Packing tight Hamilton cycles in uniform hypergraphs

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Abstract

We say that a k-uniform hypergraph C is a Hamilton cycle of type ℓ, for some $1 \leq \ell \leq k$, if there exists a cyclic ordering of the vertices of C such that every edge consists of k consecutive vertices and for every pair of consecutive edges E_{i-1}, E_i in C (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. We define a class of (ϵ, p)-regular hypergraphs, that includes random hypergraphs, for which we can prove the existence of a decomposition of almost all edges into type ℓ Hamilton cycles, where $\ell < k/2$.

1 Introduction

This paper follows a line of work initiated by Frieze and Krivelevich [3] and continued by Frieze, Krivelevich and Loh [5]. We are given a k-uniform hypergraph H (k-graph) with certain pseudo-random properties and we show that almost all of the edges of H can be packed into edge disjoint Hamilton cycles of a particular type.

The paper [5] begins with a good survey of this question which we will only give a sketch here. When $k = 2$ we are dealing with graphs. Frieze and Krivelevich [4] showed that the edge set of dense graphs with a certain pseudo-random structure typified by random graphs could be almost decomposed into edge disjoint Hamilton cycles. Knox, Kühn and Osthus [8] tightened the implied result when restricted to random graphs. Most recently, Knox, Kühn and Osthus [9] and Krivelevich and Samotij [11] have given best possible results for random graphs. I.e. it is possible to pack $\lfloor \delta/2 \rfloor$ Hamilton cycles, where δ denotes minimum degree. Random regular graphs are also known to have perfect packings of Hamilton cycles, see e.g. Kim and Wormald [10]. Christofides, Kühn and Osthus [2] proved strong results for decompositions of dense graphs.

The paper [3] discussed packing Hamilton cycles in pseudo-random hypergraphs. There are various definitions of a Hamilton cycle in a hypergraph. We will use the following: Let

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$H = (V = [n], E)$ be a k-graph i.e. $E = \{e_1, e_2, \ldots, e_m\}$ where e_j is a k-subset of V for $j = 1, 2, \ldots, m$. and let $\ell < k$ be given where $\ell \mid n$. A Hamilton cycle of *type* ℓ is a sequence $f_1, f_2, \ldots, f_{\nu_\ell}$, $\nu_\ell = n/\ell$ of edges where $|g_i = f_{i+1} \setminus f_i| = \ell$ for $i = 1, 2, \ldots, \nu_\ell$ ($f_{\nu_\ell+1} = f_1$) and $V = \bigcup_{i=1}^{\nu_\ell} g_i$. The paper [3] deals with the case $\ell \geq k/2$ and described conditions under which almost all of the edges of a hypergraph could be partitioned into Hamilton cycles. The case $\ell < k/2$ could not be handled by the methods in [3], but [5] shows how to deal with the case $k = 3, \ell = 1$. The purpose of this paper is to extend the analysis of [5] to the case where $k \geq 4$ and $\ell < k/2$.

We first give our notion of pseudo-randomness: It is somewhat technical and is what is needed for our proof. On the other hand, $H_{n,p;k}$ (the random k-uniform hypergraph on n vertices with edge probability p) will satisfy the definition for p not too small, see Section 2.0.1. We use the following notation throughout.

$$2 \leq z = \left\lceil \frac{k-\ell}{\ell} \right\rceil \text{ and } q = \ell z \qquad \text{satisfies } k/2 < k - \ell \leq q < k.$$

Definition 1. *We say that an n-vertex k-graph H, is (ϵ, p)-regular if the following holds. Let $d \in \{1, 2, \ldots, \ell\}$ and let $s \in \{1, 2, \ldots, 2z+2\}$. Given any s distinct $(k-d)$-sets, $A_1, \ldots, A_s$, such that $|\bigcup_i A_i| \leq k + 2q$, there are $(1 \pm \epsilon)\frac{n^d}{d!}p^s$ sets of d vertices, D, such that all of $A_1 \cup D, \ldots, A_s \cup D$ are edges of H.* [1]

This definition basically says that in an (ϵ, p)-regular k-graph, the "generalized co-degrees" should approximately equal their expected values in $H_{n,p;k}$. Taking $s = 1$ and $A_1 = \{x_1, \ldots, x_{k-1}\}$ (so that $d = 1$), the definition says that $x_1, \ldots x_{k-1}$ are contained together in $(1 \pm \epsilon)np$ edges. In $H_{n,p;k}$, the expected degree number of egdes containing a fixed $(k-1)$-set is np.

As a slightly more complicated example, suppose $k = 5$, $\ell = 2, d = 2$ and $s = 2$ with $A_1 = \{x_1, x_2, x_3\}, A_2 = \{x_3, x_4, x_5\}$. Then the definition says there are $(1 \pm \epsilon)\frac{n^2}{2}p^2$ pairs of vertices $\{y, z\}$ such that both $\{x_1, x_2, x_3, y, z\}$ and $\{x_3, x_4, x_5, y, z\}$ are present. In $H_{n,p;k}$, the expected number of such pairs would be $\binom{n}{2}p^2$ which is asymptotically $\frac{n^2}{2}p^2$.

The values $\ell, 2z+2$ and $k+2q$ which appear in the restrictions of d, s and $|\bigcup_i A_i|$ are not particularly important; they are merely the minimal values which we need for our proof. The fact that they are constants is all that really matters.

We now give our main theorem: [2]

Theorem 1. *Let k and $\ell < k/2$ be given. Let $\alpha = \frac{1}{9+7z^3}$. Suppose that n is a sufficiently large multiple of $2q$ and that ϵ, n and p satisfy*

$$\epsilon^{16z+12} np^{8z} \gg \log^{8z+5} n.$$

Let H be an (ϵ, p)-regular k-graph with n vertices. Then H contains a collection of edge disjoint Hamilton cycles of type ℓ that contains all but at most ϵ^α-fraction of its edges.

[1] $A = (1 \pm \epsilon)B$ if $(1-\epsilon)B \leq A \leq (1+\epsilon)B$

[2] The notation $a_n \gg b_n$ is short for $a_n/b_n \to \infty$ as $n \to \infty$.

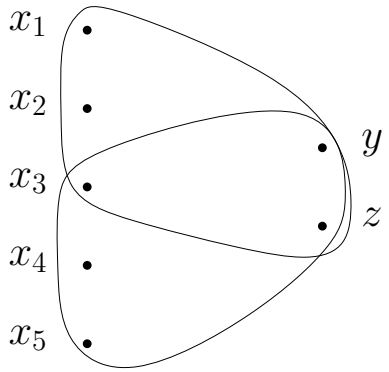


Figure 1: An example with $k = 5$, $\ell = 2, d = 2$ and $s = 2$ with $A_1 = \{x_1, x_2, x_3\}, A_2 = \{x_3, x_4, x_5\}$

Our bounds on parameters ϵ, p are unlikely to be tight and it would be interesting to sharpen our bounds. In which case, we will not fight too hard for our bounds. In particular, we will replace products $(1 \pm a\epsilon)(1 \pm b\epsilon)$ and $(1 \pm a\epsilon)(1 \pm b\epsilon)^{-1}$ by $(1 \pm (a + b + 1)\epsilon)$ without further comment. Furthermore, we are really only interested in the case where ϵ is small and so we will always assume that ϵ is sufficiently small for all such simplifications.

2 Proof overview and organization

The key insight in the proof of Theorem 1 is the following connection between type ℓ Hamilton cycles in H and Hamilton cycles in an associated digraph.

Definition 2. *Given two ordered q-tuples of vertices $\mathbf{v}_1 = (v_1, \ldots, v_q), \mathbf{v}_2 = (v_{q+1}, \ldots, v_{2q})$ of a k-uniform hypergraph H, we define*

$$\mathbf{e}_i = \mathbf{e}_i(\mathbf{v}_1, \mathbf{v}_2) = \{v_{i\ell+1}, v_{i\ell+2}, \ldots, v_{i\ell+k}\} \qquad \text{for all } i = 0, \ldots, z-1. \tag{1}$$

We say that $\mathbf{v}_1$ precedes $\mathbf{v}_2$ if the edges $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{z-1}$ are all present in H. We say that $(\mathbf{v}_1, \mathbf{v}_2)$ owns *these edges.*

Notice that the edges $\mathbf{e}_0, \ldots, \mathbf{e}_{z-1}$ are all contained in $\{v_1, v_2, \ldots, v_{2q}\}$. $\mathbf{e}_0$ consists of the first k vertices of $\mathbf{v}_1\mathbf{v}_2$. We shift ℓ places to the right to get $\mathbf{e}_1$. We continue shifting by ℓ places until a further shift would take us outside $\mathbf{v}_1\mathbf{v}_2$.

