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ABSTRACT

In this thesis, we present a number of results in set theory, particularly in the areas of forcing, large cardinals, and combinatorial set theory.

Chapter 2 concerns covering matrices, combinatorial structures introduced by Viale in his proof that the Singular Cardinals Hypothesis follows from the Proper Forcing Axiom. In the course of this proof and subsequent work with Sharon, Viale isolated two reflection principles, CP and S, which can hold of covering matrices. We investigate covering matrices for which CP and S fail and prove some results about the connections between such covering matrices and various square principles.

In Chapter 3, motivated by the results of Chapter 2, we introduce a number of square principles intermediate between the classical $\square_\kappa$ and $\square(\kappa^+)$. We provide a detailed picture of the implications and independence results which exist between these principles when κ is regular.

In Chapter 4, we address three questions raised by Cummings and Foreman regarding a model of Gitik and Sharon. We first analyze the PCF-theoretic structure of the Gitik-Sharon model, determining the extent of good and bad scales. We then classify the bad points of the bad scales existing in both the Gitik-Sharon model and various other models containing bad scales. Finally, we investigate the ideal of subsets of singular cardinals of countable cofinality carrying good scales.

In Chapter 5, we prove that, assuming large cardinals, it is consistent that there are many singular cardinals μ such that every stationary subset of μ^+ reflects but there are stationary subsets of μ^+ that do not reflect at ordinals of arbitrarily high cofinality. This answers a question raised by Todd Eisworth and is joint work with James Cummings.

In Chapter 6, we extend a result of Gitik, Kanovei, and Koepke regarding intermediate models of Prikry-generic forcing extensions to Radin-generic forcing extensions. Specifically, we characterize intermediate models of forcing extensions by Radin forcing at a large cardinal κ using measure sequences of length less than κ.

In the final brief chapter, we prove some results about iterations of ω_1-Cohen forcing with ω_1-support, answering a question of Justin Moore.

CONTENTS

Acknowledgements i

Abstract ii

Contents iii

1 Introduction 1
1.1 Notation . . . 1
1.2 Combinatorial principles . . . 2
1.3 Forcing . . . 5

2 Covering matrices 7

3 Squares 22

4 Scales 39
4.1 Diagonal supercompact Prikry forcing . . . 39
4.2 Scales in the Gitik-Sharon model . . . 41
4.3 Very weak square in the Gitik-Sharon model . . . 50
4.4 Classifying bad points . . . 51
4.5 Down to $\aleph_{\omega^2}$ . . . 55
4.6 The good ideal . . . 57

5 Bounded stationary reflection 70
5.1 Preliminaries on stationary reflection . . . 70
5.2 Forcing constructions . . . 72
5.3 The main theorem . . . 74

6 Radin forcing 83
6.1 The forcing . . . 83
6.2 Preliminaries on indiscernibles . . . 85
6.3 Coding fresh subsets . . . 88
6.4 Subsets of larger cardinals . . . 93
6.5 Products of Radin forcing . . . 95

6.6 Coding arbitrary subsets of κ 96

7 Forcing iterations with uncountable support **101**

Bibliography **107**

CHAPTER 1

INTRODUCTION

In this thesis, we present a number of results in set theory. Many of the results are directly motivated by the fundamental tension that exists in set theory between reflection phenomena and large cardinals on one hand and combinatorial principles (e.g. various square principles) witnessing incompactness on the other. Reflection and large cardinals place limits on the type of combinatorial structures which can exist, and vice versa, and the interaction between these opposing phenomena is a fertile area of mathematical study. In Chapters 4 and 5, we will be particularly interested in questions of this sort regarding successors of singular cardinals. Combinatorial problems at successors of singular cardinals are inextricably bound up with questions about canonical inner models and large cardinals and often have significant implications for cardinal arithmetic.

In this introduction, we will introduce some of the basic objects of study and present some of the major previous results.

1.1 Notation

Our notation is for the most part standard. Unless otherwise specified, the reference for all notation and definitions is [19]. If A is a set of ordinals, then A' denotes the set of limit ordinals of A, i.e. the set of α such that $A \cap \alpha$ is unbounded in α, and $\mathrm{otp}(A)$ denotes the order type of A. If λ is a regular cardinal, then $\mathrm{cof}(\lambda)$ denotes the class of ordinals of cofinality λ. If $\lambda < \kappa$, then S^κ_λ denotes $\kappa \cap \mathrm{cof}(\lambda)$. If A is a set of cardinals, then $\prod A$ is the set of functions f such that $\mathrm{dom}(f) = A$ and, for every $\lambda \in A$, $f(\lambda) \in \lambda$. If $A = \{\kappa_i \mid i < \eta\}$, we will often write $\prod_{i<\eta} \kappa_i$ instead of $\prod A$ and will write $f(i)$ instead of $f(\kappa_i)$. If A is cofinal in κ and $f, g \in \prod A$, then we write $f <^* g$ to mean that there is $\gamma < \kappa$ such that, for every $\lambda \in A \setminus \gamma$, $f(\lambda) < g(\lambda)$. If f is a function and $X \subset \mathrm{dom}(f)$, then $f[X]$ and $f``X$ both denote the image of X under f.

1.2 Combinatorial principles

Squares

One of the most important set-theoretic combinatorial principle is Jensen's square.

Definition. *Let κ be an infinite cardinal. $\vec{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$ is a $\square_\kappa$-*sequence *if, for all limit ordinals $\alpha < \beta < \kappa^+$,*

1. *C_β is a club in β.*
2. *$\mathrm{otp}(C_\beta) \leq \kappa$.*
3. *If $\alpha \in C'_\beta$, then $C_\beta \cap \alpha = C_\alpha$.*

$\square_\kappa$ is the assertion that there is a $\square_\kappa$-sequence.

A $\square_\kappa$-sequence can be seen as a canonical way of witnessing that the ordinals between κ and κ^+ are singular. The following spectrum of weakenings of Jensen's square principle was introduced by Schimmerling [25].

Definition. *Let κ and λ be cardinals, with κ infinite. $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \rangle$ is a $\square_{\kappa,\lambda}$-sequence if, for all limit ordinals $\alpha < \beta < \kappa^+$, $\mathcal{C}_\beta \neq \emptyset$, $|\mathcal{C}_\beta| \leq \lambda$, and, for all $C \in \mathcal{C}_\beta$,*

1. *C is a club in β.*
2. *$\mathrm{otp}(C) \leq \kappa$.*
3. *If $\alpha \in C'$, then $C \cap \alpha \in \mathcal{C}_\alpha$.*

$\square_{\kappa,\lambda}$ is the assertion that there is a $\square_{\kappa,\lambda}$-sequence. $\square_{\kappa,<\lambda}$ is defined in the same way, requiring that $|\mathcal{C}_\beta| < \lambda$ for all limit $\beta < \kappa^+$.

Remark $\square_{\kappa,\kappa}$ is often denoted $\square^*_\kappa$ and is referred to as *weak square*. $\square_{\kappa,\kappa^+}$ is sometimes referred to as *silly square*. It is a theorem of ZFC that $\square_{\kappa,\kappa^+}$ holds for every infinite cardinal κ.

Theorem 1.1. *(Jensen [20]) If $V = L$, then $\square_\kappa$ holds for every infinite cardinal κ.*

Theorem 1.2. *(Burke, Kanamori (see [25])) If κ is a strongly compact cardinal, then $\square_{\lambda,<\mathrm{cf}(\lambda)}$ fails for every $\lambda \geq \kappa$.*

Approachability

The notion of approachability was introduced by Shelah and has played a very important role in modern combinatorial set theory.

