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A CONCAVE PROPERTY OF THE HYPERGEOMETRIC FUNCTION WITH RESPECT TO A PARAMETER*

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Abstract. The hypergeometric function is shown to be logarithmically concave in integer values of one of its parameters. The methods used are probabilistic.

THEOREM. *Let m, i and g be positive integers satisfying* $3 \leqq i + 1 \leqq g$, *and let z be a negative real number. Then*

$$\{{}_2F_1[-m, i : g : z]\}^2 > {}_2F_1[-m, i + 1 ; g ; z]\, {}_2F_1[-m, i - 1 ; g ; z].$$

We first establish the following lemma concerning the evaluation of the generating function of the negative hypergeometric distribution.

LEMMA.

$$\sum_{j=0}^{k} \binom{b+j-1}{j}\binom{k+a-j-1}{k-j} s^j \Big/ \binom{a+b+k-1}{k} = {}_2F_1[-k, b ; a + b ; 1 - s] \tag{1}$$

for all positive integers k, and all real s, and positive real values of a and b.

Proof of Lemma. Skellam [2] has shown that if X follows a binomial distribution with parameters p and k, and if p is integrated with respect to the normalized beta function

$$\frac{p^{b-1}(1-p)^{a-1}\,dp}{B(a,b)},$$

then the unconditional distribution of X is negative hypergeometric, that is,

$$\Pr\{X = j\} = \binom{b+j-1}{j}\binom{k+a-j-1}{k-j} \Big/ \binom{a+b+k-1}{k}.$$

The left-hand side of (1), denoted below by I, is then the probability generating function of the negative hypergeometric distribution. Thus

$$\begin{aligned} I = \mathscr{E}(s^X) &= \mathscr{E}_p\{(s^X|p)\} = \mathscr{E}_p(1 - p + ps)^k \\ &= \frac{1}{B(a,b)}\int_0^1 [1 - p(1-s)]^k p^{b-1}(1-p)^{a-1}\,dp \\ &= {}_2F_1[-k, b ; a + b ; 1 - s]. \end{aligned}$$

See [3, p. 20]. This proves the lemma.

Proof of Theorem. The essence of the proof is to use two theorems proved elsewhere [1], one on the existence of a probability distribution with a certain property, the other giving an inequality relating to such a distribution.

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Let $s = 1 + z$, $h = g - 1$ and $n = m + g - 1$. Theorem 3 of [1] states that there is a probability distribution F such that $a_{i,n}$, the expected value of the ith largest of a sample of size n drawn independently from F, satisfies

$$a_{i,n} = s^{i-1} \quad \text{for all } i, \quad 1 \leqq i \leqq n.$$

By use of a standard recurrence relation, quoted in [1, (4)], the expected value of the ith largest of some smaller sample of size h can be deduced as follows: For $1 \leqq i \leqq g \leqq n$,

$$\begin{aligned} a_{i,h} &= \sum_{j=0}^{n-h} \binom{i+j-1}{j} \binom{n-j-i}{n-h-j} a_{i+j,n} \Big/ \binom{n}{h} \\ &= \sum_{j=0}^{n-h} \binom{n-j-i}{n-h-j} \binom{i+j-1}{j} s^{i+j-1} \Big/ \binom{n}{h} \qquad (2) \\ &= s^{i-1}\, {}_2F_1[h-n, i; h+1; 1-s] \end{aligned}$$

on using the lemma with $k = n - h$, $b = i$ and $a = h - i + 1$.

Theorem 4 of [1] states that if $a_{i,n} = s^{i-1}$ for all $i = 1, \cdots, n$, then

$$a_{i,h}^2 > a_{i-1,h} a_{i+1,h} \quad \text{for } i = 2, \cdots, h-1 \quad \text{and } h \leqq n-1.$$

Applying (2), we obtain

$$\begin{aligned} &s^{2i-2}\{{}_2F_1[h-n, i; h+1; 1-s]\}^2 \\ &> s^{i-2}\{{}_2F_1[h-n, i-1; h+1; 1-s]\}\, s^{i}\{{}_2F_1[h-n, i+1; h+1; 1-s]\}. \end{aligned}$$

The theorem now follows by substituting for h, n and s.

Remark. An analytic proof of the theorem has been shown to the author by Dr. Tyson of the Center for Naval Analyses.

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