For a permutation $\sigma = (v_1 = \sigma(1), \ldots, v_i = \sigma(i), \ldots, v_n = \sigma(n))$ of the vertices of H, define a $\nu_q = n/q$-vertex digraph D_σ with vertex set $V_\sigma = \{\mathbf{v}_i = (v_{(i-1)q+1}, \ldots, v_{iq}) : \ i = 1, 2, \ldots, \nu_q\}$. Place an arc (directed edge) from $\mathbf{v}_i$ to $\mathbf{v}_j$ if and only if $\mathbf{v}_i$ precedes $\mathbf{v}_j$. In this construction, Hamilton cycles in D_σ give rise to type ℓ Hamilton cycles in H. Indeed the Hamilton cycle $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n/q})$ of D_σ where $\mathbf{w}_i = (w_{(i-1)q+1}, \ldots, w_{iq})$ yields a Hamilton cycle in H made up from the edges owned by the arcs $(\mathbf{w}_i, \mathbf{w}_{i+1})$, $i = 1, \ldots, \nu_q$. This

cycle is $(e_1, e_2, \ldots, e_{\nu_\ell})$ where $e_{az+b} = \{w_{((a-1)z+b)\ell+1}, \ldots, w_{((a-1)z+b)\ell+k}\}$ for $a \in [\nu_q]$ and $b \in \{0, \ldots, z-1\}$.

We want disjoint Hamilton cycles in D_σ to yield disjoint cycles in H. This follows from the fact that the sets of edges owned by distinct arcs $(\mathbf{v}_a, \mathbf{v}_b)$ and $(\mathbf{v}_c, \mathbf{v}_d)$ are disjoint. Suppose then that some edge e of H is owned by both pairs. It follows from the definition of precedes that the first element of e (in the order defined by σ) is in $\mathbf{v}_a$ and $\mathbf{v}_c$ and so $a = c$. The $q+1$st element of e is in $\mathbf{v}_b$ and $\mathbf{v}_d$ and so $b = d$, contradiction.

The basic idea of the proof is to take a large number of random permutations $\sigma_1, \sigma_2, \ldots, \sigma_r$ and construct the digraphs $D_{\sigma_1}, D_{\sigma_2}, \ldots, D_{\sigma_r}$. Then take subgraphs $D'_{\sigma_i} \subseteq D_{\sigma_i}$ for $i = 1, 2, \ldots, r$ so that the edges of H owned by $D'_{\sigma_i}, D'_{\sigma_j}$ are disjoint for $i \neq j$. It will be argued that each D'_{σ_i} has certain regularity properties implying that its arc set can be almost decomposed into edge disjoint Hamilton cycles. We then take the edges owned by the arcs of all the Hamilton cycles in all the D'_{σ_i} and remove them to create a new hypergraph H'. We then argue that **whp** H' is (ϵ', p')-regular. We repeat this process until we have covered almost all of the edges of H by Hamilton cycles.

We now give the regularity properties that we require of our digraphs D_σ:

Definition 3. *We say that a ν-vertex digraph is (ϵ, p)-regular if it satisfies the following properties:*

- **(i)** *Every vertex a has out-degree $d^+(a) = (1 \pm \epsilon)\nu p$ and in-degree $d^-(a) = (1 \pm \epsilon)\nu p$.*
- **(ii)** *For every pair of distinct vertices a, b, all three of the following quantities are $(1 \pm \epsilon)\nu p^2$: the number of common out neighbors $d^+(a, b)$, the number of common in neighbors $d^-(a, b)$, and the number $d^{+-}(a, b)$ of out-neighbors of a which are also in-neighbors of b.*
- **(iii)** *Given any four vertices a, b, c, d, which are all distinct except for the possibility $b = c$, there are $(1 \pm \epsilon)\nu p^4$ vertices x such that $\overrightarrow{ax}, \overrightarrow{xb}, \overrightarrow{cx}, \overrightarrow{xd}$ are all directed edges.*

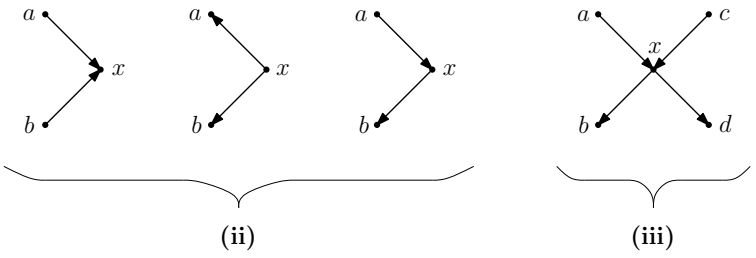


Figure 2: Properties (ii) and (iii) of digraph (ϵ, p)-regularity. The number of such vertices, x, should be approximately what it is in the random digraph $D_{\nu,p}$.

In this context, we have the following Theorem of Frieze, Krivelevich and Loh [5]:

Theorem 2. *Suppose that $\epsilon^{11}np^8 \gg \log^5 n$, and n is a sufficiently large even integer. Then every (ϵ, p)-regular digraph can have its edges partitioned into a disjoint union of directed Hamilton cycles, except for a set of at most $\epsilon^{1/8}$-fraction of its edges.*

We next describe our procedure for generating the D'_{σ_i}:

Procedure 1. This takes as input an (ϵ, p)-regular k-graph H with number of vertices divisible by $2q$. Let

$$\kappa = \frac{6(k+1)\log n}{\epsilon^2} \text{ and } r = \frac{\ell q n^{k-2}}{k! p^{z-1}} \cdot \kappa. \tag{2}$$

(1) Generate permutations $\sigma_1, \sigma_2, \ldots, \sigma_r$ of $[n]$ independently from the uniform distribution on S_n

(2) Let H_i be the k-graph made up of the edges of H that are owned by the arcs of D_{σ_i}.

(3) For each edge $e \in H$, let $I_e = \{i : e \in H_i\}$. If $I_e \neq \emptyset$, independently select a uniformly random index in I_e to label e with.

(4) For each i, define the subgraph D'_{σ_i} as follows: For each arc $e = (\mathbf{v}, \mathbf{v}')$ of D_{σ_i}, keep the arc e if and only if all z of the edges owned by e are labeled with i.

(5) For each i, let H'_i be the k-graph containing all hyperedges which are owned by the arcs of D'_{σ_i}.

Our main task is to prove

Lemma 1. *Suppose that n, p, and ϵ satisfy*

$$\epsilon^{8z+2}np^{8z} \gg \log^{4z+1} n.$$

Let H be an (ϵ, p)-regular k-graph on n vertices (n divisible by $2q$). Suppose that we carry out Procedure 1. Then, with probability $1 - o(n^{-1})$:

(a) *Every D'_{σ_i} is $(12z^2\epsilon, (p/\kappa)^z)$-regular.*

(b) *H' is an (ϵ', p')-regular k-graph where $H' = H \setminus \bigcup_{i=1}^{r} H'_i$ is the subgraph of H obtained by deleting the edges of the H'_is. Here*

$$\epsilon' = \epsilon\left(1 + \frac{7z^3}{\kappa^{z-1}}\right) \text{ and } p' = p\left(1 - \frac{1}{\kappa^{z-1}}\right)$$

Part (a) enables us to find many edge disjoint Hamilton cycles and it is proved in Section 3. Part (b) enables us to repeat the construction many times and is proved in Section 4. Section 5 shows how to use the above lemma to prove the main theorem.

2.0.1 Random k-graphs

We now check that random k-graphs are (ϵ, p)-regular for suitable ϵ, p.

$$\begin{aligned}\mathbb{P}\left[H_{n,p;k} \text{ is not } (\epsilon,p)\text{-regular}\right] &= O(n^{k+2q})\sum_{d=1}^{\ell}\sum_{s=1}^{2z+2}\mathbb{P}\left[\mathrm{Bin}\left[\binom{n}{d}, p^s\right] \neq (1\pm\epsilon)\frac{n^d}{d!}p^s\right] \\ &= o(1)\end{aligned}$$

as long as $\epsilon^2 np^{2z+2} \gg \log n$. (The hidden constant in $O(n^{k+2q})$ allows us to use $\binom{n}{d}$ in place of $\binom{n-O(1)}{d}$).

So, from Theorem 1, if $p \gg \left(\frac{\log^{8z+5} n}{\epsilon^{16z+12} n}\right)^{1/(8z)}$ then **whp** $H_{n,p,k}$ has a packing of edge disjoint type ℓ Hamilton cycles that covers all but $\epsilon^{1/(9+7z^3)}$ proportion of edges.