Definition. *Let λ be a regular cardinal.*

1. *Let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be a sequence of bounded subsets of λ. A limit ordinal $\beta < \lambda$ is* approachable with respect to $\vec{a}$ *if there is an unbounded $A \subseteq \beta$ of order type $\mathrm{cf}(\beta)$ such that, for every $\gamma < \beta$, there is $\alpha < \beta$ such that $A \cap \gamma = a_\alpha$.*

2. *Let $S \subseteq \lambda$. $S \in I[\lambda]$ if there is a sequence $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ of bounded subsets of λ and a club $C \subseteq \lambda$ such that every $\beta \in C \cap S$ is approachable with respect to $\vec{a}$.*

3. *If $\lambda = \mu^+$ for some cardinal μ, we say that* the approachability property holds at μ *if $\lambda \in I[\lambda]$. The approachability property at μ is denoted by AP_μ.*

$I[\lambda]$ turns out to be a normal ideal on λ, and so AP_μ is the same as the statement that $I[\lambda]$ is an improper ideal. If $\lambda^{<\lambda} = \lambda$ then, letting $\vec{a}$ and $\vec{b}$ be two enumerations of the bounded subsets of λ in order type λ, the set $\{\beta \mid \{a_\alpha \mid \alpha < \beta\} = \{b_\alpha \mid \alpha < \beta\}\}$ is easily seen to be a club in λ. Thus, the set of ordinals approachable with respect to $\vec{a}$ is equal, modulo clubs, to the set of ordinals approachable with respect to $\vec{b}$. In this case, if we fix an enumeration $\vec{a}$ of the bounded subsets of λ in order type λ, then the set S of ordinals approachable with respect to $\vec{a}$ is a maximal set in $I[\lambda]$ in the sense that that if $T \subseteq \lambda$, then $T \in I[\lambda]$ if and only if $T \setminus S$ is nonstationary. If such a maximal set exists, it is referred to as *the set of approachable points of* λ. See [11] for proofs of these facts and other information on $I[\lambda]$.

Fact 1.3. *If $\mu < \kappa$ are regular cardinals, then $S^{\kappa^+}_\mu \in I[\kappa^+]$.*

If μ is an infinite cardinal, then AP_μ can be seen as a kind of weak square principle. In fact, $\square^*_\mu \Rightarrow AP_\mu$.

Scales

PCF theory, the study of reduced products of sets of regular cardinals, was introduced by Shelah and has found a wide array of applications in various fields of mathematics. One of the central PCF-theoretic notions is that of a scale.

Definition. *Suppose κ is a singular cardinal and A is a cofinal subet of κ consisting of regular cardinals. $\vec{f} = \langle f_\alpha \mid \alpha < \mu \rangle$ is a* scale of length μ in $\prod A$ *if the following hold:*

1. *For all $\alpha < \mu$, $f_\alpha \in \prod A$.*
2. *For all $\alpha < \beta < \mu$, $f_\alpha <^* f_\beta$.*
3. *For all $h \in \prod A$, there is $\alpha < \mu$ such that $h <^* f_\alpha$.*

In other words, $\vec{f}$ is increasing and cofinal in $(\prod A, <^*)$. We say that A carries a scale of length μ if there is a scale of length μ in $\prod A$.

We note that we will typically consider scales in $\prod A$ when $\mathrm{otp}(A) = \mathrm{cf}(\kappa)$, but this need not necessarily be the case. A simple diagonalization argument shows that, if $2^\kappa = \kappa^+$, then every cofinal $A \subset \kappa$ consisting of regular cardinals carries a scale of length κ^+. One of the fundamental results of PCF theory is a theorem of Shelah stating that for every singular cardinal κ, there is an $A \subseteq \kappa$ that carries a scale of length κ^+ [28].

Definition. *Let κ be a singular cardinal, $A \subset \kappa$ a cofinal set of regular cardinals, and $\vec{f} = \langle f_\alpha \mid \alpha < \mu \rangle$ a scale in $\prod A$.*

1. *$\alpha < \mu$ is a* good point for $\vec{f}$ (very good point for $\vec{f}$) *if $\mathrm{cf}(\kappa) < \mathrm{cf}(\alpha) < \kappa$ and there are an unbounded (club) $C \subseteq \alpha$ and $\eta < \kappa$ such that, for all $\gamma < \gamma'$, both in C, $f_\gamma \restriction (A \setminus \eta) < f_{\gamma'} \restriction (A \setminus \eta)$.*
2. *$\alpha < \mu$ is a* bad point for $\vec{f}$ *if $\mathrm{cf}(\kappa) < \mathrm{cf}(\alpha) < \kappa$ and α is not a good point for $\vec{f}$.*
3. *$\vec{f}$ is a* good scale (very good scale) *if $\mu = \kappa^+$ and there is a club $C \subseteq \kappa^+$ such that every $\alpha \in C \cap \mathrm{cof}(> \mathrm{cf}(\kappa))$ is a good point (very good point) for $\vec{f}$.*
4. *$\vec{f}$ is a* bad scale *if $\mu = \kappa^+$ and $\vec{f}$ is not a good scale.*

An intricate web of implications connects the existence of good and very good scales with various other combinatorial principles, including squares, approachability, and stationary reflection, at successors of singular cardinals. We record some of the relevant facts here. Let κ be a singular cardinal.

- If $A \subseteq \kappa$ carries a good scale, then every scale in $\prod A$ is good [13].

- If $\square_{\kappa,\lambda}$ holds for some $\lambda < \kappa$, then every $A \subseteq \kappa$ which carries a scale of length κ^+ carries a very good scale [6].
- If AP_κ holds, then every $A \subseteq \kappa$ which carries a scale of length κ^+ carries a good scale [13].

The interested reader is referred to [6] and [13] for more details.

1.3 Forcing

We assume familiarity with forcing. For a complete exposition of the basic facts about forcing, we refer the reader to [23] and [19]. For a treatment of iterated forcing, see [3].

Remark We observe the convention that, if $\mathbb{P}$ is a forcing poset and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is stronger, or carries more information, than p.

We now review some definitions and results that will be used later.

Definition. *Let $\mathbb{P}$ be a partial order, and let $X \subseteq \mathbb{P}$. A* lower bound *for X is a condition $q \in \mathbb{P}$ such that, for all $p \in X$, $q \leq p$. If there is a unique condition $r \in \mathbb{P}$ such that r is a lower bound for X and, if q is a lower bound for X, then $q \leq r$, then this condition r is denoted by* $\inf(X)$. *If there is no such condition, then* $\inf(X)$ *is undefined.*

Definition. *Let $\mathbb{P}$ be a partial order and let β be an ordinal.*

1. *The two-player game $G_\beta(\mathbb{P})$ is defined as follows: Players I and II alternately play entries in $\langle p_\alpha \mid \alpha < \beta \rangle$, a decreasing sequence of conditions in $\mathbb{P}$ with $p_0 = 1_{\mathbb{P}}$. Player I plays at odd stages, and Player II plays at even stages (including all limit stages). If there is an even stage $\alpha < \beta$ at which Player II can not play, then Player I wins. Otherwise, Player II wins.*
2. *$G^*_\beta(\mathbb{P})$ is defined just as $G_\beta(\mathbb{P})$ except Player II no longer plays at limit stages. Instead, if $\alpha < \beta$ is a limit ordinal, then $p_\alpha = \inf(\{p_\gamma \mid \gamma < \alpha\})$ if such a condition exists. If, for some limit $\alpha < \beta$, such a condition does not exist, then Player I wins. Otherwise, Player II wins.*
3. *$\mathbb{P}$ is* β-strategically closed *if Player II has a winning strategy for the game $G_\beta(\mathbb{P})$. $\mathbb{P}$ is said to be strongly β-strategically closed if Player II has a winning strategy for the game $G^*_\beta(\mathbb{P})$.* $< \beta$-strategically closed *and* strongly $< \beta$-strategically closed *are defined in the obvious way.*

Fact 1.4. *Let $\mathbb{P}$ be a partial order and let κ be a cardinal. If $\mathbb{P}$ is $(\kappa+1)$-strategically closed, then forcing with $\mathbb{P}$ does not add any new κ-sequences of ordinals.*

Fact 1.5. *[24] Let κ be a regular cardinal, and let $\kappa < \lambda < \mu$. Suppose that, in $V^{\mathrm{Coll}(\kappa,<\lambda)}$, $\mathbb{P}$ is a separative, strongly κ-strategically closed partial order and $|\mathbb{P}| < \mu$. Let i be the natural complete embedding of $\mathrm{Coll}(\kappa, <\lambda)$ into $\mathrm{Coll}(\kappa, <\mu)$ (namely, the identity embedding). Then i can be extended to a complete embedding j of $\mathrm{Coll}(\kappa, <\lambda) * \mathbb{P}$ into $\mathrm{Coll}(\kappa, <\mu)$ so that the quotient forcing $\mathrm{Coll}(\kappa, <\mu)/j[\mathrm{Coll}(\kappa, <\lambda) * \mathbb{P}]$ is κ-closed.*

Fact 1.6. *Let κ be a regular cardinal and let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid j < \alpha, i \leq \alpha \rangle$ be a forcing iteration in which inverse limits are taken at all limit stages of cofinality $< \kappa$ and such that, for all $i < \alpha$, $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is (strongly) κ-strategically closed". Then $\mathbb{P}_\alpha$ is (strongly) κ-strategically closed.*

One of the most important applications of approachability is its implication that the stationarity of certain sets is preserved by sufficiently closed forcing.

