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Design of Control Systems for Performance:
A Constraint Mapping Approach

by

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Abstract

Simple s-plane maps are used to graphically reveal interactions of performance requirements and constraints thus providing a control system designer insight into performance trade-offs. Two well-known but powerful ideas underlie this approach: (a) the characteristic equation of an n^-order system can be fully described by the specification of n variables and (b) any dynamic performance requirement or constraint can be expressed in terms of 2n variables of which n describe the open-loop and n describe the closed-loop characteristic equations. An example illustrates the application of the approach to controller design.

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Introduction

The design of control systems is a demonstration of the art of compromise: a system is made a millisecond slower but a millimeter more accurate using an ounce less torque. A designer is often given the task of designing a controller that will allow the closed-loop system to meet some dynamic performance requirements without violating certain constraints. It is the interplay of performance requirement against performance requirement (e.g. speed vs accuracy) and performance requirement against constraint (e.g. speed vs motor torque) that necessitates compromise. Most often, the satisfaction of one performance requirement requires changes in a different direction than the satisfaction of constraints or other performance requirements would dictate.

Current control system design methods require an iterative, trial-and-error process whereby a control law is formulated, the system simulated, performance measured, and a new control law formulated which reflects the knowledge gained. If a designer could be made to see the interactions of performance measures and constraints, to see how the choice of controllers has been narrowed by the measures and constraints, could be given insight into the tradeoffs required to design a controller for the system, then much of the iteration could be eliminated. A designer could design a controller with assurance that the system will meet performance requirements and not violate system constraints: a designer could design for performance.

This paper will discuss some basic ideas in a method for designing for performance for second-order, single-input, single-output linear, time-invariant systems under a linear state-feedback control law. Future papers will discuss extensions to higher order and multiple-input, multiple-output systems. The key ideas in designing for performance are well-known: (a) the characteristic equation of a linear, n^-order system can be fully described by the specification of n independent variables, and (b) any dynamic attribute, whether a requirement (e.g. bandwidth) or constraint (e.g. actuator effort bound), of an n^-order system can be expressed as a function of 2n variables of which n describe the open-loop and n describe the closed-loop characteristic equations. Given that all attributes can be expressed in terms of the 2n variables, all attributes may be mapped to a single suitable space, graphically illustrating the interaction of attributes and giving a designer insight into the control problem.

In order to illuminate these ideas, the first half of this paper will detail the process of mapping a control effort bound in the s-plane for a linear, time-invariant, underdamped, second-order system which is closed-loop stable under a linear state-feedback law. Control effort bounds are important constraints in the design of controllers. Many schemes for designing controllers for complex systems ignore control effort bounds, resulting in impractical methodologies. The success of optimal control can be traced in no small part to its attention to control effort. Mappings to the s-plane were chosen because designers have experience in interpreting s-plane graphs and because some performance measure maps are well understood in the s-plane. Underdamped second-order systems were chosen for their simplicity.

The second half of this paper will discuss how s-plane maps may be used as the basis for a design-for-performance methodology. An example will be worked to demonstrate the important concepts.

One result of the method is the generation of the set of all possible controllers of a particular structure which yield closed-loop systems that meet all dynamic performance requirements but do not violate system constraints. In contrast, the method of stable factorizations generates the larger set of all possible

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stabilizing[1] controllers [2] [6]. Unfortunately, the stable factorization method provides little insight into how the set **was** obtained. However, the two methods might be used together to draw on the strengths of both: **the** stable factorizations method may be used to find the set of all possible stabilizing controllers in a region defined by **the** design-for-performance method.

Mapping Control Effort Bounds in the S-Plane

Given the linear, time-invariant, underdamped, second-order system in the following canonical form

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \tag{1}$$

let

(2)

represent the corresponding closed-loop system under the linear state-feedback control law

$$u(t) = -k_x x_x(t) - k_r r(t) + H_i \tag{3}$$

where: k_x are constant feedback gains
$r(t)$ is a reference input
$x_i(0), x_2(0)$ are states
ω_n is the natural frequency of the closed-loop system
ω_n' is the natural frequency of the open-loop plant
ζ is the damping of the closed-loop system
ζ' is the damping of the open-loop plant.