2.1 Concentration bounds

Fact 1. *For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that any binomial random variable X with mean μ satisfies*

$$\mathbb{P}\left[|X-\mu| > \epsilon\mu\right] < e^{c_\epsilon \mu},$$

where c_ϵ is a constant determined by ϵ. When $\epsilon < 1$, we may take $c_\epsilon = \frac{\epsilon^2}{3}$.

For a proof see Alon and Spencer [1] or Janson, Łuczak and Ruciński [7].

Fact 2. *Let X be a random variable on the uniformly distributed space of permutations on n elements, and let C be a real number. Suppose that whenever $\sigma, \sigma' \in S_n$ differ by a single transposition, $|X(\sigma) - X(\sigma')| \leq C$. Then,*

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq t\right] \leq 2\exp\left\{-\frac{2t^2}{C^2 n}\right\}.$$

For a proof see Frieze and Pittel [6] or McDiarmid [12].

2.2 Properties of (ϵ, p)-regular k-graphs

In Lemma 4, we will confirm that each digrpah, D_σ is $(\hat{\epsilon}, \hat{p})$-regular for some $\hat{\epsilon}$ and $\hat{p}$. In order to do this we will need to confirm the properties of digraph $(\hat{\epsilon}, \hat{p})$-regularity one by one. This means showing that the correct amount of certain types of directed edges exist. Directed edges in D_σ arise from the existence of certain hyperedges in H. The following lemma is a list of properties which we will use to this end.

Lemma 2. *Every n-vertex (ϵ, p)-regular k-graph H has the following properties:*

(L1) *Given any sequence of q distinct vertices, $x_1, \ldots, x_q$, there are $(1\pm\epsilon)n^{k-q}p$ sequences of vertices, $y_1, \ldots, y_{k-q}$, such that $\{x_1, \ldots, x_q, y_1, \ldots, y_{k-q}\}$ is an edge of H.*

In terms of Definition 1 we have $d = k-q$, $s = 1$, $A_1 = \{x_1, x_2, \ldots, x_q\}$. We multiply by $(k-q)!$ because we apply these properties to ordered sequences of vertices.

(L2) *Given any sequence of $k-\ell$ distinct vertices, $x_1, \ldots, x_{k-\ell}$, there are $(1\pm\epsilon)n^\ell p$ sequences of vertices, $y_1, \ldots, y_\ell$, such that $\{x_1, \ldots, x_{k-\ell}, y_1, \ldots, y_\ell\}$ is an edge of H.*

In terms of Definition 1 we have $d = \ell$, $s = 1$, $A_1 = \{x_1, \ldots, x_{k-\ell}\}$.

(L3) *Given any sequence of $2q$ distinct vertices $x_1, \ldots, x_q, y_1, \ldots, y_q$, there are $(1 \pm \epsilon)n^{k-q}p^2$ sequences of vertices $z_1, \ldots, z_{k-q}$ vertices such that $\{x_1, \ldots, x_q, z_1, \ldots, z_{k-q}\}$ and $\{y_1, \ldots, y_q, z_1, \ldots, z_{k-q}\}$ are both edges of H.*

In terms of Definition 1 we have $d = k-q$, $s = 2$, $A_1 = \{x_1, \ldots, x_q\}$, $A_2 = \{y_1 \ldots, y_q\}$.

(L4) *Given any sequence of $2(k-\ell)$ vertices $x_1, \ldots, x_{k-\ell}, y_1, \ldots, y_{k-\ell}$ (where we demand only that $x_1 \neq y_1$), there are $(1 \pm \epsilon)n^\ell p^2$ sequences of vertices $z_1, \ldots, z_\ell$ vertices such that $\{x_1, \ldots, x_{k-\ell}, z_1, \ldots, z_\ell\}$ and $\{y_1, \ldots, y_{k-\ell}, z_1, \ldots, z_\ell\}$ are both edges of H.*

In terms of Definition 1 we have $d = \ell$, $s = 2$, $A_1 = \{x_1, \ldots, x_{k-\ell}\}$, $A_2 = \{y_1 \ldots, y_{k-\ell}\}$. Note that if $\ell \mid k$, this is identical to property (L3) since in this case, $q = k - l$.

(L5) *Given any sequence of $\ell + (k-2\ell) + q$ vertices $x_1, \ldots, x_\ell, a_1, \ldots, a_{k-2\ell}, z_1, \ldots, z_q$, there are $(1 \pm \epsilon)n^\ell p^{z+1}$ sequences of vertices $b_1, \ldots, b_\ell$ such that all of the following edges are present in H:*

$$\{x_1, \ldots, x_\ell, a_1, \ldots, a_{k-2\ell}, b_1, \ldots, b_\ell\}$$

and

$$\left\{a_{i\ell+1}, \ldots, a_{k-2\ell}, b_1, \ldots, b_\ell, z_1, \ldots, z_{(i+1)\ell}\right\}$$

for all $i = 0, \ldots, z-1$.

In terms of Definition 1 we have $d = \ell$, $s = z+1$ and the sets $A_1, \ldots, A_{z+1}$ are the edges listed minus the set $\{b_1, \ldots, b_l\}$.

(L6) *Suppose $\ell \nmid k$. Given any sequence of $k - \ell + q$ distinct vertices $a_1, \ldots, a_{k-\ell}, z_1, \ldots, z_q$, there are*
$(1 \pm \epsilon)n^{q-k+\ell}p^z$ sequences of vertices $b_1, \ldots, b_{q-k+\ell}$ such that all of the following edges are present in H:

$$\left\{a_{i\ell+1}, \ldots, a_{k-\ell}, b_1, \ldots, b_{q-k+\ell}, z_1, \ldots, z_{k-q+i\ell}\right\},$$

for all $i = 0, \ldots, z-1$.

In terms of Definition 1 we have $d = q - k + \ell$, $s = z$, and the sets $A_1, \ldots, A_z$ are the edges listed minus the set $\{b_1, \ldots, b_{q-k+\ell}\}$. We require that $\ell \nmid k$ since otherwise $q - k + \ell = 0$

(L7) *Given any sequence of $2\ell + (k-2\ell) + 2q$ distinct vertices,*

$$x_1, \ldots, x_\ell, y_1, \ldots, y_\ell, a_1, \ldots, a_{k-2\ell}, z_1, \ldots, z_q, w_1, \ldots, w_q,$$

there are $(1 \pm \epsilon)n^\ell p^{2z+2}$ sequences of vertices $b_1, \ldots, b_\ell$ such that all of the following edges are present in H:

$$\{x_1,\ldots,x_\ell,a_1,\ldots,a_{k-2\ell},b_1,\ldots,b_\ell\},\{y_1,\ldots,y_\ell,a_1,\ldots,a_{k-2\ell},b_1,\ldots,b_\ell\}$$

and

$$\left\{a_{i\ell+1},\ldots,a_{k-2\ell},b_1,\ldots,b_\ell,z_1,\ldots,z_{(i+1)\ell}\right\},\left\{a_{i\ell+1},\ldots,a_{k-2\ell},b_1,\ldots,b_\ell,w_1,\ldots,w_{(i+1)\ell}\right\}$$

for all $i=0,\ldots,z-1$.

In terms of Definition 1 we have $d=\ell$, $s=2z+2$ and the sets $A_1,\ldots,A_{2z+2}$ are the edges listed minus the set $\{b_1,\ldots,b_l\}$.

(L8) *Suppose* $\ell\nmid k$. *Given any sequence of* $k-\ell+2q$ *distinct vertices* $a_1,\ldots,a_{k-\ell},z_1,\ldots,z_q$, $w_1,\ldots,w_q$, *there are* $(1\pm\epsilon)n^{q-k+\ell}p^{2z}$ *sequences of vertices* $b_1,\ldots,b_{q-k+\ell}$ *such that all of the following edges are present in* H*:*

$$\{a_{i\ell+1},\ldots,a_{k-\ell},b_1,\ldots,b_{q-k+\ell},z_1,\ldots,z_{k-q+i\ell}\},$$
$$\{a_{i\ell+1},\ldots,a_{k-\ell},b_1,\ldots,b_{q-k+\ell},w_1,\ldots,w_{k-q+i\ell}\}$$

for all $i=0,\ldots,z-1$.

In terms of Definition 1 we have $d=q-k+\ell$, $s=2z$, and the sets $A_1,\ldots,A_{2z}$ are the sets listed minus the set $\{b_1,\ldots,b_{q-k+l}\}$.

