On 2-Coverings and 2-Packings of Laminar Families

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Abstract. Let $\mathcal{H}$ be a laminar family of subsets of a groundset V. A *k-cover* of $\mathcal{H}$ is a multiset C of edges on V such that for every subset S in $\mathcal{H}$, C has at least k edges that have exactly one end in S. A *k-packing* of $\mathcal{H}$ is a multiset P of edges on V such that for every subset S in $\mathcal{H}$, P has at most $k \cdot u(S)$ edges that have exactly one end in S. Here, u assigns an integer capacity to each subset in $\mathcal{H}$.
Our main results are: (a) Given a k-cover C of $\mathcal{H}$, there is an efficient algorithm to find a 1-cover contained in C of size $\leq k|C|/(2k-1)$. For 2-covers, the factor of $2/3$ is best possible. (b) Given a 2-packing P of $\mathcal{H}$, there is an efficient algorithm to find a 1-packing contained in P of size $\geq |P|/3$. The factor of $1/3$ for 2-packings is best possible.
These results are based on efficient algorithms for finding appropriate colorings of the edges in a k-cover or a 2-packing, respectively, and they extend to the case where the edges have nonnegative weights. Our results imply approximation algorithms for some NP-hard problems in connectivity augmentation and related topics. In particular, we have a 4/3-approximation algorithm for the following problem: Given a tree T and a set of nontree edges E that forms a cycle on the leaves of T, find a minimum-size subset E' of E such that $T + E'$ is 2-edge connected.

1 Introduction

Let $\mathcal{H}$ be a laminar family of subsets of a groundset V. In detail, let V be a groundset, and let $\mathcal{H} = \{S_1, S_2, \ldots, S_q\}$ be a set of distinct subsets of V such

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that for every $1 \leq i, j \leq q$, $S_i \cap S_j$ is exactly one of $\emptyset$, S_i or S_j. A *k-cover* of $\mathcal{H}$ is a multiset of edges, C, such that for every subset S in $\mathcal{H}$, C has at least k edges (counting multiplicities) that have exactly one end in S. A *k-packing* of $\mathcal{H}$ is a multiset of edges, P, such that for every subset S in $\mathcal{H}$, P has at most $k \cdot u(S)$ edges (counting multiplicities) that have exactly one end in S. Here, u assigns an integer capacity to each subset in $\mathcal{H}$. Our main results are:

1. Given a k-cover C of $\mathcal{H}$, there is an efficient algorithm to find a 1-cover contained in C of size $\leq k|C|/(2k-1)$. For 2-covers, the factor of $2/3$ is best possible.
2. Given a 2-packing P of $\mathcal{H}$, there is an efficient algorithm to find a 1-packing contained in P of size $\geq |P|/3$. The factor of $1/3$ is best possible.

All of these results extend to the weighted case, where the edges have nonnegative weights. Also, we show that the following two problems are NP-hard: (1) Given a 2-cover C of $\mathcal{H}$, find a minimum-size 1-cover that is contained in C. (2) Given a 2-packing P of $\mathcal{H}$, u, find a maximum-size 1-packing that is contained in P.

The upper bound of $2/3$ on the ratio of the minimum size of a 1-cover versus the size of a (containing) 2-cover is tight. To see this, consider the complete graph K_3, and the laminar family $\mathcal{H}$ consisting of three singleton sets. Let the 2-cover be $E(K_3)$. A minimum 1-cover has 2 edges from K_3. The same example, with unit capacities for the three singleton sets in $\mathcal{H}$, shows that the ratio of the maximum size of a 1-packing versus the size of a (containing) 2-packing may equal $1/3$. There is an infinite family of similar examples.

An edge is said to *cover* a subset S of V if the edge has exactly one end in S. Our algorithm for finding a small-size 1-cover from a given 2-cover constructs a "good" 3-coloring of (the edges of) the 2-cover. In detail, the 3-coloring is such that for every subset S in the laminar family, at least two different colors appear among the edges covering S. The desired 1-cover is obtained by picking the two smallest (least weight) color classes. Similarly, our algorithm for finding a large-size 1-packing from a given 2-packing constructs a 3-coloring of (the edges of) the 2-packing such that for every subset S in the laminar family, at most $u(S)$ of the edges covering S have the same color. The desired 1-packing is obtained by picking the largest (most weight) color class.

1.1 A Linear Programming Relaxation

Consider the natural integer programming formulation (IP) of our minimum 1-cover problem. Let the given k-cover be denoted by E. There is a (nonnegative) integer variable x_e for each edge $e \in E$. For each subset $S \in \mathcal{H}$, there is a constraint $\sum_{e \in \delta(S)} x_e \geq 1$, where $\delta(S)$ denotes the set of edges covering S. The objective function is to minimize $\sum_e w_e x_e$, where w_e is the weight of edge e. Let (LP) be the following linear program obtained by relaxing all of the integrality

constraints on the variables.

$$(LP) \quad z_{LP} = \min \sum_e w_e x_e \quad \text{s.t.} \quad \{ \sum_{e \in \delta(S)} x_e \geq 1, \ \forall S \in \mathcal{H}; \quad x_e \geq 0, \ \forall e \in E \}.$$

Clearly, (LP) is solvable in polynomial time. The k-cover gives a feasible solution to (LP) by fixing $x_e = 1/k$ for each edge e in the k-cover.

