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First-Order Classical Modal Logic

Abstract. The paper focuses on extending to the first order case the semantical program for modalities first introduced by Dana Scott and Richard Montague. We focus on the study of neighborhood frames with constant domains and we offer a series of new completeness results for salient classical systems of first order modal logic. Among other results we show that it is possible to prove strong completeness results for normal systems without the Barcan Formula (like $\mathbf{FOL} + \mathbf{K}$) in terms of neighborhood frames with *constant* domains. The first order models we present permit the study of many epistemic modalities recently proposed in computer science as well as the development of adequate models for monadic operators of high probability. Models of this type are either difficult of impossible to build in terms of relational Kripkean semantics.

We conclude by introducing *general* first order neighborhood frames and we offer a general completeness result in terms of them which circumvents some well-known problems of propositional and first order neighborhood semantics (mainly the fact that many classical modal logics are incomplete with respect to an unmodified version of neighborhood frames). We argue that the semantical program that thus arises surpasses both in expressivity and adequacy the standard Kripkean approach, even when it comes to the study of first order normal systems.

Keywords: First-Order Modal Logic, Neighborhood Semantics, General Frames

1. Introduction

Dana Scott and Richard Montague proposed in 1970 (independently, in [45] and [42] respectively) a new semantic framework for the study of modalities, which today tends to be known as *neighborhood semantics*.

A **neighborhood frame** is a pair $\langle W, N\rangle$, where W is a set of states, or worlds, and $N : W \to 2^{2^W}$ is a neighborhood function which associates a set of neighborhoods with each world. The tuple $\langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a neighborhood frame and V a valuation is a neighborhood model. A modal necessity operator is interpreted in this context as follows: $\mathcal{M}, w \models \Box\phi$ iff $(\phi)^{\mathcal{M}} \in N(w)$, where $(\phi)^{\mathcal{M}}$ is the truth-set corresponding to ϕ in the given model.

Without imposing specific restrictions on the neighborhood function it is clear that many important principles of normal or Kripkean modal logics

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will not hold in a neighborhood model. At the same time it is possible to show that there is a class of neighborhood models, the so-called *augmented* models (see the definition below), which are elementary equivalent to the relational models for normal modal systems of propositional modal logic. So, the program of neighborhood semantics has normally been considered as a generalization of Kripke semantics, which permits the study of classical systems that fail to be normal.

Early on (in 1971) Krister Segerbeg wrote an essay [47] presenting some basic results about neighborhood models and the classical systems that correspond to them and later on Brian Chellas incorporated these and other salient results in part III of his textbook [16]. Nevertheless for more than 15 years or so after 1971, in the apparent absence of applications or in the absence of guiding intuitions concerning the role of neighborhoods, non-normal classical modal logics were studied mainly in view of their intrinsic mathematical interest. This situation has changed in important ways during the last 18 or so years. In fact, many of the normal axioms, like the additivity principle, establishing the distribution of the box over conjunction, have been found problematic in many applications. As a result many formalisms proposed to retain:

(**M**) $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

(**N**) $\Box\top$

while abandoning:

(**C**) $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

Many recently explored, and independently motivated formalisms, have abandoned the full force of additivity while retaining monotony (M).* Examples are Concurrent Propositional Dynamic Logic [30], Parikh's Game Logic [43], Pauly's Coalition Logic [44] and Alternating-Time Temporal Logic [1]. Moreover recent research [6] has shown that a large family of Non-Adjunctive logics, previously studied only syntactically or via a variety of idiosyncratic extensions of Kripke semantics can be parametrically classified neatly as members of a hierarchy of monotonic classical logics, all of which admit

*Of course there is a relatively long history of arguments favoring the abandonment of (C). For example Ruth Barcan Marcus proposed the abandonment of (C) – which she called *the factoring principle* – both for deontic and for (some) epistemic interpretations of the box.

clear and simple neighborhood models. A salient member of this family is the logic of monadic operators of high probability studied via neighborhood semantics in [36] and [5]. This is a clear case where the intended interpretation of neighborhoods is quite intuitive: the neighborhoods of a point are the propositions receiving probability higher than a fixed threshold.

More generally one can see the neighborhoods as having an epistemic role (as suggested in [5]), namely representing the knowledge, belief or certain other attitude of an agent at a certain point. This strategy permits the development of elegant and economic models of attitudes that can only be modeled alternatively either via the abandonment of the axiom of foundation in set theory (or via co-algebras) or by postulating a large array of primitive epistemic states (devoid of propositional representation). The coming subsection elaborates on this issue, which ultimately is concerned with the way in which possible worlds are conceptualized and concretely encoded in models of modalities.

Possible worlds and modalities

Moshe Vardi considered in [49] the use of neighborhood models in order to represent failures of logical omniscience, high probability operators as well as logics of knowledge, time and computation. Nevertheless, after considering the wider class of *classical modal systems* (encompassing both normal and non-normal modal systems) and their neighborhood models, Vardi discarded them without exploring them logically. Vardi gave two reasons for not utilizing neighborhood models for studying classical modalities (which he dubs *intensional logic* following Montague's terminology). The central reason is that this approach 'leaves the notion of a possible world as a primitive notion [...]. While this might be seen as an advantage by the logician whose interest is in epistemic logic, it is a disadvantage for the *user* of epistemic logic whose interest is mostly in using the framework to model belief states (page 297).' Vardi proceeds instead to establish that: 'a world consists of a truth assignment to the atomic propositions and a collection of sets of worlds. This is, of course, a circular definition...'. Barwise and Moss [12] showed how to make this strategy coherent by abandoning the axiom of foundation in set theory.

Most of the mainstream work in models of modalities has been usually done by utilizing a space of possible worlds, which are, in turn, understood as unstructured primitives. This strategy is abandoned in Vardi's proposal. Vardi seems to appreciate that the contents of a neighborhood $N(w)$ can be seen pre-systematically (for applications in epistemic logic, for example) as the *propositional* representation of the epistemic state of a given agent at world w. But Vardi wants to have as well a notion of possible world including as part of it the representation of the epistemic state of the agent (or interacting agents). This can only be done by abandoning the idea of w as an unstructured primitive point. Therefore he proposes seeing w as a structured entity with an epistemic component $N(w)$. As Vardi explains this strategy leads to circularity, which only ceases to be vicious in the context of an underlying set theory without the axiom of foundation.

There is yet a different way of facing this problem [21]. The idea is to assume that $W \subseteq O \times S_1 \times ... \times S_n$ where O is the set of *objective* states and S_i is a set of *subjective* states for agent i. Therefore worlds have the form $(o, s_1, ..., s_n)$. In multi-agent systems o is called the *environment state* and each s_i is called a *local* state for the agent in question.

Halpern [31] characterizes an agent's subjective state s_i by saying that it represents 'i's perception of the world and everything else about the agent's makeup that determines the agent's reports'. To avoid circularity it is quite crucial that both environment states and the local states of agents are now unstructured primitives. It is unclear the extent to which this strategy is conducive to concrete representational or logical advantages. It should be clear, on the other hand, that neighborhoods have equal representational power, while reducing the set of needed primitives. In fact, one can have for each world w a set of neighborhood functions $N_i(w)$ (with i ranging between 1 and n), where each neighborhood mimics the semantic behavior of the local state s_i. In fact, for each modality M_i and each structured point $(o, s_1, ..., s_n)$, such that $M_i A$ is satisfied at w, we can have a corresponding unstructured point w in a corresponding neighborhood model where the proposition expressed by A is in $N_i(w)$. This is so even if there are non-propositional elements constitutive of relevant aspects of the agent's make up determining the agent's reports. As long as these reports are propositional the neighborhoods can encode the information needed to represent the reports in question.

A concrete application where worlds or *states* are understood along the 'thick' lines just sketched is concerned with the pioneer work of J. Harsanyi devoted to model games of incomplete information played by Bayesian players [33]. In this case we have a set of *external states* S and a *state of the world* in a *type space* $\mathcal{T}$ of S is an $(n+1)$-tuple $(s, t_1, ..., t_n) \in S \times T_1 \times ... \times T_n$, where for each individual i, T_i is a finite set of types. Intuitively the $(n+1)$-tuple $(s, t_1, ..., t_n)$ specifies the relevant external state and the epistemic types for each agent, where each epistemic type corresponds, in turn, with an infinite hierarchy of (probabilistic) beliefs. Recent work has appealed to modal logic in order to formalize (hierarchies of) probabilistic beliefs of this kind via modal operators with the intended interpretation: 'i assigns probability at least p to ...' (see, for example, the review of recent work presented in [13]). As we will make clear below (via the consideration of various probabilistic applications) the use of neighborhood models offers a perfect tool for the study of type spaces of this kind where external and epistemic states can be neatly separated. Although applications of this sort already exist (see, for example, the relevant chapters in [7]) the first order case is seldom studied. Extensions of these type in models with constant domains (which are the ones considered and tacitly recommended in [21]) are nevertheless hard or impossible to study by appealing the relational models where worlds encoded as $(n+1)$ tuples. We will show below that models of this type are, in contrast, rather natural if one uses neighborhoods.

Models for first order modalities

Unfortunately the recent interest in articulating applications for neighborhood semantics has not motivated yet the systematic study of first order classical logics and first order neighborhood models. One of the first sources of insight in this area can be found in the book published by Dov Gabbay in 1976 where a variation of neighborhood semantics is systematically used [23]. More recently one of us (Arló-Costa) presented in [5] preliminary results in this area showing that the role of the Barcan schemas in this context is quite different from the corresponding role of these schemas in the relational case.[†] In fact, the use of neighborhood semantics permits the development of models *with constant domains* where neither the Barcan (BF) nor the Converse Barcan formulas (CBF) are valid. Moreover [5] provides necessary and sufficient conditions for the validity of BF and CBF.

The recent foundational debates in the area of quantified modal logic oppose, on the one hand, the so called 'possibilists' who advocate the use of quantifiers ranging over a fixed domain of possible individuals, and on the other hand, the 'actualists' who prefer models where the assumption of the constancy of domains is abandoned. A salient feature of the standard first order models of modalities is that for those models the constancy of domains requires the validity of both the BF and the CBF (see [22] for a nice proof of this fact). Many philosophers have seen the possibilist approach as the only one tenable (see for example, [18], [41], [50]), and as a matter of fact the

[†]These schemas are presented below in Definition 3.1. The schemas are named after the logician and philosopher Ruth Barcan Marcus who proposed them in [8] and [9]. Papers written before 1950 are usually referenced under 'Barcan', while papers after 1950 under 'Marcus'.

possibilist approach is the one that seems natural in many of the epistemic and computational applications that characterize the wave of recent research in modal logic (see, for example, the brief section devoted to this issue in [21]). Nevertheless, while the possibilist approach seems reasonable on its own, the logical systems that adopt the Barcan Formulas and predicate logic rules for the quantifiers might be seen as too strong for many applications. The problem is that in relational models one cannot have one without the other. This has motivated some authors to adopt more radical approaches and to construct, for example, models with individual concepts (functions from possible worlds to the domain of objects). The approach provided by Garson [25] in particular is quite ingenious although it seems limited to first order extensions of **K** = **EMNC** (and the use of individual concepts does not seem immediately motivated in some simple applications considered here, like the logic of high probability).[‡]

Notice, for example, that (as indicated in [5]) if the box operator is understood as a monadic operator of high probability the BF can be interpreted as saying that if each individual ticket of a lottery is a loser then all tickets are losers. While the CBF seems to make sense as a constraint on an operator of high probability the BF seems unreasonably strong. At the same time nothing indicates that a possibilitic interpretation of the quantifiers should be abandoned for representing a monadic operator of high probability. On the contrary the possibilistic approach seems rather natural for this application. It is therefore comforting that one can easily develop first order neighborhood semantics with constant domains where this asymmetry is neatly captured (i.e. where the CBF is validated but the BF is not). In particular we argue that the non-nested fragment of the system **FOL**+**EMN** is adequate for representing first order monadic operators of high probability.

