Graphs containing triangles are not 3-common

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A finite graph G is *k-common* if the minimum (over all k-colourings of the edges of K_n) of the number of monochromatic labelled copies of G is asymptotically equal, as n tends to infinity, to the expected number of such copies in a random k-colouring of the edges of K_n. Jagger, Šťovíček and Thomason showed that graphs which contain K_4 are not 2-common. We prove that graphs which contain K_3 are not 3-common.

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1. Introduction

In this paper we prove a result about the Ramsey multiplicity constants of certain graphs, in the context of edge colourings in three colours. Before we state the result precisely we will give some background about Ramsey multiplicities.

A *labelled copy* of a graph G in a graph G' is an injective function from $V(G)$ to $V(G')$ such that every edge of G is mapped to an edge of G'. Given a colouring of the edges of G', a *monochromatic labelled copy of G in G'* is a labelled copy of G in G' such that the images of the edges of G all have the same colour.

If G' is complete, every injective map from $V(G)$ to $V(G')$ is a labelled copy of G in G'. We note that when $v(G) = t$ and $n \geq t$ there are $t!\binom{n}{t}$ labelled copies of G in K_n, and that this quantity is asymptotic to n^t as n tends to infinity.

Let G be a finite graph with $v(G) = t$ and let $k > 1$. By Ramsey's theorem there is a natural number C such that, for every k-colouring of the edges of K_n, among every set of C vertices there is a monochromatic set of size t. It follows by a straightforward double counting argument that there

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is a constant $\epsilon > 0$ such that, for every large enough n and every k-colouring of the edges of K_n, there are at least ϵn^t monochromatic labelled copies of G. This suggest that we should study the *density* of monochromatic labelled embeddings of G in K_n, that is to say the proportion of labelled embeddings of G in K_n which are monochromatic.

Definition 1. *Let G be a graph with $v(G) = t$ and let $n \geq 1$. Let H be a colouring of the edges of K_n.*

1. *$c_{mon}(G, H)$ is the number of labelled copies of G in K_n which are monochromatic for the colouring H.*
2. *$d_{mon}(G, H) = \frac{c_{mon}(G,H)}{t!\binom{n}{t}}$.*

Definition 2. *Let G be a graph and let $k, n \geq 1$.*

$C_k(G; n) = \min\{d_{mon}(G, H) : H \text{ is a } k\text{-colouring of the edges of } K_n\}$.

The probabilistic method gives an easy upper bound on $C_k(G; n)$. Let $v(G) = t$ and $e(G) = e$, and consider a random k-colouring **H** of the edges of K_n in which the colours of the edges are independently chosen with equal probability. By linearity of expectation the expected value of $c_{mon}(G, \mathbf{H})$ is $t!\binom{n}{t}k^{1-e}$, and we conclude that $C_k(G; n) \leq k^{1-e}$.

In some cases this probabilistic bound is asymptotically tight. Goodman [3] gave an exact formula for the minimal number of monochromatic triangles in a 2-colouring of the edges of K_n. This quantity is asymptotic to $\frac{n^3}{24}$ as n tends to infinity, so that $C_2(K_3; n) \to \frac{1}{4}$. In this case $k = 2$ and $e = 3$, so that the probabilistic bound for $C_2(K_3; n)$ is also $\frac{1}{4}$. Erdős [2] conjectured that $C_2(G; n) \to 2^{1-e}$ for all complete G, and Burr and Rosta [1] conjectured that in fact this holds for all finite graphs G.

Thomason [7], showed that the conjecture by Erdős is false when $G = K_4$. Jagger, Šťovíček and Thomason [5] disproved the Burr-Rosta conjecture for all graphs which contain K_4. Thomason [8] gave a simpler counterexample for K_4. Jagger, Šťovíček and Thomason [5] also discussed the asymptotic behaviour of $C_k(G; n)$ for general values of k, and defined the notion of a *k-common graph*.

Fact 1. *[5, section 3] Let $k \geq 1$ and let G be a graph. $C_k(G; n)$ increases with n.*

Definition 3. *Let $k \geq 1$ and let G be a graph, with $e(G) = e$.*

1. *$C_k(G) = \lim_{n \to \infty} C_k(G; n)$. $C_k(G)$ is the* Ramsey multiplicity constant *(for k colours) of G.*

2. *G is k-common if $C_k(G) = k^{1-e}$, and k-uncommon if $C_k(G) < k^{1-e}$.*

In this language the Burr-Rosta conjecture states that all graphs are 2-common. Jagger, Štovíček and Thomason [5] and Jagger [4] proved several results about k-common and k-uncommon graphs.