For the minimum 1-cover problem, Theorem 1 below shows that the optimal value of the integer program (IP) is $\leq 4/3$ times the optimal value of a half-integral solution to the LP relaxation (LP). (A feasible solution x to (LP) is called *half-integral* if $x_e \in \{0, \frac{1}{2}, 1\}$, for all edges e.) There are examples where the LP relaxation has a unique optimal solution that is *not* half-integral. For the maximum 1-packing problem, Theorem 2 shows that the optimal value of the integer program is $\geq 2/3$ times the optimal value of a half-integral solution to the LP relaxation.

Recall that a laminar family $\mathcal{H}$ may be represented as a tree $T = T(\mathcal{H})$. (T has a node for V as well as for each set $A_i \in \mathcal{H}$, and T has an edge $A_i A_j$ if $A_j \in \{V\} \cup \mathcal{H}$ is the smallest set containing $A_i \in \mathcal{H}$.)

Two special cases of the minimum 1-cover problem are worth mentioning. (i) If the laminar family $\mathcal{H}$ is such that the tree $T(\mathcal{H})$ is a path, then the LP relaxation has an integral optimal solution. This follows because the constraints matrix of the LP relaxation is essentially a network matrix, see [CCPS 98, Theorem 6.28], and hence the matrix is totally unimodular; consequently, every extreme point solution (basic feasible solution) of the LP relaxation is integral. (ii) If the laminar family $\mathcal{H}$ is such that the tree $T(\mathcal{H})$ is a star (i.e., the tree has one nonleaf node, and that is adjacent to all the leaf nodes) then the LP relaxation has a half-integral optimal solution. This follows because in this case the LP relaxation is essentially the same as the linear program of the fractional matching polytope, which has half-integral extreme point solutions, see [CCPS 98, Theorem 6.13].

1.2 Equivalent Problems

The problem of finding a minimum 1-cover of a laminar family $\mathcal{H}$ from among the multiedges of a k-cover E may be reformulated as a connectivity augmentation problem. Let $T = T(\mathcal{H})$ be the tree representing $\mathcal{H}$; note that $E(T)$ is disjoint from E. Then the problem is to find a minimum weight subset of edges E' contained in E such that $T + E' = (V(T), E(T) \cup E')$ is 2-edge connected; we may assume that E' has no multiedges. Instead of taking T to be a tree, we may take T to be a connected graph. This gives the problem *CBRA* which was initially studied by Eswaran & Tarjan [ET 76], and by Frederickson & Ja'ja' [FJ 81].

Similarly, the problem of finding a maximum 1-packing of a capacitated laminar family $\mathcal{H}$, u from among the multiedges of a k-packing E may be reformulated as follows. Let $T = T(\mathcal{H})$ be the tree representing $\mathcal{H}$, and let the tree edges have (nonnegative) integer capacities $u : E(T) \rightarrow \mathbf{Z}$; the capacity of a set

$A_i \in \mathcal{H}$ corresponds to the capacity of the tree edge a_i representing A_i. The k-packing E corresponds to a set of demand edges. The problem is to find a maximum integral multicommodity flow $x : E \rightarrow \mathbf{Z}$ where the source-sink pairs (of the commodities) are as specified by E. In more detail, the objective is to maximize the total flow $\sum_{e \in E} x_e$, subject to the capacity constraints, namely, for each tree edge a_i the sum of the x-values over the demand edges in the cut given by $T - a_i$ is $\leq u(a_i)$, and the constraints that x is integral and ≥ 0.

1.3 Approximation Algorithms for NP-hard Problems in Connectivity Augmentation

Our results on 2-covers and 2-packings imply improved approximation algorithms for some NP-hard problems in connectivity augmentation and related topics. Frederickson and Ja'ja' [FJ 81] showed that problem *CBRA* is NP-hard and gave a 2-approximation algorithm. Later, Khuller and Vishkin [KV 94] gave another 2-approximation algorithm for a generalization, namely, find a minimum-weight k-edge connected spanning subgraph of a given weighted graph. Subsequently, Garg et al [GVY 97, Theorem 4.2] showed that problem *CBRA* is max SNP-hard, implying that there is no polynomial-time approximation scheme for *CBRA* modulo the P$\neq$NP conjecture. Currently, the best approximation guarantee known for *CBRA* is 2.

Our work is partly motivated by the question of whether or not the approximation guarantee for problem *CBRA* can be improved to be strictly less than 2 (i.e., to $2-\epsilon$ for a constant $\epsilon > 0$). We give a 4/3-approximation algorithm for an NP-hard problem that is a special case of *CBRA*, namely, the tree plus cycle (*TPC*) problem. See Section 4.