In this article we present a series of representation results that intend to give a preliminary insight on the scope and interest of first order neighborhood semantics. We will not limit our study to the analysis of non-normal systems. On the contrary one of our main results shows that a strong completeness result in terms of first order neighborhood models *with constant domains* can be offered for the normal system **FOL**+**K**. Relational models (with constant domains) cannot characterize syntactic derivability in **FOL** + **K**. The problem motivating this gap is that the CBF is syntactically derivable from **FOL** + **K** but the BF is not. So, one needs to have relational models with varying domains in order to characterize the system

[‡]**E** is the weakest system of classical logic.

in question.[§] But, again, there are many interesting applications, ranging from the modeling of contextual modals in linguistics [38] to the logic of finitely additive conditional probability where the use of varying domains is not immediately motivated and where the asymmetry between the CBF (as valid) and the BF (as invalid) holds.

Neighborhood semantics is usually considered as a mild extension of standard Kripke semantics in part because it is usually assumed that the augmented neighborhood models are mappable to the standard models of normal systems, and because it is usually assumed as well that these standard models provide the models one needs in order to characterize the normal systems. We claim here that the latter assumption is dubious at best. In fact, we show that there are indeed Scott-Montague models of important normal systems constructible within a possibilistic approach and without standard counterparts.

Martin Gerson [26] has argued convincingly that an unmodified version of the neighborhood approach has some of the same problems regarding incompleteness than the standard approach. He showed this by proving incompleteness with respect to neighborhood semantics of two normal systems, one between T and S4 and another which is an extension of S4. We adopt a similar remedy concerning these problems than the one adopted in [15] and [32], namely the adoption of *general first order neighborhood frames*. Therefore we introduce general frames for first order modalities (not previously studied in the literature) by utilizing algebraic techniques reminiscent to the ones employed in the algebraic study of first order logic with operators. We then present a general completeness result covering the entire family of first order classical modal logics. We conclude by discussing some examples and suggesting topics for future research.

2. Classical systems of propositional modal logic

This section reviews some basic results about classical systems of propositional modal logic. The reader is referred to the textbook [16] for a complete discussion.

Let Φ_0 be a countable set of propositional variables. Let $\mathcal{L}(\Phi_0)$, be the standard propositional modal language. That is $\phi \in \mathcal{L}(\Phi_0)$ iff ϕ has one of the following syntactic form,

[§] One needs to use, for example, the models with undefined formulas presented in [34], page 277-280; which are equivalent to the models TK-models used by Giovanna Corsi in [17], page 1478.

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi$$

where $p \in \Phi_0$. Use the standard definitions for the propositional connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ and the modal operator $\Diamond$. The standard propositional language may be denoted $\mathcal{L}$ when Φ_0 is understood.

Definition 2.1. A **neighborhood frame** is a pair $\langle W, N\rangle$, where W is a set of states, or worlds, and $N : W \rightarrow 2^{2^W}$ is a function.

Given a neighborhood frame, $\mathcal{F} = \langle W, N\rangle$, the function N is called a **neighborhood function**. The intuition is that at each state $w \in W$, $N(w)$ is the set of propositions, i.e. set of sets of states, that are either "necessary" or "known" or "believed", etc. at state w.

Definition 2.2. Given a neighborhood frame $\mathcal{F} = \langle W, N\rangle$, a **classical model** based on $\mathcal{F}$ is a tuple $\langle \mathcal{F}, V\rangle$, where $V : \Phi_0 \rightarrow 2^W$ is a valuation function.

Given a classical model $\mathcal{M} = \langle W, N, V\rangle$, truth is defined as follows, let $w \in W$ be any state:

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$ where $p \in \Phi_0$
2. $\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
3. $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Box\phi$ iff $(\phi)^{\mathcal{M}} \in N(w)$

where $(\phi)^{\mathcal{M}} \subseteq W$ is the set of all states in which ϕ is true. The dual of the modal operator $\Box$, denoted $\Diamond$, will be treated as a primitive symbol. The definition of truth for $\Diamond$ is

$$\mathcal{M}, w \models \Diamond\phi \text{ iff } W - (\phi)^{\mathcal{M}} \not\in N(w)$$

It is easy to see that given this definition of truth, the axiom scheme $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is valid in any neighborhood frame. Thus, in the presence of the E axiom scheme (see below) and a rule allowing substitution of equivalent formulas (which can be proven using the RE rule given below), we can treat $\Diamond$ as a defined symbol. As a consequence, in what follows we will not provide separate definitions for the $\Diamond$ operator since they can be easily derived. We say ϕ is valid in $\mathcal{M}$ iff $\mathcal{M}, w \models \phi$ for each $w \in W$. We say that ϕ is valid in a neighborhood frame $\mathcal{F}$ iff ϕ is valid in all models based on$\mathcal{F}$.

The following axiom schemes and rules have been widely discussed.

PC Any axiomatization of propositional calculus

E $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

M $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

C $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

N $\Box\top$

RE $\dfrac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$

MP $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$

Let **E** be the smallest set of formulas closed under instances of PC, E and the rules RE and MP. **E** is the smallest classical modal logic. The logic **EC** extends **E** by adding the axiom scheme C. Similarly for **EM**, **EN**, **ECM**, and **EMCN**. It is well known that the logic **EMCN** is equivalent to the normal modal logic **K** (see [16] page 237). Let **S** be any of the above logics, we write $\vdash_{\mathbf{S}} \phi$ if $\phi \in \mathbf{S}$.

Let N be a neighborhood function, $w \in W$ be an arbitrary state, and $X, Y \subseteq W$ be arbitrary subsets.

(m) If $X \cap Y \in N(w)$, then $X \in N(w)$ and $Y \in N(w)$

(c) If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$

(n) $W \in N(w)$

It is easy to show (see [16] page. 215) that (m) is equivalent to

(m') If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$

We say that a neighborhood function N is **supplemented**, **closed under intersection**, or **contains the unit** if it satisfies (m) (equivalently if it satisfies (m')), (c) and (n) respectively.

Definition 2.3. A frame $\langle W, N\rangle$ is **augmented** if N is supplemented and for each $w \in W$,

$$\bigcap N(w) \in N(w)$$

Call a frame supplemented if its neighborhood function is supplemented, similarly for the other semantic properties above. It is well-known that the logic **E** is sound and complete with respect to the class of all neighborhood frames. The other semantic conditions correspond to the obvious syntactic counterparts. For example, the logic **EMC** is sound and complete with respect to the class of all frames that are supplemented and closed under intersection. The completeness proofs are straightforward and are discussed in [16]. One final note about the propositional case will be important for this paper. The class of augmented frames is equivalent to the class of Kripke frames in the following sense.

THEOREM 2.4 ([16] page 221). *For every Kripke model $\langle W, R, V\rangle$, there is an pointwise equivalent classical model $\langle W, N, V\rangle$, and vice versa*

The proof can be found in [16] page 221. We only sketch the main points. Let $\mathcal{M}^K = \langle W, R, V\rangle$ be any Kripke model. To define a pointwise equivalent classical model $\mathcal{M} = \langle W, N, V\rangle$, we stipulate that for each $X \subseteq W$ and each $w \in W$,

$$X \in N(w) \text{ iff } \{w' \mid wRw'\} \subseteq X$$

It is easy to see that N is augmented and that $\mathcal{M}$ is pointwise equivalent to $\mathcal{M}^K$. To define $\mathcal{M}^K$ from a classical model $\mathcal{M} = \langle W, R, V\rangle$, stipulate that for each $w \in W$,

$$wRw' \text{ iff } w' \in \bigcap N(w)$$

Again, it is easy to see that $\mathcal{M}$ is pointwise equivalent to $\mathcal{M}^K$.

3. Classical systems of first order modal logic

The language of first order modal logic is defined as follows. Let $\mathcal{V}$ be a countable collection of individual variables. For each natural number $n \geq 1$, there is a (countable) set of n-place predicate symbols. These will be denoted by $F, G, \ldots$. In general, we will not write the arity of a predicate F. A formula of *first order modal logic* will have the following syntactic form

$$\phi := F(x_1, \ldots, x_n) \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \forall x\phi$$

Let $\mathcal{L}_1$ be the set of well-formed first order modal formulas. The other standard Boolean connectives, the diamond modal operator and the existential quantifier are defined as usual. The usual rules about free variables apply.

We write $\phi(x)$ when x (possibly) occurs free in ϕ. Denote by $\phi[y/x]$, ϕ in which free variable x is replaced with free variable y. The following axioms are taken from [34]. Let $\mathbf{S}$ be any classical propositional modal logic, let $\mathbf{FOL}+\mathbf{S}$ be the set of formulas closed under the following rules and axiom schemes:

S All axiom schemes and rules from $\mathbf{S}$.

$\forall$ $\forall x\phi(x) \rightarrow \phi[y/x]$ is an axiom scheme. ¶

Gen $\dfrac{\phi \rightarrow \psi}{\phi \rightarrow \forall x\psi}$, where x is not free in ϕ.

For example, $\mathbf{FOL}+\mathbf{E}$ contains the axiom scheme $PC, E, \forall$ and the rules Gen, MP. Given any classical propositional modal logic $\mathbf{S}$, we write $\vdash_{\mathbf{FOL}+\mathbf{S}} \phi$ if $\phi \in \mathbf{FOL}+\mathbf{S}$ (equivalently ϕ can be derived using the above axiom schemes and rules).

Notice that in the above axiom system there is no essential interaction between the modal operators and the first-order quantifiers. Two of the most widely discussed axiom schemes that allow interaction between the modal operators and the first-order quantifiers are the so-called Barcan formula and the converse Barcan formula.

Definition 3.1. Any formula of the form

$$\forall x\Box\phi(x) \rightarrow \Box\forall x\phi(x)$$

will be called a **Barcan formula** (BF). The **converse Barcan formula** (CBF) will be any formula of the form

$$\Box\forall x\phi(x) \rightarrow \forall x\Box\phi(x)$$

Technically, the Barcan and converse Barcan formulas are schemes not formulas, but we will follow standard terminology. For simplicity, we will write $\mathbf{S}+BF$ for the logic that includes all axiom schemes and rules of $\mathbf{FOL}+\mathbf{S}$ plus the Barcan formula BF. Similarly for $\mathbf{S}+CBF$.

Definition 3.2. A **constant domain neighborhood frame** for classical first-order modal logic is a tuple $\langle W, N, D\rangle$, where W is a set of possible worlds, N is a neighborhood function and D is any non-empty set, called the **domain**.

¶ According to the notation used in [34], which we are following here, this axiom follows from two additional principles called the principles of *replacement* and *agreement*. See [34] page 241. These principles guarantee that y is free for x occurring in $\phi(x)$.

Definition 3.3. A **constant domain neighborhood model** based on a frame $\mathcal{F} = \langle W, N, D\rangle$ is a tuple $\mathcal{M} = \langle W, N, D, I\rangle$, where I is a classical first-order interpretation function. Formally, for each n-ary predicate symbol F, $I(F, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$. A substitution σ' is said to be an x**-variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$. Truth is defined at a state relative to a substitution. Let $\mathcal{M} = \langle W, N, D, I\rangle$ be any constant domain neighborhood model and σ any substitution.

