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3D-2D Dimension Reduction of Homogenized Thin Films

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Contents

Chapter 1

Introduction

In this thesis we study homogenization of heterogeneous structures modeling elastic thin films with microstructure. The search for lower dimensional models describing thin three-dimensional structures is a classical problem in mechanics of materials, and the process of obtaining 2D models as limits issuing from 3D frameworks is referred to as dimension reduction.

Since the early '90s this problem has been tackled successfully by means of variational techniques. For both for 3D-1D and 3D-2D, the theory is well established within the linear elastic setting, and work stemming from Ciarlet and Destuynder [27] has lead to research in areas ranging from linear elastic plates [23] to beam models [47, 48, 82], shells [26], and various constitutive behaviors [14]. However, the linearized elasticity model is not the proper setting for describing the properties of hyperelasticity, which drove researchers to study the nonlinear elasticity case. Motivated by the one-dimensional work in [3], the first paper in this context was written by Le Dret and Raoult in [61] for a two-dimensional model as a variational limit of a three-dimensional material, and this led to several others including [7, 13, 44, 59, 60, 62, 63]. Resulting from these works, hierarchies of limit models have been deduced using Γ–convergence techniques (for a treatment of Γ–convergence see, e.g. [17, 18, 31]). To make these ideas precise, let ω be an open, connected domain in $\mathbb{R}^2$,

$$\Omega_h := \omega \times (-h/2, h/2) \text{ for } h > 0, \quad \text{and} \quad \Omega := \omega \times (-1/2, 1/2). \tag{1.0.1}$$

In the preceding papers, the authors consider a sequence of functionals of the form

$$\mathcal{I}_h(u) := \frac{1}{h^k} \int_{\Omega_h} W\left(x, \nabla u(x)\right) dx, \tag{1.0.2}$$

where $u : \Omega_h \to \mathbb{R}^3$ is a suitable deformation of Ω_h. Different characterizations of the limiting functional are obtained according to the scaling of the elastic energy, h^k, relative to the thickness parameter h.

In this thesis we undertake the derivation of 2D models for thin composite materials. Understanding microstructure is paramount to the study of many important problems including biological tissue growth and the development of new materials. As thin film technology has advanced, scientists are able to have precise control over the thickness and composition of a film, as noted in [40], and in turn this has motivated the mathematical study of highly heterogeneous material models. It is generally assumed that the microstructure is periodic on a scale ε, and the limit of the energy as $\varepsilon \to 0$ is taken in order to obtain an effective energy for the material at a macroscopic scale, or the homogenized material.

The first homogenization results in nonlinear elasticity, without dimension reduction, were proved in [16] and [69]. In these two papers, A. Braides and S. Müller assume p-growth of a stored energy density W_ε that oscillates periodically, e.g. $W_\varepsilon(x;\cdot) = W(x, \frac{x}{\varepsilon};\cdot)$. They show that as the periodicity scale goes to zero, the elastic energy W_ε converges to a homogenized energy, whose density is obtained by means of an infinite-cell homogenization formula. Later, the growth conditions were relaxed somewhat, such as in [51, 52] by Hafsa et al.

The analysis of problems involving simultaneous dimension reduction and homogenization was initiated in [8, 19] for the membrane energy regime with p-growth assumptions on the stored energy density. It should be noted that these growth assumptions on the energy W do not prevent interpenetration of matter, in that the energy remains finite as the Jacobian of the transformation tends to 0, which renders the model necessarily nonphysical, as explained in [10]. More recently, in [57,73,83] models for homogenized plates have been derived under physical growth conditions for the energy density.

The focus of this thesis is the study of a problem within the realm of homogenized thin structures: multiscale homogenization in the Kirchhoff nonlinear plate theory. The Kirchhoff plate model refers to the energy in (1.0.2) with $k = 3$. We analyze an energy of this form with two periodicity scales.

This thesis is organized as follows: in Chapter 3 we discuss a compactness result for two-scale convergence in $L^1(\Omega)$. Two-scale convergence is an important tool in the study of homogenization, and plays an important role in the arguments found in Chapter 4. Chapter 4 addresses multiscale generalizations of [57,73,83] in which we consider two microscopic scales, $\varepsilon(h)$ and $\varepsilon^2(h)$. In particular, we are able to characterize the Γ-limit for the Kirchhoff scaling under physically realistic growth assumptions through a careful application of the rigidity estimates of Friesecke, James and Müller in [45] and through the study of the interaction of the two scales.

1.1 A Note on Two Scale Compactness for p=1

The method of two-scale convergence, introduced by G. Nguetseng in [74] and further developed by G. Allaire in [4], is an important tool in the study of homogenization theory. Although periodicity poses constraints on physically realistic models, it is generally agreed that understanding the effective behavior of periodically structured composite materials may aid in the study of more complex media. Accordingly, the theory of two-scale convergence has played an important role in the study of PDEs and their applications in homogenization.

Both Nguetseng and Allaire restricted most of their interest to the case of two-scale convergence in $L^2(U)$, where U is an open subset of $\mathbb{R}^N$. The proof by Allaire in [4] of two-scale compactness in $L^2(U)$ relies on duality and the separability of $L^2(U)$. As stated in his paper [4], this proof easily extends to the case of two-scale compactness in $L^p(U)$ with $1 < p \leq +\infty$. This is the form of two-scale compactness that is most commonly used in the literature. The arguments used for the case when $1 < p \leq +\infty$ cannot be applied to the case when $p = 1$, rarely mentioned explicitly, due to a lack of separability of the dual of $L^1(U)$, $L^\infty(U)$. A few authors have touched on the problem, including Holmbom, Silfver, Svanstedt and Wellander in [54] and A. Visintin in [84], although detailed arguments seem to be unavailable in the literature. In [24, 25] the authors address a related case of two-scale convergence in generalized Besicovitch spaces where there is also lack of separability.

In Chapter 3 we present three proofs for the two-scale compactness of bounded sequences in $L^1(U)$ under appropriate assumptions. To be precise,

Theorem 1.1.1. *Let U be an open subset of $\mathbb{R}^N$. Let $\{u_\varepsilon\} \subset L^1(U)$ be a bounded sequence in $L^1(U)$, equi-integrable, and assume that for all $\eta > 0$ there exists an open set $E \subset U$ such that $|E| < +\infty$ and*

$$\sup_{\varepsilon>0} \int_{U \setminus E} |u_\varepsilon(x)| dx < \eta.$$

Then there exists a subsequence (not relabeled) such that $\{u_\varepsilon\}$ two-scale converges to some $u_0 \in L^1(U \times Y)$. In particular, $u_\varepsilon \rightharpoonup \bar{u}_0$ in $L^1(U)$, with $\bar{u}_0(x) := \int_Y u_0(x, y) dy$.

The first proof of this theorem uses a truncation argument in order to apply two-scale compactness results for $p > 1$. The second makes use of the two-scale compactness proved for Radon measures by M. Amar in [6]. The last approach relies on the periodic unfolding characterization of two-scale limits, as introduced in [28] (see also [30, 32, 84]). The latter proof is the simplest and most intuitive, due to the fact that the periodic unfolding method reduces two-scale convergence in $L^p(U)$ to standard weak L^p convergence in $U \times Y$ (where Y is the period of the oscillations) of the unfolded functions, thus allowing us to replace rapidly oscillating test functions with non-oscillatory test functions. This method has been used in many contexts, including electromagnetism, homogenization in a domain with oscillating boundaries, and thin junctions in linear elasticity [11, 15, 29, 33, 34, 49, 50, 67].

This work, jointly with Irene Fonseca, will appear in Port. Math., [22].

1.2 Multiscale Homogenization in Kirchhoff's Nonlinear Plate Theory

In Chapter 4 we present a multiscale generalization of the problems posed in [57, 73, 83]. We briefly describe the previous results. Let Ω_h, defined as in (1.0.1), be the reference configuration of a nonlinearly elastic thin plate, where ω is a bounded domain in $\mathbb{R}^2$, and $h > 0$ is the thickness parameter. Let $x' := (x_1, x_2) \in \omega$. Assume that the physical structure of the plate is such that an in-plane homogeneity scale $\varepsilon(h)$ arises, where $\{h\}$ and $\{\varepsilon(h)\}$ are monotone decreasing sequences of positive numbers, $h \to 0$, and $\varepsilon(h) \to 0$ as $h \to 0$. In [57, 73, 83] the rescaled nonlinear elastic energy associated to a deformation $v \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ is given by

$$\mathcal{I}^h(v) := \frac{1}{h} \int_{\Omega_h} W\Big(\frac{x'}{\varepsilon(h)}, \nabla v(x)\Big)\, dx,$$

where the stored energy density W is periodic in its first argument and satisfies the commonly adopted assumptions in nonlinear elasticity, as well as a non-degeneracy condition in a neighborhood of the set of proper rotations.

In [73] the authors focus on the scaling of the energy corresponding to Von Kármán plate theory, that is they consider deformations $v^h \in W^{1,2}(\Omega_h; \mathbb{R}^3)$ such that

$$\limsup_{h \to 0} \frac{\mathcal{I}^h(v^h)}{h^4} < +\infty.$$

Under the assumption that the limit

$$\gamma_1 := \lim_{h \to 0} \frac{h}{\varepsilon(h)}$$

exists, different homogenized limit models are identified, depending on the value of $\gamma_1 \in [0, +\infty]$.

A parallel analysis is carried out in [57], where the scaling of the energy associated to Kirchhoff's plate theory is studied, i.e., the deformations under consideration satisfy

$$\limsup_{h \to 0} \frac{\mathcal{I}^h(v^h)}{h^2} < +\infty.$$

In this situation a lack of compactness occurs when $\gamma_1 = 0$ (the periodicity scale tends to zero much more slowly than the thickness parameter). A partial solution to this problem, in the case in which

$$\gamma_2 := \lim_{h \to 0} \frac{h}{\varepsilon^2(h)} = +\infty,$$

is proposed in [83], by means of a careful application of Friesecke, James and Müller's quantitative rigidity estimate, and a construction of piecewise constant rotations (see [45, Theorem 4.1], [46, Theorem 6] and [83, Lemma 3.11]). The analysis of simultaneous homogenization and dimension reduction for Kirchhoff's plate theory in the remaining regimes is still an open problem.

In Chapter 4 we deduce a multiscale version of the results in [57] and [83]. We focus on the scaling of the energy which corresponds to Kirchhoff's plate theory, and we assume that the plate undergoes the action of two homogeneity scales - a coarser one and a finer one - i.e., the rescaled nonlinear elastic energy is given by

$$\mathcal{J}^h(v) := \frac{1}{h}\int_{\Omega_h} W\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla v(x)\Big)\,dx$$

for every deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$, where the stored energy density W is periodic in its first two arguments and, again, satisfies the usual assumptions in nonlinear elasticity, as well as the nondegeneracy condition (see Section 4.1) adopted in [57, 73, 83]. We consider sequences of deformations $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ verifying

$$\limsup_{h\to 0}\frac{\mathcal{J}^h(v^h)}{h^2} < +\infty, \tag{1.2.1}$$

and we seek to identify the effective energy associated to the rescaled elastic energies $\big\{\frac{\mathcal{J}^h(v^h)}{h^2}\big\}$ for different values of γ_1 and γ_2, i.e., depending on the interaction of the homogeneity scales with the thickness parameter.

As in [57], a sequence of deformations satisfying (1.2.1) converges, up to the extraction of a subsequence, to a limit deformation $u \in W^{1,2}(\omega;\mathbb{R}^3)$ satisfying the isometric constraint

$$\partial_{x_\alpha} u(x')\cdot\partial_{x_\beta} u(x') = \delta_{\alpha,\beta} \quad \text{for a.e. } x' \in \omega, \qquad \alpha,\beta \in \{1,2\}. \tag{1.2.2}$$

We prove that the effective energy for $u \in W^{1,2}(\omega;\mathbb{R}^3)$ is given by

$$\mathcal{E}^{\gamma_1}(u) := \begin{cases} \frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\hom}(\Pi^u(x'))\,dx' & \text{if } u \text{ satisfies (1.2.2)},\\ +\infty & \text{otherwise,}\end{cases}$$

where Π^u is the second fundamental form associated to u (see (4.2.4)), and $\overline{\mathscr{Q}}^{\gamma_1}_{\hom}$ is a quadratic form dependent on the value of γ_1, with explicit characterization provided in (4.3.2)–(4.3.4). To be precise, our main result is the following.

Theorem 1.2.1. *Let $\gamma_1 \in [0,+\infty]$ and let $\gamma_2 = +\infty$. Let $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ be a sequence of deformations satisfying the uniform energy estimate* (1.2.1). *There exists a map $u \in W^{2,2}(\omega;\mathbb{R}^3)$ verifying* (1.2.2) *such that, up to the extraction of a (not relabeled) subsequence, there holds*

$$v^h(x',hx_3) - \fint_{\Omega_1} v^h(x',hx_3)\,dx \to u \quad \textit{strongly in } L^2(\Omega_1;\mathbb{R}^3),$$
$$\nabla_h v^h(x',hx_3) \to (\nabla' u|n_u) \quad \textit{strongly in } L^2(\Omega_1;\mathbb{M}^{3\times 3}),$$

with

$$n_u(x') := \partial_{x_1} u(x') \wedge \partial_{x_2} u(x') \quad \textit{for a.e. } x' \in \omega,$$

and

$$\liminf_{h\to 0} \frac{\mathcal{J}^h(v^h)}{h^2} \geq \mathcal{E}^{\gamma_1}(u). \tag{1.2.3}$$

Moreover, for every $u \in W^{2,2}(\omega;\mathbb{R}^3)$ *satisfying* (1.2.2)*, there exists a sequence* $\{v^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ *such that*

$$\limsup_{h\to 0} \frac{\mathcal{J}^h(v^h)}{h^2} \leq \mathcal{E}^{\gamma_1}(u). \tag{1.2.4}$$

We remark that our main theorem is consistent with the results proved in [57] and [83]. Indeed, in the presence of a single homogeneity scale, it follows directly from (4.3.2)–(4.3.4) that $\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}$ reduces to the effective energy identified in [57, 83] for $\gamma_1 \in (0, +\infty]$ and $\gamma_1 = 0$, respectively. The main difference with respect to [57, 83] is in the structure of the homogenized energy density $\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}$, which is obtained by means of a double pointwise minimization, first with respect to the faster periodicity scale, and then with respect to the slower one and the x_3 variable (see (4.3.2)–(4.3.4)).

The quadratic behavior of the energy density around the set of proper rotations together with the linearization occurring due to the high scalings of the elastic energy yield a convex behavior for the homogenization problem, so that, despite the nonlinearity of the three-dimensional energies, the effective energy does not have an infinite-cell structure, in contrast with [69]. The main techniques for the proof of the liminf inequality (1.2.3) are the notion of multiscale convergence introduced in [5], and its adaptation to dimension reduction (see [71]). The proof of the limsup inequality (1.2.4) follows that of [57, Theorem 2.4].

The crucial part of the argument is the characterization of the three-scale limit of the sequence of linearized elastic stresses (see Section 4.2). We deal with sequences having unbounded L^2 norms but whose oscillations on the scale ε or ε^2 are uniformly controlled. As in [57, Lemmas 3.6–3.8], to enhance their multiple-scales oscillatory behavior we work with suitable oscillatory test functions having vanishing average in their periodicity cell.

The presence of three scales increases the technical difficulty of the problem in all scaling regimes. For $\gamma_1 \in (0, +\infty]$, Friesecke, James and Müller's rigidity estimate ([45, Theorem 4.1]) leads us to work with sequences of rotations that are piecewise constant on cubes of size $\varepsilon(h)$ with centers in $\varepsilon(h)\mathbb{Z}^2$. However, in order to identify the three-scale limit of the linearized stresses, we must consider sequences oscillating on a scale $\varepsilon^2(h)$. This problem is solved in Step 1 of the proof of Theorem 4.2.1, by subdividing the cubes of size $\varepsilon^2(h)$, with centers in $\varepsilon^2(h)\mathbb{Z}^2$, into "good cubes" lying completely within a bigger cube of size $\varepsilon(h)$ and center in $\varepsilon(h)\mathbb{Z}^2$ and "bad cubes", and by showing that the measure of the intersection between ω and the set of "bad cubes" converges to zero faster than or comparable to $\varepsilon(h)$, as $h \to 0$.

The opposite problem arises in the case in which $\gamma_1 = 0$. By Friesecke, James and Müller's rigidity estimate ([45, Theorem 4.1]), it is natural to work with sequences of piecewise constant rotations which are constant on cubes of size $\varepsilon^2(h)$ having centers in the grid $\varepsilon^2(h)\mathbb{Z}^2$, whereas in order to identify the limit multiscale stress we need to deal with oscillating test functions with vanishing averages on a scale $\varepsilon(h)$. The identification of "good cubes" and "bad cubes" of size $\varepsilon^2(h)$ is thus not helpful in this latter framework as the contribution of the oscillating test functions on cubes of size $\varepsilon^2(h)$ is not negligible anymore. Therefore, we are only able to perform an identification of the multiscale limit in the case $\gamma_2 = +\infty$, extending to the multiscale setting the results in [83]. The identification of the effective energy in the case in which $\gamma_1 = 0$ and $\gamma_2 \in [0, +\infty)$ remains an open question.

Chapter 4 is organized as follows: in Section 4.1 we set the problem and introduce the assumptions on the energy density. In Section 2.4 we recall a few compactness results and the definition and some properties of multiscale convergence. Sections 4.2 and 4.3 are devoted to the identification of the limit linearized stress and to the proof of the liminf inequality (1.2.3). In Section 4.4 we show the optimality of the lower bound deduced in Section 4.3, and we exhibit a recovery sequence satisfying (1.2.4).

This work, in collaboration with Elisa Davoli and Irene Fonseca, has been accepted for publication in Math. Models Methods Appl. Sci., [21].

Chapter 2

Preliminaries

2.1 Notation

In this thesis $\{\varepsilon\} = \{\varepsilon_n\}_{n=1}^{\infty}$ stands for a generic decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Y will denote a unit cube in $\mathbb{R}^N$ with faces parallel to the coordinate axes, i.e., $Y = (0,1)^N$.

Whenever a map $v \in L^2, C^\infty, ...,$ is Y-periodic, that is

$$v(x + e_i) = v(x) \quad i = 1, 2, ..., N$$

for a.e. $x \in \mathbb{R}^N$, where $\{e_1, ..., e_N\}$ is the canonical othonormal basis of $\mathbb{R}^N$, we write $v \in L^2_{\mathrm{per}}, C^\infty_{\mathrm{per}}, ...,$ respectively. We implicitly identify the spaces $L^2(Y)$ and $L^2_{\mathrm{per}}(\mathbb{R}^N)$. We denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$ by $|A|$.

We use $Q := \big(-\frac{1}{2}, \frac{1}{2}\big)^2$ to denote the unit cube in $\mathbb{R}^2$ centered at the origin and with sides parallel to the coordinate axes. We will write a point $x \in \mathbb{R}^3$ as

$$x = (x', x_3), \qquad \text{where } x' \in \mathbb{R}^2 \text{ and } x_3 \in \mathbb{R},$$

and we will use the notation ∇' to denote the gradient with respect to x'. For every $r \in \mathbb{R}$, $\lceil r \rceil$ is its greatest integer part. With a slight abuse of notation, for every $x' \in \mathbb{R}^2$, $\lceil x' \rceil$ and $\lfloor x' \rfloor$ are the points in $\mathbb{R}^2$ whose coordinates are given by the greatest and least integer parts of the coordinates of x', respectively. Given a map $\phi \in W^{1,2}(\mathbb{R}^2)$, $(y \cdot \nabla')\phi(x')$ stands for

$$(y \cdot \nabla')\phi(x') := y_1 \partial_{x_1} \phi(x') + y_2 \partial_{x_2} \phi(x') \quad \text{for a.e. } x' \in \mathbb{R}^2 \text{ and } y \in Q.$$

We write $(\nabla')^\perp \phi$ to indicate the map

$$(\nabla')^\perp \phi(x') := (-\partial_{x_2} \phi(x'), \partial_{x_1} \phi(x')) \quad \text{for a.e. } x' \in \mathbb{R}^2.$$

We denote by $\mathbb{M}^{n\times m}$ the set of real-valued matrices with n rows and m columns and by $SO(3)$ the set of proper rotations, that is

$$SO(3) := \{R \in \mathbb{M}^{3\times 3} : \; R^T R = Id \text{ and } \det R = 1\}.$$

Given a matrix $M \in \mathbb{M}^{3\times 3}$, M' stands for the 3×2 submatrix of M given by its first two columns. For every $M \in \mathbb{M}^{n\times n}$, $\operatorname{sym} M$ is the the $n \times n$ symmetrized matrix defined as

$$\operatorname{sym} M := \frac{M + M^T}{2}.$$

We adopt the convention that C designates a generic constant, whose value may change from expression to expression in the same formula.

2.2 The Direct Method of the Calculus of Variations and Γ–Convergence

In this section we present a brief introduction of the direct method of the calculus of variations and basic properties of Γ–convergence as motivation for work in Chapter 4.

Let (X, d) be a metric space and let $f : X \to \overline{\mathbb{R}}$ be a function not identically equal to ∞, where $\overline{\mathbb{R}}$ is the extended real line, $[-\infty, \infty]$. The direct method provides conditions on X and f to ensure the existence of a minimum point for f. Tonelli's direct method may be summarized in four steps ([41]) :

Step 1: Consider a *minimizing sequence* $\{u_n\} \subset X$, that is, a sequence such that

$$\lim_{n\to\infty} f(u_n) = \inf_{u\in X} f.$$

Step 2: Prove that $\{u_n\}$ admits a subsequence $\{u_{n_j}\}$ that converges with respect to some (possibly weaker) topology τ to some point $u_0 \in X$.

Step 3: Establish the sequential lower semi-continuity of F with respect to τ.

Step 4: In view of Steps 1-3, conclude that u_0 is a minimum of f because

$$\inf_{u\in X} f = \lim_{n\to\infty} f(u_n) = \lim_{j\to\infty} f(u_{n_j}) \geq f(u_0) \geq \inf_{u\in X} f.$$

We now turn our attention to describing the behavior of a family of minimum problems depending on a parameter, for example,

$$\inf\{f_h(u) : u \in X_h\}$$

for $h > 0$. The goal is to approximate these problems by using a limit theory as $h \to 0$ leading to an "effective energy" f, with the limiting problem being described by

$$\min\{f(u) : u \in X\}.$$

A suitable notion of convergence for the family of functionals f_h so that the limiting functional may be treated using the direct method, as outlined above, is Γ–convergence. A more detailed explanation can be found in [17].

Next we define De Giorgi's notion of Γ–convergence and state some of its basic properties (see [17, 31, 35, 36]).

Definition 2.2.1. *(Γ–convergence of a sequence of functionals). We say that a sequence* $f_n : X \to \overline{\mathbb{R}}$ **Γ–converges** *in* X *to* $f : X \to \overline{\mathbb{R}}$ *if for all* $x \in X$ *we have*

(i.) (lim inf inequality) for every sequence $\{x_n\}$ *converging to* x

$$f(x) \leq \liminf_{n\to\infty} f_n(x_n);$$

(ii.) (lim sup inequality) there exists a sequence $\{x_n\}$ *converging to* x *such that*

$$f(x) \geq \limsup_{n\to\infty} f_n(x_n).$$

The function f *is called the* Γ*–limit of* $\{f_n\}$ *and we write* $f = \Gamma - \lim_{n\to\infty} f_n$*.*

For every $h > 0$ let f_h be a functional over X with values in $\overline{\mathbb{R}}$, $f_h : X \to \overline{\mathbb{R}}$.

Definition 2.2.2. *(Γ–convergence of a family of functionals). A functional* f *is said to be the* $\Gamma - \lim$ *of* $\{f_h\}_h$ *with respect to the metric* d*, as* $h \to 0$*, if for every sequence* $h_n \to 0$

$$f = \Gamma - \liminf_{n\to\infty} f_{h_n},$$

and we write

$$f = \Gamma - \liminf_{h\to 0} f_h.$$

Proposition 2.2.3. [17, Prop.1.28] *If* $f = \Gamma - \lim_{h\to 0} f_h$*, then* f *is lower semi-continuous.*

If (X, d) is a separable metric space then the following compactness property holds.

Theorem 2.2.4. [17, Prop. 1.42] *For each sequence* $\{h_n\}_n$ *there exists a subsequence* $\{h_{n_j}\}_j$ *such that* $\Gamma - \lim_{j\to\infty} f_{n_j}$ *exists.*

Definition 2.2.5. *A family of functionals* $\{f_h\}$ *is said to be* equi-coercive *if for every real number* λ *there exists a compact set* K_λ *in* X *such that for each sequence* $\varepsilon_n \to 0$*,*

$$\{x \in X \,:\, f_{h_n}(u) \leq \lambda\} \subset K_\lambda \quad \text{for every } n \in \mathbb{N}.$$

The following is one of the most important properties of Γ–convergence: the convergence of (almost) minimizers of a family of functionals to the minimum of a limiting functional, under appropriate assumptions.

Theorem 2.2.6. [35, Thm. 2.6] *If* $\{f_h\}_h$ *is a family of equi-coercive functionals on* X *and if*

$$f = \Gamma - \lim_{h\to 0} f_h,$$

then the functional f *has a minimum on* X *and*

$$\min_{x\in X} f(x) = \lim_{h\to 0} \inf_{x\in X} f_h(x).$$

Moreover, given $h_n \to 0$ *and* $\{x_n\}_n$ *a converging sequence such that*

$$\lim_{n\to\infty} f_{h_n}(x_n) = \lim_{n\to\infty} \inf_{x\in X} f_{h_n}(x), \tag{2.2.1}$$

then its limit is a minimum point for f *on* X*.*

If (2.2.1) holds, then $\{u_n\}_n$ is said to be a sequence of **almost-minimizers** for f.