Garg, Vazirani and Yannakakis [GVY 97] show that the above maximum 1-packing problem (equivalently, the above multicommodity flow problem) is NP-hard and they give a 2-approximation algorithm. In fact, they show that the optimal value of an integral 1-packing z_{IP} is $\geq 1/2$ times the optimal value of a fractional 1-packing z_{LP}. We do not know whether the factor 1/2 here is tight.

It should be noted that the maximum 1-packing problem for the special case of unit capacities (i.e., $u(A_i) = 1$, $\forall A_i \in \mathcal{H}$) *is* polynomial-time solvable. If the capacities are either one or two, and the tree $T(\mathcal{H})$ representing the laminar family $\mathcal{H}$ has height two (i.e., every tree path has length ≤ 4), then the problem may be NP-hard, see [GVY 97, Lemma 4.3].

Further discussion on related topics may be found in the survey papers by Frank [F 94], Hochbaum [Hoc 96], and Khuller [Kh 96]. Jain [J 98] has interesting recent results, including a 2-approximation algorithm for an important generalization of problem *CBRA*.

We close this section by introducing some notation. For a multigraph $G = (V, E)$ and a node set $S \subseteq V$, let $\delta_E(S)$ denote the multiset of edges in E that have exactly one end node in S, and let $d_E(S)$ denote $|\delta_E(S)|$; so $d_E(S)$ is the number of multiedges in the cut $(S, V - S)$.

2 Obtaining a 1-Cover from a k-Cover

This section has our main result on k-covers, namely, there exists a 1-cover whose size (or weight) is at most $k/(2k-1)$ times the size (or weight) of a given k-cover. The main step (Proposition 1) is to show that there exists a "good" $(2k-1)$-coloring of any k-cover. We start with a preliminary lemma.

Lemma 1. *Let V be a set of nodes, and let $\mathcal{H}$ be a laminar family on V. Let E be a minimal k-cover of $\mathcal{H}$. Then there exists a set $X \in \mathcal{H}$ such that $d_E(X) = k$ and no proper subset Y of X is in $\mathcal{H}$.*

Proof. Since E is minimal, there exists at least one set $X \in \mathcal{H}$ with $d_E(X) = k$. We call a node set $X \subseteq V$ a *tight set* if $d_E(X) = k$. Consider an inclusionwise minimal tight set X in $\mathcal{H}$. Suppose there exists a $Y \subset X$ such that $Y \in \mathcal{H}$. If each edge of E that covers Y also covers X, then we have $d_E(Y) = k$. But this contradicts our choice of X. Thus there exists an edge $xy \in E$ covering Y with $x, y \in X$. By the minimality of E, xy must cover a tight set $Z \in \mathcal{H}$. Since $\mathcal{H}$ is a laminar family, Z must be a proper subset of X. This contradiction to our choice of X proves the lemma. □

Proposition 1. *Let V be a set of nodes, and let $\mathcal{H}$ be a laminar family on V. Let E be a minimal k-cover of $\mathcal{H}$. Then there is a $(2k-1)$-coloring of (the edges in) E such that*

(i) each set $X \in \mathcal{H}$ is covered by edges of at least k different colors, and
(ii) for every node v with $d_E(v) \leq k$, all of the edges incident to v have distinct colors.

Proof. The proof is by induction on $|\mathcal{H}|$. For $|\mathcal{H}| = 1$ the results holds since there are k edges in E (since E is minimal) and these can be assigned different colors. (For $|\mathcal{H}| = 0$, $|E| = 0$ so the result holds. However, even if E is nonempty, it is easy to color the edges in an arbitrary order to achieve property (ii).)

Now, suppose that the result holds for laminar families of cardinality $\leq N$. Consider a laminar family $\mathcal{H}$ of cardinality $N+1$, and let E be a minimal k-cover of $\mathcal{H}$. By Lemma 1, there exists a tight set $A \in \mathcal{H}$ (i.e., $d_E(A) = k$) such that no $Y \subset A$ is in $\mathcal{H}$. We contract the set A to one node v_A, and accordingly update the laminar family $\mathcal{H}$. Then we remove the singleton set $\{v_A\}$ from $\mathcal{H}$. Let the resulting laminar family be $\mathcal{H}'$, and note that it has cardinality N. Clearly, E is a k-cover of $\mathcal{H}'$. Let $E' \subseteq E$ be a minimal k-cover of $\mathcal{H}'$. By the induction hypothesis, E' has a $(2k-1)$-coloring that satisfies properties (i) and (ii), i.e., E' has a good $(2k-1)$-coloring.

If the node v_A is incident to $\geq k$ edges of E', then note that E' with its $(2k-1)$-coloring is good with respect to $\mathcal{H}$ (i.e., properties (i) and (ii) hold for $\mathcal{H}$ too). To see this, observe that $k \leq d_{E'}(v_A) \leq d_E(v_A) = k$, so $d_{E'}(v_A) = k$, hence, the k edges of E' incident to v_A get distinct colors by property (ii). Then, for the original node set V, the k edges of E' covering A get k different colors.

