Succinct Relaxations for Some Discrete Problems

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Abstract

A discrete problem can be relaxed by taking the continuous relaxation of an integer programming formulation. An equivalent relaxation is obtained by projecting this relaxation onto the original continuous variables. The projection is simple for piecewise linear functions, fixed charge problems, and some disjunctive constraints. This allows one to solve much smaller relaxations without sacrificing the quality of bounds. In particular the projected relaxations for some classical network design and warehouse location problems are minimum cost network flow problems, a fact that can dramatically accelerate their solution.

A relaxation for a problem with discrete elements is often obtained by adding integer variables to the model. The integrality constraints not only capture the discrete element but can be dropped in order to obtain a continuous relaxation of the model.

An equivalent relaxation can be obtained, however, by projecting the traditional continuous relaxation onto the original continuous variables. Projection can generate a large number of inequality constraints, but in some important special cases it does not. In such cases one can obtain a relaxation that is of the same quality as the conventional one but much more succinct, due to the absence of integer variables. Occasionally the projected relaxation has special structure that the convention relaxation lacks and can be more easily solved for that reason as well.

Once integer variables are removed from the relaxation, they can be eliminated entirely. Logical expressions can be used to express the discrete elements of a problem, perhaps more naturally, as noted in [3]. Branching on logical possibilities or propositional variables can replace branching on integer variables. This may even allow branching to terminate sooner than in an integer programming context, as argued in [3]. In any case, the use of projected relaxations almost certainly accelerates the solution of the problem, because it provides the same bounds as traditional integer programming, and does so more rapidly because the relaxations are smaller.

This paper examines three projected relaxations. First, Beaumont's use of elementary inequalities [2] to relax logical disjunctions is briefly reviewed and strengthened, due to the usefulness of disjunctions for expressing the discrete aspect of a problem. Next a simple convex hull relaxation of piecewise linear functions is presented. Finally, it is noted that fixed charge problems also have a compact projected relaxation. In particular, the projected relaxation of fixed charge network flow problems, including warehouse location problems, has the structure of a minimum cost network flow problem, whereas the conventional relaxation does not. This permits a far more rapid solution of the relaxation than is otherwise possible. Because solving the relaxation consumes nearly all the solution time in branch-and-bound algorithms, projected relaxations can dramatically accelerate the solution of these problems.

The relaxations presented here are simple. Nonetheless they and their advantages are normally overlooked.

1 Elementary Inequalities

Beaumont [2] showed that a single *elementary inequality* relaxes a disjunction of linear inequalities,

$$\bigvee_{i \in I} a^i x \geq \alpha_i, \tag{1}$$

where the disjunction operator $\vee$ states that at least one of the inequalities should be satisfied. If it is assumed that $0 \leq x \leq m$, the traditional big-M relaxation is,

$$a^i x \geq \alpha^i - M_i(1 - y_i), \quad i \in I \tag{2}$$

$$\sum_{i \in I} y_i \geq 1 \tag{3}$$

$$0 \leq x \leq m \tag{4}$$

$$0 \leq y_i \leq 1, \quad i \in I. \tag{5}$$

Each M_i is chosen so that $\alpha_i - M_i$ is a lower bound on the value of $a^i x$:

$$\alpha_i - M_i = \sum_j \min\{0, a^i_j\} m_j. \tag{6}$$

It can be assumed without loss of generality that $M_i > 0$, because otherwise the inequality is vacuous and can be dropped. Beaumont obtains the elementary inequality by taking a linear combination of the inequalities in (2)-(5), where each inequality i receives weight $1/M_i$. This yields,

$$\left(\sum_{i \in I} \frac{a^i}{M_i} \right) x \geq \sum_{i \in I} \frac{\alpha_i}{M_i} - |I| + 1. \tag{7}$$

Beaumont showed that (7), together with the bounds $0 \leq x \leq m$, is equivalent to the integer programming relaxation (2)-(5).

