NAMT
95-011

Coexistent Phases in Nonlinear Thermoelasticity: Radially Symmetric Equilibrium States of Aeolotropic Bodies

Mikhail M. Shvartsman
Carnegie Mellon University

Stuart S. Antman
University of Maryland

Research Report No. 95-NA-011

May 1995

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550

COEXISTENT PHASES IN NONLINEAR THERMOELASTICITY: RADIALLY SYMMETRIC EQUILIBRIUM STATES OF AEOLOTROPIC BODIES

MIKHAIL M. SHVARTSMAN

Center for Nonlinear Analysis
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213
U. S. A.

and

STUART S. ANTMAN

Department of Mathematics and
Institute for Physical Science and Technology
University of Maryland
College Park, Maryland 20742
U. S. A.

ABSTRACT

We determine the detailed qualitative behavior of radially symmetric equilibrium states with two or more phases for general classes of nonlinear thermoelastic materials. We treat the cases of structured and non-structured interface. Methods are based on a combination of geometric constructions with phase-plane analyses of the governing equations. A special feature of the solutions is that they admit nonplanar interfaces.

KEY WORDS

Phase transformations, nonlinear elasticity, equilibrium states.

1. Introduction

For aeolotropic nonlinearly thermoelastic disks and balls, we determine the detailed qualitative behavior of radially symmetric equilibrium states with two or more phases. Much of our treatment is based on a combination of simple geometric constructions with phase-plane analyses of the governing equations. We treat very general classes of materials. A host of novel effects are due to the lack of isotropy, which causes the innocuous polar singularity at the center for an isotropic body to be replaced by a very different singularity, one which strongly influences the behavior near the center. The richness of the phenomena we find reflects the richness of the material response we consider. A special feature of our solutions is that they admit nonplanar interfaces.

In Section 2 we record the equations governing the equilibrium of coexistent phases separated by smooth interfacial surfaces, including the case when the interfaces can be structured. In Section 3 we specialize these equations to a dual form especially appropriate for radially symmetric problems. In Section 4 we describe the properties of the phase portraits corresponding to isothermal equilibria of homogeneous materials. In Section 5 we solve the isothermal problem first for any finite number of simple interfaces. In Section 6 we treat the same problems for structured interfaces. In Section 7 we specialize some of these results to linearly elastic materials, which cannot sustain multiple radially symmetric phases in the absence of interfacial structure. We introduce heat conduction in Sections 8 and 9. Its presence typically prevents the governing ordinary differential equations from being autonomous. Since there is but slight advantage to treating homogeneous materials, we do not restrict our attention to these in Section 9. In Section 10 we develop a version of the theory of asymptotically autonomous equations applicable to our thermoelastic problems, which enables us to obtain detailed information about solutions for these problems from solutions of an appropriate autonomous system.

Notation. We often denote the function $u \mapsto f(u)$ by $f(\cdot)$. The partial derivative of a function f with respect to a scalar argument t is denoted by either f_t or $\partial_t f$. Obvious analogs of these notations will also be used.

2. Basic Equations for Coexistent Phases

In this section we state the equations governing the equilibrium of coexistent phases separated by smooth structured interfaces, following the formulation of Gurtin [6,7]. The purpose of this presentation is merely to identify the variables that enter the jump conditions of radially symmetric problems, which we study in the rest of the paper.

Gibbs notation. In our formulation of the three-dimensional theory, we employ Gibbs notation for vectors and tensors: Vectors, which are elements of Euclidean 3-space $\mathbf{E}^3$, and vector-valued functions are denoted by lower-case, italic, bold-face symbols. The dot product of (vectors) $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$. The value of tensor $\boldsymbol{A}$ at vector $\boldsymbol{v}$ is denoted $\boldsymbol{A} \cdot \boldsymbol{v}$ (in place of the more usual $\boldsymbol{A}\boldsymbol{v}$) and the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted $\boldsymbol{A} \cdot \boldsymbol{B}$ (in place of the more usual $\boldsymbol{A}\boldsymbol{B}$). The transpose of $\boldsymbol{A}$ is denoted $\boldsymbol{A}^*$. We write $\boldsymbol{u} \cdot \boldsymbol{A} = \boldsymbol{A}^* \cdot \boldsymbol{u}$. The inner product of $\boldsymbol{A}$ and $\boldsymbol{B}$ (which

equals the trace of $A \cdot B^*$) is denoted $A : B$. The dyadic product of vectors a and b is denoted ab (in place of the more usual $a \otimes b$). It is defined by $(ab)\cdot u = (b\cdot u)a$ for all u. Thus $(ab) : (uv) = (a\cdot u)(b\cdot v)$. Twice-repeated lower-case Latin indices are summed from 1 to 3 and twice-repeated lower-case Greek indices are summed from 1 to 2.

In Section 10, we employ traditional notation for matrices.

The (Gâteaux) differential of $u \mapsto f(u)$ at v in the direction h is $\frac{d}{dt} f(v + th)\big|_{t=0}$. When it is linear in h we denote this differential by $\frac{\partial f}{\partial u}(v) \cdot h$ or $f_u(v) \cdot h$. If x represents a point in Euclidean 3-space, then the divergence of a tensor T is defined to be $\nabla \cdot T^* \equiv (\partial T/\partial x) : I$ where I is the identity tensor or, equivalently, $(\nabla \cdot T^*) \cdot c \equiv \operatorname{tr}[\partial(T^* \cdot c)/\partial x]$ for all c, where tr denotes trace.

The surface gradient and divergence. An invariant definition of the surface gradient $\bar{\nabla}$ and surface divergence $\bar{\nabla}\cdot$ is given by Gurtin and Murdoch [8] and Gurtin [6]; we give a useful coordinate version: Let $\eta \equiv (\eta^1, \eta^2)$ be a pair of surface coordinates for a C^1 interface, which we represent by $\eta \mapsto \pi(\eta)$. By definition, the vectors $g_\alpha(\eta) \equiv \frac{\partial \pi(\eta)}{\partial \eta^\alpha}$ are independent and span the tangent space to the interface π at $\pi(\eta)$. Let $\{g^\alpha\}$ be the basis dual to $\{g_\alpha\}$. Then the surface gradient $\bar{\nabla}$ is the operator $g^\gamma \frac{\partial}{\partial \eta^\gamma}$ and the surface divergence is the operator $g^\gamma \frac{\partial}{\partial \eta^\gamma}\cdot$. (The same technique yields a coordinate version of the ordinary gradient ∇.) At the end of this section, we compute these operations in the polar coordinates we use in the rest of this paper.

Let x denote a typical material point in a body Ω and let $p(x)$ denote the position of x in a deformed configuration. Let $F(x)$ denote the deformation gradient at x, $\theta(x)$ the absolute temperature at x, $T(x)$ the first Piola-Kirchhoff stress tensor at x, $q(x)$ the (negative of) the material heat-flux vector at x, and $\psi(x)$ the Helmholtz free energy per unit reference volume at x. A *phase* of a thermoelastic material is a maximal connected set of material points on which the deformation gradient and the temperature gradient are continuous. Suppose that a body has a finite number of phases separated by smooth surfaces, called *interfaces*, having continuously differentiable unit normal fields n. To account for effects like surface tension we can endow an interface with a surface free energy $\bar{\psi}$ and a surface Piola-Kirchhoff stress tensor $\bar{T}$, which is a tensor that is a derivative of $\bar{\psi}$ and that annihilates n. When $\bar{\psi} \neq 0$, the interface is said to have *structure*.

We limit our attention to coherent interfaces, across which p is continuous. The tensor $P \equiv I - nn$ is the projector onto the tangent plane to the interface at any point. The coherency of an interface ensures that $\bar{F} \equiv F \cdot P$ (which is another surface tensor in the sense that it annihilates n) is continuous across the interface.

The body is in (*Maxwell*) *equilibrium* under zero body force and zero heat source if

$$\nabla \cdot T^* = 0, \tag{2.1}$$

$$\nabla \cdot q = 0 \tag{2.2}$$

in each phase, and

$$[\boldsymbol{T}]\cdot\boldsymbol{n}=-\bar{\nabla}\cdot\bar{\boldsymbol{T}}^{*}, \tag{2.3}$$

$$[\psi-(\boldsymbol{T}\cdot\boldsymbol{n})\cdot(\boldsymbol{F}\cdot\boldsymbol{n})]=(\bar{\psi}\boldsymbol{P}-\bar{\boldsymbol{F}}^{*}\cdot\bar{\boldsymbol{T}}):\bar{\nabla}\boldsymbol{n}-\bar{\nabla}\cdot\boldsymbol{c}, \tag{2.4}$$

$$[\boldsymbol{q}]\cdot\boldsymbol{n}=0 \tag{2.5}$$

on each interface. Here $[f]$ denote the jump across an interface of the function f, i.e., the limiting value of f on the interface in the phase into which $\boldsymbol{n}$ points minus the limiting value of f for the other phase. The vector $\boldsymbol{c}$ is associated with shear forces acting on the interface; we ignore it because it will not appear in the problems we treat.

We supplement these jump conditions with the the jump condition coming from the requirement that the interface be coherent:

$$[p]=0. \tag{2.6}$$

We limit our attention to interfaces across which the temperature is continuous:

$$[\theta]=0. \tag{2.7}$$

We study thermoelastic materials for which there are constitutive functions $\hat{\psi}$, $\hat{\boldsymbol{T}}$, $\hat{\boldsymbol{q}}$, $\mathring{\psi}$, $\mathring{\boldsymbol{T}}$, such that the functions delivering the free energy, stress, free energy, heat flux, surface free energy, surface stress, and surface shear at a material point are the compositions

$$\psi=\hat{\psi}(\boldsymbol{F},\theta,\boldsymbol{x}), \tag{2.8}$$

$$\boldsymbol{T}=\hat{\boldsymbol{T}}(\boldsymbol{F},\theta,\boldsymbol{x})=\hat{\psi}_{\boldsymbol{F}}(\boldsymbol{F},\theta,\boldsymbol{x}), \tag{2.9}$$

$$\boldsymbol{q}=\hat{\boldsymbol{q}}(\boldsymbol{F},\theta,\nabla\theta,\boldsymbol{x}), \tag{2.10}$$

$$\bar{\psi}=\mathring{\psi}(\bar{\boldsymbol{F}},\boldsymbol{n},\theta,\boldsymbol{x}), \tag{2.11}$$

$$\bar{\boldsymbol{T}}=\mathring{\boldsymbol{T}}(\bar{\boldsymbol{F}},\boldsymbol{n},\theta,\boldsymbol{x})=\mathring{\psi}_{\boldsymbol{F}}(\bar{\boldsymbol{F}},\boldsymbol{n},\theta,\boldsymbol{x}), \tag{2.12}$$

$$\boldsymbol{c}=-\mathring{\boldsymbol{T}}(\bar{\boldsymbol{F}},\boldsymbol{n},\theta,\boldsymbol{x})^{*}\cdot\boldsymbol{E}\cdot\boldsymbol{n}-\mathring{\psi}_{\boldsymbol{n}}(\bar{\boldsymbol{F}},\boldsymbol{n},\theta,\boldsymbol{x}). \tag{2.13}$$

Here $\boldsymbol{E}$ is the average of the limits of $\boldsymbol{F}$ on each side of the interface. We do not bother to put these constitutive equations into frame-indifferent form. We shall not use the fact that $\hat{\boldsymbol{T}}$ and $\mathring{\boldsymbol{T}}$ are derivable from potentials.