Fact 1.7. *Let μ and κ be cardinals. Suppose that S is a stationary subset of $S^{\kappa^+}_{<\mu}$, $S \in I[\kappa^+]$, and $\mathbb{P}$ is a μ-closed forcing poset. Then S remains stationary in $V^{\mathbb{P}}$.*

CHAPTER 2

COVERING MATRICES

Definition. *Let $\theta < \lambda$ be regular cardinals. $\mathcal{D} = \{D(i,\beta) \mid i < \theta, \beta < \lambda\}$ is a θ-*covering matrix* for λ if:*

1. *For all $\beta < \lambda$, $\beta = \bigcup_{i<\theta} D(i,\beta)$.*
2. *For all $\beta < \lambda$ and all $i < j < \theta$, $D(i,\beta) \subseteq D(j,\beta)$.*
3. *For all $\beta < \gamma < \lambda$ and all $i < \theta$, there is $j < \theta$ such that $D(i,\beta) \subseteq D(j,\gamma)$.*

$\beta_{\mathcal{D}}$ is the least β such that for all $\gamma < \lambda$ and all $i < \theta$, $\mathrm{otp}(D(i,\gamma)) < \beta$. $\mathcal{D}$ is normal *if $\beta_{\mathcal{D}} < \lambda$.*

$\mathcal{D}$ is transitive *if, for all $\alpha < \beta < \lambda$ and all $i < \theta$, if $\alpha \in D(i,\beta)$, then $D(i,\alpha) \subseteq D(i,\beta)$.*

$\mathcal{D}$ is uniform *if, for all $\beta < \lambda$, there is $i < \theta$ such that $D(j,\beta)$ contains a club in β for all $j \geq i$. (Note that this is equivalent to the statement that there is $i < \theta$ such that $D(i,\beta)$ contains a club in β.)*

$\mathcal{D}$ is closed *if, for all $\beta < \lambda$, all $i < \theta$, and all $X \in [D(i,\beta)]^{\leq\theta}$, $\sup X \in D(i,\beta)$.*

Much of this chapter will be concerned with constructing covering matrices for which the following two reflection properties fail.

Definition. *Let $\theta < \lambda$ be regular cardinals, and let $\mathcal{D}$ be a θ-covering matrix for λ.*

1. *$\mathrm{CP}(\mathcal{D})$ holds if there is an unbounded $T \subseteq \lambda$ such that, for every $X \in [T]^\theta$, there are $i < \theta$ and $\beta < \lambda$ such that $X \subseteq D(i,\beta)$ (in this case, we say that $\mathcal{D}$* covers *$[T]^\theta$).*
2. *$\mathrm{S}(\mathcal{D})$ holds if there is a stationary $S \subseteq \lambda$ such that, for every family $\{S_j \mid j < \theta\}$ of stationary subsets of S, there are $i < \theta$ and $\beta < \lambda$ such that, for every $j < \theta$, $S_j \cap D(i,\beta) \neq \emptyset$.*

Definition. *Let $\theta < \lambda$ be regular cardinals. $\mathrm{R}(\lambda,\theta)$ is the statement that there is a stationary $S \subseteq \lambda$ such that for every family $\{S_j \mid j < \theta\}$ of stationary subsets of S, there is $\alpha < \lambda$ of uncountable cofinality such that, for all $j < \theta$, S_j reflects at α.*

If $\mathcal{D}$ is a nice enough covering matrix, then $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ are equivalent and $\mathrm{R}(\lambda, \theta)$ implies both. The following is proved in [26]:

Lemma 2.1. *Let $\theta < \lambda$ be regular cardinals, and let $\mathcal{D}$ be a θ-covering matrix for λ.*

1. *If $\mathcal{D}$ is transitive, then $\mathrm{S}(\mathcal{D})$ implies $\mathrm{CP}(\mathcal{D})$.*
2. *If $\mathcal{D}$ is closed, then $\mathrm{CP}(\mathcal{D})$ implies $\mathrm{S}(\mathcal{D})$.*
3. *If $\mathcal{D}$ is uniform, then $\mathrm{R}(\lambda, \theta)$ implies $\mathrm{S}(\mathcal{D})$.*

We note briefly that scales give natural examples of covering matrices for which CP and S hold.

Proposition 2.2. *Let μ be a singular cardinal with $\mathrm{cf}(\mu) = \theta$. Let $\lambda = \mu^+$, let $\langle \mu_i \mid i < \theta \rangle$ be an increasing sequence of regular cardinals cofinal in μ such that $\theta < \mu_0$. Suppose $\vec{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ is a scale in*

$$\prod_{i<\theta} \mu_i.$$

Then, letting $\mathcal{D} = \{D(i, \beta) \mid i < \theta, \beta < \lambda\}$ be defined by $D(i, \beta) = \{\alpha < \beta \mid$ for all $j \geq i, f_\alpha(j) < f_\beta(j)\}$, we have that $\mathcal{D}$ is a transitive covering matrix, and $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ hold.

Proof. The fact that $\mathcal{D}$ is a transitive covering matrix follows easily from the fact that $\vec{f}$ is a scale. We now show that, for every $X \in [\lambda]^\theta$, there are $i < \theta$ and $\beta < \lambda$ such that $X \subseteq D(i, \beta)$. This will immediately imply that $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ both hold. To this end, fix $X \in [\lambda]^\theta$. Define

$$g \in \prod_{i<\theta} \mu_i$$

by $g(i) = \sup(\{f_\alpha(i) \mid \alpha \in X\})$. Find $\beta < \lambda$ such that $g <^* f_\beta$, and let $i < \theta$ be such that $g(j) < f_\beta(j)$ for all $j \geq i$. Then $X \subseteq D(i, \beta)$. $\square$

However, the following proposition shows that, under certain cardinal arithmetic assumptions (for example, if $\theta = \omega$ or if μ is strong limit), the covering matrix $\mathcal{D}$ defined in the preceding proposition cannot be normal.

Proposition 2.3. *Assume the same hypotheses as in Proposition 2.2 hold, and let $\mathcal{D}$ be the covering matrix defined in the statement of that proposition. Assume moreover that, for all $i < \theta$, $\mu_i^{|i|} \leq \mu$. Then $\mathcal{D}$ is not normal.*

Proof. Suppose for sake of contradiction that $\beta_{\mathcal{D}} < \lambda$. For each $j < \theta$, use the fact that $\mu_j^{|j|} \leq \mu$ to find an unbounded $B_j \subseteq \lambda$ and

$$f^j \in \prod_{i \leq j} \mu_i$$

such that, for every $\alpha \in B_j$ and every $i \leq j$, $f_\alpha(i) = f^j(i)$. Find $\{B_j^* \mid j < \theta\}$ such that, for every $j < \theta$, $B_j^* \subseteq B_j$ and $|B_j| = \mu_j$, and, letting

$$B^* = \bigcup_{j<\theta} B_j,$$

we have $\mathrm{otp}(B^*) > \beta_{\mathcal{D}}$. Now, for each $j < \theta$, define

$$g_j \in \prod_{i<\theta} \mu_i$$

by letting $g_j(i) = f^j(i) + 1$ for $i \leq j$ and $g_j(i) = \sup(\{f_\alpha(i) + 1 \mid \alpha \in B_j^*\})$ for $i > j$. Define

$$g \in \prod_{i<\theta} \mu_i$$

by letting $g(i) = \sup(\{g_j(i) \mid j < \theta\})$ for all $i < \theta$. Finally, find $\beta < \lambda$ and $i < \theta$ such that $g(j) < f_\beta(j)$ for all $j \geq i$. It follows that $B^* \subseteq D(i, \beta)$, contradicting the fact that $\mathrm{otp}(B^*) > \beta_{\mathcal{D}}$. $\square$

The following lemma is a key component of Viale's proof that SCH follows from PFA.