The open-loop plant is not necessarily stable but must be expressible in the state-space canonical form for underdamped systems of (1). For simplicity, it is assumed that systems (1) and (2) have been non-dimensionalized.

Underdamped second-order systems have two complex-conjugate poles in the s-plane. Since s-plane maps are symmetric about the real axis and stable systems have poles confined to the left half of the s-plane, the full dynamic behavior of a closed-loop-stable, underdamped, second-order system may be characterized by studying the behavior of a single pole in the upper-left quadrant of the s-plane:

$$s = a + ib \quad \text{for } a<0 \text{ and } b>0. \tag{4}$$

In general, the dynamic performance of a second-order system is completely determined by the specification of two independent[2] variables. Systems (1) and (2) are both second order and have the same canonical form. The system descriptions, and hence the system dynamics, differ only in the specification of the variables ω_n' and ζ' for (1) and ω_n and ζ for (2). For future reference, let any pair of

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[1] In stable factorizations parlance, a stablizing controller is one that confines the closed-loop system to a desired region of the left-half s-plane. In effect, the designer is left to define "stablizing" to fit the problem.

[2] In mechanical engineering, second-order plants are often specified by *three* variables: mass, stiffness, and coefficient of damping. However, the three variables are not independent For example, two second-order plants with different mass, stiffness, and coefficient of damping will exhibit the same dynamic behavior if the stiffness to mass ratio (and thus the natural frequency) and the coefficient of damping to mass ratio (and thus damping) are the same for both plants.

independent variables that describe the dynamic performance of a second-order system be called *describing variables*, denote v_1 and v_2 as the variables, and define the *describing vector* as $\mathbf{v} \equiv [v_1, v_2]^T$. One pair of describing variables are system parameters of damping and natural frequency, as in (1) and (2) above. Another possible pair is the real and imaginary parts of the system poles, as in (4). Necessarily, one pair of describing variables can be expressed as a function of another pair since both completely describe the system. For example, damping and natural frequency are related to pole location

$$\omega_n = \sqrt{a^2 + b^2}$$
$$\zeta = \tan^{-1}(b/a) \tag{5}$$

and pole location is related to damping and natural frequency

$$a = -\zeta\omega_n$$
$$b = \omega_n\sqrt{1-\zeta^2}. \tag{6}$$

Equation (5) indicates that, in the s-plane, damping is constant along straight lines passing through the origin and natural frequency is constant along circular arcs centered at the origin.

It is well known that state feedback allows the arbitrary placement of closed-loop poles for any controllable open-loop plant [3]. The effort needed to control the closed-loop system is related to the open- and closed-loop pole locations. Defining

$$u_f(t) \equiv -k_1x_1(t) - k_2x_2(t), \tag{7}$$

then

$$u(t) = u_f(t) + r(t). \tag{8}$$

As can be seen from (8), the effort needed to control the closed-loop system is composed of two parts: state-feedback input $u_f(t)$ and reference input r(t). The state-feedback term may be thought of as the effort required to change the dynamics of (1) to the dynamics of (2).

A procedure for describing the set of all possible closed-loop poles -- and hence all possible closed-loop dynamics -- given an open-loop plant and a bound on the state-feedback control effort $|u_f(t)| < u_{bound}$ for an arbitrary reference input r(t) will be described. Though quite general, the procedure will be illustrated by considering a step reference input r(t) = 1. It will be noted when the procedure is specific to the choice of r(t).

Procedure

Step 1: Obtain the feedback control gain matrix $\mathbf{k} = [k_1\ k_2]$ as a function of convenient open-loop and closed-loop describing vectors. Using system damping and natural frequency as an example, by substitution of (3) into (1) and comparison to (2)

$$k_1 = \omega_n^2 - \omega_n'^2$$
$$k_2 = 2\zeta\omega_n - 2\zeta'\omega_n'. \tag{9}$$

If $\mathbf{v}$ and $\mathbf{v}'$ are describing vectors defined as

$$\mathbf{v} \equiv [\omega_n, \zeta]^T$$
$$\mathbf{v}' \equiv [\omega_n', \zeta']^T, \tag{10}$$

then the functional dependence of the controller gains on the describing vectors expressed in (9) may be compactly written as

$$k = f^*(v,v'). \tag{11}$$

Since the damping and natural frequency are only one possible set of variables, the same procedure could be repeated for a second set, say the open- and closed-loop eigenvalues by substituting of equalities (5) into (9).