In many cases the elementary inequality can be strengtened by using a better lower bound on $a^i x$ than that in (6). One can minimize $a^i x$ subject to each of the other disjuncts and $0 \leq x \leq m$ and pick the smallest of the minimum values. M_i is therefore chosen so that

$$\alpha_i - M_i = \min_{i' \neq i} \left\{ \min_x \{ a^i x \mid a^{i'} x \geq \alpha_{i'}, \ 0 \leq x \leq m \} \right\}. \tag{8}$$

The resulting inequality can sometimes be further tightened by increasing the right-hand side, according to a closed-form formula presented in [3].

If there are k disjuncts and n continuous variables, projection reduces the $n + k$ variables and k constraints of the big-M relaxation to n variables and one constraint. The convex hull relaxation [1] is generally stronger than the big-M relaxation, but it is even larger, because it contains additional continuous variables as well as 0-1 variables.

The projected relaxation is generally much more complex for a disjunction of linear systems that contain several inequalities. In such cases it may be better to use the traditional relaxation.

2 Piecewise Linear Functions

The conventional integer programming formulation of a piecewise linear function introduces a 0-1 variable for each linear segment of the function. Fortunately, a compact projected relaxation is available.

Let $f(x)$ be a piecewise linear function to be represented, where x is a scalar variable. Let $v_0, \ldots, v_K$ be the endpoints of the linear segments, so that $f(x) = f(v_{k-1}) + (u - v_{k-1})p_k$ for $u \in [v_{k-1}, v_k]$, where p_k is the slope of the k-th linear segment.

A traditional 0-1 relaxation can be written by viewing $f(x)$ as a convex combination of two consecutive values in the sequence $f(v_0), \ldots, f(v_K)$. Every occurrence of $f(x)$ is replaced by $\sum_{k=0}^{K} \alpha_k f(v_k)$, and the following constraints are added to the model.

$$
\begin{aligned}
&\sum_{k=1}^{K} y_k = 1 \\
&\sum_{k=0}^{K} \alpha_k = 1 \\
&\alpha_0 \leq y_1, \;\; \alpha_K \leq y_K \\
&\alpha_k \leq y_{k-1} + y_k, \quad k = 1, \ldots, K-1 \\
&y_k \geq 0, \quad k = 1, \ldots, K.
\end{aligned}
\tag{9}
$$

When $y_k = 1$, $f(x)$ becomes a linear combination of $f(v_{k-1})$ and $f(v_k)$; in other words, x lies in the interval $[v_{k-1}, v_k]$.

To write a projected relaxation that does not involve 0-1 variables y_k, assume for the sake of definiteness that $f(x)$ occurs

a) in an objective function to be minimized that is nondecreasing with respect to $f(x)$, or

b) on the left-hand side of an inequality $G \leq 0$, where G is nondecreasing with respect to $f(x)$.

For example, $f(x)$ might occur as a term in a summation. Let the *epigraph* of $f(x)$ be the area that lies above the graph of the $f(x)$; i.e., the set $\{(x, z) \mid z \geq f(x)\}$. A relaxation of the optimization problem can be created by replacing $f(x)$ with a lower approximation; i.e., a function $\tilde{f}(x)$ whose epigraph is the convex hull of the epigraph of $f(x)$. This is illustrated in Fig. 1. The relaxation produced by the lower approximation is, by definition, a convex hull relaxation and is therefore just as good as the traditional relaxation. In fact it is easily shown to be equivalent to the traditional relaxation.

Because the lower approximation is piecewise linear and convex, it can be given a linear formulation. Suppose that the piecewise linear segments of $\tilde{f}(x)$ terminate at $x = w_0, \ldots, w_{K'}$,

