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DOCTORAL THESIS

Convex Analysis of an Equation arising in Oil Reservoir Models

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...in fact, the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.

Tyrrell Rockafellar

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Chapter 1

Introduction

We consider a two-phase flow equation in a porous medium with capillarity. Our approach is novel because it is the first to take advantage of the convex structure of the capillary pressure. This formulation allows us to take advantage of the extremely well studied field of convex optimization in order to get fast numerical schemes. Convex optimization is a highly developed area of numerical analysis, and due to its importance in machine learning and "big data" is an active area of research [1, 2, 3, 4, 5]. Moreover, this formulation can be used to simplify the theory of multiphase flow [6, 7]

The field of convex optimization has been growing rapidly in recent years in order to provide applications for "big data" analysis and so there are many powerful theoretical and software tools available. One example is Nesterov's Accelerated gradient descent algorithm which we use in the numerical example below.

Another benefit of the convex approach is that we can prove existence and uniqueness of a general class of two-phase flow equations which exhibit very degenerate properties. Moreover, we can numerically approximate the solutions by minimizing an unconstrained convex function on $\mathbb{R}^n$. By a more general class of problems we mean that we allow the relative permeability curves to vanish even while the respective fluid content is positive. Physically, this represents the situation when the oil content of a pore becomes small enough so that the water no longer displaces

it. Due to the properties of the reservoir, the oil will stick to the rocks and become immobile once the relative volume of oil becomes small enough.

It is often assumed in the literature, see the example in [8], that the initial conditions for the oil and water fraction are above the immobile levels. The extremely degenerate situation which occures when the initial fluid content relative volumes can take on a range of values below the immobile level is not considered in the literature. However, our convex formulation can handle this problem under some natural conditions. We provide a proof of existence of uniqueness of minimizers to the convex functional which is topological in nature.

Perhaps the most well known numerical method to study these equations is the Implicit Pressure Explicit Saturation (IMPES) method [9]. This is a time-stepping scheme which first computes the saturation for the next step explicitly and then solves for one of the pressure by typically using a Newton scheme. This method is simpler and more efficient that other methods such as the SS method [10]. However, in order for the IMPES method to be stable it requires very small time steps for the saturation. This can be prohibitive for long time integration problems [9]. Our method eliminates this problem because we do not need any time stepping schemes for the saturation. Additionally, this eliminates the need for a Kirchoff transformation [11]. Instead, by locating a minimum of a convex functional, we solve for both the oil and water pressure at the same time. The saturation is then computed as a function of the pressure difference.

On the numerical side, much work has been done on this problem using finite elements methods. In particular, Riviere and Yotov have studied discontinuous Galerkin schemes and mixed finite element methods in this direction, respectively [12, 13]. Mary Wheeler, Dick Ewing, and Todd Arbogast have also made many contributions to this area [14, 15]. On the theoretical side, we have existence, uniqueness and regularity results by Chen [16, 17]. Another existence result has been shown by Alt and DiBennedetto [18]. Ralph Showalter has applied Monotone Operators to the

theory of two-phase flow [19].

1.1 Fluid Description

We consider a porous medium to be the closure of an open and bounded set $\Omega \subseteq \mathbb{R}^n$. Each point $x \in \Omega$ represents an average of the porous medium in a small region around x. In other words, a point x is not meant to represent a very tiny part of the medium (e.g. perhaps one water molecule), rather x is a large enough region in the medium so that we may talk about how much water and/or oil there is at x at a particular time.

We will consider the pressures p_i for $i = 1, 2$ of both the water and the oil. Due to the microscopic properties of water and oil molecules (e.g. cohesion) the pressures in the fluids do not tend to distribute evenly as in the case of a single pressure with a single fluid. Therefore, we must consider the pressure of each fluid separately. The functions $s_i(x)$, $i = 1, 2$ represent the volume fraction of each fluid at the point x. Further details about these equations can be found in the book by Bear [20].

The functions $k_i(s_i(x), x)$ represent the permeability curves of the fluids. These curves model how easily each fluid flows through the reservoir and depend on both the location of the reservoir and also the volume distribution of the fluids. They depend on the reservoir location because this is how we model the geometric properties of the pores. They depend on the fluid volume distribution because, for example, it is much more difficult for a small amount of water to flow through a region which is dominated by oil than through a region which is dominated by water. In fact, if the water volume content reaches a critically low level it will not flow at all and in that case we have $k_1(s_1) = 0$, if s_1 represents water. The same is true for the oil. These curves are determined experimentally and are approximately equal to $k_i(s_i) = (s - s_c)^2$ when $s_i > s_c$ and 0 otherwise, where s_c is the critical level which

is often around .2, depending of the type of reservoir which is being modeled. The terms q_i are the wells terms which model the pumping of water into the reservoir or the pumping of oil out of the reservoir. The term $1 - s_0(x)$ represents the porosity of the medium, or how much empty volume space there is in the rock at the point x. This is needed because the size of the holes and cracks which hold the fluids varies throughout the medium. We assume that $1 - s_0(x) \geq c > 0$. It is also assumed that $s^1 + s^2 = 1 - s_0$. This means that the fluid always fills the pores, i.e. there is no space left over.

1.2 Equations

The flow of water and oil through a porous medium is described by

$$\partial_t s^i - div(k_i(\nabla p^i + g)) = q^i, \;\; i = 1, 2$$

$$k_i = k_i(s^i(x), x)$$

$$s^1(x) + s^2(x) = 1 - s_0(x)$$

$$p_c(s^1) = p^1 - p^2$$

The meaning of the last equation is that the pressure difference is a monotone decreasing function of s^1. It is important to note here that this means that actually the terms s^i are functions of the pressure difference $p^1 - p^2$.

Remark : Since the s^i are actually functions of $p^1 - p^2$ it is very helpful to consider these equations only in the variable $\mathbf{p} = (p^1, p^2)$ and consider the equations with s as just constraints on p^1, p^2.

We will mainly be concerned with the discretized version of these equations, which we write below. For simplicity, we ignore the gravity term and note that this

can easily be put back in later. Also note that we have frozen the s^i term from the previous timestep.

$$(s^1_{t_{n+1}} - s^1_{t_n}) - \tau \, div(k_1(s^1_{t_n})\nabla p^1_{t_{n+1}}) = q_1$$

$$(s^2_{t_{n+1}} - s^2_{t_n}) - \tau \, div(k_2(s^2_{t_n})\nabla p^2_{t_{n+1}}) = q_2$$

$$s^1_{t_{n+1}} + s^2_{t_{n+1}} = 1 - s_0$$

$$p_c(s^1_{t_{n+1}}) = p^1_{t_{n+1}} - p^2_{t_{n+1}}$$

where $\tau = t_{n+1} - t_n$.

For simplicity, we will write s_n instead of s_{t_n} from now on.

1.3 Example

In order to illustrate the degeneracy of the equations, in this section we will give an example of a solution in one dimension.

Let $x, t \geq 0$, and let p_c be such that $p_c(1) = -c$ and $p_c(0) = c$

For $x < t$,

Let $p^1(x,t) = x$

$p^2(x,t) = x + c$

$s^1(x,t) = 1$

$s^2(x,t) = 0$

$k^1(x, s^1) = 1$

$k^2(x, s^2) = 0$

For $x \geq t$,

Let $p^1(x, t) = x + c$

$p^2(x, t) = x$

$s^1(x, t) = 0$

$s^2(x, t) = 1$

$k^1(x, s^1) = 0$

$k^2(x, s^2) = 1$

One can compute directly that this example is a solution to the equations in the sense of distributions.

1.4 Outline of Main Results

The main focus of this work is to develop a general convex analysis framework with which we can prove existence and uniqueness results about two-phase flow. This framework can be used to solve a wider class of problems than the previous theory was able to solve. In particular, the wider class of problems includes equations where the permeability curves k_i are more degenerate than before.

In chapter 3 we we will show how the solution of the equations above can seen as the minimum of a convex functional. Chapter 4 will discuss our chosen method of discretization. In chapter 5 we will prove uniqueness of minimizers of the functional when the permeability curves are only partially degenerate. We do this case separately because there are some additional results for the partially degenerate case which do not hold for the fully degenerate case. These terms will be defined later. In chapter 6 we will build the tools necessary to handle the fully degenerate case

and the remaining two chapters will prove existence and uniqueness in the fully degenerate case.

Chapter 2

Convex Analysis of Equations

In this chapter we will show how to set up a convex functional whose minimum is the solution of the equation above. First we will derive the convex term which controls the capillary pressure. The motivation for the function below comes from the constraint $s^1 + s^2 = 1 - s_0$, where $1 - s_0(x)$ is the porosity of the medium. The next two sections are **mostly for motivational purposes** and to see how to go about finding the right convex functional. The desired convex functional is given in section 2.3 and we show how it's minimum solves the discretized equations. See Rockafellar [21] for details about convex conjugates.

2.1 Computing the Conjugate

Let $\gamma : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be convex. Consider $\Psi(\mathbf{s}) := \gamma((s_1 - s_2 - 1 - s_0)/2) + I_{\{s_1+s_2=1-s_0\}}(\mathbf{s})$, where

$$I_{\{s_1+s_2=1-s_0\}}(\mathbf{s}) = \begin{cases} \infty \text{ if } s_1 + s_2 \neq 1 - s_0 \\ 0 \text{ if } s_1 + s_2 = 1 - s_0 \end{cases}$$

Then we have $\Psi(\mathbf{s}) < \infty$ if and only if $\mathbf{s} = (\xi, 1 - s_0 - \xi)$ for some $\xi \in \mathbb{R}$ so,

$$\begin{aligned}\Psi^*(\mathbf{p}) &= \sup_{\mathbf{s}}\{\mathbf{p}\cdot\mathbf{s} - \Psi(\mathbf{s})\} \\ &= \sup_{\xi\in\mathbb{R}}\{p_1\xi + p_2(1-\xi) - \gamma(\xi)\} \\ &= p_2 + \sup_{\xi\in\mathbb{R}}\{(p_1-p_2)\xi - \gamma(\xi)\} \\ &= p_2 + \gamma^*(p_1 - p_2).\end{aligned}$$

By the chain rule, we have

$$\partial\Psi^*(\mathbf{p}) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) + \partial\gamma^*(p_1-p_2)\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right).$$

Example: If $\gamma(\xi) = 1-\sqrt{\xi}+I_{[0,1-s_0]}(\xi)$ then the domain of $\partial\,\gamma$ is $D(\partial\gamma) = (0, 1-s_0]$ and

$$\partial\gamma(\xi) = \begin{cases} -1/(2\sqrt{\xi}) \text{ if } \xi \in (0, 1-s_0) \\ [-1/2\sqrt{1-s_0}, \infty) \quad \text{if } \xi = 1 - s_0 \end{cases}$$

Then $\partial\gamma^*$ is the inverse of $\partial\gamma$, so

$$\partial\gamma^*(\eta) = \begin{cases} 1/(4\eta^2) \text{ if } \eta \le -1/(2\sqrt{1-s_0}) \\ 1 - s_0 \qquad \text{if } \eta \ge -1/(2\sqrt{1-s_0}) \end{cases}$$