2.3 Some Useful Compactness Results

While in Chapter 4 we deal with weak convergence in reflexive spaces, in Chapter 3 we work in L^1, and we will need to invoke equi-integrability:

Definition 2.3.1. *A family $\mathcal{F}$ of measurable functions $f : U \to [-\infty, +\infty]$ is said to be* **equi-integrable** *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E |f| \, dx \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for every measurable set $E \subset U$ such that $|E| \leq \delta$

We recall the Dunford-Pettis criterion for compactness in L^1 (see e.g. [41, Prop. 2.82]).

Theorem 2.3.2 (Dunford-Pettis)**.** *A family $\mathcal{F} \subset L^1(U)$ is weakly sequentially precompact if and only if*

(i) $\mathcal{F}$ is bounded in $L^1(U)$,

(ii) $\mathcal{F}$ is equi-integrable,

(iii) for every $\eta > 0$ there exists a measurable set $E \subset U$ with $|E| < +\infty$ such that

$$\sup_{f \in \mathcal{F}} \int_{U \setminus E} |f| \, dx \leq \eta. \tag{2.3.1}$$

Remark 2.3.3. *By the regularity properties of $\mathcal{L}^N$, it can be shown that assuming conditions (ii) and (iii) is equivalent to assuming (ii) and (iii'), where in (iii') E is an open bounded set of finite measure.*

Next we present a preliminary result which will allow us to deduce compactness for sequences of deformations satisfying the uniform energy estimate (4.1.1). We recall [45, Theorem 4.1], which provides a characterization of limits of deformations whose scaled gradients are uniformly close in the L^2-norm to the set of proper rotations.

Theorem 2.3.4. *Let $\{u^h\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be such that*

$$\limsup_{h \to 0} \frac{1}{h^2} \int_\Omega \mathit{dist}^2(\nabla_h u^h(x), SO(3)) \, dx < +\infty. \tag{2.3.2}$$

Then, there exists a map $u \in W^{2,2}(\omega; \mathbb{R}^3)$ such that, up to the extraction of a (not relabeled) subsequence,

$$u^h - \fint_\Omega u^h(x) \, dx \to u \quad \textit{strongly in } L^2(\Omega; \mathbb{R}^3)$$
$$\nabla_h u^h \to (\nabla' u | n_u) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}),$$

with

$$\partial_{x_\alpha} u(x') \cdot \partial_{x_\beta} u(x') = \delta_{\alpha,\beta} \quad \textit{for a.e. } x' \in \omega, \quad \alpha, \beta \in \{1, 2\} \tag{2.3.3}$$

and

$$n_u(x') := \partial_{x_1} u(x') \wedge \partial_{x_2} u(x') \quad \textit{for a.e. } x' \in \omega. \tag{2.3.4}$$

2.4 Multiscale Convergence

We recall the definitions of *two-scale convergence* and *three-scale convergence*, as introduced in [4]. For a detailed treatment of two-scale convergence we refer to, *e.g.*, [4,66,74]. The main results on multiscale convergence may be found in [5,9,38,39]. We recall that U is an open subset of $\mathbb{R}^N$ and $Y = (0,1)^N$.

Definition 2.4.1. *Let $\{u_\varepsilon\} \subset L^p(U)$ where $1 \leq p \leq +\infty$. Then $\{u_\varepsilon\}$* **two-scale converges** *to a function $u_0 \in L^p(U \times Y)$, and we write $u_\varepsilon \overset{2-s}{\rightharpoonup} u_0$, if*

$$\lim_{\varepsilon \to 0} \int_U u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_U \int_Y u_0(x,y)\, \psi(x,y)\; dy\, dx \tag{2.4.1}$$

for all $\psi \in C_c^\infty[U; C_{\text{per}}^\infty(Y)]$.

We say that $\{u_\varepsilon\}$ converges **strongly two-scale** *to $u_0 \in L^p(U \times Y)$, and we write and we write $u_\varepsilon \overset{2-s}{\longrightarrow} u$, if*

$$u_\varepsilon \overset{2-s}{\rightharpoonup} u \quad \textit{weakly two-scale}$$

and

$$\|u_\varepsilon\|_{L^p(U)} \to \|u\|_{L^p(U \times Y)}.$$

Let $\{u_\varepsilon\} \subset L^p(U)$ where $1 \leq p \leq +\infty$. Then $\{u_\varepsilon\}$ **three-scale converges** *to a function $u_0 \in L^p(U \times Y \times Y)$, and we write $u_\varepsilon \overset{3-s}{\rightharpoonup} u_0$, if*

$$\lim_{\varepsilon \to 0} \int_U u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx = \int_U \int_Y \int_Y u_0(x,y,z)\, \psi(x,y,z)\; dz\, dy\, dx \tag{2.4.2}$$

for all $\psi \in C_c^\infty[U; C_{\text{per}}^\infty(Y \times Y)]$.

We say that $\{u_\varepsilon\}$ converges **strongly three-scale** *to $u_0 \in L^p(U \times Y \times Y)$, and we write and we write $u_\varepsilon \overset{3-s}{\longrightarrow} u$, if*

$$u_\varepsilon \overset{3-s}{\rightharpoonup} u \quad \textit{weakly three-scale}$$

and

$$\|u_\varepsilon\|_{L^p(U)} \to \|u\|_{L^p(U \times Y \times Y)}.$$

Remark 2.4.2. *In the above Y may be replaced by $(-1/2, 1/2)^N$. In the standard literature Y is generally used, however, in Chapter 4 is it convenient for the use of other theorems to consider periodicity cells centered at the grid point of $\mathbb{R}^2$. Thus, we will use the convention of taking the periodicity cell to be Y in Chapter 3, and to be Q in Chapter 4.*

We now present the standard two-scale compactness result for $p > 1$, as proved by Allaire in [4, Corollary 1.15].

Theorem 2.4.3. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(U)$, with $1 < p \leq +\infty$. There exists a function $u_0(x, y)$ in $L^p(U \times Y)$ such that, up to a subsequence, $\{u_\varepsilon\}$ two-scale converges to u_0.*

An analogous result holds for three-scale convergence.

In order to simplify the statement of Theorem 4.2.1 and its proof, we introduce the definition of *dr-3-scale convergence* (dimension reduction three-scale convergence), i.e., 3-scale convergence adapted to dimension reduction, inspired by S. Neukamm's 2-scale convergence adapted to dimension reduction (see [71]).

Definition 2.4.4. *Let $u \in L^2(\Omega \times Q \times Q)$ and $\{u_\varepsilon\} \subset L^2(\Omega)$. We say that $\{u_\varepsilon\}$* converges weakly dr-3-scale to *u in $L^2(\Omega \times Q \times Q)$, and we write $u^h \overset{dr-3-s}{\longrightarrow} u$, if*

$$\int_\Omega u_\varepsilon(x)\varphi\Big(x, \frac{x'}{\varepsilon}, \frac{x'}{\varepsilon^2}\Big)\, dx \to \int_\Omega \int_Q \int_Q u(x, y, z)\varphi(x, y, z)\, dz\, dy\, dx$$

for every $\varphi \in C_c^\infty(\Omega; C^\infty_{\mathrm{per}}(Q \times Q))$.

Remark 2.4.5. *We point out that* dr-3-scale convergence *is just a particular case of classical 3-scale convergence. Indeed, what sets apart* dr-3-scale convergence *from the classical 3-scale convergence is solely the fact that the test functions in Definition 2.4.4 depend on x_3 but oscillate only in the cross-section ω. In particular, if $\{u_\varepsilon\} \in L^2(\Omega)$ and*

$$u_\varepsilon \overset{dr-3-s}{\longrightarrow} u \quad \textit{weakly dr 3 scale}$$

then $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$. Therefore, by [5, Theorem 1.1] there exists $\xi \in L^2(\Omega \times (Q \times (-\frac{1}{2}, \frac{1}{2})) \times (Q \times (-\frac{1}{2}, \frac{1}{2})))$ such that, up to the extraction of a (not relabeled) subsequence,

$$u_\varepsilon \overset{3-s}{\longrightarrow} \xi \quad \textit{weakly 3-scale}\ ,$$

that is u_ε weakly 3-scale converges to ξ in $L^2(\Omega \times (Q \times (-\frac{1}{2}, \frac{1}{2})) \times (Q \times (-\frac{1}{2}, \frac{1}{2})))$ (in the sense of classical 3-scale convergence). Hence, the dr-3-scale limit, u, and the classical 3-scale limit, ξ, are related by

$$u(x, y, z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi(x, y, z, \eta, \tau)\, d\eta\, d\tau \quad \textit{for a.e. } x \in \omega \textit{ and } y, z \in Q.$$

We now characterize the limits of scaled gradients in the multiscale setting adapted to dimension reduction.

Theorem 2.4.6. *Let $u, u^h \in W^{1,2}(\Omega)$ be such that*

$$u^h \rightharpoonup u \quad \textit{weakly in } W^{1,2}(\Omega)$$

and

$$\limsup_{h\to 0} \int_\Omega |\nabla_h u^h(x)|^2\, dx < \infty.$$

Then u is independent of x_3. Moreover, there exist $u_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q))$, $u_2 \in L^2(\Omega \times Q; W^{1,2}_{\mathrm{per}}(Q))$, and $\bar{u} \in L^2\big(\omega \times Q \times Q; W^{1,2}\big(-\frac{1}{2}, \frac{1}{2}\big)\big)$ such that, up to the extraction of a (not relabeled) subsequence,

$$\nabla_h u^h \overset{dr-3-s}{\rightharpoonup} \big(\nabla' u + \nabla_y u_1 + \nabla_z u_2 \big| \partial_{x_3} \bar{u}\big) \quad \textit{weakly dr-3-scale}.$$

Moreover,

*(i) if $\gamma_1 = \gamma_2 = +\infty$ (*i.e. *$\varepsilon(h) << h$), then $\partial_{y_i}\bar{u} = \partial_{z_i}\bar{u} = 0$, for $i = 1, 2$;*

*(ii) if $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$ (*i.e. *$\varepsilon(h) \sim h$), then*

$$\bar{u} = \frac{u_1}{\gamma_1};$$

*(iii) if $\gamma_1 = 0$ and $\gamma_2 = +\infty$ (*i.e. *$h << \varepsilon(h) << h^{\frac{1}{2}}$), then*

$$\partial_{x_3} u_1 = 0 \quad \textit{and} \quad \partial_{z_i}\bar{u} = 0,\ i = 1, 2.$$

We omit the proof of Theorem 2.4.6 as it is a simple generalization of the arguments in [71, Theorem 6.3.3].

Next we present some properties of sequences having unbounded L^2 norms but whose oscillations on the scale ε or ε^2 are uniformly controlled. Arguing as in [57, Lemmas 3.6–3.8], we highlight the multiscale oscillatory behavior of our sequences by testing them against products of maps with compact support and oscillatory functions with vanishing average in their periodicity cell. In the proof of Theorem 4.2.1 we refer to [57, Proposition 3.2] and [83, Proposition 3.2], so for simplicity we introduce the notation needed in those papers.

Definition 2.4.7. *Let $\tilde{f} \in L^2(\omega \times Q)$ be such that*

$$\int_Q \tilde{f}(\cdot, y)\, dy = 0 \quad \textit{a.e. in } \omega.$$

We write

$$f^h \overset{osc,Y}{\rightharpoonup} \tilde{f}$$

if

$$\lim_{h\to 0}\int_\omega f^h(x')\varphi(x')g\Big(\frac{x'}{\varepsilon(h)}\Big)\,dx' = \int_\omega\int_Q \tilde{f}(x',y)\varphi(x')g(y)\,dy\,dx'$$

for every $\varphi \in C_c^\infty(\omega)$ and $g \in C^\infty_{\mathrm{per}}(Q)$, with $\int_Q g(y)\,dy = 0$.

Let $\{f^h\} \subset L^2(\omega)$ and let $\tilde{\tilde{f}} \in L^2(\omega \times Q \times Q)$ be such that

$$\int_Q \tilde{\tilde{f}}(\cdot,\cdot,z)\,dz = 0 \quad \textit{a.e. in } \omega\times Q.$$

We write

$$f^h \overset{osc,Z}{\rightharpoonup} \tilde{\tilde{f}}$$

if

$$\lim_{h\to 0}\int_\omega f^h(x')\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\,dx' = \int_\omega\int_Q\int_Q \tilde{\tilde{f}}(x',y,z)\psi(x',y)\varphi(z)\,dz\,dy\,dx'$$

for every $\psi \in C_c^\infty(\omega; C^\infty_{\mathrm{per}}(Q))$ and $\varphi \in C^\infty_{\mathrm{per}}(Q)$, with $\int_Q \varphi(z)\,dz = 0$.

Remark 2.4.8. *As a direct consequence of the definition of multiscale convergence and density arguments, if $\{f^h\} \subset L^2(\omega)$, then*

$$f^h \overset{2-s}{\rightharpoonup} f \quad \textit{weakly 2-scale}$$

if and only if

$$f^h(x) \overset{osc,Y}{\rightharpoonup} f(x) - \int_Q f(x,y)\,dy.$$

Analogously,

$$f^h \overset{3-s}{\rightharpoonup} \tilde{f} \quad \textit{weakly 3-scale}$$

if and only if

$$f^h(x) \overset{osc,Z}{\rightharpoonup} \tilde{f} - \int_Q \tilde{f}(x,y,z)\,dz.$$

We recall finally [57, Lemma 3.7 and Lemma 3.8].

Lemma 2.4.9. *Let $\{f^h\} \subset L^\infty(\omega)$ and $f^0 \in L^\infty(\omega)$ be such that*

$$f^h \overset{*}{\rightharpoonup} f^0 \quad \textit{weakly-* in } L^\infty(\omega).$$

Assume that f^h are constant on each cube $Q(\varepsilon(h)z, \varepsilon(h))$, with $z \in \mathbb{Z}^2$. If $f^0 \in W^{1,2}(\omega)$, then

$$\frac{f^h}{\varepsilon(h)} \overset{osc,Y}{\rightharpoonup} -(y\cdot\nabla')f^0.$$

Lemma 2.4.10. *Let $\{f^h\} \subset W^{1,2}(\omega)$, $f^0 \in W^{1,2}(\omega)$, and $\phi \in L^2(\omega; W^{1,2}_{\mathrm{per}}(Q))$ be such that*

$$f^h \rightharpoonup f^0 \quad \textit{weakly in } W^{1,2}(\omega),$$

and

$$\nabla' f^h \xrightharpoonup{2-s} \nabla' f^0 + \nabla_y \phi \quad \textit{weakly 2-scale},$$

with $\int_Q \phi(x', y)\, dy = 0$ for a.e. $x' \in \omega$. Then,

$$\frac{f^h}{\varepsilon(h)} \xrightharpoonup{osc,Y} \phi.$$

We conclude this section with a result that will play a key role in the identification of the limit elastic stress, and in the proof of the liminf and limsup inequalities (1.2.3) and (1.2.4). We omit its proof, as it follows by [73, Lemma 4.3].

Lemma 2.4.11. *Let $\mathscr{Q} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0, +\infty)$ be such that*

(i) $\mathscr{Q}(y, z, \cdot)$ is continuous for a.e. $y, z \in \mathbb{R}^2$,

(ii) $\mathscr{Q}(\cdot, \cdot, F)$ is $Q \times Q$-periodic and measurable for every $F \in \mathbb{M}^{3\times 3}$,

(iii) for a.e. $y, z \in \mathbb{R}^2$, the map $\mathscr{Q}(y, z, \cdot)$ is quadratic on $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, and satisfies

$$\frac{1}{C}|\mathrm{symF}|^2 \leq \mathscr{Q}(y, z, F) = \mathscr{Q}(y, z, \mathrm{symF}) \leq C|\mathrm{symF}|^2$$

for all $F \in \mathbb{M}^{3\times 3}$, and some $C > 0$.

Let $\{E^h\} \subset L^2(\Omega; \mathbb{M}^{3\times 3})$ and $E \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3})$ be such that

$$E^h \xrightharpoonup{dr-3-s} E \quad \textit{weakly dr-3-scale}.$$

Then

$$\liminf_{h\to 0} \int_\Omega \mathscr{Q}\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, E^h(x)\Big)\, dx \geq \int_\Omega \int_Q \int_Q \mathscr{Q}(y, z, E(x, y, z))\, dz\, dy\, dx.$$

If in addition

$$E^h \xrightarrow{dr-3-s} E \quad \textit{strongly dr-3-scale},$$

then

$$\lim_{h\to 0} \int_\Omega \mathscr{Q}\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, E^h(x)\Big)\, dx = \int_\Omega \int_Q \int_Q \mathscr{Q}(y, z, E(x, y, z))\, dz\, dy\, dx.$$

Chapter 3

A Note on Two Scale Compactness for p=1

If a sequence $\{u_\varepsilon\}$ in $L^p(U)$ two-scale converges for $1 < p \leq +\infty$, since we may consider $C_c^\infty(U)$ as a subset of $C_c^\infty[U; C_{\text{per}}^\infty(Y)]$ and as $C_c^\infty(U)$ is dense in $L^{p'}(U)$, then we also know, from [4], that $u_\varepsilon \rightharpoonup \int_Y u_0(x,y)dy$ in $L^p(U)$ ($\overset{\star}{\rightharpoonup}$ if $p = +\infty$). This argument does not apply to $p = 1$, because $C_c^\infty(U)$ is not dense in $L^\infty(U)$. However, this is a property we would like to preserve for two-scale compactness theorem for L^1 functions. Hence, we assume the Dunford-Pettis criterion for weak sequential compactness in $L^1(U)$ (see [41]) on $\{u_\varepsilon\}$ in our main theorem.

3.1 Method Using Two-Scale Compactness for $p > 1$

Our first proof relies on the two-scale compactness result for $p > 1$, Theorem 2.4.3. We are able to use this result to prove the following:

Proposition 3.1.1. *Let U be an open subset of $\mathbb{R}^N$ with $|U| < +\infty$. If $\{u_\varepsilon\} \subset L^1(U)$ is a bounded equi-integrable sequence, then there exists a subsequence (not relabeled) such that $\{u_\varepsilon\}$ two-scale converges to $u_0 \in L^1(U \times Y)$.*

Proof. For $M > 0$, let τ_M be the truncating operator $\tau_M : L^1(U) \to L^1(U)$ defined by

$$\tau_M u(x) := \begin{cases} u(x) & \text{if } |u(x)| \leq M, \\ M & \text{if } u(x) > M, \\ -M & \text{if } u(x) < -M, \end{cases}$$

where $u \in L^1(U)$. Since $|U| < +\infty$, if $u \in L^1(U)$ then $\tau_M u \in \cap_{1<p<+\infty} L^p(U)$ and (iii) in Theorem 2.3.2 is trivially satisfied. Equi-integrability of $\{u_\varepsilon\}$ and Theorem 2.3.2 imply that there exists a weakly convergent subsequence and so, without loss of generality, we assume that $u_\varepsilon \rightharpoonup \bar{u}$ in $L^1(U)$ for some $\bar{u} \in L^1(U)$.

Step 1: Consider first the case in which $u_\varepsilon \geq 0$ a.e. in U and for all $\varepsilon > 0$. Fix $M > 0$. By Theorem 1.2 in [4] we know that, up to a subsequence (not relabeled), $\{\tau_M u_\varepsilon\} \xrightarrow{2-s} u_M$ for some $u_M \in L^2(U \times Y)$, i.e.,

$$\lim_{\varepsilon\to 0} \int_U \tau_M u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_U \int_Y u_M(x,y)\, \psi(x,y)\, dydx$$

for all $\psi \in C_c^\infty[U; C_{\text{per}}^\infty(Y)]$. From this we deduce that $u_M \geq 0$.

We will extract a two-scale convergent subsequence as follows: For $M = 1$ let $\{u_{\varepsilon(1)}\}$ be a subsequence of $\{u_\varepsilon\}$ such that $\{\tau_1 u_{\varepsilon(1)}\}$ two-scale converges to a function $u_1 \in L^2(U \times Y)$. Recursively, for $M > 1$, $M \in \mathbb{N}$, apply the compactness theorem for $p = 2$ in [4] to the sequence $\{\tau_M u_{\varepsilon(M-1)}\}$ to obtain $\{u_{\varepsilon(M)}\} \subset \{u_{\varepsilon(M-1)}\}$ such that $\{\tau_M u_{\varepsilon(M)}\}$ two-scale converges to some $u_M \in L^2(U \times Y)$. Since $\{\varepsilon^{(M+1)}\}$ is a subsequence of $\{\varepsilon^{(M)}\}$, we have that $\tau_M u_{\varepsilon(M+1)} \xrightarrow{2-s} u_M$. In turn, as $u_\varepsilon \geq 0$ a.e., then $\tau_{M+1} u_{\varepsilon(M+1)} \geq \tau_M u_{\varepsilon(M+1)}$, and thus, passing the the two-scale limit we conclude that $u_{M+1} \geq u_M$ a.e. Let

$$u^+(x,y) := \sup_M u_M(x,y) = \lim_{M\to+\infty} u_M(x,y). \tag{3.1.1}$$

Next we show that $u^+ \in L^1(U \times Y)$. Consider an increasing sequence of test functions $\{\varphi_n\} \subset C_0^\infty(U; [0,1])$ such that $\varphi_n \equiv 1$ in $\{x \in U : \text{dist}(x, \partial U) > \frac{1}{n}\} \cap B(0,n)$ and $\varphi_n \equiv 0$ in $\{x \in U : \text{dist}(x, \partial U) \leq \frac{1}{2n}\} \cup (\mathbb{R}^N \setminus B(0, n+1))$. Then $\varphi_n \in C_c^\infty[U; C_{\text{per}}(Y)]$ and

$$\begin{aligned} +\infty > \limsup_{\varepsilon\to 0} \int_U u_\varepsilon\, dx &\geq \limsup_{\varepsilon^{(M)}\to 0} \int_U \tau_M u_{\varepsilon(M)}(x)\, dx \\ &\geq \lim_{\varepsilon^{(M)}\to 0} \int_U \tau_M u_{\varepsilon(M)}(x) \varphi_n(x)\, dx = \int_U \int_Y u_M(x,y)\varphi_n(x)\, dydx \end{aligned}$$

for all $M, n \in \mathbb{N}$. Taking the limit first in n as $n \to +\infty$, and applying the Monotone Convergence Theorem, we obtain

$$+\infty > \limsup_{\varepsilon\to 0} \int_U u_\varepsilon\, dx \geq \int_U \int_Y u_M(x,y)\, dydx,$$

and next taking the limit in M as $M \to +\infty$, by (3.1.1) we deduce

$$+\infty > \limsup_{\varepsilon\to 0} \int_U u_\varepsilon\, dx \geq \int_U \int_Y u^+(x,y)\, dydx. \tag{3.1.2}$$

Hence, as u^+ is non-negative, $u^+ \in L^1(U \times Y)$.

We claim that, up to a subsequence, for all $\psi \in C_c^\infty\left[U; C_{\text{per}}^\infty(Y)\right]$

$$\lim_{\varepsilon\to 0^+} \int_U u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\, dx = \int_U \int_Y u^+(x,y)\psi(x,y)\, dydx.$$

Fix $\psi \in C_c^\infty[U; C_{\text{per}}^\infty(y)]$. We have

$$\begin{aligned}
&\int_U u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_U \int_Y u^+(x,y)\psi(x,y)\, dydx \\
&\quad= \int_U \tau_M u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_U \int_Y u_M(x,y)\psi(x,y)\, dydx \\
&\qquad+ \int_U \int_Y \big(u_M(x,y) - u^+(x,y)\big)\, \psi(x,y)\, dydx \\
&\qquad+ \int_U \left(u_\varepsilon(x) - \tau_M u_\varepsilon(x)\right) \psi\left(x, \frac{x}{\varepsilon}\right) dx.
\end{aligned}$$

First, we analyze the convergence of the first difference in the right hand side above. Consider the diagonalizing sequence $\{\hat{\varepsilon}\}$ where $\hat{\varepsilon}_j := \varepsilon_j^{(j)}$, the jth element of the subsequence $\{\varepsilon^{(j)}\}$. We claim that

$$\lim_{\hat{\varepsilon}\to 0^+} \left| \int_U \tau_M u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_U \int_Y u_M(x,y)\psi(x,y)\, dydx \right| = 0 \tag{3.1.3}$$

for all M. This can be easily seen by observing that for $j > M$, $\{\hat{\varepsilon}_j\}$ is a subsequence of $\{\varepsilon^{(M)}\}$. Hence,

$$\begin{aligned}
\lim_{\hat{\varepsilon}\to 0^+} \int_U \tau_M u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx &= \lim_{\varepsilon^{(M)}\to 0^+} \int_U \tau_M u_{\varepsilon^{(M)}}(x)\psi\left(x, \frac{x}{\varepsilon^{(M)}}\right) dx \\
&= \int_U \int_Y u_M(x,y)\psi\left(x,y\right) dxdy,
\end{aligned}$$

proving (3.1.3). We conclude that

$$\begin{aligned}
&\lim_{\hat{\varepsilon}\to 0^+} \left| \int_U u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_U \int_Y u^+(x,y)\psi(x,y)\, dydx \right| \\
&\qquad\le ||\psi||_{L^\infty(U\times Y)} \lim_{M\to\infty} \left[\int_U \int_Y \big(u^+(x,y) - u_M(x,y)\big)\, dydx \right. \\
&\qquad\qquad \left. + \sup_{\varepsilon>0} \int_U \left(u_\varepsilon(x) - \tau_M u_\varepsilon(x)\right) dx \right].
\end{aligned} \tag{3.1.4}$$

From (3.1.1), taking into account that $u^+ \in L^1(U \times Y)$, we have by the Monotone Convergence Theorem,

$$\lim_{M\to\infty} \int_U \int_Y \big(u^+(x,y) - u_M(x,y)\big)\, dydx = 0.$$

Also,

$$\sup_{\varepsilon>0}\int_U (u_\varepsilon(x) - \tau_M u_\varepsilon(x))\,dx \le \sup_{\varepsilon>0}\int_{\{u_\varepsilon>M\}} u_\varepsilon(x)\,dx,$$

so using the equi-integrability of $\{u_\varepsilon\}$ and the fact that $|U| < +\infty$, we conclude that

$$\lim_{M\to\infty}\int_U (u_\varepsilon(x) - \tau_M u_\varepsilon(x))\,dx = 0.$$

By (3.1.4), this concludes the proof.