1. If a graph G contains K_4, then G is not 2-common [5, theorem 12].
2. A notable conjecture by Sidorenko [6] states that if G is a bipartite graph with $e(G) = e$, and H is a graph with $v(H) = n$ and average degree pn, then the probability that a random map from $V(G)$ to $V(H)$ is a graph homomorphism is at least p^e. This conjecture has been established for several classes of bipartite graphs including complete bipartite graphs, trees and even cycles.
   If a bipartite graph G satisfies Sidorenko's conjecture, then G is k-common for all k [5, section 5].
3. If a graph G is k-uncommon, then G is $(k+1)$-uncommon [5, theorem 13]
4. If G is a non-bipartite graph with $v(G) = t$, then G is k-uncommon when $2^{2k-2} \geq k^t$ [5, theorem 14]. Hence G is k-uncommon for all sufficiently large values of k.
5. Odd cycles are 3-uncommon [4].

Our main result is Theorem 1, which states that any graph containing K_3 is 3-uncommon. The argument is algebraic and parallels that of [5, section 4]. In section 2 we give a formula estimating the quantity $c_{mon}(G, H)$. In section 3 we define a notion of *tensor product* for k-colourings, and give a criterion for a graph to be k-uncommon. In section 4 we prove Theorem 1. Finally in section 5 we record some questions and remarks.

2. Counting monochromatic copies

In this section we give an estimate for $c_{mon}(G, H)$, the number of monochromatic labelled copies of G in a colouring H of the edges of K_n. We are most interested in the case of 3-colourings, but the analysis works for any number k of colours. For the rest of this section we will fix some value k with $k > 1$, and will consider k-colourings for which the set of colours is $\{0, \ldots, k-1\}$. We will also fix a graph G with $v(G) = t$ and $e(G) = e$.

Let $\zeta_k = \exp(\frac{2\pi i}{k})$, so that ζ_k is a primitive k^{th} root of 1. The k complex k^{th} roots of 1 are of the form ζ_k^i for $0 \leq i < k$. To lighten the notation we let $\zeta = \zeta_k$ for the rest of this section.

Let $p(x) = \sum_{j=0}^{k-1} x^j$. A trivial calculation shows that

$$p(\zeta^i) = \begin{cases} k & \text{if } i \equiv 0 \mod k, \\ 0 & \text{if } i \not\equiv 0 \mod k. \end{cases}$$

Officially a k-colouring of the edges of a complete graph with vertex set V is a function H from $[V]^2$ to $\{0, \ldots, k-1\}$. It will often be convenient to consider H as a symmetric function from V^2 to $\{0, \ldots, k-1\}$, and to make the convention that $H(v, v) = 0$ for all $v \in V$.

Definition 4. *Let H be a k-colouring of the edges of K_n. A_H is the complex-valued function with domain $V(K_n)^2$ given by the formula $A_H(u, v) = \zeta^{H(u,v)}$.*

We note that by our conventions $A_H(u, u) = 1$ for all u.

Definition 5. *Let $n > 0$ and let H be a k-colouring of the edges of K_n.*

1. *$F^*(G, H)$ is the set of $f : V(G) \to V(K_n)$ such that for some i with $0 \le i < k$, $A_H(f(u), f(v)) = \zeta^i$ for every edge $uv \in E(G)$.*
2. *$I(G, H)$ is the set of $f \in F^*(G, H)$ which are injective.*

It follows immediately from the definitions that $I(G, H)$ is the set of labelled copies of G in K_n which are monochromatic for H. By definition, $c_{mon}(G, H) = |I(G, H)|$ and $I(G, H) \subseteq F^*(G, H)$.

We now argue that $|F^*(G, H)|$ is a reasonably good approximation to $|I(G, H)|$ when n is large. The elements of $F^*(G, H) \setminus I(G, H)$ are non-injective functions from $V(G)$ to $V(K_n)$, and the number of such functions is $n^t - t!\binom{n}{t}$, so $|F^*(G, H)| \setminus |I(G, H)|$ is $O(n^{t-1})$. As we mentioned in the the introduction, it follows from Ramsey's theorem that $|I(G, H)|$ is $\Theta(n^t)$.