To compute the anti-derivative, note that $\gamma(\xi) \ge 0 = \gamma(1)$ implies that $\gamma^*(0) = 0$, so

$$\gamma^*(\eta) = \begin{cases} -\sqrt{s_0} - 1/(4\eta) \text{ if } \eta \le -1/(2\sqrt{1-s_0}) \\ \eta \qquad\qquad \text{if } \eta \ge -1/(2\sqrt{1-s_0}) \end{cases}$$

2.2 Relationship with Capillary Pressure

A typical capillary pressure used is the Brooks-Corey Capillary Pressure [12] which is

$$p_c(s_1) = p_d s_1^{-\theta} = p_2 - p_1$$

Here θ is often taken to be $\theta = \frac{1}{2}$ and p_d is the capillary pressure needed to displace the fluid from the largest pore. In general, γ^* is related to p_c in the sense that it will be the term in our convex functional that enforces the capillary pressures given by p_c. For example, taking $p_c(s_1) = \frac{1}{2}s_1^{-\frac{1}{2}} = p_2 - p_1$ and using the example for $\partial\gamma^*$ above, we have

$$\gamma^{*'}(p_1 - p_2) = \begin{cases} \dfrac{1}{4(p_1 - p_2)^2} \text{ if } p_1 - p_2 \leq -1/2 \\ \\ 1 \qquad\quad \text{if } p_1 - p_2 \geq -1/2 \end{cases}$$

But now from the capillary pressure,

$$\begin{aligned} p_c(s_1) = 1/2\sqrt{s_1} &= p_2 - p_1 \\ 1/4s_1 &= (p_2 - p_1)^2 \\ 4s_1 &= 1/(p_1 - p_2)^2 \\ s_1 &= 1/4(p_1 - p_2)^2. \end{aligned}$$

and so we may define

$$s^1 := \gamma^{*'}(p_1 - p_2)$$

and define

$$s^2 := 1 - s_0 - s^1.$$

This will become more clear after viewing the convex functional and its derivatives below.

2.3 Convex Functional

Continuing with the example from above, we consider the following convex functional:

$$\Psi^*(\mathbf{p}) = \int_\Omega \gamma^*(p_1 - p_2) - s_n^1(p_1 - p_2) + \frac{\tau}{2}k_1(s_n^1)|\nabla p_1|^2 + \frac{\tau}{2}k_2(s_n^2)|\nabla p_2|^2.$$

Observe that if we take the functional derivative of $\Psi^*(\mathbf{p})$ with respect to p^1 and set it equal to 0 we have

$$0 = \partial_{p_1}\Psi^*(\mathbf{p})[v] = \int_\Omega \gamma^{*'}(p_1 - p_2)v - s_n^1 v + k_1(s_n^1)\nabla p^1 \cdot \nabla v$$

Integrating by parts, varying over all smooth functions v, and using the relation

$$\gamma^{*'}(p_1 - p_2) = \frac{1}{4(p_1 - p_2)^2} = s_1$$

yields

$$\begin{aligned}\gamma^{*'}(p_1 - p_2) - s_n^1 &= div(k_1(s_n^1)\nabla p^1)\\ s^1 - s_n^1 &= div(k_1(s_n^1)\nabla p^1).\end{aligned}$$

Doing the same for p_2 we have

$$\begin{aligned} -\gamma^{*'}(p_1 - p_2) + s_n^1 &= div(k_2(s_n^2)\nabla p^2) \\ -s_1 + s_n^1 &= div(k_2(s_n^2)\nabla p^2) \\ -(1 - s_2) + (1 - s_n^2) &= div(k_2(s_n^2)\nabla p^2) \\ s_2 - s_n^2 &= div(k_2(s_n^2)\nabla p^2) \end{aligned}$$

Therefore the minimum of the functional above satisfies our first two equations:

$$\begin{aligned} s^1 - s_n^1 &= div(k_1(s_n^1)\nabla p^1) \\ s^2 - s_n^2 &= div(k_2(s_n^2)\nabla p^2) \end{aligned}$$

The last two equations:

$$\begin{aligned} p_1 - p_2 &= p_c(s_1) \\ s^1 + s^2 &= 1 - s_0 \end{aligned}$$

are satisfied by definition.

We now give an example of some reasonable permeability curves:

$$k_i(s^i) = \begin{cases} (s^i - c_i)^2 & s^i \geq c_i \\ 0 & s^i \leq c_i \end{cases}$$

We have shown that the Euler-Lagrange equations (see [22]) of the convex functional above are the equations for our PDE. For the rest of this thesis we will work with the fully discretized version of the PDE. Namely, we will discretize in time and space and do the discrete analog of the above. In the next chapter we will describe our discretization method.

Chapter 3

Discretization

In this chapter we describe a fully discrete approximation for the PDE in section 1.3 which we reproduce here. Additionally, we write the flux term q^i on the right hand side explcitly. This term models two types of wells, one which pumps water into the domain and another which allows both liquids to exit the domain. Here p_{bh} is a constant which is sometimes referred to as the bottom hole pressure of the well.

$$\partial_t s^\pi - div(k_\pi(s^\pi)(\nabla p^\pi + g)) = (p^\pi - p^\pi_{bh})\chi_{E^\pi}, \;\; \pi = 1, 2$$

$$k_\pi = k_\pi(s^\pi(x), x)$$

$$s^1(x) + s^2(x) = 1 - s_0(x)$$

$$p_c(s^1) = p^1 - p^2$$

Here χ is the standard characteristic function and E^1 and E^2 are the subsets of Ω which represent the wells or holes in the porous rock. A typical example might be that $\pi = 1$ represents water, $\pi = 2$ represents oil, $\Omega = [0, 1] \times [0, 1]$, E^1 is the union of two small balls one centered at $(0, 0)$ another centered at $(1, 1)$, E^2 is a small ball centered at $(1, 1)$, and $p^1_{bh} > p^2_{bh}$. The idea is that water is being pumped into a small hole near $(0, 0)$ and both oil and water (hopefully mostly oil), is coming out at $(1, 1)$.

Next we define the notion of a Voronoi diagram.

3.1 Voronoi Diagrams

Definition 3.1.1. *Let $\mathcal{N}$ be a set of n points in $\Omega \subset \mathbb{R}^m$ (also will be called nodes or Voronoi generators).*

Definition 3.1.2. *A Voronoi diagram of Ω with respect to $\mathcal{N} = \{x_1, ..., x_n\}$ is a partition of Ω into a collection of n Voronoi cells $\{C_i\}_{i\in 1,...,n}$ such that for each $x \in C_k$ we have that $d(x, x_k) \leq d(x, x_j)$ for all $j \in \{1, ..., n\}$, where d denotes the Euclidean distance function.*

Definition 3.1.3. *Let $h_{i,j}$ denote the pairwise distances between the points x_i, x_j in $\mathcal{N}$. Let $A_{i,j}$ denote the length of the (unique) edge (or face when $d = 3$) between two adjacent cells. If two Voronoi cells are not adjacent then we define $A_{i,j} = 0$. Let $|V_i|$ denote the volume of the i^{th} Voronoi cell.*

3.2 Approximation Process

Given a Voronoi diagram of Ω with respect to $\mathcal{N}$, we integrate the PDE by parts over each Voronoi cell and then approximate each of the integral terms using the approximation from [23]. We assume that there exists a classical solution. Ignoring the gravity term g and letting $e_{i,j}$ denote the edge between cells i and j, we approximate the first integral term

$$\int_{e_{i,j}} k_\pi(s_n^\pi)\nabla p^\pi \cdot n$$

by

$$\frac{k_\pi(s^\pi(t_n, x_i) + k_\pi(s^\pi(t_n, x_j)}{2}(p^\pi(t_{n+1}, x_i) - p^\pi(t_{n+1}, x_j))\frac{A_{i,j}}{h_{i,j}},$$

and we approximate the second integral term

$$\int_{C_i} (\partial_t s^\pi)$$

on each cell C_i by

$$|C_i|\frac{s^\pi(t_{n+1}, x_i) - s^\pi(t_n, x_i)}{\tau}.$$

Here $s^\pi(t_n, x_j)$ represents the value of $s^\pi_{t_n}$ at the j^{th} node in Ω.

We approximate

$$\int_{C_i} p^\pi - p^\pi_{bh}$$

by

$$|C_i|(p^\pi_i - p^\pi_{bh}).$$

Therefore, we define our approximation as the solution of the following system of equations, for $i = 1, ..., n$. Note that there is one equation for each Voronoi cell. Here $j \in N(i)$ means that cell C_j is a neighbor of cell C_i. Here λ is the Lebesgue measure.

$$|C_i|(s^1(t_{m+1}, x_i) - s^1(t_m, x_i)) - \tau \sum_{j \in N(i)} \frac{k_1(s^1(t_m, x_i)) + k_1(s^1(t_m, x_j))}{2} (p^1_{i,t_{m+1}} - p^1_{j,t_{m+1}}) \frac{A_{i,j}}{h_{i,j}}$$

$$= \lambda(E^1 \cap C_i)(p^1_i - p^1_{bh}).$$

$$|C_i|(s^2(t_{m+1}, x_i) - s^2(t_m, x_i)) - \tau \sum_{j \in N(i)} \frac{k_2(s^2(t_m, x_i)) + k_2(s^2(t_m, x_j))}{2} (p^2_{i,t_{m+1}} - p^2_{j,t_{m+1}}) \frac{A_{i,j}}{h_{i,j}}$$

$$= \lambda(E^2 \cap C_i)(p^2_i - p^2_{bh}).$$

We now make the analogy between this discrete setting and the previous chapter. We observe that the system of equations above are simply the Euler-Lagrange equations of the following convex function on $\mathbb{R}^{2n}$. Here we will write s_i in place of s_{i,t_n} and similarly for p.

$$\begin{aligned}\Psi^*(\mathbf{p}) = &\sum_{i=1}^{n} |C_i|[\gamma(p_i^1 - p_i^2) - s_i^1(p_i^1 - p_i^2)] \\ &+ \frac{\tau}{4} \sum_{1\le i<j\le n} \big(k_1(s_i^1) + k_1(s_j^1)\big)(p_i^1 - p_j^1)^2 \frac{A_{i,j}}{h_{i,j}} \\ &+ \frac{\tau}{4} \sum_{1\le i<j\le n} \big(k_2(s_i^2) + k_2(s_j^2)\big)(p_i^2 - p_j^2)^2 \frac{A_{i,j}}{h_{i,j}} \\ &+ \tau \sum_{i=1}^{n} \lambda(E^1 \cap C_i)\big(\frac{1}{2}(p_i^1)^2 - p_{bh}^1 p_i^1\big) \\ &+ \tau \sum_{i=1}^{n} \lambda(E^2 \cap C_i)\big(\frac{1}{2}(p_i^2)^2 - p_{bh}^2 p_i^2\big)\end{aligned} \tag{*}$$

One can see that this convex function does indeed produce the above equations at its minimum by consulting the similar non-discrete proof in the previous section and taking the analogous derivative.