Step 2: In this case the sequence $\{u_\varepsilon\}$ may take both positive and negative values. The positive and negative parts of these functions can be considered separately, precisely, let $u_\varepsilon^+ := u_\varepsilon \chi_{\{u_\varepsilon \ge 0\}}$ and $u_\varepsilon^- := -u_\varepsilon \chi_{\{u_\varepsilon \le 0\}}$. Then for all ε, $u_\varepsilon = u_\varepsilon^+ - u_\varepsilon^-$, where $u_\varepsilon^+ \ge 0$ and $u_\varepsilon^- \ge 0$. From the previous step, there exists a subsequence $\{\hat\varepsilon^+\} \subset \{\varepsilon\}$ such that $\{u_{\hat\varepsilon^+}\}$ two-scale converges to some $u^+ \in L^1(U \times Y)$. Applying that step again we can extract an additional subsequence $\{\hat\varepsilon^-\} \subset \{\hat\varepsilon^+\}$ such that $\{u_{\hat\varepsilon^-}\}$ two-scale converges to some $u^- \in L^1(U \times Y)$. Let $u_0 := u^+ - u^-$. Then $u_0 \in L^1(U \times Y)$, and for all $\psi \in C_c^\infty[U; C_{\text{per}}^\infty(Y)]$

$$\begin{aligned}
&\lim_{\hat\varepsilon^-\to 0^+}\left|\int_U u_{\hat\varepsilon^-}(x)\psi\left(x, \frac{x}{\hat\varepsilon^-}\right)\,dx - \int_U\int_Y u_0(x,y)\psi(x,y)\,dydx\right| \\
&\quad\le \lim_{\hat\varepsilon^-\to 0^+}\left|\int_U u_{\hat\varepsilon^-}^+\psi\left(x, \frac{x}{\hat\varepsilon^-}\right)\,dx - \int_U\int_Y u^+(x,y)\psi(x,y)\,dydx\right| \\
&\qquad + \lim_{\hat\varepsilon^-\to 0^+}\left|\int_U u_{\hat\varepsilon^-}^-(x)\psi\left(x, \frac{x}{\hat\varepsilon^-}\right)dx - \int_U\int_Y u^-(x,y)\psi(x,y)\,dydx\right| \\
&\quad= 0.
\end{aligned}$$

□

Now that we have established two-scale compactness assuming that U is of finite measure, we may use this result in order to prove Theorem 1.1.1.

First Proof of Theorem 1.1.1. For each $k \in \mathbb{N}$ let $E_k \subset U$ be open and such that $|E_k| < +\infty$, $E_{k-1} \subset E_k$, and

$$\sup_{\varepsilon>0}\int_{U\setminus E_k} |u_\varepsilon(x)|\,dx < \frac{1}{k}.$$

Step 1: Again, we first address the case in which $u_\varepsilon \ge 0$ a.e. in U. By Proposition 3.1.1, for $k = 1$ there exists a subsequence $\{\varepsilon^{(1)}\} \subset \{\varepsilon\}$ such that $\{u_{\varepsilon^{(1)}}\}$ two scale converges in L^1 to some $u_1^+ \in L^1(E_1 \times Y)$ with

$$\int_{E_1\times Y} u_1^+(x,y)\,dydx \le \limsup_{\varepsilon\to 0}\int_{E_1} u_\varepsilon\,dx \le \limsup_{\varepsilon\to 0}\int_U u_\varepsilon\,dx =: C < +\infty$$

where we have used (3.1.2). For $k > 1$ extract $\{\varepsilon^{(k)}\} \subset \{\varepsilon^{(k-1)}\}$ such that $u_{\varepsilon^{(k)}} \xrightarrow{2-s} u_k^+$ for some $u_k^+ \in L^1(E_k \times Y)$ with

$$\int_{E_k \times Y} u_k^+(x,y)\, dydx \leq C. \tag{3.1.5}$$

The function u_k^+ can be extended to be a function in $L^1(U \times Y)$ by setting it to be zero on $(U \setminus E_k) \times Y$. Consider the diagonalizing subsequence $\{\hat{\varepsilon}\}$ where $\hat{\varepsilon}_j := \varepsilon_j^{(j)}$, the jth element of the subsequence $\{\varepsilon^{(j)}\}$. We prove that $u_k^+ \leq u_j^+$ if $k \leq j$. Let $\psi \in C_c^\infty\left[E_k; C_{\text{per}}^\infty(Y)\right]$ and define $\hat{\psi} \in C_c^\infty[E_j; C_{\text{per}}^\infty(Y)]$ by $\hat{\psi} = \psi$ in E_k and $\hat{\psi} = 0$ on $E_j \setminus E_k$. Then

$$\begin{aligned}
\int_{E_k} \int_Y u_j^+(x,y)\psi(x,y)\, dydx &= \int_{E_j} \int_Y u_j^+(x,y)\hat{\psi}(x,y)\, dydx \\
&= \lim_{\hat{\varepsilon} \to 0} \int_{E_j} u_{\hat{\varepsilon}}(x)\hat{\psi}\left(x, \frac{x}{\hat{\varepsilon}}\right)\, dx = \lim_{\hat{\varepsilon} \to 0} \int_{E_k} u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx \\
&= \int_{E_k} \int_Y u_k^+(x,y)\psi(x,y)\, dydx
\end{aligned}$$

and we conclude that $u_j^+ = u_k^+$ a.e. in $E_k \times Y$. The claim now follows by observing that $0 = u_k^+ \leq u_j^+$ on $U \setminus E_k$. Set $u^+ := \sup_k u_k^+ = \lim_{k\to\infty} u_k^+$. By (3.1.5),

$$\begin{aligned}
\int_{U \times Y} u^+(x,y)\, dydx &= \lim_{k \to \infty} \int_{U \times Y} u_k^+(x,y)\, dydx \\
&= \lim_{k \to \infty} \int_{E_k \times Y} u_k^+(x,y)\, dydx \leq C < +\infty.
\end{aligned}$$

Hence $u^+ \in L^1(U \times Y)$.

We claim that $\{u_{\hat{\varepsilon}}\}$ two-scale converges to u^+. Let $\psi \in C_c^\infty[U; C_{\text{per}}^\infty]$ be given. We have for all $k \in \mathbb{N}$

$$\begin{aligned}
&\left| \int_U u_{\hat{\varepsilon}}(x)\ \psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_U \int_Y u^+(x,y)\, \psi(x,y)\, dydx \right| \\
&\qquad \leq \left| \int_{E_k} u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_{E_k} \int_Y u_k^+(x,y)\, \psi(x,y)\, dydx \right| \\
&\qquad\quad + \left| \int_{E_k} \int_Y u^+(x,y) - u_k^+(x,y) dydx \right| \\
&\qquad\quad + \left| \int_{U \setminus E_k} u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_{U \setminus E_k} \int_Y u^+(x,y)\, \psi(x,y)\, dydx \right|.
\end{aligned} \tag{3.1.6}$$

For a fixed $k \in \mathbb{N}$, recall that $\{\hat{\varepsilon}_j\}_{j=k}^\infty \subset \{\varepsilon_j^{(k)}\}_{j=k}^\infty$ so $u_{\hat{\varepsilon}} \xrightarrow{2-s} u_k^+$, and thus

$$\lim_{\hat{\varepsilon} \to 0} \left| \int_{E_k} u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_{E_k} \int_Y u_k^+(x,y)\, \psi(x,y)\, dydx \right| = 0.$$

Also, recall that $u_k^+ = u^+$ in E_k, therefore

$$\left| \int_{E_k} \int_Y u^+(x,y) - u_k^+(x,y)\, dydx \right| = 0.$$

Moreover, by the Monotone Convergence Theorem and as $u_{\hat{\varepsilon}} \rightharpoonup u_j^+$ in $L^1(E_j)$,

$$\begin{aligned}\int_{U\setminus E_k} \int_Y u^+(x,y)\, dydx &= \lim_{j\to\infty} \int_{(U\setminus E_k)\cap E_j} \int_Y u_j^+(x,y)\, dydx \\ &= \lim_{j\to\infty} \lim_{\hat{\varepsilon}\to 0} \int_{(U\setminus E_k)\cap E_j} u_{\hat{\varepsilon}}(x)\, dx \le \sup_{\varepsilon} \int_{U\setminus E_k} u_\varepsilon(x)\, dx \le \frac{1}{k}.\end{aligned}$$

Also,

$$\begin{aligned}&\left| \int_{U\setminus E_k} u_{\hat{\varepsilon}}(x) \psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_{U\setminus E_k} \int_Y u^+(x,y)\, \psi(x,y)\, dydx \right| \\ &\qquad \le 2||\psi||_{L^\infty(U\times Y)} \sup_{\varepsilon>0} \int_{U\setminus E_k} u_\varepsilon(x)\, dx \le \frac{2||\psi||_{L^\infty(U\times Y)}}{k}.\end{aligned}$$

By first taking the limit $\hat{\varepsilon} \to \infty$ and then the limit $k \to \infty$ in (3.1.6) we obtain

$$\lim_{\hat{\varepsilon}\to\infty} \left| \int_U u_{\hat{\varepsilon}}(x) \psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_U \int_Y u^+(x,y)\, \psi(x,y)\, dydx \right| = 0,$$

and thus $\{u_{\hat{\varepsilon}}\}$ two-scale converges in L^1 to u^+.

Step 2: A similar argument as in the previous proof can be used for the case in which u_ε may also take negative values.

Lastly, we notice that should a weak limit, $\bar{u}_0$, of $\{u_\varepsilon\}$ exist, then for all $\varphi \in C_c^\infty(U)$

$$\int_U \varphi(x)\bar{u}_0(x)\, dx = \lim_{\varepsilon\to 0} \int_U \varphi(x) u_\varepsilon(x)\, dx = \int_U \varphi(x) \int_Y u_0(x,y)\, dydx. \tag{3.1.7}$$

From this we see that $\bar{u}_0(x) = \int_Y u_0(x,y)dy$ a.e. $x \in U$. From Theorem 2.3.2 we know that $\{u_\varepsilon\}$ is weakly sequentially precompact and from (3.1.7) it is easy to see that $\bar{u}_0(x) \rightharpoonup \int_Y u_0(x,y)dy$. $\square$

3.2 The Measure Approach

In [6] Amar defines two-scale convergence of measures. We will denote by $\mathcal{M}(U)$ the set of all signed Radon measures on U, $C_0(U;\mathbb{R}^N)$ is the space of all continuous functions that vanish on the boundary of U, and $C_0[U;C_{\text{per}}(Y)]$ is the space of all continuous functions $\varphi : U \to C_{\text{per}}(Y)$ that vanish on ∂U (see [6]).

Definition 3.2.1. *A sequence of measures $\{\mu_\varepsilon\} \subset \mathcal{M}(U)$ is said to* **two-scale converge** *to a measure $\mu_0 \in \mathcal{M}(U \times Y)$ if for any function $\varphi \in C_0[U; C_{\text{per}}(Y)]$ we have*

$$\lim_{\varepsilon \to 0^+} \int_U \varphi\left(x, \frac{x}{\varepsilon}\right) d\mu_\varepsilon(x) = \int_{U\times Y} \varphi(x,y)\, d\mu_0(x,y). \tag{3.2.1}$$

Using an argument similar to that of Allaire in [4], if U is an open bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary ∂U, then the following compactness result for measures is obtained in Theorem 3.5 in [6].

Theorem 3.2.2. *Every bounded sequence of measures $\{\mu_\varepsilon\}_\varepsilon \in \mathcal{M}(U)$, admits a subsequence $\{\mu_{\varepsilon_h}\}_h$ which two-scale converges to a measure $\mu_0 \in \mathcal{M}(U \times Y)$.*

We remark that the boundedness of U and the Lipschitz continuity of its boundary are not used in the proof of this result in Theorem 3.5 in [6]. Using this theorem we provide an alternate proof for the two-scale compactness of sequences bounded in $L^1(U)$. We will use the following lemma.

Lemma 3.2.3. *Let λ be a finite positive Radon measure on an open subset $U \subset \mathbb{R}^N$. Then λ is absolutely continuous with respect to $\mathcal{L}^N$ ($\lambda << \mathcal{L}^N$) if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\varphi \in C_c^\infty(U;[0,1])$ with $|\text{supp}(\varphi)| < \delta$*

$$\int_U \varphi(x)\, d\lambda(x) < \varepsilon.$$

Proof. First assume that $\lambda << \mathcal{L}^N$. Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda(A) < \varepsilon$ for all measurable sets A such that $|A| < \delta$. Let $\varphi \in C_c^\infty(U;[0,1])$ be such that $|\text{supp}(\varphi)| < \delta$, and let $A := \text{supp}(\varphi)$. Then

$$\int_U \varphi(x) d\lambda(x) \le \int_A d\lambda(x) = \lambda(A) < \varepsilon.$$

Now, assume the alternate condition. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that for all $\varphi \in C_c^\infty(U;[0,1])$ with $|\text{supp}(\varphi)| < \delta$

$$\int_U \varphi(x)\, d\lambda(x) < \frac{\varepsilon}{2}.$$

As λ is a Radon measure, every Borel set in U is outer regular and every open set in U is inner regular. Let $A \subseteq U$ be a Borel set such that $|A| < \frac{\delta}{2}$. We claim that $\lambda(A) \le \varepsilon$. By the outer regularity of λ, there exists an open set E, with $A \subseteq E$ and $|E| < \frac{2}{3}\delta$. We may use the inner regularity of λ to find a compact set $K \subset E$ such that

$$\lambda(E) \le \lambda(K) + \frac{\varepsilon}{2}. \tag{3.2.2}$$

As K is compact and E is open, there exists a function $\varphi \in C_c^\infty(U;[0,1])$ such that $\varphi = 1$ on K and $\varphi = 0$ outside E. Then

$$\lambda(K) \leq \int_U \varphi(x)\, d\lambda(x) \leq \frac{\varepsilon}{2}, \tag{3.2.3}$$

where in the second inequality we have used the assumption. Hence, by (3.2.2) and (3.2.3),

$$\lambda(A) \leq \lambda(E) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and this concludes the proof that λ is absolutely continuous with respect to $\mathcal{L}^N$. □

Second Proof of Theorem 1.1.1. ***Step 1:*** Assume that $u_\varepsilon \geq 0$ a.e. in U. By Theorem 3.2.2 there exists a finite Radon measure λ such that $u_\varepsilon \mathcal{L}^N \lfloor U \overset{2-s}{\longrightarrow} \lambda$. Note that, by (3.2.1), $\int_{U\times Y} \varphi(x,y) d\lambda(x,y) \geq 0$ for all $\varphi \in C_c^\infty[U; C_{\mathrm{per}}(Y)]$ such that $\varphi \geq 0$, so λ is a positive Radon measure.

We claim that λ is absolutely continuous with respect to $\mathcal{L}^{2N}\lfloor U \times Y$, and for this purpose we will use Lemma 3.2.3. Fix $\eta > 0$. By equi-integrability there exists $\delta > 0$ such that for all measurable $A \subset U$ such that $|A| < \delta$,

$$\sup_{\varepsilon>0} \int_A u_\varepsilon(x)\, dx < \eta. \tag{3.2.4}$$

Let $\varphi \in C_c^\infty(U \times Y; [0,1])$.

First let us assume that supp$(\varphi) \subseteq A \times B \subseteq U \times Y$ where $A := x_0 + (-\rho,\rho)^N$ for some $x_0 \in U$, $B := y_0 + (-\rho,\rho)^N$ for some $y_0 \in Y$, and $\rho > 0$. We can take, without loss of generality, $\rho > \varepsilon$. Note also that $|A| = 2^N \rho^N$. Extend φ periodically in the variable y with period Y so that φ is an admissible test function for two-scale convergence. Then, for any $\varepsilon > 0$

$$\text{supp}\left(\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right) \subseteq \left\{x \in U : x \in A \text{ and } \frac{x}{\varepsilon} \in k + B, k \in \mathbb{Z}^N\right\},$$

so

$$\left|\text{supp } \varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right| \leq \sum_{k\in\mathbb{Z}^N : A\cap\{\varepsilon k + \varepsilon B\}\neq\emptyset} |\varepsilon k + \varepsilon B| = \sum_{k\in\mathbb{Z}^N : A\cap\{\varepsilon k + \varepsilon B\}\neq\emptyset} \varepsilon^N |B|. \tag{3.2.5}$$

Now, if $k = (k_1, k_2, ..., k_N) \in \mathbb{Z}^N$ is such that $A \cap \{\varepsilon k + \varepsilon B\} \neq \emptyset$, then there exists $b \in B$, with $b = (b_1, ..., b_N)$ such that $\varepsilon k \in A - \varepsilon b$ or, equivalently, there exist $s_i \in (-\rho, \rho)$ (where $b_i = y_{0,i} + s_i$) such that

$$\varepsilon k_i \in (x_{0,i} - \rho - \varepsilon(y_{0,i} + s_i), x_{0,i} + \rho - \varepsilon(y_{0,i} + s_i)).$$

This is equivalent to

$$k_i \in X_{i,\varepsilon} + \left(\frac{-\rho}{\varepsilon} - s_i, \frac{\rho}{\varepsilon} - s_i\right) \qquad \text{with} \qquad X_{i,\varepsilon} := \frac{x_{0,i}}{\varepsilon} - y_{0,i}.$$

Therefore, the number of integer valued vectors in $\mathbb{Z}^N$ such that $A \cap \{\varepsilon k + \varepsilon B\} \neq \emptyset$ is at most the number of integer valued vectors in $X + \left(-\frac{\rho}{\varepsilon} - \rho, \frac{\rho}{\varepsilon} + \rho\right)^N$ with $X \in \mathbb{R}^N$, which is the number of $k \in \mathbb{Z}^N$ such that $k \in \left(-\frac{\rho}{\varepsilon} - \rho, \frac{\rho}{\varepsilon} + \rho\right)^N \subset \left(\frac{-2\rho}{\varepsilon}, \frac{2\rho}{\varepsilon}\right)^N$ (here we used the fact that $\rho > \varepsilon$), and this is at most $\left(\frac{4\rho}{\varepsilon}\right)^N$. In view of (3.2.5), we deduce that

$$\left|\text{supp}\left(\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right)\right| \leq 4^N \frac{\rho^N}{\varepsilon^N} \varepsilon^N |B| = 2^N |A||B|. \tag{3.2.6}$$

Now let φ be a function in $C_c^\infty(U \times Y; [0,1])$ such that for $K := \text{supp}(\varphi)$, $|K| < \frac{\delta}{2^{N+1}}$. By the construction of the Lebesgue measure and the compactness of K, there exists a finite cover of K by sets of the type $A_i \times B_i := (x_i + (-\rho_i, \rho_i)^N) \times (y_i + (-\rho_i, \rho_i)^N)$, $\rho_i > 0$, $x_i \in U$, $y_i \in Y$, such that $E \subseteq \bigcup_{i=1}^m A_i \times B_i \subset U \times Y$ and

$$\sum_{i=1}^m |A_i \times B_i| = \sum_{i=1}^m |A_i||B_i| < \frac{\delta}{2^N}.$$

Set $\rho_0 := \min\{\rho_i : i = 1, 2, ..., m\}$. Then, for $\varepsilon < \rho_0$ and by (3.2.6),

$$\begin{aligned}\left|\text{supp}\left(\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right)\right| &\leq \sum_{i=1}^m \left|\{z \in U : z \in A_i \text{ and } z \in \varepsilon k + \varepsilon B_i, k \in \mathbb{Z}^N\}\right| \\ &\leq 2^N \sum_{i=1}^m |A_i||B_i| < \delta, \end{aligned} \tag{3.2.7}$$

and so, in view of (3.2.4) and (3.2.7), we have

$$\begin{aligned}\int_{U \times Y} \varphi(x,y)\, d\lambda(x,y) &= \lim_{\varepsilon \to 0} \int_U u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx \\ &\leq \limsup_{\varepsilon \to 0} \int_{\text{supp}\left(\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right)} u_\varepsilon(x)\, dx < \eta.\end{aligned}$$

By Lemma 3.2.3 we deduce $\lambda << \mathcal{L}^{2N} \lfloor U \times Y$, and so by the Radon-Nikodym Theorem there exists a function $u_0 \in L^1(U \times Y)$ such that $\lambda = u_0 \mathcal{L}^{2N} \lfloor U \times Y$. Thus, for all $\varphi \in C_c^\infty[U; C_{\text{per}}(Y)]$

$$\lim_{\varepsilon \to 0} \int_U \varphi\left(x, \frac{x}{\varepsilon}\right) u_\varepsilon(x) dx = \int_{U \times Y} \varphi(x,y) u_0(x,y) dy dx,$$

and $\{u_\varepsilon\}$ two scale converges in L^1 to u_0.

Step 2: The proof for the general case in which u_ε are allowed to take both positive and negative values can be completed in the same manner as Step 2 of the proof of Proposition 3.1.1. From here, it is also easy to see that $\bar{u}_0(x) = \int_Y u_0(x,y) dy$. $\square$

3.3 Periodic Unfolding Approach

Recall that $\{u_\varepsilon\}$ is a family of functions in $L^1(U)$. For the following we will extend u_ε by zero outside of U for convenience of notation. An alternate approach to the study of two-scale convergence, the periodic unfolding introduced in [28], involves defining a family of scale transformations $S_\varepsilon : \mathbb{R}^N \times [0,1)^N \to \mathbb{R}^N$ which, for $\varepsilon > 0$, are defined by

$$S_\varepsilon(x,y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \quad \text{for } (x,y) \in \mathbb{R}^N \times [0,1)^N, \tag{3.3.1}$$

where

$$\mathcal{N}(x) := (\hat{n}(x_1), ..., \hat{n}(x_N)) \quad \text{for } x = (x_1, ..., x_N) \in \mathbb{R}^N,$$

and

$$\hat{n}(s) := \max\{n \in \mathbb{Z} : n \leq s\} \quad \text{for } s \in \mathbb{R}.$$

Furthermore, let

$$\hat{r}(s) := s - \hat{n}(s) \in [0,1) \quad \text{for } s \in \mathbb{R}$$

and

$$\mathcal{R}(x) := x - \mathcal{N}(x) \in [0,1)^N \quad \text{for } x \in \mathbb{R}^N.$$

Using these scale transformations it is possible to define obtain a characterization of two-scale convergence as follows (see [84] Proposition 2.5 and (1.9)).

Proposition 3.3.1. *Let $\{u_\varepsilon\} \subset L^1(U)$. Then*

$$u_\varepsilon \xrightarrow{2-s} u_0 \qquad \textit{if and only if} \quad u_\varepsilon \circ S_\varepsilon \rightharpoonup u_0 \ \ \textit{in } L^1(\mathbb{R}^N \times [0,1)^N). \tag{3.3.2}$$

Additionally, we will use the following result in [84]:

Lemma 3.3.2. *Let f be a function in $L^1(\mathbb{R}^N)$. Then for any $\varepsilon > 0$*

$$\int_{\mathbb{R}^N} f(x)\,dx = \int_{\mathbb{R}^N \times [0,1)^N} f(S_\varepsilon(x,y))\,dxdy. \tag{3.3.3}$$

For our last proof of Theorem 1.1.1, we present a modified version of the proof of two-scale compactness by Visintin in [84], Proposition 3.2 (iii).

Third Proof of Theorem 1.1.1. From the periodic unfolding characterization (3.3.2) of two-scale limits, it is sufficient to prove that if $\{u_\varepsilon\}$ is weakly sequentially precompact in $L^1(\mathbb{R}^N)$ then so is $\{u_\varepsilon \circ S_\varepsilon\}$ in $L^1(\mathbb{R}^N \times [0,1)^N)$. In turn, the latter condition is equivalent to $\{u_\varepsilon \circ S_\varepsilon\}$ being weakly sequentially precompact in $L^1(\mathbb{R}^N \times Y)$, therefore, in view of Theorem 2.3.2 it is sufficient to check that conditions (i), (ii) and (iii) hold for the sequence $\{u_\varepsilon \circ S_\varepsilon\}$. Additionally, we may assume, without loss of generality, that $0 < \varepsilon < 1$.

(i) From (3.3.3) it is easily seen that if $\{u_\varepsilon\}$ is bounded in $L^1(U)$ then so is $\{u_\varepsilon \circ S_\varepsilon\}$ in $L^1(\mathbb{R}^N \times Y)$.

(ii) By the de la Vallée-Poussin criterion, $\{u_\varepsilon\}$ is equi-integrable if and only if there exists a Borel function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \qquad \text{and} \qquad \sup_\varepsilon \int_{\mathbb{R}^N} \varphi(|u_\varepsilon(x)|)\,dx < +\infty. \tag{3.3.4}$$

By (3.3.3),

$$\int_{\mathbb{R}^N} \varphi(|u_\varepsilon(x)|)\,dx = \int_{\mathbb{R}^N\times Y} \varphi(|u_\varepsilon(S_\varepsilon(x,y))|)\,dydx.$$

Therefore, the criterion (3.3.4) holds for $\{u_\varepsilon\}$ in $\mathbb{R}^N$ if and only if it holds for $\{u_\varepsilon \circ S_\varepsilon\}$ in $\mathbb{R}^n \times Y$.