Step 2: Solve for the time response x(t) • [x^t), $x_2(t)]^r$ due to input r(t). Since the closed-loop system is linear and time-invariant, the time response x(t) of the system to a general input may be (at least in theory) solved explicitly and is a function only of the describing variables of the closed-loop system, the reference input r(t), and time t. The time response to a step input in terms of damping and natural frequency [4] is

$$x_1(t) = \frac{1}{\omega_n^2}\{1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin[\omega_n\sqrt{1-\zeta^2}t - \tan^{-1}(\frac{\sqrt{1-\zeta^2}}{-\zeta})]\}$$

$$x_2(t) = \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}} \sin(\omega_n\sqrt{1-\zeta^2}t). \tag{12}$$

Equation (12) may be written in the form

$$x(0 = f_x(v,r). \tag{13}$$

Again, a substitution like (5) may be used to express the response as a function of different describing variables.

Step 3: Obtain $u_f(t)$ as a function of the describing vectors v and v* and time t. Using the results of steps 1 and 2 and equation (7), we see that

$$u_f(t) = -k_1x_1(t) - k_2x_2(t)$$

$$= -kx$$

$$= -f_k(v,v')f_x(v,t).$$

Thus, $u_f(t)$ is a function only of the open- and closed-loop describing vectors and time:

$$«/« = f_u(v,v^{/},O. \tag{14}$$

Step 4: Obtain an expression for the maximum value of the state-feedback input over all time u_{max} as a function of only the open- and closed-loop poles. The maximum may be found algebraically by solving for the time t_{max} for which the derivative of the state-feedback input $u_f(t)$ with respect to time is zero and substituting the value of t_{max} into the state-feedback input to find the maximum. Since $u_f(t)$ is a function only of the open- and closed-loop describing vectors v and v', the maximum of $u_f(t)$ over all time u_{max} is a function of only v and v*:

$$u_{max} = '(v.V). \tag{15}$$

While there is no guarantee that ^^ can be explicitly solved or that there is only one time for which the derivative is zero, the procedure is straightforward for a step input r(t) = 1. Equation (15) can be used to map a control bound to any space spanned by the set of all possible describing vectors. For example, equations (10) and equations (15) describe a mapping to the parameter space of £ vs co_n. If the open-loop plant is described by the pole s' - a' + ib' (for b' > 0), then equalities (5) and (6) and similar equalities for the open-loop plant may be used to transform u^ from a function of v and v' to a function of a, b, a',

and b'; that is, to a function of the open- and closed-loop poles in the s-plane. For a step input r(t), the result is:

$$u_{max} = -\frac{1}{\omega_n^2}\left[1 + \sqrt{k_2^2\omega_n^2 - 2k_1k_2\zeta\omega_n + k_1^2}\; e^{-\zeta\omega_n\gamma}\right] \tag{16}$$

where: $\gamma = \frac{1}{\omega_n\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{k_2\omega_n\sqrt{1-\zeta^2}}{k_2\zeta\omega_n - k_1}\right)$

$k_1 = (a^2 + b^2) - (a'^2 + b'^2)$

$k_2 = 2(a' - a)$

$\omega_n^2 = a^2 + b^2$

$\zeta\omega_n = -a$

$\omega_n\sqrt{1-\zeta^2} = b.$

Although the outlined procedure is general with respect to the choice of reference input, the dynamic response due to a step input has often been used as a standard against which to compare the performance of widely differing systems. Therefore, the remainder of this section will discuss the implications of equation (16), which describes s-plane maps of control effort bounds for step reference inputs.

To be useful in the design of control systems, the maps should display regular behavior. Unfortunately, equation (16) is so cumbersome that few properties have been analytically extracted. Analysis has revealed the asymptotic behavior of u_{max} in two directions.

The value of u_{max} increases without bound as ω_n approaches zero

$$\lim_{\omega_n \to 0} u_{max} = \infty \tag{17}$$

In mechanical engineering, a system with zero natural frequency is a rigid body with zero damping and zero stiffness. If a step reference input is applied to a system with zero natural frequency, the system will increase displacement and velocity without bound. Any linear combination of the displacement and velocity of the body, such as the feedback control law, will also be unbounded. This is the behavior described by equation (17).