For the rest of this section we fix the value of n, and we also fix a k-colouring H of the edges of K_n. We will find an exact formula for $|F^*(G, H)|$, and in the following section we will exploit this formula and the fact that $c_{mon}(G, H)$ is close to $|F^*(G, H)|$.

Definition 6. *Let $f : V(G) \to V(K_n)$ and $0 \le i < k$.*

1. *$P(i, f) = \prod_{uv \in E(G)} p(\zeta^{-i} A_H(f(u), f(v)))$.*
2. *$Q(f) = \sum_{i=0}^{k-1} P(i, f)$.*

Lemma 1. $|F^*(G, H)| = k^{-e} \sum_{f : V(G) \to V(K_n)} Q(f)$.

Proof. Fix $f : V(G) \to V(K_n)$. By the discussion of the properties of ζ and $p(x)$ at the start of this section, if $A_H(f(u), f(v)) = \zeta^i$ for all $uv \in E(G)$ then $P(i, f) = k^e$; otherwise $P(i, f) = 0$. It follows that if $f \in F^*(G, H)$ then $P(i, f) = k^e$ for one value of i and $P(i, f) = 0$ for all other values, while if $f \notin F^*(G, H)$ then $P(i, f) = 0$ for all i. Therefore $Q(f) = k^e$ if $f \in F^*(G, H)$, and $Q(f) = 0$ if $f \notin F^*(G, H)$. Summing over all $f : V(G) \to V(K_n)$, we conclude that $\displaystyle\sum_{f:V(G)\to V(K_n)} Q(f) = k^e |F^*(G, H)|.$ $\qquad\square$

Expanding the product $P(i, f)$, we find that

$$
\begin{aligned}
P(i, f) &= \sum_{h:E(G)\to\{0,1,\dots,k-1\}} \prod_{uv\in E(G)} \zeta^{-ih(uv)} A_H(f(u), f(v))^{h(uv)} \\
&= \sum_{h:E(G)\to\{0,1,\dots,k-1\}} \zeta^{-i\sum_{uv\in E(G)} h(uv)} \prod_{uv\in E(G)} A_H(f(u), f(v))^{h(uv)}.
\end{aligned}
$$

Summing over i and recalling the definition of $p(x)$,

$$
Q(f) = \sum_{h:E(G)\to\{0,1,\dots,k-1\}} p(\zeta^{-\sum_{uv\in E(G)} h(uv)}) \prod_{uv\in E(G)} A_H(f(u), f(v))^{h(uv)}
$$

The term $p(\zeta^{-\sum_{uv\in E(G)} h(uv)})$ has the value k when $\displaystyle\sum_{uv\in E(G)} h(uv) \equiv 0$ mod k, and the value zero when $\displaystyle\sum_{uv\in E(G)} h(uv) \not\equiv 0 \mod k$. This motivates the following definition.

Definition 7. *A k-colouring h of the edges of G is* balanced *if and only if* $\displaystyle\sum_{uv\in E(G)} h(uv) \equiv 0 \mod k$. *$B_k(G)$ is the set of balanced k-colourings of the edges of G.*

It follows that

$$
Q(f) = k \sum_{h\in B_k(G)} \prod_{uv\in E(G)} A_H(f(u), f(v))^{h(uv)},
$$

and so by Lemma 1 that

$$
|F^*(G, H)| = k^{1-e} \sum_{f:V(G)\to V(K_n), h\in B_k(G)} \prod_{uv\in E(G)} A_H(f(u), f(v))^{h(uv)}.
$$

Now we write the results of this calculation of $|F^*(G, H)|$ in a more palatable form.

Definition 8. *Let h be a k-colouring of the edges of G.*

$$\rho(h, H) = n^{-t} \sum_{f:V(G) \to V(K_n)} \prod_{uv \in E(G)} A_H(f(u), f(v))^{h(uv)}.$$

We note that the quantity $\rho(h, H)$ is the average value of the product $\prod_{uv \in E(G)} A_H(f(u), f(v))^{h(uv)}$ taken over all $f : V(G) \to V(K_n)$.

Definition 9. $\Psi(G, H) = \sum\limits_{h \in B_k(G)} \rho(h, H).$

We now have a suggestive formula for $|F^*(G, H)|$.