Lemma 2.4. *Let $\lambda > \aleph_2$ be a regular cardinal. PFA implies that $\mathrm{CP}(\mathcal{D})$ holds for every ω-covering matrix $\mathcal{D}$ for λ.*

Let κ be an uncountable regular cardinal. We first consider the question of what types of κ covering matrices for κ^+ can exist. In particular, we will be interested in the existence of a covering matrix that is transitive, normal, and uniform. It turns out that one can always ask for any two of these three properties.

Proposition 2.5. *There is a uniform, transitive κ-covering matrix for κ^+.*

Proof. Simply let $D(i, \beta) = \beta$ for every $i < \kappa$ and $\beta < \kappa^+$. $\square$

Proposition 2.6. *There is a transitive, normal κ-covering matrix for κ^+.*

Proof. For each $\alpha < \kappa^+$, let $\phi_\alpha : \kappa \to \alpha$ be a surjection. For $i < \kappa$ and $\beta < \kappa^+$, recursively define

$$D(i,\beta) = \phi_\beta[i] \cup \bigcup_{\alpha \in \phi_\beta[i]} D(i,\alpha).$$

Let $\mathcal{D} = \{D(i,\beta) \mid i < \kappa, \beta < \kappa^+\}$.

It is clear that $\mathcal{D}$ is a covering matrix and, inductively, $|D(i,\beta)| < \kappa$ for every i and β. Thus, $\beta_{\mathcal{D}} \leq \kappa$, so $\mathcal{D}$ is normal. For each $i < \kappa$, we show by induction on $\beta < \kappa^+$ that if $\alpha \in D(i,\beta)$, then $D(i,\alpha) \subseteq D(i,\beta)$. Indeed, if $\alpha \in \phi_\beta[i]$, then the conclusion holds by definition, while if $\alpha \in D(i,\gamma)$ for some $\gamma \in \phi_\beta[i]$, then by induction $D(i,\alpha) \subseteq D(i,\gamma) \subseteq D(i,\beta)$. Thus, $\mathcal{D}$ is transitive. $\square$

The following lemma on ordinal arithmetic will be quite useful in our construction of covering matrices.

Lemma 2.7. *Let θ be a regular cardinal, $\mu < \theta$, and $m < \omega$. Suppose that for each $i < \mu$, X_i is a set of ordinals such that $\mathrm{otp}(X_i) < \theta^m$. Let $X = \bigcup_{i<\mu} X_i$. Then $\mathrm{otp}(X) < \theta^m$.*

Proof. By induction on m. The conclusion is immediate for $m = 0$ and $m = 1$. Let $m \geq 2$ and suppose for sake of contradiction that $\mathrm{otp}(X) \geq \theta^m$. Fix $A \subseteq X$ of order type exactly θ^m. Enumerate A in increasing order as $A = \{a_\alpha \mid \alpha < \theta^m\}$. For each $\beta < \theta$, let $A_\beta = \{a_{\theta^{m-1}\cdot\beta+\gamma} \mid \gamma < \theta^{m-1}\}$. Then $\mathrm{otp}(A_\beta) = \theta^{m-1}$, so by the induction hypothesis, there is $i_\beta < \mu$ such that $\mathrm{otp}(X_{i_\beta} \cap A_\beta) = \theta^{m-1}$. Thus, there is an $i^* < \mu$ such that $i_\beta = i^*$ for unboundedly many $\beta < \theta$. But then $\mathrm{otp}(X_{i^*}) \geq \theta^m$. Contradiction. $\square$

Proposition 2.8. *There is a uniform, normal κ-covering matrix for κ^+.*

Proof. For each $\alpha < \kappa^+$, let C_α be a club in α such that $\mathrm{otp}(C_\alpha) \leq \kappa$, and let $\phi_\alpha : \kappa \to \alpha$ be a surjection. We define $\mathcal{D} = \{D(i,\beta) \mid i < \kappa, \beta < \kappa^+\}$ by recursion on β and, for fixed β, by recursion on i. For each $\beta < \kappa^+$, let $D(0,\beta) = C_\beta$. If $i < \kappa$ is a limit ordinal, let

$$D(i,\beta) = \bigcup_{j<i} D(j,\beta).$$

Finally, let

$$D(i+1,\beta) = D(i,\beta) \cup \phi_\beta[i] \cup \bigcup_{\alpha \in \phi_\beta[i]} D(i+1,\alpha).$$

It is easily verified that $\mathcal{D}$ is a κ-covering matrix for κ^+ and, by construction, $D(0, \beta)$ contains a club in β for each $\beta < \kappa^+$. It remains to show that $\mathcal{D}$ is normal. We in fact prove by induction on $\beta < \kappa^+$ and, for fixed β, by induction on $i < \kappa$, that $\mathrm{otp}(D(i, \beta)) < \kappa^2$ for all i and β. Fix $i < \kappa$ and $\beta < \kappa^+$. By the inductive hypothesis, $D(i, \beta)$ is a union of fewer than κ-many sets, all of which have order type less than κ^2. Then, by the previous lemma, $\mathrm{otp}(D(i, \beta)) < \kappa^2$. Thus, $\beta_{\mathcal{D}} \leq \kappa^2 < \kappa^+$, so $\mathcal{D}$ is normal. $\square$

However, we can not always get all three properties, since $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ necessarily fail for a transitive, normal, uniform κ-covering matrix for κ^+.

Lemma 2.9. *If $\mathcal{D}$ is a normal κ-covering matrix for κ^+, then $\mathrm{CP}(\mathcal{D})$ fails.*

Proof. Let T be an unbounded subset of κ^+. Then any $X \in [T]^\kappa$ whose order type is greater than $\beta_{\mathcal{D}}$ can not be contained in any $D(i, \beta)$. $\square$

Since $\mathrm{S}(\mathcal{D})$ implies $\mathrm{CP}(\mathcal{D})$ whenever $\mathcal{D}$ is transitive, $\mathrm{S}(\mathcal{D})$ fails for every transitive, normal κ-covering matrix for κ^+.

Proposition 2.10. *If $\mathrm{R}(\kappa^+, \kappa)$ holds, then there are no transitive, normal, uniform κ-covering matrices for κ^+.*

Proof. $\mathrm{R}(\kappa^+, \kappa)$ implies that every uniform κ-covering matrix $\mathcal{D}$ for κ^+ satisfies $\mathrm{S}(\mathcal{D})$. But we saw above that $\mathrm{S}(\mathcal{D})$ fails for every transitive, normal κ-covering matrix for κ^+. $\square$

As a corollary, we see that the forcing axiom Martin's Maximum, introduced in [14], places limitations on the existence of certain ω_1-covering matrices for ω_2.

Corollary 2.11. *If Martin's Maximum holds, then there are no transitive, normal, uniform ω_1-covering matrices for ω_2.*

Proof. Martin's Maximum implies (see [14] for details) that $\mathrm{R}(\aleph_n, \aleph_1)$ holds for every $1 < n < \omega$ as witnessed by $S^{\aleph_n}_{\aleph_0}$. $\square$

The existence of a transitive, normal, uniform covering matrix does follow, however, from sufficiently strong square principles.