We study equilibrium states with coexistent phases. We do not concern ourselves with how these states could be reached. In particular, we assume that a set of constitutive equations is given for each phase. We need not explicitly regard these constitutive equations as corresponding to different energy wells of a single free-energy function. That our problems do in fact describe coexistent phases resulting from a free-energy function with multiple wells, rather than describe some standard equilibrium configurations, is embodied in (2.3) and the *Maxwell condition* (2.4). Since these conditions account for the instabilities associated with the spinodal regions produced by a free-energy function with multiple wells, we can assume that the constitutive equations in each phase are reasonably stable, namely, that they are strongly elliptic.

For accounts of the underlying theory of coexistent phases, see Abeyaratne and Knowles [1,2], Gurtin [6,7], and Truskinovsky [17].

Polar coordinates. Let the interface be a sphere of radius r. Let $\{i, j, k\}$ be a right-handed orthonormal basis for Euclidean space. We identify η with the spherical coordinates (θ, ϕ), in which case

$$\pi(\theta, \phi) = r[\sin\theta(\cos\phi i + \sin\phi j) + \cos\theta k] \equiv rn. \tag{2.14}$$

Set

$$a_1 = \cos\theta(\cos\phi i + \sin\phi j) - \sin\theta k, \quad a_2 = -\sin\phi i + \cos\phi j, \quad a_3 = n. \tag{2.15}$$

Then $\{a_k\}$ is orthonormal, and $g_1 = ra_1$, $g_2 = r\sin\theta a_2$, $g^1 = a_1/r$, $g_2 = a_2/r\sin\theta$. Hence,

$$\bar{\nabla} = \frac{1}{r}\left(a_1\frac{\partial}{\partial\theta} + \frac{a_2}{\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{2.16}$$

In particular, if we represent $\bar{T}$ by $\bar{T}^{ij}a_i a_j$, then

$$\bar{\nabla}\cdot\bar{T}^* = \frac{1}{r}\left(a_1\frac{\partial}{\partial\theta} + \frac{a_2}{\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\bar{T}^{ij}a_j a_i, \tag{2.17}$$

$$\bar{\nabla}n = \frac{1}{r^2}\left(a_1\frac{\partial}{\partial\theta} + \frac{a_2}{\sin\theta}\frac{\partial}{\partial\phi}\right)\pi(\theta,\phi) = \frac{1}{r}(a_1 a_1 + a_2 a_2) = \frac{1}{r}P. \tag{2.18}$$

If the interface is a circular cylinder of radius r, we identify η with the cylindrical coordinates (ϕ, z), in which case

$$\pi(\phi, z) = r(\cos\phi i + \sin\phi j) + zk. \tag{2.19}$$

We set $a_1 = -\sin\phi i + \cos\phi j$, $a_2 = k$, and $a_3 = n = \cos\phi i + \sin\phi j$ and find that

$$\bar{\nabla} = \frac{a_1}{r}\frac{\partial}{\partial\phi} + k\frac{\partial}{\partial z}, \tag{2.20}$$

$$\bar{\nabla}\cdot\bar{T}^* = \left(\frac{a_1}{r}\frac{\partial}{\partial\phi} + k\frac{\partial}{\partial z}\right)\cdot\bar{T}^{ij}a_j a_i, \tag{2.21}$$

$$\bar{\nabla}n = \left(\frac{a_1}{r}\frac{\partial}{\partial\phi} + k\frac{\partial}{\partial z}\right)n = \frac{1}{r}a_1 a_1. \tag{2.22}$$

3. Equations for Radially Symmetric Configurations

We study radially symmetric equilibrium problems, which are governed by ordinary differential equations, because they can be readily analyzed and because they describe interesting phenomena that arise in a variety of real materials. For example, the casting of a metal cylinder typically results in a configuration with a reasonable approximation of cylindrical radially symmetric aeolotropy (see Walker [18]). Spherical radially symmetric aeolotropy occurs in spherulite (see Sinha [15]).

We accordingly take our body to be a disk or a ball of outer radius 1 and assume that the constitutive equations respect this symmetry. Let s denote the radial

coordinate in the reference configuration. Let $r(s)$ be the radius of a material point originally at distance s from the origin. The following development is restricted to radially symmetric problems. We use the notation of (2.14)–(2.22).

The only strain variables are the radial and azimuthal stretches ν and τ defined by

$$\nu(s) = r'(s), \qquad \tau(s) = \frac{r(s)}{s}. \tag{3.1a,b}$$

In particular, for the ball, $\boldsymbol{F} = \tau(\boldsymbol{a}_1\boldsymbol{a}_1+\boldsymbol{a}_2\boldsymbol{a}_2)+\nu\boldsymbol{n}\boldsymbol{n}$, and for the disk, $\boldsymbol{a}_1\cdot\boldsymbol{F}\cdot\boldsymbol{a}_1 = \tau$, $\boldsymbol{a}_3\cdot\boldsymbol{F}\cdot\boldsymbol{a}_3 = \nu$.

The only nonzero components of $\boldsymbol{T}$ are the radial normal component N and the azimuthal normal component(s) T. The temperature θ depends only on s, and the only nonzero component of $\boldsymbol{q}$ is its radial component q.

Equations (2.1) and (2.2) reduce to

$$\tfrac{d}{ds}[s^\alpha N(s)] = \alpha s^{\alpha-1}T(s), \tag{3.2}$$

$$\tfrac{d}{ds}[s^\alpha q(s)] = 0 \tag{3.3}$$

where $\alpha = 1$ for cylindrical coordinates and $\alpha = 2$ for spherical coordinates.

The surface stress tensor has the form $\bar{\boldsymbol{T}} = \bar{T}(\boldsymbol{a}_1\boldsymbol{a}_1 + \boldsymbol{a}_2\boldsymbol{a}_2)$ for the ball and $\bar{\boldsymbol{T}} = \bar{T}\boldsymbol{a}_1\boldsymbol{a}_1 + \bar{T}^{33}\boldsymbol{k}\boldsymbol{k}$ for the disk. Using (2.17) and (2.21) we find that $\bar{\nabla}\cdot\bar{\boldsymbol{T}} = -\alpha\bar{T}\boldsymbol{n}/r$. The surface shear $\boldsymbol{c} = \boldsymbol{0}$ and $\mathring{\bar{\psi}}$ is independent of $\dot{\boldsymbol{n}}$. It is convenient to introduce the *Eshelby function*

$$\chi \equiv \psi - N\nu \tag{3.4}$$

(the constitutive function for which is a partial Legendre transform of that for ψ, so that χ represents a thermodynamic variable between the free energy and the free enthalpy). The interface conditions (2.3)–(2.7) reduce to

$$[N] = \frac{\alpha\bar{T}}{r} = \frac{\alpha\bar{T}}{s\tau}, \tag{3.5}$$

$$[\chi] = \frac{\alpha}{r}(\bar{\psi} - \bar{T}\tau), \tag{3.6}$$

$$[q] = 0, \tag{3.7}$$

$$[\tau] = 0, \tag{3.8}$$

$$[\theta] = 0, \tag{3.9}$$

In each phase the constitutive equations (2.8)–(2.10) reduce to equations of the form

$$T(s) = \hat{T}(\tau(s),\nu(s),\theta(s),s) = \frac{\partial\hat{\psi}}{\partial\tau}(\tau(s),\nu(s),\theta(s),s), \tag{3.10}$$

$$N(s) = \hat{N}(\tau(s),\nu(s),\theta(s),s) = \frac{\partial\hat{\psi}}{\partial\nu}(\tau(s),\nu(s),\theta(s),s), \tag{3.11}$$

$$\psi(s) = \hat{\psi}(\tau(s),\nu(s),\theta(s),s), \tag{3.12}$$

$$q(s) = \hat{q}(\tau(s),\nu(s),\theta(s),\theta'(s),s). \tag{3.13}$$

in each phase. Let

$$s = e^{\xi-1}, \quad \tau(s) = \tilde{\tau}(\xi), \quad n(s) = \tilde{n}(\xi), \quad \theta(s) = \tilde{\theta}(\xi), \tag{3.20}$$
$$\chi^{\natural}(\tau, n, \theta, s) = \tilde{\chi}^{\natural}(\tau, n, \theta, \xi).$$

Let us now drop the superposed tildes. Then (3.19) is equivalent to

$$\dot{\tau} = \nu^{\natural}(\tau, n, \theta, e^{\xi-1}) - \tau, \tag{3.21a}$$

$$\dot{n} = \alpha T^{\natural}(\tau, n, \theta, e^{\xi-1}) - \alpha n, \tag{3.21b}$$

$$\dot{\theta} = e^{\xi-1} f^{\natural}(\tau, n, \theta, \mu e^{\alpha(1-\xi)}, e^{\xi-1}) \tag{3.21c}$$

where the superposed dot denotes differentiation with respect to ξ. If the material in each phase is homogeneous, so that the constitutive functions do not depend explicitly on s, then equations (3.21a,b) are autonomous. But even under these conditions, (3.21c) for $\alpha = 1$ is not autonomous unless $f^{\natural}$ is affine in q, i.e., unless the material satisfies a version of the Fourier heat conduction law, and (3.21c) for $\alpha = 2$ is not autonomous unless $f^{\natural}$ is affine in $\sqrt{q}$.

The corresponding forms of (3.5), (3.6) are

$$[n] = \bar{N}(\tau, \theta, \xi) \equiv \frac{\alpha \mathring{T}(\tau, \theta, e^{\xi-1})}{e^{\xi-1}\tau}, \tag{3.22}$$

$$[\chi^{\natural}(\tau, n, \theta, \xi)] = \bar{X}(\tau, \theta, \xi) \equiv \frac{\alpha}{e^{\xi-1}\tau}[\mathring{\psi}(\tau, \theta, e^{\xi-1}) - \tau \mathring{T}(\tau, \theta, e^{\xi-1})], \tag{3.23}$$

while (3.7)–(3.9) remain the same. (Since there can be no interface at the center, the presence of $e^{\xi-1}$ in the denominators of (3.22), (3.23) causes no difficulty.)

We identify a constant reference temperature $\mathring{\theta}$. We assume that the reference configuration is natural for this temperature, i.e., that

$$\hat{N}(1, 1, \mathring{\theta}, s) = 0 = \hat{T}(1, 1, \mathring{\theta}, s) \tag{3.24a}$$

or, equivalently,

$$\nu^{\natural}(1, 0, \mathring{\theta}, s) = 1, \quad T^{\natural}(1, 0, \mathring{\theta}, s) = 0. \tag{3.24b}$$

We now discuss mechanical boundary conditions, postponing until Section 8 a discussion of thermal boundary conditions.