Lemma 2. $|F^*(G, H)| = k^{1-e} n^t \Psi(G, H).$

We will see in the next section that G is k-uncommon if and only if there exist n and a k-colouring H of the edges of K_n with $\Psi(G, H) < 1$.

Remark 1. *The quantities $\rho(h, H)$ are complex numbers which may have nonzero imaginary parts or irrational real parts. They are algebraic integers lying in the number field $\mathbb{Q}(\zeta)$. When we form the sum $\Psi(G, H)$ the result will always be a rational number.*

Remark 2. *When $k = 2$ these formulae reduce to those of [5, section 4]. In particular the set of balanced 2-colourings corresponds to the set $ESSUB(G)$ of even spanning subgraphs appearing in the formulae of [5].*

Remark 3. *We could use similar ideas to count copies of coloured graphs in H.*

3. A criterion for a graph to be *k*-common

In this section we will define a product construction for k-colourings of the edges of complete graphs, and use this construction to relate the quantity $\Psi(G, H)$ defined in the preceding section to the question whether G is k-common. The calculations here are along the same lines as those of [8, section 4].

Before defining the product construction, we recall our convention that k-colouring of the edges of a complete graph with vertex set V can be regarded as a symmetric function H from V^2 to $\{0, \ldots, k-1\}$ with $H(v, v) = 0$ for all $v \in V$.

Definition 10. *Let G_1 and G_2 be complete graphs. Let $V_i = V(G_i)$ and H_i be a k-colouring of the edges of V_i for $i = 1, 2$. $H_1 \otimes H_2$ is a k-colouring*

of the edges of the complete graph with vertex set $V_1 \times V_2$, whose values are given by

$$H_1 \otimes H_2((u_1, u_2), (v_1, v_2)) \equiv H_1(u_1, v_1) + H_2(u_2, v_2) \mod k$$

for all $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$.

We observe that

$$A_{H_1 \otimes H_2}((u_1, u_2), (v_1, v_2)) = A_{H_1}(u_1, v_1) A_{H_2}(u_2, v_2).$$

Lemma 3. *Let G be a graph and h be a mapping from $E(G)$ to $\{0, 1, \ldots, k-1\}$. Let $\overline{K}_m$ be the colouring of the edges of K_m such that all of the edges have colour 0. Let H_1 be a k-colouring of the edges of K_n and H_2 be a k-colouring of the edges of K_m.*

1. *$\rho(h, H_1 \otimes H_2) = \rho(h, H_1)\rho(h, H_2)$.*
2. *If $h(uv) = 0$ for all $uv \in E(G)$, then $\rho(h, H_1) = 1$.*
3. *$\rho(h, \overline{K}_m) = 1$.*
4. *$\Psi(G, H_1 \otimes \overline{K}_m) = \Psi(G, H_1)$.*

The proof of Lemma 3 is a routine calculation. We note that if H is a k-colouring of the edges of K_n, then $H \otimes \overline{K}_m$ can be described in a very simple way; each vertex v of K_n is replaced by a block of vertices S_v with size m, a complete graph is formed whose vertex set is the union of these blocks, and every edge which joins a member of S_v to a member of S_w is given colour $H(v, w)$.

Lemma 4. *Let G be a graph with $e(G) = e$ and let H be a k-colouring of the edges of K_n. Then $C_k(G) \leq k^{1-e}\Psi(G, H)$.*

Proof. Consider the sequence of k-colourings $H_m = H \otimes \overline{K}_m$, where by definition H_m is a k-colouring of the edges of K_{nm}. Since

$$c_{mon}(G, H_m) = |I(G, H_m)| \leq |F^*(G, H_m)| = k^{1-e}(nm)^t \Psi(G, H_m)$$

by Lemma 2, and $\Psi(G, H) = \Psi(G, H_m)$ by Lemma 3, we see that

$$\frac{c_{mon}(G, H_m)}{t!\binom{nm}{t}} \leq \frac{k^{1-e}(nm)^t \Psi(G, H)}{t!\binom{nm}{t}}.$$

As m tends to ∞, $\frac{(nm)^t}{t!\binom{nm}{t}}$ tends to 1. So by taking the limit in the above inequality, $C_k(G) \leq k^{1-e}\Psi(G, H)$. $\square$

Lemma 4 implies that if $\Psi(G, H) < 1$ for some H then G is not k-common, which is all we will need for the proof of Theorem 1. It is an interesting fact that the converse implication also holds.

