The critical window for the classical Ramsey-Turán problem

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Abstract

The first application of Szemerédi's powerful regularity method was the following celebrated Ramsey-Turán result proved by Szemerédi in 1972: any K_4-free graph on n vertices with independence number $o(n)$ has at most $(\frac{1}{8}+o(1))n^2$ edges. Four years later, Bollobás and Erdős gave a surprising geometric construction, utilizing the isoperimetric inequality for the high dimensional sphere, of a K_4-free graph on n vertices with independence number $o(n)$ and $(\frac{1}{8}-o(1))n^2$ edges. Starting with Bollobás and Erdős in 1976, several problems have been asked on estimating the minimum possible independence number in the critical window, when the number of edges is about $n^2/8$. These problems have received considerable attention and remained one of the main open problems in this area. In this paper, we give nearly best-possible bounds, solving the various open problems concerning this critical window.

More generally, it remains an important problem to determine if, for certain applications of the regularity method, alternative proofs exist which avoid using the regularity lemma and give better quantitative estimates. In their survey on the regularity method, Komlós, Shokoufandeh, Simonovits, and Szemerédi surmised that the regularity method is likely unavoidable for applications where the extremal graph has densities in the regular partition bounded away from 0 and 1. In particular, they thought this should be the case for Szemerédi's result on the Ramsey-Turán problem. Contrary to this philosophy, we develop new regularity-free methods which give a linear dependence, which is tight, between the parameters in Szemerédi's result on the Ramsey-Turán problem.

1 Introduction

Szemerédi's regularity lemma [41] is one of the most powerful tools in extremal combinatorics. Roughly speaking, it says that every graph can be partitioned into a small number of parts such that the bipartite subgraph between almost every pair of parts is random-like. The small number of parts is at most an integer $M(\epsilon)$ which depends only on an approximation parameter ϵ. The exact statement of the regularity lemma is given in the beginning of Section 2. For more background on the regularity lemma, the interested reader may consult the well-written surveys by Komlós and Simonovits [28] and Rödl and Schacht [31].

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In the regularity lemma, $M(\epsilon)$ can be taken to be a tower of twos of height $\epsilon^{-O(1)}$, and probabilistic constructions of Gowers [24] and Conlon and Fox [9] show that this is best possible. Unfortunately, this implies that the bounds obtained by applications of the regularity lemma are usually quite poor. It remains an important problem to determine if new proofs giving better quantitative estimates for certain applications of the regularity lemma exist (see, e.g., [25]). Some progress has been made, including the celebrated proof of Gowers [26] of Szemerédi's theorem using Fourier analysis, the new proofs [8, 10, 19, 27] that bounded degree graphs have linear Ramsey numbers, the new proof [18] of the graph removal lemma, and the new proofs [7, 30] of Pósa's conjecture for graphs of large order.

The earliest application of the regularity method[1] is a celebrated result of Szemerédi from 1972 in Ramsey-Turán theory; see Theorem 1.1. For a graph H and positive integers n and m, the Ramsey-Turán number $\mathbf{RT}(n, H, m)$ is the maximum number of edges a graph G on n vertices with independence number less than m can have without containing H as a subgraph. The study of Ramsey-Turán numbers was introduced by Sós [37]. It was motivated by the classical theorems of Ramsey and Turán and their connections to geometry, analysis, and number theory. Ramsey-Turán theory has attracted a great deal of attention over the last 40 years; see the nice survey by Simonovits and Sós [36].

Theorem 1.1 (Szemerédi [39])**.** *For every $\epsilon > 0$, there is a $\delta > 0$ for which every n-vertex graph with at least $\big(\frac{1}{8} + \epsilon\big)n^2$ edges contains either a K_4 or an independent set larger than δn.*

Four years later, Bollobás and Erdős [5] gave a surprising geometric construction, utilizing the isoperimetric inequality for the high dimensional sphere, of a K_4-free graph on n vertices with independence number $o(n)$ and $(\frac{1}{8} - o(1))n^2$ edges. Roughly speaking, the Bollobás-Erdős graph consists of two disjoint copies of a discretized Borsuk graph, which connect nearly antipodal points on a high dimensional sphere, with a dense bipartite graph in between which connects points between the two spheres which are close to each other. For details of this construction and its proof, see Section 8.

Bollobás and Erdős asked to estimate the minimum possible independence number in the critical window, when the number of edges is about $n^2/8$. This remained one of the main open problems in this area, and, despite considerable attention, not much progress has been made on this problem. In particular, Bollobás and Erdős asked the following question.

Problem 1.2 (From [5])**.** *Is it true that for each $\eta > 0$ there is an $\epsilon > 0$ such that for each n sufficiently large there is a K_4-free graph with n vertices, independence number at most ηn, and at least $(\frac{1}{8} + \epsilon)n^2$ edges?*

They also asked the following related problem, which was later featured in the Erdős paper [12] from 1990 entitled "Some of my favourite unsolved problems".

[1] We remark that Szemerédi [39] first developed regularity lemmas which were weaker than what is now commonly known as Szemerédi's regularity lemma as stated above. Original proofs of several influential results, including Theorem 1.1, Szemerédi's theorem [40] on long arithmetic progressions in dense subsets of the integers, and the Ruzsa-Szemerédi theorem [32] on the $(6, 3)$-problem, used iterative applications of these weak regularity lemmas. Typically, the iterative application of these original regularity lemma is of essentially the same strength as Szemerédi's regularity lemma as stated above and gives similar tower-type bounds. However, for Szemerédi's proof of the Ramsey-Turán result, only two iterations were needed, leading to a double-exponential bound.

Problem 1.3 (From [5]). *Is it true that for every n, there is a K_4-free graph with n vertices, independence number $o(n)$, and at least $\frac{n^2}{8}$ edges?*

Erdős, Hajnal, Simonovits, Sós, and Szemerédi [14] noted that perhaps replacing $o(n)$ by a slightly smaller function, say by $\frac{n}{\log n}$, one could get smaller upper bounds on Ramsey-Turán numbers. Specifically, they posed the following problem.

Problem 1.4 (From [14]). *Is it true for some constant $c > 0$ that $\mathbf{RT}(n, K_4, \frac{n}{\log n}) < (1/8 - c)n^2$?*

This problem was further discussed in the survey by Simonovits and Sós [36] and by Sudakov [38]. Motivated by this problem, Sudakov [38] proved that if $m = e^{-\omega\left((\log n)^{1/2}\right)}n$, then $\mathbf{RT}(n, K_4, m) = o(n^2)$.

In this paper, we solve the Bollobás-Erdős problem to estimate the minimum independence number in the critical window. In particular, we solve the above problems, giving positive answers to Problems 1.2 and 1.3, and a negative answer to Problem 1.4. We next discuss these results in depth.

The bound on δ as a function of ϵ in the now standard proof of Theorem 1.1 (sketched in Section 2) strongly depends on the number of parts in Szemerédi's regularity lemma. In particular, it shows that δ^{-1} can be taken to be a tower of twos of height $\epsilon^{-O(1)}$. However, the original proof of Szemerédi [39], which used two applications of a weak regularity lemma, gives a better bound, showing that δ^{-1} can be taken to be double-exponential in $\epsilon^{-O(1)}$.

In their survey on the regularity method, Komlós, Shokoufandeh, Simonovits, and Szemerédi surmised that the regularity method is likely unavoidable for applications where the extremal graph has densities in the regular partition bounded away from 0 and 1. In particular, they thought this should be the case for Theorem 1.1. Contrary to this philosophy, our first result is a new proof of Theorem 1.1 which gives a much better bound and completely avoids using the regularity lemma or any notion similar to regularity. More precisely, it gives a linear bound for δ on ϵ in Theorem 1.1, in stark contrast to double-exponential dependence given by the original proof.

Theorem 1.5. *For every α and n, every n-vertex graph with at least $\frac{n^2}{8} + 10^{10}\alpha n$ edges contains either a copy of K_4 or an independent set of size greater than α.*

It is natural to wonder whether one must incur a constant factor of 10^{10}. Our second result sharpens the linear dependence down to a very reasonable constant. Its proof uses the regularity lemma with an absolute constant regularity parameter (independent of n and α).

Theorem 1.6. *There is an absolute positive constant γ_0 such that for every $\alpha < \gamma_0 n$, every n-vertex graph with at least $\frac{n^2}{8} + \frac{3}{2}\alpha n$ edges contains a copy of K_4 or an independent set of size greater than α.*

We also prove the following corresponding lower bound, which shows that the linear dependence in Theorem 1.6 is best possible, matching the dependence on α to within a factor of $3 + o(1)$. Starting with the Bollobás-Erdős graph, the construction finds a slightly denser K_4-free graph without increasing the independence number much. It also gives a positive answer to Problem 1.2 of Bollobás and Erdős with the linear dependence that our previous theorems now reveal to be correct. Here, we write $f(n) \ll g(n)$ when $f(n)/g(n) \to 0$ as $n \to \infty$.

Theorem 1.7. *For* $\frac{(\log\log n)^{3/2}}{(\log n)^{1/2}} \cdot n \ll m \leq \frac{n}{3}$*, we have*

$$\mathbf{RT}(n, K_4, m) \geq \frac{n^2}{8} + \left(\frac{1}{3} - o(1)\right) mn.$$

Remarks. The tripartite Turán graph has independent sets of size $(1+o(1))\frac{n}{3}$, so once m exceeds $\frac{n}{3}$, the Ramsey-Turán problem for K_4 asymptotically coincides with the ordinary Turán problem. Also, in the sublinear regime, our proof actually produces the slightly stronger asymptotic lower bound $\mathbf{RT}(n, K_4, m) \geq \frac{n^2}{8} + \left(\frac{1}{2} - o(1)\right) mn$ when $\frac{(\log\log n)^{3/2}}{(\log n)^{1/2}} \cdot n \ll m \ll n$.

Bollobás and Erdős drew attention to the interesting transition point of exactly $\frac{n^2}{8}$ edges. Thus far, the best result for this regime was a lower bound on the independence number of $ne^{-O(\sqrt{\log n})}$ by Sudakov [38]. The proof relies on a powerful probabilistic technique known as dependent random choice; see the survey by Fox and Sudakov [20].

By introducing a new twist on the dependent random choice technique, we substantially improve this lower bound on the independence number at the critical point. We think that this new variation may be interesting in its own right, and perhaps could have other applications elsewhere, as the main dependent random choice approach has now seen widespread use. Our key innovation is to exploit a very dense setting, and to select not the common neighborhood of a random set, but the set of all vertices that have many neighbors in a random set; then, we apply a "dispersion" bound on the binomial distribution in addition to the standard Chernoff "concentration" bound.

Theorem 1.8. *There is an absolute positive constant c such that every n-vertex graph with at least $\frac{n^2}{8}$ edges contains a copy of K_4 or an independent set of size greater than $cn \cdot \frac{\log\log n}{\log n}$.*

We also prove an upper bound on this problem, giving a positive answer to Problem 1.3 of Bollobás and Erdős. The proof is again by modifying the Bollobás-Erdős graph to get a slightly denser K_4-free graph whose independence number does not increase much.

Theorem 1.9. *There is an absolute positive constant c' such that for each positive integer n, there is an n-vertex K_4-free graph with at least $\frac{n^2}{8}$ edges and independence number at most $c'n \cdot \frac{(\log\log n)^{3/2}}{(\log n)^{1/2}}$.*

Recall that Bollobás and Erdős [5] constructed a K_4-free graph on n vertices with $(1-o(1))\frac{n^2}{8}$ edges with independence number $o(n)$. The various presentations of the proof of the Bollobás-Erdős result in the literature [3], [4], [5], [14], [15], [36] do not give quantitative estimates on the little-o terms. By finding good quantitative estimates for the relevant parameters, we can use the Bollobás-Erdős graphs to prove the following theorem. This result gives a negative answer to Problem 1.4 of Erdős, Hajnal, Simonovits, Sós, and Szemerédi [14]. It also complements the result of Sudakov [38], showing that the bound coming from the dependent random choice technique is close to optimal.

Theorem 1.10. *If $m = e^{-o\left((\log n/\log\log n)^{1/2}\right)} n$, then*

$$\mathbf{RT}(n, K_4, m) \geq (1/8 - o(1))\, n^2.$$

We summarize the results in the critical window in the following theorem. All of the bounds, except for the first result in the first part, which is due to Sudakov [38], are new. As before, we write $f(n) \ll g(n)$ to indicate that $f(n)/g(n) \to 0$ as $n \to \infty$.

Theorem 1.11. *We have the following estimates. Here c and c' are absolute constants.*

1. *If $m = e^{-\omega\left((\log n)^{1/2}\right)} n$, then $\mathbf{RT}(n, K_4, m) = o(n^2)$;*

   while if $m = e^{-o\left((\log n/\log\log n)^{1/2}\right)} n$, then $\mathbf{RT}(n, K_4, m) \geq (1/8 - o(1))\, n^2$.

2. *If $m = cn \cdot \frac{\log\log n}{\log n}$, then $\mathbf{RT}(n, K_4, m) \leq n^2/8$;*

   while if $m = c'n \cdot \frac{(\log\log n)^{3/2}}{(\log n)^{1/2}}$, then $\mathbf{RT}(n, K_4, m) \geq n^2/8$.