(iii) Last we show that $\{u_\varepsilon \circ S_\varepsilon\}$ also inherits property (iii) from $\{u_\varepsilon\}$, i.e., we claim that for all $\eta > 0$ there exists a set $E \subset \mathbb{R}^N \times Y$ such that $|E| < +\infty$

$$\sup_\varepsilon \int_{(\mathbb{R}^N\times Y)\setminus E} |u_\varepsilon(S_\varepsilon(x,y))|\,dydx < \eta.$$

Fix $\eta > 0$ and in view of Remark 2.3.3 let E_0 be open, bounded and such that

$$\sup_\varepsilon \int_{\mathbb{R}^N\setminus E_0} |u_\varepsilon(x)|\,dx < \eta. \tag{3.3.5}$$

Let $E := (E_0 + [-1,1]^N) \times Y$. Clearly $|E| < +\infty$, and we show that if $(x,y) \in (\mathbb{R}^N \times Y)\setminus E$ then $S_\varepsilon(x,y) \in \mathbb{R}^N \setminus E_0$. Indeed, $(\mathbb{R}^N\times Y)\setminus E = (\mathbb{R}^N \setminus (E_0 + [-1,1]^N)) \times Y$, thus the claim reduces to proving that if $x \in \mathbb{R}^N \setminus (E_0 + [-1,1]^N)$ then $S_\varepsilon(x,y) \in \mathbb{R}^N \setminus E_0$, or, equivalently,

$$\text{if} \qquad S_\varepsilon(x,y) \in E_0 \qquad \text{then} \qquad x \in E_0 + [-1,1]^N. \tag{3.3.6}$$

From the definition of S_ε we know that

$$x = S_\varepsilon(x,y) - \varepsilon\left[y - \mathcal{R}\left(\frac{x}{\varepsilon}\right)\right],$$

and $\varepsilon[y - \mathcal{R}(\frac{x}{\varepsilon})] \in [-1,1]^N$, and this asserts (3.3.6). We conclude that

$$\begin{aligned}
\int_{(\mathbb{R}^N\times Y)\setminus E} |u_\varepsilon(S_\varepsilon(x,y))|\,dydx &= \int_{(\mathbb{R}^N\setminus(E_0+[-1,1]^N))\times Y} |u_\varepsilon(S_\varepsilon(x,y))|\,dydx \\
&\le \int_{\mathbb{R}^N\times Y} \chi_{\mathbb{R}^N\setminus E_0}(S_\varepsilon(x,y))\,|u_\varepsilon(S_\varepsilon(x,y))|\;dydx \\
&= \int_{\mathbb{R}^N\setminus E_0} |u_\varepsilon(x)|\,dx < \eta
\end{aligned}$$

where in the last equality we have used (3.3.3) and (3.3.5).

We have shown that $\{u_\varepsilon \circ S_\varepsilon\}$ is relatively weakly sequentially compact in $L^1(\mathbb{R}^N)$, therefore it admits a subsequence that converges weakly in $L^1(\mathbb{R}^N)$ which, by (3.3.2), is equivalent to $\{u_\varepsilon\}$ admitting, up to a subsequence, a two-scale limit. $\square$

Chapter 4

Multiscale Homogenization in Kirchhoff's Nonlinear Plate Theory

4.1 Setting of the Problem

Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain whose boundary is piecewise C^1. This regularity assumption is only needed in Section 4.4, while the results in Sections 2.4–4.3 continue to hold for every bounded Lipschitz domain $\omega \subset \mathbb{R}^2$. We assume that the set

$$\Omega_h := \omega \times (-\tfrac{h}{2}, \tfrac{h}{2})$$

is the reference configuration of a nonlinearly elastic thin plate. In the sequel, $\{h\}$ and $\{\varepsilon(h)\}$ are monotone decreasing sequences of positive numbers, $h \to 0$, $\varepsilon(h) \to 0$ as $h \to 0$, such that the following limits exist

$$\gamma_1 := \lim_{h \to 0} \frac{h}{\varepsilon(h)} \quad \text{and} \quad \gamma_2 := \lim_{h \to 0} \frac{h}{\varepsilon^2(h)},$$

with $\gamma_1, \gamma_2 \in [0, +\infty]$. There are five possible regimes: $\gamma_1, \gamma_2 = +\infty$; $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$; $\gamma_1 = 0$ and $\gamma_2 = +\infty$; $\gamma_1 = 0$ and $0 < \gamma_2 < +\infty$; $\gamma_1 = 0$ and $\gamma_2 = 0$. We focus here on the first three regimes, that is, on the cases in which $\gamma_2 = +\infty$.

For every deformation $v \in W^{1,2}(\Omega_h; \mathbb{R}^3)$, we consider its rescaled elastic energy

$$\mathcal{J}^h(v) := \frac{1}{h} \int_{\Omega_h} W\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla v(x)\Big)\, dx,$$

where $W : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0, +\infty)$ represents the stored energy density of the plate, and $(y, z, F) \mapsto W(y, z, F)$ is measurable and Q-periodic in its first two variables, i.e., with respect to y and z. We also assume that for a.e. y and z, the map $W(y, z, \cdot)$ is continuous and satisfies the following assumptions:

(H1) $W(y, z, RF) = W(y, z, F)$ for every $F \in \mathbb{M}^{3\times 3}$ and for all $R \in SO(3)$ (frame indifference), (H2) $W(y,z,F) \geq C_1 \operatorname{dist}^2(F;SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ and for some $C_1 > 0$ (nondegeneracy),

(H3) there exists $\delta > 0$ such that $W(y,z,F) \leq C_2 \operatorname{dist}^2(F;SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$ with $\operatorname{dist}(F;SO(3)) < \delta$ and some $C_2 > 0$,

(H4) $\lim_{|G|\to 0} \frac{W(y,z,Id+G)-\mathscr{Q}(y,z,G)}{|G|^2} = 0$, where $\mathscr{Q}(y,z,\cdot)$ is a quadratic form on $\mathbb{M}^{3\times 3}$.

By assumptions (H1)–(H4) we obtain the following lemma, which guarantees the continuity of the quadratic map $\mathscr{Q}$ introduced in (H4).

Lemma 4.1.1. *Let $W : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,+\infty)$ satisfy (H1)–(H4) and let $\mathscr{Q} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,+\infty)$ be defined as in (H4). Then,*

(i) $\mathscr{Q}(y,z,\cdot)$ is continuous for a.e. $y,z \in \mathbb{R}^2$,

(ii) $\mathscr{Q}(\cdot,\cdot,F)$ is $Q\times Q$-periodic and measurable for every $F \in \mathbb{M}^{3\times 3}$,

(iii) for a.e. $y,z \in \mathbb{R}^2$, the map $\mathscr{Q}(y,z,\cdot)$ is quadratic on $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, and satisfies

$$\frac{1}{C}|\mathrm{symF}|^2 \leq \mathscr{Q}(y,z,F) = \mathscr{Q}(y,z,\mathrm{symF}) \leq C|\mathrm{symF}|^2$$

for all $F \in \mathbb{M}^{3\times 3}$, and some $C > 0$. In addition, there exists a monotone function

$$r : [0,+\infty) \to [0,+\infty],$$

such that $r(\delta) \to 0$ as $\delta \to 0$, and

$$|W(y,z,Id+F) - \mathscr{Q}(y,z,F)| \leq |F|^2 r(|F|)$$

for all $F \in \mathbb{M}^{3\times 3}$, for a.e. $y,z \in \mathbb{R}^2$.

We refer to [72, Lemma 2.7] and to [73, Lemma 4.1] for a proof of Lemma 4.1.1 in the case in which $\mathscr{Q}$ is independent of z. The proof in the our setting is a straightforward adaptation.

As it is usual in dimension reduction analysis, we perform a change of variables in order to reformulate the problem on a domain independent of the varying thickness parameter. We set

$$\Omega := \Omega_1 = \omega \times (-\tfrac{1}{2}, \tfrac{1}{2})$$

and we consider the change of variables $\psi^h : \Omega \to \Omega^h$, defined as

$$\psi^h(x) = (x', hx_3) \quad \text{for every } x \in \Omega.$$

To every deformation $v \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ we associate a function $u \in W^{1,2}(\Omega;\mathbb{R}^3)$, defined as $u := v \circ \psi^h$, whose elastic energy is given by

$$\mathcal{E}^h(u) = \mathcal{J}^h(v) = \int_\Omega W\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla_h u(x)\Big)\,dx,$$

where

$$\nabla_h u(x) := \Big(\nabla' u(x)\Big|\frac{\partial_{x_3} u(x)}{h}\Big) \quad \text{for a.e. } x \in \Omega.$$

In this chapter we focus on the asymptotic behavior of sequences of deformations $\{u^h\} \subset W^{1,2}(\Omega_h;\mathbb{R}^3)$ satisfying the uniform energy estimate

$$\mathcal{E}^h(u^h) := \int_\Omega W\Big(\frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}, \nabla_h u^h(x)\Big)\, dx \leq Ch^2 \quad \text{for every } h > 0. \tag{4.1.1}$$

We remark that in the case in which W is independent of y and z, such scalings of the energy lead to Kirchhoff's nonlinear plate theory, which was rigorously justified by means of $\Gamma-$convergence techniques in the seminal paper [45].

4.2 Identification of the Limit Stresses

Due to the linearized behavior of the nonlinear elastic energy around the set of proper rotations, a key point in the proof of the liminf inequality (1.2.3) is to establish a characterization of the weak limit, in the sense of 3-scale-dr convergence, of the sequence of linearized elastic stresses

$$E^h := \frac{\sqrt{(\nabla_h u^h)^T \nabla_h u^h} - Id}{h}.$$

We introduce the following classes of functions:

$$\begin{aligned}\mathcal{C}_{\gamma_1,+\infty} := \Big\{&U \in L^2(\Omega \times Q \times Q;\mathbb{M}^{3\times 3}) : \\ &\text{there exist } \phi_1 \in L^2\big(\omega; W^{1,2}\big(\big(-\tfrac{1}{2},\tfrac{1}{2}\big); W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\big)\big) \\ &\text{and } \phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\rm per}(Q;\mathbb{R}^3)) \\ &\text{such that } U = \mathrm{sym}\Big(\nabla_y \phi_1\Big|\frac{\partial_{x_3}\phi_1}{\gamma_1}\Big) + \mathrm{sym}(\nabla_z \phi_2|0)\Big\},\end{aligned} \tag{4.2.1}$$

$$\begin{aligned}\mathcal{C}_{+\infty,+\infty} := \Big\{&U \in L^2(\Omega \times Q \times Q;\mathbb{M}^{3\times 3}) : \\ &\text{there exist } d \in L^2(\Omega;\mathbb{R}^3),\ \phi_1 \in L^2\big(\Omega; W^{1,2}_{\rm per}(Q;\mathbb{R}^3)\big) \\ &\text{and } \phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\rm per}(Q;\mathbb{R}^3)) \\ &\text{such that } U = \mathrm{sym}(\nabla_y \phi_1|d) + \mathrm{sym}(\nabla_z \phi_2|0)\Big\},\end{aligned} \tag{4.2.2}$$

and

$$\begin{aligned}\mathcal{C}_{0,+\infty} := \Big\{&U \in L^2(\Omega \times Q \times Q;\mathbb{M}^{3\times 3}) : \\ &\text{there exist } \xi \in L^2(\Omega; W^{1,2}_{\rm per}(Q;\mathbb{R}^2)),\ \eta \in L^2(\omega; W^{2,2}_{\rm per}(Q)),\end{aligned} \tag{4.2.3}$$

$$g_i \in L^2(\Omega \times Y),\ i = 1,2,3, \text{ and } \phi \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3)) \text{ such that}$$
$$U = \operatorname{sym}\begin{pmatrix} \nabla_y \xi + x_3 \nabla_y^2 \eta & g_1 \\ & g_2 \\ g_1 \quad g_2 & g_3 \end{pmatrix} + \operatorname{sym}(\nabla_z \phi | 0) \Bigg\}.$$

We now state the main result of this section.

Theorem 4.2.1. *Let* $\gamma_1 \in [0, +\infty]$ *and* $\gamma_2 = +\infty$. *Let* $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *be a sequence of deformations satisfying* (2.3.2) *and converging to a deformation* u *in the sense of Theorem 2.3.4. Then there exist* $E \in L^2(\Omega \times Q \times Q; \mathbb{M}^{3\times 3}_{\text{sym}})$, $B \in L^2(\omega; \mathbb{M}^{2\times 2})$, *and* $U \in C_{\gamma_1, +\infty}$, *such that, up to the extraction of a (not relabeled) subsequence,*

$$E^h \overset{dr-3-s}{\rightharpoonup} E \quad \textit{weakly dr-3-scale},$$

where

$$E(x,y,z) = \begin{pmatrix} x_3 \Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0 \end{pmatrix} + U(x,y,z),$$

for almost every $(x,y,z) \in \Omega \times Q \times Q$, *with*

$$\Pi^u_{\alpha,\beta}(x') := -\partial^2_{\alpha,\beta} u(x') \cdot n_u(x') \quad \textit{for } \alpha, \beta = 1,2, \tag{4.2.4}$$

and

$$n_u(x') := \partial_1 u(x') \wedge \partial_2 u(x') \tag{4.2.5}$$

for every $x' \in \omega$.

A crucial point in the proof is to approximate the scaled gradients of deformations with uniformly small energies, by sequences of maps which are either piecewise constant on cubes of size comparable to the homogenization parameters with values in the set of proper rotations, or have Sobolev regularity and are close in the L^2-norm to piecewise constant rotations. The following lemma has been stated in [83, Lemma 3.3], and its proof follows by combining [46, Theorem 6] with the argument in [45, Proof of Theorem 4.1, and Section 3]. We remark that the additional regularity of the limit deformation u in Theorem 2.3.4 is a consequence of Lemma 4.2.2, and in particular of the approximation of scaled gradients by $W^{1,2}$ maps.

Lemma 4.2.2. *Let* $\gamma_0 \in (0,1]$ *and let* $h, \delta > 0$ *be such that*

$$\gamma_0 \leq \frac{h}{\delta} \leq \frac{1}{\gamma_0}.$$

There exists a constant C, *depending only on* ω *and* γ_0, *such that for every* $u \in W^{1,2}(\omega;\mathbb{R}^3)$ *there exists a map* $R : \omega \to SO(3)$ *piecewise constant on each cube* $x + \delta Y$, *with* $x \in \delta\mathbb{Z}^2$, *and there exists* $\tilde{R} \in W^{1,2}(\omega; \mathbb{M}^{3\times 3})$ *such that*

$$\|\nabla_h u - R\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} + \|R - \tilde{R}\|^2_{L^2(\omega;\mathbb{M}^{3\times 3})} + h^2 \|\nabla' \tilde{R}\|^2_{L^2(\omega;\mathbb{M}^{3\times 3}\times\mathbb{M}^{3\times 3})}$$

$$\leq C\|\textit{dist}(\nabla_h u; SO(3))\|^2_{L^2(\Omega)}.$$

Moreover, for every $\xi \in \mathbb{R}^2$ *satisfying*

$$|\xi|_\infty := \max\{|\xi \cdot e_1|, |\xi \cdot e_2|\} < h,$$

and for every $\omega' \subset \omega$, *with dist*$(\omega', \partial\omega) > Ch$, *there holds*

$$\|R(\cdot) - R(\cdot + \xi)\|_{L^2(\omega';\mathbb{M}^{3\times 3})} \leq C\|\textit{dist}(\nabla_h u; SO(3))\|_{L^2(\Omega)}.$$

Proof of Theorem 4.2.1. Let $\{u^h\}$ be as in the statement of the theorem. By Theorem 2.3.4 the map $u \in W^{2,2}(\omega;\mathbb{R}^3)$ is an isometry, and

$$\nabla_h u^h \to (\nabla' u|n_u) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}), \tag{4.2.6}$$

where n_u is defined in (4.2.5).

For simplicity, we subdivide the proof into three cases, corresponding to the three regimes $0 < \gamma_1 < +\infty$, $\gamma_1 = +\infty$, and $\gamma_1 = 0$, and each case will be treated in multiple steps.

Case 1: $0 < \gamma_1 < +\infty$ **and** $\gamma_2 = +\infty$.
Applying Lemma 4.2.2 with $\delta(h) = \varepsilon(h)$, we construct two sequences $\{R^h\} \subset L^\infty(\omega; SO(3))$ and $\{\tilde{R}^h\} \subset W^{1,2}(\omega;\mathbb{M}^{3\times 3})$ such that R^h is piecewise constant on every cube of the form $Q(\varepsilon(h)z, \varepsilon(h))$, with $z \in \mathbb{Z}^2$, and

$$\begin{aligned}&\|\nabla_h u^h - R^h\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} + \|R^h - \tilde{R}^h\|^2_{L^2(\omega;\mathbb{M}^{3\times 3})} \\ &\quad + h^2\|\nabla' \tilde{R}^h\|^2_{L^2(\omega;\mathbb{M}^{3\times 3}\times\mathbb{M}^{3\times 3})} \leq C\|\mathrm{dist}(\nabla_h u^h; SO(3))\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.2.7}$$

By (2.3.2) and (4.2.7), there holds

$$\nabla_h u^h - R^h \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}),$$

$$R^h - \tilde{R}^h \to 0 \quad \text{strongly in } L^2(\omega;\mathbb{M}^{3\times 3}),$$

and $\{\tilde{R}^h\}$ is bounded in $W^{1,2}(\omega;\mathbb{M}^{3\times 3})$. Therefore, by (4.2.6) and the uniform boundedness of the sequence $\{R^h\}$ in $L^\infty(\omega;\mathbb{M}^{3\times 3})$, and in particular in $L^2(\omega;\mathbb{M}^{3\times 3})$,

$$R^h \to R \quad \text{strongly in } L^2(\omega;\mathbb{M}^{3\times 3}), \qquad R^h \rightharpoonup^* R \quad \text{weakly* in } L^\infty(\omega;\mathbb{M}^{3\times 3}), \tag{4.2.8}$$

and

$$\tilde{R}^h \rightharpoonup R \quad \text{weakly in } W^{1,2}(\omega;\mathbb{M}^{3\times 3}), \tag{4.2.9}$$

where

$$R(x') := (\nabla' u(x')|n_u(x')). \tag{4.2.10}$$

In order to identify the multiscale limit of the linearized stresses, we argue as in [57, Proof of Proposition 3.2], and we introduce the scaled linearized strains

$$G^h := \frac{(R^h)^T \nabla_h u^h - Id}{h}. \tag{4.2.11}$$

By (2.3.2) and (4.2.7) the sequence $\{G^h\}$ is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$. By standard properties of 3-scale convergence (see [5, Theorem 2.4]) there exists $G \in L^2(\Omega\times Q\times Q;\mathbb{M}^{3\times 3})$ such that, up to the extraction of a (not relabeled) subsequence,

$$G^h \xrightarrow{3-s} G \quad \text{weakly 3-scale.} \tag{4.2.12}$$

By the identity

$$\sqrt{(Id+hF)^T(Id+hF)} = Id + h\,\mathrm{sym}\,F + O(h^2),$$

and observing that

$$E^h = \frac{\sqrt{(\nabla_h u^h)^T\nabla_h u^h} - Id}{h} = \frac{\sqrt{(Id+hG^h)^T(Id+hG^h)} - Id}{h},$$

there holds

$$E = \mathrm{sym}\,\mathrm{G}. \tag{4.2.13}$$

By (4.2.12), it follows that

$$G^h \xrightarrow{2-s} \int_Q G(x,y,z)\,dz \quad \text{weakly 2-scale.}$$

Therefore, by [57, Proposition 3.2] there exist $B \in L^2(\omega;\mathbb{M}^{2\times 2})$ and $\phi_1 \in L^2\big(\omega; W^{1,2}\big(\big(-\frac{1}{2},\frac{1}{2}\big); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\big)\big)$ such that

$$\begin{aligned}
&\mathrm{sym}\int_Q G(x,y,\xi)\,d\xi \\
&\quad = \begin{pmatrix} x_3\Pi^u(x') + \mathrm{sym}\,B(x') & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{sym}\left(\nabla_y\phi_1(x,y)\Big|\frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\right)
\end{aligned} \tag{4.2.14}$$

for a.e. $x \in \Omega$ and $y \in Y$. Thus, by (4.2.13) and (4.2.14) to complete the proof we only need to prove that

$$\mathrm{sym}\,G(x,y,z) - \mathrm{sym}\int_Q G(x,y,\xi)\,d\xi = \mathrm{sym}\,(\nabla_z\phi_2(x,y,z)|0) \tag{4.2.15}$$

for some $\phi_2 \in L^2(\Omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$.

Set

$$\bar{u}^h(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} u^h(x',x_3)\,dx_3 \quad \text{for a.e. } x' \in \omega \tag{4.2.16}$$

and define $r^h \in W^{1,2}(\Omega;\mathbb{R}^3)$ as

$$u^h(x) =: \bar{u}^h(x') + hx_3\tilde{R}^h(x')e_3 + hr^h(x',x_3) \quad \text{for a.e. } x \in \Omega. \tag{4.2.17}$$

We remark that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} r^h(x', x_3)\, dx_3 = 0, \tag{4.2.18}$$

and

$$\frac{\nabla_h u^h - R^h}{h} = \Big(\frac{\nabla' \bar{u}^h - (R^h)'}{h} + x_3 \nabla' \tilde{R}^h e_3 \Big| \frac{(\tilde{R}^h - R^h)}{h} e_3 \Big) + \nabla_h r^h. \tag{4.2.19}$$

We first notice that by (2.3.2), (4.2.7), (4.2.9), and (4.2.18), the sequence $\{r^h\}$ is uniformly bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$. Hence, by Theorem 2.4.6 (ii) there exist $r \in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1 \in L^2\big(\omega; W^{1,2}\big(\big(-\frac{1}{2},\frac{1}{2}\big); W^{1,2}_{\text{per}}(Q;\mathbb{R}^3)\big)\big)$ and $\hat{\phi}_2 \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$ such that, up to the extraction of a (not relabeled) subsequence,

$$\nabla_h r^h \xrightarrow{dr-3-s} \Big(\nabla' r + \nabla_y \hat{\phi}_1 + \nabla_z \hat{\phi}_2 \Big| \frac{\partial_{x_3} \hat{\phi}_1}{\gamma_1} \Big) \quad \text{weakly dr-3-scale.} \tag{4.2.20}$$

By (2.3.2) and (4.2.7), and since R^h does not depend on x_3, $\big\{\frac{\nabla_h \bar{u}^h - (R^h)'}{h}\big\}$ is bounded in $L^2(\omega;\mathbb{M}^{3\times 2})$. Therefore by [5, Theorem 2.4] there exists $V \in L^2(\omega \times Q \times Q; \mathbb{M}^{3\times 2})$ such that, up to the extraction of a (not relabeled) subsequence,

$$\frac{\nabla' \bar{u}^h - (R^h)'}{h} \xrightarrow{3-s} V \quad \text{weakly 3-scale.} \tag{4.2.21}$$

Case 1, Step 1: Characterization of V.
In view of (4.2.15), we provide a characterization of

$$V(x', y, z) - \int_Q V(x', y, \xi)\, d\xi.$$

We claim that there exists $v \in L^2(\omega \times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$ such that

$$V(x', y, z) - \int_Q V(x', y, \xi)\, d\xi = \nabla_z v(x', y, z) \quad \text{for a.e. } x' \in \omega, \text{ and } y, z \in Q. \tag{4.2.22}$$

Arguing as in [57, Proof of Proposition 3.2], we first notice that by [5, Lemma 3.7] to prove (4.2.22) it is enough to show that

$$\int_\omega \int_Q \int_Q \Big(V(x', y, z) - \int_Q V(x', y, \xi)\, d\xi \Big) : (\nabla')^\perp \varphi(z) \psi(x', y)\, dz\, dy\, dx' = 0 \tag{4.2.23}$$

for every $\varphi \in C^1_{\text{per}}(Q;\mathbb{R}^3)$ and $\psi \in C^\infty_c(\omega; C^\infty_{\text{per}}(Q))$. Fix $\varphi \in C^1_{\text{per}}(Q;\mathbb{R}^3)$ and $\psi \in C^\infty_c(\omega; C^\infty_{\text{per}}(Q))$. We set

$$\tilde{\varphi}^\varepsilon(x') := \varepsilon^2(h) \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \quad \text{for every } x' \in \omega.$$

Then,

$$
\begin{aligned}
&\int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\, dx' \qquad (4.2.24)\\
&\quad = \int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \tilde{\varphi}^\varepsilon(x')\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\, dx'\\
&\quad = \int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \Big[\tilde{\varphi}^\varepsilon(x')\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big]\, dx'\\
&\qquad - \int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : \Big[\tilde{\varphi}^\varepsilon(x') \otimes \Big((\nabla')^\perp_x \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big) + \frac{1}{\varepsilon(h)}(\nabla')^\perp_y \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big)\Big]\, dx'.
\end{aligned}
$$

The first term in the right-hand side of (4.2.24) is equal to zero, due to the definition of $(\nabla')^\perp$. Therefore we obtain

$$
\begin{aligned}
&\int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\, dx' \qquad (4.2.25)\\
&\quad = -\frac{\varepsilon^2(h)}{h}\int_\omega \nabla' \bar{u}^h(x') : \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes (\nabla')^\perp_x \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big]\\
&\qquad - \frac{\varepsilon(h)}{h}\int_\omega \nabla' \bar{u}^h(x') : \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes (\nabla')^\perp_y \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big].
\end{aligned}
$$

By (4.2.7), the regularity of the test functions, and since $\gamma_2 = +\infty$, we get

$$
\frac{\varepsilon^2(h)}{h}\int_\omega \nabla' \bar{u}^h(x') : \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes (\nabla')^\perp_x \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big]\, dx' \to 0, \qquad (4.2.26)
$$

while by (4.2.6), (4.2.10), and the regularity of the test functions,

$$
\begin{aligned}
&\lim_{h\to 0} \frac{\varepsilon(h)}{h}\int_\omega \nabla' \bar{u}^h(x') : \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes (\nabla')^\perp_y \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big]\, dx' \qquad (4.2.27)\\
&\quad = \frac{1}{\gamma_1}\int_\omega\int_Q\int_Q R'(x') : (\varphi(z) \otimes (\nabla')^\perp_y \psi(x', y))\, dz\, dy\, dx' = 0,
\end{aligned}
$$

where the latter equality is due to the periodicity of ψ with respect to the y variable. Combining (4.2.24), (4.2.25), (4.2.26) and (4.2.27), we conclude that

$$
\lim_{h\to 0}\int_\omega \frac{\nabla' \bar{u}^h(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\, dx' = 0. \qquad (4.2.28)
$$

In view of (4.2.21), and since

$$
\int_\omega\int_Q\int_Q \Big(\int_Q V(x', y, \xi)\, d\xi\Big) : (\nabla')^\perp \varphi(z)\psi(x', y)\, dz\, dy\, dx' = 0
$$

by the periodicity of φ, (4.2.23) will be established once we show that

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx' = 0. \tag{4.2.29}$$

In order to prove (4.2.29), we adapt [57, Lemma 3.8] to our framework.