Chapter 4

Graph Description of Non-Degenerate Components

In this chapter we build some tools to prove uniqueness of minimizers in the fully degenerate setting. Let Ω be some polygonal domain and let T be some triangulation of $\Omega_h \subset \Omega$, where Ω_h is discrete. The points from Ω_h will be called nodes. We will analyze the topology of the triangulation T when certain sets of "degenerate" nodes are removed from it. Recall that the k_i are the relative permeability curves discussed in the introductory chapter. We will always assume that $k_1(s_n^1) + k_2(s_n^2) > 0$ for each node $n \in \Omega_h$.

We consider the set of nodes n for which $k_i(s_n^i) = 0$ and for each neighbor m of n, $k_i(s_m^i) = 0$, $i = 1, 2$ to be degenerate. We call this degenerate set D_i. We will consider the connected components of the $\Omega - D_i$. We will see that the fact that having $k_1(s^1) + k_2(s^2) > 0$ will give us a form of connectedness that is strong enough to get some form of uniqueness.

Definition 4.0.1.

- *Let T be some Delaunay triangulation on $\Omega_h \subset \Omega$ where Ω_h is discrete. The triangulation will sometimes be referred to as the mesh.*
- *For nodes $n, m \in \Omega_h$, define $d(n, m)$ to be the number of edges traversed along the shortest path from n to m.*

- *A subset $A \subseteq \Omega_h$ is called connected if it is connected as a subset of the triangulation.*
- *Let $A \subseteq \Omega_h$. We define $Int(A) = \{n \in A : d(n, A^c) > 1\}$.*
- *Define $D_i = Int(k_i(s^i) = 0)$, $i = 1, 2$. We refer to these as the "degenerate regions."*

Remark: We will be using the notation Int mainly for $A = \{k_i(s_i) = 0\}$. Therefore, in this case, $Int(A)$ is the set of nodes n such that $k_i(s_n) = 0$ and each neighbor m of n satisfies $k_i(s_m) = 0$.

4.1 Graph Notation and Connectedness

We now define a notion of connectedness of the nodes from Ω_h which is natural with respect to the discrete gradient.

Definition 4.1.1. *A maximally connected component in the k_i gradient-connected sense is a maximal set S of nodes such that there is at least one node $n \in S$ such that $k_i(s^i_n) > 0$ and for each pair of nodes n, m in S there is a path in the mesh from n to m which does not contain two consecutive nodes $j, j+1$ such that $k_i(s^i_j) = 0$ and $k_i(s^i_{j+1}) = 0$.*

Definition 4.1.2. *Elements of Ω_h are called nodes and elements of the graph G which we now define will be called vertices. Let V_i be the vertex set such that each vertex is identified with a maximally connected component in the k_i gradient-connected sense of $\Omega_h - Int(k_i(s^i) = 0)$, $i = 1, 2$. Let $G = V_1 \cup V_2$ and add edges to G in the following way: the vertices v_1 from V_1 and v_2 from V_2 share an edge (v_1, v_2) if and only if $v_1 \cap v_2 \neq \emptyset$, where the intersection is defined to be the node set intersection.*

Note that for a fixed i no two vertices from V_i share an edge, $i = 1, 2$. This is because by definition two vertices from V_1 have disjoint node sets.

By abuse of notation, we will refer to $v \in G$ as the element of the graph G and also as the identified set of nodes in Ω_h. We begin with a lemma in order to show the connectedness of G. We assume from now on that the sets V_1, V_2 are both nonempty.

4.2 Connectedness

Lemma 4.2.1. *Let $v_1 \in V_1$, let $n \in v_1$ and suppose that $N(n)$ is some neighbor of n such that $N(n) \notin v_1$. Then there exists $v_2 \in V_2$ such that $n \in v_2$, and $N(n) \in v_2$.*

Proof. We first claim that $k_1(s^1_{N(n)}) = 0$. This follows by definition of gradient connected and the fact that the components are maximal. Since $k_1(s^1_n) + k_2(s^2_n) > 0$, we have $k_2(s^1_{N(n)}) > 0$. By definition of gradient-connected again we have the result. □

Theorem 4.2.1. *Let G be the graph given in the previous definition. Then G is connected.*

Proof. Let $v_1 \in V_1$ and $v_2 \in V_2$. We show there is a path in G from v_1 to v_2. Select a node n_1 from v_1 and a node n_2 from v_2. The mesh is connected so there is a path from n_1 to n_2 in the mesh. Each node in the mesh is in some vertex from G. We want to show that if a node along the path is in some vertex $v \in G$ and the next node on the path is in a different vertex $w \in G$ then there is an edge between v and w (i.e. $v \cap w \neq \emptyset$). This follows directly from the lemma above. □

4.3 Tree Property

Our next goal is to show that the graph G is actually a tree. For the rest of this section, we will assume that $\Omega = \mathbb{R}^n$ and the triangulation T covers all of $\mathbb{R}^n$. This simplifies the proof of the next lemma and will be removed later.

To show that G is a tree we will suppose there are cycles and produce a contradiction. We begin by referring the reader to the following well known fact from algebraic topology.

Lemma 4.3.1. *Let $A \subset \mathbb{R}^n$ be open and connected. Assume also that A^c is connected. Then ∂A is connected.*

Our goal now is to "fill in" the triangulation so that we can apply the previous lemma. The construction in the next lemma consists of taking the set of nodes contained in a vertex $v_i \in G$ and enlarging this set as a subset of $\mathbb{R}^n$ (the number of nodes remains the same) by taking its union with all of the faces and edges from the triangulation which are adjacent to at least one of these nodes from v_i. This will be defined precisely below.

Lemma 4.3.2. *Assume that the underlying triangulation consists of simplices. That is, triangles in 2 dimensions or tetrahedrons in 3 dimensions. If A is a connected set of nodes and A^c, the set of remaining nodes from the triangulation, is connected, then the set $d_1(A) = \{n \in \Omega_h : d(n, A) = 1\}$ is connected.*

Proof. We prove the case of 2 dimensions, the proof in 3 dimensions is almost identical. Let E be the edge set and F the face set of the mesh. We consider an operation on the node set A which "fills in" some of the edges and faces of the triangulation to form a new subset A' of $\mathbb{R}^n$. That is, let

$$A' = A \cup \{f \in F : \bar{f} \text{ contains a node from } A, \text{ where } \bar{f} \text{ is the closure of} f\}$$
$$\cup \{e \in E : \bar{e} \text{ contains a node from } A\}.$$

Then A' is connected and open. It is clear that A' is open. To see that A' is connected, observe that A' still contains A which is connected. Now A is only a set of nodes (no edges) and is connected in the triangulation sense (not as a subset of $\mathbb{R}^n$). However, by construction of A', A' contains $e(A)$, the nodes A together with all of their induced edges. Since the node set A is connected in the triangulation sense, $e(A)$ is connected as a subset of $\mathbb{R}^n$. We observe that A' has the following form

$$A' = (\bigcup_{\alpha \in I} A_\alpha) \cup e(A)$$

where for each $\alpha \in I$, $A_\alpha \cap e(A) \neq \emptyset$. The fact that a set of this form is connected is an exercise from Munkres [24]. We define the sets A_α as follows: for each edge/face in A' together with the node(s) from A which induced the edge/face, form a set A_α. This completes the proof that A' is connected.

We claim that $(A')^c$ is connected, where the complement is taken in $\mathbb{R}^n$. Observe that $(A')^c$ contains the node set A^c, which is connected in the triangulation sense by assumption. Also, $(A')^c$ contains $e((A')^c)$ since none of these edges are in A'.

Observe that each edge or face in Ω is adjacent to at least one node. If the face/edge is adjacent to a node from A then that face/edge is in A' and therefore not in $(A')^c$. If it is not adjacent to a node from A then it is adjacent to a node from A^c. Therefore $(A')^c$ has the similar form

$$A' = (\bigcup_{\alpha \in I} A_\alpha) \cup e((A')^c)$$

where we the define the A_α in a similar way.

We have constructed two sets A' and $(A')^c$ which satisfy the hypothesis of the previous theorem and this implies that $\partial A'$ is connected where $\partial A'$ is the boundary of A' in the topology of $\mathbb{R}^n$. We will next discuss the relationship between the set $d_1(A)$ and $\partial A'$.

We define $n(S)$ to be the set of nodes from the triangulation which are contained in the set S.

Claim :

$$n(\partial A') = d_1(A).$$

Proof of Claim:

First direction: $n(\partial A') \subseteq d_1(A)$.

Let $n \in \partial A'$. We first show $d(n, A) > 0$. If $d(n, A) = 0$, then $n \in A$ by definition. However, if $n \in A$ then $n \in A'$ by construction. Since A' is open, $n \notin \partial A'$. This shows $d(n, A) > 0$.

Before we show $d(n, A) < 2$, we consider the following fact about our triangulation:

Since the triangulation consists of simplices we know that if F is a face of the triangulation, then each node n which is adjacent to F is distance 1 from every other node which is adjacent to F. The same holds true for edges and this is true in all dimensions.

We show $d(n, A) < 2$. If $d(n, A) \geq 2$ then A' contains a face or an edge which is adjacent to a node which is a distance of at least 2 from any node from A. By construction, the only edges and faces which A' contains are those adjacent to nodes from A. By the paragraph above, this is a contradiction. Therefore, we have shown $n(\partial A') \subseteq d_1(A)$.

Now consider the reverse direction $d_1(A) \subseteq n(\partial A')$. Let n be such that $d(n, A) = 1$. Then by construction of A', $n \in \partial A'$. This completes the proof that $n(\partial A') = d_1(A)$.

Now since $\partial A'$ consists only of nodes and edges, it follows that $n(\partial A')$ is connected in the mesh. Therefore, the previous equality implies that $d_1(A)$ is connected. □

Theorem 4.3.1. *The graph G is a tree.*

Proof. Suppose that G has a cycle containing the vertices $v_1, v_2 \in V_1$. Consider the components of v_1^c, which we list as $c_1, ..., c_n$. Note that these components c_i are NOT components(vertices) from the graph G(i.e. non degenerate sets). They are just maximally k_1 graph connected components of v_1^c where the complement is taken with respect to the set of nodes in T. Since v_2 is a connected component of $T - D_1 - v_1$ and each c_i is a connected component of $T - v_1$, we have that $v_2 \subseteq c_i$ for some c_i. We assume that $v_2 \subseteq c_1$.

To apply the previous theorem we need that c_1 and c_1^c are both connected. Note that c_1 is connected by construction. We will now show that $c_1^c = x_1 \cup (\cup_{i=2}^n c_i)$ is connected.

Let n_1, n_2 be nodes in c_1^c. We show there is a path from n_1 to n_2 which does not leave $c_1^c = v_1 \cup (\cup_{i=2}^n c_i)$. If either n_1, n_2 are both in v_1 or n_1, n_2 are both in c_i for some $c_i, i = 2, ..., n$ then since each of these is connected by definition, we are done. Therefore, assume that $n_1 \in v_1$ and $n_2 \in c_2$ (the proof for the case that $n_1 \in c_2$ and $n_2 \in c_3$ is similar and of course there is no loss of generality from changing the subscripts of the c_i.). Since T is connected, there is a path p from n_1 to n_2 in T. We will now reconstruct p so that it remains in c_1^c.