3. *If $\frac{(\log\log n)^{3/2}}{(\log n)^{1/2}} \cdot n \ll m \leq \frac{n}{3}$, we have*

$$\frac{n^2}{8} + \left(\frac{1}{3} - o(1)\right) mn \leq \mathbf{RT}(n, K_4, m) \leq \frac{n^2}{8} + \frac{3}{2} mn\,,$$

   where the constant $\frac{1}{3}$ can be replaced with $\frac{1}{2}$ in the range $m \ll n$.

Organization. In Section 2, we recall the standard proof of Theorem 1.1 using the regularity lemma. Our new proof has two main steps. First, we show that every K_4-free graph on n vertices with at least $n^2/8$ edges and small independence number must have a large cut, with very few non-crossing edges. Second, we show that having a large cut implies the desired Ramsey-Turán result.

For the first step we present two different approaches. The first approach, presented in Section 3, is conceptually simpler. Here we apply the regularity lemma with an absolute constant level of precision and then apply the stability result for triangle-free graphs to obtain a large cut that lets us obtain Theorem 1.6. The second approach, presented in Section 4, avoids using the regularity lemma completely, and leads to Theorem 1.5. Once we know that the maximum cut is large, we proceed to the second step, presented in Section 5, where we obtain either a K_4 or a large independent set. The conclusions of the proofs are found in Section 6. In Section 7 we prove Theorem 1.8 by introducing a new variant of the dependent random choice technique. In Section 8, we give a quantitative proof of the Bollobás-Erdős result, and use it to establish Theorem 1.10. In Section 9, we show how to modify the Bollobás-Erdős graph to get a slightly denser graph whose independence number is not much larger. We use this to establish Theorems 1.7 and 1.9. Finally, Section 10 contains some concluding remarks. Throughout this paper, all logarithms are base e unless otherwise indicated. For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial.

2 The standard regularity proof

In this section we recall the standard proof of Theorem 1.1. We reproduce the proof here because our proof of Theorem 1.6 starts the same way. We first need to properly state the regularity lemma,

which requires some terminology. The edge density $d(X,Y)$ between two subsets of vertices of a graph G is the fraction of pairs $(x,y) \in X \times Y$ that are edges of G. A pair (X,Y) of vertex sets is called *ϵ-regular* if for all $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have $|d(X',Y') - d(X,Y)| < \epsilon$. A partition $V = V_1 \cup \ldots \cup V_t$ is called equitable if $||V_i| - |V_j|| \leq 1$ for all i and j. The regularity lemma states that for each $\epsilon > 0$, there is a positive integer $M(\epsilon)$ such that the vertices of any graph G can be equitably partitioned $V(G) = V_1 \cup \ldots \cup V_t$ into $\frac{1}{\epsilon} \leq t \leq M(\epsilon)$ parts where all but at most ϵt^2 of the pairs (V_i, V_j) are ϵ-regular.

We next outline the standard proof of Theorem 1.1. We apply Szemerédi's regularity lemma to obtain a regular partition. The edge density between two parts cannot exceed $\frac{1}{2} + \epsilon$, or else we can find a K_4 or a large independent set. Then the reduced graph has density exceeding $\frac{1}{2}$, so by Mantel's theorem we can find three vertex sets pairwise giving dense regular pairs, from which we can obtain a K_4 or a large independent set. We follow this outline with a few simple lemmas leading to the detailed proof.

Lemma 2.1. *Let G be a K_4-free graph with independence number at most α. Let uv be an edge of G. Then u and v have at most α common neighbors.*

Proof. If we have an edge uv whose endpoints have codegree exceeding α, then there is an edge xy within the common neighborhood of u and v. This forms a K_4. □

Lemma 2.2. *Let $t, \gamma > 0$ satisfy $\gamma t \leq 1$. If G is a K_4-free graph on n vertices with independence number at most γn, and X and Y are disjoint vertex subsets of size n/t, then the edge density between X and Y is at most $\frac{1}{2} + \gamma t$.*

Proof. Let $A \subset X$ be the vertices with Y-degree greater than $\frac{n}{2t} + \frac{\gamma n}{2}$. If A contains an edge, then the endpoints of that edge will have neighborhoods in Y that overlap in more than γn vertices, contradicting the K_4-freeness of G by Lemma 2.1. Hence, A is an independent set and $|A|/|X| \leq \gamma t$. It follows that the edge density between X and Y is at most

$$\begin{aligned}\frac{|A|}{|X|} \cdot 1 + \left(1 - \frac{|A|}{|X|}\right) \cdot \frac{\frac{n}{2t} + \frac{\gamma n}{2}}{|Y|} &\leq (\gamma t)1 + (1 - \gamma t) \cdot \frac{\frac{n}{2t} + \frac{\gamma n}{2}}{n/t} \\ &= \gamma t + (1 - \gamma t) \cdot \frac{1}{2}(1 + \gamma t) \\ &< \frac{1}{2} + \gamma t\,. \end{aligned}$$

□

Lemma 2.3. *Suppose that X, Y, and Z are disjoint subsets of size m, and each of the three pairs are ϵ-regular with edge density at least 3ϵ. Then there is either a K_4 or an independent set of size at least $4\epsilon^2 m$.*

Proof. We may assume that $\epsilon < \frac{1}{3}$, as otherwise the given conditions are vacuous. By the regularity condition, at most an ϵ-fraction of the vertices of X fail to have Y-density at least 2ϵ, and at most ϵ-fraction fail to have Z-density at least 2ϵ. Select one of the other vertices $x \in X$, and let Y' and Z' be x's neighborhoods in Y and Z. At most an ϵ-fraction of the vertices of Y fail to have Z'-density at least 2ϵ, so among the vertices of Y', there are still at least ϵm of them that have Z'-density at least 2ϵ. Pick one such $y \in Y'$. Now x and y have at least $(2\epsilon)^2 m$ common neighbors

in Z, and that is either an independent set, or it contains an edge uv which forms a K_4 together with x and y. □

Now we recall the standard proof of Theorem 1.1 using the regularity lemma.

Proof of Theorem 1.1. Suppose we have a K_4-free graph G on n vertices with at least $(\frac{1}{8}+\epsilon)n^2$ edges. Let $\beta = \epsilon/6$, $M = M(\beta)$ be the bound on the number of parts for Szemerédi's regularity lemma with regularity parameter β, and $\delta = \epsilon^2/(9M)$. So M and δ^{-1} are at most a tower of height $\epsilon^{-O(1)}$. We apply Szemerédi's regularity lemma with regularity parameter β to get a regularity partition into $\frac{1}{\beta} \leq t \leq M$ parts. For clarity of presentation, we ignore floor signs here and assume all parts have exactly n/t vertices. At most $\beta t^2(n/t)^2 \leq \epsilon n^2/6$ edges go between pairs of parts which are not β-regular, and at most $\epsilon n^2/4$ edges go between parts which have edge density less than $\epsilon/2$ between them. The number of edges within individual parts is less than $t \cdot (n/t)^2/2 = n^2/(2t) \leq \beta n^2/2 = \epsilon n^2/12$. Thus, more than $\left(\frac{1}{8}+\frac{\epsilon}{2}\right)n^2$ edges of G go between pairs of parts which are β-regular and have edge density at least $\epsilon/2$ between them.

Since $\epsilon > \delta t$, by Lemma 2.2, if there is no independent set of size δn, then the edge density between each pair of parts in the regularity partition is less than $\frac{1}{2}+\epsilon$. Consider the t-vertex reduced graph R of the regularity partition, whose vertices are the parts of the regularity partition, and two parts are adjacent if the pair is β-regular and the edge density between them is at least $\epsilon/2$. As there are more than $\left(\frac{1}{8}+\frac{\epsilon}{2}\right)n^2$ edges between pairs of parts which form edges of R, and the edge density between each pair is less than $\frac{1}{2}+\epsilon$, the number of edges of R is more than $\left(\frac{1}{8}+\frac{\epsilon}{2}\right)n^2 \Big/ \left[\left(\frac{1}{2}+\epsilon\right)(n/t)^2\right] > \left(\frac{1}{4}+\frac{\epsilon}{4}\right)t^2$. By Mantel's theorem, R must contain a triangle. That is, the regularity partition has three parts each pair of which is β-regular and of density at least $\epsilon/2$. As $\beta = (\epsilon/2)/3$, Lemma 2.3 tells us that there is an independent set of size greater than $\frac{\epsilon^2}{9}\frac{n}{t} \geq \delta n$. □

We note that the above proof can be modified to use the weak regularity lemma by Frieze and Kannan [21, 22] to give a singly exponential dependence between δ and ϵ, i.e., $\delta = 2^{-\operatorname{poly}(\epsilon^{-1})}$. This observation was made jointly with David Conlon. Here is a rough sketch. We apply the weak regularity lemma with parameter $\beta = \operatorname{poly}(\epsilon)$ to obtain a weakly regular partition of the graph into $t \leq 2^{O(\beta^{-2})}$ parts, and let $\delta^{-1} = \operatorname{poly}(t/\epsilon)$. As before, no pair of parts can have density exceeding $\frac{1}{2}+\epsilon$, so the reduced graph has at least $\left(\frac{1}{4}+\frac{\epsilon}{16}\right)t^2$ edges. Using Goodman's triangle supersaturation result [23], there are at least $\Omega(\epsilon t^3)$ triangles in the reduced graph. Applying the triangle counting lemma associated to the weak regular partition [6] (i.e., counting lemma with respect to the cut norm) we see that G has at least $\Omega(\epsilon^4 n^3)$ triangles. We then conclude as before to show that G must contains a large independent set.

In each of the above proofs, we needed to apply a regularity lemma with the input parameter depending on ϵ, so the dependency of δ on ϵ is at the mercy of the regularity lemma, which cannot be substantially improved (see [9]). In the next section, we start a new proof where we only need to apply the regularity lemma with an absolute constant regularity parameter, so that we can obtain a very reasonable linear dependence between δ and ϵ. In Section 4 we provide an alternate approach which completely avoids the use of regularity.

3 Large cut via regularity lemma

Now we begin the proof of Theorem 1.6. It is conceptually easier than the regularity-free approach (Theorem 1.5), so we start with it. The proof follows the same initial lines as the original argument in Section 2, except that we only use as much regularity as we need to find a large cut. Importantly, the cut is deemed satisfactory once its size is within an absolute constant approximation factor of the true maximum cut, which asymptotically contains $1 - o(1)$ proportion of all of the graph's edges. We only need to apply the regularity lemma with a prescribed absolute constant level of precision, and this is key to developing the sharper dependence on the independence number.

We need the following stability version of Mantel's theorem to obtain our large cut.

Theorem 3.1 (Erdős [11], Simonovits [35])**.** *For every $\epsilon > 0$, there is a $\delta > 0$ such that every n-vertex triangle-free graph with more than $\left(\frac{1}{4} - \delta\right)n^2$ edges is within edit distance ϵn^2 from a complete bipartite graph.*

We use this result to obtain the following lemma.

Lemma 3.2. *For every $c > 0$ there is a $\gamma > 0$ such that every K_4-free graph G on n vertices with at least $\frac{n^2}{8}$ edges and independence number at most γn has a cut which has at most cn^2 non-crossing edges.*

Proof. Let ν be the δ produced by Theorem 3.1 when applied with $\frac{c}{2}$ as the input. Let $\epsilon = \min\left\{\frac{\nu}{7}, \frac{c}{6}\right\}$. We apply Szemerédi's regularity lemma to G with parameter ϵ, to find a partition of the vertex set into t parts of equal size, where all but at most ϵt^2 pairs of parts are ϵ-regular, and $\frac{1}{\epsilon} \leq t \leq M$. Importantly, M depends only on ϵ, and is completely independent of n. Let $\gamma = 4\epsilon^2/M$.

Let H be an auxiliary t-vertex graph where each vertex corresponds to one of the parts of the regularity partition. Place an edge between a pair of vertices in H if and only if the corresponding pair of parts is ϵ-regular with edge density greater than 3ϵ. The total number of edges of G not represented in H is at most

$$t\binom{n/t}{2} + \epsilon t^2 \left(\frac{n}{t}\right)^2 + \binom{t}{2}(3\epsilon)\left(\frac{n}{t}\right)^2 < \frac{n^2}{2t} + \epsilon n^2 + \frac{3}{2}\epsilon n^2 \leq 3\epsilon n^2 \,. \tag{1}$$

The first term came from the edges within individual parts of the regularity partition, the second term came from pairs that were not ϵ-regular, and the third term came from pairs that had density at most 3ϵ.