The requirement that the center of the body be intact, i.e., that $r(0) = 0$, is equivalent to

$$\lim_{\xi \to -\infty} e^{\xi-1} r(\xi) = 0 \tag{3.25}$$

by virtue of (3.1b) and (3.20).

If the center is not required to be intact, then the body could suffer cavitation, in which a hole forms about the center. In this case, $r(0)$ is positive and there

is no traction acting on the boundary of the hole. This situation at the center corresponds to the point

$$\lim_{\xi \to -\infty} (\tau(\xi), n(\xi)) = (\infty, 0). \tag{3.26}$$

Here we follow [3] in characterizing cavitation by the vanishing of the Piola-Kirchhoff traction n at the center. Alternatively, one can adopt the weaker, but eminently reasonable, assumption that the Cauchy traction n/τ vanish at the center (cf. Ball [5]). The advantage of our approach is that there is at most one phase-plane trajectory satisfying (3.26), whereas otherwise there could be a whole family of admissible trajectories, to which a stabilty criterion must be applied to select the physically reasonable one.

We can prescribe the radius $\tau(1)$ of the outer boundary of the body, or prescribe the normal Piola-Kirchhoff traction $n(1)$, or prescribe the normal stress per unit actual area, etc. We handle all such possibilities by requiring $(\tau(1), n(1))$ to lie on a prescribed curve $\S\mathcal{B}$ in the (τ, n)-plane:

$$(\tau(1), n(1)) \in \S\mathcal{B}. \tag{3.27}$$

4. Phase Portraits for Isothermal Autonomous Problems

We now limit our attention to isothermal problems for homogeneous media, so that the behavior in each phase is given by the autonomous version of (3.21a,b) for constant θ, which we write as

$$\dot{\tau} = \nu^{\natural}(\tau, n) - \tau, \quad \dot{n} = \alpha T^{\natural}(\tau, n) - \alpha n. \tag{4.1a,b}$$

Under certain constitutive assumptions (which are slightly stronger than ours, which can be weakened, and which we do not spell out) the methods of Antman and Negrón-Marrero [3] show that the phase portraits[1] for (4.1) have special properties illustrated in Figure 4.2: *The point $(\tau, n) = (1, 0)$ is a singular point. All nondegenerate singular points are either saddle points or stable nodes. All singular points lie along a simple curve $\S\mathcal{S}$ that is asymptotic to the negative n-axis and that is unbounded in the positive n-direction for $\tau > 1$. The curve $\S\mathcal{S}$ is the union of the singular points and trajectories (separatrices) joining the singular points.*

The following result is central for our analysis: *For the phase containing the center, any trajectory satisfying* (3.25) *must begin at a saddle point (possibly at infinity) and must be confined to a separatrix leaving the saddle point, or else must be confined to a singular point.* See [3] for a proof. For an isotropic material, the curve $\S\mathcal{S}$ degenerates into a curve of singular points. If the phase containing the center is isotropic, then any trajectory satisfying (3.25) is merely a point of $\S\mathcal{S}$. The treatment of the degenerate problems in which the phase containing the center is isotropic is much easier than those in which this phase is not isotropic, and is accordingly omitted. (Our methods automatically account for cases in which other phases are isotropic.)

There can be a cavitation, satisfying (3.18), if the constitutive equations for the phase containing the center admit a trajectory, like the $\S\mathcal{C}$'s of Figure 4.2, that is asymptotic to the positive τ-axis and originates at $(\tau, n) = (\infty, 0)$. See the discussion and references in [3].

[1] Although the *phase* in *phase portrait* has nothing to do with the *phase* of a material, we encounter no ambiguity because each material phase has its own phase portrait.

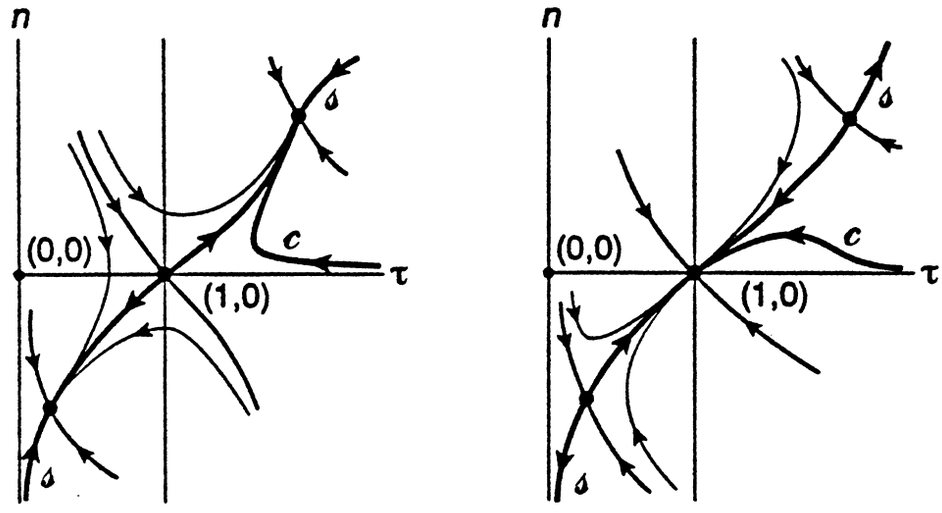

Fig. 4.2. Typical phase portraits for (4.1) showing the separatrix curve $\S\mathcal{S}$ and the trajectory $\S\mathcal{C}$ asymptotic to the positive τ-axis.

5. The Isothermal Problem with Unstructured Interfaces

We seek equilibrium states in which there are a finite number of phases, each of which touch along circular interfaces for $\alpha = 1$ and spherical interfaces for $\alpha = 2$. We prescribe the number $k+1$ of phases and the constitutive functions in each phase (but not the locations of the phase boundaries). In this section, we assume that the interfaces are not structured, so that the right-hand sides of (3.5) and (3.6) are each 0. We use the very simple geometric construction shown in Figure 5.1 to characterize solutions:

First suppose that the center is intact. Let χ_0 be the Eshelby constitutive function $\chi^{\natural}$ for the phase containing the center, which we call the phase 0. (This phase is a disk or a ball.) In (τ, n, χ)-space construct the curve γ_0 formed by the graph of χ_0 restricted to the curve $\S\mathcal{S}$ for the phase 0, which is denoted $\S\mathcal{S}_0$. Thus

$$\gamma_0 = \{(\tau, n, \chi) : (\tau, n) \in \S\mathcal{S}_0, \chi = \chi_0(\tau, n)\}. \tag{5.2}$$

Segments of γ_0 can be parametrized by the ξ that parametrizes each separatrix constituting $\S\mathcal{S}_0$.

Let χ_1 be the Eshelby constitutive function $\chi^{\natural}$ for the phase, called phase 1, adjacent to that containing the center. (This phase is an annulus or a shell.) In

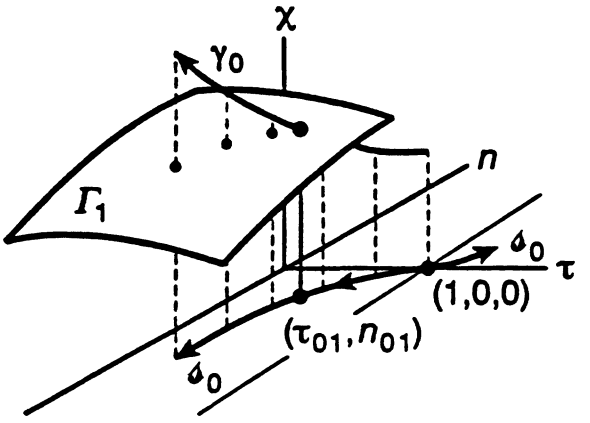

Fig. 5.1. The graphs of the curve γ_0 and the surface Γ_1. Their intersection points, if any, correspond to the (τ, n) at the interface between the phases 0 and 1.

the same (τ, n, χ)-space construct the graph Γ_1 of χ_1:

$$\Gamma_1 = \{(\tau, n, \chi) : \chi = \chi_1(\tau, n)\}. \tag{5.3}$$

The curve γ_0 might intersect the surface at one or more points. If there are no such intersections, then there cannot be equilibria with phase 1 touching phase 0 along an interface satisfying (3.5), (3.6), (3.8). Otherwise, denote the projection of a typical intersection point onto the (τ, n)-plane by (τ_{01}, n_{01}). This point determines the state on the interface. A segment of a separatrix in the portrait of phase 0 that begins at a saddle point and terminates at (τ_{01}, n_{01}) describes the state of phase 0. For each (τ_{01}, n_{01}) there is different state of phase 0. We do not (yet) know the value of ξ corresponding to (τ_{01}, n_{01}) because (4.1), being autonomous, is invariant under changes of the independent variable ξ. The separatrix determines an appropriate state because an infinite amount of ξ is exhausted in going from the saddle point to the terminal point (τ_{01}, n_{01}), as is required by (3.25).

Now we locate (τ_{01}, n_{01}) in the portrait of phase 1. Issuing from it is at most one semi-orbit $\S\mathcal{C}_1$ along which ξ increases. (It could happen that (τ_{01}, n_{01}) is a singular point, in which case $\S\mathcal{C}_1$ lies on (τ_{01}, n_{01}).)

First, let us suppose that the only phases are phases 0 and 1. Then we get a solution to the equilibrium problem if and only if $\S\mathcal{C}_1$ intersects the boundary curve $\S\mathcal{B}$. (In the rare case that (τ_{01}, n_{01}) is a singular point, it must lie on $\S\mathcal{B}$. Then phase 1 is degenerate, being confined to the boundary of the body. We choose

not to regard the solution as a two-phase solution.) The semi-orbit $\S\mathcal{C}_1$ determines the state of phase 1. In consonance with (3.27), we identify the value of ξ at the intersection of $\S\mathcal{C}_1$ with $\S\mathcal{B}$ as 1. We then obtain from (4.1) an integral expression for the amount of the independent variable ξ used up along the $\S\mathcal{C}_1$ from (τ_{01}, n_{01}) to $(\tau(1), n(1))$. Since there can be no singular points on $\S\mathcal{C}_1$ (or else there would be no two-phase solution), this amount is finite. It determines the ξ at the interface. All the conditions of our problem are met, and the portraits of phases 0 and 1 determine the solutions in those phases.