Claim: The last component that the path p was in before it hit c_2 was v_1.
Suppose for a contradiction that this is not the case. Say that the last component that p was in before c_2 was some c_j. We consider a subpath of p which has some nodes in c_j and some in c_2. That is, some subpath p' with $p' \subset c_2 \cup c_j$ and $p' \not\subset c_2$ and $p' \not\subset c_j$. This means that by definition of the graph G, the vertices c_2 and c_j are connected and thus they are not distinct maximally connected components of x_1^c. In other words, c_2 and c_j are the same component and we have a contradiction. This proves the "last component of the path" **claim**.

Therefore, using the above claim, suppose that n_a is the last node of p from v_1 and define the following path p':

$$n_1 - ... - n_a - n_{a+1} - ... - n_2.$$

This path p' begins at n_1 and ends at n_2. We define p' so that it remains in v_1 from n_1 to n_a and remains in c_2 from n_{a+1} to n_2. This is enough to show that c_1^c is connected. Therefore, by the previous lemma, $d_1(c_1^c)$ is connected.

Claim: We have $d_1(c_1^c) \subset d_1(v_1)$.

Suppose for a contradiction that there exists a node n where $n \in d_1(c_1^c)$ but $n \notin d_1(v_1)$. Then we have $d(n, c_1^c) = 1$ and $d(n, v_1) \neq 1$. The equality yields the following:

First, that $n \in c_1$, for otherwise we have $d(n, c_1^c) = 0$

Second, that there is a neighbor $N(n)$ of n where $N(n) \in c_1^c$.

The inequality means that this $N(n)$ cannot be in v_1. Note carefully that we rule out the possibility that $d(n, v_1) = 0$ since, indeed, $n \in c_1$.

Therefore $N(n)$ must be in some c_j, $j \neq 1$. But this contradicts that the c_j are maximally connected components. This completes the **claim**.

Recall that we assumed that G has a cycle containing v_1 and $v_2 \subseteq c_1$. We will now produce a contradiction. By definition of a cycle, there are two paths p_1, p_2 from v_1 to v_2 in G such that the only common vertices on the paths are v_1 and v_2. Moreover, p_1 and p_2 must each contain some vertex v, v', respectively, from V_2 which contains a node from the connected set $d_1(c_1^c) \subseteq d_1(v_1)$, by definition of the distance

function. Since $d_1(c_1^c) \subseteq d_1(v_1)$, we have that $k_1(s_n^1) = 0$ and therefore $k_2(s_n^2) > 0$ for each $n \in d_1(c_1^c)$. By the previous theorem, $d_1(c_1^c)$ is connected and this together with the previous sentence, implies that $d_1(c_1^c)$ is actually k_2 graph connected. Therefore, $d_1(c_1^c)$ is covered by a single element from V_2. Therefore, $v = v'$ and so there cannot be a cycle. □

Theorem 4.3.2. *Assume that the hypothesis of the previous lemma except that we remove the restriction that the triangulation covers $\mathbb{R}^n$ and assume that Ω is any polygonal domain. Then G is a tree.*

Proof. Extend T to be a triangulation T' of $\mathbb{R}^n$ and extend s^1, s^2 to be functions on the extended triangulation. Consider the tree G_1 which is furnished by the previous theorem with respect to the extended triangulation T' and compare it to the graph G_2 which is given by the original triangulation T. By definition, G_2 is a subgraph of G_1 and therefore G_2 is a forest. However, the connectedness proof above still applies to G_2 and therefore G_2 is a tree. □

4.4 Inter-Component Properties

In this section we prove a lemma about the relationship between the function s^1 and the overlapping regions between vertices from V_1 and V_2. Of course, no two vertices from the same V_i may overlap since their node sets are disjoint. However, if the node set from some $v_1 \in V_1$ and some $v_2 \in V_2$ overlap then there must be a node n on this overlapping region where $s_n^1 \in (0, 1)$. This crucial property is what causes our convex function to diverge if the pressure values get too far apart on different components. Therefore, this fact will be needed to prove coercivity and hence existence results later on. The results from the previous section about trees will not be used in this section.

In the following we assume that $k_i(s^i) = 0$ if and only if $s^i \leq c_i$.

Definition 4.4.1. *For $v \in G$ define $\partial v = v \cap d_1(d_1(v))$.*

Lemma 4.4.1. *Let $v_1 \in V_1$. For each $n \in \partial v_1$, we have $k_1(s_n^1) = 0$.*

Proof. By definition of the k_1 graph-connected sense, for each $n \in d_1(v_1)$, $k_1(s_n^1) = 0$, otherwise we contradict the maximality of the component. Because of this, each $n \in v \cap d_1(d_1(v))$ is distance 1 from some $m \in d_1(v_1)$ such that $k_1(s_m^1) = 0$. Therefore $k_1(s_n^1) = 0$ or otherwise, by definition of k_1 graph connected, we actually have that $n \in v_1$ which is a contradiction. □

Lemma 4.4.2. *For each pair $(v_1, v_2) \in V_1 \times V_2$ such that $v_1 \cap v_2 \neq \emptyset$, there is a node $n \in v_1 \cap v_2$ such that $s_n^1 < 1$.*

Proof. If $v_2 \subseteq v_1$ we are done since v_2 must contain the required node. The remaining case is that the $\partial v_1 \cap v_2 \neq \emptyset$, since v_2 is connected. By the previous lemma, each $n \in \partial v_1$ satisfies $k_1(s_n^1) = 0$ (i.e. $s_n < 1$). Thus any node from $\partial v_1 \cap v_2$ will satisfy the requirements of the lemma. □

4.5 Restricted Functional

In this section we define a restricted version $\Psi^R$of the functional Ψ^*. This will be helpful in proving existence results in the next chapter. The idea is to prove the results for the restricted functional and then show how the proof can be extended to the original functional.

The functional Ψ^R has as its domain functions which take values in $\mathbb{R}^2 \cup \mathbb{R}^1$. This is because there are some nodes from Ω_h for which the oil and water contents s^1, s^2 are sufficiently high so that both $k_1(s^1)$ and $k_2(s^2)$ are positive (non-degenerate). However, there are some nodes in Ω_h for which one of the $k_i(s^i)$ is degenerate. In

this case only the pressure of the non-degenerate fluid is considered in Ψ^R.

Before looking at the definition of our restricted functional, the reader should consult equation $(*)$ at the end of chapter 3. Our restricted function here is a modification of that functional where the modification consists of removing the terms from the last two sums

Definition 4.5.1. *Define* $\Omega^{\mathcal{R}} := Int(k_1(s^1) = 0)^c \times Int(k_2(s^2) = 0)^c$

Definition 4.5.2. *We define the restricted functional* $\Psi^R : (\mathbb{R}^2 \cup \mathbb{R}^1)^{\Omega^{\mathcal{R}}} \to \mathbb{R}$ *by*

$$\begin{aligned}\Psi^R(p) := &\sum_{i \in Int(k_1(s^1)=0)^c \cap Int(k_2(s^2)=0)^c} [\gamma^*(p_i^1 - p_i^2) - s_n^1(p_i^1 - p_i^2)] \\ &+ \frac{1}{4} \sum_{\substack{1 \le i < j \le n \\ i,j \in Int(k_1(s^1)=0)^c}} [k_1(s_i^1) + k_1(s_j^1)](p_i^1 - p_j^1)^2 \\ &+ \frac{1}{4} \sum_{\substack{1 \le i < j \le n \\ i,j \in Int(k_2(s^2)=0)^c}} [k_2(s_i^2) + k_2(s_j^2)](p_i^2 - p_j^2)^2\end{aligned}$$

Remark : *Note that we are considering function from* $\Omega^{\mathcal{R}}$ *into* $\mathbb{R}^2 \cup \mathbb{R}^1$ *because of the following. We have* $Int(k_1(s^1) = 0)^c \subseteq \Omega_h$ *and* $Int(k_2(s^2) = 0)^c \subseteq \Omega_h$*. Also, if* $n \in Int(k_1(s^1) = 0)^c \cap Int(k_2(s^2) = 0)^c$ *then we consider both* p_n^1 *and* p_n^2 *but if, say,* $n \in Int(k_1(s^1) = 0)^c$ *but* $n \notin Int(k_2(s^2) = 0)^c$ *then we only consider the value* p_n^1 *but we do not consider the value* p_n^2*.*

Definition 4.5.3. *Let* $R : (\mathbb{R}^2)^{\Omega \times \Omega} \to (\mathbb{R}^2 \cup \mathbb{R}^1)^{\Omega^{\mathcal{R}}}$ *denote the natural restriction mapping. Therefore, we may talk about* $\Psi^R(R(p))$ *where* $p : \Omega \times \Omega \to \mathbb{R}^2$*.*

Chapter 5

Partially Degenerate Setting: Uniqueness Results

In this work we are considering two different degenerate settings which we call the partially degenerate setting and the fully degenerate setting. In this chapter we consider the partially degenerate setting. That is:

Definition 5.0.1. Partially Degenerate Setting :

$k_1(s^1) > 0$

$k_2(s^2) = 0$ *if and only if* $s^2 \leq c_2$ *where* $c_2 > 0$.

$k_1(s_n^1) + k_2(s_n^2) > 0$.

In a later chapter we will discuss results on the fully degenerate setting. That is:

Definition 5.0.2. Fully Degenerate Setting :

$k_1(s^1) = 0$ *if and only if* $s^1 \leq c_1$ *where* $c_1 > 0$ *and*

$k_2(s^2) = 0$ *if and only if* $s^2 \leq c_2$ *where* $c_2 > 0$.

$k_1(s_n^1) + k_2(s_n^2) > 0$.

The reason we do this is because we have some stronger uniqueness results in the partially degenerate setting. We continue now with our discussion of uniqueness

in the partially degenerate setting after first discussing some considerations about uniqueness properties of the problem in general.

5.1 Uniqueness Considerations

The question of uniqueness of minima for Ψ^* is delicate. In the absolute sense, there is no uniqueness. However, as we shall see, this non-uniqueness occurs in a trivial way. If we partially quotient out some of the "degenerate regions" we do, in fact, achieve uniqueness. The important point is that all of the minima correspond to essentially the same solution of the Euler-Lagrange equations.

In order to deal with abundance of minima, we consider the following modified functional. **The reader should compare this functional with the one from the end of chapter 3.** For the definition of Int, see the previous chapter. Here, I is the indicator function and the C_i are the Voronoi cells.

$$\begin{aligned}
\tilde{\Psi^*}(p) = &\sum_{i=1}^{n}[\gamma^*(p_i^1 - p_i^2) - s_n^1(p_i^1 - p_i^2)]|C_i| \\
&+ \frac{\tau}{4} \sum_{\substack{1\le i<j\le n \\ i,j\in Int(k_1(s^1)=0)^c}} [k_1(s_i^1) + k_1(s_j^1)](p_i^1 - p_j^1)^2 \frac{A_{i,j}}{h_{i,j}} \\
&+ \frac{\tau}{4} \sum_{\substack{1\le i<j\le n \\ i,j\in Int(k_2(s^2)=0)^c}} [k_2(s_i^2) + k_2(s_j^2)](p_i^2 - p_j^2)^2 \frac{A_{i,j}}{h_{i,j}} \\
&+ \frac{1}{2} \sum_{i\in Int(k_1(s_i^1)=0)\cup Int(k_2(s_i^2)=0)} (p_i^1 - p_i^2 - \hat{\xi})_+^2 |C_i| \\
&+ \tau \sum_{i=1}^{n} \alpha \Big[\lambda(E^1 \cap C_i)(\frac{1}{2}(p_i^1)^2 - p_{bh}^1 p_i^1) + \lambda(E^2 \cap C_i)(\frac{1}{2}(p_i^2)^2 - p_{bh}^2 p_i^2)\Big]
\end{aligned}$$

Take $\alpha = 0$ for no wells or $\alpha = 1$ for wells. We will work with this functional for the time being and explain how to remove the additional term later.