Since $\psi \in C_c^\infty(\omega; C^\infty_{\mathrm{per}}(Q))$ and $h \to 0$, we can assume, without loss of generality, that for h small enough

$$\text{dist}(\operatorname{supp}\psi; \partial\omega \times Q) > \Big(1 + \frac{3}{\gamma_1}\Big)h.$$

We define

$$\mathbb{Z}^\varepsilon := \Big\{z \in \mathbb{Z}^2 :\ Q(\varepsilon(h)z, \varepsilon(h)) \times Q \cap \operatorname{supp}\psi \neq \emptyset\Big\},$$

and

$$Q_\varepsilon := \bigcup_{z\in\mathbb{Z}^\varepsilon} Q(\varepsilon(h)z, \varepsilon(h)).$$

Since $0 < \gamma_1 < +\infty$, for h small enough we have $\sqrt{2}\varepsilon(h) < \frac{2h}{\gamma_1}$, so that

$$\text{dist}(Q_\varepsilon; \partial\omega) \geq \Big(1 + \frac{3}{\gamma_1}\Big)h - \sqrt{2}\varepsilon(h) \geq \Big(1 + \frac{1}{\gamma_1}\Big)h.$$

We subdivide

$$\mathcal{Q}_{\varepsilon^2} := \Big\{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) :\ \lambda \in \mathbb{Z}^2 \text{ and } Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \cap Q_\varepsilon \neq \emptyset\Big\}$$

into two subsets:

(a) "good cubes of size $\varepsilon^2(h)$", i.e., those which are entirely contained in a cube of size $\varepsilon(h)$ belonging to Q_ε, and where $(R^h)'$ is hence constant,

(b) "bad cubes of size $\varepsilon^2(h)$", i.e., those intersecting more than one element of Q_ε.

We observe that, as $\gamma_2 = +\infty$,

$$\text{dist}(\mathcal{Q}_{\varepsilon^2}; \partial\omega) \geq \text{dist}(Q_\varepsilon; \partial\omega) - \sqrt{2}\varepsilon^2(h) > h \tag{4.2.30}$$

for h small enough, and

$$\#\mathbb{Z}^\varepsilon \leq C\frac{|\omega|}{\varepsilon^2(h)}. \tag{4.2.31}$$

Moreover, if $z \in \mathbb{Z}^\varepsilon$, $\lambda \in \mathbb{Z}^2$, and

$$\varepsilon^2(h)\lambda \in Q(\varepsilon(h)z, \varepsilon(h) - \varepsilon^2(h)),$$

then $Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))$ is a "good cube", therefore the boundary layer of $Q(\varepsilon(h)z, \varepsilon(h))$, that could possibly intersect "bad cubes" (see Fig 1) has measure given by

$$|Q(\varepsilon(h)z, \varepsilon(h))| - |Q(\varepsilon(h)z, \varepsilon(h) - \varepsilon^2(h))| = \varepsilon(h)^2 - (\varepsilon(h) - \varepsilon(h)^2)^2 = 2\varepsilon(h)^3 - \varepsilon(h)^4.$$

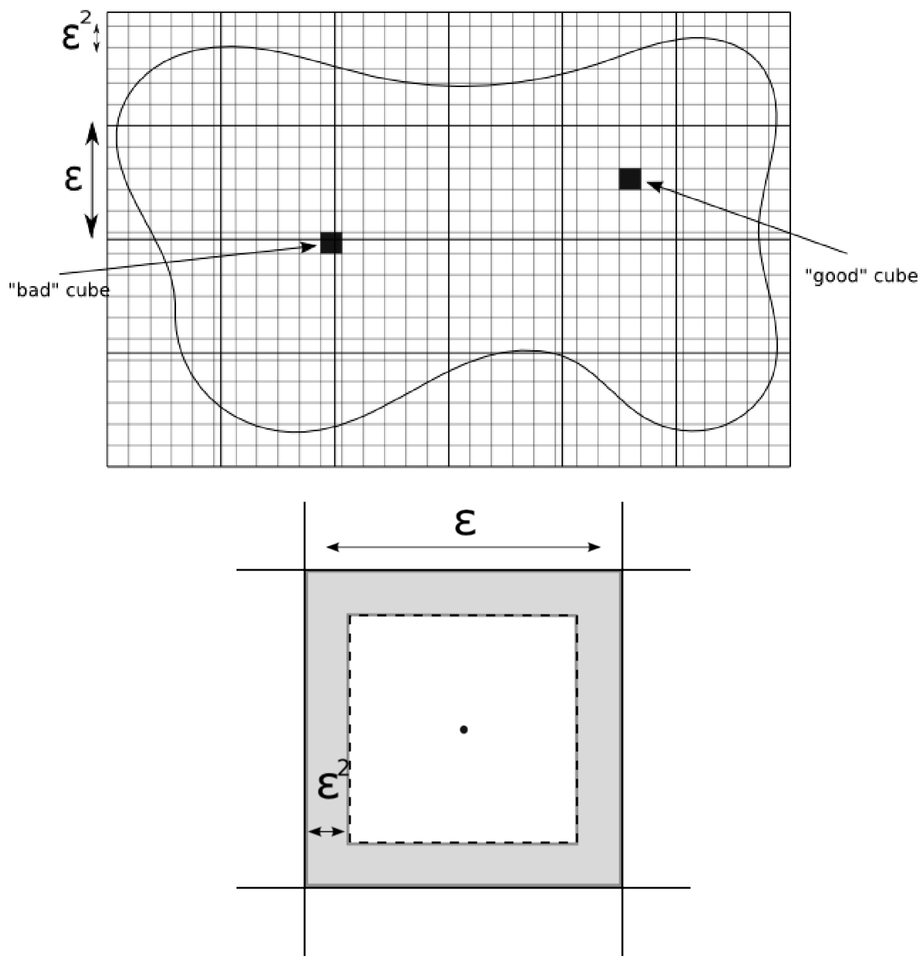


Fig 1. The layer $Q(\varepsilon(h)z, \varepsilon(h)) \setminus Q(\varepsilon(h)z, \varepsilon(h) - \varepsilon^2(h))$

By (4.2.31) we conclude that the sum of all areas of "bad cubes" intersecting Q_ε is bounded from above by

$$C\frac{|\omega|}{\varepsilon^2(h)}(2\varepsilon^3(h) - \varepsilon^4(h)) \leq C\varepsilon(h). \tag{4.2.32}$$

We define the sets

$$\mathbb{Z}_g^\varepsilon := \Big\{\lambda \in \mathbb{Z}^2 : \exists z \in \mathbb{Z}^\varepsilon \text{ s.t. } Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \subset Q(\varepsilon(h)z, \varepsilon(h))\Big\},$$

and

$$\mathbb{Z}_b^\varepsilon := \Big\{\lambda \in \mathbb{Z}^2 : Q(\varepsilon(h)^2\lambda, \varepsilon^2(h)) \cap Q_\varepsilon \neq \emptyset \text{ and } \lambda \notin \mathbb{Z}_g^\varepsilon\Big\}$$

(where 'g' and 'b' stand for "good" and "bad", respectively). We rewrite (4.2.29) as

$$\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\, dx'$$

$$= \sum_{\lambda\in\mathbb{Z}_g^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx'$$
$$+ \sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx'.$$

Since the maps $\{(R^h)'\}$ are piecewise constant on "good cubes", by the periodicity of φ we have

$$\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx' \tag{4.2.33}$$
$$= \sum_{\lambda\in\mathbb{Z}_g^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h}$$
$$: (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\left(\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big) - \psi(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda)\right)dx'$$
$$+ \sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h}$$
$$: (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\left(\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big) - \psi(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda)\right)dx'$$
$$+ \sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda)\,dx'.$$

We claim that

$$\lim_{h\to0}\Big|\sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda\Big)\,dx'\Big| = 0. \tag{4.2.34}$$

Indeed, by the periodicity of φ,

$$\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\,dx' = 0 \quad \text{for every } \lambda\in\mathbb{Z}^2,$$

and we have

$$\Big|\sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x')}{h} : (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda\Big)\,dx'\Big|$$
$$= \Big|\sum_{\lambda\in\mathbb{Z}_b^\varepsilon} \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} \frac{(R^h)'(x') - (R^h)'(\varepsilon^2(h)\lambda)}{h}$$
$$: (\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda\Big)\,dx'\Big|.$$

Therefore, by Hölder's inequality,

$$\Big|\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h}:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda\Big)\,dx'\Big| \tag{4.2.35}$$
$$\leq\frac{C}{h}\int_{\cup_{\lambda\in\mathbb{Z}_b^\varepsilon}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|\,dx'$$
$$\leq\frac{C}{h}\Big|\cup_{\lambda\in\mathbb{Z}_b^\varepsilon}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\Big|^{\frac{1}{2}}\|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)\|_{L^2(\cup_{\lambda\in\mathbb{Z}_b^\varepsilon}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h)))}.$$

Every cube $Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))$ in the previous sum intersects at most four elements of Q_ε (see Fig 2). For every $\lambda\in\mathbb{Z}_b^\varepsilon$, let $Q(\varepsilon(h)z_i^\lambda,\varepsilon)$, $i=1,\cdots,4$, be such cubes, where

$$\#\{z_i^\lambda : i=1,\cdots,4\}\leq 4.$$

Without loss of generality, for every $\lambda\in\mathbb{Z}_b^\varepsilon$ we can assume that

$$\varepsilon^2(h)\lambda\in Q(\varepsilon(h)z_4^\lambda,\varepsilon(h)),$$

so that

$$|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|=0\quad\text{a.e. in } Q(\varepsilon(h)z_4^\lambda,\varepsilon(h)).$$

Hence,

$$\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|^2\,dx'$$
$$=\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\sum_{i=1}^{3}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\cap Q(\varepsilon(h)z_i^\lambda,\varepsilon(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|^2\,dx'.$$

Since the maps $\{R^h\}$ are piecewise constant on each set

$$Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\cap Q(\varepsilon(h)z_i^\lambda,\varepsilon(h)),$$

there holds

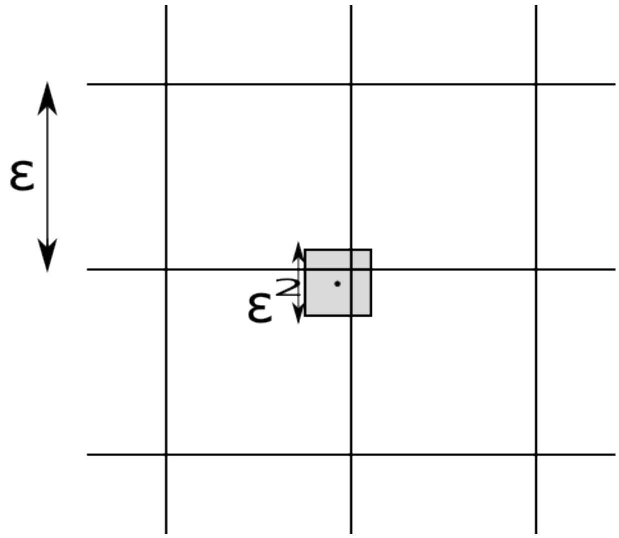
$$|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|=|(R^h)'(x')-(R^h)'(x'+\xi)|$$

for some $\xi\in\{\pm\varepsilon^2(h)e_1,\pm\varepsilon^2(h)e_2,\pm\varepsilon^2(h)e_1\pm\varepsilon^2(h)e_2\}$.

Fig 2. For every $\lambda\in\mathbb{Z}_b^\varepsilon$, the cube $Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))$ intersects at most four element of Q_ε.

Therefore, by (4.2.30) and Lemma 4.2.2, and since $\gamma_1\in(0,+\infty)$, we have

$$\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|^2\,dx' \tag{4.2.36}$$



$$\leq C\|\mathrm{dist}(\nabla_h u^h; SO(3))\|^2_{L^2(\Omega)}.$$

Combining (2.3.2), (4.2.32), (4.2.35), and (4.2.36), we finally get the inequality

$$\Big|\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h}:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda\Big)\,dx'\Big|$$
$$\leq\frac{C}{h}\Big|\cup_{\lambda\in\mathbb{Z}_b^\varepsilon}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\Big|^{\frac12}\|\mathrm{dist}(\nabla_h u^h; SO(3))\|_{L^2(\Omega)}$$
$$\leq C\Big|\cup_{\lambda\in\mathbb{Z}_b^\varepsilon}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\Big|^{\frac12}\leq C\sqrt{\varepsilon(h)},$$

and this concludes the proof of (4.2.34).

Estimates (4.2.33) and (4.2.34) yield

$$\lim_{h\to0}\int_\omega\frac{(R^h)'(x')}{h}:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx'$$
$$=\lim_{h\to0}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h}$$
$$:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\Big(\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)-\psi(\varepsilon^2(h)\lambda,\varepsilon(h)\lambda)\Big)\,dx'$$
$$=\lim_{h\to0}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\frac{(R^h)'(x')}{h}$$
$$:(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\Big(\int_0^1\frac{d}{dt}\phi_\varepsilon(\varepsilon^2(h)\lambda+t(x'-\varepsilon^2(h)\lambda))\,dt\Big)\,dx',$$

where $\phi_\varepsilon(x') := \psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)$ for every $x' \in \omega$. Therefore, by the periodicity of φ

$$\begin{aligned}
&\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx' \\
&\quad = \lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)} \frac{\varepsilon^2(h)}{h}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (R^h)'(x') : (\nabla')^\perp\varphi \\
&\quad\Big(\frac{x'-\varepsilon^2(h)\lambda}{\varepsilon^2(h)}\Big)\Big(\int_0^1 \nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda + t(x'-\varepsilon^2(h)\lambda))\cdot\frac{(x'-\varepsilon^2(h)\lambda)}{\varepsilon^2(h)}\,dt\Big)\,dx'\Bigg].
\end{aligned} \tag{4.2.37}$$

Changing coordinates in (4.2.37) we get

$$\begin{aligned}
&\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx' \\
&\quad = \lim_{h\to 0}\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)} \frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) \\
&\quad\quad : (\nabla')^\perp\varphi(z)\Big(\int_0^1 \nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z)\,dt\cdot z\Big)\,dz \\
&\quad = \lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)} \frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) \\
&\quad\quad : (\nabla')^\perp\varphi(z)\Big(\int_0^1 (\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda))\,dt\cdot z\Big)\,dz \\
&\quad\quad + \sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)} \frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) \\
&\quad\quad : (\nabla')^\perp\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\Bigg].
\end{aligned} \tag{4.2.38}$$

We notice that

$$\begin{aligned}
&\lim_{h\to 0}\Bigg[\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)} \frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) \\
&\quad : (\nabla')^\perp\varphi(z)\Big(\int_0^1 (\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda))\,dt\Big)\cdot z\,dz\Bigg] = 0.
\end{aligned} \tag{4.2.39}$$

Indeed, since $\|(\nabla')^2\phi_\varepsilon\|_{L^\infty(\omega\times Q;\mathbb{M}^{3\times 3})} \leq \frac{C}{\varepsilon^2(h)}$, we have

$$\Bigg|\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)} \frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)$$

$$
\begin{aligned}
&: (\nabla')^{\perp}\varphi(z)\Big(\int_0^1 (\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda))\,dt\cdot z\Big)\,dz\Bigg| \\
&\le C\frac{\varepsilon^6(h)}{h}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_Q |(R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)|\|(\nabla')^2\phi_\varepsilon\|_{L^\infty(\Omega\times Q)}|\varepsilon^2(h)z|\,dz \\
&\le C\frac{\varepsilon^6(h)}{h}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_Q |(R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)|\,dz \\
&= C\frac{\varepsilon^2(h)}{h}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')|\,dx' \le C\frac{\varepsilon^2(h)}{h}\|(R^h)'\|_{L^1(\omega;\mathbb{M}^{3\times3})}
\end{aligned}
$$

which converges to zero by (4.2.8) and because $\gamma_2 = +\infty$.

By (4.2.39), estimate (4.2.38) simplifies as

$$
\begin{aligned}
&\lim_{h\to0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx' \qquad (4.2.40)\\
&\quad= \lim_{h\to0}\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) \\
&\qquad : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz \\
&\quad= \lim_{h\to0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q((R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) - (R^h)'(\varepsilon^2(h)\lambda)) \\
&\qquad : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz \\
&\quad+ \sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)\lambda) : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\Bigg].
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\lim_{h\to0}\Bigg[\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q((R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) - (R^h)'(\varepsilon^2(h)\lambda)) \qquad (4.2.41)\\
&\quad : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\Bigg] = 0.
\end{aligned}
$$

Indeed, since $\varphi \in C^1_{\text{per}}(\mathbb{R}^2;\mathbb{M}^{3\times3})$ and $\|(\nabla')\phi_\varepsilon\|_{L^\infty(\omega\times Q)} \le \frac{C}{\varepsilon(h)}$, recalling the definition of the sets $\mathbb{Z}_b^\varepsilon$ and $\mathbb{Z}_g^\varepsilon$, and applying Hölder's inequality, (2.3.2), (4.2.32), and (4.2.36), we obtain

$$
\Bigg|\sum_{\lambda\in(\mathbb{Z}_b^\varepsilon\cup\mathbb{Z}_g^\varepsilon)}\frac{\varepsilon^6(h)}{h}\int_Q((R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda) - (R^h)'(\varepsilon^2(h)\lambda))
$$

$$
\begin{aligned}
&\qquad : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\Big|\\
&\leq C\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)}\int_Q |(R^h)'(\varepsilon^2(h)z+\varepsilon^2(h)\lambda)-(R^h)'(\varepsilon^2(h)\lambda)|\,dz\\
&=\frac{C\varepsilon(h)}{h}\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|\,dx'\\
&\leq\frac{C\varepsilon(h)}{h}\Big|\cup_{\lambda\in\mathbb{Z}^\varepsilon_b}Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))\Big|^{\frac12}\|\mathrm{dist}(\nabla_h u^h;SO(3))\|_{L^2(\Omega)}\leq C\varepsilon(h)^{\frac32}.
\end{aligned}
$$

Collecting (4.2.40) and (4.2.41), we deduce that

$$
\begin{aligned}
&\lim_{h\to0}\int_\omega\frac{(R^h)'(x')}{h}:(\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx' \qquad (4.2.42)\\
&\quad=\lim_{h\to0}\Bigg[\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)}\frac{\varepsilon^6(h)}{h}\int_Q(R^h)'(\varepsilon^2(h)\lambda):(\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\Bigg].
\end{aligned}
$$

Since $0<\gamma_1<+\infty$ and $\gamma_2=+\infty$, by (4.2.8) we have

$$
\begin{aligned}
&\lim_{h\to0}\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)}\frac{\varepsilon^6(h)}{h}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x'):(\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\,dz\\
&\quad=\lim_{h\to0}\frac{\varepsilon^2(h)}{h}\int_\omega\int_Q(R^h)'(x')\\
&\qquad:(\nabla')^{\perp}\varphi(z)\Big[\Big(\nabla_x\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)+\frac{1}{\varepsilon(h)}\nabla_y\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\Big)\cdot z\Big]\,dz\,dx'\\
&\quad=\frac{1}{\gamma_1}\int_\omega\int_Q\int_Q R'(x'):(\nabla')^{\perp}\varphi(z)(\nabla_y\psi(x',y)\cdot z)\,dz\,dy\,dx'=0,
\end{aligned}
$$

by the periodicity of ψ with respect to y. We observe that if $\lambda\in\mathbb{Z}^\varepsilon_g$, then

$$
\begin{aligned}
&\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x'):(\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\\
&\quad=(R^h)'(\varepsilon^2(h)\lambda):\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx',
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
&\lim_{h\to0}\Bigg[\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)}\frac{\varepsilon^6(h)}{h}\int_Q(R^h)'(\varepsilon^2(h)\lambda):(\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\\
&\qquad-\sum_{\lambda\in(\mathbb{Z}^\varepsilon_b\cup\mathbb{Z}^\varepsilon_g)}\frac{\varepsilon^6(h)}{h}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x')
\end{aligned}
$$

$$
\begin{aligned}
&: (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\,dz\Bigg]\\
&= \lim_{h\to 0}\Bigg[\sum_{\lambda\in\mathbb{Z}_g^\varepsilon}\frac{\varepsilon^6(h)}{h}(R^h)'(\varepsilon^2(h)\lambda)\\
&\quad : \int_Q (\nabla')^{\perp}\varphi(z)\Big[\Big(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda) - \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi_\varepsilon(x')\,dx'\Big)\cdot z\Big]\,dz\\
&\quad + \sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)\lambda) : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz\\
&\quad - \sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\frac{\varepsilon^6(h)}{h}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x') : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\,dz\Bigg].
\end{aligned}
$$

By the regularity of φ and ψ, and the boundedness of $\{R^h\}$ in $L^\infty(\omega;\mathbb{M}^{3\times 3})$,

$$
\begin{aligned}
&\Bigg|\sum_{\lambda\in\mathbb{Z}_g^\varepsilon}\frac{\varepsilon^6(h)}{h}(R^h)'(\varepsilon^2(h)\lambda) \qquad (4.2.43)\\
&\quad : \int_Q (\nabla')^{\perp}\varphi(z)\Bigg[\Big(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda) - \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi_\varepsilon(x')\,dx'\Big)\cdot z\Bigg]\,dz\Bigg|\\
&\le C\frac{\varepsilon^2(h)}{h}\sum_{\lambda\in\mathbb{Z}_g^\varepsilon}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda) - \nabla'\phi_\varepsilon(x')|\,dx'\\
&\le C\frac{\varepsilon^4(h)}{h}\|\nabla^2\phi_\varepsilon\|_{L^\infty(\omega\times Q;\mathbb{M}^{3\times 3})} \le C\frac{\varepsilon^2(h)}{h},
\end{aligned}
$$

which converges to zero, because $\gamma_2 = +\infty$. On the other hand,

$$
\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\frac{\varepsilon^6(h)}{h}\int_Q\Bigg[(R^h)'(\varepsilon^2(h)\lambda) : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda)\cdot z)\,dz \qquad (4.2.44)\\
&\quad - \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}(R^h)'(x') : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\Bigg]\,dz\\
&= \sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\frac{\varepsilon^6(h)}{h}\int_Q (R^h)'(\varepsilon^2(h)\lambda)\\
&\quad : (\nabla')^{\perp}\varphi(z)\Big[\Big(\nabla'\phi_\varepsilon(\varepsilon^2(h)\lambda) - \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi_\varepsilon(x')\,dx'\Big)\cdot z\Big]\,dz\\
&\quad + \sum_{\lambda\in\mathbb{Z}_b^\varepsilon}\frac{\varepsilon^6(h)}{h}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}((R^h)'(\varepsilon^2(h)\lambda) - (R^h)'(x'))
\end{aligned}
$$

$$: (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\,dz.$$

Therefore, arguing as in (4.2.43), the first term on the right hand side of (4.2.44) is bounded by $C\frac{\varepsilon^2(h)}{h}$, whereas by (4.2.32) and the boundedness of $\{R^h\}$ in $L^\infty(\omega;\mathbb{M}^{3\times 3})$,

$$\begin{aligned}
&\Bigg|\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\frac{\varepsilon^6(h)}{h}\int_Q \fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}((R^h)'(\varepsilon^2(h)\lambda)-(R^h)'(x')) \qquad (4.2.45)\\
&\qquad : (\nabla')^{\perp}\varphi(z)(\nabla'\phi_\varepsilon(x')\cdot z)\,dx'\,dz\Bigg|\\
&\quad\le C\frac{\varepsilon(h)}{h}\sum_{\lambda\in\mathbb{Z}^\varepsilon_b}\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}|(R^h)'(x')-(R^h)'(\varepsilon^2(h)\lambda)|\,dx'\\
&\quad\le C\frac{\varepsilon^2(h)}{h},
\end{aligned}$$

which converges to zero as $\gamma_2=+\infty$.

Combining (4.2.42)–(4.2.45) we conclude that

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\,dx'=0. \tag{4.2.46}$$

By (4.2.21), (4.2.28), and (4.2.46), we obtain

$$\int_\omega\int_Q\int_Q\Big(V(x',y,z)-\int_Z V(x',y,\xi)\,d\xi\Big) : (\nabla')^{\perp}\varphi(z)\psi(x',y)\,dz\,dy\,dx'=0$$

for all $\varphi\in C^1_{\mathrm{per}}(Q;\mathbb{R}^3)$ and $\psi\in C^\infty_c(\omega;C^\infty_{\mathrm{per}}(Q))$.

This completes the proof of (4.2.22).

Case 1, Step 2: Characterization of the limit linearized strain G.

In order to identify the multiscale limit of the sequence of linearized strains G^h, by (4.2.13), (4.2.15), (4.2.19)–(4.2.21) we now characterize the weak 3-scale limits of the sequences $\{x_3\nabla'\tilde{R}^h e_3\}$ and $\{\frac{1}{h}(\tilde{R}^h e_3-R^h e_3)\}$.