Now, suppose that there is at least one more phase, phase 2, surrounding phase 1. Then over the entire semi-orbit $\S\mathcal{C}_1$ in the portrait of phase 1 we construct the curve

$$\gamma_1 = \{(\tau, n, \chi) : (\tau, n) \in \S\mathcal{C}_1, \chi = \chi_1(\tau, n)\}. \tag{5.4}$$

Let χ_2 be the Eshelby constitutive function $\chi^{\natural}$ for the phase 2. In the same (τ, n, χ)-space construct the graph Γ_2 of χ_2:

$$\Gamma_2 = \{(\tau, n, \chi) : \chi = \chi_2(\tau, n)\}. \tag{5.5}$$

We denote intersection points of γ_1 with Γ_2 by (τ_{12}, n_{12}). We then repeat our preceding arguments with obvious modifications, according as phase 2 is the last phase or not.

Now consider problems in which there is cavitation at the center. (This can only occur for certain materials in phase 0.) We then carry out the preceding argument, but replace $\S\mathcal{S}$ in the definition of γ_0 with the $\S\mathcal{C}$ of Figure 4.2.

Note that it can happen that there are no solutions or there are be several solutions. All the steps just described are based on straightforward geometric constructions of phase portraits and graphs over phase portraits from the constitutive equations. (Given analytical expressions for the constitutive equations, all these constructions could be carried out on available graphical packages for computers.) The most important feature of the equilibrium configurations is the state at the center of the body, which corresponds to a saddle point in the portrait of phase 0. In general, the particular saddle point corresponding the the boundary condition (3.27) cannot be determined until all the phases are determined. As in [3] and [4], there are critical values of boundary conditions, i.e., critical dispositions of $\S\mathcal{B}$, across which the state at the center jumps. For example, the stresses N and T at the center can be zero for all normal pressures up to a threshold and then jump to $-\infty$ as this threshold is crossed.

6. The Isothermal Problem with Structured Interfaces

To obtain equilibrium solutions with structured interfaces we need a construction more complicated than that of the last section because the presence of the right-hand sides in the jump conditions (3.22), (3.23) distorts the simple geometry there and because the material coordinate ξ enters the interface conditions explicitly.

Let phase k be the outermost phase. In its phase portrait all orbits that terminate on $\S\mathcal{B}$ sweep out a region $\mathcal{D}_k$ of the (τ, n)-plane. (See [4] for sketches of $\mathcal{D}_k$.) At each point (τ, n) of $\mathcal{D}_k$ we compute from integral expressions obtained from (4.1)

the value $\xi_k(\tau,n)$ of ξ with the property that the orbit from (τ,n) to $\S\mathcal{B}$ uses up exactly $1-\xi_k(\tau,n)$ units of the independent variable. In particular, we note that under mild constitutive equations, Antman and Negrón-Marrero [3] showed that the horizontal and vertical isoclines of (4.1) have graphs for which τ is an increasing function of n. The curve $\S\mathcal{S}$ lies between these isoclines and has a graph of the same sort. It follows from these facts that a nonsingular trajectory can cross at most one isocline. Therefore each trajectory can be parametrized by τ or by n. Let $n \mapsto \tilde{\tau}_k(n)$ be the graph of a trajectory in the kth phase that is nowhere parallel to the τ axis. Then (4.1b) implies that

$$1-\xi_k(\tilde{\tau}_k(n),n) = \int_n^{n(1)} \frac{dm}{\alpha T^k(\tilde{\tau}_k(m),m) - \alpha m} \tag{6.1a}$$

where $(\tilde{\tau}_k(n(1)),n(1)) \in \S\mathcal{B}$. Likewise, Let $\tau \mapsto \tilde{n}_k(\tau)$ be the graph of a trajectory in the kth phase that is nowhere parallel to the n axis. Then (4.1a) implies that

$$1-\xi_k(\tau,\tilde{n}_k(\tau)) = \int_\tau^{\tau(1)} \frac{dt}{\nu^k(t,\tilde{n}_k(t)) - t} \tag{6.1b}$$

where $(\tau(1),\tilde{n}_k(\tau(1))) \in \S\mathcal{B}$. When the hypotheses leading to (6.1a) or (6.1b) are not in force, we can represent ξ_k by a sum of integrals of the kind that occur in (6.1). These integral expressions show that ξ_k depends continuously on its arguments, except when bifurcations occur. It could happen that the positive semi-orbit through (τ,n) in $\mathcal{D}_k$ intersects $\S\mathcal{B}$ at more than one point. For each such intersection we define a different function ξ_k.

Let (τ^+,n^+) denote the limit of (τ,n) as ξ approaches the interface between the kth and the $(k-1)$st phase from the kth phase, and (τ^-,n^-) denote the same limit as ξ approaches the interface from the $(k-1)$st phase. Then (3.22), (3.23), (3.8) yield

$$n^- = n^+ - \bar{N}_k(\tau^+,\xi_k(\tau^+,n^+)), \tag{6.2}$$

$$\chi_{k-1}(\tau^-,n^-) = \chi_k(\tau^+,n^+) - \bar{X}_k(\tau^+,\xi_k(\tau^+,n^+)), \tag{6.3}$$

$$\tau^- = \tau^+. \tag{6.4}$$

Here $\bar{N}_k$ and $\bar{X}_k$ denote the given constitutive functions for the interface between the kth and the $(k-1)$st phase, and, as before, χ_j denotes the given constitutive function for the the jth phase. We substitute (6.2) and (6.4) into (6.3) to find that (τ^+,n^+) must satisfy

$$\begin{aligned} H_{k-1}(\tau,n) &\equiv \chi_{k-1}(\tau,n-\bar{N}_k(\tau,\xi_k(\tau,n))) \\ &= \chi_k(\tau,n) - \bar{X}_k(\tau,\xi_k(\tau,n)) \equiv G_k(\tau,n). \end{aligned} \tag{6.5}$$

We construct the graphs $\mathcal{G}_k$ of G_k and $\mathcal{H}_{k-1}$ of H_{k-1} over $\mathcal{D}_k$:

$$\begin{aligned} \mathcal{G}_k &= \{(\tau,n,\chi) : (\tau,n) \in \mathcal{D}_k,\ \chi = G_k(\tau,n)\}, \\ \mathcal{H}_{k-1} &= \{(\tau,n,\chi) : (\tau,n) \in \mathcal{D}_k,\ \chi = H_{k-1}(\tau,n)\}. \end{aligned} \tag{6.6}$$

(Note that $\mathcal{H}_{k-1}$ is a distortion of the graph of χ_{k-1} in the n-direction; it is satisfying to note that there is no distortion in the τ-direction, because such a distortion would wreak havoc with the the inequality $\tau > 0$.) The graphs $\mathcal{G}_k$ and $\mathcal{H}_{k-1}$ might intersect, on a (curve-like[2]) set whose projection onto the (τ, n)-plane is $\S\mathcal{B}_{k-1,k}$. The pairs (τ^+, n^+), if any, satisfying (6.5) lie on $\S\mathcal{B}_{k-1,k}$. If there is no intersection of $\mathcal{G}_k$ and $\mathcal{H}_{k-1}$, then there can be no solution.

Now we construct the curve-like set

$$\S\mathcal{A}_{k-1,k} \equiv \{(\tau, n) : n = n^+ - \bar{N}_k(\tau, \xi_k(\tau, n^+)),\ (\tau, n^+) \in \S\mathcal{B}_{k-1,k}\}, \tag{6.7}$$

which consists of all points (τ^-, n^-) satisfying the interface conditions (6.2) and (6.4). All orbits in the portrait of phase $k-1$ that terminate on $\S\mathcal{A}_{k-1,k}$ sweep out a region $\mathcal{D}_{k-1}$ of the (τ, n)-plane. It can happen that no orbits terminate on $\S\mathcal{A}_{k-1,k}$, so that $\mathcal{D}_{k-1}$ is empty, whence there can be no equilibrium state with these phases. If $(\tau, n) \in \S\mathcal{A}_{k-1,k}$, then we define $\xi_{k-1}(\tau, n)$ to be the value of ξ_k at the corresponding point of $\S\mathcal{B}_{k-1,k}$, namely,

$$\xi_{k-1}(\tau, n^+ - \bar{N}_k(\tau, \xi_k(\tau, n^+))) \equiv \xi_k(\tau, n^+), \quad (\tau, n^+) \in \S\mathcal{B}_{k-1,k}. \tag{6.8}$$

At each point (τ, n) of $\mathcal{D}_{k-1}$, we compute from integral expressions obtained from (4.1) for phase $k-1$ the value $\xi_{k-1}(\tau, n)$ of ξ with the property that the orbit from (τ, n) to (τ^-, n^-) on $\S\mathcal{A}_{k-1,k}$ uses up exactly $\xi_{k-1}(\tau^-, n^-) - \xi_{k-1}(\tau, n)$ units of the independent variable. (As before, it could happen that the positive semi-orbit through (τ, n) in $\mathcal{D}_{k-1}$ intersects $\S\mathcal{A}_{k-1,k}$ at more than one point. For each such intersection, we define a different function ξ_{k-1}, which would lead to a different solution.)

We define G_{k-1} and H_{k-2}, their graphs $\mathcal{G}_{k-1}$ and $\mathcal{H}_{k-2}$, and the sets $\S\mathcal{B}_{k-2,k-1}$ and $\S\mathcal{A}_{k-2,k-1}$ just as above, and we repeat the process just described, with the obvious changes of indices, until we obtain the curve-like set $\S\mathcal{A}_{01}$. (If any of the sets $\S\mathcal{A}_{j-1,j}$ is empty, there can be no radially symmetric equilibrium state with these phases.)

Each segment of the separatrix $\S\mathcal{S}_0$ that terminates on $\S\mathcal{A}_{01}$ determines the state in phase 0 for which the center is intact. (For cavitation problems, replace $\S\mathcal{S}_0$ with $\S\mathcal{C}_0$.) The point at which $\S\mathcal{S}_0$ and $\S\mathcal{A}_{01}$ intersect determines via (6.2) the initial point on $\S\mathcal{B}_{01}$ of an orbit in phase 1 and therefore determines the whole orbit in this phase. By repeating this process, we determine an orbit in each phase; these orbits give the equilibrium solution. (This procedure bears a vague resemblance to that used to construct solutions of shrink-fit problems; see [4].)