3 Proof of Lemma 1(a)

We will follow the convention that a factor $1+o(1)$ will be absorbed into the $1\pm\epsilon$ factors when the $o(1)$ term is clearly small enough. This will simplify several expressions.

Lemma 3. *Let* S *be a set of ordered* q*-tuples of distinct vertices with* $\epsilon^2|S|^2/n^{2q-1}\gg\log n$. *Let* σ *be a random permutation of* $[n]$. *Let* $N=|S\cap V(D_\sigma)|$. *Then* $N=(1\pm\epsilon)\frac{|S|}{qn^{q-1}}$ **qs**[3].

Proof. If $\mathbf{v}=(v_1,\ldots,v_q)$ then

$$\Pr(\mathbf{v}\in V(D_\sigma))=\frac{1}{q}\cdot\frac{1}{n-1}\cdot\frac{1}{n-2}\cdots\frac{1}{n-q+1}=\left(1\pm\frac{q^2}{2n}\right)\frac{1}{qn^{q-1}},$$

So

$$\mathbb{E}[N]=\left(1\pm\frac{q^2}{2n}\right)\frac{|S|}{qn^{q-1}}.$$

Suppose the permutation σ is converted to σ' by a single transposition. Then this changes at most 2 of the vertices of D_σ. So N can change by at most 2. Then Fact 2 implies that the probability that N deviates from its mean by more than $\frac{\epsilon}{2}\frac{|S|}{qn^{q-1}}$ is at most

$$2\exp\left\{-\frac{2\left(\frac{\epsilon}{2}\frac{|S|}{qn^{q-1}}\right)^2}{2^2n}\right\}=O(n^{-K})$$

[3]A sequence of events $\mathcal{E}_n,n\geq0$ is said to occur *quite surely* (**qs**) if $\Pr(\mathcal{E}_n)=1-O(n^{-K})$ for any positive constant K

for any positive constant K. The lemma follows since $q^2/n = O(1/n) \ll \epsilon$.

□

Lemma 4. *Suppose n, p, and ϵ satisfy $\epsilon^2 np^{8z} \gg \log n$. Let H be an (ϵ, p)-regular k-graph on n vertices (n divisible by q). Let σ be a random permutation of $[n]$. Then $D = D_\sigma$ is $((2z+5)\epsilon, p^z)$-regular,* **qs**.

Proof. We verify the properties of D one at a time, starting with out-degrees. Fix any q vertices, $v_1, \ldots, v_q$. Let $\mathbf{v} = (v_1, \ldots, v_q)$. Let $N_\mathbf{v}$ be the number of q-tuples $\mathbf{w}$ such that (a) $\mathbf{w} \in V(D)$ and (b) $\mathbf{v}$ precedes $\mathbf{w}$. It suffices to show that with probability $1 - o(n^{-(q+1)})$, $N_\mathbf{v} = (1 \pm (2z+5)\epsilon)\, p^z \nu_q$. Let $S_\mathbf{v}$ be the set of q-tuples $\mathbf{w}$, such that $\mathbf{v}$ precedes $\mathbf{w}$.

Apply property (L1) of Lemma 2 to $\{v_1, \ldots v_q\}$ and fix one of the $(1 \pm \epsilon)\, n^{k-q} p$ sequences $(v_{q+1}, \ldots, v_k)$ such that $\{v_1, \ldots, v_k\} \in H$. Let $\mathbf{u} = (v_1, \ldots, v_k)$ and do the following $z - 1$ times:

1. Apply property (L2) of Lemma 2 to the trailing $k - \ell$ elements of $\mathbf{u}$.
2. Fix one of the $(1 \pm \epsilon)\, n^\ell p$ sequences of ℓ vertices.
3. Append this sequence of ℓ vertices to the end of $\mathbf{u}$.

At the end of this process, $k - q + (z-1)\ell = k - \ell$ distinct vertices, $(v_{q+1}, \ldots, v_{q+k-\ell})$, have been fixed and appear at the trailing end of $\mathbf{u}$. Fix any $q - k + \ell \geq 0$ distinct vertices to give the q tuple $\mathbf{w} = (v_{q+1}, \ldots, v_{2q})$.

Combining our estimates from each step tells us that

$$\begin{aligned} |S_\mathbf{v}| &= (1 \pm \epsilon)\, n^{k-q} p \cdot \left((1 \pm \epsilon)\, n^\ell p\right)^{z-1} \cdot n^{q-k+\ell} \\ &= (1 \pm (2z+3)\epsilon)\, n^q p^z \end{aligned}$$

and so

$$\mathbb{E}\left[N_\mathbf{v}\right] = \frac{\mathbb{E}\left[|S_\mathbf{v}|\right]}{q(n-1)\cdots(n-q+1)} = \frac{(1 \pm (2z+4)\epsilon)\, n^q p^z}{q n^{q-1}} = (1 \pm (2z+4)\epsilon)\, p^z \nu_q.$$

Since $\epsilon^2 p^z n \gg \log n$, we can apply Lemma 3 to $S_\mathbf{v}$ to conclude that **qs**

$$N_\mathbf{v} = (1 \pm (2z+5)\epsilon)\, p^z \nu_q.$$

For in-degrees, fix a q-tuple

$$\mathbf{u} = \mathbf{v} = (v_{q+1}, \ldots, v_{2q}).$$

do the following z times:

1. Apply property (L2) of Lemma 2 to the leading $k - \ell$ elements of $\mathbf{u}$.
2. Fix one of the $(1 \pm \epsilon)\, n^\ell p$ sequences of ℓ vertices.
3. Prepend this sequence to the beginning of $\mathbf{u}$.

At the end of this process, q vertices have been fixed and appear in the first q positions of $\mathbf{u}$. Call this q-tuple $\mathbf{w}$. Combining estimates from each step of the process tells us that the number of such $\mathbf{w}$ that precede $\mathbf{v}$ is

$$\left((1 \pm \epsilon)\, n^{\ell} p\right)^{z} = (1 \pm (2z+1)\epsilon)\, n^{q} p^{z}.$$

Applying Lemma 3 as before gives us that **qs** the in-degree of $\mathbf{v}$ in D is

$$(1 \pm (2z+3)\epsilon)\, p^{z} \nu_q.$$

The remaining properties are dealt with in a similar manner. For each, we will state what properties from Lemma 2 to apply and compute the number of satisfying q-tuples. In all cases, an application of Lemma 3 completes the argument.

For $d^{+}(\mathbf{x}, \mathbf{y})$ in D, fix 2 q-tuples of distinct vertices, $\mathbf{x} = (x_1, \ldots, x_q)$ and $\mathbf{y} = (y_1, \ldots, y_q)$ and apply property (L3) to obtain $(z_1, z_2, \ldots, z_{k-q})$ in $(1 \pm \epsilon) n^{k-q} p^2$ ways. Follow by $z-1$ applications of property (L4). Our first iteration applies (L4) to $x_{\ell+1}, \ldots, x_q, z_1, \ldots, z_{k-q}$ and $y_{\ell+1}, \ldots, y_q, z_1, \ldots, z_{k-q}$ to obtain $(z_{k-q+1}, \ldots, z_{k-q+\ell})$ in $(1 \pm \epsilon) n^{k-q} p^2$ ways. We then shift right ℓ terms along both sequences and apply (L4) again. In our last application we feed sequences that begin with $x_{(z-1)\ell+1} \neq y_{(z-1)\ell+1}$ using the fact that $(z-1)\ell + 1 < q+1$. Arbitrarily choose $q - k + \ell \geq 0$ more vertices to fill out $z_1, \ldots, z_q$. The estimate in this case is

$$(1 \pm \epsilon) n^{k-q} p^2 \cdot \left((1 \pm \epsilon) n^{\ell} p^2\right)^{z-1} \cdot (n - (k-\ell)) \cdots (n - (q-1))$$

Simplifying and applying Lemma 3 gives that $d^{+}(\mathbf{x}, \mathbf{y})$ in D is **qs**

$$(1 \pm (2z+5)\epsilon)\, p^{2z} \nu_q.$$

Similarly $d^{-}(\mathbf{x}, \mathbf{y})$ is **qs**

$$(1 \pm (2z+5)\epsilon)\, p^{2z} \nu_q.$$

For $d^{+-}(\mathbf{x}, \mathbf{y})$ in D, fix $2q$ distinct vertices arranged in 2 q-tuples, $\mathbf{x} = (x_1, \ldots, x_q)$ and $\mathbf{y} = (y_1, \ldots, y_q)$. If ℓ divides k, (so that $q = k - \ell$), apply property (L2) $z-1$ times starting with $\mathbf{x}$. After the first iteration, we obtain $(z_1, \ldots, z_\ell)$ in $(1 \pm \epsilon)\, n^{l} p$ ways. We shift right by ℓ in the sequence for each subsequent application of property (L2) to obtain $(z_1, \ldots, z_{q-\ell})$. Note here that $q - \ell = k - 2\ell$. Property (L5) is then applied to $x_{q-\ell+1}, \ldots, x_q, z_1, \ldots, z_{q-\ell}, y_1, \ldots, y_q$. The estimate in this case is

$$\left((1 \pm \epsilon)\, n^{\ell} p\right)^{z-1} \cdot (1 \pm \epsilon)\, n^{\ell} p^{z+1}$$