Now focus on the case when $d_{E'}(v_A) < k$. Clearly, each edge in $E - E'$ is incident to v_A, since each edge in E not incident to v_A covers some tight set that

is in both $\mathcal{H}$ and $\mathcal{H}'$. We claim that the remaining edges of $E-E'$ incident to v_A can be colored and added to E' in such a way that E with its $(2k-1)$-coloring is good with respect to $\mathcal{H}$.

It is easy to assign colors to the edge (or edges) of $E-E'$ such that the k edges of E incident to v_A get different colors. The difficulty is that property (ii) has to be preserved, that is, we must not "create" nodes of degree $\leq k$ that are incident to two edges of the same color. It turns out that this extra condition is easily handled as follows. Let $e \in E-E'$ be an edge incident to v_A, and let $w \in V$ be the other end node of e. If w has degree $\leq k$ for the current subset of E, then e is incident to $\leq (2k-2)$ other edges; since $(2k-1)$ colors are available, we can assign e a color different from the colors of all the edges incident to e. Otherwise (w has degree $> k$ for the current subset of E), the other edges incident to w impose no coloring constraint on e, and we assign e a color different from the colors of the other edges incident to v_A; this is easy since $d_E(v_A) = k$. □

Theorem 1. *Let V be a node set, and let $\mathcal{H}$ be a laminar family on V. Let E be a k-cover of $\mathcal{H}$, and let each edge $e \in E$ have a nonnegative weight $w(e)$. Then there is a 1-cover of $\mathcal{H}$, call it E', such that $E' \subseteq E$ and $w(E') \leq k\, w(E)/(2k-1)$. Moreover, there is an efficient algorithm that given E finds E'; the running time is $O(\min(k|V|^2,\ k^2|V|))$.*

Proof. We construct a good $(2k-1)$-coloring of the k-cover E by applying Proposition 1 to a minimal k-cover $\tilde{E} \subseteq E$ and then "extending" the good $(2k-1)$-coloring of $\tilde{E}$ to E. That is, we partition E into $(2k-1)$ subsets such that each set X in $\mathcal{H}$ is covered by edges from at least k of these subsets. We take E' to be the union of the cheapest k of the $(2k-1)$ subsets. Clearly, the weight of E' is at most $k/(2k-1)$ of the weight of E, and (by property (i) of Proposition 1) E' is a 1-cover of $\mathcal{H}$.

Consider the time complexity of the construction in Proposition 1. Let $n = |V|$; then note that $|\mathcal{H}| \leq 2n$ and $|E| \leq 2kn$. The construction is easy to implement in time $O(|\mathcal{H}| \cdot |E|) = O(kn^2)$. Also, for $k < n$, the time complexity can be improved to $O(k^2 \cdot |\mathcal{H}|) = O(k^2 n)$. To see this, note that for each set $A \in \mathcal{H}$ we assign colors to at most k of the edges covering A after we contract A to v_A, and for each such edge e we examine at most $(2k-2)$ edges incident to e. □

3 Obtaining a 1-Packing from a 2-Packing

This section has our main result on 2-packings, namely, there exists a 1-packing whose size (or weight) is at least 1/3 times the size (or weight) of a given 2-packing. First, we show that there is no loss of generality in assuming that the 2-packing forms an Eulerian multigraph. Then we give a 3-coloring for the edges of the 2-packing such that for each set S in the laminar family at most $u(S)$ edges covering S have the same color. We take the desired 1-packing to be the biggest color class.

Lemma 2. *Let V be a set of nodes, let $\mathcal{H}$ be a laminar family on V, and let $u : \mathcal{H} \to \mathbf{Z}$ assign an integral capacity to each set in $\mathcal{H}$. Let E be a 2-packing of $\mathcal{H}, u$, i.e., for all sets $A_i \in \mathcal{H}$, $d_E(A_i) \leq 2\,u(A_i)$. If E is a maximal 2-packing, then the multigraph $G = (V, E)$ is Eulerian.*

Proof. If G is not Eulerian, then it has an even number (≥ 2) of nodes of odd degree. Let $A \in \{V\} \cup \mathcal{H}$ be an inclusionwise minimal set that contains ≥ 2 nodes of odd degree. For every proper subset S of A that is in $\mathcal{H}$ and that contains an odd-degree node, note that $d_E(S)$ is odd, hence, this quantity is strictly less than the capacity $2\,u(S)$. Consequently, we can add an edge (or another copy of the edge) vw where v, w are odd-degree nodes in A to get $E \cup \{vw\}$ and this stays a 2-packing of $\mathcal{H}, u$. This contradicts our choice of E, since E is a maximal 2-packing. Consequently, G has no nodes of odd degree, i.e., G is Eulerian. □

Proposition 2. *Given an Eulerian multigraph $G = (V, E)$, an arbitrary pairing $\mathcal{P}$ of the edges such that for every edge-pair the two edges have a common end node, and a laminar family of node sets $\mathcal{H}$, there is a 3-coloring of E such that*

(i) for each cut $\delta_E(A_i)$, $A_i \in \mathcal{H}$, at most half of the edges have the same color, and
(ii) for each edge-pair e, f in $\mathcal{P}$, the edges e and f have different colors.