By (4.2.9) and [5, Theorem 1.2] there exist $S\in L^2(\omega;W^{1,2}_{\mathrm{per}}(Q;\mathbb{M}^{3\times 3}))$ and $T\in L^2(\omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{M}^{3\times 3}))$ such that

$$\nabla'\tilde{R}^h \xrightarrow{3-s} \nabla' R+\nabla_y S+\nabla_z T\quad\text{weakly 3-scale,} \tag{4.2.47}$$

where $\int_Q S(x',y)\,dy=0$ for a.e. $x'\in\omega$, and $\int_Q T(x',y,z)\,dz=0$ for a.e. $x'\in\omega$, and $y\in Y$. By (2.3.2) and (4.2.7), there exists $w\in L^2(\omega\times Q\times Q;\mathbb{R}^3)$ such that

$$\frac{1}{h}(\tilde{R}^h e_3-R^h e_3)\xrightarrow{3-s} w\quad\text{weakly 3-scale}$$

and hence,

$$\frac{1}{h}(\tilde{R}^h e_3 - R^h e_3) \rightharpoonup w_0 \quad \text{weakly in } L^2(\omega;\mathbb{R}^3)$$

where

$$w_0(x') := \int_Q \int_Q w(x', y, z)\, dy\, dz,$$

for a.e. $x' \in \omega$. We claim that

$$\frac{1}{h}(\tilde{R}^h e_3 - R^h e_3) \xrightarrow{3-s} w_0(x') + \frac{1}{\gamma_1} S(x', y) e_3 + \frac{(y \cdot \nabla')R(x')e_3}{\gamma_1}, \tag{4.2.48}$$

weakly 3-scale. We first remark that the same argument as in the proof of (4.2.29) yields

$$\frac{R^h e_3}{h} \xrightarrow{osc,Z} 0.$$

Moreover, since $\gamma_1 \in (0, +\infty)$, by (4.2.8), Lemmas 2.4.9 and 2.4.10, there holds

$$\frac{R^h e_3}{h} \xrightarrow{osc,Y} -\frac{(y \cdot \nabla')Re_3}{\gamma_1}$$

and

$$\frac{\tilde{R}^h e_3}{h} \xrightarrow{osc,Y} \frac{Se_3}{\gamma_1},$$

where in the latter we used the fact that $\int_Q \nabla_z T(x', y, z)\, dz = 0$ for a.e. $x' \in \omega$ and $y \in Y$ by periodicity, and $\int_Q S(x', y)\, dy = 0$ for a.e. $x' \in \omega$. Therefore, by Remark 2.4.8, to prove (4.2.48) we only need to show that

$$\frac{\tilde{R}^h e_3}{h} \xrightarrow{osc,Z} 0. \tag{4.2.49}$$

To this purpose, fix $\varphi \in C^\infty_{\mathrm{per}}(Q)$, with $\int_Q \varphi(z)\, dz = 0$, and $\psi \in C^\infty_c(\Omega; C^\infty_{\mathrm{per}}(Q))$, and let $g \in C^2(Q)$ be the unique periodic solution to

$$\begin{cases} \Delta g(z) = \varphi(z) \\ \int_Q g(z)\, dz = 0. \end{cases}$$

Set

$$g^\varepsilon(x') := \varepsilon^2(h) g\Big(\frac{x'}{\varepsilon^2(h)}\Big) \quad \text{for every } x' \in \omega, \tag{4.2.50}$$

so that

$$\Delta g^\varepsilon(x') = \frac{1}{\varepsilon^2(h)} \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \quad \text{for every } x' \in \omega. \tag{4.2.51}$$

By (4.2.50) and (4.2.51), and for $i \in \{1,2,3\}$, we obtain

$$\int_\omega \frac{\tilde{R}^h_{i3}(x')}{h}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx' = \frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Delta g^\varepsilon(x')\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx'.$$

Integrating by parts, we have

$$\begin{aligned}
&\int_\omega \frac{\tilde{R}^h_{i3}(x')}{h}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx' \\
&= -\frac{\varepsilon^2(h)}{h}\int_\omega \nabla'\tilde{R}^h_{i3}(x')\cdot\nabla'\Big(g^\varepsilon(x')\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big)\,dx' \\
&\quad - \frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Big(2\nabla' g^\varepsilon(x')\cdot(\nabla_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big) \\
&\quad + g^\varepsilon(x')(\Delta_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big)\,dx' \\
&\quad - \frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Big[2\nabla' g^\varepsilon(x')\cdot\nabla_y\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big) \\
&\quad + 2g^\varepsilon(x')(\operatorname{div}_y\nabla_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big]\,dx' \\
&\quad - \frac{1}{h\varepsilon(h)}\int_\omega \tilde{R}^h_{i3}(x')g^\varepsilon(x')\Delta_y\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx'.
\end{aligned} \tag{4.2.52}$$

Since $\nabla'\Big(g^\varepsilon(\cdot)\psi\Big(\cdot, \frac{\cdot}{\varepsilon(h)}\Big)\Big) \in L^\infty(\omega;\mathbb{R}^2)$,

$$\lim_{h\to0}\frac{\varepsilon^2(h)}{h}\int_\omega \nabla'\tilde{R}^h_{i3}(x')\cdot\nabla'\Big(g^\varepsilon(x')\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big)\,dx' = 0, \tag{4.2.53}$$

where we used the fact that $\gamma_2 = +\infty$, and similarly,

$$\lim_{h\to0}\frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Big(2\nabla' g^\varepsilon(x')\cdot(\nabla_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big) + g^\varepsilon(x')(\Delta_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big)\,dx' = 0. \tag{4.2.54}$$

Regarding the third term in the right-hand side of (4.2.52), we write

$$\begin{aligned}
&\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Big[2\nabla' g^\varepsilon(x')\cdot\nabla_y\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big) + 2g^\varepsilon(x')(\operatorname{div}_y\nabla_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big)\Big]\,dx' \qquad (4.2.55) \\
&= 2\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\nabla' g\Big(\frac{x'}{\varepsilon^2(h)}\Big)\cdot\nabla_y\psi\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx' \\
&\quad + \frac{2\varepsilon^3(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')g\Big(\frac{x'}{\varepsilon^2(h)}\Big)(\operatorname{div}_y\nabla_{x'}\psi)\Big(x', \frac{x'}{\varepsilon(h)}\Big)\,dx'.
\end{aligned}$$

By the regularity of g and ψ,

$$\nabla' g\Big(\frac{x'}{\varepsilon^2(h)}\Big)\cdot\nabla_y\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\xrightarrow{3-s}\nabla g(z)\nabla_y\psi(x',y)\quad\text{strongly 3-scale.}$$

Therefore, by (4.2.9), and since $0<\gamma_1<+\infty$ and $\gamma_2=+\infty$, we obtain

$$\begin{aligned}\lim_{h\to 0}\bigg[\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h_{i3}(x')\Big[&2\nabla' g^\varepsilon(x')\cdot\nabla_y\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\\ &+2g^\varepsilon(x')(\operatorname{div}_y\nabla_{x'}\psi)\Big(x',\frac{x'}{\varepsilon(h)}\Big)\Big]\,dx'\bigg]\\ =\frac{2}{\gamma_1}\int_\omega\int_Q\int_Q &R_{i3}(x')\nabla g(z)\cdot\nabla_y\psi(x',y)\,dz\,dy\,dx'=0,\end{aligned}\tag{4.2.56}$$

where the last equality is due to the periodicity of ψ in the y variable.

Again by the regularity of g and ψ,

$$g\Big(\frac{x'}{\varepsilon^2(h)}\Big)\Delta_y\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\xrightarrow{3-s}g(z)\Delta_y\psi(x',y)\quad\text{strongly 3-scale,}$$

hence, by (4.2.9), and since $0<\gamma_1<+\infty$ and $\psi\in C^\infty_c(\omega;C^\infty_{\rm per}(Q))$, the fourth term in the right-hand side of (4.2.52) satisfies

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h\varepsilon(h)}\int_\omega \tilde{R}^h_{i3}(x')g^\varepsilon(x')\Delta_y\psi\Big(x',\frac{x'}{\varepsilon(h)}\Big)\Big)\,dx'&\\ =\frac{1}{\gamma_1}\int_\omega\int_Q\int_Q R_{i3}(x')g(z)\Delta_y\psi(x',y)\,dz\,dy\,dx'&=0.\end{aligned}\tag{4.2.57}$$

Claim (4.2.49), and thus (4.2.48), follow now by combining (4.2.52) with (4.2.53)–(4.2.57).

Case 1, Step 3: Characterization of E.

By (4.2.8), and by collecting (4.2.19), (4.2.20), (4.2.21), (4.2.47), and (4.2.48), we deduce the characterization

$$\begin{aligned}R(x')G(x,y,z)=&\Big(\nabla' r(x')+\nabla_y\hat{\phi}_1(x,y)+\nabla_z\hat{\phi}_2(x,y,z)\Big|\frac{1}{\gamma_1}\partial_{x_3}\hat{\phi}_1(x,y)\Big)\\ &+\Big(V(x',y,z)\Big|w_0(x')+\frac{1}{\gamma_1}S(x',y)e_3+\frac{(y\cdot\nabla')R'(x')}{\gamma_1}e_3\Big)\\ &+x_3\Big(\nabla' R(x')e_3+\nabla_y S(x',y)e_3+\nabla_z T(x',y,z)e_3|0\Big)\end{aligned}$$

for a.e. $x\in\Omega$ and $y,z\in Q$, where $r\in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1\in L^2(\omega;W^{1,2}((-\frac12,\frac12);W^{1,2}_{\rm per}(Q;\mathbb{R}^3)))$, $w_0\in L^2(\omega;\mathbb{R}^3)$, $S\in L^2(\omega;W^{1,2}_{\rm per}(Q;\mathbb{M}^{3\times3}))$, $V\in L^2(\omega\times Q\times Q;\mathbb{M}^{3\times2})$, $\hat{\phi}_2\in L^2(\Omega\times Q;W^{1,2}_{\rm per}(Q;\mathbb{R}^3))$, and $T\in L^2(\omega\times Q;W^{1,2}_{\rm per}(Q;\mathbb{M}^{3\times3}))$. Therefore, by (4.2.22)

$$\operatorname{sym}G(x,y,z)-\int_Q\operatorname{sym}G(x,y,\xi)\,d\xi$$

$$= \operatorname{sym}\left[R(x')^T\Big(V(x',y,z) - \int_Q V(x',y,z) + \nabla_z\hat{\phi}_2(x,y,z)\Big|0\Big)\right.$$

$$\left. + x_3 R(x')^T\Big(\nabla_z T(x',y,z)e_3\Big|0\Big)\right]$$

$$= \operatorname{sym}\left[R(x')^T\Big(\nabla_z v(x',y,z) + \nabla_z\hat{\phi}_2(x,y,z) + x_3\nabla_z T(x',y,z)e_3\Big|0\Big)\right],$$

where $Te_3, \tilde{\tilde{v}} \in L^2(\omega\times Q; W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$. The thesis follows now by (4.2.13), (4.2.14), and by setting

$$\phi_2 := R^T(v + \hat{\phi}_2 + x_3 Te_3).$$

for a.e. $x\in\Omega$, and $y,z\in Q$.

Case 2: $\gamma_1 = +\infty$ **and** $\gamma_2 = +\infty$.
The proof is very similar to the first case where $0 < \gamma_1 < +\infty$. We only outline the main modifications. In order to complete the proof we will need the following lemma.

Lemma 4.2.3. *Let $a \in \mathbb{N}_0$. Then for every $z\in\mathbb{Z}$ there exists $z'\in\mathbb{Z}$ such that*

$$Q(\varepsilon(h)z, \varepsilon(h)) \subset Q(\delta(h)z', \delta(h))$$

or, equivalently, with $m := \frac{\delta(h)}{\varepsilon(h)} \in \mathbb{N}$,

$$\Big(z-\frac{1}{2}, z+\frac{1}{2}\Big) \subset m\Big(z'-\frac{1}{2}, z'+\frac{1}{2}\Big). \tag{4.2.58}$$

holds with $m = 2a+1$.

Proof. Without loss of generality we may assume that $z\in\mathbb{N}_0$ (the case in which $z<0$ is analogous). Solving (4.2.58) is equivalent to finding $z'\in\mathbb{Z}$ such that

$$\begin{cases} z - \frac{1}{2} \geq (2a+1)z' - \frac{(2a+1)}{2}, \\ z + \frac{1}{2} \leq (2a+1)z' + \frac{(2a+1)}{2}, \end{cases} \tag{4.2.59}$$

that is

$$\begin{cases} z \geq (2a+1)z' - a, \\ z \leq (2a+1)z' + a. \end{cases} \tag{4.2.60}$$

Let $n, l \in \mathbb{N}_0$ be such that $z = n(2a+1) + l$ and

$$l < 2a+1. \tag{4.2.61}$$

Then (4.2.60) is equivalent to

$$\begin{cases} n(2a+1) + l + a \geq (2a+1)z', \\ n(2a+1) + l - a \leq (2a+1)z'. \end{cases} \tag{4.2.62}$$

Now, if $0 \leq l \leq a$ it is enough to choose $z' = n$. If $l > a$, the result follows setting $z' := n + 1$. Indeed, with $a + 1 > r > 1 \in \mathbb{N}$ such that $l = a + r$, (4.2.62) simplifies as

$$\begin{cases} n(2a+1) + 2a + r \geq (2a+1)(n+1), \\ n(2a+1) + r \leq (2a+1)(n+1), \end{cases}$$

that is,

$$\begin{cases} 2a + r \geq 2a + 1 \\ r \leq 2a + 1, \end{cases}$$

which is trivially satisfied. $\square$

Remark 4.2.4. *By Lemma 4.2.3 it follows that, setting $p := \frac{\delta(h)}{\varepsilon^2(h)}$ and provided p is odd, for every $z \in \mathbb{Z}^2$ there exists $z' \in \mathbb{Z}^2$ such that*

$$Q(\varepsilon^2(h)z, \varepsilon^2(h)) \subset Q(\delta(h)z, \delta(h)).$$

This observation allowed us to construct the sequence $\{R^h\}$ in Case 3 of the proof of Theorem 1.2.1.

Remark 4.2.5. *We point out that if m is even there may be $z \in \mathbb{Z}$ such that* (4.2.58) *fails to be true for every $z' \in \mathbb{Z}$, i.e.,*

$$\Big(z - \frac{1}{2}, z + \frac{1}{2}\Big) \not\subseteq \Big(mz' - \frac{m}{2}, mz' + \frac{m}{2}\Big).$$

Indeed, if m is even, then $z = \frac{3}{2}m \in \mathbb{N}$ and (4.2.59) *becomes*

$$\begin{cases} \frac{3}{2}m - \frac{1}{2} \geq mz' - \frac{m}{2}, \\ \frac{3}{2}m + \frac{1}{2} \leq mz' + \frac{m}{2}, \end{cases}$$

which in turn is equivalent to

$$z' \in \Big[1 + \frac{1}{2m}, 2 - \frac{1}{2m}\Big].$$

This last condition leads to a contradiction as

$$\Big[1 + \frac{1}{2m}, 2 - \frac{1}{2m}\Big] \cap \mathbb{Z} = \emptyset \quad \textit{for every } m \in \mathbb{N}.$$

We now return to the proof of Case 2. Arguing as in [57, Proof of Proposition 3.2], in order to construct the sequence $\{R^h\}$, we apply Lemma 4.2.2 with

$$\delta(h) := \Big(2\Big\lceil \frac{h}{\varepsilon(h)} \Big\rceil + 1\Big)\varepsilon(h).$$

This way,

$$\lim_{h\to 0} \frac{h}{\delta(h)} = \frac{1}{2},$$

and the maps R^h are piecewise constant on cubes of the form $Q(\delta(h)z, \delta(h))$, with $z \in \mathbb{Z}^2$. In particular, since $\left\{\frac{\delta(h)}{\varepsilon(h)}\right\}$ is a sequence of odd integers, by Lemma 4.2.3 the maps R^h are piecewise constant on cubes of the form $Q(\varepsilon(h)z, \varepsilon(h))$ with $z \in \mathbb{Z}^2$, and (4.2.7) holds true. Defining $\{r^h\}$ as in (4.2.17), we obtain equality (4.2.19). By Theorem 2.4.6(i), there exist $r \in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1 \in L^2(\Omega; W^{1,2}_{\rm per}(Q;\mathbb{R}^3))$, $\hat{\phi}_2 \in L^2(\Omega\times Q; W^{1,2}_{\rm per}(Q;\mathbb{R}^3))$, and $\bar{\phi} \in L^2\big(\omega; W^{1,2}\big(\big(-\frac{1}{2},\frac{1}{2}\big);\mathbb{R}^3\big)\big)$ such that

$$\nabla_h r^h \overset{dr-3-s}{\longrightarrow} (\nabla' r + \nabla_y \hat{\phi}_1 + \nabla_z \hat{\phi}_2 | \partial_{x_3}\bar{\phi}) \quad \text{weakly dr-3-scale.} \tag{4.2.63}$$

Moreover, (4.2.14) now becomes

$$\begin{aligned}&\operatorname{sym} \int_Q G(x,y,\xi)\, d\xi \\ &\quad = \begin{pmatrix} x_3\Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0 \end{pmatrix} + \operatorname{sym}\Big(\nabla_y \phi_1(x,y)\Big|\partial_{x_3}\bar{\phi}\Big)\end{aligned}$$

for a.e. $x \in \Omega$ and $y \in Y$, where $B \in L^2(\omega;\mathbb{M}^{2\times 2})$. Arguing as in Step 1–Step 3 of Case 1, we obtain the characterization

$$\begin{aligned}E(x,y,z) = &\begin{pmatrix} x_3\Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0 \end{pmatrix} \\ &+ \operatorname{sym}(\nabla_y \phi_1(x,y)|d(x)) + \operatorname{sym}(\nabla_z \phi_2(x,y,z)|0),\end{aligned}$$

with $d := \partial_{x_3}\bar{\phi} \in L^2(\Omega;\mathbb{R}^3)$, $\phi_1 \in L^2(\Omega; W^{1,2}_{\rm per}(Q;\mathbb{R}^3))$, and $\phi_2 \in L^2(\Omega \times Q; W^{1,2}_{\rm per}(Q;\mathbb{R}^3))$.

Case 3: $\gamma_1 = 0$ **and** $\gamma_2 = +\infty$.
We apply Lemma 4.2.2 with

$$\delta(h) := \Big(2\Big\lceil \frac{h}{\varepsilon^2(h)} \Big\rceil + 1\Big)\varepsilon^2(h),$$

and by Lemma 4.2.3 and Remark 4.2.4 we construct

$$\{R^h\} \subset L^\infty(\omega; SO(3)) \quad \text{and} \quad \{\tilde{R}^h\} \subset W^{1,2}(\omega;\mathbb{M}^{3\times 3}),$$

satisfying (4.2.7), and with R^h piecewise constant on every cube of the form $Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))$, with $\lambda \in \mathbb{Z}^2$.

Arguing as in Case 1, we obtain the convergence properties in (4.2.8) and (4.2.9), and the identification of E reduces to establishing a characterization of the weak 3-scale limit

G of the sequence $\{G^h\}$ defined in (4.2.11). In view of [83, Proposition 3.2], there exist $B \in L^2(\omega; \mathbb{M}^{2\times 2})$, $\xi \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2))$, $\eta \in L^2(\omega; W^{2,2}_{\mathrm{per}}(Q;\mathbb{R}^2))$, and $g_i \in L^2(\Omega\times Y)$, $i = 1, 2, 3$, such that

$$\begin{aligned} \int_Q E(x', y, z)\, dz &= \mathrm{sym} \int_Q G(x', y, z)\, dz \\ &= \begin{pmatrix} x_3\Pi^u(x') + \mathrm{sym}\, B(x') & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathrm{sym}\nabla_y \xi(x, y) + x_3 \nabla_y^2 \eta(x', y) & & g_1(x,y) \\ & & g_2(x,y) \\ g_1(x,y) & g_2(x,y) & g_3(x,y) \end{pmatrix} \end{aligned} \tag{4.2.64}$$

for a.e. $x \in \Omega$ and $y \in Y$. We consider the maps $\{\bar{u}^h\}$ and $\{r^h\}$, defined in (4.2.16) and (4.2.17), and the decomposition in (4.2.19). By Theorem 2.4.6 (iii) there exist maps $r \in W^{1,2}(\omega;\mathbb{R}^3)$, $\hat{\phi}_1 \in L^2(\omega; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, $\hat{\phi}_2 \in L^2(\Omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\bar{\phi} \in L^2\big(\omega \times Q; W^{1,2}\big(\big(-\tfrac{1}{2}, \tfrac{1}{2}\big);\mathbb{R}^3\big)\big)$ such that

$$\nabla_h r^h \overset{dr-3-s}{\longrightarrow} (\nabla' r + \nabla_y \hat{\phi}_1 + \nabla_z \hat{\phi}_2 | \partial_{x_3}\bar{\phi}) \quad \text{weakly dr-3-scale.}$$

Defining V as in (4.2.21), we need to identify the quantity

$$V(x', y, z) - \int_Q V(x', y, z)\, dz.$$

Case 3, Step 1: Characterization of V
We claim that

$$V(x', y, z) - \int_Q V(x', y, z)\, dz = \nabla_z v(x', y, z) \tag{4.2.65}$$

for a.e. $x' \in \omega$, and $y, z \in Q$, for some $v \in L^2(\omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$.

As in Case 1–Step 1, by [5, Lemma 3.7] and by a density argument, to prove (4.2.65) it is enough to show that

$$\int_\omega \int_Q \int_Q \Big(V(x', y, z) - \int_Q V(x', y, z)\, dz \Big) : (\nabla')^\perp \varphi(z)\phi(y)\psi(x')\, dz\, dy\, dx' = 0 \tag{4.2.66}$$

for every $\varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^3)$, $\phi \in C^\infty_{\mathrm{per}}(Q)$ and $\psi \in C^\infty_c(\omega)$. Without loss of generality, we may assume in addition that $\int_Q \varphi(z)\, dz = 0$.

Fix $\varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^3)$, $\phi \in C^\infty_{\mathrm{per}}(Q)$, $\psi \in C^\infty_c(\omega)$, and set

$$\varphi^\varepsilon(x') := \varepsilon^2(h)\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \quad \text{for every } x' \in \mathbb{R}^2.$$

We notice that

$$(\nabla')^\perp \varphi^\varepsilon(x') = \nabla^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\phi\Big(\frac{x'}{\varepsilon(h)}\Big) + \varepsilon(h)\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes \nabla^\perp \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \tag{4.2.67}$$

for every $x' \in \mathbb{R}^2$. Integrating by parts, in view of the definition of $\nabla^\perp$ we have

$$\begin{aligned}\int_\omega \frac{\nabla_h \bar{u}^h}{h} : (\nabla')^\perp \varphi^\varepsilon(x')\psi(x')\,dx' = -\int_\omega \frac{\nabla_h \bar{u}^h}{h} : \varphi^\varepsilon(x') \otimes (\nabla')^\perp \psi(x')\,dx' \\ = -\frac{\varepsilon^2(h)}{h}\int_\omega \nabla_h \bar{u}^h(x) : \phi\Big(\frac{x'}{\varepsilon(h)}\Big)\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes (\nabla')^\perp \psi(x')\,dx'.\end{aligned}$$

Therefore, by (4.2.7) and since $\gamma_2 = +\infty$, we obtain

$$\lim_{h\to 0}\int_\omega \frac{\nabla_h \bar{u}^h}{h} : (\nabla')^\perp \varphi^\varepsilon(x')\psi(x')\,dx' = 0.$$

By the regularity of ϕ and φ,

$$\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\nabla^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \xrightarrow{3-s} \phi(y)\nabla^\perp\varphi(z) \quad \text{strongly 3-scale}$$

and

$$\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes \nabla^\perp \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \xrightarrow{3-s} \varphi(z) \otimes \nabla^\perp\phi(y) \quad \text{strongly 3-scale.}$$

By the definition of V (see (4.2.21)), (4.2.67), and since R^h does not depend on x_3, we deduce that

$$\begin{aligned}\lim_{h\to 0}\int_\omega \frac{\nabla_h \bar{u}^h - (R^h)'}{h} : (\nabla')^\perp \varphi^\varepsilon(x')\psi(x')\,dx' \\ = \int_\omega\int_Q\int_Q V(x',y,z) : \nabla^\perp\varphi(z)\phi(y)\psi(x')\,dz\,dy\,dx'.\end{aligned} \tag{4.2.68}$$

In view of (4.2.68), (4.2.66) reduces to showing that

$$\lim_{h\to 0}\int_\omega \frac{(R^h)'(x')}{h} : (\nabla')^\perp \varphi^\varepsilon(x')\psi(x')\,dx' = 0. \tag{4.2.69}$$

Defining

$$\hat{\mathbb{Z}}^\varepsilon := \Big\{\lambda \in \mathbb{Z}^2 :\, Q(\varepsilon^2(h)\lambda, \varepsilon^2(h)) \cap \operatorname{supp}\psi \neq \emptyset\Big\},$$

we obtain

$$\begin{aligned}\int_\omega \frac{(R^h)'(x')}{h} : \phi\Big(\frac{x'}{\varepsilon(h)}\Big)(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi(x')\,dx' \\ = \frac{1}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda) \\ : \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\Big(\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x') - \phi(\varepsilon(h)\lambda)\psi(\varepsilon^2(h)\lambda)\Big)(\nabla')^\perp\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\,dx',\end{aligned} \tag{4.2.70}$$

where we used the fact that

$$\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\, dx' = 0 \quad \text{for every } \lambda \in \hat{\mathbb{Z}}^{\varepsilon}$$

by the periodicity of φ. The regularity of ϕ, φ and ψ yields

$$\begin{aligned}
&\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x') - \phi(\varepsilon(h)\lambda)\psi(\varepsilon^2(h)\lambda) \\
&\quad = \int_0^1 \frac{d}{dt}\Big[\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda\Big)\Big)\psi\big(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)\big)\Big]\, dt \\
&\quad = \int_0^1 \Big[\nabla'\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda\Big)\Big)\psi\big(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)\big) \\
&\qquad \cdot \Big(\frac{x' - \varepsilon^2(h)\lambda}{\varepsilon(h)}\Big)\Big]\, dt \\
&\qquad + \int_0^1 \Big[\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda\Big)\Big)\nabla'\psi\big(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)\big) \\
&\qquad \cdot (x' - \varepsilon^2(h)\lambda)\Big]\, dt.
\end{aligned}$$

Therefore (4.2.70) can be rewritten as

$$\begin{aligned}
&\int_\omega \frac{(R^h)'(x')}{h} : \phi\Big(\frac{x'}{\varepsilon(h)}\Big)(\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\psi(x')\, dx' \qquad (4.2.71)\\
&\quad = \frac{1}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^{\varepsilon}} (R^h)'(\varepsilon^2(h)\lambda) \\
&\qquad : \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\Big\{\int_0^1 \Big[\nabla'\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda\Big)\Big) \\
&\quad \psi\big(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)\big) \cdot \Big(\frac{x' - \varepsilon^2(h)\lambda}{\varepsilon(h)}\Big)\Big]\, dt\Big\}\, dx' \\
&\qquad + \frac{1}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^{\varepsilon}} (R^h)'(\varepsilon^2(h)\lambda) \\
&\qquad : \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^{\perp}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\Big\{\int_0^1 \Big[\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda\Big)\Big) \\
&\quad \nabla'\psi\big(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)\big) \cdot (x' - \varepsilon^2(h)\lambda)\Big]\, dt\Big\}\, dx'.
\end{aligned}$$

The first term in the right-hand side of (4.2.71) can be further decomposed as follows:

$$\frac{1}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^{\varepsilon}} (R^h)'(\varepsilon^2(h)\lambda) \qquad (4.2.72)$$

$$
\begin{aligned}
&: \int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))} (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\Bigg\{ \int_0^1 \Big[\nabla'\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda\Big)\Big) \\
&\psi\big(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)\big) \cdot \Big(\frac{x' - \varepsilon^2(h)\lambda}{\varepsilon(h)}\Big)\Big]\, dt\Bigg\}\, dx' \\
&= \frac{\varepsilon^5(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\
&: \int_Q \int_0^1 \big[\nabla'\phi(\varepsilon(h)\lambda + t\varepsilon(h)z) \cdot z\big]\psi(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z)(\nabla')^\perp\varphi(z)\, dt\, dz \\
&= \frac{\varepsilon^5(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\
&: \int_Q \int_0^1 \big[\nabla'\phi(\varepsilon(h)\lambda) \cdot z\big]\big(\psi(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \psi(\varepsilon^2(h)\lambda)\big)(\nabla')^\perp\varphi(z)\, dt\, dz \\
&+ \frac{\varepsilon^5(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\
&: \int_Q \Bigg\{ \int_0^1 \big[\big(\nabla'\phi(\varepsilon(h)\lambda + t\varepsilon(h)z) - \nabla'\phi(\varepsilon(h)\lambda)\big) \cdot z\big] \\
&\psi(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z)(\nabla')^\perp\varphi(z)\, dt\Bigg\}\, dz \\
&+ \frac{\varepsilon^5(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) : \int_Q \int_0^1 \big[\nabla'\phi(\varepsilon(h)\lambda) \cdot z\big]\psi(\varepsilon^2(h)\lambda)(\nabla')^\perp\varphi(z)\, dt\, dz.
\end{aligned}
$$

In view of the regularity of the maps ϕ, φ, and ψ, and the boundedness of $\{R^h\}$ in $L^\infty(\omega; \mathbb{M}^{3\times3})$, the first term in the right-hand side of (4.2.72) is estimated from above as

$$
\begin{aligned}
&\Big|\frac{\varepsilon^5(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\
&: \int_Q \int_0^1 \big[\nabla'\phi(\varepsilon(h)\lambda) \cdot z\big]\big(\psi(\varepsilon^2(h)\lambda + t\varepsilon^2(h)z) - \psi(\varepsilon^2(h)\lambda)\big)(\nabla')^\perp\varphi(z)\, dt\, dz\Big| \\
&\le \frac{\varepsilon^7(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} |(R^h)'(\varepsilon^2(h)\lambda)| : \int_Q |\nabla'\phi(\varepsilon(h)\lambda)\|\nabla\psi\|_{L^\infty(\omega;\mathbb{R}^2)}|(\nabla')^\perp\varphi(z)| dz \\
&\le C\frac{\varepsilon^3(h)}{h},
\end{aligned}
\tag{4.2.73}
$$

using the fact that $\#\hat{\mathbb{Z}}^\varepsilon \cong O\Big(\frac{1}{\varepsilon^4(h)}\Big)$ (see (4.2.31)), which converges to zero because $\gamma_2 = +\infty$.