If ℓ does not divide k, then apply (L1) to $\mathbf{x}$ to obtain $(z_1, \ldots, z_{k-q})$ in $(1 \pm \epsilon)\, n^{k-q} p$ ways. Follow this by $z-1$ applications of (L2), shifting right by ℓ in the sequence for each application to obtain $(z_1, \ldots, z_{k-\ell})$. Follow by an application of (L6) to $z_1, \ldots, z_{k-\ell}, y_1, \ldots, y_q$ to fill out $(z_1, \ldots, z_q)$. The estimate in this case is

$$(1 \pm \epsilon)\, n^{k-q} p \cdot \left((1 \pm \epsilon)\, n^{\ell} p\right)^{z-1} \cdot (1 \pm \epsilon)\, n^{q-k+\ell} p^{z}$$

Simplifying and applying Lemma 3 in both cases gives that $d^{+-}(\mathbf{x},\mathbf{y})$ is **qs**

$$(1 \pm (2z+5)\epsilon)\, p^{2z}\nu_q$$

For the third property of digraph uniformity, fix $4q$ distinct vertices arranged in 4 q-tuples, $\mathbf{x} = (x_1, \ldots, x_q)$, $\mathbf{y} = (y_1, \ldots, y_q)$, $\mathbf{z} = (z_1, \ldots, z_q)$, and $\mathbf{w} = (w_1, \ldots, w_q)$. If ℓ divides k, do $z-1$ applications of property (L4). Our first iteration applies (L4) to $x_1, \ldots, x_q, y_1, \ldots, y_q$ to obtain $(a_1, \ldots, a_\ell)$ in $(1 \pm \epsilon)n^\ell p^2$ ways. We then shift right ℓ terms along both sequences and apply (L4) to $x_{\ell+1}, \ldots, x_q, a_1, \ldots, a_\ell$ and $y_{\ell+1}, \ldots, y_q, a_1, \ldots, a_\ell$ and so on until we have obtained $(a_1, \ldots, a_{k-2\ell})$. We then apply property (L7) to

$$x_{q-\ell+1}, \ldots, x_q, y_{q-\ell+1}, \ldots, y_q, a_1, \ldots, a_{k-2\ell}, z_1, \ldots, z_q, w_1, \ldots, w_q$$

to find $(a_{k-2\ell+1}, \ldots, a_q)$. The estimate in this case is

$$\left((1 \pm \epsilon)\, n^\ell p^2\right)^{z-1} \cdot (1 \pm \epsilon)\, n^\ell p^{2z+2}.$$

If ℓ does not divide k, apply property (L3) to $x_1, \ldots, x_q, y_1, \ldots, y_q$ to obtain $(a_1, \ldots, a_{k-q})$ in $(1 \pm \epsilon)\, n^{k-q}p^2$. Follow by $z-1$ applications of (L4) as in the proof of $d^+(\mathbf{x},\mathbf{y})$ to obtain $(a_1, a_2, \ldots, a_{k-\ell})$. Then apply (L8) to $a_1, \ldots, a_{k-\ell}, z_1, \ldots, z_q, w_1, \ldots, w_q$ in order to find $(a_{k-\ell+1}, \ldots, a_q)$. The estimate in this case is

$$(1 \pm \epsilon)\, n^{k-q}p^2 \cdot \left((1 \pm \epsilon)\, n^\ell p^2\right)^{z-1} \cdot (1 \pm \epsilon)\, n^{q-k+\ell}p^{2z}.$$

Simplifying and applying Lemma 3 in both cases gives **qs**

$$(1 \pm (2z+5)\epsilon)\, p^{4z}\nu_q$$

for property (iii) of digraph uniformity. □

Lemma 5. *Suppose n, p, and ϵ satisfy $\epsilon n \gg 1$. Let H be an (ϵ, p)-regular k-graph on n vertices (n divisible by q), and randomly and independently construct digraphs $D_1, \ldots, D_r$ according to Procedure 1. Let H_i be their corresponding k-graphs. Then with probability $1 - o(n^{-1})$, every edge of H is an edge in $(1 \pm (z+2)\epsilon)\,\kappa$ of the H_i. Here κ, r are as defined in (2).*

Proof. We must first calculate the probability that an edge of H appears in an H_i after Procedure 1. This probability is

$$p_1 = \frac{k!(1 \pm z\epsilon)p^{z-1}}{\ell q n^{k-2}}.$$

To see this, first fix an edge $e = \{x_1, \ldots, x_k\}$ of H. We want the probability that this is an edge of H_1, say. For this to happen, there must be two vertices $\mathbf{v}_1 = (v_1, \ldots, v_q), \mathbf{v}_2 = (v_{q+1}, \ldots, v_{2q})$ of D_1 and $0 \le i \le z-1$ such that $e = e_i(\mathbf{v}_1, \mathbf{v}_2)$. Fix such an i. We now

have to consider the number of choices for $v_1, \ldots, v_{i\ell}, v_{i\ell+k+1}, \ldots, v_{(z-1)\ell+k+1}, \ldots, v_{2q}$. The (ϵ, p)-regularity of H implies that there will be

$$((1 \pm \epsilon)n^{\ell}p)^{z-1}n^{2q-(z-1)\ell-k} = (1 \pm (z - .5)\epsilon)p^{z-1}n^{2q-k}$$

choices for this sequence.

The probability that $\mathbf{v}_1, \mathbf{v}_2$ are vertices of H_1 is

$$p_2 = \left(\frac{1}{q} \cdot \left(\prod_{i=1}^{q-1} \frac{1}{n-i}\right)\right)^2.$$

Now there are z choices for i and $k!$ choices for the ordering of e and so the probability that e is an edge of H_1 is

$$zk!(1 \pm (z - .5)\epsilon)p^{z-1}n^{2q-k}p_2 = p_1.$$

Since the r random constructions are independent, the number Z_e of H_i that contain e is distributed as $Bin[r, p_1]$. So,

$$\mathbb{E}[Z_e] = rp_1 = (1 \pm z\epsilon)\,\kappa.$$

So the Chernoff bound tells us the probability that this Binomial deviates from its mean by more than a factor of $1 \pm \epsilon$ is at most

$$2\exp\left\{-\frac{\epsilon^2}{3} \cdot (1 - z\epsilon)\,\kappa\right\} = o(n^{-k-1}).$$

So taking a union bound over all $O(n^k)$ choices for e gives the result. □

Proof of Lemma 1(a): Our conditions on n, p and ϵ allow us to apply Lemmas 4 and 5. So with probability $1 - o(n^{-1})$, after Step 1 of Procedure 1,

(a) Every D_i is $((2z+5)\epsilon, p^z)$-regular.

(b) Every edge in H is covered $(1 \pm (z+2)\epsilon)\,\kappa$ times by the H_i.

Condition on the above outcome of Steps 1 and 2, and consider an arbitrary D_i' (as defined in Step 4 of Procedure 1. $r = o(n^{k-1})$ (since $\epsilon^2 np^{z-1} \gg \log n$), so it suffices to show that with probability $1 - o(n^{-k})$, D_1' has the desired properties.