Let m be the number of edges of H. Lemma 2.2 bounds all pairwise densities by at most

$$\frac{1}{2} + \gamma t \leq \frac{1}{2} + \gamma M = \frac{1}{2} + 4\epsilon^2 \,.$$

Therefore, the number of edges in the original graph is at most

$$e(G) \leq m\left(\frac{1}{2} + 4\epsilon^2\right)\left(\frac{n}{t}\right)^2 + 3\epsilon n^2 \,.$$

Yet we assumed that G had at least $\frac{n^2}{8}$ edges, so dividing, we find that

$$\begin{aligned} m &\geq \frac{\frac{1}{8} - 3\epsilon}{\frac{1}{2} + 4\epsilon^2} \cdot t^2 \\ &> \left(\frac{1}{4} - 6\epsilon\right)\left(1 - 8\epsilon^2\right) t^2 \\ &> \left(\frac{1}{4} - 7\epsilon\right) t^2 \,. \end{aligned}$$

We may now appeal to the Erdős-Simonovits stability (Theorem 3.1), which by our choice of ϵ implies that H is within $\frac{ct^2}{2}$ edges of being complete bipartite. In particular, there is a cut of H which has at most $\frac{ct^2}{2}$ non-crossing edges. Consider the corresponding cut of G. Even if those non-crossing edges of H corresponded to pairs of full density, after adding (1) we find that the total number of non-crossing edges of G is at most

$$\frac{ct^2}{2}\left(\frac{n}{t}\right)^2 + 3\epsilon n^2 \leq cn^2 \,,$$

as desired. □

Now we deviate from the original regularity-based approach. Our next ingredient is a minimum-degree condition.

Lemma 3.3. *Let G be a graph with n vertices and m edges, and suppose there is a vertex with degree at most $\frac{m}{n}$. Delete the vertex from G, and let the resulting graph have $n' = n - 1$ vertices and m' edges. Then $\frac{m'}{n'} \geq \frac{m}{n}$.*

Proof. After deletion, the number of edges is $m' \geq m - \frac{m}{n} = m\left(1 - \frac{1}{n}\right) = m \cdot \frac{n-1}{n} = m \cdot \frac{n'}{n}$, and therefore $\frac{m'}{n'} \geq \frac{m}{n}$. □

Lemma 3.4. *Let G be an n-vertex graph with at least m edges. Then G contains an induced subgraph G' with $n' > 2m/n$ vertices, at least $n'\frac{m}{n}$ edges, and minimum degree at least $\frac{m}{n}$.*

Proof. Repeat the following procedure: as long as the graph contains a vertex v of degree at most $\frac{m}{n}$, remove v. Let G' be the resulting induced subgraph when this process terminates, and n' be the number of vertices of G'. Note that at the very beginning, the ratio of edges to vertices is at least $\frac{m}{n}$, and by Lemma 3.3, this ratio does not decrease in each iteration. Therefore, throughout the process, the ratio of the number of edges to the number of vertices is always at least $\frac{m}{n}$. Yet this ratio is precisely half of the average degree of the graph, which is less than the number of vertices of the graph, so we must have $n' > 2\frac{m}{n}$. Also, the number of edges of G' is at least $m - (n - n')\frac{m}{n} = n'\frac{m}{n}$. Finally, as no more vertices are deleted, G' has minimum degree at least $\frac{m}{n}$. □

At this point, we switch gears, and introduce our regularity-free approach, which will also reach this same point. After both approaches have arrived here, we will complete both proofs with the same argument.

4 Large cut without regularity

In this section, we assume the conditions of Theorem 1.5.

Lemma 4.1. *Theorem 1.5 is trivial unless* $\alpha \leq n/(2 \cdot 10^{10})$.

Proof. Theorem 1.5 assumes that the number of edges is at least $\frac{n^2}{8}+10^{10}\alpha n$. But if $\alpha > n/(2\cdot 10^{10})$, then this number already rises above $\binom{n}{2}$, and the theorem becomes vacuous because there are no graphs with that many edges. □

Lemma 4.2. *When we are proving Theorem 1.5, we may assume that all degrees are at least* $\frac{n}{4} + (10^{10} - 1)\alpha$, *or else we are already done.*

Proof. Let $C = 10^{10}$, so that we are proving that every graph with no K_4 and no independent set of size greater than α must contain fewer than $\frac{n^2}{8} + C\alpha n$ edges. We proceed by induction on n. Theorem 1.5 is trivial unless $\alpha \geq 1$, in which case $C\alpha n$ is already at least Cn. This exceeds $\binom{n}{2}$ for all $n \leq 2C$, so those serve as our base cases.

For the induction step, let G be a graph with at least $\frac{n^2}{8} + C\alpha n$ edges, and assume that the result is known for $n-1$. Suppose for the sake of contradiction that G has no K_4 or independent set of size greater than α. Let δ be its minimum degree, and delete its minimum degree vertex. The resulting graph also has no K_4 or independent set of size greater than α, so by the induction hypothesis,

$$e(G) - \delta < \frac{(n-1)^2}{8} + C\alpha(n-1)\,.$$

Yet we assumed that $e(G) \geq \frac{n^2}{8} + C\alpha n$. Combining these, we find that

$$\begin{aligned} \frac{n^2}{8} + C\alpha n - \delta &< \frac{(n-1)^2}{8} + C\alpha(n-1) \\ C\alpha + \frac{n}{4} - \frac{1}{8} &< \delta\,. \end{aligned}$$

Therefore, $\delta > \frac{n}{4} + (C-1)\alpha$. □

This strong minimum degree condition establishes that every neighborhood has size greater than $n/4$. The first step of our regularity-free approach associates a large set of neighbors to each vertex.

Definition 4.3. *For each vertex* v *in* G, *arbitrarily select a set of exactly* $n/4$ *neighbors of* v, *and call that set* N_v. *Define the remainder* R_v *to be the complement of* N_v.

Definition 4.4. *If a vertex* $u \in R_v$ *has density to* N_v *in the range* $[0.3, 0.34]$ *we say that* u ***trisects*** v.

The next lemma blocks an extreme case which would otherwise obstruct our proof.

Lemma 4.5. *Let* G *be a graph on* n *vertices with minimum degree at least* $n/4$. *Suppose that for every vertex* v, *all but at most* $0.03n$ *vertices of* R_v *trisect* v. *Then there is either a* K_4 *or an independent set of size at least* $n/1200$.

Proof. Assume for the contrary that G is K_4-free and the maximum independent set in G has size $\alpha < n/1200$. As the minimum degree is at least $n/4$ and G does not contain an independent set of this size, it must contain a triangle. Let abc be an arbitrary triangle in the graph. Define the three disjoint sets

$$\begin{aligned} N_a^* &= N_a \setminus (N_b \cup N_c)\,, \\ N_b^* &= N_b \setminus (N_a \cup N_c)\,, \\ N_c^* &= N_c \setminus (N_a \cup N_b)\,. \end{aligned}$$

Let $m = n/4$. Each of N_a, N_b, and N_c has size exactly m, and Lemma 2.1 ensures that their pairwise intersections are at most α. So, each of N_a^*, N_b^*, and N_c^* has size at least $m - 2\alpha$. At most $0.06n$ vertices of N_a^* fail to trisect either of b or c, so we may choose $v \in N_a^*$ which trisects both b and c.

Since we selected $v \in N_a^*$, it is adjacent to a, and therefore Lemma 2.1 implies that v has at most α neighbors in N_a^*. By above, there are still at least $m - 3\alpha$ non-neighbors of v in N_a^*, of which at most $0.09n$ fail to trisect any of v, b, or c. Therefore, we may now select $u \in N_a^*$ which is non-adjacent to v, and trisects each of v, b, and c.

Let $B = N_v \cap N_b^*$ and $C = N_v \cap N_c^*$. We will establish two claims: first, that N_u intersects B in more than $0.17m$ vertices, and second, that N_u intersects C in more than $0.17m$ vertices. This is a contradiction, because B and C are disjoint subsets of N_v, and the condition that u trisects v forces $|N_u \cap N_v| \leq 0.34m$. By symmetry between b and c, it suffices to prove only the first claim.

For this, suppose for the sake of contradiction that N_u intersects B in at most $0.17m$ vertices. Since u trisects b, u has at least $0.3m$ neighbors in N_b, hence at least $0.3m - 2\alpha$ neighbors in N_b^*, hence at least $0.13m - 2\alpha > 0.03n + \alpha$ neighbors in $N_b^* \setminus N_v$. (Here, we used $\alpha < \frac{n}{1200}$.) Of these, at most $0.03n$ fail to trisect v, and since the resulting number is more than α, there is an edge xy such that $x, y \in N_b^* \setminus N_v$, they both trisect v, and they both are adjacent to u.

Since x and y are adjacent, N_x and N_y overlap in at most α vertices by Lemma 2.1. Since they both trisect v, we conclude that $(N_x \cup N_y) \cap N_v$ has size at least $0.6m - \alpha$. Also by Lemma 2.1, all but at most 2α of these vertices lie outside N_b^*, because x and y are adjacent to b. Thus, we have already identified at least $0.6m - 3\alpha$ vertices of $N_v \setminus B$ that are adjacent to x or y. Yet u is adjacent to both x and y, so by Lemma 2.1, N_u can only include up to 2α of these vertices. Hence

$$\begin{aligned} |N_u \cap (N_v \setminus B)| &\leq |N_v \setminus B| - (0.6m - 3\alpha) + 2\alpha \\ &= (m - |B|) - (0.6m - 3\alpha) + 2\alpha\,. \end{aligned}$$

Since v trisects b, we must have $B = N_v \cap N_b^*$ of size at least $0.3m - 2\alpha$. Thus,

$$\begin{aligned} |N_u \cap (N_v \setminus B)| &\leq (0.7m + 2\alpha) - (0.6m - 3\alpha) + 2\alpha \\ &= 0.1m + 7\alpha\,. \end{aligned}$$

Since u trisects v, we must have $|N_u \cap N_v| \geq 0.3m$. Therefore, $|N_u \cap B| \geq 0.2m - 7\alpha$, which exceeds $0.17m$ because $\alpha < n/1200$. This establishes the claim, and completes the proof of this lemma. □

The next lemma is a simple averaging argument which will be useful in the lemma that follows.

Lemma 4.6. *Let $a_1, \ldots, a_m$ be a sequence of real numbers from $[0,1]$ whose average exceeds $1/3$. Suppose that at most 0.1% of them exceed 0.3334. Then at most 3% of them lie outside the range $[0.3, 0.34]$.*

Proof. On the contrary, if at least 2.9% of them fall below 0.3, then the average of the sequence is at most

$$0.029 \cdot 0.3 + 0.97 \cdot 0.3334 + 0.001 \cdot 1 = 0.333098 < \frac{1}{3},$$

because the maximum value is at most 1. □

Using the preceding two lemmas, we deduce the next lemma, which shows that if the independence number is small in a K_4-free graph, then there is a vertex v such that a substantial fraction of the vertices in R_v have density substantially larger than $1/3$ to N_v.

Lemma 4.7. *Suppose that $\alpha < n/1200$. In every K_4-free graph on n vertices with independence number at most α and minimum degree greater than $\frac{n}{4} + \alpha$, there exists a vertex v for which over 0.1% of the vertices of R_v have density greater than 0.3334 to N_v.*

Proof. For every vertex v, every $u \in N_v$ has at most α neighbors in N_v by Lemma 2.1. Then u has more than $n/4$ neighbors in R_v. In particular, the density of the bipartite subgraph between N_v and R_v is strictly greater than $1/3$. Therefore, by Lemma 4.6, each vertex v which fails the property produces a situation where all but 3% of the vertices of R_v trisect v. If this occurs for every vertex v, then we satisfy the main condition of Lemma 4.5. □

Lemma 4.8. *For any $c > 0$, the following holds with $C = \frac{9}{8c} + \frac{1}{2}$. Let G be an n-vertex graph with no K_4, in which all independent sets have size at most α, and suppose that $\alpha \le cn/3$. Let R be a subset of $3n/4$ vertices, and let $T \subset R$ have size cn. Suppose that every vertex of T has degree (in R) at least $\frac{n}{8} + C\alpha$. Then there is a subset $U \subset R$ (not necessarily disjoint from T) of size at least $n/4$ such that every vertex of U has more than α neighbors in T.*

Proof. Greedily pull out a matching from $G[T]$ of $\frac{cn}{3}$ edges. This is possible because $G[T]$ has independence number at most $\alpha \le \frac{cn}{3}$. Create an auxiliary bipartite graph H with two sides A and B as follows. Set $|A| = \frac{cn}{3}$, with one vertex for each of the matching edges. Let B be a copy of R. Place an edge between $a \in A$ and $b \in B = R$ whenever the vertex $b \in R$ is adjacent to at least one of the endpoints of the matching edge corresponding to a. Since every vertex in T has degree in R at least $\frac{n}{8} + C\alpha$, and no independent set of size larger than α, Lemma 2.1 implies that in the auxiliary bipartite graph H, every vertex of A has degree at least $\frac{n}{4} + (2C-1)\alpha$.