7. The Isothermal Problem for Linearly Elastic Materials

It is instructive to specialize our results to homogeneous linearly elastic materials, because these are used in most studies of coexistent phases, and more importantly, because they yield some surprising results. Let the free energy in a typical phase have the form:

$$\psi(\tau, \nu) = \tfrac{1}{2}A(\tau - 1)^2 + B(\tau - 1)(\nu - 1) + \tfrac{1}{2}C(\nu - 1)^2, \tag{7.1}$$

[2] We use *curve-like* tautologically as an adjective describing any set that is the intersection of two-dimensional surfaces; this use is suggestive of the generic local behavior of such intersections.

where the elastic moduli A, B, C are numbers satisfying $A, C > 0$, $AC > B^2$. (In a specific problem these moduli would bear indices identifying the phase.) The material is isotropic if $A = C$. Let a typical surface free energy have the the form

$$\mathring{\psi}(\tau) = \tfrac{1}{2}D(\tau - 1)^2, \tag{7.2}$$

where D is a positive number. Then the stresses in the phase characterized by (7.1) are

$$\hat{N}(\tau, \nu) = B(\tau - 1) + C(\nu - 1), \tag{7.3a}$$

$$\hat{T}(\tau, \nu) = A(\tau - 1) + B(\nu - 1), \tag{7.3b}$$

and the surface stress corresponding to (7.2) is

$$\mathring{T}(\tau) = D(\tau - 1). \tag{7.4}$$

We explicitly invert the equation $\hat{N}(\tau, \nu) = n$ and thus obtain

$$\begin{aligned} C[\nu^{\mathrm{I}}(\tau, n) - 1] &= n - B(\tau - 1), \\ C\mathrm{T}^{\mathrm{I}}(\tau, n) &= (AC - B^2)(\tau - 1) + Bn, \\ C\chi^{\mathrm{I}}(\tau, n) &= \tfrac{1}{2}(AC - B^2)(\tau - 1)^2 - \tfrac{1}{2}n^2 + Bn(\tau - 1) - Cn \end{aligned} \tag{7.5}$$

(cf. (3.16)). Then our autonomous system (4.1) reduces to

$$\begin{aligned} C\dot{\tau} &= n - (B + C)(\tau - 1), \\ C\dot{n} &= \alpha\left[(AC - B^2)(\tau - 1) + (B - C)n\right]. \end{aligned} \tag{7.6}$$

The interface conditions (3.8), (3.22), (3.23) reduce to

$$[\tau] = 0, \qquad [n] = \alpha D \frac{\tau(\eta) - 1}{e^{\eta - 1}\tau(\eta)}, \qquad [\chi] = \alpha D \frac{1 - \tau(\eta)^2}{2e^{\eta - 1}\tau(\eta)}, \tag{7.7}$$

where η corresponds to the reference preimage of the interface.

The general solution of (7.6) is

$$\begin{bmatrix} \tau(\xi) - 1 \\ n(\xi) \end{bmatrix} = -Ae^{\lambda\xi}\begin{bmatrix} 1 \\ l \end{bmatrix} + Me^{\mu\xi}\begin{bmatrix} 1 \\ m \end{bmatrix} \tag{7.8}$$

where

$$\begin{aligned} \lambda &= -1 + \sqrt{A/C}, \qquad \mu = -1 - \sqrt{A/C} < 0, \\ l &= B + \sqrt{AC} > 0, \quad m = B - \sqrt{AC} < 0, \end{aligned} \tag{7.9}$$

and A, M are arbitrary constants of integration.

Now we study a material with two phases. Since we are considering only linear elasticity, the only meaningful solutions are those that stay near the natural state $(\tau, n) = (1, 0)$. We accordingly suppose that the singular point corresponding to the center must be a saddle at $(1, 0)$. In this case, the eigenvalue λ_0 for the solution for the phase 0 containing the center must be positive, so that $A_0 > C_0$; thus the material must be stronger in the azimuthal directions than in the radial direction. (In the degenerate case that the material of phase 0 is isotropic, so that $A_0 = C_0$, the analysis is simplified, and is left as an exercise for the interested reader.) Moreover, $\S S_0$ must be the ray spanning the eigenvector corresponding to λ_0, and the solution in the phase 0 containing the center must take values on this ray. Thus we conclude that in phase 0 the solution (7.8) reduces to

$$\begin{bmatrix} \tau(\xi) - 1 \\ n(\xi) \end{bmatrix} = -A_0 e^{\lambda_0 \xi}\begin{bmatrix} 1 \\ l_0 \end{bmatrix}. \tag{7.10}$$

The solution in the outer phase 1 is given by (7.8) with all the material constants indexed with 1.

Let $\xi_{01} = \eta$, so that $\sigma \equiv e^{\eta-1}$ is the radius of material points forming the interface. Let $L \equiv \Lambda_0 e^{\lambda_0 \eta}$. Substituting our solutions into the interface conditions (7.7) we obtain

$$-\Lambda_1 e^{\lambda_1 \eta} + M_1 e^{\mu_1 \eta} + L = 0, \tag{7.11}$$

$$-\Lambda_1 e^{\lambda_1 \eta} l_1 + M_1 e^{\mu_1 \eta} m_1 + L l_0 = \frac{\alpha D L}{\sigma(L-1)}, \tag{7.12}$$

$$\begin{aligned} K \equiv{}& \frac{1}{2}\left[\!\left[\frac{AC-B^2}{C}\right]\!\right] - \frac{l_0^2}{2}\left[\!\left[\frac{1}{C}\right]\!\right] + l_0\left[\!\left[\frac{B}{C}\right]\!\right] \\ ={}& \frac{\alpha D}{\sigma(1-L)}\left\{\frac{l_0}{C_1} - \frac{1}{2} - \frac{B_1}{C_1} + \frac{\alpha D}{2C_1\sigma(1-L)}\right\}. \end{aligned} \tag{7.13}$$

To be specific, suppose that $n(1)$ is prescribed to equal $\bar{n}$. Then (7.8) yields

$$-\Lambda_1 e^{\lambda_1} l_1 + M_1 e^{\mu_1} m_1 = \bar{n}. \tag{7.14}$$

Equations (7.11)–(7.14) are to be solved for the four unknowns Λ_1, M_1, L, and $\sigma \equiv e^{\eta-1}$, with σ confined to $(0,1)$.

Let us first consider the unstructured interface, for which $D = 0$. Then (7.13) implies: *For phases consisting of linearly elastic materials and for unstructured interfaces there can only be a solution in the unlikely case that $K = 0$, i.e., only if there is a very special relation between the elastic moduli of each phase. In this case, the interface can be located anywhere*, because the remaining equations do not restrict σ. Indeed, let σ be arbitrary in $(0,1)$. The determinant of coefficients of Λ_1, M_1, L in (7.11), (7.12), (7.14) vanishes if and only if

$$\sigma^2\sqrt{\Lambda_1/C_1} = \frac{l_1}{m_1}\frac{l_0 - m_1}{l_0 - l_1}. \tag{7.15}$$

Since $l_0, l_1 > 0 > m_1$, the right-hand side of (7.15) cannot take values in $(0,1)$. Thus this determinant never vanishes for σ in $(0,1)$, so that for each such σ, the system (7.11), (7.12), (7.14) has a unique solution for Λ_1, M_1, L.

This result indicates the paramount importance of interfacial structure for linearly elastic phases, provided we can show that (7.11)–(7.14) can be solved under natural circumstances when there is structure. Let us briefly sketch how this could be done. We rewrite (7.13) as

$$g(\sigma, L) \equiv \left(l_0 - \frac{1}{2}C_1 - B_1\right)\sigma(1-L) + \frac{\alpha D}{2} - \frac{KC_1\sigma^2(1-L)^2}{\alpha D} = 0. \tag{7.16}$$

We solve (7.11), (7.12) uniquely for Λ_1, M_1 in terms of L, σ, and we substitute these solutions into (7.14) to obtain

$$\begin{aligned} h(\sigma, L) \equiv{}& \left[l_1(l_0 - m_1)\sigma^{1+\mu_1} + m_1(l_1 - l_0)\sigma^{1+\lambda_1}\right] L(1-L) \\ &+ (l_1\sigma^{\mu_1} - m_1\sigma^{\lambda_1})\alpha D L + \bar{n}(l_1 - m_1)\sigma^{1+\lambda_1+\mu_1}(1-L) = 0. \end{aligned} \tag{7.17}$$

Now we determine the vector field (g, h) on the rectangle consisting of the lines $\sigma = 0$, $\sigma = 1$, $L = 1$, $L = L^+$ and the rectangle consisting of the lines $\sigma = 0$, $\sigma = 1$, $L = 0$, $L = L^-$, where L^+ is a large positive number and L^- is a large negative number. We can readily obtain reasonable conditions on the elastic moduli and on $\bar{n}$ to show that the rotation of this vector field on at least one of these rectangles is not 0; this condition ensures that (7.16), (7.17) has at least one solution within a rectangle having nonzero rotation. (See [11].)

8. The Thermoelastic Problem with Unstructured Interfaces

Our governing equations for (τ, n, θ) are the ordinary differential equations (3.21) and the interface conditions (3.22), (3.23), (3.8), (3.9). Recall that μ, which is the

heat flux on the outer boundary, is a pure constant by virtue of (3.7). If $\mu \neq 0$, then the heat flux, which is given by $q(\xi) = \mu e^{\alpha(1-\xi)}$, blows up as $\xi \to -\infty$. We admit this flux source at the center so that thermal effects can enter our problem. (One adopts a similar approach in studying sources and sinks in fluids.)

In this section we restrict our attention to unstructured interfaces for homogeneous materials and adopt the obvious analogs of the notation used there. In our endeavor to imitate the development of Section 5, we encounter a few technical difficulties occasioned by the failure of our equations to be autonomous. We first consider the case in which $q(1) = \mu$ is prescribed. We introduce the temperature β at the center:

$$\theta(-\infty) = \beta \tag{8.1}$$

We denote the solution of (3.21) satisfying (8.1) and either (3.25) or the cavitation conditions that $\tau(-\infty) = \infty$, $n(-\infty) = 0$ in the phase 0 containing the center by $(\tau_0(\cdot,\beta), n_0(\cdot,\beta), \theta_0(\cdot,\beta))$. We discuss conditions ensuring the existence and the consequent behavior of (τ_0, n_0, θ_0) below. Let χ_0, χ_1, ..., denote the constitutive function $\chi^\flat$ in the phases 0,1, ... For the time being, we merely assume that it is defined for ξ in some interval containing $-\infty$.

In the 4-dimensional space of (τ, n, θ, χ) we construct the curve

$$\xi \mapsto (\tau_0(\xi,\beta), n_0(\xi,\beta), \theta_0(\xi,\beta), \chi_0(\tau_0(\xi,\beta))) \tag{8.2}$$

and the 3-dimensional surface

$$(\tau, n, \theta) \mapsto ((\tau, n, \theta, \chi_1(\tau, n, \theta)). \tag{8.3}$$

The homogeneity of the material implies that no computations need be made in finding this surface. Any point where (8.2) intersects (8.3) defines a point $(\tau_{01}, n_{01}, \theta_{01})$ at which the interface conditions (3.22), (3.23), (3.8), (3.9) are met. If there are no such intersections, there cannot be equilibria with phase 1 touching phase 0 along an interface satisfying (3.22), (3.23), (3.8), (3.9). We do not have to make special provision for the possibility that solutions of the differential equations blow up: A solution that blows up either generates a (8.2) that intersects (8.3) or it does not. A trajectory of (3.21) in phase 0 terminating at a $(\tau_{01}, n_{01}, \theta_{01})$ defines the state in this phase that meets the intactness condition and the interface conditions.