Lemma 5. *Let G be a graph and let $k > 1$. The following are equivalent.*

1. *G is k-uncommon.*
2. *There exist n and a k-colouring H of the edges of K_n such that $\Psi(G, H) < 1$.*

Proof. Let G be a graph with $v(G) = t$ and $e(G) = e$. If there is H such that $\Psi(G, H) < 1$, then by Lemma 4 $C_k(G) < k^{1-e}$ and G is k-uncommon.

Conversely suppose that G is k-uncommon. By definition we have $C_k(G) < k^{1-e}$, and we let $\epsilon = k^{1-e} - C_k(G)$. Then $\epsilon > 0$, and since $C_k(G; n)$ increases with n and has limit $C_k(G)$ we have $C_k(G; n) \leq k^{1-e} - \epsilon$ for all n.

We recall from Section 2 that for any n and any k-colouring H of the edges of K_n, $I(G, H) \subseteq F^*(G, H)$ and $|F^*(G, H) \setminus I(G, H)| \leq n^t - t!\binom{n}{t}$. The polynomial $n^t - t!\binom{n}{t}$ has degree $t - 1$, so we may choose n so large that $n^t - t!\binom{n}{t} < \epsilon n^t$.

By definition $C_k(G; n)$ is the minimum of $\frac{|I(G,H)|}{t!\binom{n}{t}}$ over all k-colourings H of the edges of K_n, so we may choose such an H with $C_k(G; n) = \frac{|I(G,H)|}{t!\binom{n}{t}}$. Then

$$|I(G, H)| \leq (k^{1-e} - \epsilon)t!\binom{n}{t} \leq (k^{1-e} - \epsilon)n^t,$$

and

$$|F^*(G, H)| - |I(G, H)| \leq n^t - t!\binom{n}{t} < \epsilon n^t,$$

so that

$$|F^*(G, H)| < k^{1-e}n^t.$$

It follows from Lemma 2 that $\Psi(G, H) < 1$. $\qquad\square$

Remark 4. *Lemma 5 implies that the property of being k-common is computably enumerable. That is to say there is an algorithm which takes G as its input, and eventually halts if and only if G is not k-common. The algorithm is easy to describe: enumerate all possible colourings, compute $\Psi(G, H)$ for each such colouring H, and halt if we find an H such that $\Psi(G, H) < 1$. It is not clear that the property of being k-common is computable, that is that there is an algorithm which halts on every G and tells us whether G is k-common or not.*

4. Graphs containing K_3

In this section we prove our main result, that every graph which contains a triangle is 3-uncommon. Our approach is similar to that used in [5], and involves quadratic forms over finite fields. We use the field $GF(3)$ where $GF(2)$ is used in [5]. Compared with $GF(2)$, the field $GF(3)$ has the advantage that every quadratic form is diagonalizable, but the disadvantage that a diagonal matrix can have both $+1$ and -1 as entries on the diagonal.

We will consider the field $GF(3)$ as the set $\{0, 1, 2\}$ with the operations of addition and multiplication modulo 3. As in Section 2 we will be doing computations with a complex root of unity, in this case the primitive third root $\zeta_3 = \exp(\frac{2\pi i}{3}) = \frac{-1+i\sqrt{3}}{2}$. We note that if $a, b \in GF(3)$ then $\zeta_3^a \zeta_3^b = \zeta_3^{a+b}$ and $(\zeta_3^a)^b = \zeta_3^{ab}$, where the sum $a + b$ and the product ab are computed in $GF(3)$.

Let s be a positive integer and let $m = 4s + 2$. We will construct a 3-colouring of the edges of K_n where $n = 3^m$, starting from a certain quadratic form q with coefficients in $GF(3)$.

$$q = x_1^2 + x_2^2 + \ldots + x_{2s+2}^2 - x_{2s+3}^2 - x_{2s+4}^2 - \ldots - x_{4s+2}^2.$$

Now let J_s be the following 3-colouring of the edges of a complete graph with n vertices: the vertex set is $GF(3)^m$, the set of colours is $GF(3)$, and the colour of edge xy is $q(x - y)$. It is clear that $q(x - y) = q(y - x)$, so that the colouring J_s is well-defined.