For out-degrees: A vertex $\mathbf{v} \in D_1'$ corresponds to a q-tuple of vertices in H. An edge e of D_1 remains in D_1' if and only if all the z hyperedges of H owned by e receive label 1 in Step 3. This happens with probability

$$\frac{1}{[(1 \pm (z+2)\epsilon)\,\kappa]^z} = \left(1 \pm (z^2 + 2z + 1)\epsilon\right)\frac{1}{\kappa^z}$$

There are $(1 \pm (2z+5)\epsilon)\,\nu_q p^z$ neighbors of $\mathbf{v}$ in D_1, so the expected out-degree of $\mathbf{v}$ in D_1' is

$$(1 \pm (2z+5)\epsilon)\left(1 \pm (z^2 + 2z + 1)\epsilon\right)\nu_q\left(\frac{p}{\kappa}\right)^z = \left(1 \pm (z^2 + 4z + 7)\epsilon\right)\frac{n}{q}\left(\frac{p}{\kappa}\right)^z.$$

For concentration, the Chernoff inequality tells us that the probability that the out-degree of vertex $\mathbf{v}$ in D_1' deviates from its expectation by more than a factor of $1 \pm \epsilon$ is at most

$$2\exp\left\{\frac{\epsilon^2}{3}\cdot\left(1-(z^2+4z+7)\epsilon\right)\frac{n}{q}\left(\frac{p}{\kappa}\right)^z\right\} \leq o(n^{-k-1})$$

as long as

$$\frac{\epsilon^2 np^z}{\kappa^z} = \Theta\left(\frac{\epsilon^{2z+2}np^z}{\log^z n}\right) \gg \log n.$$

This is true by our assumptions on n, p and ϵ. Therefore with probability $1 - o(n^{-k-1})$, the out degree of $\mathbf{v}$ in D_i' is $(1 \pm (z^2+4z+9)\epsilon)\,\nu_q\left(\frac{p}{\kappa}\right)^z$. Taking a union bound over all $O(n)$ vertices in D_1' establishes uniformity for out-degrees.

The other properties follow from a similar argument. The smallest mean we deal with is in property (iii) of digraph regularity:

$$\frac{n(1\pm(2z+5)\epsilon)p^{4z}}{q((1\pm(z+2)\epsilon)\kappa)^{4z}} = \left(1\pm(4z^2+10z+7)\epsilon\right)\frac{n}{q}\left(\frac{p}{\kappa}\right)^{4z}.$$

So the error in concentration is at most

$$2\exp\left\{\frac{\epsilon^2}{3}\cdot\left(1-(4z^2+10z+7)\epsilon\right)\frac{n}{q}\left(\frac{p}{\kappa}\right)^{4z}\right\} \leq o(n^{-k-4})$$

as long as $\epsilon^{8z+2}np^{4z}/\log^{4z} n \gg \log n$ which it is by assumption. Taking a union bound over all $O(n^4)$ choices for vertices in D_1' gives the result. □

4 Proof of Lemma 1(b)

We will be applying the principle of inclusion-exclusion to get an estimate on the regularity of H'. So we use the next two Lemmas to compute a first order estimate and a second order upper bound on several quantities.

Given a hyperedge e and a digraph D_i from Procedure 1, edge e is owned by at most one directed edge in D_i. If this edge exists, let it be denoted $u_i(e)$. Now $u_i(e)$ owns exactly z hyperedges in H_i. If e is is an edge of H_i, let $\phi_i(e)$ be the set of z hyperedges owned by $u_i(e)$. Note that $\phi_i(e)$ includes the edge e. We call $\phi_i(e)\setminus\{e\}$ the *partner* edges of e in H_i.

Lemma 6. *Condition on $|I_e| = (1\pm(z+2)\epsilon)\,\kappa$ for each edge of H. Fix $d \in \{1,\dots,\ell\}$ and any set of $k-d$ vertices, $A = \{a_1,\dots,a_{k-d}\} \subset V(H)$. Fix a family $\mathcal{B}$ of d-sets of vertices such that $A\cup B$ is a hyperedge of H for all $B \in \mathcal{B}$. Suppose $\epsilon^2\,|\mathcal{B}|\,/\kappa^{2z-1} \gg \log n$. Then with probability $1 - o(n^{-(k+2q)-1})$, the number $N_{\mathcal{B}}$ of $B \in \mathcal{B}$ such that $A\cup B \in \bigcup_i E(H_i')$ satifies $N_{\mathcal{B}} = (1\pm(z^2+z)\epsilon)\,\frac{|\mathcal{B}|}{\kappa^{z-1}}$*

Proof. Let $\mathcal{B} = \{B_1,\dots,B_t\}$. Because we are conditioning on $|I_e|, e \in E(H)$, the relevant probability space is the choice of labels in Step 3 of Procedure 1. Define $F = F(A)$, the set

of relevant edges, as follows: For each j such that $A \cup B_i \in E(H_j)$ there are exactly $z-1$ partner edges $F_{i,j}$ such that $A \cup B_i \in E(H'_j)$ if and only if all of these edges as well as $A \cup B_i$ receive label j. Let $F = \bigcup_{i,j} F_{i,j}$. Since we assume that each edge is in $(1 \pm (z+2)\epsilon)\,\kappa$ of the H_j, we have that $|F| \leq 2z\kappa\,|\mathcal{B}|$. The labels outside of F do not affect the count N, so we may condition on an arbitrary setting of those labels leaving only the labels of F to be exposed.

Now

$$\Pr\left[A \cup B_i \in \bigcup_j E(H'_j)\right] = \left[(1 \pm (z+2)\epsilon)\,\kappa\right]^{-(z-1)}.$$

To see this, expose the label of an edge $A \cup B_i$. Suppose that it receives label j. Then all of its partner edges must also receive label j. Each of them is an edge of $(1 \pm (z+2)\epsilon)\,\kappa$ of the H_k, and since their labelings are independent, the probability that each of them receive label j is as claimed above. So

$$\mathbb{E}\left[N_{\mathcal{B}}\right] = |\mathcal{B}|\left[(1 \pm (z+2)\epsilon)\,\kappa\right]^{-(z-1)} = \left(1 \pm (z^2+z-1)\epsilon\right)\frac{|\mathcal{B}|}{\kappa^{z-1}}$$

Our probability space is a product space of dimension $|F|$. We use the Hoeffding-Azuma inequality to show that $N_{\mathcal{B}}$ is concentrated. Suppose the label of an edge $e \in F$ is changed from i to j. Suppose that e is owned by the edge $(\mathbf{v}_1 = (v_1, \ldots, v_q), \mathbf{v}_2 = (v_{q+1}, \ldots, v_{2q}))$ of D_i. Let $S = \{v_1, \ldots, v_{2q}\}$. The definition of F implies that $S \supseteq A$. So at most $\binom{2q-(k-d)}{d}$ sets from $\mathcal{B}$ will be removed from the count $N_{\mathcal{B}}$ by this switch in labels. Similarly, at most $\binom{2q-(k-d)}{d}$ sets from $\mathcal{B}$ will be added to the count $N_{\mathcal{B}}$. Hence $N_{\mathcal{B}}$ is $\binom{2q-(k-d)}{d}$-Lipschitz and the Hoeffding-Azuma inequality implies that the probability that $N_{\mathcal{B}}$ deviates from its mean by more than $\epsilon\,|\mathcal{B}|\,/\kappa^{z-1}$ is at most

$$2\exp\left\{-\frac{\left(\epsilon\,|\mathcal{B}|\,/\kappa^{z-1}\right)^2}{2\binom{2q-(k-d)}{d}^2|F|}\right\} \leq 2\exp\left\{-\frac{\epsilon^2\,|\mathcal{B}|}{4z\binom{2q-(k-d)}{d}^2\kappa^{2z-1}}\right\} \leq o(n^{-(k+2q)-1})$$

as long as $\epsilon^2\,|\mathcal{B}|\,/\kappa^{2z-1} \gg \log n$, which we assumed. Therefore $N = (1 \pm (z^2+z)\epsilon)\,\frac{|\mathcal{B}|}{\kappa^{z-1}}$ with the desired probability. □

Let $1 \leq t \leq 2q-k$. Let D_i be a digraph constructed from Procedure 1. Say that a set S of $k+t$ vertices is *condensed* in D_i if there exist edges $e_1 \neq e_2$ of H such that $S = e_1 \cup e_2$ and $\phi_i(e_1) \cap \phi_i(e_2) \neq \emptyset$.

Lemma 7. *Suppose $r \ll n^{k-\frac{3}{2}}$ Construct r independent D_i according to Procedure 1. Then with probability $1 - o(n^{-1})$, every set of S of $k+t$ vertices, $1 \leq t \leq 2q-k$, is condensed in at most $4q+1$ of the D_i.*

Proof. Fix a set of $k+t$ vertices $S = \{x_1, x_2, \ldots, x_{k+t}\} = e_1 \cup e_2$ where e_1, e_2 are edges of H. The probability that S is condensed in D_1 is at most

$$(k+t)! \cdot \frac{1}{q} \cdot (z-1) \cdot \left(\prod_{i=1}^{q+t-1} \frac{1}{n-i}\right) < \frac{(2q)!}{\ell n^{k+t-1}}$$

This calculation is very similar to the one in Lemma 5.