But now we encounter what distinguishes this construction from that of Section 5: The failure of our system to be autonomous means that the value $\xi_{01}(\beta)$ of ξ at this interface is determined. We illustrate this failure by a simple example: Suppose that the function $f^\flat$, which appears in (3.21c), is independent of (τ, n, θ, s), as it would be for Fourier's Law. Then (3.21c) reduces to an equation of the form $\dot{\theta} = e^{\xi-1} f^\flat(\mu e^{\alpha(1-\xi)})$. To be specific, assume that $f^\flat(q) = Aq^\delta$ where A and δ are positive constants. Then the last equation becomes $\dot{\theta} = Be^{\gamma(1-\xi)}$, where B and γ are suitable constants. When we solve this equation and set the solution equal to θ_{01}, we find that $\xi_{01}(\beta) = 1 + \frac{1}{\gamma} \ln\left[\frac{\gamma}{B}(\theta_{01} - \beta)\right]$.

Thus the amount of ξ available for the remaining phases is fixed. It may happen that $\xi_{01}(\beta) > 1$, in which case there can be no equilibrium state for this β. In

general, for a fixed β there is thus little hope of satisfying the boundary condition (3.27). We shall accordingly wish to vary β to satisfy (3.27). We discuss this matter below.

Now we solve (3.21) in the phase 1 subject to the initial conditions $\tau(\xi_{01}(\beta)) = \tau_0(\xi_{01}(\beta),\beta)$, $n(\xi_{01}(\beta)) = n_0(\xi_{01}(\beta),\beta)$, $\theta(\xi_{01}(\beta)) = \theta_0(\xi_{01}(\beta),\beta)$, denote the resulting solution by $(\tau_1(\cdot,\beta),n_1(\cdot,\beta),\theta_1(\cdot,\beta))$ and repeat the process beginning with (8.2) with the obvious change of indices. We continue in this manner until we find the solution $(\tau_k(\cdot,\beta),n_k(\cdot,\beta),\theta_k(\cdot,\beta))$ in the kth phase.

Now we plot the curve $\beta \mapsto (\tau_k(1,\beta),n_k(1,\beta))$. In view of our preceding remarks, it may happen that parts of this curve are not defined for certain ranges of β. A value of β at which this curve intersects $\S\mathcal{B}$ generates a solution of the equilibrium problem.

Now suppose that in place of μ we prescribe

$$\theta(1) = \gamma. \tag{8.4}$$

In this case μ is an unknown parameter. We imitate the entire preceding development, merely replacing the single parameter β with the pair (μ,β). In particular, we denote the solution of (3.21) satisfying (8.1) and either (3.25) or the cavitation conditions in the phase 0 containing the center by $(\tau_0(\cdot,\mu,\beta),n_0(\cdot,\mu,\beta),\theta_0(\cdot,\mu,\beta))$, etc. We plot the two-dimensional surface $(\mu,\beta) \mapsto (\tau_k(1,\mu,\beta),n_k(1,\mu,\beta),\theta_k(1,\mu,\beta))$ in the three-dimensional (τ,n,θ)-space. A value of (μ,β) at which this surface intersects the curve $\{(\tau,n,\theta) : (\tau,n) \in \S\mathcal{B},\theta(1) = \gamma\}$ generates a solution of the equilibrium problem.

Finally we could prescribe the β of (8.1) and leave μ free to accommodate (8.4). (The artificiality inherent in prescribing the temperature at the center is comparable to that in prescribing μ.)

Let us now discuss the feasibility of choosing parameters β and μ to satisfy (3.27) and (8.4). For thermoelastic materials that are sensitive to temperature changes, the curve $\beta \mapsto (\tau_k(1,\beta),n_k(1,\beta))$ in (τ,n)-space and the surface $(\mu,\beta) \mapsto (\tau_k(1,\mu,\beta),n_k(1,\mu,\beta),\theta_k(1,\mu,\beta))$ may be expected to vary considerably with the parameters μ and β. Indeed, it would not be difficult to give sufficient conditions on the constitutive functions to ensure that β or μ or both could be found at which the requisite intersections take place. To do so, we could first observe that (3.21c) and (8.1) imply that

$$\theta(\xi) = \beta + \int_{-\infty}^{\xi} e^{\eta-1} f^1(\tau(\eta),n(\eta),\theta(\eta),\mu e^{\alpha(1-\eta)},e^{\eta-1})\,d\eta. \tag{8.5}$$

By the remarks following (3.16), the integrand in (8.5) has the same sign as μ. If $\mu > 0$, then (8.5) implies that $\theta(\xi) \to \infty$ as $\beta \to \infty$ for each ξ. If we can show that (our constitutive assumptions imply that) the solutions (τ,n,θ) of (3.21) are under control, so that the integrand of (8.5) has an upper bound on solutions, then we obtain $\theta(\xi) \to -\infty$ as $\beta \to -\infty$. (To do this, we could follow [3] to show that solutions (τ,n) are confined to certain invariant regions. When the resulting bounds are introduced into (8.5), we get integral inequalities for θ alone. Under favorable constitutive hypotheses (which are mild) we could use the available theory for such

inequalities to control θ and thus control the integrand.) We omit the technical details (cf. [14]). Of course, we get analogous results for $\mu < 0$. When we have to choose μ, we use the assumption following (3.16) to show that we can push the integrand of (8.5) to $\pm\infty$ by pushing μ to $\pm\infty$, provided the other variables are under control.

We need further constitutive restrictions on the dependence of f^1 on q in order to ensure that the solution of (3.21) for phase 0 supports the construction we have just described. These have the following character: For (τ, n, θ, s) in a given subset $\mathcal{E}$ of $(0,\infty)\times(-\infty,\infty)\times(0,\infty)\times[0,1]$, there is a positive number ε (depending on $\mathcal{E}$) such that

$$\frac{f^1(\tau,n,\theta,q,s)}{q^{(1/\alpha)-\varepsilon}} \to 0 \quad \text{as} \quad |q| \to \infty. \tag{8.6}$$

When we invoke (8.6), we must prescribe the set $\mathcal{E}$. This assumption is equivalent to the requirement that the constitutive function $\hat{q}$ of (3.13) be asymptotically superlinear in θ' when $\alpha = 1$ and be asymptotically superquadratic in θ' when $\alpha = 2$.

In Section 10 we shall show that a suitable version of (8.6) together with mild constitutive conditions imply that (τ, n) for a solution of the system (3.21) for the phase 0 containing the center, subject to (8.1) and to either the intactness condition (3.25) or the cavitation condition (3.26), behaves just like solutions of the autonomous system

$$\dot{\tau} = \nu^1(\tau,n,\beta,0) - \tau, \quad \dot{n} = \alpha T^1(\tau,n,\beta,0) - \alpha n \tag{8.7a,b}$$

for the phase containing the center; this system is just a version of (4.1). (In particular, solutions are defined for all ξ in a neighborhood of $-\infty$.) This means that when β is determined, the state of the center is completely known from a simple computation of the corresponding singular point for (8.7). The state at the center and the way it can jump at critical thresholds of the data are the most important physical aspects of the problems we treat.

9. The Thermoelastic Problem with Structured Interfaces

We now extend the methods of Section 6 to thermoelastic problems by accounting for the failure of our system to be autonomous. Let us first suppose that (8.4) is prescribed, so that the parameter μ is at our disposal. It is illuminating to parametrize the terminal curve $\S\mathcal{B}$ as $\sigma \mapsto (\tau^*(\sigma), n^*(\sigma))$, so that (3.27) and (8.4) have the form

$$\tau(1) = \tau^*(\sigma), \quad n(1) = n^*(\sigma), \quad \theta(1) = \gamma. \tag{9.1a}$$

We denote (9.1a) by

$$(\tau(1), n(1), \theta(1)) \in \S\mathcal{B}^*. \tag{9.1b}$$

Let

$$\xi \mapsto (\tau_k(\xi,\sigma,\mu), n_k(\xi,\sigma,\mu), \theta_k(\xi,\sigma,\mu)) \tag{9.2}$$

denote the solution of (3.21) for the kth phase for $\xi < 1$ subject to the terminal conditions (9.1). For each fixed μ, (9.2) defines a two-dimensional surface in the three-dimensional space of (τ, n, θ) parametrized by ξ, σ. This surface contains the curve (9.1).

Let

$$J_k(\xi,\sigma,\mu) \equiv \chi_{k-1}(\tau, n - \bar{N}_k(\tau,\theta,\xi),\theta,\xi) - \chi_k(\tau,n,\theta,\xi) + \bar{X}_k(\tau,\theta,\xi) \tag{9.3}$$

where (τ, n, θ) are replaced with their values given by (9.2). The set of (ξ, σ) for which

$$J_k(\xi,\sigma,\mu) = 0 \tag{9.4}$$

lie on a curve-like set $\S\mathcal{Z}_{k-1,k}$ in the (ξ,σ)-plane. Equation (9.4) is the analog of (6.5). For (ξ,σ) in $\S\mathcal{Z}_{k-1,k}$, $\tau_k(\xi,\sigma,\mu), n_k(\xi,\sigma,\mu), \theta_k(\xi,\sigma,\mu)$ lies in a curve-like set $\S\mathcal{B}^*_{k-1,k}$ in (τ,n,θ)-space.

As in (6.7) we now define

$$\begin{aligned} \S\mathcal{A}^*_{k-1,k} \equiv \{(\tau,n,\theta) : \tau &= \tau_k(\xi,\sigma,\mu), \\ n &= n_k(\xi,\sigma,\mu) - \bar{N}_k(\tau_k(\xi,\sigma,\mu),\theta_k(\xi,\sigma,\mu),\xi), \\ \theta &= \theta_k(\xi,\sigma,\mu), \quad (\xi,\sigma) \in \S\mathcal{Z}_{k-1,k}\}. \end{aligned} \tag{9.5}$$

The members of $\S\mathcal{B}^*_{k-1,k}$ and $\S\mathcal{A}^*_{k-1,k}$ are precisely those that satisfy the interface conditions (3.22), (3.23).

We now take $\S\mathcal{A}^*_{k-1,k}$ to be an terminal curve just like $\S\mathcal{B}^*$ and construct the solutions of (3.21) for the $(k-1)$st phase that ends on $\S\mathcal{A}^*_{k-1,k}$. But here the terminal values of ξ are not constant, but are those for $\S\mathcal{Z}_{k-1,k}$. By exactly the same methods used to construct $\S\mathcal{A}^*_{k-1,k}$, we obtain the curve-like sets $\S\mathcal{A}^*_{k-1,k-2}, \ldots, \S\mathcal{A}^*_{01}$.