Let U contain all vertices of B that have degree (in H) greater than α. Since the sum of all degrees of B equals the sum of all degrees of A, this sum is at least $\left(\frac{cn}{3}\right)\left(\frac{n}{4} + (2C-1)\alpha\right)$. At the same time, it is also at most $|R \setminus U|\alpha + |U|\left(\frac{cn}{3}\right)$. Putting these together, we find that

$$(|R| - |U|)\alpha + |U|\left(\frac{cn}{3}\right) \ge \left(\frac{cn}{3}\right)\left(\frac{n}{4} + (2C-1)\alpha\right)$$

so that

$$\begin{aligned}
|U| &\geq \frac{\left(\frac{cn}{3}\right)\left(\frac{n}{4}+(2C-1)\alpha\right)-\left(\frac{3n}{4}\right)\alpha}{\frac{cn}{3}-\alpha} \\
&> \frac{\left(\frac{cn}{3}\right)\left(\frac{n}{4}+(2C-1)\alpha\right)-\left(\frac{3n}{4}\right)\alpha}{\frac{cn}{3}} \\
&= \frac{n}{4}+(2C-1)\alpha-\frac{9}{4c}\alpha \\
&= \frac{n}{4}.
\end{aligned}$$

Finally, from the definition of U, it follows that (in G) every vertex in U has at least α neighbors in T. □

Lemma 4.9. *In a graph, let L be a subset of vertices, and let xyz be a triangle. (The vertices x, y, and z each may or may not lie in L.) Suppose that the L-degrees of x, y, and z sum up to more than $|L|+3\alpha$. Then the graph contains a K_4 or an independent set of size greater than α.*

Proof. Let X, Y, and Z be the neighborhoods of x, y, and z within L. By inclusion-exclusion,

$$\begin{aligned}
|L| \geq |X \cup Y \cup Z| &\geq (|X|+|Y|+|Z|) - |X \cap Y| - |Y \cap Z| - |Z \cap X| \\
&> (|L|+3\alpha) - |X \cap Y| - |Y \cap Z| - |Z \cap X|\,.
\end{aligned}$$

Thus at least one of the pairwise intersections between X, Y, and Z exceeds α; without loss of generality, suppose it is between the L-neighborhoods of x and y. If this intersection is an independent set, then we have found an independent set of size greater than α. Otherwise, it spans an edge uv, and $xyuv$ forms a copy of K_4. □

Corollary 4.10. *In a K_4-graph with independence number at most α, let L and X be disjoint subsets of vertices. Suppose that every vertex of X has L-degree greater than $\frac{|L|}{3}+\alpha$. Then the induced subgraph on X has maximum degree at most α.*

Proof. Suppose a vertex $x \in X$ has more than α neighbors in X. This neighborhood cannot be an independent set, so it spans an edge yz. Now xyz is a triangle whose vertices have L-degree sum greater than $|L|+3\alpha$, and Lemma 4.9 completes the proof. □

Our next lemma establishes a major milestone toward constructing a cut which contains almost all of the edges. Such a bipartition spans few edges within each part, and the following lemma achieves this for one part.

Lemma 4.11. *For any $0 < c < \frac{1}{2}$, the following holds with $C = \frac{9 \cdot 10^5}{8c} + 1$. In a K_4-free graph on n vertices with minimum degree $\frac{n}{4} + C\alpha$ and independence number at most $\alpha \leq 10^{-5} \cdot \frac{cn}{3}$, there must exist a subset X of $\left(\frac{1}{2}-c\right)n$ vertices for which its induced subgraph has maximum degree at most α.*

Proof. Use Lemma 4.7 to select a vertex v for which over 0.1% of the vertices of R_v have density greater than 0.3334 to N_v. Let $L = N_v$ and let $R = R_v$. Let $c_1 = 10^{-5}c$, so that $C - \frac{1}{2}$ is the constant obtained from Lemma 4.8 with parameter c_1.

Each vertex $u \in L$ has degree at least $\frac{n}{4} + C\alpha$ by assumption, but by Lemma 2.1, since u is adjacent to v, at most α of this degree can go back to L. Therefore, every vertex of L has more than $|R|/3$ neighbors in R, which implies that the total number of edges between L and R exceeds $|L||R|/3$.

Let $A \subset R$ be the vertices of R whose L-degree exceeds $\frac{|L|+\alpha}{2}$. If $|A| > \alpha$, then A cannot be an independent set, so it induces an edge wx; each endpoint has L-degree greater than $\frac{|L|+\alpha}{2}$, so w and x have more than α common neighbors in L. That common neighborhood is too large to be an independent set, so it must induce an edge yz, and $wxyz$ forms a copy of K_4. Therefore, $|A| \leq \alpha$.

Every vertex of $R \setminus A$ has total degree at least $\frac{n}{4} + C\alpha$ by assumption, at most $\frac{|L|+\alpha}{2}$ of which goes to L by construction. At least $\frac{n}{8} + \left(C - \frac{1}{2}\right)\alpha$ remains within R. Let T be the $c_1 n$ vertices of $R \setminus A$ of highest L-degree. We may now apply Lemma 4.8 on R and T, and find $U \subset R$ of size exactly $\frac{n}{4} = \frac{|R|}{3}$, each of whose vertices has more than α neighbors in T.

Let x be the vertex in U with highest L-degree, and let its L-degree be a. Let b be the smallest L-degree of a vertex in T. Since $|T \cup A| \leq c_1 n + \alpha < 0.1\%|R|$, we must have

$$b > 0.3334|L|\,. \tag{2}$$

by the initial choice of v. Since x has more than α neighbors in T, its neighborhood in T spans an edge yz, forming a triangle xyz. The sum of the L-degrees of its vertices is at least $a + 2b$. If this exceeds $|L| + 3\alpha$, then we are already done by Lemma 4.9, so we may now assume that

$$a + 2b \leq |L| + 3\alpha\,. \tag{3}$$

To put α in perspective, note that our initial assumption on α translates into

$$\alpha \leq 10^{-5} \cdot \frac{cn}{3} = \frac{c_1 n}{3} = \frac{4}{3} \cdot c_1 |L|\,. \tag{4}$$

If U and T overlap at all, then we also have $a \geq b$, so inequality (2) then forces both $a, b > 0.3334|L|$. Combining this with inequality (3), we find that $0.0002|L| < 3\alpha$, and since $|L| = \frac{n}{4}$, we have $\frac{n}{60000} < \alpha$. This is impossible, because we assumed that $\alpha < 10^{-5} \cdot \frac{cn}{3}$, and $c < \frac{1}{2}$. Therefore, U and T are disjoint, and we may upper bound the sum of all L-degrees from R by

$$e(L, R) \leq |U|a + (|R| - |U|)b + |T \cup A|(|L| - b)\,. \tag{5}$$

This is because all vertices of U have L-degree at most a, and of the remaining vertices of R, only those in $T \cup A$ may have L-degree exceeding b; even then, all L-degrees are at most $|L|$. Simplifying this expression with $|U| = |R|/3$, $|T \cup A| \leq c_1 n + \alpha$, and inequalities (3) and (4), we find that the

total L-degree sum from R is

$$\begin{aligned} e(L,R) &\leq \left(\frac{|R|}{3}\right)a + \left(\frac{2|R|}{3}\right)b + (c_1 n + \alpha)(|L| - b) \\ &< \frac{|R|}{3}(a+2b) + (c_1 n + \alpha)|L| \\ &\leq \frac{|R|}{3}(|L| + 3\alpha) + (c_1 n + \alpha)|L| \\ &= \frac{|L||R|}{3} + \alpha(|L| + |R|) + c_1 n|L| \\ &= \frac{|L||R|}{3} + \alpha(n) + c_1 n|L| \\ &\leq \frac{|L||R|}{3} + \frac{7}{3}\cdot c_1 n|L|\,. \end{aligned}$$

Yet one of our first observations was that $e(L,R) > |L||R|/3$. Therefore, the total amount of slack in inequality (5) is at most $\frac{7}{3}\cdot c_1 n|L|$.

This is a very small gap. To take advantage of it, let S be the subset of vertices in $R \setminus (A \cup U)$ whose L-degree is at most $\frac{|L|}{3} + \alpha$. Note that inequalities (4) and (2) show that this bound is less than b, and hence S is entirely contained in $R \setminus (A \cup T \cup U)$. We may then sharpen inequality (5) to

$$e(L,R) \leq |U|a + (|R| - |U|)b + |T \cup A|(|L| - b) - |S|\left(b - \frac{|L|}{3} - \alpha\right). \tag{6}$$

In particular, the new summand cannot exceed the amount of slack we previously determined, and so

$$|S|\left(b - \frac{|L|}{3} - \alpha\right) < \frac{7}{3}\cdot c_1 n|L|.$$

Hence

$$|S| < \frac{7c_1 n|L|}{3b - |L| - 3\alpha} = \frac{7c_1 n}{\frac{3b}{|L|} - 1 - \frac{3\alpha}{|L|}}.$$

Combining this with inequalities (2) and (4), we conclude that

$$|S| < \frac{7c_1 n}{1.0002 - 1 - 4c_1} < \frac{7c_1 n}{1.0002 - 1 - 2\cdot 10^{-5}} = \frac{7\cdot 10^{-5}}{1.0002 - 1 - 2\cdot 10^{-5}}\cdot cn < 0.39cn\,,$$

and so if we define $X = R \setminus (U \cup S \cup A)$, the size of X is at least $\left(\frac{1}{2} - c\right)n$. Furthermore, every vertex of X has L-degree greater than $\frac{|L|}{3} + \alpha$, and so Corollary 4.10 implies that the induced subgraph $G[X]$ has all degrees at most α. □

Lemma 4.12. *For any $0 < c^* < \frac{2}{5}$, the following holds with $c = c^*/4$. Let G be a K_4-free graph on n vertices with independence number at most $\alpha < \frac{c^* n}{50}$ and minimum degree at least $\frac{n}{4}$. Suppose it has a set X of $\left(\frac{1}{2} - c\right)n$ vertices, which induces a subgraph of maximum degree at most α. Then in the max-cut of G, the total number of non-crossing edges is at most $c^* n^2$.*

Proof. We will use $c < \frac{1}{10}$. Let Y be the complement of X. It suffices to show that the total number of edges spanned within each of X and Y is at most $c^* n^2$, because the max-cut can only do better. Since $G[X]$ has maximum degree at most α, we clearly have $e(X) \leq \frac{\alpha|X|}{2}$.

By the minimum degree condition, each vertex of X must have degree at least $\frac{n}{4}$, and at most α of its neighbors can fall back in X. Therefore, the total number of edges from X to Y is already

$$e(X,Y) \geq \left(\frac{1}{2} - c\right) n \cdot \left(\frac{n}{4} - \alpha\right) > \frac{n^2}{8} - \frac{cn^2}{4} - \frac{\alpha n}{2}\,. \tag{7}$$

Let $A \subset Y$ be the vertices of Y which have more than $\frac{|X|+\alpha}{2}$ neighbors in X. As in the beginning of the proof of Lemma 4.11, we must have $|A| \leq \alpha$. Summing the X-degrees of the vertices in Y, we find that

$$\begin{aligned} e(X,Y) &\leq \alpha|X| + (|Y| - \alpha) \cdot \frac{|X| + \alpha}{2} \\ &= \alpha\left(\frac{1}{2} - c\right) n + \frac{1}{2}\left(\frac{n}{2} + cn - \alpha\right)\left(\frac{n}{2} - cn + \alpha\right) \\ &= \frac{\alpha n}{2} - c\alpha n + \frac{1}{2}\left(\frac{n^2}{4} - (cn - \alpha)^2\right) \\ &< \frac{\alpha n}{2} + \frac{n^2}{8} \end{aligned} \tag{8}$$

The amount of slack between the bounds for $e(X,Y)$ in (7) and (8) is at most $\alpha n + \frac{cn^2}{4}$.

Let S be the subset of vertices in Y whose X-degree is at most $\frac{|X|}{3} + \alpha$. Just as in the proof of Lemma 4.11, we may use our bound on the slack to control the size of S. Indeed, in our upper bound (8), we used a bound of at least $\frac{|X|+\alpha}{2}$ for every vertex of Y. Each vertex of S now reduces the bound of (8) by

$$\frac{|X|}{6} - \frac{\alpha}{2} \geq \frac{n}{15} - \frac{\alpha}{2} \geq \frac{n}{20}\,. \tag{9}$$

Here, we used $c < \frac{1}{10}$ to bound $|X| \leq 0.4n$, and $\alpha \leq \frac{n}{30}$. Therefore, the size of S is at most the slack divided by (9):

$$|S| \leq \left(\alpha n + \frac{cn^2}{4}\right) / \left(\frac{n}{20}\right) = 20\alpha + 5cn\,. \tag{10}$$

Using this, we may finally bound the number of edges in Y. The key observation is that Corollary 4.10 forces the induced subgraph on $Y \setminus S$ to have maximum degree at most α. Therefore, even if S were complete to itself and to the rest of Y,

$$e(Y) \leq \frac{\alpha(|Y| - |S|)}{2} + \frac{|S|^2}{2} + |S| \cdot (|Y| - |S|) < |Y||S| + \frac{\alpha|Y|}{2}\,.$$

Combining this with (10) and our initial bound on $e(X)$, we obtain

$$\begin{aligned} e(X) + e(Y) &< |Y||S| + \frac{\alpha n}{2} \\ &\leq (0.6n)(20\alpha + 5cn) + \frac{\alpha n}{2} \\ &= 3cn^2 + 12.5\alpha n \\ &\leq 4cn^2 = c^* n^2\,. \end{aligned}$$

Here, we used $c < \frac{1}{10}$ to bound $|Y| \leq 0.6n$, and $\alpha \leq \frac{c^* n}{50} = \frac{cn}{12.5}$. This completes the proof. □

5 Refinement of stability

Both arguments have now found very good cuts. In this section, we show how to finish the argument from this point.