Proof. Let $\mathcal{P}$ be a set of triples $[v, e, f]$, where e and f are paired edges incident to the node v. Note that an edge $e = vw$ may occur in two triples $[v, e, f]$ and $[w, e, g]$. W.l.o.g. assume that $\mathcal{P}$ gives, for each node v, a pairing of all the edges incident to v. Then $\mathcal{P}$ partitions E into one or more (edge disjoint) subgraphs $Q_1, Q_2, \ldots$, where each subgraph Q_j is a connected Eulerian multigraph. To see this, focus on the Eulerian tour given by fixing the successor of any edge $e = vw$ to be the other edge in the triple $[w, e, f] \in \mathcal{P}$, assuming e is oriented from v to w; each such Eulerian tour gives a subgraph Q_j.

If $\mathcal{H} = \emptyset$, then we color each subgraph Q_j with 3 colors such that no two edges in the same edge-pair in $\mathcal{P}$ get the same color. This is easy: We traverse the Eulerian tour of Q_j given by $\mathcal{P}$, and alternately assign the colors red and blue to the edges in Q_j, and if necessary, we assign the color green to the last edge of Q_j.

Otherwise, we proceed by induction on the number of sets in $\mathcal{H}$. We take an inclusionwise minimal set $A \in \mathcal{H}$, shrink it to a single node v_A, and update $G = (V, E)$, $\mathcal{H}$ and $\mathcal{P}$ to $G' = (V', E')$, $\mathcal{H}'$ and $\mathcal{P}'$. Here, $\mathcal{H}' = \mathcal{H} - \{A\}$, i.e., the singleton set $\{v_A\}$ is not kept in $\mathcal{H}'$. Also, we add new edge pairs to $\mathcal{P}'$ to ensure that all edges incident to v_A are paired. For a node $v \notin A$, all its triples $[v, e, f] \in \mathcal{P}$ are retained in $\mathcal{P}'$. Consider the pairing of all the edges incident to v_A in G'. For each triple $[v, e, f]$ in $\mathcal{P}$ such that $v \in A$ and each of e, f has one end node in $V - A$ (so e, f are both incident to v_A in G'), we replace the triple by $[v_A, e, f]$. We arbitrarily pair up the remaining edges incident to v_A in G'.

By the induction hypothesis, there exists a good 3-coloring for G', $\mathcal{H}'$, $\mathcal{P}'$. It remains to 3-color the edges with both ends in A. For this, we shrink the nodes in $V - A$ to a single node v_B, and update $G = (V, E), \mathcal{P}, \mathcal{H}$, to $G'' =$

$(V'', E''), \mathcal{P}'', \mathcal{H}''$; note that $\mathcal{H}''$ is the empty family and so may be ignored. We also keep the 3-coloring of $\delta_{E'}(v_A) = \delta_{E''}(v_B)$. Our final goal is to extend this 3-coloring to a good 3-coloring of E'' respecting $\mathcal{P}''$. We must check that this can always be done. Consider the differently-colored edge pairs incident to v_B. Consider any connected Eulerian subgraph Q_j containing one of these edge pairs e_1, e_2; the corresponding triple in $\mathcal{P}''$ is $[v_B, e_1, e_2]$. Let $\tilde{Q}_j$ be a minimal walk of (the Eulerian tour of) Q_j starting with e_2 and ending with an edge f incident to v_B (possibly, $f = e_1$). The number of internal edges in $\tilde{Q}_j$ is $\equiv 0$ or $1 \pmod 2$, and the two terminal edges either have the same color or not. If the number of internal edges in $\tilde{Q}_j$ is nonzero, then it is easy to assign one, two, or three colors to these edges such that every pair of consecutive edges gets two different colors. The remaining case is when $\tilde{Q}_j$ has *no* internal edges, say, $\tilde{Q}_j = v_B, e_2, w, f, v_B$, where w is a node in A. Then edges e_2, f are paired via the common end-node w, i.e., the triple $[w, e_2, f]$ is present in both $\mathcal{P}''$ and $\mathcal{P}$. Then, by our construction of $\mathcal{P}'$ from $\mathcal{P}$, the triple $[v_A, e_2, f]$ is in $\mathcal{P}'$, and so edges e_2 and f (which are paired in $\mathcal{P}'$ and present in $\delta_{E'}(v_A) = \delta_{E''}(v_B)$) must get different colors. Hence, a good 3-coloring of $G', \mathcal{H}', \mathcal{P}'$ can always be extended to give a good 3-coloring of $\tilde{Q}_j$, and the construction may be repeated to give a good 3-coloring of Q_j.