5.2 Results when $s_1 > 0$ and $k_1(s^1) > 0$

Assume the following conditions.

$\gamma^*(\eta) = \eta$ for $\eta \geq \hat{\xi}$

$\gamma^*(\eta) \to M$ as $\eta \to -\infty$, for some constant M.

$\gamma^*(\eta)$ is strictly convex on $\eta \leq \hat{\xi}$.

$k_1(s^1) > 0$

$k_2(s^2) = 0$ if and only if $s^2 \leq c_2$

$0 \leq c_2 < 1$

$0 < s_n^1 \leq 1$.

$0 \leq s_n^2 < 1$.

We will now prove a uniqueness result for the case when $\alpha = 0$, that is, the case when there are no well terms.

Theorem 5.2.1. *Consider the functional $\tilde{\Psi}^*$ defined above and suppose $\alpha = 0$. Then if $\tilde{\Psi}^*$ has a minimizer it is unique up to shifting $p^1 - p^2$ by a constant.*

Proof. Let $\mathbf{p} = (p^1, p^2)$ and $\tilde{\mathbf{p}} = (\tilde{p^1}, \tilde{p^2})$ be minimizers. We will show that if $(p^1, p^2) \neq (\tilde{p^1} + c, \tilde{p^2} + c)$, for some constant c then we can construct $\tilde{\tilde{p}}$ such that $\Psi(\tilde{\tilde{\mathbf{p}}}) < \Psi(\mathbf{p})$. Define $\xi = p^1 - p^2$ and $\tilde{\xi} = \tilde{p^1} - \tilde{p^2}$.

Case 1. Assume that $\xi_n \neq \tilde{\xi}_n$ for some $n \in Int(k_2(s^2) = 0)$. Since γ^* is strictly convex on $(\infty, \hat{\xi}]$ and linear on $[\hat{\xi}, \infty)$ and the term containing the positive part is strictly convex on $[\hat{\xi}, \infty)$ and since the remaining terms are convex, we have that $\tilde{\Psi}^*(\frac{\xi+\tilde{\xi}}{2}) < \frac{\tilde{\Psi}^*(\tilde{\xi})+\tilde{\Psi}^*(\tilde{\xi})}{2}$. Threfore, we can take $\tilde{\tilde{\mathbf{p}}}$ to be the average of $\mathbf{p}$ and $\tilde{\mathbf{p}}$.

Case 2. Assume that $\xi_n \neq \tilde{\xi}_n$ for some fixed $n \notin Int(k_2(s^2) = 0)$. Assume that $\xi_n = \tilde{\xi_n} + d$, where $d > 0$. Assume that $\xi_n, \tilde{\xi_n} \geq \hat{\xi}$ since otherwise we can use the fact that γ^* is strictly convex on $(\infty, \hat{\xi}]$ and linear on $[\hat{\xi}, \infty)$ and take an average as before.

Let C be the maximal k_2 gradient-connected component of $\Omega - Int(k_2(s^2) = 0)$ containing n. By the strict convexity of the terms containing the k_i and the fact that $\tilde{\mathbf{p}}$ and $\mathbf{p}$ are minima, it follows that

$\tilde{p}_i^1 - \tilde{p}_j^1 = p_i^1 - p_j^1$, for all i, j with $j \in N(i)$ where $N(i)$ means neighbor of i and

$\tilde{p}_i^2 - \tilde{p}_j^2 = p_i^2 - p_j^2$, for all $i, j \in C$, with $j \in N(i)$.

This implies that for all $i \in C$ we have $\xi_i = \tilde{\xi}_i + d$. Define

$$\tilde{\tilde{p}}_i = \begin{cases} (p_n^1, p_n^2 + d) & \text{if } i \in C \\ (p_i^1, p_i^2) & \text{if } i \notin C \end{cases}$$

and define $\tilde{\tilde{\xi}} = \tilde{\tilde{p}}^1 - \tilde{\tilde{p}}^2$, noting that $\tilde{\tilde{\xi}}_i = \tilde{\xi}_i$ for $i \in C$.

We claim that $\tilde{\Psi}^*(\tilde{\tilde{\mathbf{p}}}) < \tilde{\Psi}^*(\mathbf{p})$. Consider the first two terms of $\tilde{\Psi}^*$. Observe that for $i \in C$, we have

$$\begin{aligned} \sum_{i \in C} \gamma^*(\tilde{\tilde{\xi}}_i) - s_i^1 \tilde{\tilde{\xi}}_i &= \sum_{i \in C} \gamma^*(\tilde{\xi}_i) - s_i^1 \tilde{\xi}_i \\ &= \sum_{i \in C} (1 - s_i^1) \tilde{\xi}_i \\ &< \sum_{i \in C} (1 - s_i^1) \xi_i \\ &= \sum_{i \in C} \gamma^*(\xi_i) - s_i^1 \xi_i \end{aligned}$$

The first equality holds since $\tilde{\tilde{\xi}} = \tilde{\xi}$. The second equality holds since $\xi_i, \tilde{\xi}_i \geq \hat{\xi}$ for each $i \in C$ since otherwise we can use the strict convexity of γ^* to take an average as in case 1 and $\gamma^*(\eta) = \eta$ for $\eta \geq \hat{\xi}$. The third equality holds since $s_m^1 < 1$ for some m in C (otherwise $s_m^2 = 0$ for all $m \in C$ and then C is not a k_2 component) and $\xi_i = \tilde{\xi}_i + d$, $d > 0$ for each i.

We claim that the remaining terms from $\Psi(\tilde{p})$ are identical to those from $\Psi(\tilde{p})$. To see this for the terms containing the k_i, observe that if $i \in C$ and $j \in N(i)$ with $j \notin C$ then we have $k_2(s_j^2) = 0$ and $k_2(s_i^2) = 0$, by definition of k_2 gradient-connectedness.

To see that the term containing the positive part is the same for both $\tilde{\Psi}^*(\tilde{\mathbf{p}})$ and $\tilde{\Psi}^*(\mathbf{p})$ observe that $\tilde{\mathbf{p}}$ only differs from $\mathbf{p}$ on C and the term in question vanishes on C. □

Theorem 5.2.2. *Consider the functional $\tilde{\Psi}^*$ defined above and suppose that $\alpha = 1$. If $\tilde{\Psi}^*$ has a minimum then that minimum is unique.*

Proof. Since $\frac{1}{2}(p_{n_1}^1)^2$, $\frac{1}{2}(p_{n_2}^1)^2$ are strictly convex and all the terms of $\tilde{\Psi}*$ are at least non-strictly convex, the values of $p_{n_1}^1$, $p_{n_2}^1$ are unique over all minimizers. Since $k_1 > 0$, and by the strict convexity of the terms containing k_1, it follows that the p_n^1 are unique for each $n \in \Omega_h$. Also, the gradients of p^2 are unique on each component and since the value $p_{n_2}^2$ is fixed, p^2 is fixed on the component containing n_2. We now adapt the two cases in the proof above for when the wells are present.

Suppose we are in Case 1. There is a node n such that $\xi_n = \tilde{\xi}_n + d$. If $n = n_2$ then by the paragraph above we actually must have $\xi_n = \tilde{\xi}_n$ and so this case cannot happen. If $n = n_1$ or n is some other node we may proceed as in the theorem above.

Suppose we are in Case 2. There is a component C with $n \in C$ and $\xi_n = \tilde{\xi}_n + d$. If $n_2 \in C$ then first paragraph of this proof gives a contradiction. If $n_2 \notin C$, we may proceed as in the theorem above. □

Chapter 6

Existence Results

In this chapter we prove that $\tilde{\Psi}^*$ is coercive in both the partially degenerate and fully degenerate settings. We will use the graph notation and the restricted functional notation described in the chapter on graph description. In particular, we will begin with showing the the restricted functional Ψ^R is coercive. We will then see how to extend this result to get the coercivity of $\tilde{\Psi}^*$. We begin with some definitions and will then show that if $|p| \to \infty$ then $\Psi^R(p) \to \infty$. Note that the domains of $\tilde{\Psi}^*$ and Ψ^R are finite dimensional.

6.1 Coercivity of Restricted Functional

We reproduce the definition of the restricted functional Ψ^R here for convenience and then give some definitions which will be helpful for the existence proof.

$$\begin{aligned}\Psi^R(\mathbf{p}) := &\sum_{i \in Int(k_1(s^1)=0)^c \cap Int(k_2(s^2)=0)^c} [\gamma^*(p_i^1 - p_i^2) - s_n^1(p_i^1 - p_i^2)] \\ &+ \frac{1}{4} \sum_{\substack{1 \le i < j \le n \\ i,j \in Int(k_1(s^1)=0)^c}} [k_1(s_i^1) + k_1(s_j^1)](p_i^1 - p_j^1)^2 \frac{A_{i,j}}{h_{i,j}} \\ &+ \frac{1}{4} \sum_{\substack{1 \le i < j \le n \\ i,j \in Int(k_2(s^2)=0)^c}} [k_2(s_i^2) + k_2(s_j^2)](p_i^2 - p_j^2)^2 \frac{A_{i,j}}{h_{i,j}}\end{aligned}$$

Definition 6.1.1.

In the following definitions we consider a sequence of functions on the set of nodesp^k, $k = 1, 2, ...$ We use the notation $p_n^1 \to +\infty$ to mean that for the underlying sequence p^k we have that the value of p^k at the node n or p_n^k is tending to ∞. We drop the index k from now on and assume that the arrows below refer to convergence as $k \to \infty$.

- *For $v \in G$ we will write $p_v^1 \to +\infty$ if for at least one node $n \in v$ we have $p_n^1 \to +\infty$ and for the remaining nodes either $p_n^1 \to +\infty$ or $|p_n^1| \leq M$.*
- *For $v \in G$ we will write $p_v^1 \to -\infty$ if for at least one node $n \in v$ we have $p_n^1 \to -\infty$ and for the remaining nodes either $p_n^1 \to -\infty$ or $|p_n^1| \leq M$.*

- *We write $p^1 \to +\infty$ if for at least one $v \in G$ we have $p_v^1 \to +\infty$ and for the remaining $v \in G$, either $p_v^1 \to +\infty$ or $|p_v^1| \leq M$.*
- *We write $p^1 \to -\infty$ if for at least one $v \in G$ we have $p_v^1 \to -\infty$ and for the remaining $v \in G$, either $p_v^1 \to -\infty$ or $|p_v^1| \leq M$.*

The same definitions hold when 1 is replaced with 2.