Lemma 5.1. *Let G be a K_4-free graph on n vertices, at least $\frac{n^2}{8}$ edges,and independence number at most $\alpha \leq cn$. Suppose its vertices have been partitioned into $L \cup R$, and $e(L) + e(R) \leq cn^2$. Then $|L|$ and $|R|$ are both within the range $\left(\frac{1}{2} \pm \sqrt{3c}\right)n$.*

Proof. Without loss of generality, suppose that $|L| \leq |R|$, and let $|L| = \frac{n}{2} - l$. The same argument that yielded (8) implies that

$$\begin{aligned} e(L,R) &\leq \alpha|L| + (|R| - \alpha) \cdot \frac{|L| + \alpha}{2} \\ &= \alpha|L| + \frac{1}{2}\left(\frac{n}{2} + l - \alpha\right)\left(\frac{n}{2} - l + \alpha\right) \\ &< \alpha|L| + \frac{n^2}{8} - \frac{l^2}{2} + l\alpha \\ &= \frac{n^2}{8} - \frac{l^2}{2} + \frac{\alpha n}{2}. \end{aligned}$$

Combining this with the assumed lower bound on $e(G)$, assumed upper bound on $e(L)+e(R)$, and $\alpha \leq cn$, we find that

$$\frac{n^2}{8} \leq e(G) \leq cn^2 + \frac{n^2}{8} - \frac{l^2}{2} + \frac{\alpha n}{2},$$

and hence

$$\frac{l^2}{2} \leq \frac{3cn^2}{2},$$

and $l \leq \sqrt{3c} \cdot n$, as desired. □

The next result actually uses an extremely weak condition on the minimum degree. It leverages it by taking a max-cut, which has the nice property that every vertex has at least as many neighbors across the cut as on its own side. This local optimality property immediately translates the minimum degree condition to a minimum cross-degree condition, which is very useful. Although it may seem like we are re-using many of the techniques that we introduced for earlier parts of this proof, we are not re-doing the same work, because we are now proving properties for the max-cut, which *a priori* could be somewhat different from the partitions obtained thus far.

Lemma 5.2. *Let G be a K_4-free graph on n vertices with minimum degree at least cn and independence number at most $\alpha \leq \frac{cn}{36}$. Let $L \cup R$ be a max-cut with $\frac{n}{3} \leq |R| \leq \frac{2n}{3}$. Let $T \subset L$ be the vertices with R-degree greater than $\left(\frac{1}{2} - \frac{c}{8}\right)|R|$. Then every vertex of L has at most α neighbors in T.*

Proof. The minimum degree condition and the local optimality property of the max-cut implies that every vertex of L has R-degree at least $\frac{cn}{2} > \frac{c|R|}{2}$. Suppose that L contains a triangle which has at least two vertices in T. Then, the sum of the triangle's R-degrees would exceed

$$2\left(\frac{1}{2} - \frac{c}{8}\right)|R| + \frac{c|R|}{2} = |R| + \frac{c|R|}{4} \geq |R| + \frac{cn}{12} \geq |R| + 3\alpha\,.$$

This is impossible by Lemma 4.9.

Now, suppose for the sake of contradiction that some vertex $v \in L$ has more than α neighbors in T. This neighborhood is too large to be an independent set, and therefore it contains an edge with both endpoints in T. That edge, together with v, forms one of the triangles prohibited above. □

Lemma 5.3. *For any $0 < c < 1$, the following holds with $c' = c^2/800$. Assume $\alpha < cn/300$. Let G be a K_4-free graph on n vertices with at least $\frac{n^2}{8} + \frac{3\alpha n}{2}$ edges, and minimum degree at least cn. Suppose that the max-cut of G partitions the vertex set into $L \cup R$ such that $e(L) + e(R) \leq c'n^2$. Then G either has a copy of K_4, or an independent set of size greater than α.*

Proof. Assume for the sake of contradiction that G has no K_4 or independent sets larger than α. Let $A_L \subset L$ be the vertices whose R-degree exceeds $\frac{|R|+\alpha}{2}$, and let $A_R \subset R$ be the vertices whose L-degree exceeds $\frac{|L|+\alpha}{2}$. As in the beginning of the proof of Lemma 4.11, we must have $|A_L|, |A_R| \leq \alpha$.

Next, let $S_L \subset L$ be the vertices whose R-degree is at most $\left(\frac{1}{2} - \frac{c}{8}\right)|R|$, and let $S_R \subset R$ be the vertices whose L-degree is at most $\left(\frac{1}{2} - \frac{c}{8}\right)|L|$. We first show that S_L and S_R must be small. For this, we count crossing edges in two ways. If we add all R-degrees of vertices in L, and all L-degrees of vertices in R, then we obtain exactly $2e(L,R)$. Since $|A_L|, |A_R| \leq \alpha$, we can bound this sum by

$$\begin{aligned} 2e(L,R) \leq &\left[\alpha|R| + |S_L|\left(\frac{1}{2} - \frac{c}{8}\right)|R| + (|L| - \alpha - |S_L|)\frac{|R|+\alpha}{2}\right] \\ &+ \left[\alpha|L| + |S_R|\left(\frac{1}{2} - \frac{c}{8}\right)|L| + (|R| - \alpha - |S_R|)\frac{|L|+\alpha}{2}\right]. \end{aligned}$$

The first bracket simplifies to

$$\frac{|L||R|}{2} - |S_L|\left(\frac{c|R|}{8} + \frac{\alpha}{2}\right) + \alpha\left(|R| + \frac{|L|}{2} - \frac{|R|}{2}\right) - \frac{\alpha^2}{2}\,.$$

Since $c' < \frac{1}{300}$ and $\alpha < \frac{n}{300}$, Lemma 5.1 bounds $|R| > 0.4n$. Therefore, the first bracket is less than

$$\frac{|L||R|}{2} - |S_L|\left(\frac{cn}{20}\right) + \alpha \cdot \frac{|L| + |R|}{2} = \frac{|L||R|}{2} - |S_L|\left(\frac{cn}{20}\right) + \frac{\alpha n}{2}\,,$$

and similarly with the second bracket. Hence

$$e(L,R) < \frac{|L||R|}{2} - |S_L|\left(\frac{cn}{40}\right) - |S_R|\left(\frac{cn}{40}\right) + \frac{\alpha n}{2}\,. \tag{11}$$

On the other hand, we were given that $e(L) + e(R) \leq c'n^2$, while also $e(G) \geq \frac{n^2}{8} + \frac{3\alpha n}{2}$. Therefore, we must also have

$$e(L,R) \geq \frac{n^2}{8} - c'n^2 + \frac{3\alpha n}{2}\,. \tag{12}$$

Combining (12), (11), and $|L||R| \leq \frac{n^2}{4}$, we find that

$$|S_L|\left(\frac{cn}{40}\right) + |S_R|\left(\frac{cn}{40}\right) < c'n^2\,, \tag{13}$$

and in particular, both $|S_L|$ and $|S_R|$ are at most $\frac{40c'n}{c}$. Since we defined $c' = \frac{c^2}{800}$, we have

$$|S_L|, |S_R| < \frac{cn}{20}\,. \tag{14}$$

Finally, we derive more precise bounds on $e(L)$ and $e(R)$, and combine them with (11). We start with $e(L)$. By Lemma 5.2, every vertex of L can only send at most α edges to $L \setminus S_L$, so the number of edges that are incident to $L \setminus S_L$ is at most $|L|\alpha$. All remaining edges in L must have both endpoints in S_L, and even if they formed a complete graph there, their number would be bounded by $\frac{|S_L|^2}{2}$. Thus $e(L) < |L|\alpha + \frac{|S_L|^2}{2}$. Combining this with a similar estimate for $e(R)$, and with inequality (11), we find that

$$\begin{aligned} e(G) &< (|L| + |R|)\alpha + \frac{|S_L|^2}{2} + \frac{|S_R|^2}{2} + \frac{|L||R|}{2} - |S_L|\left(\frac{cn}{40}\right) - |S_R|\left(\frac{cn}{40}\right) + \frac{\alpha n}{2} \\ &= \frac{|L||R|}{2} + \frac{3\alpha n}{2} + \frac{|S_L|}{2}\left(|S_L| - \frac{cn}{20}\right) + \frac{|S_R|}{2}\left(|S_R| - \frac{cn}{20}\right). \end{aligned} \tag{15}$$

Inequality (14) shows that the quadratics in $|S_L|$ and $|S_R|$ are nonpositive. The maximum possible value of $\frac{|L||R|}{2}$ is $\frac{n^2}{8}$. This contradicts our given $e(G) \geq \frac{n^2}{8} + \frac{3\alpha n}{2}$, thereby completing the proof. □

6 Putting everything together

Now we finish the proofs by putting the parts together. Combining the results of Sections 3 and 5, we obtain Theorem 1.6 which involves an application of regularity with an absolute constant regularity parameter as input.

Proof of Theorem 1.6. Let γ be the result of feeding $c = \frac{1}{51200}$ into Lemma 3.2, and let $\gamma_0 = 4\gamma$. We are given an n-vertex graph with $m \geq \frac{n^2}{8} + \frac{3}{2}\alpha n$ edges, with no K_4 and with all independent sets of size at most α, where $\alpha < \gamma_0 n$. By Lemma 3.4, we may extract a subgraph G' on n' vertices which has at least $n'\frac{m}{n} \geq n'\left(\frac{n^2}{8} + \frac{3}{2}\alpha n\right)/n \geq \frac{(n')^2}{8} + \frac{3}{2}\alpha n'$ edges, no K_4, independence number at most $\alpha < \gamma n'$, and also minimum degree at least $\frac{m}{n} > \frac{n'}{8}$.

By Lemma 3.2, G' has a cut with at most $\frac{(n')^2}{51200}$ non-crossing edges. Finally, the minimum degree condition of $\frac{n'}{8}$ allows us to apply Lemma 5.3 with $c = \frac{1}{8}$, as $\frac{1}{51200} = \frac{(1/8)^2}{800}$ then is the corresponding c'. This completes the proof. □

Next, by combining the results of Sections 4 and 5, we prove Theorem 1.5 without any regularity at all.

Proof of Theorem 1.5. By Lemma 4.2, we may assume that the minimum degree is at least $\frac{n}{4} + (10^{10}-1)\alpha$. Lemma 4.1 lets us assume that $\alpha \le n/(2\cdot 10^{10})$. This satisfies the conditions of Lemma 5.3 with $c = \frac{1}{4}$, so it suffices to show that the max-cut leaves only at most $c'n^2 = \frac{(1/4)^2}{800}\cdot n^2 = \frac{n^2}{12800}$ crossing edges. To establish this, we use Lemma 4.12, with $c^* = \frac{1}{12800}$. This requires that $\alpha < \frac{c^* n}{50}$, which we have, as well as a sparse set X of $\big(\frac{1}{2} - c\big)n$ vertices, where $c = c^*/4 = \frac{1}{51200}$. This is provided by Lemma 4.11, which then requires that all degrees are at least $\frac{n}{4} + C\alpha$, with $C = \frac{9}{8}\cdot 51200\cdot 10^5 + 1 < 6\cdot 10^9$, as well as requiring that $\alpha \le 10^{-5}\cdot\frac{n}{3\cdot 51200}$. As $10^{-5}\cdot\frac{n}{3\cdot 51200} \approx 6.5\cdot 10^{-11}$, our bound from Lemma 4.1 is indeed sufficient. □

7 Dependent random choice

In this section, we use the good cut discovered by our constant-parameter regularity approach to find a pair of large disjoint sets of vertices which has density extremely close to $\frac{1}{2}$. Then, we introduce our variant of the dependent random choice technique, and use this to find a large independent set or a K_4.

Lemma 7.1. *For any constant $c > 0$, there is a constant $c' > 0$ such that the following holds. Suppose that $\alpha \le \frac{c^2 n}{1600}$. Let G be a graph with at least $\frac{n^2}{8}$ edges, minimum degree at least cn, no K_4, and independence number at most α. Suppose that the max-cut of G partitions the vertex set into $L \cup R$ such that $e(L) + e(R) \le c'n^2$. Then all of the following hold:*

(i) *Each of $|L|$ and $|R|$ are between $0.4n$ and $0.6n$.*

(ii) *At most α vertices of L have R-degree greater than $\frac{|R|+\alpha}{2}$.*

(iii) *At most α vertices of R have L-degree greater than $\frac{|L|+\alpha}{2}$.*

(iv) *Both induced subgraphs $G[L]$ and $G[R]$ have maximum degree at most $\big(\frac{120}{c} + 1\big)\alpha$.*

Proof. We may assume $c < 1$, or else there is nothing to prove. Let $c' = \frac{c^2}{3200}$. Now proceed exactly as in the proof of Lemma 5.3, and again obtain inequality (11). Note that along the way, parts (i)–(iii) are established. But after reaching (11), this time, we only know $e(G) \ge \frac{n^2}{8}$, so instead of (12), we now have

$$e(L,R) \ge \frac{n^2}{8} - c'n^2\,. \tag{16}$$

Combining (11) and (16), we obtain the following instead of (13):

$$|S_L|\left(\frac{cn}{40}\right) + |S_R|\left(\frac{cn}{40}\right) < c'n^2 + \frac{\alpha n}{2},$$

so

$$|S_L| < \frac{40c'n}{c} + \frac{20\alpha}{c} \le \frac{cn}{40}\,,$$

since $c' = \frac{c^2}{3200}$ and $\alpha \le \frac{c^2 n}{1600}$. Note that this is twice as strong as (14). The same argument as in the proof of Lemma 5.3 leads again to (15), which we copy here for the reader's convenience.

$$e(G) < \frac{|L||R|}{2} + \frac{3\alpha n}{2} + \frac{|S_L|}{2}\left(|S_L| - \frac{cn}{20}\right) + \frac{|S_R|}{2}\left(|S_R| - \frac{cn}{20}\right).$$

This time, we only have $e(G) \geq \frac{n^2}{8}$. As before, the maximum possible value of $\frac{|L||R|}{2}$ is $\frac{n^2}{8}$, so the nonpositive quadratics in $|S_L|$ and $|S_R|$ are permitted to cost us up to $\frac{3\alpha n}{2}$ of slack. However, as we established in (7) that $|S_L| < \frac{cn}{40}$, the value of $\left(\frac{cn}{20} - |S_L|\right)$ is between $\frac{cn}{40}$ and $\frac{cn}{20}$. Therefore, we must have

$$\frac{|S_L|}{2}\left(\frac{cn}{40}\right) \leq \frac{|S_L|}{2}\left(\frac{cn}{20} - |S_L|\right) < \frac{3\alpha n}{2}$$

and hence $|S_L| < \frac{120\alpha}{c}$.