The value of u_{max} is bounded as ω_n becomes large, approaching a limiting value dependent on the open-loop plant damping

$$\lim_{\omega_n \to \infty} u_{max} = 1 + e^{-\zeta\gamma_\infty} \tag{18}$$

where $\gamma_\infty = \frac{1}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right)$.

Increasing closed-loop natural frequency increases closed-loop stiffness. Because increasing stiffness limits closed-loop system response (measured by displacement and velocity due to a step input), the feedback control effort is bounded.

These two analytical results help explain the map characteristics revealed through numerical investigation and shown in Figures 1 through 4. Figures 1 and 2 show the dependence of the mapping on open-loop plant natural frequency and damping, respectively. The effect of increasing the control bound value u ^ ^ is shown in Figure 3 and the differences between maps for stable and unstable open-loop plants are illustrated in Figure 4. As discussed earlier with equation (4), s-plane maps are symmetric about the real axis; the full dynamic behavior of a stable, linear system can be described by the behavior of a single pole in the upper-left quadrant of the s-plane. All four figures show maps confined to the upper-left quadrant.

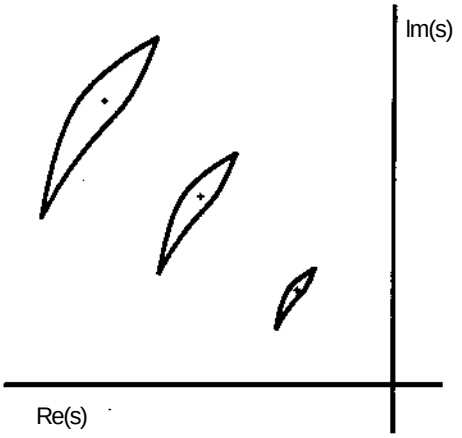


Figure 1: The control bound maps for open-loop plants with the same damping are the same shape, but have sizes which increase with natural frequency.

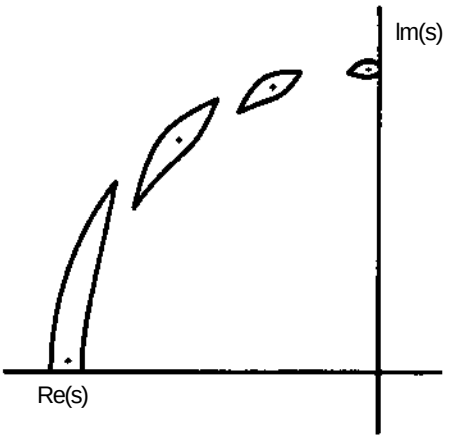


Figure 2: Control bound maps are elongated as the open-loop damping is increased while the natural frequency is constant.

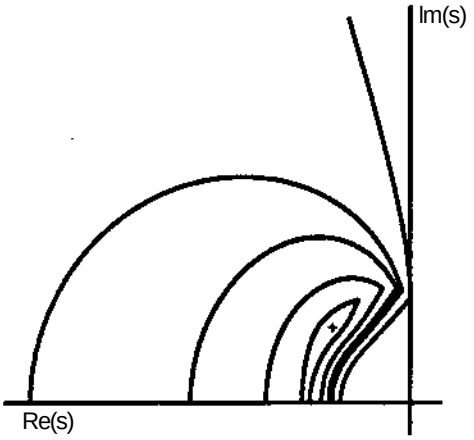


Figure 3: **As** the bound on control effort increases, the region of obtainable closed-loop systems increases in size and distorts in shape. Past a limiting value, the region becomes semi-infinite.

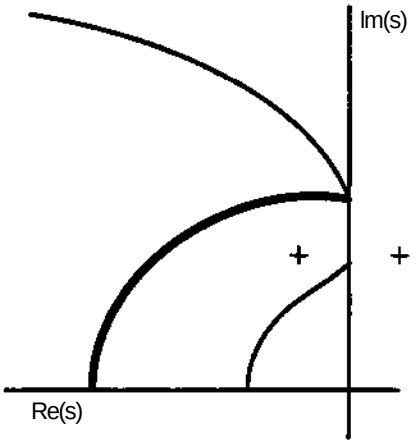


Figure 4: Maps for unstable open-loop plants are always semi-infinite regions. The region of obtainable closed-loop systems for the unstable open-loop plant is to the upper left of the thicker line.