1. $\mathcal{M}, w \models_\sigma F(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in I(F, w)$ for each n-place predicate symbol F.
2. $\mathcal{M}, w \models_\sigma \neg\phi$ iff $\mathcal{M}, w \not\models_\sigma \phi$
3. $\mathcal{M}, w \models_\sigma \phi \wedge \psi$ iff $\mathcal{M}, w \models_\sigma \phi$ and $\mathcal{M}, w \models_\sigma \psi$
4. $\mathcal{M}, w \models_\sigma \Box\phi$ iff $(\phi)^{\mathcal{M},\sigma} \in N(w)$
5. $\mathcal{M}, w \models_\sigma \forall x\phi(x)$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} \phi(x)$

where $(\phi)^{\mathcal{M},\sigma} \subseteq W$ is the set of states $w \in W$ such that $\mathcal{M}, w \models_\sigma \phi$.

Before looking at completeness for classical systems of first-order modal logic, we survey the situation with respect to first order relational structures (deriving from seminal work by Saul Kripke). A relational frame is a tuple $\langle W, R\rangle$, where W is a set of states and $R \subseteq W \times W$ is an accessibility relation. A (constant domain) first order relational model based on a relational frame $\mathcal{F} = \langle W, R\rangle$ is a tuple $\langle W, R, D, I\rangle$ where D is a set and I is a first-order classical interpretation (defined above).

Truth is defined as above except for the modal case:

$$\mathcal{M}, w \models_\sigma \Box\phi \text{ iff for each } w' \in W, \text{ if } wRw' \text{ then } \mathcal{M}, w' \models_\sigma \phi$$

The following observations are well-known and easily checked (see [34] page 245).

Observation 3.4 ([34]). *The converse Barcan formula is provable in the logic* **FOL + K**

OBSERVATION 3.5 ([34]). *The Barcan formula is valid in all first order relational models with constant domains.*[‖]

Since the weakest propositional modal logic sound and complete for all relational frames is $\mathbf{K}$, we will focus on the logic $\mathbf{FOL} + \mathbf{K}$. Given Observation 3.5, if we want a completeness theorem with respect to all constant domain relational structures, we need to consider the logic $\mathbf{K} + BF$. Soundness is shown via the following theorem (Corollary 13.3, page 249 in [34]). Given any Kripke frame $\mathcal{F}$, we say that $\mathcal{F}$ is a frame for a logic $\mathbf{S}$ iff every theorem of $\mathbf{S}$ is valid on $\mathcal{F}$.

In order to make this paper self-contained, we now review some well-known techniques and results concerning first-order normal modal logics. The reader already familiar with such results may want to skip to Section 3. Recall, that a normal modal logic is a propositional logic that contains at least the axiom scheme K and the rule Nec.

THEOREM 3.6 ([34] page 249). *Let* $\mathbf{S}$ *be any propositional normal modal logic, then a Kripke frame* $\mathcal{F}$ *is a frame for* $\mathbf{S}$ *iff* $\mathcal{F}$ *is a frame for* $\mathbf{S} + BF$*.*

The proof of the only if direction is a straightforward induction on a derivation in the logic $\mathbf{S} + BF$. As for the if direction, the main idea is that given $\mathcal{F}$ which is not a frame for $\mathbf{S}$, one can construct a frame $\mathcal{F}^*$ that is not a frame for $\mathbf{S} + BF$.

Essentially this theorem shows that the notion of a frame for first order Kripke models is *independent* of the domain D. That is, the proof of the above theorem goes through *whatever* D *may be*.

We need some more definitions before proving a completeness theorem. Let Λ of formulas of first order modal logic.

DEFINITION 3.7. A set Λ has the $\forall-$**property**[**] iff for each formula $\phi \in \Lambda$ and each variable x, there is some variable y, called the witness, such that $\phi[y/x] \rightarrow \forall x \phi(x) \in \Lambda$.

The proof of the following Lindenbaum-like Lemma is a straightforward and can be found in [34] page 258

[‖]There are, of course, many different ways in which a relational semantics can be presented and not all are inspired by Kripke's early work. For example, Barcan Marcus offered in [10] a model theoretic semantics with constant domains where both Barcan schemas are valid.

[**]The terminology follows the one used in [34]. The property is also known as Henkin's property.

LEMMA 3.8. *If X is a consistent set of formulas of $\mathcal{L}_1$, then there is a consistent set of formulas Y of $\mathcal{L}_1^+$ with the $\forall$-property such that $X \subseteq Y$, where $\mathcal{L}_1^+$ is the language $\mathcal{L}_1$ with countably many* new *variables.*

We can now define the canonical model for first order Kripke structures. The canonical model, $\mathcal{M}_C = \langle W_C, R_C, D_C, I_C \rangle$, is defined as follows. Given the first- order modal language $\mathcal{L}_1$, let $\mathcal{L}_1^+$ be the extension of $\mathcal{L}_1$ used in Lemma 3.8 and $\mathcal{V}^+$ the variables in this extended language. Finally let $MAX(\Gamma)$ indicate that the set Γ is a maximally consistent set of formulas of $\mathcal{L}_1^+$.

$$W_C = \{\Gamma \mid \ MAX(\Gamma) \text{ and } \Gamma \text{ has the } \forall\text{-property}\}$$

$$\Gamma R_C \Delta \text{ iff } \{\phi \mid \Box\phi \in \Gamma\} \subseteq \Delta$$

$$D_C = \mathcal{V}^+$$

$$\langle x_1, \ldots, x_n \rangle \in I_C(\phi, \Gamma) \text{ iff } \phi(x_1, \ldots, x_n) \in \Gamma$$

Let σ be the canonical substitution: for each $x \in \mathcal{V}^+$, $\sigma(x) = x$. The truth Lemma is

LEMMA 3.9. *For any $\Gamma \in W_C$, and any formula $\phi \in \mathcal{L}^+$,*

$$\phi \in \Gamma \textit{ iff } \mathcal{M}_C, \Gamma \models_\sigma \phi$$

Of course, the proof is by induction on the formula ϕ. The base case and propositional connectives are as usual. The 'if' direction for the modal and quantifier case is straightforward. We will discuss two cases. Suppose that $\forall x\phi(x) \notin \Gamma$. Then since Γ is maximal, $\neg\forall x\phi(x) \in \Gamma$, and so by the $\forall$-property, there is some variable $y \in \mathcal{V}^+$ such that $\neg\phi[y/x] \in \Gamma$, and so $\phi[y/x] \notin \Gamma$. Thus by the induction hypothesis, $\mathcal{M}_C, \Gamma \not\models_{\sigma'} \phi[y/x]$, where σ' is the x-variant of σ in which $\sigma(x) = y$. Hence, $\mathcal{M}_C, \Gamma \not\models_\sigma \forall x\phi(x)$. The modal case relies on the following Lemma, which requires the Barcan formula in its proof.

LEMMA 3.10. *If Γ is a maximally consistent set of formulas (of $\mathcal{L}^+$) with the $\forall$-property and ϕ is a formula such that $\Box\phi \notin \Gamma$, then there is a consistent set of formulas of $\mathcal{L}^+$ with the $\forall$-property such that $\{\psi \mid \Box\psi \in \Gamma\} \cup \{\neg\phi\} \subseteq \Delta$.*

Given this Lemma, the proof of the truth Lemma above is straightforward.

3.1. First-order neighborhood frames

As with Kripke frames, validity of certain axiom schemes in a frame corresponds to properties on that frame. In this section, we discuss the connections between validity of formulas and properties of the corresponding frame. We first need some notation. Recall that a neighborhood frame is a tuple $\langle W, N\rangle$ where W is a set of states and $N : W \to 2^{2^W}$ is the neighborhood function. A first-order constant domain neighborhood frame is a triple $\langle W, N, D\rangle$ where W and N are as above and D is an arbitrary non-empty set called the domain. By a frame, we mean either a neighborhood frame or a first-order neighborhood frame.

A **filter** is any collection of sets that is closed under finite intersections and supersets. A filter is non-trivial if it does not contain the empty set. We say that a frame $\mathcal{F}$ is a filter if for each $w \in W$, $N(w)$ is a filter. A frame $\mathcal{F}$ is closed under infinite intersections if for each $w \in W$, $N(w)$ is closed under infinite intersections. Finally, a frame is augmented (recall Definition 2.3) if N is supplemented and for each $w \in W$, $\bigcap N(w) \in N(w)$. Obviously, every augmented frame is closed under infinite intersections, but there are supplemented frames closed under infinite intersections that are not augmented (provided that W is infinite).

In [5] it is shown that the presence of the Barcan and the Converse Barcan formulas implies interesting properties on the corresponding frame. Before reporting these results, we need some definitions.

Definition 3.11. A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$ and $\{\emptyset\} \notin N(w)$.

Definition 3.12. A first-order neighborhood frame $\mathcal{F} = \langle W, N, D\rangle$ is **non-trivial** iff $|D| > 1$

First of all, a couple of facts about frames which are trivial or not consistent. First of all, if $\mathcal{F}$ is a trivial first-order neighborhood model with constant domain, then $\Box\forall x\phi(x) \leftrightarrow \forall x\Box\phi(x)$ is valid. Thus both the Barcan and converse Barcan formulas are valid on trivial domains. Secondly if $\mathcal{F}$ is not consistent, then for trivial reasons both the Barcan and converse Barcan are valid.

Observation 3.13 ([5]). *Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.*

Proof. Suppose that $\mathcal{F}$ is a consistent constant domain first-order neighborhood model. If $\mathcal{F}$ is trivial, then, as noted above, the converse Barcan formula is valid. Suppose that $\mathcal{F}$ is supplemented and that $\mathcal{M}$ is an arbitrary model based on $\mathcal{F}$. Let w be an arbitrary state, we will show that $\mathcal{M}, w \models \Box\forall x\phi(x) \rightarrow \forall x\Box\phi(x)$. Suppose that $\mathcal{M}, w \models_\sigma \Box\forall x\phi(x)$. Therefore $(\forall x\phi(x))^{\mathcal{M},\sigma} \in N(w)$.

Since $(\forall x\phi(x))^{\mathcal{M},\sigma} = \bigcap_{\sigma'\sim_x\sigma}(\phi(x))^{\mathcal{M},\sigma'}$ and $N(w)$ is supplemented, for each σ' which is an x-variant of σ, $(\phi(x))^{\mathcal{M},\sigma'} \in N(w)$. But this implies $\mathcal{M}, w \models_\sigma \forall x\Box\phi(x)$. Hence the right to left implication is proven.

For the other direction, we must show that if $\mathcal{F}$ is a consistent, nontrivial and not supplemented, then the converse Barcan formula is not valid. Since $\mathcal{F}$ is not supplemented, there is some state w and sets X and Y such that $X \cap Y \in N(w)$ but either $X \notin N(w)$ or $Y \notin N(w)$. We need only construct a model $\mathcal{M}$ in which an instance of the converse Barcan formula is not true. Let F be a unary predicate symbol. We will construct a model $\mathcal{M}$ based on $\mathcal{F}$ where $\mathcal{M}, w \models_\sigma \Box\forall xF(x)$ but $\mathcal{M}, w \not\models_\sigma \forall x\Box F(x)$. WLOG assume that $X \notin N(w)$.

Then we have two cases: 1. $Y \subseteq X$ and 2. $Y \not\subseteq X$. Suppose we are in the first case. That is $Y \subseteq X$, then $Y = X \cap Y \in N(w)$. Suppose that 1. for each $v \in X \cap Y$ and for all $d \in D$, $\langle d\rangle \in I(F, v)$ and 2. $I(F, v') = \emptyset$ for all $v' \in W - X$.

Then for any substitution σ, $(\forall xF(x))^{\mathcal{M},\sigma} = \bigcap_{\sigma'\sim_x\sigma}(F(x))^{\mathcal{M},\sigma'} = X \cap Y \in N(w)$. Hence $\mathcal{M}, w \models_\sigma \Box\forall xF(x)$.