By Lemma 5.2, every vertex of L has at most α neighbors in $L \setminus S_L$. Therefore, every vertex of L has at most $|S_L| + \alpha < \frac{120\alpha}{c} + \alpha$ neighbors in L. A similar argument holds in R, establishing part (iv) of this theorem, and completing the proof. □

Corollary 7.2. *There is an absolute constant γ_0 such that for every $\gamma < \gamma_0$, every n-vertex graph with at least $\frac{n^2}{8}$ edges, no copy of K_4, and independence number at most γn, has two disjoint subsets of vertices X and Y with $|X| \geq \frac{n}{16}$, $|Y| \geq \frac{n}{10}$, and where every vertex of X has Y-degree at least $\left(\frac{1}{2} - 20000\gamma\right)|Y|$.*

Proof. Let $\alpha = \gamma n$. We begin in the same way as in our proof of Theorem 1.6 in Section 6, except we use Lemma 7.1 instead of Lemma 5.3. Indeed, let γ_1 be the result of feeding $c = \frac{1}{204800}$ into Lemma 3.2, and let $\gamma_0 = \gamma_1/4$. We are given an n-vertex graph with at least $\frac{n^2}{8}$ edges, with no K_4 and with all independent sets of size at most α, where $\alpha < \gamma_0 n$. By Lemma 3.4, we may extract a subgraph G' on $n' \geq \frac{n}{4}$ vertices which has at least $\frac{(n')^2}{8}$ edges, no K_4, independence number at most $\gamma n < \gamma_1 n'$, and also minimum degree at least $\frac{n'}{8}$.

By Lemma 3.2, G' has a cut with at most $\frac{(n')^2}{204800}$ non-crossing edges. Finally, the minimum degree condition of $\frac{n'}{8}$ allows us to apply Lemma 7.1 with $c = \frac{1}{8}$, as $\frac{1}{204800} = \frac{(1/8)^2}{3200}$ then is the corresponding c'. This gives a bipartition $L \cup R$ of G'. Without loss of generality, assume that $|L| \geq |R|$, so that $|L| \geq \frac{n'}{2} \geq \frac{n}{8}$. Part (i) of that Lemma gives $|R| \geq 0.4n' \geq \frac{n}{10}$. Part (iv) establishes that all degrees in $G[L]$ and $G[R]$ are at most $\left(\frac{120}{1/8} + 1\right)\alpha = 961\alpha$. Hence

$$e(L,R) \geq \frac{(n')^2}{8} - \frac{961\alpha n'}{2} \geq \frac{|L||R|}{2} - \frac{961\alpha n'}{2} \geq \frac{|L||R|}{2} - 961\alpha|L|\,. \tag{17}$$

By part (ii), at most α vertices of L can have R-degree greater than $\frac{|R|+\alpha}{2}$. Let $Y = R$, and let $X \subset L$ be the vertices that have R-degree at least $\frac{|R|}{2} - 1923.5\alpha$. We claim that $|X| \geq \frac{|L|}{2}$. Indeed, if this were not the case, then by summing up the R-degrees of the vertices of L, we would find

$$\begin{aligned} e(L,R) &\leq \frac{|L|}{2}\left(\frac{|R|}{2} - 1923.5\alpha\right) + \left(\frac{|L|}{2} - \alpha\right)\left(\frac{|R|}{2} + 0.5\alpha\right) + \alpha|R| \\ &= \frac{|L||R|}{2} - 961.5\alpha|L| + \frac{\alpha|R|}{2} - \frac{\alpha^2}{2} \\ &< \frac{|L||R|}{2} - 961\alpha|L|\,, \end{aligned}$$

contradicting (17). Thus $|X| \geq \frac{|L|}{2} \geq \frac{n'}{4} \geq \frac{n}{16}$. Finally, observe that since $\alpha = \gamma n$ and $|R| \geq \frac{n}{10}$ as noted above, we have

$$1923.5\alpha = 1923.5\gamma n \leq 19235\gamma|R|\,,$$

and so every vertex of X indeed has Y-degree at least $\left(\frac{1}{2} - 20000\gamma\right)|Y|$. □

We will present two proofs of Theorem 1.8 that every sufficiently large graph with more than $\frac{n^2}{8}$ edges contains either a copy of K_4, or an independent set of order $\Omega\left(n \cdot \frac{\log\log n}{\log n}\right)$. The first proof is shorter. However, the second proof introduces a new twist of the Dependent Random Choice technique, which may find applications elsewhere.

The *odd girth* of a graph is the length of the shortest odd cycle in the graph. Both proofs start by applying Corollary 7.2 with $\gamma = c\frac{\log\log n}{\log n}$, where $c > 0$ is an absolute constant. In the first proof, we take $c = 10^{-6}$, and show that the induced subgraph on X has odd girth at least $1/(40020\gamma)$. Together with a lemma of Shearer [34], which implies that every graph on n vertices with odd girth $2k+3$ has independence number at least $\frac{1}{2}n^{1-1/k}$, we obtain the desired result.

To prove a lower bound on the odd girth of the subgraph induced on X, consider first a walk $v_1, \ldots, v_t$ in X, and let Y_i denote the set of neighbors of v_i in Y. Since consecutive vertices on the path are adjacent, Lemma 2.1 implies that $|Y_i \cap Y_{i+1}| \leq \gamma n \leq 10\gamma|Y|$. Roughly, since the cardinality of each Y_i is almost at least half the order of Y, this forces Y_i, Y_j to be nearly complementary if $j - i$ is odd. We next make this claim rigorous. Let $\gamma_k = (10 + 40020(k-1))\gamma$. We will show by induction on k that

$$|Y_i \cap Y_{i+2k-1}| \leq \gamma_k|Y|. \tag{18}$$

holds for each positive integer k. The base $k = 1$ is satisifed as $|Y_i \cap Y_{i+1}| \leq \gamma n \leq \gamma_1|Y|$.

The induction hypothesis is $|Y_i \cap Y_{i+2k-1}| \leq \gamma_k|Y|$. It follows that

$$|(Y_i \cup Y_{i+2k}) \cap Y_{i+2k-1}| \leq |Y_i \cap Y_{i+2k-1}| + |Y_{i+2k-1} \cap Y_{i+2k}| \leq \gamma_k|Y| + \gamma n \leq (\gamma_k + 10\gamma)\,|Y|$$

and

$$\begin{aligned} |Y| &\geq |Y_i \cup Y_{i+2k} \cup Y_{i+2k-1}| \geq |Y_i \cup Y_{i+2k}| + |Y_{i+2k-1}| - |(Y_i \cup Y_{i+2k}) \cap Y_{i+2k-1}| \\ &\geq |Y_i \cup Y_{i+2k}| + \left(\frac{1}{2} - 20000\gamma\right)|Y| - (\gamma_k + 10\gamma)\,|Y|. \end{aligned}$$

We conclude that

$$|Y_i \cup Y_{i+2k}| \leq \left(\frac{1}{2} + 20010\gamma + \gamma_k\right)|Y|.$$

Finally, we have

$$\begin{aligned} |Y_i \cap Y_{i+2k+1}| &\leq |Y_i \setminus Y_{i+2k}| + |Y_{i+2k} \cap Y_{i+2k+1}| = |Y_i \cup Y_{i+2k}| - |Y_{i+2k}| + |Y_{i+2k} \cap Y_{i+2k+1}| \\ &\leq \left(\frac{1}{2} + 20010\gamma + \gamma_k\right)|Y| - \left(\frac{1}{2} - 20000\gamma\right)|Y| + \gamma n \leq (40020\gamma + \gamma_k)\,|Y| \\ &= \gamma_{k+1}|Y|. \end{aligned}$$

This completes the claimed inequality (18) by induction on k. Now suppose the graph has odd girth $2k-1$. So there is a closed walk of that length from a vertex to itself, in which case $Y_{i+2k-1} = Y_i$. Hence, we must have

$$\left(\frac{1}{2} - 20000\gamma\right)|Y| \leq |Y_i| = |Y_{i+2k-1} \cap Y_i| \leq (10 + 40020(k-1))\gamma|Y|$$

This implies the odd girth $2k-1$ must satisfy $2k-1 \geq 1/(40020\gamma)$. This completes the first proof of Theorem 1.8.

We next present the second proof of Theorem 1.8, starting again with Corollary 7.2 . We use a twist of the Dependent Random Choice technique to find either a K_4 or a large independent set in $G[X \cup Y]$. The traditional technique is to select a random subset $T \subset Y$ by sampling t vertices of Y uniformly at random, with replacement, and then to define $U \subset X$ as those vertices that are adjacent to every single vertex of T. Straightforward analysis establishes the following lemma, which was used in this precise setting by Sudakov [38] to prove a lower bound of $ne^{-O(\sqrt{\log n})}$ for this Ramsey-Turán problem.

Lemma 7.3 (As formulated in [20])**.** *For every n, d, s, and k, every n-vertex graph with average degree d contains a subset U of at least*

$$\max\left\{\frac{d^t}{n^{t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t : t \in \mathbb{Z}^+\right\}$$

vertices, such that every subset $S \subset U$ of size s has at least k common neighbors.

The significance of this lemma is that if it is applied to $G[X \cup Y]$ with $s = 2$ and a suitably chosen t, one immediately finds a moderately sized subset U from which every pair of vertices has many common neighbors. Then, either U is an independent set, or it contains an edge uv. The common neighborhood of u and v is now guaranteed to be large, and it is either an independent set, or it contains an edge, creating a K_4. It is worth noting that this approach works even if the density between X and Y is only bounded away from zero by an arbitrarily small constant. However, the lower bound on the independence number that it gives is only $ne^{-\Theta(\sqrt{\log n})}$.

Yet one might suspect that there is room for improvement, because, for example, if every vertex of X had Y-degree greater than $\left(\frac{1}{2}+5\gamma\right)|Y| \geq \frac{|Y|+\alpha}{2}$, then it is already even guaranteed that *every* pair of vertices in X has common neighborhood larger than α, finishing the argument outright. Our minimum Y-degree condition is very close, at $\left(\frac{1}{2}-20000\gamma\right)|Y|$.

It turns out that we can indeed capitalize on this, by adjusting the Dependent Random Choice procedure. We will still sample t vertices of Y with replacement, but this time we will place a vertex $u \in X$ into U if and only if at least $\left(\frac{1}{2}+\epsilon\right)t$ of the sampled vertices are adjacent to u. Relaxing our common adjacency requirement from t to just over half of t allows us to take many more vertices into U. In order to analyze this procedure, we will use the usual Chernoff upper bounds on large Binomial deviations, but we will also need lower bounds on Binomial tail probabilities. The second type guarantees "dispersion," in addition to the usual "concentration."