Similarly, the second term in the right-hand side of (4.2.72) is bounded by

$$\frac{\varepsilon^6(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}|(R^h)'(\varepsilon^2(h)\lambda)|:\int_Q\|(\nabla')^2\phi\|_{L^\infty(\omega;\mathbb{M}^{2\times 2})}\|\psi\|_{L^\infty(\omega)}|(\nabla')^\perp\varphi(z)|dz \leq C\frac{\varepsilon^2(h)}{h}, \tag{4.2.74}$$

and hence it converges to zero, as $\gamma_2=+\infty$.

The third term in the right-hand side of (4.2.72) is further decomposed into

$$\begin{aligned}&\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda):\int_Q\int_0^1\big[\nabla'\phi(\varepsilon(h)\lambda)\cdot z\big]\psi(\varepsilon^2(h)\lambda)(\nabla')^\perp\varphi(z)\,dt\,dz \\ &= \frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda):\int_Q(\nabla')^\perp\varphi(z)\Big\{\Big[\nabla'\phi(\varepsilon(h)\lambda)\psi(\varepsilon^2(h)\lambda) \\ &\quad -\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx'\Big]\cdot z\Big\}\,dz \\ &\quad +\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q(R^h)'(x') \\ &\quad :(\nabla')^\perp\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\,dz\,dx'.\end{aligned} \tag{4.2.75}$$

We first study the second term in the right-hand side of (4.2.75). We add and subtract the function $\frac{\tilde{R}^h}{h}$, and we obtain

$$\begin{aligned}&\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q(R^h)'(x'):(\nabla')^\perp\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\,dz\,dx' \\ &= \frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\int_Q\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\Big\{\big((R^h)'(x')-(\tilde{R}^h)'(x')\big) \\ &\quad :(\nabla')^\perp\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\Big\}\,dz\,dx' \\ &\quad +\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q(\tilde{R}^h)'(x') \\ &\quad :(\nabla')^\perp\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\,dz\,dx'.\end{aligned} \tag{4.2.76}$$

By (4.2.7) and by the regularity of the test functions ϕ,φ, and ψ, we have

$$\Big|\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q\Big\{\big((R^h)'(x')-(\tilde{R}^h)'(x')\big) \tag{4.2.77}$$

$$: (\nabla')^{\perp}\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\,dz\Big\}\,dx'$$
$$\leq C\varepsilon(h)\Big\|\frac{(R^h)'-(\tilde{R}^h)'}{h}\Big\|_{L^2(\omega;\mathbb{M}^{3\times 2})}\leq C\varepsilon(h).$$

Hence the first term in the right-hand side of (4.2.76) is infinitesimal.

Regarding the second term in the right-hand side of (4.2.76), by (4.2.9) and [5, Theorem 1.2], there exist $S\in L^2(\omega;W^{1,2}_{\mathrm{per}}(Q;\mathbb{M}^{3\times 3}))$, and $T\in L^2(\omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ such that

$$\nabla'\tilde{R}^h \overset{dr-3-s}{\rightharpoonup} \nabla' R+\nabla_y S+\nabla_z T \quad \text{weakly 3-scale,} \tag{4.2.78}$$

where $\int_Q S(x',y)\,dy=0$ for a.e. $x'\in\omega$, and $\int_Q T(x',y,z)\,dz=0$ for a.e. $x'\in\omega$ and $y\in Q$. By Lemma 2.4.10, there holds

$$\frac{\tilde{R}^h}{\varepsilon(h)} \overset{osc,Y}{\rightharpoonup} S,$$

and hence

$$\lim_{h\to 0}\int_\omega \frac{(\tilde{R}^h)'(x')}{\varepsilon(h)}\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx'=\int_\omega\int_Q S'(x',y)\nabla'\phi(y)\psi(x')\,dx'\,dy. \tag{4.2.79}$$

Since $\gamma_2=+\infty$, (4.2.79) yields

$$\lim_{h\to 0}\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q\Big\{(\tilde{R}^h)'(x') \tag{4.2.80}$$
$$: (\nabla')^{\perp}\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\Big\}\,dz\,dx'=0.$$

Combining (4.2.76) with (4.2.77) and (4.2.80), we conclude that

$$\lim_{h\to 0}\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\int_Q\Big\{(R^h)'(x') \tag{4.2.81}$$
$$: (\nabla')^{\perp}\varphi(z)\Big[\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot z\Big]\psi(x')\Big\}\,dz\,dx'=0.$$

It remains to estimate the first term in the right-hand side of (4.2.75). Since

$$\Big|\nabla'\phi(\varepsilon(h)\lambda)\psi(\varepsilon^2(h)\lambda)-\fint_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx'\Big|$$
$$\leq 2\big(\|(\nabla')^2\phi\|_{L^\infty(Q;\mathbb{M}^{2\times 2})}\|\psi\|_{L^\infty(\omega)}+\|\nabla'\psi\|_{L^\infty(\omega;\mathbb{R}^2)}\|\nabla'\phi\|_{L^\infty(Q;\mathbb{R}^2)}\big)\varepsilon(h),$$

there holds

$$\Big|\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda) \tag{4.2.82}$$

$$: \int_Q \Big\{ (\nabla')^\perp \varphi(z) \Big[\Big(\nabla' \phi(\varepsilon(h)\lambda) \psi(\varepsilon^2(h)\lambda) \\ - \fint_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} \nabla' \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \psi(x')\, dx' \Big) \cdot z \Big] \Big\}\, dz \Big| \leq C \frac{\varepsilon^2(h)}{h}.$$

Therefore, by (4.2.73), (4.2.74), (4.2.75), (4.2.81), and (4.2.82), and since $\gamma_2 = +\infty$, we conclude that (see (4.2.72))

$$\lim_{h\to 0} \Bigg\{ \frac{1}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\ : \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \int_0^1 \Big[\nabla' \phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda \Big)\Big) \\ \psi(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)) \cdot \Big(\frac{x' - \varepsilon^2(h)\lambda}{\varepsilon(h)} \Big) \Big]\, dt\, dx' \Bigg\} = 0. \tag{4.2.83}$$

The same argument as in the proof of (4.2.83) yields also

$$\lim_{h\to 0} \Bigg\{ \frac{1}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\ : \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} (\nabla')^\perp \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \int_0^1 \Big[\phi\Big(\varepsilon(h)\lambda + t\Big(\frac{x'}{\varepsilon(h)} - \varepsilon(h)\lambda \Big)\Big) \\ \nabla' \psi(\varepsilon^2(h)\lambda + t(x' - \varepsilon^2(h)\lambda)) \cdot (x' - \varepsilon^2(h)\lambda) \Big]\, dt\, dx' \Bigg\} = 0,$$

which, in turn, implies that (4.2.71) (and thus (4.2.70)) is infinitesimal as $h \to 0$.

To complete the study of the asymptotic behavior of (4.2.69), we observe that

$$\frac{\varepsilon(h)}{h} \int_\omega (R^h)'(x') : \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes \nabla^\perp \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \Big] \psi(x')\, dx' \\ = \frac{\varepsilon(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\ : \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes \nabla^\perp \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \Big] \big(\psi(x') - \psi(\varepsilon^2(h)\lambda) \big)\, dx' \\ + \frac{\varepsilon(h)}{h} \sum_{\lambda \in \hat{\mathbb{Z}}^\varepsilon} (R^h)'(\varepsilon^2(h)\lambda) \\ : \int_{Q(\varepsilon^2(h)\lambda, \varepsilon^2(h))} \Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \otimes \nabla^\perp \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \Big] \psi(\varepsilon^2(h)\lambda)\, dx'. \tag{4.2.84}$$

We bound from above the first term in the right-hand side of (4.2.84) by

$$\begin{aligned}&\Big|\frac{\varepsilon(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda) \\ &\quad:\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\Big[\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\otimes\nabla^\perp\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\Big]\big(\psi(x')-\psi(\varepsilon^2(h)\lambda)\big)\,dx'\Big| \\ &\leq C\frac{\varepsilon^3(h)}{h}\|R^h\|_{L^\infty(\omega;\mathbb{M}^{3\times3})}\|\nabla^\perp\phi\|_{L^\infty(\omega;\mathbb{R}^2)}\|\varphi\|_{L^\infty(\omega;\mathbb{R}^3)}\|\nabla\psi\|_{L^\infty(\omega;\mathbb{R}^2)}\end{aligned} \tag{4.2.85}$$

therefore, it goes to zero due to the boundedness of $\{R^h\}$ in $L^\infty(\omega;\mathbb{M}^{3\times3})$, the regularity of the test functions, and the fact that $\gamma_2=+\infty$.

Regarding the second term in (4.2.84), we observe that

$$\begin{aligned}&\lim_{h\to0}\frac{\varepsilon(h)}{h}\int_\omega(R^h)'(x'):\Big[\Big(\int_Q\varphi(z)\,dz\Big)\otimes\nabla^\perp\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\Big]\psi(x')\,dx' \\ &=\lim_{h\to0}\Big\{\frac{\varepsilon(h)}{h}\int_\omega\Big((R^h)'(x')-(\tilde{R}^h)'(x')\Big) \\ &\quad:\Big[\Big(\int_Q\varphi(z)\,dz\Big)\otimes\nabla^\perp\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\Big]\psi(x')\,dx' \\ &\quad+\frac{\varepsilon(h)}{h}\int_\omega(\tilde{R}^h)'(x'):\Big[\Big(\int_Q\varphi(z)\,dz\Big)\otimes\nabla^\perp\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\Big]\psi(x')\,dx'\Big\}=0,\end{aligned}$$

due to (4.2.7), (4.2.79), and because $\gamma_2=+\infty$.

Therefore, and since $\int_Q\varphi(z)\,dz=0$, by changing coordinates we deduce

$$\begin{aligned}&\left|\frac{\varepsilon(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda)\psi(\varepsilon^2(h)\lambda):\int_{Q(\varepsilon^2(h)\lambda,\varepsilon^2(h))}\varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big)\otimes\nabla^\perp\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\,dx'\right| \\ &=\Big|\frac{\varepsilon^5(h)}{h}\sum_{\lambda\in\hat{\mathbb{Z}}^\varepsilon}(R^h)'(\varepsilon^2(h)\lambda)\psi(\varepsilon^2(h)\lambda) \\ &\quad:\int_Q\varphi(z)\otimes\Big[\nabla^\perp\phi(\varepsilon(h)\lambda+\varepsilon(h)z)-\nabla^\perp\phi(\varepsilon(h)\lambda)\Big]\,dz\Big|\leq C\frac{\varepsilon^2(h)}{h}.\end{aligned} \tag{4.2.86}$$

In view of (4.2.86) and since $\gamma_2=+\infty$, we conclude that the second term in the right-hand side of (4.2.84) is infinitesimal. This completes the proof of (4.2.65).

Case 3, Step 2: Characterization of the limit linearized strain G.

To identify E we need to characterize the weak 3-scale limit of the scaled linearized strains G^h (see (4.2.11), (4.2.12) and (4.2.13)). By (4.2.19) this reduces to study the weak 3-scale limit of the sequence

$$\Big\{\frac{R^he_3-\tilde{R}^he_3}{h}\Big\}.$$

By (4.2.7) there exists $w \in L^2(\omega \times Q \times Q; \mathbb{R}^3)$ such that

$$\frac{(\tilde{R}^h - R^h)}{h} \xrightarrow{3-s} w(x', y, z) \quad \text{weakly 3-scale.}$$

We claim that

$$w(x', y, z) - \int_Q w(x', y, z)\, dz = 0 \tag{4.2.87}$$

for a.e. $x' \in \omega$, and $y, z \in Q$.

To prove (4.2.87), by Remark 2.4.8 we have to show that

$$\frac{\tilde{R}^h e_3 - R^h e_3}{h} \xrightarrow{osc, Z} 0.$$

A direct application of the argument in the proof of (4.2.69) yields

$$\frac{R^h e_3}{h} \xrightarrow{osc, Z} 0. \tag{4.2.88}$$

Therefore, (4.2.87) is equivalent to proving that

$$\frac{\tilde{R}^h e_3}{h} \xrightarrow{osc, Z} 0, \tag{4.2.89}$$

that is, to characterize

$$\lim_{h \to 0} \int_\omega \frac{\tilde{R}^h(x')_{i3}}{h} \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \psi(x')\, dx'$$

for every $\varphi \in C^\infty_{\mathrm{per}}(Q)$, $\psi \in C^\infty_c(\omega)$, and $\phi \in C^\infty_{\mathrm{per}}(Q)$, for $i = 1, 2, 3$.

Fix $\varphi \in C^\infty_{\mathrm{per}}(Q)$, $\psi \in C^\infty_c(\omega)$, $\phi \in C^\infty_{\mathrm{per}}(Q)$, and define g as the unique periodic solution to

$$\begin{cases} -\Delta g = \varphi & \text{in } Q \\ \quad g = 0 & \text{on } \partial Q, \end{cases}$$

and set

$$g^\varepsilon(x') := \varepsilon^2(h) g\Big(\frac{x'}{\varepsilon^2(h)}\Big) \quad \text{for every } x' \in \mathbb{R}^2. \tag{4.2.90}$$

Then

$$\Delta g^\varepsilon(x') = -\frac{1}{\varepsilon^2(h)} \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \text{ and } \nabla g^\varepsilon(x') = \nabla g\Big(\frac{x'}{\varepsilon^2(h)}\Big)$$

for every $x' \in \mathbb{R}^2$. Integrating by parts, we obtain

$$\int_\omega \frac{\tilde{R}^h(x')_{i3}}{h} \varphi\Big(\frac{x'}{\varepsilon^2(h)}\Big) \phi\Big(\frac{x'}{\varepsilon(h)}\Big) \psi(x')\, dx' \tag{4.2.91}$$

$$
\begin{aligned}
&= -\frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h(x')_{i3}\Delta g^\varepsilon(x')\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx' \\
&= \frac{\varepsilon^2(h)}{h}\int_\omega \nabla'\tilde{R}^h(x')_{i3}\cdot\nabla' g^\varepsilon(x')\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx' \\
&\quad + \frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h(x')_{i3}\nabla' g^\varepsilon(x')\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot\nabla'\psi(x')\,dx' \\
&\quad + \frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h(x')_{i3}\nabla' g^\varepsilon(x')\cdot(\nabla'\phi)\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx'.
\end{aligned}
$$

Since $\gamma_2 = +\infty$, by (4.2.9) and by the regularity of the test functions, there holds

$$
\lim_{h\to 0}\frac{\varepsilon^2(h)}{h}\int_\omega \nabla'\tilde{R}^h(x')_{i3}\cdot\nabla' g^\varepsilon(x')\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx' = 0, \tag{4.2.92}
$$

and

$$
\lim_{h\to 0}\frac{\varepsilon^2(h)}{h}\int_\omega \tilde{R}^h(x')_{i3}\nabla' g^\varepsilon(x')\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\cdot\nabla'\psi(x')\,dx' = 0. \tag{4.2.93}
$$

The third term in the right-hand side of (4.2.91) can be rewritten as

$$
\begin{aligned}
&\frac{\varepsilon(h)}{h}\int_\omega \tilde{R}^h(x')_{i3}\nabla' g\Big(\frac{x'}{\varepsilon^2(h)}\Big)\cdot(\nabla'\phi)\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx' \\
&\quad = \frac{\varepsilon(h)}{h}\int_\omega R^h(x')_{i3}\nabla' g\Big(\frac{x'}{\varepsilon^2(h)}\Big)\cdot\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx' \\
&\qquad + \varepsilon(h)\int_\omega \frac{\big(\tilde{R}^h(x')_{i3} - R^h(x')_{i3}\big)}{h}\nabla' g\Big(\frac{x'}{\varepsilon^2(h)}\Big)\cdot\nabla'\phi\Big(\frac{x'}{\varepsilon(h)}\Big)\psi(x')\,dx'.
\end{aligned}
$$

Therefore, it is infinitesimal due to (4.2.9), (4.2.88) and the fact that $\gamma_2 = +\infty$.

Claim (4.2.89) follows now by (4.2.92) and (4.2.93).

Case 3, Step 3: Identification of E.

Performing the same computation as in Case 1, Step 3, and combining (4.2.65) with (4.2.78), and (4.2.87), we obtain

$$
\begin{aligned}
&R(x')G(x,y,z) - \int_Q R(x')G(x,y,z)\,dz \\
&\quad = \big(\nabla_z v(x',y,z) + x_3\nabla_z T(x',y,z)e_3 + \nabla_z\hat{\phi}_2(x,y,z)|0\big)
\end{aligned}
$$

for a.e. $x \in \Omega$, and $y, z \in Q$, where $v, Te_3 \in L^2(\omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\hat{\phi}_2 \in L^2(\Omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$.

Thus, by (4.2.13),

$$
E(x,y,z) - \int_Q E(x,y,z)\,dz = \mathrm{sym}\,(\nabla_z\phi(x,y,z)|0)
$$

for a.e. $x \in \Omega$, and $y, z \in Q$, where $\phi := R^T(v + x_3 Te_3 + \hat{\phi}_2)$. In view of (4.2.64) we conclude that

$$E(x,y,z) = \begin{pmatrix} x_3\Pi^u(x') + \operatorname{sym} B(x') & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \operatorname{sym}\nabla_y\xi(x,y) + x_3\nabla_y^2\eta(x',y) & g_1(x,y) \\ & g_2(x,y) \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y) \end{pmatrix} + \operatorname{sym}(\nabla_z\phi(x,y,z)|0)$$

for a.e. $x \in \Omega$, and $y, z \in Q$, where $B \in L^2(\omega;\mathbb{M}^{2\times 2})$, $\xi \in L^2(\Omega; W^{1,2}_{\rm per}(Q;\mathbb{R}^2))$, $\eta \in L^2(\omega; W^{2,2}_{\rm per}(Q))$, $g_i \in L^2(\Omega \times Y)$, $i = 1,2,3$, and $\phi \in L^2(\Omega\times Q; W^{1,2}_{\rm per}(Q;\mathbb{R}^3))$. The thesis follows now by (4.2.3). □

4.3 The Γ-liminf inequality

With the identification of the limit linearized stress obtained in Section 4.2, we now find a lower bound for the effective limit energy associated to sequences of deformations with uniformly small three-dimensional elastic energies, satisfying (1.2.3).

Theorem 4.3.1. *Let $\gamma_1 \in [0,+\infty]$ and let $\gamma_2 = +\infty$. Let $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations satisfying the uniform energy estimate* (4.1.1) *and converging to $u \in W^{2,2}(\omega;\mathbb{R}^3)$ as in Theorem 2.3.4. Then,*

$$\liminf_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} \geq \frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\rm hom}(\Pi^u(x'))\,dx',$$

where Π^u is the map defined in (4.2.4)*, and*

(a) if $\gamma_1 = 0$, for every $A \in \mathbb{M}^{2\times 2}_{\rm sym}$

$$\begin{aligned}\overline{\mathscr{Q}}^{0}_{\rm hom}(A) := \inf\Bigg\{ \int_{\left(-\frac12,\frac12\right)\times Q} \mathscr{Q}_{\rm hom}\Bigg(y, \begin{pmatrix} x_3 A + B & 0 \\ 0 & 0\end{pmatrix} \\ + \operatorname{sym}\begin{pmatrix} \operatorname{sym}\nabla_y\xi(x_3,y) + x_3\nabla^2_y\eta(y) & g_1(x_3,y) \\ & g_2(x_3,y) \\ g_1(x_3,y) \quad g_2(x_3,y) & g_3(x_3,y)\end{pmatrix}\Bigg)\, dx_3 dy : \\ \xi \in L^2\big(\big(-\tfrac12,\tfrac12\big); W^{1,2}_{\rm per}(Q;\mathbb{R}^2)\big),\ \eta \in W^{2,2}_{\rm per}(Q), \\ g_i \in L^2\big(\big(-\tfrac12,\tfrac12\big)\times Q\big), i = 1,2,3,\ B \in \mathbb{M}^{2\times 2}_{\rm sym}\Bigg\};\end{aligned} \tag{4.3.1}$$

(b) if $0 < \gamma_1 < +\infty$, for every $A \in \mathbb{M}^{2\times 2}_{\rm sym}$

$$\overline{\mathscr{Q}}^{\gamma_1}_{\rm hom}(A) := \inf\Bigg\{ \int_{\left(-\frac12,\frac12\right)\times Q} \mathscr{Q}_{\rm hom}\Big(y, \begin{pmatrix} x_3 A + B & 0 \\ 0 & 0\end{pmatrix} \tag{4.3.2}$$

$$+ \operatorname{sym}\left(\nabla_y \phi_1(x_3, y)\Big|\frac{\partial_{x_3}\phi_1(x_3, y)}{\gamma_1}\right)\Bigg) dx_3 dy :$$

$$\phi_1 \in W^{1,2}\left((-\tfrac{1}{2}, \tfrac{1}{2}); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right),\ B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}\Bigg\};$$

(c) if $\gamma_1 = +\infty$, for every $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$

$$\overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(A) := \inf\Bigg\{ \int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \mathscr{Q}_{\mathrm{hom}}\Bigg(y, \begin{pmatrix} x_3 A + B & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3.3}$$

$$+ \operatorname{sym}\left(\nabla_y \phi_1(x_3, y)|d(x_3)\right)\Bigg) dx_3 dy :\ d \in L^2((-\tfrac{1}{2}, \tfrac{1}{2}); \mathbb{R}^3),$$

$$\phi_1 \in L^2((-\tfrac{1}{2}, \tfrac{1}{2}); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)),\ \textit{and } B \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}\Bigg\}$$

where

$$\mathscr{Q}_{\mathrm{hom}}(y, C) := \inf\left\{ \int_Q \mathscr{Q}\big(y, z, C + \operatorname{sym}\left(\nabla\phi_2(z)\big|0\right)\big)\, dz : \phi_2 \in W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right\} \tag{4.3.4}$$

for a.e. $y \in Q$, and for every $C \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}$.

Proof. The proof is an adaptation of [57, Proof of Theorem 2.4]. For the convenience of the reader, we briefly sketch it in the case $0 < \gamma_1 < +\infty$. The proof in the cases $\gamma_1 = +\infty$ and $\gamma_1 = 0$ is analogous.