We now prove a lemma which shows that if for some n we have $p_n^i \to \infty$, for either $i = 1$ or $i = 2$, then the same must be true for all nodes in the same k_i graph-connected component as n (or Ψ^R will diverge)

Lemma 6.1.1. *Let $v \in G$ and suppose there exists $n \in v$ such that $p^1(n) \to +\infty$. Then if the following statement in quotes is not true:*

"for all $n' \in v$, $p_{n'}^1 \to +\infty$,"

we have that Ψ^R blows up.

The same is true if we replace ∞ with $-\infty$

Proof. Suppose $v \in V_1$. Then if the statement in quotes is not true, the term from Ψ^R containing k_1 blows up. The same is true if $v \in V_2$. □

Observe that Ψ^R only depends on the differences $p^1 - p^2$ and the gradients of p^1, p^2. Therefore, if we consider the translation $\hat{p^1} := p^1 + c$ and $\hat{p^2} := p^2 + c$ then $\Psi^R(\hat{p}) = \Psi^R(p)$. Therefore, we shall prove that there exists a minimum over the constrained set $\int_{V_1} p^1 = 0$, where the integral is taken over the nodes $n \in V_1$. See the definition above for the notation used in the proof of the following theorem.

Theorem 6.1.1. *Let Ψ^R be the restricted functional defined in previous chapter. If V_1 is nonempty, then Ψ^R is coercive on $\int_{V_1} p^1 = 0$. If V_1 is empty, Ψ^R is coercive on $\int_{V_2} p^2 = 0$.*

Proof.

We first assume that V_1 is nonempty. Since the domain of Ψ^R is finite dimensional, and since all norms on a finite dimensional vector space are equivalent, we do not write any norms here. Since Ψ^R is bounded below, we only need to show that one term from Ψ^R blows up in order to show $\Psi^R \to \infty$.

Case 1: Suppose $p^1 \to +\infty$ or $p^1 \to -\infty$ (see definition above). This violates $\int_{V_1} p^1 = 0$. Therefore, we consider the other ways in which $|p| \to \infty$ on $\int_{V_1} p^1 = 0$.

Case 2: Suppose that for some $v_+, v_- \in V_1$, $p^1_{v_+} \to +\infty$ and $p^1_{v_-} \to -\infty$ (see definition above). We proceed by induction. For the base case, assume there is a vertex $v_2 \in V_2$ such that $v_+ - v_2 - v_-$ is a path of length 3. By lemma 4.4.2 there is $n_+ \in v_+ \cap v_2$ with $s^1_{n_+} \in (0,1)$ and $n_- \in v_- \cap v_2$ with $s^1_{n_-} \in (0,1)$.

Claim 1:

If $|p^1_{n_+} - p^2_{n_+}| \leq M$ does not hold for some M, then the term

$$\gamma^*(p^1_{n_+} - p^2_{n_+}) - s^1_{n+}(p^1_{n_+} - p^2_{n_+})$$

blows up.

Similarly, if $|p^1_{n_-} - p^2_{n_-}| \leq M$ does not hold for some M, then

$$\gamma^*(p^1_{n_-} - p^2_{n_-}) - s^1_{n-}(p^1_{n_-} - p^2_{n_-})$$

blows up.

Suppose that $|p^1_{n_+} - p^2_{n_+}| \to \infty$. If $(p^1_{n_+} - p^2_{n_+}) \to \infty$ then since $\gamma^*(\eta) - s^1_{n_+}(\eta) = (1-s^1_{n_+})\cdot\eta$ for $\eta \geq \hat{\xi}$, we have $\gamma^*(p^1_{n_+}-p^2_{n_+})-s^1_{n+}(p^1_{n_+}-p^2_{n_+}) \to \infty$. On the other hand, if $(p^1_{n_+} - p^2_{n_+}) \to -\infty$, then since $\gamma^*(\eta) - s^1_{n_+}(\eta) > \epsilon \cdot \eta$ for large negative values of η (since $s^1_{n_+} > 0$ and $\gamma^*(\eta) \to L$ as $\eta \to -\infty$), we have $\gamma^*(p^1_{n_+}-p^2_{n_+})-s^1_{n+}(p^1_{n_+}-p^2_{n_+}) \to \infty$. This completes the proof of the **Claim 1**.

We continue with the base case. As shown above, if one of the terms $|p^1_{n_+} - p^2_{n_+}|$ or $|p^1_{n_-} - p^2_{n_-}|$ blows up, then Ψ^R blows up so we are done. Assume, therefore, that both $|p^1_{n_+} - p^2_{n_+}| \leq M$ and $|p^1_{n_-} - p^2_{n_-}| \leq M$. Since we are still in Case 2, $p^1_{v_+} \to +\infty$ and $p^1_{v_-} \to -\infty$. By lemma 6.1.1 applied to both v_+ and v_-, if we do not have both $p^2_{n+} \to +\infty$ and $p^2_{n-} \to -\infty$, then Ψ^R blows up. Thus we assume that both $p^2_{n+} \to +\infty$ and $p^2_{n-} \to -\infty$. Then by lemma 6.1.1 applied to v_2, recalling that $n_+ \in v_2$ and $n_- \in v_2$, we have that Ψ^R blows up.

We proceed with the induction step of Case 2. As before, suppose that for some $v_+, v_- \in V_1$, $p^1_{v_+} \to +\infty$ and $p^1_{v_-} \to -\infty$. This time, however, we have a path between v_+ and v_- with length larger than 3. Let $v_+ \to w_+ \to ... \to w_- \to v_-$ be a path from v_+ to v_-.

Claim 2:

Suppose that Ψ^R does not blow up, then both $p^2_{w_+} \to +\infty$ and $p^2_{w_-} \to -\infty$.

By lemma 4.4.2, there exists $n_+ \in v_+ \cap w_+$ such that $s_{n_+} \in (0,1)$.

Similarly, there exists $n_- \in v_- \cap w_-$ such that $s_{n_-} \in (0,1)$.

By still being in Case 2, we have $p^1_{v_+} \to +\infty$ and $p^1_{v_-} \to -\infty$.

By lemma 6.1.1, $p^1_{n_+} \to +\infty$ and $p^1_{n_-} \to -\infty$.

By **Claim 1**, $p^2_{n_+} \to +\infty$ and $p^2_{n_-} \to -\infty$.

By lemma 6.1.1 again, $p^1_{w_+} \to +\infty$ and $p^1_{w_-} \to -\infty$.

This completes the proof of **Claim 2** and also completes the induction step which finishes Case 2.

Case 3: Assume that $|p^2_{v_+}| \to \infty$ for some $v \in V_2$. Then by similar arguments from Case 2, we can see that we are actually again in the situation of Case 2.

The above cases complete the proof for when $V_1 \neq \emptyset$. We now assume that $V_1 = \emptyset$. Then $V_2 = \{\Omega_h\}$. In this case Ψ^R is reduced to

$$\frac{1}{4}\sum_{i<j}[k_2(s^2_i) + k_2(s^2_j)](p^2_i - p^2_j)^2.$$

To complete the proof we must show that if $|p^2| \to \infty$ while $\int_{\Omega_h} p^2 = 0$ then the above term diverges. Since $V_1 = \emptyset$, it follows that $k_1(s^1_n) = 0$ for each $n \in \Omega_h$. Therefore, since $k_1(s^1_n) + k_2(s^2_n) > 0$ for each n, we have $k_2(s^2_n) > 0$ for each n. The divergence of the above now follows easily. □

6.2 Extension of Coercivity to Original Functional

See the chapter on graph description for the definition of the restriction function R.

Theorem 6.2.1. *The functional $\tilde{\Psi}^*$ is coercive.*

Proof. We have shown in the previous theorem that if $|R(p)| \to \infty$, then $\Psi^R \to \infty$. Since $\tilde{\Psi}^* \geq \Psi^R$ it also follows that if $|R(p)| \to \infty$, then $\tilde{\Psi}^* \to \infty$. Thus we take let $|p| \to \infty$, while keeping $|R(p)|$ bounded, and show that still $\tilde{\Psi}^* \to \infty$, this will be enough to show coercivity of $\tilde{\Psi}^*$.

Therefore assume that $|R(p)|$ is bounded and that $|p| \to \infty$. First we recall some key terms from $\tilde{\Psi}^*$ for convenience and then we split up the remainder of the proof into cases.

$$\sum_{n\in\Omega_h} [\gamma^*(p_n^1 - p_n^2) - s_n^1(p_n^1 - p_n^2)] \tag{6.1}$$

$$\sum_{n\in\Omega_h} (p_n^1 - p_n^2 - \hat{\xi})_+^2 I(Int(\{k_1(s_n^1) = 0\}) \cup I(Int(\{k_2(s_n^2) = 0\})) \tag{6.2}$$

In each of the following cases when we write, say, for some $n \in Int(\{k_1(s^1) = 0\})$ that $p_n^1 \to \infty$, it is always assumed that p_n^2 is bounded. This is because we have assumed that $|R(p)|$ is bounded and p_n^2 is in this case in the domain of Ψ^R since $D_1 \cap D_2 \neq \emptyset$. The analogous statement is assumed for the indices 1 and 2 swapped.

CASE 1: $p_n^1 \to +\infty$ for some $n \in Int(\{k_1(s^1) = 0\})$. The second term above blows up in this case.

CASE 2: $p_n^1 \to -\infty$ for some $n \in Int(\{k_1(s^1) = 0\})$. Recall that $s_n^1 > 0$ is a global assumption. This together with the fact that $\gamma^*(\eta) \to L$ as $\eta \to -\infty$ forces the first term to blow up.

Case 3: $p_n^2 \to +\infty$ for some $n \in \mathit{Int}(\{k_2(s^2) = 0\})$. Apply the same argument from Case 2.

Case 4: $p_n^2 \to -\infty$ for some $n \in \mathit{Int}(\{k_2(s^2) = 0\})$. Apply the same argument from Case 1.

This completes the proof that $\tilde{\Psi}^*$ is coercive. □

Chapter 7

Fully Degenerate Setting: Uniqueness Results

In this chapter we deal with the question of uniqueness for the case when

$k_1(s^1) = 0$ for $s^1 \leq c_1$

$k_2(s^2) = 0$ for $s^2 \leq c_2$,

where $c_1 + c_2 < 1$ and $c_1, c_2 > 0$.

In addition to assuming that γ^* has the properties listed at the beginning of section 5.2, we also assume that

$$\gamma^{*''}(\eta) > 0 \text{ on } \{\eta : \gamma^{*'}(\eta) < 1\}$$

$$\gamma^{*''} \geq 0$$

$$\gamma^{*'}(\eta) \to 0 \textit{ as } \eta \to -\infty$$

$\gamma^{*'}$ is continuous.

All of these properties are satisfied by our example case which is

$$\gamma^{*'}(\eta) = \begin{cases} \frac{1}{4\eta^2} & \eta \leq -\frac{1}{2} \\ 1 & \eta \geq -\frac{1}{2} \end{cases}$$

and

$$\gamma^{*''}(\eta) = \begin{cases} -\frac{1}{2\eta^3} & \eta \leq -\frac{1}{2} \\ 0 & \eta \geq -\frac{1}{2} \end{cases}$$

and so $\gamma^{*''}(\eta) \geq 0$.