Lemma 7.4. *For any constant $C > 0$, there are $\epsilon_0 > 0$ and $n_0 < \infty$ such that the following holds for all $\epsilon < \epsilon_0$ and $n > n_0$:*

$$\mathbb{P}\left[\mathrm{Bin}\left(n, \frac{1}{2} - C\epsilon\right) \geq \left(\frac{1}{2}+\epsilon\right)n\right] > \frac{\epsilon\sqrt{n}}{2} \cdot e^{-n\epsilon^2(4C^2+20C+16)}.$$

Proof. Throughout, we will implicitly assume that n is large and ϵ is small. Define

$$p_i = \binom{n}{i}\left(\frac{1}{2}-C\epsilon\right)^i\left(\frac{1}{2}+C\epsilon\right)^{n-i}.$$

Observe that

$$\frac{\binom{n}{i+1}}{\binom{n}{i}} = \frac{\frac{n!}{(i+1)!(n-i-1)!}}{\frac{n!}{i!(n-i)!}} = \frac{n-i}{i+1},$$

and therefore

$$\frac{p_{i+1}}{p_i} = \frac{n-i}{i+1}\cdot\frac{\frac{1}{2}-C\epsilon}{\frac{1}{2}+C\epsilon}.$$

Since the following inequalities are equivalent:

$$\begin{aligned}(n-i)\left(\frac{1}{2}-C\epsilon\right) &\le (i+1)\left(\frac{1}{2}+C\epsilon\right)\\ n\left(\frac{1}{2}-C\epsilon\right)-\left(\frac{1}{2}+C\epsilon\right) &\le i\left[\left(\frac{1}{2}+C\epsilon\right)+\left(\frac{1}{2}-C\epsilon\right)\right] = i,\end{aligned}$$

we know that in particular, p_i is a decreasing sequence for all $i \ge \frac{n}{2}$. Hence

$$\begin{aligned}\mathbb{P}\left[\mathrm{Bin}\left(n,\frac{1}{2}-C\epsilon\right)\ge\left(\frac{1}{2}+\epsilon\right)n\right] &> \epsilon n p_{(\frac{1}{2}+2\epsilon)n}\\ &= \epsilon n\binom{n}{(\frac{1}{2}+2\epsilon)n}\left(\frac{1}{2}-C\epsilon\right)^{(\frac{1}{2}+2\epsilon)n}\left(\frac{1}{2}+C\epsilon\right)^{(\frac{1}{2}-2\epsilon)n}.\end{aligned}$$

By Stirling's formula,

$$\binom{n}{(\frac{1}{2}+2\epsilon)n} = (1+o(1))\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(\frac{1}{2}+2\epsilon)n}\left(\frac{(\frac{1}{2}+2\epsilon)n}{e}\right)^{(\frac{1}{2}+2\epsilon)n}\sqrt{2\pi(\frac{1}{2}-2\epsilon)n}\left(\frac{(\frac{1}{2}-2\epsilon)n}{e}\right)^{(\frac{1}{2}-2\epsilon)n}},$$

so our Binomial probability is at least

$$\mathbb{P} > \frac{\epsilon\sqrt{n}}{2}\cdot e^{n\left[\left(\frac{1}{2}+2\epsilon\right)\log(1-2C\epsilon)+\left(\frac{1}{2}-2\epsilon\right)\log(1+2C\epsilon)-\left(\frac{1}{2}+2\epsilon\right)\log(1+4\epsilon)-\left(\frac{1}{2}-2\epsilon\right)\log(1-4\epsilon)\right]}.$$

Since $\log(1+x)\le x$, we have

$$\left(\frac{1}{2}+2\epsilon\right)\log(1+4\epsilon)+\left(\frac{1}{2}-2\epsilon\right)\log(1-4\epsilon) \le \left(\frac{1}{2}+2\epsilon\right)(4\epsilon)+\left(\frac{1}{2}-2\epsilon\right)(-4\epsilon) \le 16\epsilon^2.$$

Also, since $\log(1-x) > -2x$ for all sufficiently small positive x, we have

$$\begin{aligned}\left(\frac{1}{2}+2\epsilon\right)\log(1-2C\epsilon)+\left(\frac{1}{2}-2\epsilon\right)\log(1+2C\epsilon) &= \frac{1}{2}\log(1-4C^2\epsilon^2)+2\epsilon\log\frac{1-2C\epsilon}{1+2C\epsilon}\\ &> -4C^2\epsilon^2+2\epsilon\log(1-5C\epsilon)\\ &> -4C^2\epsilon^2-20C\epsilon^2.\end{aligned}$$

This completes the proof. □

We are now ready to prove that every sufficiently large graph with more than $\frac{n^2}{8}$ edges contains either a copy of K_4, or an independent set of size $\Omega\big(n \cdot \frac{\log\log n}{\log n}\big)$.

Proof of Theorem 1.8. Define

$$C = 2000\,, \quad K = 4C^2 + 20C + 16\,, \quad \gamma = \frac{\log\log n}{200K\log n}\,, \quad t = \frac{200K\log^2 n}{\log\log n}\,, \quad \text{and} \quad \epsilon = 10\gamma\,.$$

We will show that as long as n is sufficiently large, there must be an independent set larger than γn. Assume for the sake of contradiction that there is no K_4 and all independent sets have size at most γn. By Corollary 7.2, there are disjoint subsets of vertices A and B with $|A| \geq \frac{n}{16}$, $|B| \geq \frac{n}{10}$, where every vertex of A has B-degree at least $\big(\frac{1}{2} - C\epsilon\big)|B|$.

Select a random multiset T of t vertices of B by independently sampling t vertices uniformly at random. Let $U_0 \subset A$ be those vertices who each have at least $\big(\frac{1}{2} + \epsilon\big)t$ neighbors in T. Note that different vertices of U_0 are permitted to have different neighborhoods in T. Next, for each pair of vertices of U_0 which has at most $\epsilon|B|$ common neighbors in B, remove one of the vertices, and let U be the resulting set.

We claim that U must be an independent set. Indeed, if not, then there is an edge in U, whose endpoints must have more than $\epsilon|B| \geq \gamma n$ common neighbors, and hence their common neighborhood contains an edge, which creates a K_4. It remains to show that U can be large. Define the random variable $X = |U_0|$, and let Y be the number of pairs of vertices in U_0 that have at most $\epsilon|B|$ common neighbors in B.

We start by estimating $\mathbb{E}[Y]$. Let $u, v \in A$ be a pair of vertices whose common neighborhood in B has size at most $\epsilon|B|$. The only way in which they could both enter U_0 is if both u and v had at least $\big(\frac{1}{2} + \epsilon\big)t$ elements of T in their neighborhoods. In particular, this requires that at least $2\epsilon t$ elements of T fell in their common neighborhood. Since elements of T are sampled uniformly from B with replacement, the probability of this is at most

$$\mathbb{P}\left[\mathrm{Bin}\,(t, \epsilon) \geq 2\epsilon t\right] < e^{-\frac{1}{3}t\epsilon} < e^{-3\gamma t}\,.$$

Here, we used the well-known Chernoff bound (see, e.g., Appendix A of the book [1] for a reference). Therefore, by linearity of expectation,

$$\mathbb{E}[Y] < n^2 e^{-3\gamma t} = \frac{1}{n}\,.$$

Next, we move to estimate $\mathbb{E}[X]$. Since every vertex of A has B-degree at least $\big(\frac{1}{2} - C\epsilon\big)|B|$, the probability that a particular vertex of A is selected for U_0 is at least

$$\mathbb{P}\left[\mathrm{Bin}\left(t, \frac{1}{2} - C\epsilon\right) \geq \left(\frac{1}{2} + \epsilon\right)t\right] > \frac{\epsilon\sqrt{t}}{2} \cdot e^{-t\epsilon^2 K} = \frac{\epsilon\sqrt{t}}{2} \cdot e^{-100K\gamma^2 t}\,,$$

by Lemma 7.4. As $|A| \geq \frac{n}{16}$, linearity of expectation gives

$$\mathbb{E}[X] > \frac{n}{16} \cdot \frac{\epsilon\sqrt{t}}{2} \cdot e^{-100K\gamma^2 t} = \frac{n}{16} \cdot 5\gamma \cdot \sqrt{\frac{200K\log^2 n}{\log\log n}} \cdot e^{-\frac{1}{2}\log\log n}\,,$$

which has higher order than γn. Therefore, a final application of linearity of expectation gives $\mathbb{E}[X - Y] > \gamma n$, and hence there is an outcome of our random sampling which produces $|U| \geq X - Y > \gamma n$, so U is too large to be an independent set, a contradiction. □

8 Quantitative bounds on the Bollobás-Erdős construction

Recall that Bollobás and Erdős [5] constructed a K_4-free graph on n vertices with $(1-o(1))\frac{n^2}{8}$ edges with independence number $o(n)$. The various presentations of the proof of this result in the literature [3], [4], [5], [14], [15], [36] do not give quantitative estimates on the little-o terms. In this section, we present the proof with quantitative estimates. It shows that the Bollobás-Erdős graph gives a good lower bound for the Ramsey-Turán numbers in the lower part of the critical window, nearly matching the upper bounds established using dependent random choice. The presentation here closely follows the proof sketched in [36]. The next result is the main theorem of this section, which gives the quantitative estimates for the Bollobás-Erdős construction. Call a graph $G=(V,E)$ on n vertices *nice* if it is K_4-free and there is a bipartition $V=X\cup Y$ into parts of order $n/2$ such that each part is K_3-free.

Theorem 8.1. *There exists some universal constant $C>0$ such that for every $0<\epsilon<1$, positive integer $h\geq 16$ and even integer $n\geq (C\sqrt{h}/\epsilon)^h$, there exists a nice graph on n vertices, with independence number at most $2ne^{-\epsilon\sqrt{h}/4}$, and minimum degree at least $(1/4-2\epsilon)n$.*

This graph comes from the Bollobás-Erdős construction, which we describe now. Let $\mu=\epsilon/\sqrt{h}$. Feige and Schechtman [17] show that, for every even integer $n\geq (C/\mu)^h$, the unit sphere $\mathbb{S}^{h-1}$ in $\mathbb{R}^h$ can be partitioned into $n/2$ pieces $D_1,\dots,D_{n/2}$ of equal measure so that each piece has diameter at most $\mu/4$.[2] Choose a vertex $x_i\in D_i$ and an $y_i\in D_i$ for each i. Let $X=\{x_1,\dots,x_{n/2}\}$ and $Y=\{y_1,\dots,y_{n/2}\}$. Construct the graph $BE(n,h,\epsilon)$ on vertex set $X\cup Y$ as follows:

(a) Join $x_i\in X$ to $y_j\in Y$ if $|x_i-y_j|<\sqrt{2}-\mu$.

(b) Join $x_i\in X$ to $x_j\in X$ if $|x_i-x_j|>2-\mu$.

(c) Join $y_i\in Y$ to $y_j\in Y$ if $|y_i-y_j|>2-\mu$.

Theorem 8.1 then follows from next four claims.

Claim 8.2. *The subsets X and Y both induce triangle-free subgraphs in $BE(n,h,\epsilon)$.*

Claim 8.3. *The graph $BE(n,h,\epsilon)$ is K_4-free.*

Claim 8.4. *The independence number of the graph $BE(n,h,\epsilon)$ is at most $2ne^{-\epsilon\sqrt{h}/4}$.*

Claim 8.5. *The minimum degree of the graph $BE(n,h,\epsilon)$ is at least $(1/4-2\epsilon)n$.*

Proof of Claim 8.2. Suppose $x_i,x_j,x_k\in X$ form a triangle. Then

$$0\leq |x_i+x_j+x_k|^2=9-|x_i-x_j|^2-|x_i-x_k|^2-|x_j-x_k|^2<9-3(2-\mu)^2<0\,,$$

which is a contradiction. So the subgraph induced by X is triangle-free, and similarly with Y. □

[2] Lemma 21 in their paper states this for a single value of n_0, but their proof actually shows it for all $n\geq n_0$.

Proof of Claim 8.3. By Claim 8.2, any K_4 must come from four vertices $x, x' \in X$ and $y, y' \in Y$, and if they do form a K_4, then

$$\begin{aligned}
0 &\leq |x + x' - y - y'|^2 \\
&= |x-y|^2 + |x-y'|^2 + |x'-y|^2 + |x'-y'|^2 - |x-x'|^2 - |y-y'|^2 \\
&< 4(\sqrt{2}-\mu)^2 - 2(2-\mu)^2 \\
&= (2\mu + 8 - 8\sqrt{2})\mu \\
&< 0\,,
\end{aligned}$$

which is impossible. □

The following isoperimetric theorem on the sphere shows that, of all subsets of the sphere of a given diameter, the cap has the largest measure. It plays a crucial role in the proof, as any independent set which is a subset of X or Y will have diameter at most $2-\mu$.

Theorem 8.6 (Schmidt [33], see also [36])**.** *Let $\ell \in [0,2]$ and h be a positive integer. If $A \subset \mathbb{S}^{h-1}$ is an arbitrary measurable set with diameter at most ℓ and B a spherical cap in $\mathbb{S}^{h-1}$ with diameter ℓ, then $\lambda(A) \leq \lambda(B)$ (where λ is the Lebesgue measure).*

We have the following corollary of this theorem and a standard concentration of measure inequality for spherical caps (see, e.g., Lemma 2.2 in [2]).

Corollary 8.7 (Corollary 30 in [36])**.** *Let $\mu \in [0,1)$. If $A \subset \mathbb{S}^{h-1}$ is any measurable set with diameter at most $2-\mu$, then $\lambda(A) \leq 2e^{-\mu h/2}$ (here λ is normalized to 1 on $\mathbb{S}^{h-1}$).*

Proof of Claim 8.4. We show that the largest independent set contained in each of X and Y has size at most $ne^{-\epsilon\sqrt{h}/4}$. Let $X_I \subset X$ be an independent set. Then the diameter of X_I on $\mathbb{S}^{h-1}$ is at most $2-\mu$. Let $D_I = \bigcup_{i \in X_I} D_i$. Since the regions D_i all have diameter at most $\mu/4$, the diameter of D_I is at most $2-\mu/2$. By Corollary 8.7 we have $\lambda(D_I) \leq 2e^{-\mu h/4}$. Since $\lambda(D_I) = |X_I|/(n/2)$, we have $|X_I| \leq ne^{-\mu h/4}$. □

The next lemma gives a lower bound on the measure of spherical caps.