Finally, note that E'' is partitioned by $\mathcal{P}''$ into several connected Eulerian subgraphs $Q_1, Q_2, \ldots$, where some of these subgraphs contain edges of $\delta_{E''}(v_B)$ and others do not. Clearly, the good 3-coloring of $G', \mathcal{H}', \mathcal{P}'$ can always be extended to give a good 3-coloring of each of $Q_1, Q_2, \ldots$, and thus we obtain a good 3-coloring of $G, \mathcal{H}, \mathcal{P}$. □

Theorem 2. *Let V be a node set, let $\mathcal{H}$ be a laminar family on V, and let $u : \mathcal{H} \to \mathbf{Z}$ assign an integer capacity to each set in $\mathcal{H}$. Let E be a 2-packing of $\mathcal{H}$, and let each edge $e \in E$ have a nonnegative weight $w(e)$. Then there is a 1-packing of $\mathcal{H}$, call it E', such that $E' \subseteq E$ and $w(E') \geq w(E)/3$. Moreover, there is an efficient algorithm that given E finds E'; the running time is $O(|V| \cdot |E|)$.*

Proof. If the multigraph (V, E) is not Eulerian, then we use the construction in Lemma 2 to add a set of edges to make the resulting multigraph Eulerian without violating the 2-packing constraints. We assign a weight of zero to each of the new edges. Let us continue to use E to denote the edge set of the resulting multigraph. We construct a good 3-coloring of the 2-packing E by applying Proposition 2. Let F be the most expensive of the three "color classes;" so, the weight of F, $w(F)$, is $\geq w(E)/3$. Note that F is a 1-packing of $\mathcal{H}$, u by property (i) in the proposition since for every set $A_i \in \mathcal{H}$, we have $d_F(A_i) \leq d_E(A_i)/2 \leq 2u(A_i)$. Finally, we discard any new edges in F (i.e., the edges added by the construction in Lemma 2) to get the desired 1-packing.

Consider the time complexity of the whole construction. It is easy to see that the construction in Proposition 2 for the minimal set $A \in \mathcal{H}$ takes linear time. This construction may have to be repeated $|\mathcal{H}| = O(|V|)$ times. Hence, the overall running time is $O(|V| \cdot |E|)$. □

4 Applications to Connectivity Augmentation and Related Topics

This section applies our covering result (Theorem 1) to the design of approximation algorithms for some NP-hard problems in connectivity augmentation and related topics. The main application is to problem *CBRA*, which is stated below. Problem *CBRA* is equivalent to some other problems in this area, and so we immediately get some more applications.

Recall problem *CBRA*: given a connected graph $T = (V, F)$, and a set of "supply" edges E with nonnegative weights $w : E \to \Re_+$, the goal is to find a minimum-weight subset E' of E such that $T + E' = (V, F \cup E')$ is 2-edge-connected. One application of Theorem 1 is to give a 4/3-approximation algorithm for the special case of *CBRA* when the LP relaxation has an optimal solution that is half-integral.

Theorem 3. *Given a half-integral solution to the LP relaxation of CBRA of weight z, there is an $O(|V|)$-time algorithm to find an integral solution (i.e., a feasible solution of CBRA) whose weight is $\leq \frac{4}{3}z$.*

Proof. Problem *CBRA* may be restated as the problem of finding a minimum-weight 1-cover of a laminar family $\mathcal{H}$, where the 1-cover must be chosen from the set of supply edges E and each supply edge has a nonnegative weight. To specify $\mathcal{H}$, fix any node $r \in V$ to be the root of T, and focus on the cut edges of T, call them $f_1, f_2, \ldots$. For each of these cut edges $f_1, f_2, \ldots$, let A_i be the (node set of the) component of $T - f_i$ that does not contain r. We take $\mathcal{H} = \{A_1, A_2, \ldots\}$.

Let $x : E \to \{0, \frac{1}{2}, 1\}$ be a half-integral solution to the LP relaxation of *CBRA*, and let $z = \sum_e w_e x_e$. Then x corresponds to a 2-cover C of $\mathcal{H}$, where C has zero, one or two copies of a supply edge e iff $x_e = 0,\ 1$, or 2. By Theorem 1, C contains a 1-cover C' whose weight is $\leq 4z/3$, and moreover, C' can be computed in time $O(|V|)$. □

We have sharper results for the following (NP-complete) special case of problem *CBRA*.

Tree Plus Cycle Problem (TPC):
INSTANCE: A tree $T = (W, F)$ whose set of leaf nodes is $V \subseteq W$, a "supply" cycle $Q = (V, E)$ on the leaves of T (i.e., $d_E(v) = 2,\ \forall v \in V$), and a positive integer N.
QUESTION: Is there a set of edges $E' \subseteq E$ with $|E'| \leq N$ such that $T + E' = (W, F \cup E')$ is 2-edge-connected?