Now to complete the description of the model, choose an element $a \in D$ and set $a \in I(F, y)$ for each $y \in X - (X \cap Y)$ and such that for no other $b \in D$, $b \in I(F, y)$. Then if $\sigma(x) = a$, $(F(x))^{\mathcal{M},\sigma} = X \notin N(w)$. Hence, $\mathcal{M}, w \not\models_\sigma \forall x\Box F(x)$. The non-triviality of $\mathcal{F}$ is needed to ensure that $(F(x))^{\mathcal{M},\sigma} \neq (\forall xF(x))^{\mathcal{M},\sigma}$. As for case 2 ($Y \not\subseteq X$), we can require that $I(F, v) = \emptyset$ for all $v \in Y - (X \cap Y)$, reducing to the first case.

$\blacksquare$

OBSERVATION 3.14. $\mathbf{FOL} + \mathbf{EM} \vdash CBF$

We now turn our attention to BF. In [5] it is shown that the Barcan formula corresponds to interesting properties on first-order constant domain neighborhood frames. We first need some notation. Let κ be a cardinal. We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i\in I}X_i \in N(w)$.

DEFINITION 3.15. A consistent first-order neighborhood frame $\langle W, N, D\rangle$ is **monotonic** iff the number of objects in its domain is as large as the number of sets in the neighborhood with the largest number of sets in the frame.

[5] shows the following result about monotonic frames: Let $\mathcal{F}$ be a constant, consistent and monotonic domain neighborhood frame.

$$\forall x \Box \phi(x) \rightarrow \Box \forall x \phi(x)$$

is valid on every model based on $\mathcal{F}$ if and only if $\mathcal{F}$ is either trivial or closed under infinite intersections. Monotony cannot be dropped from an unrevised formulation of this result. In fact, consider the following counterexample:

Counterexample Let the neighborhoods of $\mathcal{F}$ contain exactly an infinite (but countable) family of sets $X = X_i$ with $i = 1, 2, ...$ The domain of worlds is also infinite (but countable). Let $\mathcal{F}$ be closed under finite intersections but not under infinite intersections.

Let now the domain of objects contain exactly two objects a, b. It is clear that $\mathcal{F}$ is non-trivial (its cardinality is strictly greater than 1). Therefore if the revised formulation of the result from [5] (where monotony is dropped) were true this entails that there is a model $\mathcal{M}$ based on $\mathcal{F}$, world w and substitution σ, such that: $\mathcal{M}, w \models_\sigma \forall x \Box \phi(x)$; but $\mathcal{M}, w \not\models_\sigma \Box \forall x \phi(x)$.

Since we have $\mathcal{M}, w \models_\sigma \forall x \Box \phi(x)$ we know that for each x-variant σ' of σ,

$\mathcal{M}, w \models_{\sigma'} \Box \phi(x)$;

Given the constitution of the domain for each substitution σ there is only one x-variant of it, which we will cal σ'. Therefore we have that both $(\forall x \phi(x))^{\mathcal{M},\sigma}$ and $(\forall x \phi(x))^{\mathcal{M},\sigma'}$ are in $N(w)$. Since $\mathcal{M}$ based on $\mathcal{F}$, and $\mathcal{F}$ in closed under finite intersections then we have that:

$\cap_{\sigma' \sim_x \sigma} (\phi(x))^{\mathcal{M},\sigma'} \in N(w)$.

contradicting the assumption that $\mathcal{M}, w \not\models_\sigma \Box \forall x \phi(x)$

•

Nevertheless the example shows that there are non-monotonic frames (not closed under infinite intersections) where the Barcan schema is validated. Nevertheless one can adapt the method of proof used in [5] to prove the following result giving necessary and sufficient conditions for the frame validity of the Barcan formula (where the cardinality restriction imposed by monotony is dropped):

Observation 3.16. *Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either 1. $\mathcal{F}$ is trivial or 2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.*

Proof. Suppose that $\mathcal{F}$ is a consistent constant domain first-order neighborhood model. If $\mathcal{F}$ is trivial, then, as noted above, the Barcan formula is valid. Suppose condition 2. holds and let $\mathcal{M} = \langle W, N, D, I\rangle$ be any model based on $\mathcal{F}$. Given any state $w \in W$ and substitution σ, we must show that $\mathcal{M}, w \models_\sigma \forall x \Box \phi(x) \rightarrow \Box \forall x \phi(x)$. Suppose that $\mathcal{M}, w \models_\sigma \forall x \Box \phi(x)$. If D is finite, then $\{(\phi(x))^{\mathcal{M},\sigma'} \mid \sigma' \sim_x \sigma\}$ is finite and since $(\phi(x))^{\mathcal{M},\sigma'} \in N(w)$ for each $\sigma' \sim_x \sigma$ and N is closed under finite intersections $\bigcap\{(\phi(x)^{\mathcal{M},\sigma'} \mid \sigma' \sim_x \sigma\} \in N(w)$. Therefore, $\mathcal{M}, w \models_\sigma \Box \forall x \phi(x)$. The proof is similar if D is is infinite.

For the other direction suppose that $\mathcal{F} = \langle W, N, D\rangle$ is non-trivial. Since D is nontrivial it contains at least two distinct elements, say $d, c \in D$ such that $d \neq c$. Suppose that D is finite and $\mathcal{F}$ is not closed under finite intersections. Since $\mathcal{F}$ is not closed under finite interesections there is a state w such that $N(w)$ is not closed under finite intersections. This means that there are two sets X, Y such that $X, Y \in N(w)$ but $X \cap Y \notin N(w)$. To see this, let n be the size of the smallest collection of sets $\mathbb{C}$ such that each element of $\mathbb{C}$ is in $N(w)$ but $\cap\mathbb{C} \notin N(w)$. If $n = 2$ we are done. Otherwise, suppose that $n > 2$. In this case we can partition $\mathbb{C}$ into two subclasses, $\mathbb{C}_1$ and $\mathbb{C}_2$ such that each element of $\mathbb{C}_i$ is in $N(w)$ and $\cap\mathbb{C}_i \in N(w)$ (for $i = 1, 2$). Note that $\cap\mathbb{C}_1 \cap \cap\mathbb{C}_2 \notin N(w)$ by assumption. But then $\{\cap\mathbb{C}_1, \cap\mathbb{C}_2\}$ is a collection of size two both of whose elements are in $N(w)$ and whose intersection is not in $N(w)$. This contradicts the assumption that $n > 2$. Thus there are two sets X and Y such that $X \cap Y \notin N(w)$.

Given $X, Y \in N(w)$ such that $X \cap Y \notin N(w)$ we construct a model based on $\mathcal{F}$ that invalidates the Barcan formula. Let F be a unary predicate symbol. The idea is to define I such that $(F(x))^{\mathcal{M},\sigma} = X$ if $\sigma(x) = c$ and $(F(x))^{\mathcal{M},\sigma} = Y$ if $\sigma(x) = d$. Hence for each $v \in X$, let $c \in I(F, v)$. Then if $\sigma(x) = c$, $(F(x))^{\mathcal{M},\sigma} = X$. For each $v \in Y$, let $d \in I(F, v)$. Then if $\sigma(x) = d$, $(F(x))^{\mathcal{M},\sigma} = Y$. For the other elements c of the domain (if they exist), fix a set, say Y, and let $c \in I(F, v)$ for each $v \in Y$. Let σ be any arbitrary assignment. We have $\mathcal{M}, w \models_\sigma \forall x \Box F(x)$. However, $(\forall x F(x))^{\mathcal{M},\sigma} = X \cap Y \notin N(w)$. Hence, $\mathcal{M}, w \not\models \Box \forall x F(x)$ and so the Barcan formula is not valid.

Suppose that D is infinite and of cardinality κ. Since $\mathcal{F}$ is not closed under $\leq \kappa$ intersections there is a state w and a collection $\{X_i \mid i \in I\}$ where

$|I| \leq \kappa$ such that $\bigcap_{i \in I} X_i \notin N(w)$. Since $|I| \leq |D|$ there is a 1-1 function $f : I \to D$. Thus we can find for each X_i a unique $c \in D$, call it c^{X_i}. The argument is similar to the above argument. Let F be a unary predicate. For each X_i, define $F(x)$ so that $F(x)^{\mathcal{M},\sigma} = X_i$ provided $\sigma(x) = c^{X_i}$. That is for each X_i, for each $v \in X_i$, let $c^{X_i} \in I(F, v)$. For the other elements c of the domain (if they exist), fix any set, say X_j, and let $c \in I(F, v)$ for each $v \in X_j$. Now for any σ, $F(x)^{\mathcal{M},\sigma} = X_i$ for some $i \in I$, hence $(F(x))^{\mathcal{M},\sigma} \in N(w)$. However $(\forall x F(x))^{\mathcal{M},\sigma} = \bigcap_{\sigma \sim_x \sigma'} F(x)^{\mathcal{M},\sigma'} = \bigcap_{i \in I} X_i \notin N(w)$. Hence the Barcan formula is not valid.

■

3.2. Completeness of classical systems of first-order modal logic

In this section we discuss the completeness of various classical systems of first-order modal logic. We start by defining the smallest canonical model for classical first-order modal logic. Let Λ be any first-order classical modal logic. Define $\mathcal{M}_\Lambda = \langle W_\Lambda, N_\Lambda, D_\Lambda, I_\Lambda \rangle$ as follows. Let $MAX_\Lambda(\Gamma)$ indicate that the set Γ is a Λ-maximally consistent set of formulas of $\mathcal{L}_1^+$.

$$W_\Lambda = \{\Gamma \mid MAX_\Lambda(\Gamma) \text{ and } \Gamma \text{ has the } \forall\text{-property}\}$$

$$X \in N_\Lambda(\Gamma) \text{ iff for some } \Box\phi \in \Gamma,\ X = \{\Delta \in W_\Lambda \mid \phi \in \Delta\}$$

$$D_\Lambda = \mathcal{V}^+$$

$$\langle x_1, \ldots, x_n \rangle \in I_\Lambda(\phi, \Gamma) \text{ iff } \phi(x_1, \ldots, x_n) \in \Gamma$$

$$\text{For every variable } x \in \mathcal{V}^+, \sigma(x) = x$$

where $\mathcal{V}^+$ is the extended set of variables used in Lemma 3.8. The definition of the neighborhood function N_Λ essentially says that a set of states of the canonical model is necessary at a world Γ precisely when Γ *claims that it should*. For any formula $\phi \in \mathcal{L}_1$, let $|\phi|_\Lambda$ be the proof set of ϕ in the logic Λ, that is,

$$|\phi|_\Lambda = \{\Gamma \mid \Gamma \in W_\Lambda \text{ and } \phi \in \Gamma\}$$

The fact that N_Λ is a well-defined function follows from the fact that Λ contains the rule *RE*.

Definition 3.17. Let $\mathcal{M} = \langle W, N, D, I\rangle$ be any first-order constant domain neighborhood model. $\mathcal{M}$ is said to be **canonical for** a first-order classical system Λ provided $W = W_\Lambda$, $D = D_\Lambda$, $I = I_\Lambda$ and

$$|\phi|_\Lambda \in N(\Gamma) \text{ iff } \Box\phi \in \Gamma$$

Thus the model $\mathcal{M}_\Lambda$ is the smallest canonical model for a logic Λ. It is shown in Chellas ([16]) that if $\mathcal{M} = \langle W, N, D, I\rangle$ is a canonical model, then so is $\mathcal{M}' = \langle W, N', D, I\rangle$,where for each $\Gamma \in W$, $N'(\Gamma) = N(\Gamma) \cup \{X \subseteq W \mid X \neq |\phi|_\Lambda$ for any $\phi \in \mathcal{L}_1\}$. That is N' is N with all of the non-proof sets.