In the figures, crosses represent open-loop plants, solid lines represent the boundary at which $u_{max} = u_{bound}$ • and" obtainable closed-loop systems for a given open-loop plant lie inside (or to the left of) the boundary. That is, given an open-loop plant and a control effort bound, the set of all closed-loop systems for which $|u_f(t)| \leq u$ ^ ^ for all t £ 0 is given by the interior of (or leftmost plane defined by) the region.

The observations below have been shown to be true for the mapping of many different combinations of control bounds and open-loop plants, though analysis of (16) has not yet proved these claims in general.

(1) Regions of obtainable closed-loop systems are either simply connected[3] or semi-infinite, an important and not altogether unexpected characteristic. Since intuitively (a) no feedback control effort is required if the closed-loop system is identical to the open-loop plant ($u_f(t) - 0$ if $s \ll s^f$) and (b) the control effort increases as the closed-loop poles are moved further from the open-loop poles, it is not unexpected that the increase is regular enough that the regions are simply connected. Regions become semi-infinite if the control bound becomes too great. Since u_{max} approaches a limiting value as ω_n increases, the mapping of control bounds that are greater than the limiting value cannot be simple, closed regions; rather, the regions are semi-infinite.

(2) For the same value of u^nd*thereg lon increases in size while retaining the same shape as the open-loop plant increases in natural frequency while retaining the same damping regardless of the value of damping (see Figure 1). Size increases are desirable, indicating that a wider range of closed-loop dynamic behavior, and thus an increase in performance, is possible.

(3) For the same value of $u_{boun\&}$ the region is elongated as the open-loop plant increases in damping while retaining the same natural frequency (see Figure 2). Elongated regions allow for closed-loop systems with a broad range of damping.

(4) For the same open-loop plant, the region increases in size as u^und increases (see Figure 3). The shape of the region is distorted; boundary points to the left of the open-loop plant (those representing closed-loop systems with higher natural frequency than the open-loop plant) increase in displacement from the open-loop pole as u^und increases more than boundary points to the right. The distortion reflects the fact that u , ^ is bounded to the left of the open-loop plant and unbounded to the right. Since higher natural frequency is usually associated with increased performance, increasing the control bound allows for systems with higher performance, as intuitively must be true. Surprisingly, the distortion indicates that it is easier to increase the performance of a plant than to decrease the performance; that is, a lesser maximum control effort is needed to increase natural frequency by 10% than to decrease plant natural frequency by 10%.

(5) Figure 4 shows the control bound maps for two open-loop plants, one stable and the other unstable but both with the same natural frequency, under the same control bound. The region of obtainable closed-loop systems for the unstable plant is semi-infinite while the region for the stable plant, although not fully shown, is simply connected. The map illustrates that, in general, higher closed-loop performance may be obtained by an unstable plant than by the equivalent stable plant. Control effort bound maps for unstable plants are always semi-infinite regions in the s-plane.

The regular characteristics of the maps provides a chance that the maps may be of more than only of theoretical use: the maps might be used in the analysis and synthesis of dynamic system. Some of the possible uses will be discussed next.

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[3]Loose(y speaking, regions are simply connected if they contain no holes. More formally, "a region is said to be simply connected if *e^ery* dosed curve in the region can be shrunk continuously to a point in the region [5]".

Designing for Performance with S-Plane Maps

Equation (15) shows that a control effort bound may be expressed as a function of only the open- and closed-loop describing variables, a fact necessitated by the definition of the describing variables as fully describing the dynamic behavior of a system. The key idea of this paper is that, by analogy, *any dynamic attribute of a closed-loop system may be expressed as a function of the same open- and closed-loop describing vectors.* Thus, any dynamic requirement or constraint may be similarly expressed, providing an easy vehicle for studying the interactions of requirements and constraints. This fact has many interesting consequences.