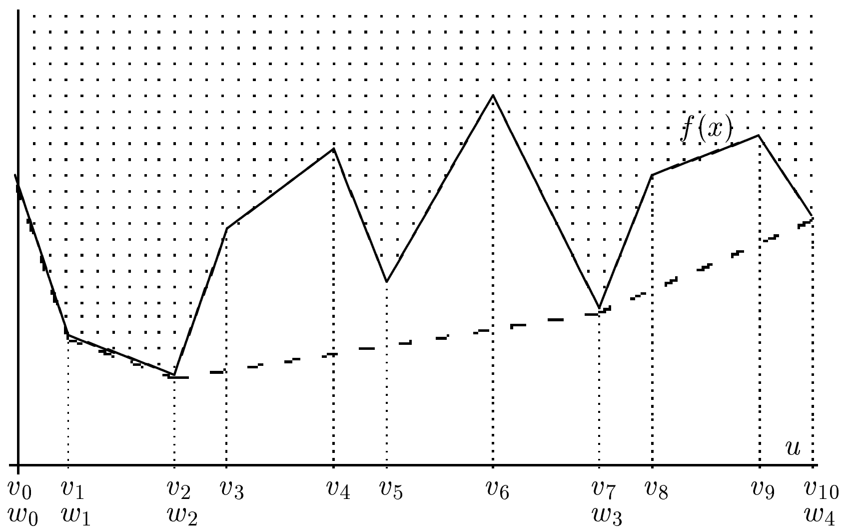

Figure 1: *The shaded area is the epigraph of the piecewise linear function shown, $f(x)$. The area above the dashed line is the convex hull of the epigraph, and the dashed line itself represents the lower approximation of $f(x)$.*

so that $\tilde{f}(x) = f(w_{k-1}) + (x - w_{k-1})q_k$ for $x \in [w_{k-1}, w_k]$. Every occurrence of $f(x)$ in the model is replaced by

$$f(w_0) + \sum_{k=1}^{K'} x_k q_k,$$

and the constraints

$$\begin{aligned} & x = \textstyle\sum_{k=1}^{K'} x_k \\ & 0 \leq x_k \leq w_k - w_{k-1}, \quad k = 1, \ldots, K'. \end{aligned} \tag{10}$$

are added to the model. Because $\tilde{f}(x)$ is convex, the slopes q_k are nondecreasing; i.e., $q_0 \leq q_1 \leq \ldots \leq q_{K'}$. Due to this and the assumption (a) or (b) above, any optimal solution of the constraints can be replaced by one in which $x_k > 0$ only if $x_{k-1} = w_{k-1}$, without affecting feasibility or the value of the objective function.

This relaxation requires $K' + 2$ inequalities and K' additional continuous variables, but K' may be considerably less than the number K of linear segments in the original function $f(x)$.

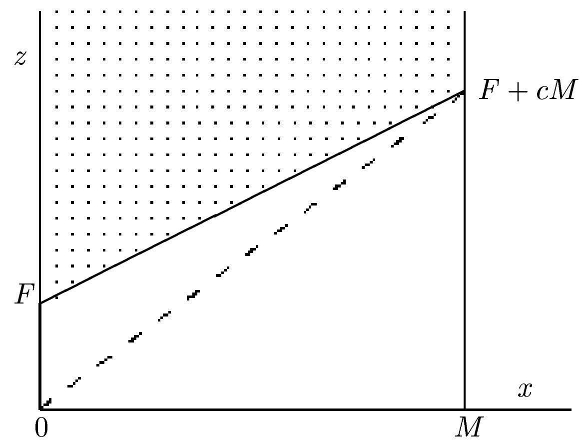

Figure 2: *A fixed charge problem. The feasible set is the shaded area, plus the ray that extends vertically from the origin. The convex hull of the feasible set is the area above the dashed line.*

3 Fixed Charge Problems

Fixed charge problems typically contain functions of the form,

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ F + cx & \text{if } x > 0, \end{cases}$$

where $0 \leq x \leq M$. Again it may be assumed that $f(x)$ occurs as in (a) or (b) above. The epigraph is depicted in Fig. 2. It is the union of two polyhedra: a ray that extends upward from the origin, and the shaded trapezoidal area that lies above the sloped line.

The traditional relaxation is obtained by replacing each occurrence of $f(x)$ by $Fy + cx$ and adding the constraints,

$$\begin{aligned} &0 \leq x \leq My \\ &0 \leq y \leq 1. \end{aligned} \tag{11}$$

In is clear from Fig. 2, however, that a convex hull relaxation can be written by replacing each occurrence of $f(x)$ by $\left(\frac{F}{M} + c\right) x$ and adding the constraint $0 \leq x \leq M$.