Lemma 3.18 (Truth Lemma). *For each $\Gamma \in W_\Lambda$ and formula $\phi \in \mathcal{L}_1$,*

$$\phi \in \Gamma \textit{ iff } \mathcal{M}_\Lambda, \Gamma \models_\sigma \phi$$

Proof. The proof is by induction on ϕ. The base case and propositional connectives are as usual. The quantifier case is exactly as in the Kripke model case (refer to Lemma 3.9). We need only check the modal cases. The proof proceeds easily by construction of N_Λ and definition of truth: $\Box\phi \in \Gamma$ iff (by construction) $|\phi| \in N_\Lambda(\Gamma)$ iff (by definition of truth) $\mathcal{M}_\Lambda, \Gamma \models_\sigma \Box\phi$. ■

Notice that in the above proof, as opposed to the analogous result for relational models, we can construct a constant domain model without making use of the Barcan formula. The following corollary follows from the truth Lemma via a standard argument.

Theorem 3.19. *For any canonical model $\mathcal{M}$ for a classical first-order modal logic Λ, ϕ is valid in the canonical model $\mathcal{M}$ iff $\vdash_\Lambda \phi$*

Corollary 3.20. *The class of all first-order neighborhood constant domain frames is sound and complete for* **FOL + E**.

Notice that in the canonical model $\mathcal{M}_\Lambda$ constructed above, for any state Γ, the set $\mathcal{N}_\Lambda(\Gamma)$ contains only proof sets, i.e., sets of the form $\{\Delta \mid \phi \in \Delta\}$ for some formula $\phi \in \mathcal{L}_1$. For this reason, even if Λ contains the M axiom scheme, N_Λ may not be supplemented. Essentially the reason is if $X \in N_\Lambda(\Gamma)$, and $X \subseteq Y$, Y may not be a proof set, so cannot possibly be in $N_\Lambda(\Gamma)$. The **supplementation** of a frame $\mathcal{F} = \langle W, N\rangle$ ($\langle W, N, D\rangle$), denoted $\mathcal{F}^+$, is a tuple $\langle W, N^+\rangle$ ($\langle W, N^+, D\rangle$) where for each $w \in W$, $N^+(w)$ is the smallest collection of sets containing $N(w)$ that is closed under superset. It can be shown that the supplementation $\mathcal{M}_\Lambda^+$ is a canonical for **FOL + EM** (by adapting to the first order case the proof offered in [16] page 257).

THEOREM 3.21. **FOL + EM** *is sound and complete with respect to the class of supplemented first-order constant domain neighborhood frames.*

Example: Qualitative probability defined over rich languages The system **EMN** and first order extensions of it seem to play a central role in characterizing monadic operators of high probability. These operators have been studied both in [36] and in [6]. Roughly the idea goes as follows: let W be a set of states and Σ_W a σ-algebra generated by W. Let $P : \Sigma_W \to [0, 1]$ be a probability measure and $t \in (0.5, 1)$. Let $\mathcal{H}_t \subseteq \Sigma_W$ be the set of events with "high" probability with respect to t, that is $\mathcal{H}_t = \{X \mid P(X) > t\}$. It is easy to see that $\mathcal{H}_t$ is closed under superset (**M**) and contains the unit (**N**). A similar construction is offered in [36], where the authors claim that the resulting propositional logic is **EMN**.[††]

The results offered in [36] show that the approach in terms of neighborhoods has interesting (and potentially rich) applications. These applications go beyond the study of monadic operators of high probability. In [6] Arlo-Costa considers the more general case of non-adjunctive modalities and he develops a measure of the level of coherence of neighborhoods. These models can, in turn, be generalized in order to classify parametrically other families of paraconsistent logics, aside from the non-adjunctive ones.

The probabilistic applications considered in [36] can also be generalized. In fact, they cover only the case of propositional languages. But, as Henry Kyburg has pointed out in [35] researchers in various communities (most recently in Artificial Intelligence) have been interested in having a qualitative notion of probability defined over a language at least as reach as first order logic. As Kyburg correctly points out in [35] one of the (misguided) reasons advanced by the researchers in the 'logical' branch of AI for *not* using probability in knowledge representation was related to the alleged difficulties in defining a notion of probability over a rich language. Kyburg reminds the reader of the seminal work already done in this area by Gaifman and Snir [24] and by Scott and Krauss [46]. Nevertheless Kyburg and Teng do not appeal to these accounts of probability over first order languages in their own model of qualitative (monadic) probability. This might have been caused by the fact that first order classical modal logics have not been studied carefully in the previous literature.

[††]The model offered in [36] differs from the one sketched here in various important manners. First it attributes probability to sentences, not to events in field. Second it works with a notion of primitive conditional probability which is finitely additive. Third probability in Kyburg and Teng's model is not personal probability but objective chance, which is interval-valued.

We claim that the accounts of monadic operators of high probability presented in [35] and [5] can be generalized by appealing to the tools presented here. We will sketch in this example how this can be done by appealing to the account offered in [24]. This is one the most detailed models of how to attribute probability to first order sentences in the literature. There are, nevertheless, various discontinuities between this account and Kyburg and Teng's. For example Gaifman's account intends to axiomatize a notion of probability which does not conflict with the standard Kolmogorovian account, while Kyburg and Teng work with finitely additive primitive conditional probability. These foundational issues might have some logical repercussions but the semantic framework utilized here is largely neutral with respect to those issues. So, the proposal below can be seen as a model of high probability operators for a Kolmogorovian notion of probability defined over a first order language.

Let L_0 be a first order language for arithmetic. So L_0 has names for the members of N, aside from symbols for addition and multiplications, variables that take values on N and quantifiers, etc. Notice that this language is richer than the one used above. For example, it has constants, which in this case are numerals $n_1, \ldots$. We use 'n' ambiguously for natural numbers and their numerals. Let in addition L be an extension of L_0 containing a finite amount of atomic formulas of the form $R(t_0, \ldots, t_k)$ where t_i is either a variable or a numeral. Let Pr be a nonnegative real-valued function defined for the sentences of L and such that the following conditions hold:

1 If $\models \psi \leftrightarrow \phi$ then $Pr(\psi) = Pr(\phi)$

2 If $\models \psi$, then $Pr(\psi) = 1$.

3 If $\models \neg(\psi \wedge \phi)$, then $Pr(\psi \vee \phi) = Pr(\psi) + Pr(\phi)$

4 $Pr(\exists x \phi(x)) = Sup\{Pr(\phi(n_1) \vee \ldots \vee \phi(n_k)) : n_1 \ldots n_k \in N,\ k = 1,\ 2,\ \ldots\}$

Condition (4) is the substantive condition, which in [46] is called ' Gaifman's condition'. It is clear that we can use this notion of probability in order to define a modality 'ψ is judged as highly probable by individual i' modulo a threshold t. So, for example, for (a finite stock of) atomic formulas we will have

$\mathcal{M}, w \models_\sigma \Box R(t_1, \ldots, t_n)$ if and only if $(R(t_1, \ldots, t_n))^{\mathcal{M},\sigma} \in N(w)$ if and only if $Pr_w(R(\sigma(t_1), \ldots, \sigma(t_n)) > t$.

And in general neighborhoods contain the propositions expressed by sentences of L to which an agent of reference assigns high probability. Of course, other interesting modalities, like 'sequence s is random' or 'ψ is judged as more probable than ϕ', etc. can also be defined. In view of the previous results we conjecture that the logic that thus arises should be an extension of the *non-nested fragment* of $\mathbf{FOL}+\mathbf{EMN}$ – we refer to the logic encoding the valid formulas determined by the aforementioned high probability neighborhoods. Of course, BF should fail in this case, given that it instantiates cases of what is usually known as the 'lottery paradox'$\bullet$

The situation is more complicated in the case of $\mathbf{FOL}+\mathbf{E}+CBF$. We need some definitions in order to consider this case.

DEFINITION 3.22. A classical system of first order modal logic Λ is *canonical* if and only if the frame of at least one of its canonical models is a frame for Λ.

Now we can establish a result showing that $\mathbf{FOL}+\mathbf{E}+CBF$ is not canonical (in the strong sense we just defined). That is we will show that the frame of the smallest canonical model for $\mathbf{FOL}+\mathbf{E}+CBF$ not a frame for $\mathbf{FOL}+\mathbf{E}+CBF$.

OBSERVATION 3.23. $\mathbf{FOL}+\mathbf{E}+CBF$ *is not canonical.*

PROOF. Assume by contradiction that $\mathbf{FOL}+\mathbf{E}+CBF$ is canonical. If this were so, given that (by construction) the frame of any canonical model for $\mathbf{FOL}+\mathbf{E}+CBF$ is non-trivial (see Definition 3.12 for non-triviality), there is at least a frame of one of the canonical models of $\mathbf{FOL}+\mathbf{E}+CBF$ (which is also a frame for $\mathbf{FOL}+\mathbf{E}+CBF$) and that frame should also be supplemented (by Observation 3.13) as CBF is obviously valid. And if this were the case then every first order instance of M should also be valid in the canonical model. But this implies that $M \in \mathbf{FOL}+\mathbf{E}+CBF$, which is a contradiction. $\blacksquare$

It is easy to see that the previous argument can be extended to show that $\mathbf{FOL}+\mathbf{E}+CBF$ is not complete with respect to any class of non-trivial frames (i.e. a class where each frame is non-trivial). However, it is not difficult to see that if we take the class of non-trivial supplemented first-order neighborhood frames and we add to it a trivial and non-supplemented frame (of the sort used in Observation 3.17) this class fully characterizes $\mathbf{FOL}+\mathbf{E}+CBF$. This is so given that CBF continues to be valid with respect to the widened class, but the addition of the trivial frame guarantees that $\mathbf{M}$ is no longer valid in the widened class of frames. We first need a definition. The **augmentation** of a frame $\mathcal{F}=\langle W,N\rangle$ ($\langle W,N,D\rangle$), denoted $aug(\mathcal{F})$, is the supplementation of $\mathcal{F}'$, where $\mathcal{F}'$ is the tuple $\langle W, N^!\rangle$ ($\langle W, N^!, D\rangle$) and for each $w \in W$, $N^!(w) = N(w) \cup \{\bigcap N(w)\}$. The supplementation of a model $\mathcal{M}$ based on a frame $\mathcal{F}$, denoted $\mathcal{M}^+$, is the corresponding model based on $\mathcal{F}^+$. Similarly for augmentation.

OBSERVATION 3.24. *The augmentation of the smallest canonical model for* $\mathbf{FOL}+\mathbf{K}$ *is not a canonical model for* $\mathbf{FOL}+\mathbf{K}$. *In fact, the closure under infinite intersection of the minimal canonical model for* $\mathbf{FOL}+\mathbf{K}$ *is not a canonical model for* $\mathbf{FOL}+\mathbf{K}$.

PROOF. Let $\mathcal{M}_\Lambda$ where $\Lambda = \mathbf{FOL}+\mathbf{K}$ be the smallest canonical model for $\mathbf{FOL}+\mathbf{K}$ and $aug(\mathcal{M}_\Lambda) = \langle W_\Lambda, N^!, D_\Lambda, I_\Lambda\rangle$ its augmentation. Suppose that $\forall x\Box\phi(x) \in \Gamma$. Then for each $y \in \mathcal{V}^+$, $\Box\phi(y) \in \Gamma$, and so for each $y \in \mathcal{V}^+$, $|\phi(y)|_\Lambda \in N^!_\Lambda(\Gamma)$. Hence $\bigcap_{x\in\mathcal{V}^+}|\phi(x)|_\Lambda \in N^!_\Lambda(\Gamma)$. This follows if we assume N_Λ is closed under infinite intersections. Since $|\forall x\phi(x)|_\Lambda = \bigcap_{x\in\mathcal{V}^+}|\phi(x)|_\Lambda \in N^!_\Lambda(\Gamma)$, we have $\Box\forall x\phi(x) \in \Gamma$. Therefore, for each Γ, $BF \in \Gamma$. But this is a contradiction since BF is not provable in $\mathbf{FOL}+\mathbf{K}$.