Lemma 8.8. *Let $h \geq 5$ be positive integer, and $\epsilon > 0$. Let B be the spherical cap in $\mathbb{S}^{h-1}$ consisting of all points with distance at most $\sqrt{2} - \frac{\epsilon}{\sqrt{h}}$ from some fixed point. Then $\lambda(B) \geq \frac{1}{2} - \sqrt{2}\epsilon$.*

Proof. Let $\delta = \epsilon\sqrt{2/h} - \epsilon^2/(2h)$ so that $(1-\delta)^2 + (1-\delta^2) = (\sqrt{2} - \epsilon/\sqrt{h})^2$, and thus B can be taken to be $\mathbb{S}^{h-1} \cap \{x_1 \geq \delta\}$. Let A be the intersection of the (h-dimensional) unit ball with the cone determined by the origin and the boundary $\mathbb{S}^{h-1} \cap \{x_1 = \delta\}$ of our spherical cap. Note that A is the convex hull of the origin and the spherical cap B. Since we have normalized the total surface area of the sphere to be 1, and the cone contains the same fraction of each concentric sphere around the origin, the surface area of this spherical cap B is precisely the ratio between the (h-dimensional) volumes of A and the entire unit ball. We may lower bound this by replacing A with the simpler intersection of the unit ball and the half-space $x_1 \geq \delta$. It therefore suffices to show that the volume

of the part of the unit ball within the slice $0 \le x_1 \le \delta$ is at most $\sqrt{2}\epsilon$ times the volume of the entire unit ball. This final ratio is exactly

$$\frac{\int_0^\delta \left(\sqrt{1-x^2}\right)^{h-1}\ dx}{\int_{-1}^1 \left(\sqrt{1-x^2}\right)^{h-1}\ dx},$$

because the $(h-1)$-dimensional intersection between any hyperplane $x_1 = c$ and the unit ball is always an $(h-1)$-dimensional ball, whose measure is an absolute constant multiplied by its radius to the $(h-1)$-st power, and the constants cancel between the numerator and denominator.

The numerator is at most δ. Using $1 - t \ge e^{-2t}$ for $t \in [0, 1/2]$, we see that the denominator is at least

$$\int_{-1/2}^{1/2} e^{-(h-1)x^2}\ dx = \frac{1}{\sqrt{h-1}} \int_{-\sqrt{h-1}/2}^{\sqrt{h-1}/2} e^{-x^2}\ dx \ge \frac{1}{\sqrt{h}} \int_{-1}^{1} e^{-x^2}\ dx > \frac{1}{\sqrt{h}}.$$

Thus

$$\lambda(\mathbb{S}^{h-1} \cap \{0 \le x_1 \le \delta\}) \le \frac{\delta}{1/\sqrt{h}} \le \sqrt{2}\epsilon. \qquad \square$$

Proof of Claim 8.5. Take any $x \in X$. We show that x is joined to at least $(1/4 - 2\epsilon)n$ vertices in Y. The spherical cap containing all points within distance at most $\sqrt{2} - 2\mu$ from x has measure at least $1/2 - 4\epsilon$ by applying Lemma 8.8 with 2ϵ. Thus this cap must intersect at least $(1/2 - 4\epsilon)n/2$ regions D_i, and we have $|y_i - x| < \sqrt{2} - \mu$ for each D_i that the cap intersects, so that xy_i is an edge of the graph $BE(n, h, \epsilon)$. $\square$

Having completed the proof of Theorem 8.1, we may now easily obtain Theorem 1.10.

Proof of Theorem 1.10. We apply Theorem 8.1 with $h = \log n / \log\log n$ and ϵ tending to 0 sufficiently slowly with n, so that $n \ge (C\sqrt{h}/\epsilon)^h$ is satisfied for sufficiently large n. Theorem 1.10 then follows as an immediate corollary. $\square$

We next formulate another useful corollary of Theorem 8.1. As the applications in the next section will rely on it, it will be helpful to to study a variant of the Ramsey-Turán numbers $\mathbf{RT}(n, K_4, m)$. Recall that a graph $G = (V, E)$ on n vertices is nice if it is K_4-free and there is a bipartition $V = X \cup Y$ into parts of order $n/2$ such that each part is K_3-free. Let $S(n, m)$ be the maximum number of edges of a nice graph on n vertices with independence number less than m. Note that

$$\mathbf{RT}(n, K_4, m) \ge S(n, m).$$

Also note that Theorem 8.1 provides a lower bound on $S(n, m)$, as the Bollobás-Erdős graph is nice.

For our next corollary, we will pick h to be the largest positive integer so that $n \ge h^h$, and hence $h > \frac{\log n}{\log\log n}$. Then, by picking $\epsilon = 4(\log\log n)h^{-1/2}$, the condition $n \ge (C\sqrt{h}/\epsilon)^h$ in Theorem 8.1 is satisfied. In this case, $2ne^{-\epsilon\sqrt{h}/4} = 2n/(\log n)$ and the minimum degree $(1/4 - 2\epsilon)n$ implies that the graph has at least $(1/4 - 2\epsilon)n^2/2 = (1/8 - \epsilon)n^2$ edges.

Corollary 8.9. *For n sufficiently large and $\delta = 4(\log\log n)^{3/2}/(\log n)^{1/2}$, we have* $\mathbf{RT}(n, K_4, \delta n) \geq S(n, \delta n) \geq (1/8 - \delta)n^2$.

Corollary 8.9 will be used in combination with the densifying construction in the next section to prove Theorems 1.7 and 1.9, which give lower bounds on Ramsey-Turán numbers that are significantly larger than the number of edges in the Bollobás-Erdős graph.

9 Above the Bollobás-Erdős density

If G is a nice graph with edge density less than $1/2$, we will find another nice graph G' on the same vertex set which is a hybrid of G and a complete bipartite graph. The graph G' is denser than G, and its independence number is not much larger than that of G. Specifically, with V_1, V_2 being the triangle-free parts of G of equal size, we will take some $U_1 \subset V_1$ and $U_2 \subset V_2$, and, for $i = 1, 2$, we add all edges between U_i and V_{3-i} and delete all edges in V_i which contain at least one vertex in U_i. Starting with G being a Bollobás-Erdős graph, we will be able to get a denser nice graph G' without increasing the independence number too much.

Lemma 9.1. *For positive integers d, m, n with $n \geq 6$ even and $d \leq n/2$, we have*

$$S(n, m + d) \geq \left(1 - \frac{2d}{n}\right)^2 S(n, m) + dn - d^2 - n.$$

Proof. Let $G = (V, E)$ be a nice graph on n vertices and $S(n, m)$ edges with independence number less than m. So there is a bipartition $V = V_1 \cup V_2$ into parts of order $n/2$ with each part K_3-free.

Let $U_i \subset V_i$ for $i = 1, 2$ be such that $|U_i| = d$ and the induced subgraph of G with vertex set $V \setminus (U_1 \cup U_2)$ has the maximum number of edges. Denote this induced subgraph by G_0. By deleting randomly chosen vertex subsets of V_1 and V_2 each of order d, each edge of G survives in this resulting induced subgraph with probability at least $\binom{n/2-d}{2}/\binom{n/2}{2}$. Hence, the number of edges of G_0 satisfies

$$\begin{aligned} e(G_0) &\geq S(n, m)\binom{n/2 - d}{2}/\binom{n/2}{2} = S(n, m)\left(1 - \frac{2d}{n}\right)\left(1 - \frac{2d}{n-2}\right) \\ &\geq \left(1 - \frac{2d}{n}\right)^2 S(n, m) - n, \end{aligned}$$

where in the last inequality we used $S(n, m) \leq n^2/3$, which follows from Turán's theorem for K_4-free graphs, $d \leq n/2$, and $n \geq 6$.

Modify G to obtain G' by first deleting all edges in V_i which contain at least one vertex in U_i, and then adding all edges from U_i to V_{3-i}. The number of edges of G' satisfies

$$e(G') = e(G_0) + |U_1||V_2| + |U_2||V_1 \setminus U_1| = e(G_0) + dn - d^2 \geq \left(1 - \frac{2d}{n}\right)^2 S(n, m) + dn - d^2 - n.$$

We next show that G' is nice. Since, for $i = 1, 2$, the induced subgraph of G' with vertex set V_i is a subgraph of the induced subgraph of G on the same vertex set, then the induced subgraph

of G' with vertex set V_i is triangle-free. Assume for the sake of contradiction that G' contained a K_4. As G is K_4-free, the K_4 must contain at least one vertex in $U_1 \cup U_2$. If the K_4 contained a vertex u in U_i, as all the neighbors of u are in V_{3-i}, then the other three vertices in the K_4 must be contained in V_{3-i}, contradicting that it is triangle-free. Hence G' is K_4-free, and hence nice.

As U_1 is complete to U_2 in G', any independent set in G' cannot contain a vertex in both U_1 and U_2. As also $|U_1| = |U_2| = d$ and G has independence number less than m, the independence number of G' is less than $m + d$. □

We have the following simple corollaries.

Corollary 9.2. *For even $n \geq 6$, if $S(n,m) \geq \left(\frac{1}{8} - \delta\right) n^2$ with $n^{-1/2} \leq \delta \leq \frac{1}{4}$, then $S(n, m+2\delta n) \geq \frac{n^2}{8}$.*

Proof. Let $d = 2\delta n$. By Lemma 9.1, we have

$$\begin{aligned} S(n,m+d) &\geq \left(1 - \frac{2d}{n}\right)^2 S(n,m) + dn - d^2 - n \\ &= (1-4\delta)^2 \left(\frac{1}{8} - \delta\right) n^2 + 2\delta n^2 - 4\delta^2 n^2 - n \\ &= \frac{n^2}{8}\left(1 + 48\delta^2 - \frac{8}{n} - 128\delta^3\right) \\ &\geq \frac{n^2}{8}, \end{aligned}$$

where the last inequality uses $n^{-1/2} \leq \delta \leq \frac{1}{4}$. □

Proof of Theorem 1.9. This is an immediate consequence of Corollary 8.9, Corollary 9.2, and $\mathbf{RT}(n, K_4, m) \geq S(n,m)$. □

The next corollary allows us to get a lower bound on Ramsey-Turán numbers greater than $n^2/8$.

Corollary 9.3. *For even $n \geq 6$, if $S(n,m) \geq \left(\frac{1}{8} - \delta\right) n^2$ and $\frac{1}{\delta n} \leq a \leq \frac{1}{2}$, then $S(n, m+an) \geq \frac{n^2}{8}\left(1 + 4a - 4a^2 - 8\delta\right)$.*

Proof. Let $d = an$. By Lemma 9.1, we have

$$\begin{aligned} S(n,m+d) &\geq \left(1 - \frac{2d}{n}\right)^2 S(n,m) + dn - d^2 - n \\ &= (1-2a)^2 \left(\frac{1}{8} - \delta\right) n^2 + an^2 - a^2 n^2 - n \\ &= \frac{n^2}{8}\left(1 + 4a - 4a^2 - 8\delta + 32\delta a - 32\delta a^2 - \frac{8}{n}\right) \\ &\geq \frac{n^2}{8}\left(1 + 4a - 4a^2 - 8\delta\right), \end{aligned}$$

where the last inequality uses $\frac{1}{\delta n} \leq a \leq \frac{1}{2}$. □

When $a \gg \delta$, Corollary 9.3 produces a construction with substantially more than $\frac{n^2}{8}$ edges.

Proof of Theorem 1.7. This is an immediate consequence of Corollary 8.9, Corollary 9.3, and $\mathbf{RT}(n, K_4, m) \geq S(n, m)$. □

10 Concluding remarks

In this paper, we solve the Bollobás-Erdős problem of providing estimates on the independence number of K_4-free graphs in the critical window; see Theorem 1.11. There is still some room to improve the bounds further. For example, in the third part of Theorem 1.11 we showed that for m just $o(n)$, we have $\mathbf{RT}(n, K_4, m) - n^2/8 = \Theta(mn)$, where the implied constants in the lower and upper bound are within a factor $3 + o(1)$. It would be interesting to close the gap.

The asymptotic behavior for the Ramsey-Turán numbers for odd cliques were determined by Erdős and Sós [16] in 1969. They proved, if q is odd, then

$$\mathbf{RT}(n, K_q, o(n)) = \frac{1}{2}\left(1 - \frac{2}{q-1}\right) n^2 + o(n^2).$$

Even after the Bollobás-Erdős-Szemerédi result, it still was years before it was generalized by Erdős, Hajnal, Sós, and Szemerédi [13] to all even cliques. They proved, if q is even, then

$$\mathbf{RT}(n, K_q, o(n)) = \frac{1}{2}\left(1 - \frac{6}{3q-4}\right) n^2 + o(n^2).$$

It would be nice to extend the results of this paper concerning the critical window for every even q.

Finally, it is quite remarkable that the old construction of Bollobás and Erdős can be tweaked to produce lower bounds which nearly reach our new upper bounds. Perhaps a further variation using high dimensional geometry (e.g., changing the underlying space or metric) could further close the gap.

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