Proposition 2.12. *Suppose κ is a regular cardinal and $\square_{\kappa, <\kappa}$ holds. Then there is a transitive, normal, uniform κ-covering matrix for κ^+.*

Proof. Let $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ be a $\square_{\kappa,<\kappa}$-sequence. We construct a transitive, normal, uniform κ-covering matrix for κ^+, $\mathcal{D} = \{D(i,\beta) : i < \kappa,\ \beta < \kappa^+\}$, by recursion on β as follows:

- $D(i,\beta+1) = D(i,\beta) \cup \{\beta\}$
- If β is a limit ordinal and $\mathrm{cf}(\beta) < \kappa$, fix E, a club in β of order type less than κ, and let

$$D(i,\beta) = \begin{cases} \emptyset & \text{if } \sup_{C \in \mathcal{C}_\beta} \mathrm{otp}(C) \geq \omega \cdot i \\ E \cup \bigcup_{\alpha \in E} D(i,\alpha) & \text{if } \sup_{C \in \mathcal{C}_\beta} \mathrm{otp}(C) < \omega \cdot i \end{cases}$$

- If $\mathrm{cf}(\beta) = \kappa$, fix $C \in \mathcal{C}_\beta$, and let

$$D(i,\beta) = C' \cup \bigcup_{\alpha \in C'} D(i,\alpha)$$

It is routine to check that $\mathcal{D}$ is a uniform κ-covering matrix for κ^+, and an easy induction shows that it is transitive. We claim that $\mathcal{D}$ is normal. We prove that $\mathrm{otp}(D(i,\beta)) < \kappa^2$ for every $i < \kappa$ and $\beta < \kappa^+$ by induction on β. If β is a successor ordinal or a limit ordinal of cofinality less than κ, then $D(i,\beta)$ is the union of fewer than κ-many sets, each, by the induction hypothesis, of order type less than κ^2. Thus, $\mathrm{otp}(D(i,\beta)) < \kappa^2$. Suppose $\mathrm{cf}(\beta) = \kappa$. Let $C \in \mathcal{C}_\beta$ be the club used in the construction of $D(i,\beta)$. Enumerate C' in increasing order as $\{\alpha_\gamma \mid \gamma < \kappa\}$. For each $\gamma < \kappa$, $C \cap \alpha_\gamma \in \mathcal{C}_{\alpha_\gamma}$, so for $\gamma \geq i$, $D(i,\alpha_\gamma) = \emptyset$. Thus, $D(i,\beta)$ is itself a union of fewer than κ-many sets, each of order type less than κ^2, so $\mathrm{otp}(D(i,\beta)) < \kappa^2$. $\square$

We now show that $\square_{\kappa,<\kappa}$ is the optimal hypothesis in the previous proposition by producing, via a standard argument due originally to Baumgartner [2], a model in which $\square^*_\kappa$ and $\mathrm{R}(\kappa^+,\kappa)$ both hold. Recall that, if $\kappa^{<\kappa} = \kappa$, then $\square^*_\kappa$ holds.

Proposition 2.13. *Let $\kappa < \lambda$, with κ regular and λ measurable. Let $\mathbb{P} = \mathrm{Coll}(\kappa, < \lambda)$, and let G be $\mathbb{P}$-generic over V. Then, in $V[G]$, $\kappa^{<\kappa} = \kappa$ and $R(\lambda,\kappa)$ hold.*

Proof. First note that, since $\mathbb{P}$ is κ-closed, it doesn't add any new bounded subsets of κ, so, since λ is measurable in V, $\kappa^{<\kappa} = \kappa$ in $V[G]$, so $\square^*_\kappa$ holds in $V[G]$.

We now show that $R(\lambda,\kappa)$ holds in $V[G]$ as witnessed by $S^\lambda_{<\kappa}$. Let $j : V \to M$ be an elementary embedding with M transitive and $\mathrm{crit}(j) = \lambda$, and let H be $\mathrm{Coll}(\kappa, < j(\lambda))$-generic over V such that $G \subset H$. We can then lift the embedding to $j : V[G] \to M[H]$. Let $\{S_\alpha \mid \alpha < \kappa\}$ be a family of stationary subsets of $S^\lambda_{<\kappa}$. Note that $j(\{S_\alpha \mid \alpha < \kappa\}) = \{j(S_\alpha) \mid \alpha < \kappa\}$ and, for each $\alpha < \kappa$, $j(S_\alpha) \cap \lambda = S_\alpha$. Since $\mathrm{Coll}(\kappa, [\lambda, j(\lambda))$ is κ-closed, Facts 1.3 and 1.7 imply that each S_α remains stationary in $V[H]$ and therefore also in $M[H]$. Thus, in $M[H]$, the sets $\{j(S_\alpha) \mid \alpha < \kappa\}$ reflect simultaneously to a point of uncountable cofinality below $j(\lambda)$, namely λ, so, by elementarity, in $V[G]$, the sets $\{S_\alpha \mid \alpha < \kappa\}$ reflect simultaneously to a point of uncountable cofinality below λ. Thus, in $V[G]$, $R(\lambda, \kappa) = R(\kappa^+, \kappa)$ holds. $\square$

Thus, $\square^*_\kappa$ does not imply the existence of a transitive, normal, uniform κ-covering matrix for κ^+.

We now prove that the converse of Proposition 2.12 does not hold in general by showing that, if κ is regular and not strongly inaccessible, one can force to add a transitive, normal, uniform κ-covering matrix $\mathcal{D}$ for κ^+ without adding a $\square_{\kappa,<\kappa}$-sequence. The argument is similar to that introduced by Jensen to distinguish between various weak square principles (see [20]). We first introduce a forcing poset, which we will denote here by $\mathbb{Q}$, which will add a transitive, normal, uniform κ-covering matrix for κ^+. Elements of $\mathbb{Q}$ will be of the form $q = \{D^q(i,\beta) \mid i < \kappa,\ \beta \leq \beta^q\}$ and will satisfy the following:

- $\beta^q < \kappa^+$.
- For all $\beta \leq \beta^q$, $\beta = \bigcup_{i<\kappa} D^q(i,\beta)$.
- For all $\beta \leq \beta^q$ and all $i < j < \kappa$, $D^q(i,\beta) \subseteq D^q(j,\beta)$.
- For all $\alpha < \beta \leq \beta^q$ and all $i < \kappa$, if $\alpha \in D^q(i,\beta)$, then $D^q(i,\alpha) \subseteq D^q(i,\beta)$.
- For all $i < \kappa$ and all $\beta \leq \beta^q$, $\mathrm{otp}(D^q(i,\beta)) < \kappa^2$.
- For all $\beta \leq \beta^q$, $D^q(i,\beta)$ contains a club in β for sufficiently large $i < \kappa$.

For $p, q \in \mathbb{Q}$, $p \leq q$ if and only if p end-extends q, i.e. $\beta^p \geq \beta^q$ and $D^p(i,\beta) = D^q(i,\beta)$ for every $i < \kappa$ and $\beta \leq \beta^q$.

Proposition 2.14. *$\mathbb{Q}$ is κ-closed.*

Proof. Suppose $\mu < \kappa$ and $\langle q_\alpha \mid \alpha < \mu \rangle$ is a descending sequence of conditions from $\mathbb{Q}$. We will define $q \in \mathbb{Q}$ such that for all $\alpha < \mu$, $q \leq q_\alpha$. Let $\beta^q = \sup(\{\beta^{q_\alpha} \mid \alpha < \mu\})$. We may assume without loss of generality that the β^{q_α} were strictly increasing, so, for all $\alpha < \mu$, $\beta^{q_\alpha} < \beta^q$. For all $\beta < \beta^q$ and $i < \kappa$, let $D^q(i, \beta) = D^{q_\alpha}(i, \beta)$ for some $\alpha < \mu$ such that $\beta \leq \beta^{q_\alpha}$. It remains to define $D^q(i, \beta^q)$ for $i < \kappa$. To this end, fix a club $C \subseteq \beta^q$ whose order type is cf(β^q). Note that $|C| < \kappa$. For $i < \kappa$, let $D^q(i, \beta^q) = C \cup \bigcup_{\beta \in C} D^q(i, \beta)$. Since $D^q(i, \beta^q)$ is the union of fewer than κ-many sets of order type less than κ^2, the order type of $D^q(i, \beta^q)$ is also less than κ^2. It easily follows that $q \in \mathbb{Q}$ and, for all $\alpha < \mu$, $q \leq q_\alpha$. □

Proposition 2.15. *$\mathbb{Q}$ is $(\kappa+1)$-strategically closed.*

Proof. We need to exhibit a winning strategy for Player II in the game $G_{\kappa+1}(\mathbb{Q})$. Suppose $\gamma \leq \kappa$ is an even ordinal and that $\langle q_\alpha \mid \alpha < \gamma \rangle$ has been played. We specify Player II's next move, q_γ. Let $C_\gamma = \{\beta^{q_\alpha} \mid \alpha < \gamma \text{ is an even or limit ordinal}\}$ (C_γ is thus the set of the top points of the conditions played by Player II thus far). We assume the following induction hypotheses are satisfied:

1. C_γ is closed beneath its supremum.
2. If $\alpha < \alpha' < \gamma$ and α, α' are even ordinals, then $\beta^{q_\alpha} < \beta^{q_{\alpha'}}$.
3. For all even ordinals $\alpha < \gamma$ and all $i < \alpha$, $D^{q_\alpha}(i, \beta^{q_\alpha}) = \emptyset$.