Equation (16) may be used to size actuators for use in a closed-loop system. By specifying the open- and closed-loop poles, (16) yields the value of maximum control effort u_{max}. An actuator able to provide an input at least as great as $1 + u_{max}$ would be chosen.[4]

Control bound maps may be used to analyze how an open-loop plant should be designed, given an unchangeable bound on control effort. Since different open-loop plants display different maps, some open-loop plants will display more favorable closed-loop characteristics under the same control bound. It has already been shown that the greater the damping and/or natural frequency of an open-loop plant, the larger the region of obtainable closed-loop systems, and hence the larger selection of closed-loop performance. If a designer cannot accept an increase in control effort but requires higher performance, re-design of the open-loop plant using control bound maps is one way of achieving his goals.

Finally, and most importantly, the maps may be used as the basis of a control system design method. Given an open-loop plant and dynamic attribute (requirements and/or constraints) on the closed-loop performance of the system, a controller may be designed such that the system will meet all requirements without violating constraints. Since any dynamic attribute may be expressed as a function of the same open- and closed-loop describing vectors, the attributes may be mapped to the same space, overlaid, and the interactions of the attributes studied. A convenient space, and the one used throughout this paper, is the s-plane. Selection of a closed-loop system from the intersection of the sets of all obtainable closed-loop systems for all attributes will ensure that the system meet all requirements without violating constraints. Furthermore, if two or more attributes are in conflict (cannot be satisfied within problem formulation, thus yielding a null intersection of obtainable sets), the mapping will reveal the conflict. If well-understood, the maps will show how the relaxation of a requirement or constraint can resolve a conflict. Frequently, it is the responsibility of the designer to not blindly accept but to evaluate and refine performance requirements.

An example will be worked to illustrate how maps may be applied to the design of a controller.

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[4] By equation (8), an actuator must provide input of u(t) = u^t) + r(t). Since r(t) = 1 for a step input and the maximum of $u_f(t)$ is ITWC **• m a * i m u m °f W ls guaranteed to be less than **or equal** to 1 + u ^ . **Remember that** u^ **and 1 are** non-dimensional, having been consistently scaled in the formulation of (1) and (2).

Example

The design-for-performance method is inherently graphical, providing insight into design problems using simple graphs in the s-plane which show the interactions of requirements and constraints. Accordingly, this example shall illustrate the graphical nature of the procedure. Although contrived, the example does highlight important ideas in the method.

Design Problem: For the given open-loop plant, design a controller such that the closed-loop system (consisting of the open-loop plants and the controller) will meet the given requirements and not violate the given constraint. The closed-loop system must be second-order, linear, time-invariant, and underdamped. The control law must be linear state-feedback in form (3).

Open-loop plant:

$$mx'' + cx' + kx = f(t) \tag{19}$$

where: $m = 10.0$ *kg is mass of plant*
$c = 20.0$ *Nslm is coefficient of damping*
k *m* 260.0 *N/m is stiffness*
$f(t)$ *is actuator input*
the plant output of interest is position x
' denotes differentiation with respect to time t.

Requirements:

- In response to a step reference input, the system output must settle to within 5% of steady-state in less than t_8 - 0.40 seconds.
- In response to a step reference input, the system output must have maximum peak overshoot of less than M_p * 10%.
- The system bandwidth must be greater than o^ - 2.25 hz.

Constraints:

- The control effort u(t) must be less than u_{max} « 2.10 N in magnitude.

Simulation and analysis of the open-loop plant (19) show a settling time of 2.7 seconds, a maximum peak overshoot of 53%, and a bandwidth of 1.2 hz.

To use the design-for-performance method, the plant is first expressed in non-dimenstonalized form:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t) \tag{20}$$

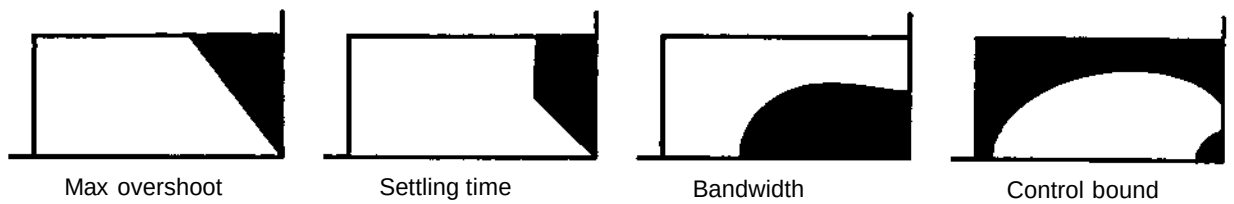
$$y = (\omega_n^2 \; 0 \; 0)$$

where: $x_1 = x$
$x_2 = x'$
$u(t) = f(t)/m$
$\omega_B = \sqrt{c/m} = 5.1$

$C = c/2\sqrt{km} = 0.20$

Next, the dynamic constraint and requirements are mapped to the s-plane. Figure 5 below shows the