Corollary 1. *There exists a $\frac{4}{3}$-approximation algorithm for TPC. Moreover, there exists a feasible solution $E' \subseteq E(Q)$ of size $\leq 2|V(Q)|/3$.*

Proof. Consider the LP relaxation of problem *TPC*; it is easy to verify that an optimal solution is given by $x_e = 1/2$ for all supply edges $e \in E(Q)$. Now, the result follows directly from Theorem 3. □

Now consider the following problem: given a $(2k-1)$-edge-connected graph $T=(V,F)$ and a set of "supply" edges E with nonnegative weights $w: E \to \Re_+$, the goal is to find a minimum weight subset E' of E such that $G'=(V,F+E')$ is $(2k)$-edge-connected. Since the edge connectivity of T is odd, this problem is equivalent to problem *CBRA* because all the $(2k-1)$-cuts (minimum cuts) of T can be represented by means of a laminar family. (This follows easily from the fact that the node sets of two minimum cuts do not cross in this case.)

5 NP-completeness Results

First, we show that problem *TPC* (tree plus cycle) is NP-complete. It is convenient to reformulate *TPC* in terms of a laminar family rather than a tree.

Laminar Family Plus Cycle Problem (LPC):
INSTANCE: A laminar family $\mathcal{H}$ on a node set V, a cycle $Q=(V,E)$ on V, and a positive integer N. (Assume $\emptyset, V \notin \mathcal{H}$.)
QUESTION: Is there a 1-cover E' of $\mathcal{H}$ such that $E' \subseteq E$ and $|E'| \leq N$?

We give a polynomial-time reduction from the 3-dimensional matching problem to problem *LPC*. Our reduction is based on the proof of [FJ 81, Theorem 2].

Theorem 4. *Problem LPC is NP-complete.*

Proof. It is easy to see that *LPC* is in NP. Given an instance of *3DM* (that is, three disjoint sets W, X, Y, of cardinality q each, and a set M of 3-edges (triples) $(w_i x_j y_k) \in W \times X \times Y$), construct a connected graph T as follows. First build a star with a "root" r and $3q$ leaves $\{w_1, \ldots, w_q, x_1, \ldots, x_q, y_1, \ldots, y_q\}$ corresponding to the elements of $W \cup X \cup Y$. Then for each 3-edge $(w_i x_j y_k)$ of M add two nodes a_{ijk} and $\bar{a}_{ijk}$ to T and add the edges $w_i a_{ijk}$, $w_i \bar{a}_{ijk}$. Now replace each of the $2q$ nodes corresponding to elements of X and Y by complete graphs (or arbitrary 2-edge-connected graphs) denoted by $X_1, \ldots, X_q, Y_1, \ldots, Y_q$ as follows. Each complete subgraph of this type has $d_M(x_j)8q$ ($d_M(y_k)8q$) nodes and is partitioned into $d_M(x_j)$ ($d_M(y_k)$) parts (so-called "lanes") of size $8q$ each. (Here, $d_M(x_j)$ and $d_M(y_k)$ denote the number of 3-edges of M containing x_j, respectively, y_k.) The graph constructed is connected and has $2p+2q$ "leaves" (that is, leaf 2-edge-connected components), where $p := |M|$.

The next step is to define the cycle Q. The nodes of Q are the nodes of the leaves of T. Hence, $|V(Q)| = p(16q+2)$. First, we define p disjoint paths of Q such that each has $16q+2$ nodes (so each of these paths has length $16q+1$). Every 3-edge $(w_i x_j y_k)$ of M defines such a path as follows: take the $8q$ nodes (and edges connecting the consecutive ones) $l_1 l_2 \ldots l_{8q}$ of a lane of X_1 in an arbitrary order, then take the edges $l_{8q} a_{ijk}$, $a_{ijk} \bar{a}_{ijk}$, $\bar{a}_{ijk} m_{8q}$ for some node m_{8q} of some lane of Y_k, in this order, and then take the other nodes of this lane $m_{8q-1}, \ldots, m_1$ in an arbitrary order. The lanes are chosen in such a way that these paths are pairwise disjoint. This can be done, since the lanes are pairwise disjoint and each X_j (or Y_k) has $d_M(x_j)$ (or $d_M(y_k)$) lanes.

Now fix a cyclic ordering $e_1, \ldots, e_p$ of the 3-edges of M and complete the cycle Q by adding the missing p edges in such a way that the end of the path corresponding to $e_s = (w_i x_j y_k)$ (that is, a node m_1 of a lane in Y_k) is connected to the first node of the path corresponding to $e_{s+1} = w_{i'} x_{j'} y_{k'}$ (that is, to a node l_1 of a lane of $X_{j'}$) for $1 \leq s \leq p$. Note that each of these edges connects a complete subgraph X_j to a complete subgraph Y_k and all the edges of Q either connect different leaves of T or connect different nodes of some leaf of T. Furthermore, $V(Q)$ equals the union of the nodes of the leaves of T.