Now we choose a β and construct the solution

$$\xi \mapsto (\tau_0(\xi,\beta,\mu), n_0(\xi,\beta,\mu), \theta_0(\xi,\beta,\mu)) \tag{9.6}$$

of (3.21) for phase 0 satisfying (8.1) and either (3.25) or (3.26). The existence of such solutions, which look like those for (8.7), is discussed in Section 10. For each fixed (β,μ), (9.6) defines a curve in (τ,n,θ)-space. If this curve intersects $\S\mathcal{A}^*_{01}$ at exactly the value of ξ corresponding to the intersection point, then our solutions in each phase determine a solution of our full problem. Of course, for fixed (β,μ) it is extremely unlikely that two curves intersect in three-dimensional space. But by the remarks at the end of Section 8, we can expect to vary θ_0 appreciably by varying β. Thus for each μ we can expect to find a $\beta = \beta^*(\mu)$ that causes (9.6) and $\S\mathcal{A}^*_{01}$ to intersect, though not necessarily at the right ξ. Since $\dot{\theta}$ has the same sign as μ by our constitutive hyperbolics on $\hat{q}$, we can equivalently parametrize solution with θ instead of ξ. Since θ is sensitive to changes in μ, we can expect to find a μ that delivers the right value for ξ at this intersection.

The adjustment of this argument when μ is prescribed follows the lines of Section 8.

Although there are appreciable computations for these nonautonomous problems in these shooting methods, the essential qualitative feature of the solutions, namely, the behavior of the solutions at the center is completely determined from an elementary study of (8.7).

10. Thermoelastic Problems as Asymptotically Autonomous Problems

We now sketch the proofs of the assertion made at the end of Section 8. These are examples of perturbation theorems for asymptotically autonomous systems, for which there is a rich theory (see Hale [9, Sec. IV.3] and Markus [12], e.g.). Since we could not find theorems that directly apply to our situations, we prove the requisite results.

It is intuitively clear that as $\xi \to -\infty$, our problem approaches an autonomous problem. We should like to take initial conditions at the center of the body at an unstable singular point of this autonomous system, but a trajectory cannot leave any singular point. We accordingly must be content with starting a trajectory on the unstable manifold near the singular point. The following development shows how to handle the consequent technicalities.

We first study the behavior of of solutions of (3.21a,b) near a saddle point $(\tau^\times, n^\times) \in (0,\infty)\times\mathbb{R}$ of the autonomous system (8.7) for the phase containing the center. We assume that (8.6) holds for (τ, n, θ) in a compact subset of $(0,\infty)\times\mathbb{R}\times(0,\infty)$ and for all s sufficiently small. For simplicity of exposition, we assume that ν^{I}, T^{I}, and f^{I} are twice continuously differentiable. (See Hale [9, Sec. IV.3] for weaker conditions that work.) We set

$$\Delta\tau \equiv \tau - \tau^\times, \quad \Delta n \equiv n - n^\times, \quad \Delta\theta \equiv \theta - \beta. \tag{10.1}$$

Let

$$\nu_\tau^\times \equiv \nu_\tau^{\mathrm{I}}(\tau^\times, n^\times, \beta, 0), \quad \text{etc.} \tag{10.2}$$

Then using (8.7), we can write (3.21a,b) as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{g}(\mathbf{z}, \Delta\theta, \xi), \tag{10.3}$$

where

$$\mathbf{z} \equiv \begin{bmatrix} \Delta\tau \\ \Delta n \end{bmatrix}, \quad \mathbf{A} \equiv \begin{bmatrix} \nu_\tau^\times - 1 & \nu_n^\times \\ \alpha T_\tau^\times & \alpha(T_n^\times - 1) \end{bmatrix},$$

$$\mathbf{g} \equiv \begin{bmatrix} \nu_\theta^\times \Delta\theta + \nu_s^\times e^{\xi-1} + O\left(|\mathbf{z}|^2 + (\Delta\theta)^2 + e^{2(\xi-1)}\right) \\ \alpha T_\theta^\times \Delta\theta + \alpha T_s^\times e^{\xi-1} + O\left(|\mathbf{z}|^2 + (\Delta\theta)^2 + e^{2(\xi-1)}\right) \end{bmatrix}.$$

From (3.21c), (8.4), and (10.1) we get a version of (8.5):

$$(\Delta\theta)(\xi) = \int_{-\infty}^{\xi} e^{\eta-1} f^{\mathrm{I}}(\tau^\times + \Delta\tau, n^\times + \Delta n, \beta + \Delta\theta, \mu e^{\alpha(1-\eta)}, e^{\eta-1})\, d\eta. \tag{10.4}$$

We want to convert (10.3) into a useful integral equation that exploits the saddle-point structure.

Since $(\tau^\times, n^\times)$ is a saddle point of (8.7), $\mathbf{A}$ has one positive eigenvalue λ^+ and one negative eigenvalue $-\lambda^-$ with corresponding unit eigenvectors $\mathbf{e}^+$ and $\mathbf{e}^-$. Let $\mathbf{\Pi}^+$ and $\mathbf{\Pi}^-$ be the projections of $\mathbb{R}^2$ onto $\mathrm{span}\{\mathbf{e}^+\}$ and $\mathrm{span}\{\mathbf{e}^-\}$, so that $\mathbf{\Pi}^+\zeta + \mathbf{\Pi}^-\zeta = \zeta$. Thus

$$e^{\mathbf{A}\xi}\mathbf{\Pi}^+\zeta = e^{\lambda^+\xi}\mathbf{\Pi}^+\zeta, \quad e^{\mathbf{A}\xi}\mathbf{\Pi}^-\zeta = e^{-\lambda^-\xi}\mathbf{\Pi}^-\zeta. \tag{10.5}$$

We now get the desired representation or solutions of (10.3):

10.6. Lemma. *Let (8.6) hold for (τ, n, θ) in a compact subset of $(0,\infty)\times\mathbb{R}\times(0,\infty)$ and for all s sufficiently small. For each solution $(\mathbf{z}, \Delta\theta)$ of (10.3), (10.4) for which $(\tau^\times, n^\times) + \mathbf{z}(\xi)$ lies in a compact subset $\mathcal{K}$ of $(0,\infty)\times\mathbb{R}$ for $\xi \le \sigma$, there is a ζ^+ in $\mathrm{span}\{\mathbf{e}^+\} \cap [\mathcal{K} \setminus (\tau^\times, n^\times)]$ such that*

$$\begin{aligned} \mathbf{z}(\xi) = e^{\mathbf{A}(\xi-\sigma)}\zeta^+ &+ \int_\sigma^\xi e^{\mathbf{A}(\xi-\eta)}\mathbf{\Pi}^+\mathbf{g}(\mathbf{z}(\eta), \Delta\theta(\eta), \eta)\, d\eta \\ &+ \int_{-\infty}^0 e^{-\mathbf{A}\eta}\mathbf{\Pi}^-\mathbf{g}(\mathbf{z}(\xi+\eta), \Delta\theta(\xi+\eta), \xi+\eta)\, d\eta \end{aligned} \tag{10.6}$$

for $\xi \le \sigma$. Conversely, any solution of (10.6), (10.4) *for which* $(\tau^\times, n^\times) + z(\xi)$ *lies in a compact subset of* $(0,\infty) \times \mathbb{R}$ *for* $\xi \le \sigma$ *is a solution of* (10.3), (10.4).

Proof. By the variation-of-constants formula, (10.3) is equivalent to

$$z(\xi) = e^{A(\xi-\sigma)}z(\sigma) + \int_\sigma^\xi e^{A(\xi-\eta)}g(z(\eta), \Delta\theta(\eta), \eta)\, d\eta. \tag{10.7}$$

Since A commutes with $\Pi^\pm$, (10.5) and (10.7) imply that

$$\Pi^- z(\xi) = e^{-\lambda^-(\xi-\rho)}\Pi^- z(\rho) + \int_\rho^\xi e^{-\lambda^-(\xi-\eta)}\Pi^- g(z(\eta), \Delta\theta(\eta), \eta)\, d\eta. \tag{10.8}$$

Since $(\tau^\times, n^\times) + z(\xi) \in \mathcal{K}$ for $\xi \le \sigma$, we use (8.6) to deduce from (10.4) that $\Delta\theta \to 0$ (monotonically) as $\xi \to -\infty$. It follows that there is a number $N > 0$ such that $|g(z(\eta), \Delta\theta(\eta), \eta)| \le N$ for $\eta \le \sigma$. Thus, for $\rho < \xi < \sigma$,

$$\left|\int_\rho^\xi e^{-\lambda^-(\xi-\eta)}\Pi^- g(z(\eta), \Delta\theta(\eta), \eta)\, d\eta\right| \le N\int_\rho^\xi e^{-\lambda^-(\xi-\eta)}\, d\eta = \frac{N}{\lambda^-}[1 - e^{-\lambda^-(\xi-\rho)}]. \tag{10.9}$$

Thus the limit of the integral in (10.8) as $\rho \to -\infty$ exists. Letting $\rho \to -\infty$ in (10.8), we get

$$\Pi^- z(\xi) = \int_{-\infty}^\xi e^{-\lambda^-(\xi-\eta)}\Pi^- g(z(\eta), \Delta\theta(\eta), \eta)\, d\eta. \tag{10.10}$$

Now we replace ξ in (10.10) by σ, substitute the resulting expression into (10.7), and use the properties of $\Pi^\pm$ to reduce (10.7) to (10.6) with $\zeta^+ = \Pi^+ z(\sigma)$. The last statement of the theorem is proved by direct computation. □

We adopt the system (10.4), (10.6) as our governing equations. We now show that this system is solvable.

10.11. Lemma. *For any* ζ^+ *in* $\operatorname{span}\{e^+\}$ *with* $|\zeta^+|$ *sufficiently small, the system* (10.6), (10.4) *has a unique solution* $\xi \mapsto (z^*(\xi, \sigma, \beta, \mu, \zeta^+), \theta^*(\xi, \sigma, \beta, \mu, \zeta^+))$ *defined on* $(-\infty, \sigma]$ *depending continuously* $\sigma, \beta, \mu, \zeta^+$.