∎

LEMMA 3.25. *The augmentation of the smallest canonical model for* $\mathbf{FOL}+\mathbf{K}+BF$ *is a canonical for* $\mathbf{FOL}+\mathbf{K}+BF$.

PROOF. Suppose that $\Lambda = \mathbf{FOL}+\mathbf{K}+BF$ and $\mathcal{M}_\Lambda$ is the smallest canonical model for Λ. Let $aug(\mathcal{M}_\Lambda) = \langle W_\Lambda, N^!_\Lambda, D_\Lambda, I_\Lambda\rangle$ be the augmentation of $\mathcal{M}_\Lambda$. We must show, for any formula $\phi \in \mathcal{L}_1$ and any state $\Gamma \in W_\Lambda$,

$$\Box\phi \in \Gamma \text{ iff } |\phi|_\Lambda \in N^!_\Lambda(\Gamma)$$

The proof is by induction on the complexity of ϕ.

- The boolean connectives and the base cases are straightforward.
- (Quantifier Case) Suppose that ϕ is $\forall x\psi(x)$. Then if $\Box\forall x\psi(x) \in \Gamma$, using CBF, $\forall x\Box\psi(x) \in \Gamma$. Since Γ is a maximally consistent set, for each $y \in \mathcal{V}^+$, $\Box\psi(y) \in \Gamma$. By the induction hypothesis, $|\phi(y)|_\Lambda \in N^!_\Lambda(\Gamma)$ for each $y \in \mathcal{V}^+$. By the construction of $N^!_\Lambda(\Gamma)$, $\bigcap_{y\in\mathcal{V}^+}|\psi(y)|_\Lambda \in N^!_\Lambda(\Gamma)$, hence $|\forall x\psi(x)|_\Lambda \in N^!_\Lambda(\Gamma)$ as desired. Suppose that $\Box\forall x\psi(x) \notin \Gamma$. Then $\neg\Box\forall x\psi(x) \in \Gamma$ and using BF, we have $\neg\forall x\Box\psi(x) \in \Gamma$. Then if $|\forall x\psi(x)|_\Lambda \in N^!_\Lambda(\Gamma)$ we have for each $y \in \mathcal{V}^+$, $|\psi(y)|_\Lambda \in N^!_\Lambda(\Gamma)$ since $N^!_\Lambda(\Gamma)$ is supplemented. By the induction hypothesis, for each $y \in \mathcal{V}^+$, $\Box\phi(y) \in \Gamma$. Hence $\forall x\Box\phi(x) \in \Gamma$. But this is a contradiction, so $|\forall x\psi(x)|_\Lambda \notin N^!_\Lambda(\Gamma)$, as desired.

- (Modal Case) Suppose that ϕ is $\Box\psi$. For the left to right direction, if $\Box\phi \in \Gamma$, then $|\phi|_\Lambda \in N_\Lambda(\Gamma)$ and so $|\phi|_\Lambda \in N^!_\Lambda(\Gamma)$ (since $N_\Lambda(\Gamma) \subseteq N^!_\Lambda(\Gamma)$). For the other direction, assume that $|\phi|_\Lambda = |\Box\psi|_\Lambda \in N^!_\Lambda(\Gamma)$. Hence $\bigcap N_\Lambda(\Gamma) \subseteq |\Box\psi|_\Lambda$. We must show $\Box\Box\psi \in \Gamma$. Now, it is easy to see that for each maximally consistent set Δ,

$$\Delta \in \bigcap N_\Lambda(\Gamma) \text{ iff } \{\alpha \mid \Box\alpha \in \Gamma\} \subseteq \Delta$$

  Thus for each Δ with $\{\alpha \mid \Box\alpha \in \Gamma\} \subseteq \Delta$, $\Box\psi \in \Delta$. Then $\{\alpha \mid \Box\alpha \in \Gamma\} \vdash \Box\psi$. Hence by compactness there are $\alpha_1, \ldots, \alpha_n$ in $\{\alpha \mid \Box\alpha \in \Gamma\}$ with $\vdash (\alpha_1 \wedge \cdots \wedge \alpha_n) \to \Box\psi$. Hence $\Box\Box\psi \in \Gamma$, as desired.

■

THEOREM 3.26. **FOL + K** *is sound and complete with respect to the class of filters.*

Example: Finitely additive conditional probability When neighborhoods encode qualitative expectations for finitely additive measures (in countable spaces) they form non-augmented filters validating **FOL + K** but not BF.

Some distinguished decision theorists (like Bruno De Finetti and Leonard J. Savage) as well as some philosophers (Isaac Levi) have advocated the use of finitely additive conditional probability in the decision sciences. Lester E. Dubins offers an axiomatic characterization of finitely additive conditional probability in [20]. As De Finetti suggested in [19] it is possible to extract a *superiority* ordering from finitely additive probability, which has, in turn, been used more recently in order to define full belief and expectations from primitively given conditional probability in a paradox-free manner ([48], [3]). Moreover finitely additive conditional probability has also been used in order to define non-monotonic notions of consequence ([3], [29]). We will show here that the resulting modalities can be represented in salient cases by neighborhood frames forming non-augmented filters.

We will proceed as follows: we will first define a qualitative structure in neighborhoods (basically a non-augmented filter). Then we will define a conditional measure satisfying Dubins' axioms by utilizing this qualitative structure. Finally we will show that the structure in question offers a characterization of belief and expectation corresponding to the measure.

Let the space be the (power set of the) positive integers and consider $E = \{2n \;:\; n = 1, \ldots\}$ be the even integers in the space, and let $O =$

$\{2n-1 \ : \ n=1,\ldots\}$ be the odd integers in the space. Let $P(i) = \{1/2^n$: if $i = 2n\}$ and let $P(i) = 0$, otherwise. So the unconditional P is countably additive, whose support is the even integers E. But $P(.|O)$ might be uniform on the odd integers. This can be reflected by the fact that there is a *core system* over O defined as follows: Let the outermost core $C_1 = O$. Then $C_2 = \{1\}^c$, $C_3 = \{1,3\}^c$, etc. For any n: 1, ...; define a rank system r_n for C_n as follows:

(ranks) $r_n = \{w \in \Omega \ : \ w \in C_n - C_{n+1}\}$

Notice that each rank contains exactly one odd number with $r_1 = \{1\}$. Now we can define conditional probability as follows. If there is a largest integer i such that $A \cap C_i \neq \emptyset$, define $Q(B|A)$ (for $A, B \subseteq O$) as: $p_i(\mathrm{A} \cap \mathrm{B})$ / $p_i(A)$. Otherwise set $Q(B|A)$ to 1 if there is r_i such that $A \cap C_i \neq \emptyset$, and $A \cap C_i \subseteq B$ and $Q(B^c|A)$ to 0 for Bs satisfying the given conditions. For the remaining infinite sets such that both B and B^c are infinite, arbitrarily set one of them to 1 and the complement to 0. Finally set $Q(B|A)$ to 0 for every other event in the space. According to this definition each co-finite set in O has measure 1 (because it is entailed by at least a core) and each number in O carries zero probability. Of two infinite but not co-finite sets, say S = {1, 5, 9,...} and H = {3, 7, 11, ...}, we can assign 1 to the set of lower rank (so S carries measure 1 and H zero).

We shall now formally introduce the notion of *probability core*. We follow here ideas presented in [3], which, in turn, slightly modify the schema first proposed in [48]. A notion that plays an important role in those works is the notion of *normality*. The basic idea is that an event A is normal for Q as long as $Q(\emptyset|A) = 0$. Conditioning on abnormal events would lead to incoherence.

A probability *core* for Q is an event K which is normal and satisfies the *strong superiority condition* (SSC) i.e. if A is a nonempty subset of K and B is disjoint from K, then $Q(B|A \cup B) = 0$ (and so $Q(A|A \cup B) = 1$). Thus any non-empty subset of K is more "believable" than any set disjoint from K.

Now it is easy to see that the system of cores C_n constructed above constitutes a system of cores for Q as defined in the previous paragraphs. Cores can be used in order to characterize *qualitative expectations* relative to Q: an event E is expected relative to Q as long as it is entailed by some core for Q. So, given Q we can construct all neighborhoods of a frame as the set of expectations for Q. A binary modality can also be intuitively characterized, namely conditional expectation: relative to any finite subset of O its largest element will always be expected. It is easy to see that these neighborhoods form non-augmented filters and that the first order logic of qualitative expectations should at least obey the axioms of the non-nested fragment of $\mathbf{FOL} + \mathbf{K}$.

Expectations have been utilized in [3] in order to define non-monotonic consequence: B is non-monotonically entailed by A relative to Q if and only if B is expected relative to $Q(.|A)$.‡‡ An important argument in [3] shows that in infinite spaces and for logically infinite languages (equipped with at least a denumerable set of atoms) the definition of non-monotonic consequence sketched above obeys standard axioms of *rational consequence* proposed by Lehman and Magidor [39] only if the underlying measure is not countable additive, so the filter structure presented above is essentially needed in order to characterize probabilistically both qualitative expectation and conditionals (relative to a conditional measure Q)•

THEOREM 3.27. $\mathbf{FOL} + \mathbf{K} + BF$ *is sound and complete with respect to the class of augmented first-order neighborhood frames.*

Example Perhaps the simplest example is constituted by the expectations induced by a measure Q defined over a finite space. In this case core systems always have an innermost core and the corresponding neighborhoods are augmented. In general [2] shows that the core systems of countable additive measures always have an innermost core, also inducing augmented neighborhoods.•

4. General Frames

The argument presented to this point shows that although central normal systems like $\mathbf{FOL} + \mathbf{K}$ cannot be fully characterized via constant domain relational models, there are many modalities obeying this logic which seem to require interpretations in terms of constant domains. Moreover the analysis of operators of high probability provides neither examples of modalities not obeying neither the Barcan Formula nor the full strength of K, but which are best interpreted in terms of frames with constant domains. The adoption of neighborhood semantics in the tradition suggested by Scott and Montague solves this and many other related problems (pointed out in the introduction) rather well.

‡‡Other notions of expectation proposed in the literature, like the one presented in [40], do not seem to exhibit the logical structure encoded in $\mathbf{FOL} + \mathbf{K}$; but they can also be analyzed with the tools offered here.

But an unmodified version of the program of neighborhood semantics (even at the propositional level) suffers also from certain important (and independent) inadequacies. The problem in question is that there are many classical modal logics which are not complete with respect to *any* class of neighborhood frames. The problem was first suggested by M. Gerson [26] who showed the existence of two incomplete logics with respect to neighborhood frames: one is the logic L (between T and $S4$) and the other the logic L' (an extension of S4 independently defined by Kit Fine).

The logics L and L' are also incomplete with respect to relational semantics à la Kripke, so Gerson's result shows that neighborhood semantics inherits other types of important inadequacies of the Kripkean program. In addition it is also well known that the first order logics strictly between $S4.4$ and $S5$ *without* the Barcan Formula are also incomplete (in relational semantics) [28]. We conjecture that the latter incompleteness can be removed by using first order neighborhood frames. But the first kind of incompleteness cannot be eradicated by utilizing first order neighborhood frames.

The solution for this kind of incompleteness in relational semantics consists in adopting the so-called *general frames* constituted by a frame together with a restricted but suitably well-behaved set of *admissible valuations* – see [15] for a textbook presentation of general frames. The terminology 'general frames' can be traced back to van Benthem's paper [14]. More complete historical references can be found in footnote 6 in chapter 9 of [34]. An identical maneuver is feasible in neighborhood semantics. [32] presents a general result of this type circumventing incompleteness results for the family of monotonic classical logics.

Hansen's general monotonic frames are nevertheless not ideal for our own purposes. First Hansen's main goal is to prove duality results between neighborhood frames and algebras with operators and second she only focuses on propositional systems. So, we will define here certain general first order neighborhood frames which we will call *general first order frames*.