Let G be a graph with $V(G) = \{1, 2 \ldots t\}$ and let h be a 3-colouring of the edges of G, where we consider the colouring h as taking values in $GF(3)$. Define $M(h)$ as the $t \times t$ matrix with entries from $GF(3)$ given by

$$M(h)_{ab} = \begin{cases} h(ab) & \text{if } ab \in E(G), \\ 0 & \text{if } ab \notin E(G) \text{ and } a \neq b, \\ -\sum_{\substack{i=1 \\ i \neq a}}^{t} h(ai) & \text{if } a = b. \end{cases}$$

Clearly $M(h)$ is symmetric, and each row and column has sum equal to zero. We denote the rank of $M(h)$ by $r(h)$. We note that since the sum of its rows is zero the matrix $M(h)$ is singular, and hence $r(h) < t$.

Lemma 6. *Let G be a graph with $V(G) = \{1, \ldots t\}$, let h be a 3-colouring of the edges of G, and let $r = r(h)$. Then*

$$\rho(h, J_s) = (-1)^r 3^{-r(2s+1)}.$$

Proof. Let $M = M(h)$. $\rho(h, J_s)$ is the average over all functions f from $V(G)$ to $GF(3)^m$ of the quantity

$$\prod_{\substack{a<b \\ ab \in E(G)}} [\zeta_3^{q(f(a)-f(b))}]^{M_{ab}} = \zeta_3^{Q^*(f)},$$

where $Q^*(f) = \sum\limits_{\substack{a<b \\ ab \in E(G)}} M_{ab} q(f(a) - f(b))$. For each $a \in V(G)$ we regard $f(a)$ as an m-tuple of variables, $(x_{1,a}, x_{2,a}, \ldots, x_{m,a})$. Now we regard $Q^*(f)$ as a quadratic form in the mt variables $x_{i,a}$ where $1 \le i \le m$ and $a \in V(G)$. In this new formulation we need to average the expression $\zeta_3^{Q^*(f)}$ over all assignments of values in $GF(3)$ to the variables $x_{i,a}$, and we will do this by diagonalising $Q^*(f)$.

Let A be the $m \times m$ matrix with entries

$$A_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } 1 \le i \le 2s+2, \\ -1 & \text{if } i = j \text{ and } 2s+3 \le i \le m, \\ 0 & \text{if } i \ne j. \end{cases}$$

This is the matrix of the quadratic form q, and by a routine calculation

$$Q^*(f) = \sum_{a<b} M_{ab} q(f(a) - f(b)) = \sum_{a<b,i} M_{ab} A_{ii}(x_{i,a}^2 + x_{i,b}^2 - 2x_{i,a}x_{i,b}).$$

Let $1 \le p \le t$ and $1 \le i \le m$. The coefficient of $x_{i,p}^2$ in $Q^*(f)$ is $\sum\limits_{a \ne p} A_{ii} M_{ap}$. By the definition of M, $\sum\limits_{a \ne p} M_{ap} = -M_{pp}$, and so

$$Q^*(f) = -\sum_{a,i} M_{aa} A_{ii} x_{i,a}^2 - 2\sum_{a<b,i} M_{ab} A_{ii} x_{i,a} x_{i,b}.$$

The matrix $M \otimes A$ is the $mt \times mt$ matrix constructed by assigning $M_{ab}A$ as the (a, b) entry of a $t \times t$ matrix with $m \times m$ matrices as entries. The (i, j) entry of the (a, b) entry is $M_{ab}A_{ij}$.

Let $P = M \otimes A$ and

$$\overline{x} = (x_{1,1}, x_{2,1}, \ldots, x_{n,1}, \ldots, x_{1,t}, x_{2,t}, \ldots, x_{m,t}).$$

An easy calculation shows that $Q^*(f) = -\overline{x} P \overline{x}^\top$.