mappings, first individually and then together.[5] In all maps, the blackened area corresponds to areas of the upper-left quadrant of the s-plane within the arbitrary bounds of Re(s) > -30 and Im(s) < 16 from which the closed-loop pole *cannot be* selected.



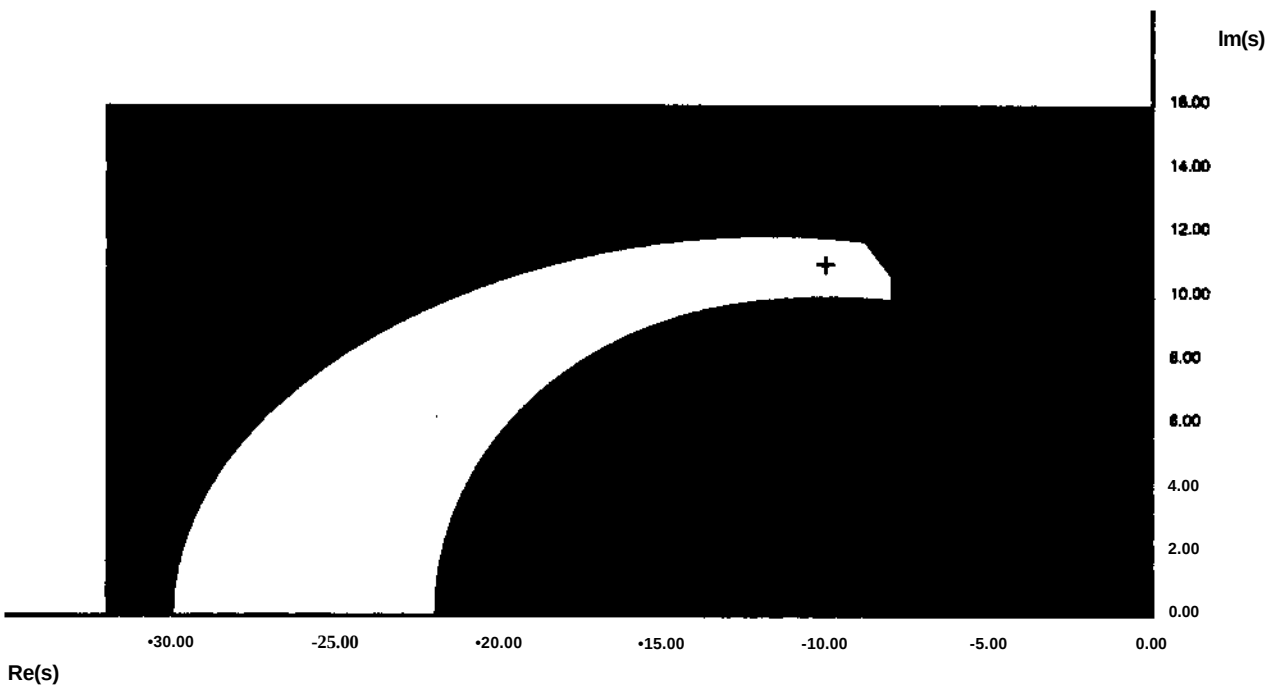


Figure 5: All closed-loop systems designed using a pole selected from the white region will meet all requirements and not violate constraints.

In Figure 5, the open-bop plant at -1 + 5i is clearly in violation of the performance requirements. It remains only to select a closed-loop pole from the white region of Figure 5. For lack of other criteria, the pole s « -10 + 11i, marked by the black cross, is selected. Using equations (9) and (5), the pole is transformed to controller gains k_1 = 195.0 and k_2 = 18.0. Simulation and analysis confirm the following characteristics of the closed-loop system:

- **Settling time of 0.33 seconds,**
- **Maximum peak overshoot of 5.8%,**
- **Bandwidth of 2.5 hz, and**
- **Maximum control effort of 2.08 N.**

[5]Settling time has traditionally been mapped as a straight vertical line in the s-plane. The more accurate mapping here is based on the value of the decay envelop, as shown in [1].