Let $\mathcal{F} = \langle W, N, D\rangle$ be a first-order neighborhood frame with constant domain. The neighborhood function induces a function $N_\Box : 2^W \to 2^W$ defined as follows. Let $X \subseteq W$, then $N_\Box(X) = \{w \in W \mid X \in N(w)\}$. Intuitively, $N_\Box(X)$ is the set of states where the proposition X is necessary.

Let $\langle W, N, D\rangle$ be a first-order neighborhood frame with constant domain. Let ${}^\omega D$ denote the set of all functions from ω to D. For $i \in \omega$, $s \in {}^\omega D$ and $d \in D$, let s^i_d denote the function which is exactly the same as s except for the ith component which is assigned d. For $s, s' \in {}^\omega D$, we say that s and s' are i-equivalent if $s' = s^i_d$ for some $d \in D$. When convenient, we will think of a function $s \in {}^\omega D$ as an infinite sequence of elements from

D. Intuitively, these sequences represent a substitution $\sigma : \mathcal{V} \to D$. To make this representation concrete, we need to fix an ordering on the set of variables $\mathcal{V}$, i.e., assume that $\mathcal{V} = \{v_1, \ldots, v_n, \ldots\}$. We fix this ordering on the set of variables for the rest of this section. Then we can be more precise about the correspondence between sequences and substitutions. For any substitution, $\sigma : \mathcal{V} \to D$, there is a unique sequence, denoted by s_σ, such that $s_\sigma(i) = \sigma(v_i)$.

Similarly, for each sequence $s \in {}^\omega\mathcal{D}$, there is a unique substitution, denoted by σ_s, such that $\sigma_s(v_i) = s_i$. Obviously, we have that $\sigma \sim_{v_i} \sigma'$ implies $s_\sigma \sim_i s_{\sigma'}$ (and vice versa).

Consider a function $f : {}^\omega\mathcal{D} \to 2^W$. Intuitively the function f should be thought of as representing the equivalence class of formulas logically equivalent to a relation $\phi(v_{i_1}, \ldots, v_{i_n})$, in the sense that for each substitution σ, $f(s_\sigma) = (\phi(v_{i_1}, \ldots, v_{i_n}))^{\mathcal{M},\sigma}$. Given a first-order neighborhood frame with constant domain $\mathcal{F} = \langle W, N, D\rangle$, we say that a collection of functions $\{f_i\}_{i\in S}$ is **appropriate for** $\mathcal{F}$ if,

1. For each $i \in S$, there is a $j \in S$ such that for all $s \in {}^\omega\mathcal{D}$, $f_j(s) = W - f_i(s)$. We will denote f_j by $\neg f_i$.
2. For each $i, j \in S$, there is a $k \in S$ such that for all $s \in {}^\omega\mathcal{D}$, $f_k(s) = f_i(s) \cap f_j(s)$. We will denote f_k by $f_i \wedge f_j$.
3. For each $i \in S$, there is a $j \in S$ such that for all $s \in {}^\omega\mathcal{D}$, $f_j(s) = N_\Box(f_i(s))$. We will denote f_j by $\Box f_i$.

Definition 4.1. $\langle W, N, D, A, \{f_i\}_{i\in S}\rangle$ is a **general neighborhood frame with constant domain** if $\langle W, N, D\rangle$ is a first-order neighborhood frame with constant domain, $A \subseteq 2^W$ is a collection of sets closed under complement, finite intersection and the $N_\Box$ operator and $\{f_i\}_{i\in S}$ is a set of functions $f : {}^\omega\mathcal{D} \to 2^W$ appropriate for $\langle W, N, D\rangle$, where for each i and each $s \in {}^\omega\mathcal{D}$, $f_i(s) \in A$ and each f_i satisfies the following condition:

$$\text{(C)} \qquad \text{For each } v_i \in \mathcal{V}, \bigcap_{s' \sim_i s} f(s') \in A$$

An interpretation I is A**-admissible for** $\{f_i\}_{i\in S}$ in a general first-order neighborhood frame $\mathcal{F} = \langle W, N, D, A, \{f_i\}_{i\in S}\rangle$ provided for each n-ary atomic formula F and each substitution σ,

1. $((F(x_1, \ldots, x_n))^{\mathcal{M},\sigma} \in A$, where $\mathcal{M}$ is the first-order neighborhood model based on $\mathcal{F}$ with interpretation I, and

2. For each n-ary atomic formula F, there is a function f_i such that for each $s \in {}^{\omega}\mathcal{D}$, $w \in f_i(s)$ iff $\langle \sigma_s(v_{i_1}), \ldots, \sigma_s(v_{i_n})\rangle \in I(w, F)$, for any set of variables, $v_{i_1}, \ldots, v_{i_n}$.

No further restrictions in the cardinality of the index set S are imposed.

A **general first-order neighborhood model with constant domain** is a structure $\mathcal{M}^g = \langle W, N, D, A, \{f_i\}_{i\in S}, I\rangle$, where $\langle W, N, D, A, \{f_i\}_{i\in S}\rangle$ is a general first-order neighborhood frame with constant domain and I is an A-admissible interpretation for $\{f_i\}_{i\in S}$. Truth and validity are defined as usual. It is easy to see that the following is true in any such model:

> $(*)$ for any scheme ϕ with n-free variables $v_{i_1}, \ldots, v_{i_n}$, there is a function f_i that corresponds to the scheme $\phi(v_1, \ldots, v_n)$ in the sense that for each substitution σ, $(\phi(v_1 \ldots, v_n))^{\mathcal{M},\sigma} = f(s_\sigma)$.

Lemma 4.2. *For each formula $\phi \in \mathcal{L}_1$ and any general first-order neighborhood model with constant domain $\mathcal{M}^g = \langle W, N, D, A, \{f_i\}_{i\in S}, I\rangle$, $(\phi)^{\mathcal{M},\sigma} \in A$ for all substitutions σ.*

Proof. Let $\mathcal{M}^g = \langle W, N, D, A, \{f_i\}_{i\in S}, I\rangle$ be a general first-order neighborhood model with constant domain and σ any substitution. The proof is by induction on ϕ. The atomic case is by definition. The Boolean cases are obvious as is the modal case. Let ϕ be $\forall v_i\psi(v_i)$. We must show $((\forall x\psi(x))^{\mathcal{M},\sigma} \in A$. Now, $(\forall v_i\psi(v_i))^{\mathcal{M},\sigma} = \bigcap_{\sigma'\sim_{v_i}\sigma}(\psi(v_i))^{\mathcal{M},\sigma'}$.

By $(*)$, there is a function f_j such that $f_j(s) = (\psi(v_{i_1}, \ldots, v_{i_n}))^{\mathcal{M},\sigma_s}$ for all sequences s and variables $v_{i_1}, \ldots, v_{i_n}$. Hence, it is easy to show that $\bigcap_{\sigma'\sim_{v_i}\sigma}(\psi(v_i))^{\mathcal{M},\sigma'} = \bigcap_{s'\sim_i s_\sigma} f_j(s_{\sigma'})$. By property C, $\bigcap_{s'\sim_i s_\sigma} f_j(s') \in A$. Therefore, $(\forall v_i\psi(v_i))^{\mathcal{M},\sigma} \in A$.

■

In order to define the general first-order canonical neighborhood model, we only need to define the canonical set of admissible sets and a set of canonical functions. The definitions of the domain D_Λ, the interpretation I_Λ and σ_Λ are as in the definitions for the canonical frame for Λ. The admissible sets are defined as follows.

$$A_\Lambda = \{|\phi|_\Lambda \mid \phi \in \mathcal{L}_1\}$$

As for the canonical functions, enumerate all n-ary predicate symbols in $\mathcal{L}_1$. Then define for each n-ary predicate symbol P with variables $v_{i_1}, \ldots, v_{i_n}$ a function $f_P : {}^{\omega}D_\Lambda \to 2^{W_\Lambda}$ as follows, for each $s \in {}^{\omega}D_\Lambda$,

$$f_P(s) = |P(s_{i_1}, \ldots, s_{i_n})|_\Lambda$$

Let the set of canonical functions $\{f_i^\Lambda\}_{i\in S}$ be the smallest set of functions containing the set $\{f_P \mid P \text{ an } n\text{-ary predicate symbol}\}$ and closed under the conditions 1, 2, and 3 above. For every formula $\phi \in \mathcal{L}_1$, there is a function f_ϕ^Λ such that $f_\phi^\Lambda(s) = |\phi(s_{i_1}, \ldots, s_{i_n})|_\Lambda$ for every sequence $s \in {}^\omega D$. It is easy to see that the set $\{f_i^\Lambda\}_{i\in S}$ satisfies condition C. Fix an arbitrary function $f_i \in \{f_i^\Lambda\}_{i\in S}$. Since the set of functions is the smallest set satisfying the closure conditions, there must be some expression ϕ such that $f_i(s) = |\phi(s_1, \ldots, s_n)|_\Lambda$. Then $\bigcap_{s'\sim_i s} f(s') = |\forall x\phi(x)|_\Lambda$, where $x = s_i$. Furthermore, it is easy to check that I_Λ is an admissible interpretation for $\{f_i^\Lambda\}_{i\in S}$. Thus, $\mathcal{M}_\Lambda^g = \langle W_\Lambda, N_\Lambda, D_\Lambda, A_\Lambda, \{f_i^\Lambda\}_{i\in S}, I_\Lambda\rangle$ is a general first-order neighborhood model with constant domain.

THEOREM 4.3. *Let Λ be any classical first-order modal logic. Λ is sound and strongly complete with respect to the class of general first-order neighborhood frames with constant domains for Λ.*

PROOF. Let $\langle W_\Lambda, N_\Lambda, A_\Lambda, \{f_i^\Lambda\}_{i\in S}, D_\Lambda\rangle$ be the minimal general canonical frame $\mathcal{F}_\Lambda^g$. Strong completeness requires showing that any Λ-consistent set of formulas is satisfiable in a modal based on $\mathcal{F}_\Lambda^g$. Since I_Λ is an admissible interpretation, the truth Lemma will hold for $\mathcal{M}_\Lambda^g$. Hence, it is easy to show that for any consistent set of formulas Σ, $\mathcal{M}_\Lambda^g \models_{\sigma_\Lambda} \phi$ for each $\phi \in \Sigma$. Thus we need only show that $\mathcal{F}_\Lambda^g \models \phi$ for each $\phi \in \Lambda$. That is, let I be an A-admissible interpretation for $\{f_i\}_{i\in S}$ and suppose $\mathcal{M}^* = \langle W_\Lambda, N_\Lambda, A_\Lambda, \{f_i^\Lambda\}_{i\in S}, D_\Lambda, I\rangle$ is a model based on $\mathcal{F}_\Lambda^g$. We must show for any substitution $\sigma : \mathcal{V}^+ \to \mathcal{D}$, $\mathcal{M}^* \models_\sigma \phi$ for each $\phi \in \Lambda$.

