$f_i(X) \cap f_j(X) = \emptyset \text{ for all } i \neq j \tag{106}$$

With these definitions and notations, we now review conditions for the invariant measure of an one-dimensional homogeneous LIFS to be singular with respect to the Lebesgue measure. An one-dimensional LIFS consists of one-dimensional affine transformations i.e. $f_i(x) = A_i x + a_i$ where $A_i \in \mathbb{R}$ and $a_i \in \mathbb{R}$ for $\forall i$. A homogeneous LIFS requires $A_i = A$ for $\forall i$.

A one-dimensional homogeneous LIFS with two-states satisfies the no-overlap property if the absolute value of its affine coefficient, $|A|$, is strictly less than $1/2$ so that there is a gap between the images of the two individual maps of the system on the smallest open interval containing its attractor. This non-overlapped LIFS has a Cantor-like Lebesgue measure zero attractor. Its invariant measure will be singular with respect to the Lebesgue measure since the probability measure concentrates on the Lebesgue measure zero set.

The intuition behind the appearance of a Cantor-like attractor from a non-overlapped LIFS with two-states can be seen in the following figure. Figure 1a shows a one-dimensional non-overlapped LIFS which has two parallel affine transformations corresponding to one of two states of a shock. The affine coefficient of the system is $1/3$. Since the LIFS has a unique invariant set in $\mathbb{R}$, we restrict both domain and range to the smallest open interval containing its invariant set. In Figure 1a, the middle one-third gap in the images of the two parallel maps on the interval eliminates the middle one-third of an interval for the next iteration of the LIFS. Through iterations of the LIFS, the middle one-third of the all intervals left disappears. Therefore, the limiting set becomes a Cantor set whose Lebesgue measure is zero as seen in Figure 1b. By the Ergodic theorem, the limiting set should be the attractor of the LIFS.

Figure 1: LIFS and Cantor set



As a generalization, when there are S number of states, a one-dimensional homogeneous LIFS will satisfy the no-overlap property if the Lipschitz constant of the LIFS is strictly less than $1/S$. A non-overlapped LIFS with more than two states will also have a singular invariant measure following the same logic in the two states case.

However, satisfying the no-overlap property is a sufficient condition to have a singular invariant measure since a LIFS with overlap can also generate such measure. This issue was first examined by Erdös (1939) for a one-dimensional homogeneous LIFS under a symmetric shock with two states. Erdős showed that if the Lipschitz constant of a LIFS is the reciprocal of a Pisot number, then its unique invariant measure is singular because the probability measure concentrates on a Lebesgue measure zero set even though it has full support.[2] This singular measure is called as an essentially singular measure. However, it is yet to be known whether the reciprocal of a Pisot number constitutes the entire set of the Lipschitz constant in the interval $(1/2, 1)$ with which a LIFS generates a singular invariant measure. However, it is known that the set for an essentially singular measure in the interval $(1/2, 1)$ has Lebesgue measure zero due to the work of Solomyak (1995).

See Mitra et al. (2003) to review important results from the IFS literature in the context of economic problems.

[2] A Pisot number is a positive algebraic integer greater than one all of whose conjugate elements (Galois conjugates) have an absolute value less than one. Here, an algebraic integer is any complex number that is a root of a non-zero polynomial in one variable whose leading coefficient is one and the other coefficients are all integers. For example, the golden ratio $\left(\sqrt{5}+1\right)/2$ is a Pisot number which is a root of $x^2 - x - 1 = 0$.

3 Numerical Algorithm

In this appendix, we explain how to compute the equilibria in the general SOLG models with a single long-lived asset. One can also apply the procedure here to solve the three-period SOLG models as a special case. Our numerical strategy is to approximate the policy functions for the asset holdings and prices with high-degree Chebyshev polynomials under the projection method. To find the coefficients of the functions, we use the fixed point iteration algorithm with dampening.

Step1. Approximation

$$\widehat{\xi_{t+1}^{s_{t+1}}} = \begin{bmatrix} \sum_{i_1=0}^{N_{i_1}} \cdots \sum_{i_{(L-1)M-1}=0}^{N_{i_{(L-1)M-1}}} \theta_{\{i_k\}_k}^{e_{(L-1)M-1},s} T^{i_1}\left(e_{t1}^t\right) \cdots T^{i_{(L-1)M-1}}\left(e_{(t-L+2),(M-1)}^t\right) \\ \vdots \\ \sum_{i_1=0}^{N_{i_1}} \cdots \sum_{i_{(L-1)M-1}=0}^{N_{i_{(L-1)M-1}}} \theta_{\{i_k\}_k}^{e_1,s} T^{i_1}\left(e_{t1}^t\right) \cdots T^{i_{(L-1)M-1}}\left(e_{(t-L+2),(M-1)}^t\right) \end{bmatrix} \tag{107}$$

$$\widehat{P_{t+1}^{s_{t+1}}} = \sum_{i_1=0}^{N_{i_1}} \cdots \sum_{i_{(L-1)M-1}=0}^{N_{i_{(L-1)M-1}}} \theta_{\{i_k\}_k}^{P,s} T^{i_1}\left(e_{t1}^t\right) \cdots T^{i_{(L-1)M-1}}\left(e_{(t-L+2),(M-1)}^t\right) \tag{108}$$