There are three cases.

Case 1: γ is a successor ordinal: Suppose $\gamma = \gamma' + 1$. Let $\beta^{q_\gamma} = \beta^{q_{\gamma'}} + 1$. For $i < \kappa$ and $\beta \leq \beta^{q_{\gamma'}}$, let $D^{q_\gamma}(i, \beta) = D^{q_{\gamma'}}(i, \beta)$. For $i < \kappa$, let

$$D^{q_\gamma}(i, \beta^{q_\gamma}) = \begin{cases} \emptyset & \text{if } i < \gamma \\ \{\beta^{q_{\gamma'}}\} \cup D^{q_{\gamma'}}(i, \beta^{q_{\gamma'}}) & \text{if } i \geq \gamma \end{cases}.$$

Case 2: $\gamma < \kappa$ is a limit ordinal: Let $\beta^{q_\gamma} = \sup(C_\gamma)$ (so C_γ is club in β^{q_γ}). For $i < \kappa$ and $\beta < \beta^{q_\gamma}$, let $D^{q_\gamma}(i, \beta) = D^{q_\alpha}(i, \beta)$ for some $\alpha < \gamma$ such that $\beta \leq \beta^{q_\alpha}$. For $i < \kappa$, let

$$D^{q_\gamma}(i, \beta^{q_\gamma}) = \begin{cases} \emptyset & \text{if } i < \gamma \\ C_\gamma \cup \bigcup_{\beta \in C_\gamma} D^{q_\gamma}(i, \beta) & \text{if } i \geq \gamma \end{cases}.$$

For all $i < \kappa$, $D^{q_\gamma}(i, \beta^{q_\gamma})$ is the union of fewer than κ-many sets of order type less than κ^2 and thus has order type less than κ^2.

Case 3: $\gamma = \kappa$**:** Let $\beta^{q_\gamma} = \sup(C_\gamma)$. For $i < \kappa$ and $\beta < \beta^{q_\gamma}$, let $D^{q_\gamma}(i, \beta) = D^{q_\alpha}(i, \beta)$ for some $\alpha < \gamma$ such that $\beta \leq \beta^{q_\alpha}$. For $i < \kappa$, let

$$D^{q_\gamma}(i, \beta^{q_\gamma}) = C_\gamma \cup \bigcup_{\beta \in C_\gamma} D^{q_\gamma}(i, \beta).$$

Since, for each $i < \kappa$, $D^{q_\gamma}(i, \beta) = \emptyset$ for all $\beta \in C_\gamma \setminus \beta^{q_i}$, each $D^{q_\gamma}(i, \beta^{q_\gamma})$ is the union of fewer than κ-many sets of order type less than κ^2 and thus has order type less than κ^2.

It is easy to check that in each case the inductive hypotheses are preserved and that this provides a winning strategy for Player II in $G_{\kappa+1}(\mathbb{Q})$. Thus, $\mathbb{Q}$ is $(\kappa + 1)$-strategically closed. $\square$

Proposition 2.16. *If $2^\kappa = \kappa^+$, then $\mathbb{Q}$ is a cardinal-preserving forcing poset that adds a transitive, normal, uniform κ-covering matrix for κ^+.*

Proof. Since $\mathbb{Q}$ is $(\kappa + 1)$-strategically closed, forcing with $\mathbb{Q}$ does not add any new κ-sequences of ordinals, so all cardinals $\leq \kappa^+$ are preserved. Since $2^\kappa = \kappa^+$, $|\mathbb{Q}| = \kappa^+$, so $\mathbb{Q}$ has the κ^{++}-chain condition and hence preserves all cardinals $\geq \kappa^{++}$. Finally, a proof similar to that of the previous proposition yields the fact that for all $\alpha < \kappa^+$, the set $E_\alpha = \{q \mid \beta^q \geq \alpha\}$ is dense in $\mathbb{Q}$. Thus, if G is $\mathbb{Q}$-generic over V, then $\bigcup G$ is a transitive, normal, uniform κ-covering matrix for κ^+. $\square$

Given a κ-covering matrix $\mathcal{D}$ for κ^+, we define a forcing notion $\mathbb{T}_\mathcal{D}$ whose purpose is to add a club in κ^+ of order type κ which interacts nicely with $\mathcal{D}$. It plays a similar role in our argument as the forcing to thread a square sequence plays in [6] and [20]. Elements of $\mathbb{T}_\mathcal{D}$ are sets t such that:

1. t is a closed, bounded subset of κ^+.

2. $|t| < \kappa$.

3. If t is enumerated in increasing order as $\langle \tau_\alpha \mid \alpha \leq \gamma_t < \kappa \rangle$, then for all $\alpha \leq \gamma_t$ and all $i < \alpha$, $D(i, \tau_\alpha) = \emptyset$.

If $t, t' \in \mathbb{T}_\mathcal{D}$, then $t' \leq t$ if and only if t' end-extends t, i.e. $\gamma_{t'} \geq \gamma_t$ and, for all $\alpha \leq \gamma_t$, $\tau'_\alpha = \tau_\alpha$.

In general, $\mathbb{T}_\mathcal{D}$ may be very poorly behaved. For example, the set it adds may not be cofinal in κ^+ and, even if it is, its order type might be less than κ. However, if $\mathcal{D}$ has been added by $\mathbb{Q}$ immediately prior to forcing with $\mathbb{T}_\mathcal{D}$, then it has some nice properties.

Proposition 2.17. *Let G be $\mathbb{Q}$-generic over V, and let $\mathcal{D} = \bigcup G$. Then, in $V[G]$, for all $\alpha < \kappa^+$, $E_\alpha = \{t \mid \alpha \leq \tau_{\gamma_t}\}$ is dense in $\mathbb{T}_\mathcal{D}$.*

Proof. This follows from the fact that, in V, for every $\alpha < \kappa^+$ and every $j < \kappa$, the set $E_{j,\alpha} = \{q \mid \alpha \leq \beta^q$ and for every $i < j, D(i, \beta^q) = \emptyset\}$ is easily seen to be dense in $\mathbb{Q}$. $\square$

Proposition 2.18. *If $\mathcal{D}$ is the covering matrix added by $\mathbb{Q}$, then $\mathbb{Q} * \dot{\mathbb{T}}_\mathcal{D}$ has a κ-closed dense subset.*

Proof. Let $\mathbb{S} = \{(q, \dot{t}) \mid q$ decides the value of $\dot{t}$ and $q \Vdash$ "$\dot{\tau}_{\gamma_t} = \beta^q$"$\}$. We first show that $\mathbb{S}$ is dense in $\mathbb{Q} * \dot{\mathbb{T}}_\mathcal{D}$. To this end, let $(q_0, \dot{t}_0) \in \mathbb{Q} * \dot{\mathbb{T}}_\mathcal{D}$. Find $q_1 \leq q_0$ such that q_1 decides the value of $\dot{t}$ to be some $\langle \tau_\alpha \mid \alpha \leq \gamma_t < \kappa \rangle$ (this is possible, since $\mathbb{Q}$ is $(\kappa + 1)$-strategically closed and hence doesn't add any new κ-sequences of ordinals). Without loss of generality, $\beta^{q_1} > \tau_{\gamma_t}$. Now form $q_2 \leq q_1$ by setting $\beta^{q_2} = \beta^{q_1} + 1$ and

$$D^{q_2}(i, \beta^{q_2}) = \begin{cases} \emptyset & \text{if } i \leq \gamma_t \\ \{\beta^{q_1}\} \cup D^{q_1}(i, \beta^{q_1}) & \text{if } i > \gamma_t \end{cases}.$$

Finally, let $\dot{t}_1$ be such that $q_2 \Vdash \dot{t}_1 = \dot{t}_0 \cup \{\beta^{q_2}\}$. Then $(q_2, \dot{t}_1) \leq (q_0, \dot{t}_0)$ and $(q_2, \dot{t}_1) \in \mathbb{S}$.