Sketch of proof. Let

$$\mathcal{G}(\sigma, \zeta^+, \delta) \equiv \{(z, \Delta\theta) \in C^0(-\infty, \sigma] : \sup_{\xi\in(-\infty,\sigma]} [|z(\xi)| + |\Delta\theta(\xi)|] \le \delta,\ \Pi^+ z(\sigma) = \zeta^+\}. \tag{10.12}$$

Note that for δ sufficiently small, if z is in $\mathcal{G}(\sigma, \zeta^+, \delta)$, then its values lie in $\mathcal{K} - (\tau^\times, n^\times)$. For δ sufficiently small, the properties of g and $f^\natural$ readily imply that the operator defined by the right-hand sides of (10.4) and (10.6) is a uniform contraction taking $\mathcal{G}(\sigma, \zeta^+, \delta)$ (which is a closed subset of the Banach space of bounded continuous functions on $(-\infty, \sigma]$) into itself. The Contraction Mapping Principle yields the conclusion. □

It immediately follows from (8.6) that there is a positive number C such that

$$|\theta^*(\xi)| \le Ce^{\varepsilon(\xi-1)} \quad \text{for} \quad \xi \le \sigma. \tag{10.13}$$

Here and below we suppress some of the arguments of θ^*. In view of (10.13) and the definition of g, we find from (10.6) that there is a $C > 0$ and a function $\delta \mapsto \kappa(\delta) > 0$ with $\kappa(\delta) \to 0$ as $\delta \to 0$ such that

$$\begin{aligned} |z^*(\xi)| \le e^{\lambda^+(\xi-\sigma)}|\zeta^+| &+ C\int_\xi^\sigma e^{\lambda^+(\xi-\eta)}[\kappa(\delta)|z^*(\eta)| + e^{\varepsilon(\eta-1)} + e^{\eta-1}]\, d\eta \\ &+ C\int_{-\infty}^0 e^{\lambda^-\eta}[\kappa(\delta)|z^*(\xi+\eta)| + e^{\varepsilon(\xi+\eta-1)} + e^{\xi+\eta-1}]\, d\eta \end{aligned} \tag{10.14}$$

for $\xi \le \sigma$. When applied to (10.14), Lemma III.6.2 of Hale [9] implies

10.15. Theorem. *Let* (8.6) *hold for* (τ, n, θ) *in a compact subset of* $(0,\infty)\times\mathbf{R}\times(0,\infty)$ *and for all* s *sufficiently small. If* δ *is small enough, then*

$$z^*(\xi)\to 0 \quad \text{as} \quad \xi\to-\infty. \tag{10.15}$$

This is the critical result of our analysis, which implies that the state at the center of the thermoelastic body is described by a saddle point of (8.7) when this saddle point lies in a compact set in $(0,\infty)\times\mathbf{R}$. By following Hale, we could show that $z^*(\xi)$ approaches 0 exponentially fast as $\xi\to-\infty$ and could show that its trajectory is tangent to e^+ at 0, but these results, while illuminating, are not crucial for our development. This development is an adaptation of that of Hale [9, Sec. IV.3].

Now we show how to handle problems in which the state at the center of the thermoelastic body is described by an initial point that does not lie in a compact set of the phase space $(0,\infty)\times\mathbf{R}$. This initial point may be $(\tau, n) = (\infty, 0)$ for cavitation, or $(\tau, n) = (0, -\infty)$ or $(\tau, n) = (\infty, \infty)$ for intact centers. The singular behavior of reasonable constitutive functions near this initial point prevents us from obtaining an equation like (10.3): The matrix $\mathbf{A}$ does not have a well-defined limit at the point; its values depend on the manner in which the initial point is approached.

Let $(\bar\tau, \bar n)$ be a solution of (8.7) emanating from the initial point. (It lies on $\S\mathcal{C}$ or $\S\mathcal{S}$.) We seek solutions of (3.21a,b) of the form

$$\tau = \bar\tau u, \quad n = \bar n v, \tag{10.16a}$$

$$u(-\infty) = 1 = v(-\infty). \tag{10.16b}$$

We indicate how to prove that there are solutions of this form; further constitutive assumptions (cf. [13]) ensure uniqueness. Substituting (10.16) into (3.21a,b) we obtain

$$\dot u = \frac{\nu^{\mathrm{I}}(\bar\tau u, \bar n v, \theta(e^{\xi-1}), e^{\xi-1}) - \nu^{\mathrm{I}}(\bar\tau, \bar n, \beta, 0)u}{\bar\tau}, \tag{10.17a}$$

$$\frac{\dot v}{\alpha} = \frac{T^{\mathrm{I}}(\bar\tau u, \bar n v, \theta(e^{\xi-1}), e^{\xi-1}) - T^{\mathrm{I}}(\bar\tau, \bar n, \beta, 0)v}{\bar n}, \tag{10.17b}$$

When the right-hand sides of (10.17) are regular, the existence theory for solutions of it subject to (10.16b) can be treated by methods like those used to treat (10.3).

We illustrate how the regularity of (10.17) can be demonstrated in a specific, but typical problem in which the state of the center corresponds to $(\tau, n) = (0, -\infty)$, which is a saddle point in the sense that $\S\mathcal{S}$ emanates from it. We now assume that (8.6) holds for (τ, n) in a neighborhood $\mathcal{N}$ of $(0, -\infty)$ (i.e., when the material is highly compressed). As before, we deduce that $\theta(\xi)\to\beta$ as $\xi\to-\infty$ if $(\tau, n)\in\mathcal{N}$. Now suppose that the constitutive equations (3.10), (3.11) in the phase containing the center have the form

$$\hat T(\tau, \nu, \theta, s) = -aG(\theta, s)\tau^{-a-1}\nu^{-b} + \cdots, \tag{10.18a}$$

$$\hat N(\tau, \nu, \theta, s) = -bG(\theta, s)\tau^{-a}\nu^{-b-1} + \cdots, \tag{10.18b}$$

where a and b are positive constants, G is a positive-valued continuous function, and the ellipses stand for terms negligible with respect to the visible terms for (τ, n) in $\mathcal{N}$. (See [3] for the treatment of constitutive equations more complicated than (10.18).) From (10.18) we immediately obtain

$$\begin{aligned} \nu^{\mathrm{I}}(\tau, n, \theta, s) &= \left[\frac{bG(\theta, s)}{-n\tau^a}\right]^{1/(b+1)} + \cdots, \\ T^{\mathrm{I}}(\tau, n, \theta, s) &= -aG(\theta, s)\tau^{-a-1}\left[\frac{-n\tau^a}{bG(\theta, s)}\right]^{b/(b+1)} + \cdots. \end{aligned} \tag{10.19}$$

The substitution of (10.19) into (10.17) yields

$$\dot u = b^{1/(b+1)}\Phi\left\{\left[\frac{G(\theta(e^{\xi-1}), e^{\xi-1})}{u^a v}\right]^{1/(b+1)} - G(\beta, 0)^{1/(b+1)}u\right\} + \cdots, \tag{10.20a}$$

$$\frac{\dot v}{\alpha} = -\frac{a}{b^{b/(b+1)}}\Phi\left\{G(\theta(e^{\xi-1}), e^{\xi-1})^{1/(b+1)}\frac{v^{b/(b+1)}}{u^{a+b+1}} - G(\beta, 0)^{1/(b+1)}v\right\} + \cdots \tag{10.20b}$$

where

(10.20c) $$\Phi = (-\bar{n}\bar{r}^{a+b+1})^{-1/(b+1)}.$$

The terms in braces in (10.20a,b) are completely regular for (u, v) near its initial value $(1, 1)$. Thus (10.20a,b) is well-behaved if Φ is. It is easy to show (cf. [13]) that if (10.18) holds, then

(10.21a) $$\bar{r} = \Omega(\beta)s^{\omega-1} + \cdots \quad \text{where} \quad \Omega(\beta) > 0, \quad \omega = \frac{b + b(a+b+1)}{a + b(a+b+1)}.$$

For $(0, -\infty)$ to be a saddle point it is necessary that $b > a$. From (3.1) and (10.18b) we immediately obtain that

(10.21b) $$\bar{v} = \omega\bar{r}, \quad \bar{n} = -bG(\beta, 0)\omega^{-b-1}\bar{r}^{-a-b-1} + \cdots,$$

so that Φ approaches a positive constant as $\xi \to -\infty$. In view of our preceding remarks, we conclude that an analog of Theorem 10.15 holds in this case.

11. Comments

Our treatment in Sections 8 and 9 are versions of the shooting method for ordinary differential equations (see Keller[10]). (Our methods are reminiscent of those used to solve the Riemann problem for hyperbolic conservation laws (cf. Smoller [16]), not surprisingly, since the Riemann problem and our problems are each concerned with the disposition of discontinuities.) What makes our problems special is that our approach exploits the underlying geometry so that we can extract useful qualitative insights about solutions with little effort. We also believe that our formulations could lead to practical numerical algorithms.

Let us assume that $\hat{q}$ of (3.13) is linear in θ' as in the Fourier heat conduction law. In this case, (3.21c) reduces to an equation of the form $\dot{\theta} = g(\tau, n, \theta)\mu$. If the solution of this equation satisfying (8.1) decays fast enough to β as $\xi \to -\infty$, then we can carry out all the analysis of Section 10. We do not get this decay if g is constant.

Our methods can handle many other kinds of jump conditions at interfaces, e.g., some allowing jumps in θ.

Acknowledgment. The work reported here, which extends a part of the doctoral dissertation of M. M. Shvartsman [14], was supported in part by a grant from the National Science Foundation. We are grateful to M. E. Gurtin for some helpful suggestions.

References

1. R. Abeyaratne and J. K. Knowles, *On the driving traction acting on a surface of strain discontinuity in a continuum*, J. Mech. Phys. Solids **38** (1990), 345–360.
2. R. Abeyaratne and J. K. Knowles, *Kinetic relations and the propagation of phase boundaries in solids*, Arch. Rational Mech. Anal. **114** (1991), 119–154.
3. S. S. Antman and P. V. Negrón-Marrero, *The remarkable nature of radially symmetric equilibrium states of aeolotropic nonlinearly elastic bodies*, J. Elasticity **18** (1987), 131–164.
4. S. S. Antman and M. M. Shvartsman, *The shrink-fit problem for aeolotropic nonlinearly elastic bodies*, J. Elasticity **37** (1995), 157–166.
5. J. M. Ball, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Phil. Trans. Roy. Soc. Lond. **A306** (1982), 557–611.

6. M. E. Gurtin, *The dynamics of solid-solid phase transitions 1. Coherent interfaces*, Arch. Rational Mech. Anal. 123 (1993), 305–335.
7. M. E. Gurtin, *The nature of configurational forces*, Arch. Rational Mech. Anal. (1995).
8. M. E. Gurtin and I. Murdoch, *A continuum theory of elastic material surfaces*, Arch. Rational Mech. Anal. 57 (1976), 291–323.
9. J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, 1969.
10. H. B. Keller, *Numerical Solution of Two-Point Boundary Value Problems*, S.I.A.M., 1976.
11. M. A. Krasnosel'skiy, A. I. Perov, A. I. Povolotskiy, P. P. Zabreiko, *Plane Vector Fields*, Academic Press, 1966.
12. L. Markus, *Asymptotically autonomous differential systems*, *Contributions to the Theory of Nonlinear Oscillations, Vol. III* (S. Lefschetz, ed.), 1956, pp. 17–29.
13. P. V. Negrón-Marrero and S. S. Antman, *Singular global bifurcation problems for buckling of anisotropic plates*, Proc. Roy. Soc. Lond. A 427 (1990), 95-137.
14. M. M. Shvartsman, *Coherent Equilibrium Phases in Aeolotropic Nonlinear Thermoelasticity*, Dissertation, University of Maryland, 1994.
15. A. K. Sinha, *Ferrous Physical Metallurgy*, Butterworth, 1989.
16. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer, 1983.
17. L. Truskinovsky, *Transition to detonation in dynamic phase change*, Arch. Rational Mech. Anal. 125 (1994), 375–397.
18. J. L. Walker, *Structure of Ingots and Castings*, *Liquid Metals and Solidification*, American Society for Metals, 1958.