1. Define the type of polynomials to approximate the recursive ME, $\left\{T^{i_k}(\cdot)\right\}_k$.
   - In this paper, we use the Chebyshev polynomials of which domain is $[-1,1]$.
2. Set the degree of polynomials for each state variable in the policy functions, $\left\{N_{i_k}\right\}_k$.
   - $\theta_{\{i_k\}_k}^{e,s}$ and $\theta_{\{i_k\}_k}^{P,s}$ are the coefficients of polynomials in the policy functions of the equity holdings and the equity price respectively. Hence, the number of unknowns is (# of policy functions) $\times$ (# of shocks) $\times \left(N_{i_1}+1\right) \times \ldots \times \left(N_{i_{(L-1)M-1}}+1\right)$.
3. To use the collocation method in finding the unknowns, generate grid points as much as the degree of polynomials plus one.
   - Create $(N_{i_1}+1) \times \ldots \times \left(N_{i_{(L-1)M-1}}+1\right)$ Chebyshev grids in $[-1,1]$: $\left\{\{x_l^{e_k}\}_{l=1}^{N_{i_k}+1}\right\}_{k=1}^{(L-1)M-1}$. Transform these Chebyshev nodes into appropriately chosen intervals for the endogenous state variables, $[e_{k,min}, e_{k,max}]_{k=1}^{(L-1)M-1}$.

Step2. Iteration

$$\begin{aligned} & P_t^{s_t} u'\left(\omega_{t-\tau+1,j} + \left(P_t^{s_t} + \delta^{s_t}\right) e_{\tau,j}^{t-1} - P_t^{s_t} e_{\tau,j}^t\right) \\ = & \beta E_t\left[\left(P_{t+1}^{s_{t+1}} + \delta^{s_{t+1}}\right) u'\left(\omega_{t-\tau+2,j} + \left(P_{t+1}^{s_{t+1}} + \delta^{s_{t+1}}\right) e_{\tau,j}^t - P_{t+1}^{s_{t+1}} e_{\tau,j}^{t+1}\right)\right] \end{aligned} \tag{109}$$

where $j \in [1,\ldots,M]$, $e_{\tau,j}^{t-1}$ and $e_{\tau,j}^{t+1}$ are zero for the first and lasst generations, respectively.

1. Construct the Euler equations combined with the budget constraints and the market clearing conditions as above.

2. Set an initial value for the coefficients of the Chebyshev polynomials, $\left\{\theta^{e,s}_{\{i_j\}_j}, \theta^{P,s}_{\{i_j\}_j}\right\}^0$.

   - Use the deterministic steady-state values to decide the initial value of the coefficients of the zero-order terms in the approximated policy functions.

3. Compute the equity holdings and the price in the Euler equations over the grid points by plugging the initial values for the coefficients of the Chebyshev polynomials into the policy functions: $\widehat{P^{s_t}_t}, \left\{\widehat{e^t_{\tau,j}}\right\}_{\tau,j} \widehat{P^{s_{t+1}}_{t+1}}$ and $\left\{\widehat{e^{t+1}_{\tau,j}}\right\}_{\tau,j}$.

   - The number of the Euler equations is as much as the number of the policy functions, $(L-1)\,M-1$. Such Euler equations exist over each shock. Hence, the number of total equations is the same as the number of unknowns.
   - Replace $P^{s_t}_t$ with $\widehat{P^{s_t}_t}$ on the left-hand side of the Euler equations all but one. In the Euler equation for the oldest and M-type agent, use $\left\{\widehat{e^t_{\tau,j}}\right\}_{\tau,j}$ and the asset market clearing condition instead of $e^t_{(t-L+2),M}$.

4. Find a new set of $P^{s_t}_t$ and $\left\{e^t_{\tau,j}\right\}_{\tau,j}$ by solving a system of linear equations derived from the computed Euler equations.

5. Find a new set of coefficients, $\left\{\theta^{e,s}_{\{i_j\}_j}, \theta^{P,s}_{\{i_j\}_j}\right\}^n$, by inverting the new set of $P^{s_t}_t$ and $\left\{e^t_{\tau,j}\right\}_{\tau,j}$. Update the coefficients with dampening as follows.

   $$\left\{\theta^{e,s}_{\{i_j\}_j}, \theta^{P,s}_{\{i_j\}_j}\right\}^{n+1} = \lambda \left\{\theta^{e,s}_{\{i_j\}_j}, \theta^{P,s}_{\{i_j\}_j}\right\}^{0} + (1-\lambda)\left\{\theta^{e,s}_{\{i_j\}_j}, \theta^{P,s}_{\{i_j\}_j}\right\}^{n} \tag{110}$$

   where λ is a dampening parameter which decides the speed of convergence.

6. Iterate the sub-steps from 1 to 5 until being converged.

   $$\sup\left|\left\{\widehat{P^{s_t}_t}, \left\{\widehat{e^t_{\tau,j}}\right\}_{\tau,j}\right\} - \left\{P^{s_t}_t, \left\{e^t_{\tau,j}\right\}_{\tau,j}\right\}\right| < \epsilon \tag{111}$$

   where ϵ is a predetermined convergence measure.

References

Erdös, P. (1939). On a family of symmetric bernoulli convolutions. *American Journal of Mathematics*, 61(4):974–976.

Hutchinson, J. E. (1981). Fractals and self similarity. *Indiana University Mathematics Journal*, 30(5):713–747.

Kim, E. and Spear, S. E. (2017). Determinate perfect foresight forecasting in overlapping generations models. Unpublished Manuscript, Carnegie Mellon University.

Mitra, T., Montrucchio, L., and Privileggi, F. (2003). The nature of the steady state in models of optimal growth under uncertainty. *Economic Theory*, 23(1):39–1.

Solomyak, B. (1995). On the random series $\sum \pm \lambda$ n (an erdos problem). *Annals of Mathematics*, 142(3):611–625.