Next, we show that $\mathbb{S}$ is κ-closed. Let $\langle (q_\alpha, \dot{t}_\alpha) \mid \alpha < \nu \rangle$ be a decreasing sequence of conditions from $\mathbb{S}$ with $\nu < \kappa$ a limit ordinal. We will find a lower bound $(q, \dot{t}) \in \mathbb{S}$. Let $\beta^q = \sup(\{\beta^{q_\alpha} \mid \alpha < \nu\})$ and let $X = \{\beta \mid$ for some $\alpha < \nu, q_\alpha \Vdash$ "$\beta \in \dot{t}_\alpha$"$\}$. Note that by our definition of $\mathbb{S}$, X is club in β^q. Let $\gamma = \mathrm{otp}(X)$. Define q as a lower bound to the q_α's by letting

$$D^q(i, \beta^q) = \begin{cases} \emptyset & \text{if } i \leq \gamma \\ X \cup \bigcup_{\beta \in X} D^q(i, \beta) & \text{if } i > \gamma \end{cases}.$$

Let $\dot{t}$ be a name forced by q to be equal to $X \cup \{\beta^q\}$. Then $(q, \dot{t}) \in \mathbb{S}$ and, for all $\alpha < \nu$, $(q, \dot{t}) \leq (q_\alpha, \dot{t}_\alpha)$. $\square$

Thus, if $\mathcal{D}$ has been added by $\mathbb{Q}$, then $\mathbb{T}_\mathcal{D}$ does in fact add a club in κ^+ and, since $\mathbb{Q} * \dot{\mathbb{T}}_\mathcal{D}$ has a κ-closed dense subset and therefore doesn't add any new sets of ordinals of order type less than κ, the club added by $\mathbb{T}_\mathcal{D}$ has order type κ.

Theorem 2.19. *Let κ be a regular cardinal that is not strongly inaccessible, and let $\lambda > \kappa$ be a measurable cardinal. Let $\mathbb{P} = \mathrm{Coll}(\kappa, < \lambda)$, and, in $V^\mathbb{P}$,* *let $\mathbb{Q}$ be the poset defined above. Let G be $\mathbb{P}$-generic over V and, in $V[G]$, let H be $\mathbb{Q}$-generic over $V[G]$. Then, in $V[G*H]$, there is a transitive, normal, uniform κ-covering matrix for κ^+, but $\square_{\kappa,<\kappa}$ fails.*

Proof. We have already shown that, in $V[G*H]$, there is a transitive, normal, uniform κ-covering matrix for κ^+, $\mathcal{D}$. In $V[G*H]$, let $\mathbb{T} = \mathbb{T}_{\mathcal{D}}$. Fix an elementary embedding $j : V \to M$ with critical point λ. Then $j(\mathbb{P}) = \mathrm{Coll}(\kappa, < j(\lambda))$, and $j \restriction \mathbb{P}$ is the identity map. $V[G] \models |\mathbb{Q} * \mathbb{T}| = \lambda$, and $|\mathbb{Q}*\mathbb{T}|$ has a κ-closed dense subset, so, by Fact 1.5, we can extend $j \restriction \mathbb{P}$ to a complete embedding of $\mathrm{Coll}(\kappa, < \lambda) * \mathbb{Q} * \mathbb{T}$ into $\mathrm{Coll}(\kappa, < j(\lambda))$ so that the quotient forcing is κ-closed. Then, letting I be $\mathbb{T}$-generic over $V[G*H]$ and J be $\mathbb{R} = \mathrm{Coll}(\kappa, < j(\lambda))/G*H*I$-generic over $V[G*H*I]$, we can further extend j to an elementary embedding $j : V[G] \to M[G*H*I*J]$.

We would now like to extend j further still to an embedding with domain $V[G*H]$. To do this, consider the partial order $j(\mathbb{Q})$. In $M[G*H*I*J]$, $j(\mathbb{Q})$ is the partial order to add a transitive, normal, uniform κ-covering matrix for $j(\lambda)$. Let $E = \bigcup I$. E is a club in λ of order type κ, and, if $\beta \in E$ is such that $\mathrm{otp}(E \cap \beta) = \gamma$, then for every $i < \gamma$, $D(i, \beta) = \emptyset$. We use E to define a "master condition" $q^* \in j(\mathbb{Q})$ as follows. Let $\beta^{q^*} = \lambda$. For $\beta < \lambda$ and $i < \kappa$, let $D^{q^*}(i, \beta) = D(i, \beta)$ and

$$D^{q*}(i, \lambda) = E \cup \bigcup_{\beta \in E} D(i, \beta).$$

$q^* \in j(\mathbb{Q})$ and, if $q \in H$, $j(q) = q \leq q^*$. Let K be $j(\mathbb{Q})$-generic over $V[G*H*I*J]$ such that $q^* \in K$. Since $j[H] \subseteq K$, we can extend j to an elementary embedding $j : V[G*H] \to M[G*H*I*J*K]$.

Suppose for sake of contradiction that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ is a $\square_{\kappa,<\kappa}$-sequence in $V[G*H]$. For $\alpha < \lambda$, $j(\mathcal{C}_\alpha) = \mathcal{C}_\alpha$, and $j(\vec{\mathcal{C}}) = \langle \mathcal{C}_\alpha \mid \alpha < j(\lambda) \rangle$ is a $\square_{\kappa,<\kappa}$-sequence in $M[G*H*I*J*K]$. Fix $F \in \mathcal{C}_\lambda$. F is a thread through $\vec{\mathcal{C}}$ (i.e., F is a club in λ and, for every $\alpha \in F'$, $F \cap \alpha \in \mathcal{C}_\alpha$) and $F \in V[G*H*I*J*K]$.

Claim 2.20. $F \in V[G*H]$.

Proof. Suppose not. Work in $V[G]$. There is a $\mathbb{Q}*\mathbb{T}*\mathbb{R}*j(\mathbb{Q})$-name $\dot{f}$ such that $\dot{f}^{H*I*J*K} = F$ and $\Vdash_{\mathbb{Q}*\mathbb{T}*\mathbb{R}*j(\mathbb{Q})} \dot{f} \notin V[G*G_{\mathbb{Q}}]$.

Subclaim 2.21. *For all $(q, \dot{t}, \dot{r}, \dot{p}) \in \mathbb{Q}*\mathbb{T}*\mathbb{R}*j(\mathbb{Q})$, there are $q' \leq q$, $(\dot{t}_0, \dot{r}_0, \dot{p}_0)$, $(\dot{t}_1, \dot{r}_1, \dot{p}_0)$, and $\alpha < \kappa^+$ such that $(q', \dot{t}_0, \dot{r}_0, \dot{p}_0), (q', \dot{t}_1, \dot{r}_1, \dot{p}_1) \leq (q, \dot{t}, \dot{r}, \dot{p})$ and such that $(q', \dot{t}_0, \dot{r}_0, \dot{p}_0)$ and $(q', \dot{t}_1, \dot{r}_1, \dot{p}_1)$ decide the statement "$\check{\alpha} \in \dot{f}$" in opposite ways.*

Proof. Suppose the subclaim fails for some $(q, \dot{t}, \dot{r}, \dot{p})$. Define a $\mathbb{Q}$-name $\dot{f}'$ such that for all $q' \leq q$ and $\alpha < \kappa^+$, $q' \Vdash_{\mathbb{Q}}$ "$\check{\alpha} \in \dot{f}$" if and only if there is $(\dot{t}', \dot{r}', \dot{p}')$ such that $(q', \dot{t}', \dot{r}', \dot{p}') \leq (q', \dot{t}, \dot{r}, \dot{p})$ and $(q', \dot{t}', \dot{r}', \dot{p}') \Vdash_{\mathbb{Q}*\mathbb{T}*\mathbb{R}*j(\mathbb{Q})}$ "$\check{\alpha} \in \dot{f}$". Then $(q, \dot{t}, \dot{r}, \dot{p}) \Vdash$ "$\dot{f} = \dot{f}'$", contradicting our choice of $\dot{f}$. This proves the subclaim. $\square$