The design is successful: all requirements are met and the constraint not violated.

Design Solution: $k_1 = 195.0$ and $k_2 = 18.0$

Summary

A design-for-performance methodology which uses simple s-plane maps to graphically illustrate the interactions of performance requirements and constraints and which yields the the design space of all closed-loop systems meeting requirements and not violating constraints has been presented. The power of the method is in its versatility: the method may be used to study performance requirement trade-offs, open-loop plant re-design trade-offs, and controller design. The method was detailed for a second-order, underdamped system under a particular control law for simplicity, not from necessity. Future work will address extensions to more varied control laws and to higher order, multiple-input, multiple-output, and real systems. The ideas underlying this work are well-known; if a system is linear, finite-dimensional, time-invariant and of order n, then: (a) its characteristic equation can be fully characterized by the specification of n independent variables and (b) any dynamic attribute of the system can be characterized by the 2n variables describing its open- and closed-loop characteristic equations. An example demonstrated the approach applied to controller design.

Acknowledgments

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Appendix A

Derivations

The following will be a more detailed derivation of the expression for the maximum over all time u_{max} of the state feedback input $u_f(t)$ as outlined in step 4 of the procedure. Using elementary calculus, the maximum of $u_f(t)$ may be found by (a) finding the time t_{max} at which the time derivative of $u_f(t)$ vanishes and (b) substituting the value of t_{max} for t: $u_{max} = u_f(t=t_{max})$.

For later convenience, define the damped natural frequency as

$$\omega_d \equiv \omega_n\sqrt{1-\zeta^2},$$

rewrite equation (12) using the damped natural frequency as

$$x_1(t) = \frac{1}{\omega_n^2}\{1 - \frac{e^{-\zeta\omega_n t}}{d}[\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t]\} \quad (21)$$

$$x_2(t) = \frac{e^{-\zeta\omega_n t}}{\omega_d}\sin(\omega_d t) \quad (22)$$

and repeat equation (7), in which the state feedback input is defined as

$$u_f(t) \equiv -k_1 x_1(t) - k_2 x_2(t). \quad (23)$$

Differentiating both sides of (23) with respect to time yields:

$$\frac{du_f(t)}{dt} = -k_1\frac{dx_x\{t)}{dt} - \dot{}_L\frac{dx.it)}{dt}. \quad (24)$$

From equations (2) and (22), it is evident that

$$\frac{dx_x(t)}{dt} = x_2(t)$$

$$= \frac{e^{-\zeta\omega_n t}}{\omega_d}\sin(\omega_d t). \quad (25)$$

The expression for $x_2(t)$ given in (22) may be differentiated with respect to time to yield:

$$\frac{\wedge(0}{\dot{dt}} = *"\dot{v}V[oos\,co\hat{}\,t\frac{C^{w}_{rt}}{co\hat{L}}\sin co\hat{}r\,]. \quad (26)$$

If equations (25) and (26) are substituted into (24), the result after rearrangement is

$$\frac{du_f(t)}{at} = -\frac{e^{-\zeta\omega_n t}\,v}{co\hat{}}[(\hat{}\hat{}Ccojsinco\hat{}r + \hat{}co\hat{}cosco\hat{}r]. \quad (27)$$

The time at which maximum of $u_f(t)$ occurs may be solved for by setting the time derivative of $u_f(t)$ to zero. From equation (27), it is evident that the time derivative of $u_f(t)$ can be zero only if the second term on the right-hand side vanishes; that is, only if

$$(k_1 - k_2\zeta\omega_n)\sin\omega_d t + k_2\omega_d\cos\omega_d t = 0,$$

or

$$t_{max} = \frac{1}{\omega_d}\tan^{-1}\left(\frac{k_2\omega_d}{k_2\zeta\omega_n - k_1}\right). \quad (28)$$

The value of the the maximum of $u_f(t)$ can be found by substituting t_{max} for t in equations (21) and (22), and substituting the resultant expressions into (23). The result, after rearrangement, is equation (16).

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