3.1 Fixed Charge Network Problem

The projected convex hull relaxation is only slightly smaller than the continuous relaxation of (11), because it eliminates a single 0-1 variable y. The size reduction could be significant, however, when there are a large number of fixed charges. Consider, for example, a network design problem in which the objective is to decide which arcs are to be present in a network so as to minimize cost. The cost includes the fixed cost of installing the arcs, plus the minimum cost of carrying a prescribed amount of flow over the resulting network. The problem can be written,

$$\begin{aligned}
\min \quad & \sum_{ij} z_{ij} \\
\text{s.t.} \quad & (z_{ij} = x_{ij} = 0) \vee (z_{ij} = F_{ij} + c_{ij} x_{ij}), \ \ \text{all } i,j \\
& \sum_i x_{ij} - \sum_i x_{ji} = S_j, \ \ \text{all } j \\
& 0 \le x_{ij} \le M_{ij}, \ \ \text{all } i,j.
\end{aligned}$$

F_{ij} is the fixed cost of installing directed arc (i,j), x_{ij} is the flow placed on the arc, c_{ij} is the unit cost of carrying the flow, and M_{ij} is the capacity of the arc. S_j is the net supply available at node j of the network.

The traditional relaxation is,

$$\begin{aligned}
\min \quad & \sum_{ij} F_{ij} y_{ij} + c_{ij} x_{ij} \\
\text{s.t.} \quad & 0 \le x_{ij} \le M_{ij} y_{ij}, \ \ \text{all } i,j \\
& \sum_i x_{ij} - \sum_i x_{ji} = S_j, \ \ \text{all } j \\
& 0 \le y_{ij} \le 1, \ \ \text{all } i,j.
\end{aligned}$$

The projected relaxation is equivalent but only about half this size.

$$\begin{aligned}
\min \quad & \sum_{ij} \left(\frac{F_{ij}}{M_{ij}} + c_{ij} \right) x_{ij} \\
\text{s.t.} \quad & 0 \le x_{ij} \le M_{ij}, \ \ \text{all } i,j \\
& \sum_i x_{ij} - \sum_i x_{ji} = S_j, \ \ \text{all } j.
\end{aligned} \tag{12}$$

In addition, the projected relaxation, unlike the traditional one, can be solved with a specialized minimum cost network flow algorithm. This can provide a substantial advantage, because network flow algorithms run much faster than general linear programming algorithms.

3.2 Warehouse Location Problem

The same device can provide a network flow relaxation of problems with fixed node charges. A node i with a fixed charge need only be replaced by an arc (h_i, t_i) with the same fixed charge, with all the incoming arcs to node i attached to h_i and all the outgoing arcs to t_i.

An important special case is a warehouse location problem, which may be written

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{n} F_i y_i \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} \le K_i, \quad i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} \ge D_j, \quad j = 1,\ldots,n \\
& 0 \le x_{ij} \le M_{ij} y_i, \quad \text{all } i,j \\
& y_i \in \{0,1\}, \quad i = 1,\ldots,m
\end{aligned}$$

Nodes $i = 1,\ldots,m$ are potential warehouse locations, each with fixed cost C_i and capacity K_i. Nodes $j = 1,\ldots,n$ are demands points with demand D_j.

The problem can be transformed into a network design problem with fixed arc costs. First a bipartite graph forms with potential warehouse sites and places supplied by warehouses. $c_{i,j}$ becomes the cost on the arc from the warehouse i to place j together with infinite capacity. Then add two nodes, a source s and a sink t with arcs from the source to the warehouse sites and from the places to the sink. An arc from s to a warehouse site i has cost F_i and capacity K_i for all $i \in \{1,\ldots,m\}$. An arc from a place j to t has zero cost and capacity D_i. Also, add an arc from the source to the sink with zero cost and capacity $\sum_{i=1}^{m} K_i - \sum_{j=1}^{n} D_i$. As before, the projected relaxation is a minimum cost network flow problem, which can substantially accelerate the solution of the problem.

References

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[3] Hooker, J. N., and M. A. Osorio, Mixed logical/linear programming, to appear in *Discrete Applied Mathematics.*