Since the field $GF(3)$ has characteristic 3, by a standard result about quadratic forms there exists a invertible $t \times t$ matrix C such that $CMC^\top$

is diagonal. Let $D = CMC^\top$, then $rank(D) = rank(M) = r$ by a standard argument from linear algebra. Since D is diagonal, r is equal to the number of nonzero entries in D. Let r_+ be the number of 1's in D, from which it follows that $r - r_+$ is the number of -1's. We may assume that

$$D_{aa} = \begin{cases} 1 & \text{if } 1 \le a \le r_+, \\ -1 & \text{if } r_+ + 1 \le a \le r, \\ 0 & \text{if } r + 1 \le a \le m. \end{cases}$$

Let $\overline{C} = C \otimes I$, where I is the $m \times m$ identity matrix. By a routine calculation $\overline{C} P \overline{C}^\top = CMC^\top \otimes IAI^\top = D \otimes A$. The matrix $D \otimes A$ is diagonal, so we have diagonalised the form $Q^*(f)$; if we define a new vector of variables $\overline{y}$ such that $\overline{x} = \overline{y}\,\overline{C}$, then $Q^*(f) = -\overline{x} P \overline{x}^\top = -\overline{y}(D \otimes A)\overline{y}^\top$.

We now return to the problem of averaging $\zeta_3^{Q^*(f)}$ over all functions from $V(G)$ to $GF(3)^m$, or equivalently averaging $\zeta_3^{Q^*(f)}$ over all $\overline{x} \in GF(3)^{mt}$.

$$\sum_{f:V(G)\to GF(3)^m} \zeta_3^{Q^*(f)} = \sum_{\overline{x}\in GF(3)^{mt}} \zeta_3^{-\overline{x}P\overline{x}^\top} = \sum_{\overline{y}\in GF(3)^{mt}} \zeta_3^{-\overline{y}(D\otimes A)\overline{y}^\top},$$

and since $D \otimes A$ is diagonal

$$\sum_{\overline{y}\in GF(3)^{mt}} \zeta_3^{-\overline{y}(D\otimes A)\overline{y}^\top} = \sum_{\overline{y}\in GF(3)^{mt}} \zeta_3^{-\sum_{a,i} D_{aa}A_{ii}y_{i,a}^2}.$$

Clearly

$$D_{aa}A_{ii} = \begin{cases} 1 & \text{if } 1 \le i \le 2s+2 \text{ and } 1 \le a \le r_+ \text{ or } 2s+3 \le i \le m \text{ and } r_++1 \le a \le r, \\ -1 & \text{if } 1 \le i \le 2s+2 \text{ and } r_++1 \le a \le r \text{ or } 2s+3 \le i \le m \text{ and } 1 \le a \le r_+, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that if we let $\alpha = 2(rs + r - r_+)$ and $\beta = 2(rs + r_+)$, then the expression $-\sum_{a,i} D_{aa}A_{ii}y_{i,a}^2$ contains α terms with coefficient 1, β terms with coefficient -1 and $mt - (\alpha + \beta)$ terms with coefficient zero.

We reorder and relabel the mt variables $y_{i,a}$ as $z_1, \ldots, z_{mt}$ so that

$$-\sum_{a,i} D_{aa}A_{ii}y_{i,a}^2 = z_1^2 + z_2^2 + \cdots + z_\alpha^2 - z_{\alpha+1}^2 - z_{\alpha+2}^2 - \cdots - z_{\alpha+\beta}^2.$$

Noting that the right hand side depends only on the variables $z_1, \ldots z_{\alpha+\beta}$,

we get

$$\sum_{f:V(G)\to GF(3)^m} \zeta_3^{Q^*(f)} = 3^{mt-(\alpha+\beta)} \sum_{z_1,\ldots,z_{\alpha+\beta}\in GF(3)} \zeta_3^{z_1^2+z_2^2+\cdots+z_\alpha^2-z_{\alpha+1}^2-z_{\alpha+2}^2-\cdots-z_{\alpha+\beta}^2}.$$

Since $\zeta_3^{-1} = \zeta_3^2$, we can rewrite the right hand side as

$$3^{mt-(\alpha+\beta)} \sum_{z_1,\ldots,z_{\alpha+\beta}\in GF(3)} \zeta_3^{z_1^2}\zeta_3^{z_2^2}\cdots\zeta_3^{z_\alpha^2}\zeta_3^{2z_{\alpha+1}^2}\zeta_3^{2z_{\alpha+2}^2}\cdots\zeta_3^{2z_{\alpha+\beta}^2}.$$

This sum is the product of α terms of the form $1 + \zeta_3 + \zeta_3^4$ and β terms of the form $1 + \zeta_3^2 + \zeta_3^8$, so using $\zeta_3^3 = 1$ we get

$$\sum_{f:V(G)\to GF(3)^m} \zeta_3^{Q^*(f)} = 3^{mt-(\alpha+\beta)}(1 + 2\zeta_3)^\alpha(1 + 2\zeta_3^2)^\beta.$$