The last part of the reduction consists of defining a laminar family $\mathcal{H}$ on $V(Q)$. We define $\mathcal{H}$ by defining two disjoint subfamilies $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $\mathcal{H}_1 := \{S \cap V(Q) \ : \ d_T(S) = 1, r \notin S, S \subseteq V(T)\}$ contain intersections of $V(Q)$ and those minimum cuts of T which do not contain the root. It is easy to see that this family is laminar. $\mathcal{H}_2$ consists of $2p$ disjoint collections, each of them defined on the nodes of a lane of a complete subgraph of the form X_j or Y_k of T as follows. Let us fix such a subgraph, say X_1. (The definition is similar for all the $2q$ subgraphs $X_1, \ldots, X_q, Y_1, \ldots, Y_q$.) Focus on a lane $l_1, \ldots, l_{8q}$ of X_1, where the numbering follows the ordering of these nodes in Q. (Hence l_{8q} is connected to some leaf a_{ijk} and l_1 is connected to some Y_k.) This lane adds the following sets to $\mathcal{H}_2$: the singletons $l_1, \ldots, l_{8q}$, the sets of nodes of the intervals of Q with end-node pairs (l_{8q-1}, l_{8q-s}) $(2 \leq s \leq 4q)$ and (l_{4q-r}, l_2) $(1 \leq r \leq 4q-3)$. Each lane of every complete subgraph X_j, Y_k $(1 \leq j, k \leq q)$ adds a similar collection to $\mathcal{H}_2$. Clearly, every collection of this type is laminar, and the collections are defined on pairwise disjoint sets of nodes, where each of these sets is included in a minimal element of $\mathcal{H}_1$. Therefore $\mathcal{H}$ is a laminar family on $V(Q)$, where $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$. Note that each node of Q belongs to $\mathcal{H}$ as a singleton set.

Observe the following important property, that follows from the structure of these collections and the fact that every node of Q belongs to $\mathcal{H}$. Let $E' \subseteq E(Q)$ be a 1-cover of $\mathcal{H}$. Then

> $(*)$ if the edge $l_{8q}l_{8q-1}$ (or similarly $l_1 l_2$, $m_{8q}m_{8q-1}$, $m_1 m_2$) for some lane in an X_j or Y_k is not in E' then $|E'| \geq |V(Q)|/2 + 2q - 1$.

It is easy to see that our reduction is polynomial. We claim that there exists a solution to the given instance of *3DM* (that is, a set of q pairwise disjoint 3-edges of M) if and only if $\mathcal{H}$ has a 1-cover of size at most $p + 8pq + q = |V(Q)|/2 + q$.

First observe that a set E' is a 1-cover if and only if $T + E'$ is 2-edge-connected and E' covers each member of $\mathcal{H}_2$. Moreover, as it was verified in [FJ 81], there is a 3-dimensional matching if and only if there is set E^* of $p + q$ edges in Q^* for which $T^* + E^*$ is 2-edge-connected, where T^* and Q^* arise from T and Q, respectively, by contracting the complete subgraphs (that is, the sets of the form X_j, Y_k, which are 2-edge-connected) to singletons and deleting the edges connecting these complete subgraphs from the cycle.

Suppose that there exists a 3-dimensional matching $M' \subseteq M$. Then there exists a set E^* of size $p + q$ which makes T^* 2-edge-connected and it is easy to see that there exists a set E'' of independent edges in Q which covers $\mathcal{H}_2$. Hence $|E''| = 16qp/2 = 8pq$. Now $E' := E^* \cup E''$ covers $\mathcal{H}$ and $|E'| = 8pq + p + q$, as required.

The proof of the other direction (which relies on $(*)$) is omitted. □

Corollary 2. *The following problem is NP-hard: given a 2-cover C of a laminar family $\mathcal{H}$, find a minimum-size 1-cover that is contained in C.*

Theorem 5. *The following problem is NP-hard: given a 2-packing P of a capacitated laminar family $\mathcal{H}$, u, find a maximum-size 1-packing that is contained in P.*

6 Conclusions

We suspect that our bounds on the ratios for 1-covers versus 2-covers and for 1-packings versus 2-packings hold in general.

1-Cover Conjecture: Consider the integer program for a minimum weight 1-cover of a laminar family and its LP relaxation (see Section 1). We conjecture that the ratio of the optimal values is at most 4/3.

1-Packing Conjecture: Consider the integer program for a maximum weight 1-packing of a capacitated laminar family and its LP relaxation (see Section 1). We conjecture that the ratio of the optimal values is at least 2/3.

Another interesting question is to find sufficient conditions on the laminar family $\mathcal{H}$ (or, on the tree $T(\mathcal{H})$ representing $\mathcal{H}$) such that the LP relaxation has $\frac{1}{k}$-integral extreme point solutions. As noted in Section 1, the LP relaxation has integral extreme point solutions iff $T(\mathcal{H})$ is a path.

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