Since the D_i are independent, the number of them which have the above property with respect to S is stochastically dominated by Bin $\left[r, \frac{(2q)!}{\ell n^{k+t-1}}\right]$. Since we assumed that $r \ll n^{k-\frac{3}{2}}$, the probability that this exceeds $4q+1$ is at most

$$\binom{r}{4q+2}\left(\frac{(2q)!}{\ell n^{k+t-1}}\right)^{4q+2} = o(n^{(k-\frac{3}{2}-k-t+1)(4q+2)}) = o(n^{-2q-1})$$

Now taking a union bound over all $O(n^{2q})$ choices for S gives the result. □

Lemma 8. *Condition on $|I_e| = (1 \pm (z+2)\epsilon)\,\kappa$ for each edge of H. Also condition on the property that every set of $k+t$ vertices, $1 \le t \le 2q-k$, is condensed in at most $4q+1$ of the D_i. Fix $d \in \{1, \dots \ell\}$ and any 2 sets, A_1 and A_2 of $k-d$ vertices. Fix a family $\mathcal{B}$ of d-sets of vertices such that $A_1 \cup B$ and $A_2 \cup B$ are both hyperedges of H for all $B \in \mathcal{B}$. Suppose $|\mathcal{B}|/\kappa^{2z+1} \gg \log n$. Then with probability $o(n^{-(k+2q)-1})$, the number $N_{\mathcal{B}}$ of $B \in \mathcal{B}$ such that $A_1 \cup B \in \bigcup_i H_i'$ and $A_2 \cup B \in \bigcup_i H_i'$ is at most $7q\,|\mathcal{B}|/\kappa^z$*

Proof. Let $\mathcal{B} = \{B_1, \dots, B_t\}$ and let $F^* = F(A_1) \cup F(A_2)$ where F is as defined in Lemma 6. Then $|F^*| \le 3z\kappa\,|\mathcal{B}|$. We would like an upper bound on the probability that a particular $B \in \mathcal{B}$ contributes to $N_{\mathcal{B}}$. Let $e_1 = A_1 \cup B$ and $e_2 = A_2 \cup B$. First, expose the label of e_1 and suppose it is j.

Case 1: e_2 receives label j.

If $\phi_k(e_1) \cap \phi_k(e_2) = \emptyset$, then the probability that $e_1, e_2 \in H_j'$ is at most

$$q_1 := \left[(1-(z+2)\epsilon)\,\kappa\right]^{-(2z-1)}.$$

To see this, note that the probability that e_2 receives label j is $((1 \pm (z+2)\epsilon)\,\kappa)^{-1}$, and since their $2(z-1)$ partner edges are distinct and labelings are independent, we get the desired probability.

If $\phi_j(e_1) \cap \phi_j(e_2) \neq \emptyset$ then the vertices of $e_1 \cup e_2$ are condensed in D_j. We have $k+1 \le |e_1 \cup e_2| \le 2q$, so by assumption, these vertices are condensed in at most $4q+1$ of the D_i. So the probability that e_1 and e_2 are both in $E(H_j')$ is bounded above by

$$q_2 = \frac{4q+1}{(1-(z+2)\epsilon)\,\kappa} \cdot \frac{1}{\left[(1-(z+2)\,\epsilon)\,\kappa\right]^{z-1}} = \frac{4q+1}{\left[(1-(z+2)\,\epsilon)\,\kappa\right]^{z}}$$

since all of the partner edges of e_1 must also receive label j.

Case 2: e_2 receives label $l \neq j$.

If $\phi_j(e_1) \cap \phi_l(e_2) = \emptyset$ then the probability that everything receives the appropriate label is at most

$$q_3 = \left[(1-(z+2)\epsilon)\,\kappa\right]^{-(2z-2)}.$$

If $\phi_j(e_1) \cap \phi_l(e_2) \neq \emptyset$, then the probability that B contributes to $N_{\mathcal{B}}$ is 0 since an edge in the intersection must receive both labels j and l.

Summing up these upper bounds, we get that the probability that B contributes to $N_{\mathcal{B}}$ is bounded above by

$$q_1 + q_2 + q_3 \leq \frac{4q+3}{\left[(1-(z+2)\,\epsilon)\,\kappa\right]^z} \leq \frac{6q}{\kappa^z}.$$

So $\mathbb{E}\left[N_{\mathcal{B}}\right] \leq \frac{6q}{\kappa^z}\left|\mathcal{B}\right|$. By a similar argument as in Lemma 6, we can see that $N_{\mathcal{B}}$ is $\binom{2q-(k-d)}{d}$-Lipschitz in the product space of dimension $\left|F^*\right| \leq 3z\kappa\left|\mathcal{B}\right|$. So the probability that $N_{\mathcal{B}}$ exceeds its expectation by more than $\left|\mathcal{B}\right|/\kappa^z$ is at most

$$2\exp\left\{-\frac{\left(\left|\mathcal{B}\right|/\kappa^z\right)^2}{2\cdot\binom{2q-(k-d)}{d}^2\left|F^*\right|}\right\} \leq 2\exp\left\{-\frac{\left|\mathcal{B}\right|}{6z\cdot\binom{2q-(k-d)}{d}^2\cdot\kappa^{2z+1}}\right\} \leq o(n^{-(k+2q)-1})$$

since we assumed that $\left|\mathcal{B}\right|/\kappa^{2z+1} \gg \log n$. Therefore $N_{\mathcal{B}} \leq \frac{7q}{\kappa^z}\left|\mathcal{B}\right|$ with the desired probability. □

Proof of Lemma 1(b): By applying Lemma 5 and Lemma 7, the conditions of which hold by our requirements on n, p and ϵ, the outcome of Steps 1 and 2 of Procedure 1 satisfies the following with probability $1 - o(n^{-1})$.

- Every edge of H is covered $(1 \pm (z+2)\epsilon)$ by the H_i.
- Every set of $k+t$, $1 \leq t \leq 2q-k$ vertices is condensed in at most $4q+1$ of the D_i.

Condition on this outcome. We will show that in the context of the choices in Step 3, (ϵ', p')-regularity is satisfied with probability $1 - o(n^{-1})$.

Fix $d \in \{1,\ldots,\ell\}$, $s \in \{1,\ldots,2z+2\}$ and a family of s distinct $(k-d)$-sets $\Gamma = \{A_1,\ldots,A_s\}$ with $|\cup_i A_i| \leq k+2q$. Let X be the number of d-sets, B, such that $A_i \cup B$ is an edge of H' for all $i = 1,\ldots,s$. It suffices to show that $X = (1\pm\epsilon')\frac{n^d}{d!}p'^s$ with probability $1 - o(n^{-(k+2q)-1})$. Then we can use the union bound over all $O(n^{k+2q})$ choices for vertices $|\cup_i A_i|$ and all $O(1)$ choices of set families on those vertices.

Let $\mathcal{B}$ be the family of all d-sets B such that $A_i \cup B$ are edges of H for all $i = 1,\ldots,s$ and $B \in \mathcal{B}$. H is (ϵ, p)-regular, so $\left|\mathcal{B}\right| = (1\pm\epsilon)\,\frac{n^d}{d!}p^s$.

For each $i \in \{1,\ldots,s\}$, let X_i be the number of elements B of $\mathcal{B}$ with $A_i \cup B \in \bigcup_l H'_l$. For every $i, j \in \{1,\ldots,s\}, i \neq j$, let X_{ij} be the number of elements, B, of $\mathcal{B}$ with both $A_i \cup B \in \bigcup_l H'_l$ and $A_j \cup B \in \bigcup_l H'_l$.

Then

$$\left|\mathcal{B}\right| - \sum_{i=1}^{s} X_i \leq X \leq \left|\mathcal{B}\right| - \sum_{i=1}^{s} X_i + \sum_{i<j} X_{ij}.$$

Note that since $d \geq 1$ and $s \leq 2z+2$, we have

$$\left|\mathcal{B}\right| = \Theta\left(n^d p^s\right) = \Omega\left(np^{2z+2}\right).$$

We apply Lemmas 6 and 8. Indeed, by our requirements on n, p and ϵ we have both

$$\frac{\epsilon^2\left|\mathcal{B}\right|}{\kappa^{2z-1}} = \Omega\left(\frac{\epsilon^{4z}np^{2z+2}}{\log^{2z-1}n}\right) \gg \log n$$

and

$$\frac{|\mathcal{B}|}{\kappa^{2z+1}} = \Omega\left(\frac{\epsilon^{4z+2}np^{2z+2}}{\log^{2z+1} n}\right) \gg \log n.$$

So we may apply Lemmas 6 and 8 to get

$$\begin{aligned} X &= |\mathcal{B}| - s\left(1 \pm (z^2+z)\epsilon\right)\frac{|\mathcal{B}|}{\kappa^{z-1}} \pm s^2\frac{7q}{\kappa^z}|\mathcal{B}| \\ &= |\mathcal{B}|\left(1 - \frac{s\left(1 \pm (z^2+z+1)\epsilon\right)}{\kappa^{z-1}}\right) \end{aligned}$$

where in the second line we use the fact that $\frac{1}{\kappa} \ll \epsilon$.