Let I be an admissible interpretation for $\{f_i^\Lambda\}_{i\in S}$ and therefore an admissible interpretation of $\mathcal{F}_\Lambda^g$ and let $\sigma : \mathcal{V}^+ \to D_\Lambda$ be any substitution. Fix a formula $\phi \in \Lambda$. Then for each atomic formula $F_i(v_{i_1}, \ldots, v_{i_n})$ in ϕ, there is some function $g \in \{f_i^\Lambda\}_{i\in S}$ such that $g(s) = (F(v_{i_1}, \ldots, v_{i_n}))^{\mathcal{M}^*, \sigma_s}$ for all sequences s. By construction of the $\{f_i^\Lambda\}_{i\in S}$, there is a formula $\psi(v_{j_1}, \ldots, v_{j_m})$ such that $g(s) = |\psi(s_{j_1}, \ldots, s_{j_m})|_\Lambda$. Note that in general, ψ be have arity $m \neq n$ (different from F). The goal is to construct a new formula ϕ' where each F_i is replaced by the correpsonding ψ. For a given atomic formula $F(v_{i_1}, \ldots, v_{i_n})$ and a corresponding formula $\psi(v_{j_1}, \ldots, v_{j_m})$, the main case that we need to consider is the one when $m \geq n$ and there is a subset $\{v_{j_1}, \ldots, v_{j_k}\} \subset \{v_{j_1}, \ldots, v_{j_m}\}$, such that $\{v_{j_1}, \ldots, v_{j_k}\} \subseteq \{v_{i_1}, \ldots, v_{i_n}\}$. In this case, replace $F(v_{i_1}, \ldots, v_{i_n})$ with $\psi(\sigma(v_{i_1}), \ldots, \sigma(v_{i_k}), v_{j_{k+1}}, v_{j_m})$.

Claim: For each $\Gamma \in W_\Lambda$, $\mathcal{M}^g_\Lambda, \Gamma \models_{\sigma_\Lambda} \phi'$ iff $\mathcal{M}^*, \Gamma \models_\sigma \phi$.

The proof is by induction on ϕ. If $\phi = F(v_{i_1}, \ldots, v_{i_n})$, then there are two cases to consider.

Suppose that $\phi' = \psi(\sigma(v_{i_1}), \ldots, \sigma(v_{i_k}), v_{j_{k+1}}, \ldots, v_{j_m})$. Then let $\pi \in {}^\omega\mathcal{D}$ be any sequence where $\pi_1 = \sigma(v_{i_1}), \ldots, \pi_n = \sigma(v_{i_n}), \pi_{n+1} = v_{j_{n+1}}, \ldots, \pi_m = v_{j_m}$. Let $\pi'_1 = \sigma(v_{j_1}), \ldots, \pi'_k = \sigma(v_{j_k})$, for the variables $\{v_{j_1}, \ldots, v_{j_k}\} \subset \{v_{i_1}, \ldots, v_{i_n}\}$ – see above where the formulas ϕ' are introduced.

We have,

$$\begin{array}{lll}
\mathcal{M}^*, \Gamma \models_\sigma F(v_{i_1}, \ldots, v_{i_n}) & \text{iff} & \langle \sigma(v_{i_1}), \ldots, \sigma(v_{i_n}) \rangle \in I(F, \Gamma) \\
& \text{iff} & \langle \pi_1, \ldots, \pi_n \rangle \in I(F, \Gamma) \\
& \text{iff} & \Gamma \in g(\pi) = |\psi(\pi'_1, \ldots, \pi'_k, \pi_{k+1}, \ldots, \pi_m)|_\Lambda \\
& \text{iff} & \psi(\pi'_1, \ldots, \pi'_k, \pi_{k+1}, \ldots, \pi_m) \in \Gamma \\
& \text{iff} & \psi(\sigma(v_{j_1}), \ldots, \sigma(v_{j_k}), v_{j_{k+1}}, \ldots, v_{j_m}) \in \Gamma \\
& \text{iff} & \mathcal{M}^g_\Lambda, \Gamma \models_{\sigma_\Lambda} \psi(\sigma(v_{j_1}), \ldots, \sigma(v_{j_k}), v_{j_{k+1}}, \ldots, v_{j_m})
\end{array}$$

The case when $\phi' = \psi(\sigma(v_{i_1}, \ldots, v_{i_m}))$ is similar to the above except use the sequence $\pi_1 = \sigma(v_{i_1}), \pi_2 = \sigma(v_{i_2}), \ldots, \pi_n = \sigma(v_{i_n})$.

The Boolean cases are straightforward. Suppose that $\phi = \Box\alpha$. By the induction hypothesis,

$$(\alpha')^{\mathcal{M}^g_\Lambda, \sigma_\Lambda} = (\alpha)^{\mathcal{M}^*, \sigma}$$

Thus $\mathcal{M}^g_\Lambda, \Gamma \models_{\sigma_\Lambda} \Box\alpha'$ iff $(\alpha')^{\mathcal{M}^g_\Lambda, \sigma_\Lambda} \in N_\Lambda(\Gamma)$ iff $(\alpha)^{\mathcal{M}^*, \sigma} \in N_\Lambda(\Gamma)$ iff $\mathcal{M}^*, \Gamma$ $\models_\sigma \Box\alpha$.

Suppose that $\phi = \forall x \alpha(x)$. Then $\mathcal{M}^*, \Gamma \models_\sigma \forall x \alpha(x)$ iff

$$\Gamma \in \bigcap_{\sigma' \sim_x \sigma} (\alpha(x))^{\mathcal{M}^*, \sigma'} = \bigcap_{\sigma' \sim_{\sigma(x)} \sigma_\Lambda} (\alpha'(\sigma(x)))^{\mathcal{M}^g_\Lambda, \sigma_\Lambda}_{I_\Lambda}$$

iff $\mathcal{M}^g_\Lambda, \Gamma, I_\Lambda \models_{\sigma_\Lambda} \forall \sigma(x) \alpha'(\sigma(x))$

■–(of Claim)

Then for any $\phi \in \Lambda$, $\phi' \in \Lambda$ since Λ is closed under universal substitution. By the truth Lemma, $\mathcal{M}^g_\Lambda, \Gamma \models_{\sigma_\Lambda} \phi'$. Hence by the above claim, $\mathcal{M}^*, \Gamma \models_\sigma \phi$ where $\mathcal{M}^*$ is any model based on $\mathcal{F}^g_\Lambda$ and σ any substituion. Hence, Λ is valid on $\mathcal{F}^g_\Lambda$.

■

5. Conclusion and Further Work

Relational semantics for first order classical modalities suffers at least from three major inadequacies. First, the semantic study of non-normal classical systems is either impossible or indirect and unnecessarily complicated (some monotonic non-normal systems can be simulated in terms of polymodal normal logics – see [37]). This is particularly important given the fact that many interesting modal logics of interest in computer science and philosophy are non-normal.

Second, the standard relational semantics with constant domains imposes the validity of both the Barcan and the Converse Barcan schemas. This is unduly restrictive given that there are many interesting modalities which are better represented with constant domains and for which the Barcan schema (or the Converse Barcan) fails to be intuitively valid. Monadic operators of high probability as well as many epistemic modalities are salient examples.

Third, even when systems like $\mathbf{FOL} + \mathbf{K}$ can be characterized in terms of relational semantics utilizing expanding domains there are many other incompleteness results for systems strictly between $S4.4$ and $S5$ without the Barcan schema. The origin of these incompleteness results seems differently motivated than in the case of other incompleteness results for purely propositional systems.

Some of these inadequacies can be removed via the adoption of general relational frames, but not all of them. For example, the adoption of general relational frames cannot give the Kripkean program the ability of characterizing certain normal systems without the Barcan schema in terms of class of relational frames with constant domains.

In contrast, the adoption of general first order neighborhood frames removes *all* these problems at once and delivers a very intuitive and appealing alternative semantical framework. Our results show that the adoption of varying domains in modal logic remains optional but it is not mandatory in order to characterize normal systems like $\mathbf{FOL}+\mathbf{K}$. A general completeness result can be proved for the entire first order classical family of modal logics in terms of (general) constant domain frames.

We have tackled here some problems that seem central but much remains to be done in this area. Concerning applications many notions of interest in distributed Artificial Intelligence and Game Theory can be studied with the tools we offer. Some examples were provided in the introduction. As an important additional example we conjecture that a strong completeness result for the type spaces in the sense of Harsanyi is provable for first order operators of the type 'individual i assigns probability at least a'. Concerning extensions an obvious further step is to consider the case of varying domains. We expect that also in this area there are significant gains by applying the neighborhood approach (particularly in the case of transition logics and in the case multi modal logics combining operators of tense and belief). There is also a promissory area of investigation related to the logic of conditionals and non-monotonic inference. Dyadic modal classical modal operators were considered in passing in [16] but this area of research remains practically unexplored.

Duality results for general first order neighborhood frames in terms of cylindric algebras with operators remain also unknown. And although [32] contains some preliminary results concerning recent work in co-algebras, this is also a topic that remains open. One should expect interesting connections with co-algebras given that knowledge operators treated in terms of neighborhoods (as is explained in passing in the introduction) can be alternatively characterized in terms of non-well founded models for modalities of the kind studied in [12].

As Fitting and Mendelsohn remarked in [22] (page 134) the lack of 'a completeness proof that can cover constant domains, varying domains, and models meeting other conditions...' has often been felt. Garson suggests as well that 'ideally, we would like to find a completely general completeness proof.' Our focus in the last sections of this paper was to offer such a general completeness proof for the entire class of first order classical modal systems.

Some authors have recently published results that aim at meeting Garson's challenge. The results offered in [17] are perhaps an example. In this work the author explains that the 'production of such a proof [a general completeness proof] is the aim of this paper' ([17], page 1483). The paper then focuses on considering '...all free and classical quantified extensions of the propositional modal logic **K** obtained by adding either the axioms of identity or the Converse Barcan Formula or the Extended Barcan Rule' ([17], page 1483). A general strategy for proving completeness results for these quantified logics is then provided. All this is done by utilizing variations of well-known relational semantics. 'Original' Kripke semantics, 'Kripke semantics' and what the author calls Tarski-Kripke semantics are utilized among others and compared. The appeal to this semantic framework has well-known limitations that seem to hamper the generality of the results presented in the paper.

First the presented results are confined to the class of classical normal first order modal systems that admit a characterization via this type of relational semantics. As a matter of fact the article includes some incompleteness results as well, marking the limits of this semantic approach. So,

[17] clearly does not present a general completeness result even if one confines attention exclusively to the subset of classical first order modal logics that are normal.

Second, as a consequence of the previous remarks, [17] does not consider first order classical non-normal systems at all.

Third the appeal in [17] to relational semantics forces the consideration of models with varying domains. The lack of serious study of alternatives to relational semantics at the first order level has perhaps created the impression that the recourse to these kinds of models is obligatory in order to consider general completeness results. We conclude here that this is not the case by offering a general completeness result for the entire class of classical first order modal systems in terms of general frames with constant domains. As we remarked above nothing precludes, nevertheless, the study of first order neighborhood frames with varying domains. A more detailed comparison of the behavior of these kind of models with the relational models (and 'unified' strategies for completeness) studied in [17] could be therefore of interest as a topic for further study.

To a large extent the many philosophical discussions that followed Quine skeptical comments about quantified modal logic, have been based on the observation that quantified modal logic seems to lead to the toleration and eventual acceptance of *possibilia*. Ruth Barcan Marcus puts the problem in a clear way:

> Since we have the option of coextensive domains, QML is not *committed* to possibilia. Yet admission of possibilia would *seem* to be a natural extension, for informally, the notion of possible worlds lends itself to framing counterfactuals not merely about the properties actual objects might have and the relations into which they might have entered but about alternative worlds that might have individuals that fail actually to exist, or fail to have individuals that do actually exist. The [Kripke-style] semantics accommodated such interpretations ([11], 195).

Barcan Marcus has always been skeptical about semantics that exhibit this tolerance:

> [...] modal discourse need not and should not admit possibilia despite the elegance of the generalization ([11], 213).

She defended this position mainly in philosophical terms, and she, of course, noticed that this philosophical attitude is consistent with the adoption of the so-called Barcan schema.

Our paper can be seen as providing further formal support for Barcan Marcus's philosophical view. Moreover we show as well that a modal semantics which rejects possibilia although consistent with the adoption of the Barcan schemas, does not require the adoption of either of them for its coherence. Modal discourse in its most general possible expression (the classical family of first order modal systems) can be coherently, completely and parametrically interpreted without any recourse to possibilia of any sort.

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