In the language of connected components, the difference between the fully degenerate setting (FDS) and the partially degenerate setting (PDS) is that in PDS the domain of p^1 has only one component and the domain of p^2 may have multiple components. In the FDS, both domains will have multiple components.

Because $\tilde{\Psi}^*$ is **not** strictly convex, for the proof of uniqueness in PDS we needed to resort to a shifting argument where we shifted p^2 on one of its components in order to produce a strictly smaller minimum. For the FDS a similar idea works but the implementation is more delicate. We will need the fact that the graph G is actually a tree.

7.1 Uniqueness of Ψ^R

We will first directly show uniqueness for Ψ^R and then use the relationship between $\tilde{\Psi}^*$ and Ψ^R to get uniqueness for $\tilde{\Psi}^*$. First we need a lemma.

Lemma 7.1.1. *Let $v_1 \in V_1, v_2 \in V_2$ such that $v_1 \cap v_2 \neq \emptyset$. Then the function*

$$g_{v_1,v_2}(c) := \sum_{n \in v_1 \cap v_2} [\gamma^*(p_n^1 - p_n^2 + c) - s_n^1(p_n^1 - p_n^2 + c)],$$

has a unique minimum at some $c^* \in \mathbb{R}$.

Proof. We will write g in place of g_{v_1,v_2} for convenience. First observe that

$$g'(c) = \sum_{n \in v_1 \cap v_2} \gamma^{*'}(p_n^1 - p_n^2 + c) - s_n^1.$$

By lemma 4.4.2 there exists $\hat{n} \in v_1 \cap v_2$ for which $s^1_{\hat{n}} \in (0,1)$. We claim there is at most one value of c for which $g'(c) = 0$. We proceed by contradiction. First observe that g' is increasing (at least non-strictly) since each of the summands is (non-strictly) increasing.

We assume there is a c such that $g'(c) = 0$ and claim that $g'(c+\epsilon) > 0$ for any $\epsilon > 0$. By assumption $\gamma^{*'}(\eta)$ is strictly increasing on the set $\{\eta : \gamma^{*'}(\eta) < 1\}$. Therefore, if for some n in the sum we have $\gamma^{*'}(p^1_n - p^2_n + c) < 1$, we are done since

$$\gamma^{*'}(p^1_n - p^2_n + (c+\delta)) > \gamma^{*'}(p^1_n - p^2_n + (c+\delta))$$

and since this is a summand of $g'(c)$ and all of its summands are increasing, $g(c+\delta) > g(c)$.

Incidentally, this must be the case. If not, then $\gamma^{*'}(p^1_n - p^2_n + c) = 1$ for each n. But then $g'(c) = (1 - s^1_{\hat{n}}) + (1 - s^1_{n_1}) + ... + (1 - s^1_{n_m}) > 0$ since $s^1_{\hat{n}} < 1$, as shown earlier, and since $s^1_{n_k} \leq 1$ for each k. But $g'(c) = 0$ so this is a contradiction.

This shows that there is at most one critical point of g. We now show that $g'(c)$ has at least one critical point. For convenience we rewrite $g'(c)$,

$$g'(c) = \sum_{n \in v_1 \cap v_2} \gamma^{*'}(p^1_n - p^2_n + c) - s^1_n.$$

Since $\gamma^{*'}(\eta) \to 0$ as $\eta \to -\infty$, and since $s^1_n > 0$ for each n, there exists c such that $g'(c) < 0$. On the other hand, notice that $\gamma^{*'}(\eta) = 1$ for $\eta \geq -\frac{1}{2}$, that $s^1_n \leq 1$ for each n and that $s^1_{\hat{n}} < 1$. Therefore, since the sum is finite, there exists c such that $g'(c) > 0$. Thus, since $g'(c)$ is continuous we have the existence of c such that $g'(c) = 0$. Finally, since $g'' \geq 0$, the critical point is a minimum and we are done. □

Theorem 7.1.1. *If $V_1 \neq \emptyset$ then there is at most one minimum for Ψ^R over the set $\int_{V_1} p^1 = 0$. If $V_1 = \emptyset$ then there is at most one minimum over the set $\int_{V_2} p^2 = 0$.*

Proof. Assume that $V_1 \neq \emptyset$ and that $\mathbf{p}$ is a minimizer. The strict convexity of the terms in Ψ^R containing the k_i implies that the gradients ∇p_i are unique on each component. Therefore we need only show that the translates of the restrictions of the p^i on each component $v \in G$ are unique up to shifting everything by a constant.

To show this, we consider the function

$$g_{v_1,v_2}(c) := \sum_{n \in v_1 \cap v_2} [\gamma^*(p_n^1 - p_n^2 + c) - s_n^1(p_n^1 - p_n^2 + c)]$$

with $v_1 \in V_1$ and $v_2 \in V_2$ for which $v_1 \cap v_2 \neq \emptyset$. This has a unique minimum over $c \in \mathbb{R}$ by lemma 7.1.1. We write g in place of g_{v_1,v_2} for convenience. We claim that $g(0) = \min_{c \in \mathbb{R}} g(c)$.

claim itself for the end.

If $g(0) = \min_{c \in \mathbb{R}} g(c)$ then the gradients of the p^i are fixed on each component and the values $p^1 - p^2$ are fixed on overlapping regions $v_1 \cap v_2$, where $v_1 \in V_1$, $v_2 \in V_2$. This together with the fact that G is connected implies that p^1, p^2 are unique up to shifting both by the same constant.

We prove that $g(0) = \min_{c \in \mathbb{R}} g(c)$. Recall that G is a tree. Let $\mathbf{p}$ be a minimizer and suppose that for some $v_1 \cap v_2 \neq \emptyset$ we have that $g(0) \neq \min_{c \in \mathbb{R}} g(c)$. We claim that we can find a strictly smaller minimum than $\mathbf{p}$ which is a contradiction.

We will construct this new minimum by shifting a large section of the values of p on the tree G by a constant. The principle is that we will shift the values of p so that for the new values we will have $g(0) = \min_{c \in \mathbb{R}} g(c)$ but we will not disturb the other terms from Ψ^R. We now describe this in detail.

Suppose that for some v_1, v_2, $g_{v_1,v_2}(0) \neq \min_{c\in\mathbb{R}} g_{v_1,v_2}(c)$. Since g has a unique minimum, we have $g(c^*) = \min_{c\in\mathbb{R}} g(c)$ for some $c^* \neq 0$. Since G is a tree, deleting the edge (v_1, v_2) will yield two maximally connected subgraphs G_1, G_2. Now on G_2, replace the values p^1, p^2 by $p^1 - c^*$ and $p^2 - c^*$. By inspection of each of the terms of Ψ^R this yields a strictly smaller minimum. This contradicts that $\mathbf{p}$ is a minimum and therefore we indeed have $g(0) = \min_{c\in\mathbb{R}} g(c)$. We have completed the proof in the case where $V_1 \neq \emptyset$.

Next assume that $V_1 = \emptyset$ and $\mathbf{p}$ is a minimum.

In this case, $V_2 = \{\Omega\}$. Therefore, since the gradient of p^2 is unique and Ω_h is connected, we have that p^2 is unique up to a constant. Therefore p^2 is unique when restricted to the set $\int_{V_2} p^2 = 0$. □

This concludes the uniqueness section. We now have existence and uniqueness of a minimum for Ψ^R. The next section will extend this result and show existence and uniqueness of a minimum for $\tilde{\Psi}^*$.

7.2 Extension of Existence and Uniqueness of Minimum of Ψ^R to $\tilde{\Psi}^*$

In this section we will see how to extend the existence and uniqueness results from Ψ^R to to $\tilde{\Psi}*$. We will see that this extension is somewhat trivial since the additional variables which are in the domain of $\tilde{\Psi}^*$ but not in Ψ^R are "free" variables. Therefore, to minimize $\tilde{\Psi}^*$ we can split the functional and minimize the pieces separately.

7.2.1 Splitting the Functional

We write the functional here for convenience. The first two terms below are $\phi(\mathbf{p})$ and the last 4 terms are $\Psi^R(\mathbf{p})$.

$$\begin{aligned}
\tilde{\Psi}^*(\mathbf{p}) &= \phi(\mathbf{p}) + \Psi^R(\mathbf{p}) \\
&= \sum_{Int(k_1(s^1)=0)\cup Int(k_2(s^2)=0)} [\gamma^*(p_n^1 - p_n^2) - s_n^1(p_n^1 - p_n^2)]|V_i| \\
&+ \frac{1}{2} \sum_{Int(k_1(s^1)=0)\cup Int(k_2(s^2)=0)} (p_n^1 - p_n^2 - \hat{\xi})_+^2 |V_i| \\
&+ \sum_{i\in Int(k_1(s^1)=0)^c \cap Int(k_2(s^2)=0)^c} [\gamma^*(p_n^1 - p_n^2) - s_n^1(p_n^1 - p_n^2)]|V_i| \\
&+ \frac{\tau}{4} \sum_{\substack{1\le i<j\le n \\ i,j\in Int(k_1(s^1)=0)^c}} [k_1(s_n^1) + k_1(s_{n+1}^1)](p_{n+1}^1 - p_n^1)^2 \frac{A_{i,j}}{h_{i,j}} \\
&+ \frac{\tau}{4} \sum_{\substack{1\le i<j\le n \\ i,j\in Int(k_2(s^2)=0)^c}} [k_2(s_n^2) + k_2(s_{n+1}^2)](p_{n+1}^2 - p_n^2)^2 \frac{A_{i,j}}{h_{i,j}}
\end{aligned}$$

The fact that $\tilde{\Psi}^*(\mathbf{p})$ has a unique minimizer follows from the fact that the terms $\phi(\mathbf{p})$ and $\Psi^R(\mathbf{p})$ may be treated separately since their domains are essentially disjoint in the sense that the domains overlap in at most one coordinate of (p_1, p_2) and $\phi(\mathbf{p})$ only depends on the difference $p_1 - p_2$. Both ϕ and Ψ^R satisfy existence and uniqueness properties for minimizers. The fact that ϕ is coercive follows from the same arguments from the proof that Ψ^R is coercive. The uniqueness of ϕ follows from its strict convexity. Of course any uniqueness claim here is understood to be up to shifting both p_1 and p_2 by the same constant.

7.3 Results for Original Functional Ψ^*

We finally return to our original functional Ψ^*. Recall that we modified Ψ^* to get $\tilde{\Psi}^*$ by adding in the extra term

$$\frac{1}{2}\sum (p_n^1 - p_n^2 - \hat{\xi})_+^2.$$

In fact, without this extra term we do not actually have uniqueness. To see this, observe that if $s_n^1 = 1$ then we have

$$\gamma^*(p_n^1 - p_n^2) - s_n^1(p_n^1 - p_n^2) = \gamma^*(p_n^1 - p_n^2 + c) - s_n^1(p_n^1 - p_n^2 + c)$$

if $p_n^1 - p_n^2 \geq \hat{\xi}$. Therefore, we can move one of the p^i around freely in the term ϕ above and achieve different functions p having the same minimum.