Note that

$$\left(1 - \frac{1}{\kappa^{z-1}}\right)^s = 1 - \frac{s}{\kappa^{z-1}} + O\left(\frac{1}{\kappa^{2z-2}}\right).$$

Then by using $\frac{1}{\kappa} \ll \epsilon$ we get that

$$\begin{aligned} X &= (1 \pm \epsilon)\frac{n^d}{d!}p^s\left(\left(1 - \frac{1}{\kappa^{z-1}}\right)^s \pm \frac{(2z+2)(z^2+z+2)}{\kappa^{z-1}} \cdot \epsilon\right) \\ &= \frac{n^d}{d!}\left(p\left(1 - \frac{1}{\kappa^{z-1}}\right)\right)^s \cdot (1 \pm \epsilon)\left(1 \pm \frac{(2z+2)(z^2+z+2)}{\kappa^{z-1}\left(1 - \frac{1}{\kappa^{z-1}}\right)^s} \cdot \epsilon\right) \\ &= \frac{n^d}{d!}\left(p\left(1 - \frac{1}{\kappa^{z-1}}\right)\right)^s \left(1 \pm \left(1 + \frac{h(z)}{\kappa^{z-1}}\right)\epsilon\right) \end{aligned}$$

where $h(z) = (2z+2)(z^2+z+3)$. Now $z \geq 2$ and so $h(z) \leq 7z^3$ which gives us the result

$$X = (1 \pm \epsilon')\frac{n^d}{d!}p'^s$$

with the desired probability. □

5 Finishing the proof of Theorem 1

Let $H_0 = H$, $\epsilon_0 = \epsilon$ and $p_0 = p$. Define ϵ_t and p_t recursively using the following recursion:

$$\epsilon_{t+1} = \epsilon_t\left(1 + 7z^3\left(\frac{\epsilon_t^2}{6(k+1)\log n}\right)^{z-1}\right)$$

and

$$p_{t+1} = p_t\left(1 - \left(\frac{\epsilon_t^2}{6(k+1)\log n}\right)^{z-1}\right).$$

Let T be the smallest index such that $p_T \leq \frac{1}{2}\epsilon^\alpha p$ where $\alpha = \frac{1}{9+7z^3}$. For $t = 0, \ldots, T$, let $x_t = \left(\frac{\epsilon_t^2}{6(k+1)\log n}\right)^{z-1}$. Then since (ϵ_t) is an increasing sequence, we have

$$\begin{aligned}\frac{1}{2}p\epsilon^\alpha \leq p_{T-1} &= \frac{p_{T-1}}{p_{T-2}} \cdot \frac{p_{T-2}}{p_{T-3}} \cdots \frac{p_2}{p_1} \cdot \frac{p_1}{p} \cdot p \\ &\leq p\left(1 - x_0\right)^{T-1} \\ &\leq pe^{-x_0(T-1)}.\end{aligned}$$

From this we can see that

$$T \leq O\left(\frac{\log^{z-1} n}{\epsilon^{2z-1}}\right) = o(n).$$

Also note that since

$$\left(1 + 7z^3x\right)(1-x)^{7z^3} \leq e^{7z^3x}\left(e^{-x}\right)^{7z^3} = 1,$$

we have in general that

$$\frac{\epsilon_{t+1}}{\epsilon_t} = (1 + 7z^3x_t) \leq \frac{1}{(1-x_t)^{7z^3}} = \left(\frac{p_t}{p_{t+1}}\right)^{7z^3}.$$

Hence

$$\begin{aligned}\epsilon_{T-1} &= \frac{\epsilon_{T-1}}{\epsilon_{T-2}} \cdot \frac{\epsilon_{T-2}}{\epsilon_{T-3}} \cdots \frac{\epsilon_2}{\epsilon_1} \cdot \frac{\epsilon_1}{\epsilon} \cdot \epsilon \\ &\leq \epsilon \cdot \left(\frac{p_{T-2}}{p_{T-1}} \cdot \frac{p_{T-3}}{p_{T-2}} \cdots \frac{p}{p_1}\right)^{7z^3} \\ &= \epsilon\left(\frac{p}{p_{T-1}}\right)^{7z^3} \\ &< \epsilon \cdot \left(2\epsilon^{-\alpha}\right)^{7z^3} = \Theta\left(\epsilon^{1-7z^3\alpha}\right)\end{aligned}$$

So we have that

$$\epsilon_{T-1}^{1/8} = \Theta\left(\epsilon^{\frac{9}{8}\alpha}\right) \ll \epsilon^\alpha.$$

We now construct $H_1, \ldots, H_T$ such that each H_t is (ϵ_t, p_t)-regular. Let $\kappa_t = \frac{6(k+1)\log n}{\epsilon_t^2}$ and $r = \frac{n^{k-2}q\ell}{k!p_t^{z-1}}\kappa_t$ and consider Procedure 1 applied to H_t with these parameters. This produces digraphs $D'_{t,i}$ and k-graphs $H'_{t,i}$ with all $H'_{t,i}$ disjoint. Let H_{t+1} be the k-graph which results from the deletion of all $H'_{t,i}$ from H_t. In order to apply Lemma 1 at each step, we must check that $\epsilon_t^{8z+2}np_t^{8z} \gg \log^{4z+1} n$. This condition follows from our assumptions on ϵ, n, p since $\epsilon_t \geq \epsilon$ and $p_t \geq \frac{1}{2}\epsilon^\alpha p$. So we may apply Lemma 1 to conclude that with probability $1 - o(n^{-1})$, an application of Procedure 1 satisfies

- Every $D'_{t,i}$ is $(12z^2\epsilon_t, (p_t/\kappa_t)^z)$-regular.

- H_{t+1} is $(\epsilon_{t+1}, p_{t+1})$-regular.

Since $T = o(n)$, we may condition on this holding at each step. In order to apply the result on packing cycles in digraphs to each $D'_{t,i}$, we must verify that $\epsilon_t^{11}\nu_q \left(p_t^z/\kappa_t^z\right)^8 \gg \log^5 n$. We have

$$\epsilon_t^{11}\nu_q \left(\frac{p_t}{\kappa_t}\right)^{8z} \geq \Theta\left(\frac{\epsilon^{11+8z\alpha+16z} np^{8z}}{\log^{8z} n}\right) \gg \log^5 n$$

by our assumption that $\epsilon^{16z+12} np^{8z} \gg \log^{8z+5} n$ since $8z\alpha \leq 1$. So every $D'_{t,i}$ can be packed with Hamilton cycles missing only $(12z^2\epsilon_t)^{1/8}$-fraction of its edges. As observed already, these edge-disjoint Hamilton cycles in $D'_{t,i}$ correspond to edge disjoint Hamilton cycles in $H'_{t,i}$. Hence the packing in $D'_{t,i}$ gives a packing in $H'_{t,i}$ missing the same fraction of edges since there is a z-to-1 correspondence between edges in $H'_{t,i}$ and $D'_{t,i}$.

The above procedure is carried out until H_T is created. Then Hamilton cycles have been packed in $H \backslash H_T$, up to an error of $(12z^2\epsilon_{T-1})^{1/8}$-fraction. Let us estimate the fraction of edges present in H_T itself. By applying (ϵ, p)-regularity to H, we see that H had at least

$$(1-\epsilon)\frac{n^k}{k!}p \geq \frac{n^k}{k!+1}p$$

edges to begin with.

Similarly, we see that H_T has at most

$$(1+\epsilon_T)\frac{n^k}{k!}p_T \leq (1+\epsilon_T)\frac{n^k}{2\cdot k!}\epsilon^\alpha p \leq \frac{n^k}{2\cdot k!-1}p\epsilon^\alpha$$

edges. Since $k \geq 3$, we have that

$$\frac{|H_T|}{|H|} \leq c\epsilon^\alpha$$

where $c < 1$ is some constant.

Hence the fraction of edges of H not covered is at most

$$(12z^2\epsilon_{T-1})^{1/8} \cdot (1 - c\epsilon^\alpha) + c\epsilon^\alpha \leq (12z^2\epsilon_{T-1})^{1/8} + c\epsilon^\alpha \leq \epsilon^\alpha,$$

since $\epsilon_{T-1}^{1/8} \ll \epsilon^\alpha$.

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