Since $1+2\zeta_3 = \imath\sqrt{3}$, $1+2\zeta_3^2 = -\imath\sqrt{3}$, and $\rho(h, J_s)$ is the average of $\zeta_3^{Q^*(f)}$ over the 3^{mt} possible values for f, we see that $\rho(h, J_s) = \frac{(-1)^\beta(\imath\sqrt{3})^{\alpha+\beta}}{3^{\alpha+\beta}}$. Since β is even, $(-1)^\beta = 1$. Because $\alpha + \beta = 2r(2s+1)$,

$$\rho(h, J_s) = (\imath)^{2r(2s+1)}\left(\frac{\sqrt{3}}{3}\right)^{2r(2s+1)} = (-1)^r 3^{-r(2s+1)}.$$

$\square$

We recall from Definition 7 that $B_3(G)$ is the set of balanced 3-colourings of the edges of G. Since we are now considering 3-colourings as taking values in $GF(3)$, $B_3(G)$ is the set of colourings where the sum of the colours of the edges is equal to zero.

Definition 11. *Let G be a graph with $V(G) = \{1,\ldots t\}$. For each i such that $0 \le i \le t-1$, $R_i(G)$ is the number of $h \in B_3(G)$ such that $r(h) = i$.*

Lemma 7. *Let G be a graph with $V(G) = \{1,\ldots t\}$. If $V(G)$ contains a triangle, then $R_0(G) = 1$, and $R_1(G) > 0$.*

Proof. Let h be the colouring in which every edge gets colour 0. Then $M(h)$ is the $t \times t$ zero matrix, which implies $r(h) = 0$. Since the zero matrix is the only $t \times t$ matrix with rank 0, $R_0(G) = 1$.

Relabelling the vertices if necessary, we may assume that vertices 1, 2 and 3 form a triangle in G. Let h be the colouring in which the edges 12, 13,

and 23 are assigned colour 1 and the remaining edges are assigned colour 0. Then h is balanced, and $M(h)$ is the $t \times t$ matrix

$$\begin{bmatrix} 1_{3\times3} & 0_{3\times t-3} \\ 0_{t-3\times3} & 0_{t-3\times t-3} \end{bmatrix}$$

where $1_{3\times3}$ is a 3×3 matrix with all entries equal to 1, and $0_{a\times b}$ is a $a \times b$ matrix with all entries equal to 0. Therefore $r(h) = 1$, which implies that $R_1(G) > 0$. □

It is not hard to see that $R_1(G) > 0$ if and only if G contains a triangle.

Theorem 1. *Let G be a graph which contains a triangle. Then G is 3-uncommon.*

Proof. Let G have t vertices, so that without loss of generality $V(G) = \{1, \ldots t\}$. Let $R_i = R_i(G)$ for $i < t$, and let $B = B_3(G)$. By Lemma 6

$$\Psi(G, J_s) = \sum_{h\in B} \rho(h, J_s) = \sum_{h\in B} (-1)^{r(h)} 3^{-r(h)(2s+1)} = \sum_{j=0}^{t-1} (-1)^j 3^{-j(2s+1)} R_j.$$

For large s the leading terms $1 - \frac{R_1}{3^{(2s+1)}}$ will dominate this sum. Since $R_1 > 0$ there exists an s such that $\Psi(G, J_s) < 1$. It follows from Lemma 5 that G is 3-uncommon. □

5. Conclusion

We have shown that there is a sequence of graph colourings J_s which witnesses that every G containing K_3 is 3-uncommon. Note that the colouring J_s that we constructed is $T_1^{\otimes(2s+2)} \otimes T_2^{\otimes(2s)}$, where T_i is the edge colouring of K_3 in which all the edges get colour i.

We finish with some open questions.

1. Can the arguments of this paper be pushed further to show that larger classes of non-bipartite graphs are 3-uncommon? In particular, is every non-bipartite graph 3-uncommon?
2. We have a very simple description of J_s in terms of the colourings T_i. It is not hard to see that $\rho(h, T_i)$ can be computed recursively by a contraction-deletion formula. Can these facts be used to give a simpler proof of Theorem 1?

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