The important thing is that even though there is no uniqueness, there is enough uniqueness in the sense to ensure that all minimizers yield the same saturation s^i values and the same gradients ∇p on the regions where ∇p is not multiplied by a zero term.

Chapter 8

Numerical Example

We will now give a numerical example. First we will list the parameters of the example and provide some contour plots. The porosity $1 - s_0(x)$ is layered in the following way: $s_0(x) = .2 * sin(2\pi(x - y))$. The countour plot of s_0 is shown in figure 8.1. The relative permeability curves k_1, k_2 have the equations $k_1(s^1, x) = min(0, (s^1 - .2(1 - s_0(x)))^2)$ and $k_2(s^2, x) = min(0, (s^2 - .2(1 - s_0(x)))^2)$. The plots are shown in figure 8.2.

Next we consider the general permeabilty of the medium. We suppose that the medium is very dense at the point $(.3, .4)$. The density is uniform with permeability value $.04$ in a disk of radius $.2$ around $(.3, .4)$. Outside the disk, the permeability increases linearly as a function of the distance from the disk. The contour plot is shown in figure 8.3 and the equation, where $p = (.3, .4)$, is

$$K(x) = .04 + max(0, 10\, d(x, p) - 2).$$

Our mesh is generated using the program Triangle by Richard Shewchuk and can be found at https://www.cs.cmu.edu/ quake/triangle.html. We use 300 grid points on the square $[0, 1] \times [0, 1]$. The wells are the quarter circles with radius $.2$ and centers $(1, 0)$, and $(0, 1)$. The pressures are $p^1_{bh} = 100$ and $p^2_{bh} = 0$. We assume that $s^1 = 0$ at time $t = 0$. We use a time step of $t = .1$. The figures 8.4-8.9 give the solutions at times $t = 1, 5, 10, 15, 20, 30$, respectively, and the colorbar is in figure 8.10. At each time step we computed the minimum of the associated

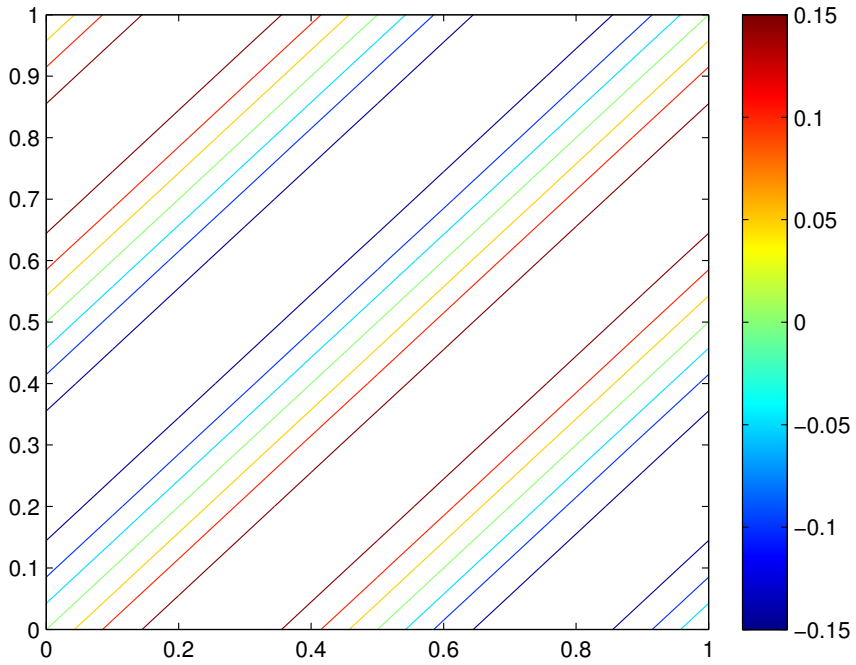

FIGURE 8.1: $s_0(x)$

convex functional by first using a Nesterov accelerated proximal gradient method with 100 iterations and then then finishing by using up to 100 Newton scheme iterations. The code for the proximal gradient method is from Brendan O'Donoghue at https://github.com/bodono/apg. This was enough to approximate the minimum to within 10^{-8} ($|\nabla\Psi|_{l^\infty} < 10^{-8}$).

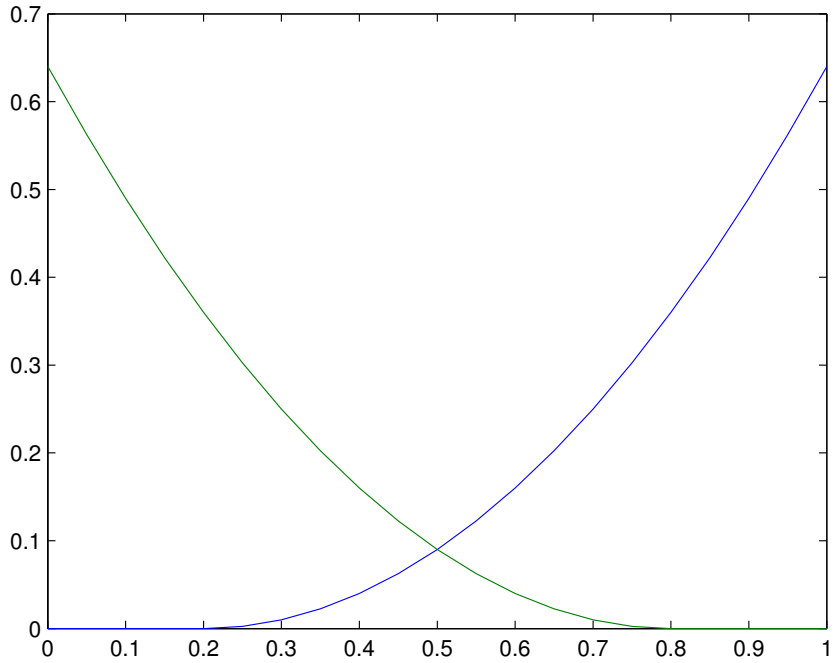

FIGURE 8.2: Relative Permeability Curves

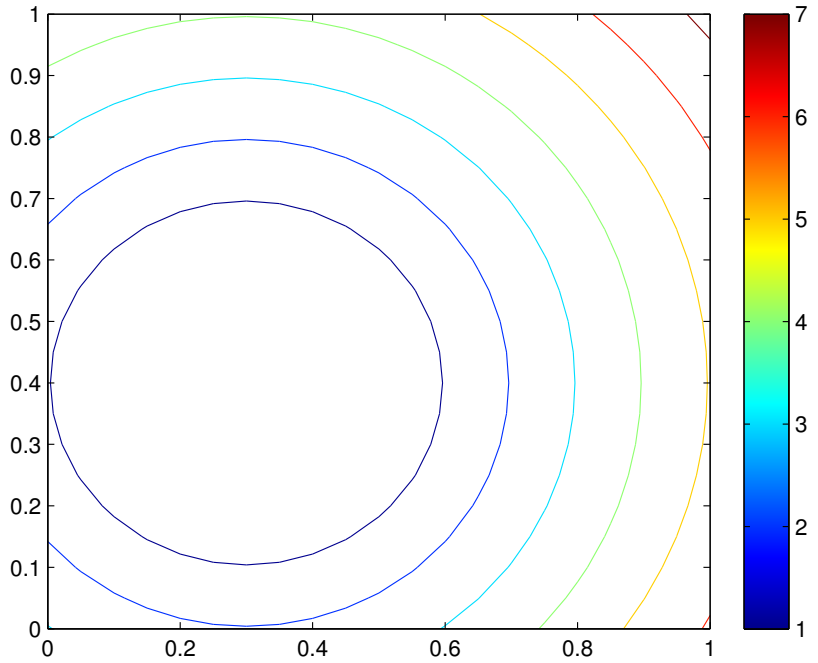

FIGURE 8.3: Permeability $K(x)$

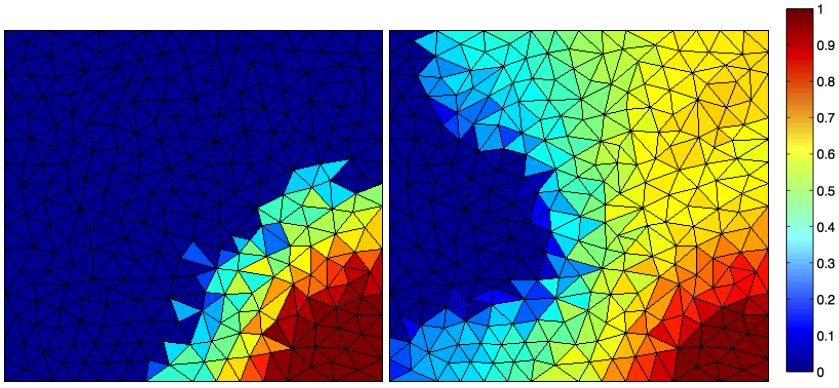

FIGURE 8.4: $t = 1, 5$

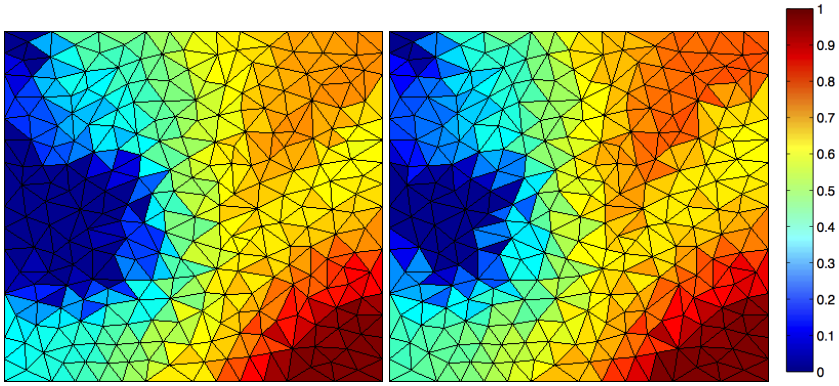

FIGURE 8.5: $t = 10, 15$

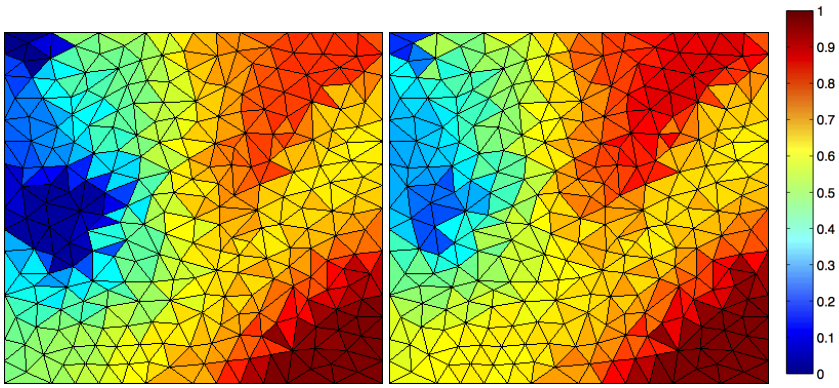

FIGURE 8.6: $t = 20, t = 30$

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