Without loss of generality, we can assume that $⨍_\Omega u^h(x)\, dx = 0$. By assumption $(H2)$ and by Theorem 2.3.4, $u \in W^{2,2}(\omega;\mathbb{R}^3)$ is an isometry, with

$$u^h \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3)$$

and

$$\nabla_h u^h \to (\nabla' u|n_u) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}),$$

where the vector n_u is defined according to (2.3.3) and (2.3.4). By Theorem 4.2.1 there exists $E \in L^2(\Omega \times Q \times Q;\mathbb{M}^{3\times 3})$ such that, up to the extraction of a (not relabeled) subsequence,

$$E^h := \frac{\sqrt{(\nabla_h u^h)^T \nabla_h u^h} - Id}{h} \overset{dr-3-s}{\longrightarrow} E \quad \text{weakly dr-3-scale},$$

with

$$E(x, y, z) = \begin{pmatrix} \operatorname{sym} B(x') + x_3 \Pi^u(x') & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3.5}$$

$$+\operatorname{sym}\Big(\nabla_y\phi_1(x,y)\Big|\frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\Big)+\operatorname{sym}\big(\nabla_z\phi_2(x,y,z)\big|0\big),$$

for a.e. $x'\in\omega$, and $y,z\in Q$, where $B\in L^2(\omega;\mathbb{M}^{2\times 2})$, $\phi_1\in L^2(\omega;W^{1,2}((-\frac{1}{2},\frac{1}{2});W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$, and $\phi_2\in L^2(\omega\times Q;W^{1,2}_{\text{per}}(Q;\mathbb{R}^3))$. Arguing as in [45, Proof of Theorem 6.1 (i)], by performing a Taylor expansion around the identity, and by Lemma 2.4.11 we deduce that

$$\begin{aligned}\liminf_{h\to 0}\frac{\mathcal{E}^h(u^h)}{h^2}&\geq\liminf_{h\to 0}\int_\Omega\mathscr{Q}\Big(\frac{x'}{\varepsilon(h)},\frac{x'}{\varepsilon^2(h)},E^h(x)\Big)\,dx\\&\geq\int_\Omega\int_Q\int_Q\mathscr{Q}(y,z,E(x,y,z))\,dz\,dy\,dx.\end{aligned}$$

By (4.3.2), (4.3.4), and (4.3.5), we finally conclude that

$$\begin{aligned}\liminf_{h\to 0}\frac{\mathcal{E}^h(u^h)}{h^2}&\geq\int_\Omega\int_Q\mathscr{Q}_{\text{hom}}\Bigg(y,\begin{pmatrix}\operatorname{sym}B(x')+x_3\Pi^u(x')&0\\0&0\end{pmatrix}\\&\qquad+\operatorname{sym}\Bigg(\nabla_y\phi_1(x,y)\Big|\frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\Bigg)\Bigg)\,dy\,dx\\&\geq\int_\Omega\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}(x_3\Pi^u(x'))\,dx=\int_\Omega x_3^2\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}(\Pi^u(x'))\,dx\\&=\tfrac{1}{12}\int_\omega\overline{\mathscr{Q}}^{\gamma_1}_{\text{hom}}(\Pi^u(x'))\,dx'.\end{aligned}$$

□

4.4 The Γ-limsup Inequality: Construction of the Recovery Sequence

Let $W^{2,2}_R(\omega;\mathbb{R}^3)$ be the set of all $u\in W^{2,2}(\omega;\mathbb{R}^3)$ satisfying (2.3.3). Let $\mathcal{A}(\omega)$ be the set of all $u\in W^{2,2}_R(\omega;\mathbb{R}^3)\cap C^\infty(\bar{\omega};\mathbb{R}^3)$ such that, for all $B\in C^\infty(\bar{\omega};\mathbb{M}^{2\times 2}_{\text{sym}})$ with $B=0$ in a neighborhood of

$$\{x'\in\omega:\Pi^u(x')=0\}$$

(where Π^u is the map defined in (4.2.4)), there exist $\alpha\in C^\infty(\bar{\omega})$ and $g\in C^\infty(\bar{\omega};\mathbb{R}^2)$ such that

$$B=\operatorname{sym}\nabla'g+\alpha\Pi^u. \tag{4.4.1}$$

Remark 4.4.1. *Note that for $u\in W^{2,2}_R(\omega;\mathbb{R}^3)\cap C^\infty(\bar{\omega};\mathbb{R}^3)$, condition* (4.4.1) *(see [57, Lemmas 4.3 and 4.4]), is equivalent to writing*

$$B=\operatorname{sym}\left((\nabla'u)^T\nabla'V\right) \tag{4.4.2}$$

for some $V \in C^\infty(\bar{\omega};\mathbb{R}^3)$ (see [83, Lemmas 4.3 and 4.4]).

Indeed, (4.4.2) *follows from* (4.4.1) *setting*

$$V := (\nabla' u)g + \alpha n_u,$$

and in view of the cancellations due to (2.3.3)*. In fact, by* (2.3.3) *we have*

$$\begin{aligned}&(\nabla' u)^T \nabla' u = Id\\ &(\nabla' u)^T \partial_1(\nabla' u) = (\nabla' u)^T \partial_2(\nabla' u) = 0\\ &(\nabla' u)^T n_u = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{sym}\big((\nabla' u)^T(\nabla' u)g\big) &= \operatorname{sym}\big((\nabla' u)^T(\partial_1(\nabla' u)g + (\nabla' u)\partial_1 g|\, \partial_2(\nabla' u)g + (\nabla' u)\partial_2 g)\big)\\ &= \operatorname{sym}\nabla' g,\end{aligned}$$

and

$$\begin{aligned}\operatorname{sym}\big((\nabla' u)^T\nabla'(\alpha n_u)\big) &= \operatorname{sym}\big((\nabla' u)^T((\partial_1\alpha)n_u|\,(\partial_2\alpha)n_u)\big) + \alpha\Pi^u\\ &= \alpha\Pi^u.\end{aligned}$$

Conversely, (4.4.1) *is obtained from* (4.4.2) *defining $g := (\nabla' u)^T V$ and $\alpha := V \cdot n_u$.*

A key tool in the proof of the limsup inequality (1.2.4) is the following lemma, which has been proved in [57, Lemma 4.3] (see also [55], [57], [56], [75], and [76]). Again, the arguments in the previous sections of this chapter continue to hold if ω is a bounded Lipschitz domain. The piecewise C^1-regularity of $\partial\omega$ is necessary for the proof of the limsup inequality (1.2.4) (although it can be slightly relaxed as in [57]), since it is required in order to obtain the following density result.

Lemma 4.4.2. *The set $\mathcal{A}(\omega)$ is dense in $W^{2,2}_R(\omega;\mathbb{R}^3)$ in the strong $W^{2,2}$ topology.*

Before we prove the limsup inequality (1.2.4), we state a lemma and a corollary that guarantee the continuity of the relaxations (defined in (4.3.2)–(4.3.4)) of the quadratic map $\mathscr{Q}$ introduced in (H4). The proof of Lemma 4.4.3 is a combination of [57, Proof of Lemma 4.2], [73, Proof of Lemma 2.10] and [83, Lemma 4.2]. Corollary 4.4.4 is a direct consequence of Lemma 4.4.3.

Lemma 4.4.3. *Let $\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}$ and $\mathscr{Q}_{\mathrm{hom}}$ be the maps defined in* (4.3.1)-(4.3.4)*, and let $\gamma_2 = +\infty$.*

(i) Let $0 < \gamma_1 < +\infty$. Then for every $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ there exists a unique pair

$$(B, \phi_1) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \times W^{1,2}((-\tfrac{1}{2},\tfrac{1}{2}); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$$

with

$$\int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \phi_1(x_3,y)\,dy\,dx_3 = 0,$$

such that

$$\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(A) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \mathscr{Q}_{\mathrm{hom}}\Big(y, \begin{pmatrix} x_3A+B & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{sym}\Big(\nabla_y\phi_1(x_3,y)\Big|\frac{\partial_{x_3}\phi_1(x_3,y)}{\gamma_1}\Big)\Big)\,dx_3dy.$$

The induced mapping

$$A\in\mathbb{M}^{2\times 2}_{\mathrm{sym}} \mapsto (B(A),\phi_1(A)) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}} \times W^{1,2}((-\tfrac{1}{2},\tfrac{1}{2});W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$$

is bounded and linear.

(ii) Let $\gamma_1=+\infty$. Then for every $A\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ there exists a unique triple

$$(B,d,\phi_1)\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}\times L^2((-\tfrac{1}{2},\tfrac{1}{2});\mathbb{R}^3)\times L^2((-\tfrac{1}{2},\tfrac{1}{2});W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$$

with

$$\int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \phi_1(x_3,y)\,dy\,dx_3 = 0,$$

such that

$$\overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(A) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3A+B & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{sym}\left(\nabla_y\phi_1(x_3,y)|d(x_3)\right)\right)dx_3dy.$$

The induced mapping $A\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}\mapsto(B(A),d(A),\phi_1(A))\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}\times L^2((-\frac{1}{2},\frac{1}{2});\mathbb{R}^3)\times L^2((-\frac{1}{2},\frac{1}{2});W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ is bounded and linear.

(iii) Let $\gamma_1=0$. Then for every $A\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ there exists a unique 6-tuple

$$(B,\xi,\eta,g_1,g_2,g_3)$$

with $B\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}$, $\xi\in L^2\big((-\frac{1}{2},\frac{1}{2});W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2)\big)$, $\eta\in W^{2,2}_{\mathrm{per}}(Q)$, $g_i\in L^2\big((-\frac{1}{2},\frac{1}{2});\times Q\big)$, $i=1,2,3$, such that

$$\overline{\mathscr{Q}}^{0}_{\mathrm{hom}}(A) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y, \begin{pmatrix} x_3A+B & 0 \\ 0 & 0 \end{pmatrix}\right.$$

$$+\mathrm{sym}\left(\begin{pmatrix} \mathrm{sym}\,\nabla_y\xi(x_3,y)+x_3\nabla_y^2\eta(y) & & g_1(x_3,y) \\ & & g_2(x_3,y) \\ g_1(x_3,y) & g_2(x_3,y) & g_3(x_3,y)\end{pmatrix}\right)\, dx_3dy.$$

The induced mapping

$$A\mapsto (B(A),\xi(A),\eta(A),g_1(A),g_2(A),g_3(A))$$

from $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ *to* $\mathbb{M}^{2\times 2}_{\mathrm{sym}}\times L^2((-\frac{1}{2},\frac{1}{2});\mathbb{R}^3)\times W^{2,2}_{\mathrm{per}}(Q)\times L^2((-\frac{1}{2},\frac{1}{2})\times Q;\mathbb{R}^3)$ *is bounded and linear.*

For a.e. $y\in Q$ *and for every* $C\in\mathbb{M}^{3\times 3}_{\mathrm{sym}}$ *there exists a unique* $\phi_2\in W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)$*, with* $\int_Q\phi_2(z)\,dz=0$*, such that*

$$\mathscr{Q}_{\mathrm{hom}}(y,C)=\int_Q\mathscr{Q}\left(y,z,C+\mathrm{sym}\left(\nabla\phi_2(z)\middle|0\right)\right).$$

The induced mapping

$$C\in\mathbb{M}^{3\times 3}_{\mathrm{sym}}\mapsto\phi_2(C)\in W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)$$

is bounded and linear. Furthermore, the induced operator

$$P:L^2\left((-\tfrac{1}{2},\tfrac{1}{2})\times Q;\mathbb{M}^{3\times 3}\right)\to L^2\left((-\tfrac{1}{2},\tfrac{1}{2})\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right),$$

defined as

$$P(C):=\phi_2(C)\quad\textit{for every } C\in L^2\left((-\tfrac{1}{2},\tfrac{1}{2})\times Q;\mathbb{M}^{3\times 3}\right)$$

is bounded and linear.

Corollary 4.4.4. *Let* $\gamma_1\in[0,+\infty]$*. The map* $\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}$ *is continuous, and there exist* $c_1(\gamma_1)\in(0,+\infty)$ *such that*

$$\frac{1}{c_1}|F|^2\leq\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(F)\leq c_1|F|^2$$

for every $F\in\mathbb{M}^{2\times 2}_{\mathrm{sym}}$*.*

(i) If $0<\gamma_1<+\infty$*, then for every* $A\in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ *there exists a unique triple* $(B,\phi_1,\phi_2)\in L^2\left(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}\right)\times L^2\left(\omega;W^{1,2}\left(\left(-\frac{1}{2},\frac{1}{2}\right);W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)\right)\right)\times L^2(\Omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ *such that*

$$\frac{1}{12}\int_\omega\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(A(x'))\,dx'=\int_\Omega\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(x_3A(x'))\,dx$$
$$=\int_{\Omega\times Q}\mathscr{Q}_{\mathrm{hom}}\left(y,\begin{pmatrix} x_3A(x')+B(x') & 0\\ 0 & 0\end{pmatrix}\right)$$

$$
\begin{aligned}
&+ \operatorname{sym}\left(\nabla_y \phi_1(x,y)\Big|\frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\right)\bigg)\, dy\, dx \\
&= \int_{\Omega\times Q\times Q} \mathscr{Q}\Bigg(y,z,\begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \operatorname{sym}\left(\nabla_y \phi_1(x,y)\Big|\frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\right) \\
&+ \operatorname{sym}\left(\nabla_z \phi_2(x,y,z)|\,0\right)\Bigg)\, dz\, dy\, dx.
\end{aligned}
$$

(ii) If $\gamma_1 = +\infty$, then for every $A \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ there exists a unique 4-tuple $(B,d,\phi_1,\phi_2) \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3)) \times L^2(\Omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ such that

$$
\begin{aligned}
&\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(A(x'))\, dx' = \int_\Omega \overline{\mathscr{Q}}^{\infty}_{\mathrm{hom}}(x_3 A(x'))\, dx' \\
&= \int_{\Omega\times Q} \mathscr{Q}_{\mathrm{hom}}\left(y,\begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} + \operatorname{sym}\left(\nabla_y \phi_1(x,y)|d(x)\right)\right) dy\, dx \\
&= \int_{\Omega\times Q\times Q} \mathscr{Q}\Bigg(y,z,\begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} + \operatorname{sym}\left(\nabla_y \phi_1(x,y)|d(x)\right) \\
&+ \operatorname{sym}\left(\nabla_z \phi_2(x,y,z)|\,0\right)\Bigg)\, dz\, dy\, dx.
\end{aligned}
$$

(iii) If $\gamma_1 = 0$, then for every $A \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ there exists a unique 7-tuple $(B,\xi,\eta,g_1,g_2,g_3,\phi) \in L^2(\omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times L^2(\Omega;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^2)) \times L^2(\Omega;W^{2,2}_{\mathrm{per}}(Q)) \times L^2(\Omega\times Q;\mathbb{R}^3) \times L^2(\Omega\times Q;W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, such that

$$
\begin{aligned}
&\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{0}_{\mathrm{hom}}(A(x'))\, dx' = \int_\Omega \overline{\mathscr{Q}}^{0}_{\mathrm{hom}}(x_3 A(x'))\, dx' \\
&= \int_{\Omega\times Q} \mathscr{Q}_{\mathrm{hom}}\Bigg(y,\begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \operatorname{sym}\begin{pmatrix} \operatorname{sym}\nabla_y \xi(x,y) + x_3 \nabla^2_y \eta(x',y) & \begin{matrix} g_1(x,y) \\ g_2(x,y) \end{matrix} \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y) \end{pmatrix}\Bigg) \\
&= \int_{\Omega\times Q\times Q} \mathscr{Q}\Bigg(y,z,\begin{pmatrix} x_3 A(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \operatorname{sym}\begin{pmatrix} \operatorname{sym}\nabla_y \xi(x,y) + x_3 \nabla^2_y \eta(x',y) & \begin{matrix} g_1(x,y) \\ g_2(x,y) \end{matrix} \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y) \end{pmatrix} \\
&+ \operatorname{sym}\left(\nabla_z \phi_2(x,y,z)\big|0\right)\Bigg)\, dz\, dy\, dx.
\end{aligned}
$$

We now prove that the lower bound obtained in Section 4.3 is optimal.

Theorem 4.4.5. *Let* $\gamma_1 \in [0, +\infty]$. *Let* $\overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}$ *and* $\mathscr{Q}_{\mathrm{hom}}$ *be the maps defined in* (4.3.1)–(4.3.4)*, let* $u \in W^{2,2}_R(\omega;\mathbb{R}^3)$ *and let* Π^u *be the map introduced in* (4.2.4)*. Then there exists a sequence* $\{u^h\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *such that*

$$\limsup_{h\to 0} \frac{\mathcal{E}^h(u^h)}{h^2} \leq \frac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(\Pi^u(x'))\,dx'. \tag{4.4.3}$$

Proof. The proof is an adaptation of [57, Proof of Theorem 2.4] and [83, Proof of Theorem 2.4]. We outline the main steps in the cases $0 < \gamma_1 < +\infty$ and $\gamma_1 = 0$ for the convenience of the reader. The proof in the case $\gamma_1 = +\infty$ is analogous.

Case 1: $0 < \gamma_1 < +\infty$ and $\gamma_2 = +\infty$.
By Lemma 4.4.2 and Corollary 4.4.4 it is enough to prove the theorem for $u \in \mathcal{A}(\omega)$. By Corollary 4.4.4 there exist $B \in L^2(\omega;\mathbb{M}^{2\times 2})$, $\phi_1 \in L^2(\omega; W^{1,2}((-\frac{1}{2},\frac{1}{2}); W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$, and $\phi_2 \in L^2(\Omega\times Q; W^{1,2}_{\mathrm{per}}(Q;\mathbb{R}^3))$ such that

$$\begin{aligned}
&\tfrac{1}{12}\int_\omega \overline{\mathscr{Q}}^{\gamma_1}_{\mathrm{hom}}(\Pi^u(x'))\,dx'\\
&\quad= \int_\Omega\int_Q\int_Q \mathscr{Q}\left(y, z, \begin{pmatrix} \operatorname{sym} B(x') + x_3\Pi^u(x') & 0\\ 0 & 0\end{pmatrix}\right.\\
&\qquad \left. +\operatorname{sym}\left(\nabla_y\phi_1(x,y)\middle|\frac{\partial_{x_3}\phi_1(x,y)}{\gamma_1}\right) + \operatorname{sym}\left(\nabla_z\phi_2(x,y,z)\middle|0\right)\right)\,dz\,dy\,dx.
\end{aligned}$$

Since B depends linearly on Π^u by Lemma 4.4.3, in particular there holds

$$\{x' : \Pi^u(x') = 0\} \subset \{x' :\ B(x') = 0\}.$$

By Lemma 4.4.3, we can argue by density and we can assume that $B \in C^\infty(\bar{\omega};\mathbb{M}^{2\times 2})$, $B = 0$ in a neighborhood of $\{x' : \Pi^u(x') = 0\}$, $\phi_1 \in C^\infty_c(\omega; C^\infty((-\frac{1}{2},\frac{1}{2}); C^\infty(Q;\mathbb{R}^3))$, and $\phi_2 \in C^\infty_c(\omega\times Q; C^\infty(Q;\mathbb{R}^3))$. In addition, since $u \in \mathcal{A}(\omega)$, by (4.4.1) there exist $\alpha \in C^\infty(\bar{\omega})$, and $g \in C^\infty(\bar{\omega};\mathbb{R}^2)$ such that

$$B = \operatorname{sym}\nabla' g + \alpha\Pi^u.$$

Set

$$\begin{aligned}
&v^h(x) := u(x') + h((x_3 + \alpha(x'))n_u(x') + (g(x')\cdot\nabla')y(x')),\\
&R(x') := (\nabla' u(x')|n_u(x')),\\
&b(x') := -\begin{pmatrix}\partial_{x_1}\alpha(x')\\ \partial_{x_2}\alpha(x')\end{pmatrix} + \Pi^u(x')g(x'),
\end{aligned}$$

and let

$$u^h(x) := v^h(x') + h\varepsilon(h)\tilde{\phi}_1\Big(x, \frac{x'}{\varepsilon(h)}\Big) + h\varepsilon^2(h)\tilde{\phi}_2\Big(x, \frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}\Big)$$

for a.e. $x \in \Omega$, where

$$\tilde{\phi}_1 := R\Big(\phi_1 + \gamma_1 x_3 \begin{pmatrix} b \\ 0 \end{pmatrix}\Big) \quad \text{and} \quad \tilde{\phi}_2 := R\phi_2.$$

Arguing similarly to [57, Proof of Theorem 2.4 (upper bound)], it can be shown that (4.4.3) holds.

Case 2: $\gamma_1 = 0$ **and** $\gamma_2 = +\infty$.
By Lemma 4.4.2 and Corollary 4.4.4 it is enough to prove the theorem for $u \in \mathcal{A}(\omega)$. By Corollary 4.4.4 there exist $B \in L^2(\omega; \mathbb{M}^{2\times 2}_{\text{sym}})$, $\xi \in L^2(\Omega; W^{1,2}_{\text{per}}(Q; \mathbb{R}^2))$, $\eta \in L^2(\Omega; W^{2,2}_{\text{per}}(Q))$, $g_i \in L^2(\Omega \times Y)$, $i = 1, 2, 3$, and $\phi \in L^2(\Omega \times Q; W^{1,2}_{\text{per}}(Q; \mathbb{R}^3))$ such that

$$\begin{aligned}
&\frac{1}{12}\int_\omega \overline{\mathscr{Q}}^0_{\text{hom}}(\Pi^u(x'))\,dx' \\
&\quad = \int_{\Omega\times Q\times Q} \mathscr{Q}\left(y, z, \begin{pmatrix} x_3\Pi^u(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix}\right. \\
&\quad\quad + \text{sym}\begin{pmatrix} \text{sym}\,\nabla_y \xi(x,y) + x_3 \nabla_y^2 \eta(x', y) & g_1(x,y) \\ & g_2(x,y) \\ g_1(x,y) \quad g_2(x,y) & g_3(x,y) \end{pmatrix} \\
&\quad\quad \left. + \,\text{sym}\,(\nabla_z \phi_2(x,y,z)|\,0)\right) dz\,dy\,dx.
\end{aligned}$$

By the linear dependence of B on Π^u, in particular there holds

$$\{x' : \Pi^u(x') = 0\} \subset \{x' :\ B(x') = 0\}.$$

By density, we can assume that $B \in C^\infty(\bar{\omega}; \mathbb{M}^{2\times 2})$, $\xi \in C^\infty_c(\omega; C^\infty_{\text{per}}(Q; \mathbb{R}^2))$, $\eta \in C^\infty_c(\omega; C^\infty_{\text{per}}(Q))$, and $g_i \in C^\infty_c(\omega; C^\infty_{\text{per}}\big(\big(-\frac{1}{2}, \frac{1}{2}\big) \times Q\big))$, $i = 1, 2, 3$. Since $u \in \mathcal{A}(\omega)$, by (4.4.2) there exists a displacement $V \in C^\infty(\bar{\omega}; \mathbb{R}^2)$ such that

$$B = \text{sym}\,((\nabla' u)^T \nabla' V).$$

Set

$$\begin{aligned}
&v^h(x) := u(x') + hx_3 n_u(x') = h(V(x') + hx_3\mu(x')), \\
&\mu(x') := (Id - n_u(x') \otimes n_u(x'))(\partial_1 V(x') \wedge \partial_2 u(x') + \partial_1 u(x') \wedge \partial_2 V(x')), \\
&R(x') := (\nabla' u(x')|n_u(x')),
\end{aligned}$$

and let

$$u^h(x) := v^h(x) - \varepsilon^2(h) n_u(x') \eta\Big(x', \frac{x'}{\varepsilon(h)}\Big)$$

$$
\begin{aligned}
&+ h\varepsilon^2(h)x_3R(x')\begin{pmatrix} \partial_{x_1}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) + \frac{1}{\varepsilon(h)}\partial_{y_1}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) \\ \partial_{x_2}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) + \frac{1}{\varepsilon(h)}\partial_{y_2}\eta\left(x', \frac{x'}{\varepsilon(h)}\right) \end{pmatrix} \\
&+ h\varepsilon(h)R(x')\begin{pmatrix} \xi\left(x', \frac{x'}{\varepsilon(h)}\right) \\ 0 \end{pmatrix} \\
&+ h^2\int_{-\frac{1}{2}}^{x_3} R(x')g\left(x', t, \frac{x'}{\varepsilon(h)}\right)dt + h\varepsilon^2(h)R(x')\phi\left(x, \frac{x'}{\varepsilon(h)}, \frac{x'}{\varepsilon^2(h)}\right),
\end{aligned}
$$

for a.e. $x \in \Omega$. The proof of (4.4.3) is a straightforward adaptation of [83, Proof of Theorem 2.4 (Upper bound)]. □

Proof of Theorem 1.2.1. Theorem 1.2.1 follows now by Theorem 4.3.1 and Theorem 4.4.5. □

Chapter 5

Future Research Directions

Future research related to the topics in Chapter 4 will include:

- Studying time dependent dimension reduction problem with homogenization of the material, in the sense of [1,2]. In these papers the authors study the dynamic equation of nonlinear elasticity

$$\partial_\tau^2 u - \mathrm{div}_x DW(\nabla u) = f^h \quad \text{in } [0, \tau_h] \times \Omega_h.$$

  It would be interesting to modify the equation such that W also depends on $x/\varepsilon(h)$ and study the limit at $h \to 0$.

- Studying varied preferred strain spatially instead of a varied elastic energy as related to tissue growth and inhomogeneous swelling of materials. Some analysis of these problems can be found in [37, 58, 64, 65, 68], however it has yet to be studied for fast periodically varying preferred strains.

- Considering minimizing over functions other than gradients in the sense of $\mathcal{A}$–quasiconvexity (see [20, 42, 43, 70]). In continuum mechanics and electromagnetism, linear PDEs other than curl $v = 0$ naturally arise (see [77–81]). One can work with $\mathcal{A}$ free fields, where the differential operator $\mathcal{A}$ is given by

$$\mathcal{A} : L^q(\Omega; \mathbb{R}^d) \to W^{(-1,q)}(\Omega; \mathbb{R}^l), \qquad \mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial u}{\partial x_i},$$

  $A^{(i)}$ are linear operators and $\mathcal{A}$ has constant rank. In particular, it would be interesting to explore these operators under appropriate rigidity bound growth assumptions, rather than the standard p-growth assumption that are generally found in the literature.

- Studying the resulting 2D models for the effect of wrinkling [12, 53] and folding for these homogenized material models.

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