Since κ is not strongly inaccessible, letting γ be the least cardinal such that $2^\gamma \geq \kappa$, we have $\gamma < \kappa$. Recall that $\mathbb{S}$ is the previously defined κ-closed dense subset of $\mathbb{Q} * \mathbb{T}$. We will construct $\langle q_i \mid i \leq \gamma \rangle$, $\langle (\dot{t}_s, \dot{r}_s, \dot{p}_s) \mid s \in {}^{\leq\gamma}2 \rangle$ and $\langle \alpha_i \mid i \leq \gamma \rangle$ such that:

1. $(q_0, \dot{t}_{\langle\rangle}, \dot{r}_{\langle\rangle}, \dot{p}_{\langle\rangle}) \Vdash$ "$\dot{f}$ is a thread through $\mathcal{C}$".
2. For each $s \in {}^{\leq\gamma}2$, $(q_{|s|}, \dot{t}_s, \dot{r}_s, \dot{p}_s) \in \mathbb{S} * \mathbb{R} * j(\mathbb{Q})$ and $\beta^{q_{|s|}} = \alpha_{|s|}$.
3. If $s, u \in {}^{\leq\gamma}2$ and $s \subseteq u$, then $(q_{|u|}, \dot{t}_u, \dot{r}_u, \dot{p}_u) \leq (q_{|s|}, \dot{t}_s, \dot{r}_s, \dot{p}_s)$.
4. If $i < \gamma$ and $s \in {}^i2$, then there is $\alpha \in [\alpha_i, \alpha_{i+1})$ such that $(q_{i+1}, \dot{t}_{s^\frown\langle 0\rangle}, \dot{r}_{s^\frown\langle 0\rangle}, \dot{p}_{s^\frown\langle 0\rangle})$ and $(q_{i+1}, \dot{t}_{s^\frown\langle 1\rangle}, \dot{r}_{s^\frown\langle 1\rangle}, \dot{p}_{s^\frown\langle 1\rangle})$ decide the statement "$\check{\alpha} \in \dot{f}$" in opposite ways.
5. If $i < \gamma$ and $s \in {}^i2$, then $(q_{i+1}, \dot{t}_{s^\frown\langle 0\rangle}, \dot{r}_{s^\frown\langle 0\rangle}, \dot{p}_{s^\frown\langle 0\rangle})$ and $(q_{i+1}, \dot{t}_{s^\frown\langle 1\rangle}, \dot{r}_{s^\frown\langle 1\rangle}, \dot{p}_{s^\frown\langle 1\rangle})$ both force that $\dot{f} \cap [\alpha_i, \alpha_{i+1}) \neq \emptyset$.

Suppose for a moment that we have successfully completed this construction. Find $q^* \leq q_\gamma$ such that q^* decides the set $\mathcal{C}_{\alpha_\gamma}$ (this can be done, since $\mathbb{Q}$ is $(\kappa+1)$-strategically closed). Then, for each $s \in {}^\gamma 2$, $(q^*, \dot{t}_s, \dot{r}_s, \dot{p}_s) \in \mathbb{Q} * \mathbb{T} * \mathbb{R} * j(\mathbb{Q})$ and $(q^*, \dot{t}_s, \dot{r}_s, \dot{p}_s) \Vdash$ "α_γ is a limit point of $\dot{f}$". Moreover, if $s, u \in {}^\gamma 2$ and $s \neq u$, then $(q^*, \dot{t}_s, \dot{r}_s, \dot{p}_s)$ and $(q^*, \dot{t}_u, \dot{r}_u, \dot{p}_u)$ force contradictory information about $\dot{f} \cap \alpha_\gamma$, which is forced to be an element of $\mathcal{C}_{\alpha_\gamma}$. But $2^\gamma \geq \kappa$, contradicting the fact that $\mathcal{C}$ is a $\square_{\kappa,<\kappa}$-sequence.

We now turn to the construction. Fix $(q_0, \dot{t}_{\langle\rangle}, \dot{r}_{\langle\rangle}, \dot{p}_{\langle\rangle})$ such that $(q_0, \dot{t}_{\langle\rangle}, \dot{r}_{\langle\rangle}, \dot{p}_{\langle\rangle}) \Vdash$ "$\dot{f}$ is a thread through $\mathcal{C}$", and let $\alpha_0 = \beta^{q_0}$. We first consider the successor case. Fix $i < \gamma$ and suppose that q_i, $\langle (\dot{t}_s, \dot{r}_s, \dot{p}_s) \mid s \in {}^i2 \rangle$, and α_i have been defined. Enumerate ${}^i2$ as $\langle s_j \mid j < 2^i \rangle$, noting that $2^i < \kappa$. Now, using the κ-closure of $\mathbb{Q}$, $\mathbb{S}$, $\mathbb{R}$, and $j(\mathbb{Q})$, the density of $\mathbb{S}$ in $\mathbb{Q}*\mathbb{T}$, the subclaim, and the fact that $(q_0, \dot{t}_{\langle\rangle}, \dot{r}_{\langle\rangle}, \dot{p}_{\langle\rangle}) \Vdash$ "$\dot{f}$ is unbounded in κ^+", it is straightforward to construct $\langle q^i_j \mid j < 2^i \rangle$ and $\langle (((\dot{t}^*_{s_j^\frown\langle 0\rangle}, \dot{r}^*_{s_j^\frown\langle 0\rangle}, \dot{p}^*_{s_j^\frown\langle 0\rangle}), (\dot{t}^*_{s_j^\frown\langle 1\rangle}, \dot{r}^*_{s_j^\frown\langle 1\rangle}, \dot{p}^*_{s_j^\frown\langle 1\rangle})) \mid j < 2^i \rangle$ such that:

- $\langle q^i_j \mid j < 2^i \rangle$ is a decreasing sequence of conditions from $\mathbb{Q}$ below q_i.

- For all $j < 2^i$, $(q^i_j, \dot{t}^*_{s_j \frown \langle 0 \rangle}, \dot{r}^*_{s_j \frown \langle 0 \rangle}, \dot{p}^*_{s_j \frown \langle 0 \rangle})$, $(q^i_j, \dot{t}^*_{s_j \frown \langle 1 \rangle}, \dot{r}^*_{s_j \frown \langle 1 \rangle}, \dot{p}^*_{s_j \frown \langle 1 \rangle}) \leq (q^i_j, \dot{t}_s, \dot{r}_s, \dot{p}_s)$ are both in $\mathbb{S} * \mathbb{R} * j(\mathbb{Q})$ and both force that $\dot{f} \cap [\alpha_i, \beta^{q^i_j}) \neq \emptyset$.
- For all $j < 2^i$, there is $\alpha \in [\alpha_i, \beta^{q^i_j})$ such that $(q^i_j, \dot{t}^*_{s_j \frown \langle 0 \rangle}, \dot{r}^*_{s_j \frown \langle 0 \rangle}, \dot{p}^*_{s_j \frown \langle 0 \rangle})$ and $(q^i_j, \dot{t}^*_{s_j \frown \langle 1 \rangle}, \dot{r}^*_{s_j \frown \langle 1 \rangle}, \dot{p}^*_{s_j \frown \langle 1 \rangle})$ decide the statement "$\check{\alpha} \in \dot{f}$" in opposite ways.

Now let $\xi = \sup(\{\delta \mid$ for some $j < 2^i$ and $\ell \in \{0, 1\}$, $q^i_j \Vdash$ "$\mathrm{otp}(\dot{t}^*_{s_j \frown \langle \ell \rangle}) = \check{\delta}$"$\})$. Since $2^i < \kappa$, we know that $\xi < \kappa$. Let $\alpha_{i+1} = \sup(\{\beta^{q^i_j} \mid j < 2^i\})$. Let E_{i+1} be a club in α_{i+1} of order type cf$(\alpha_{i+1}) < \kappa$. Define q_{i+1} to be a lower bound of $\langle q^i_j \mid j < 2^i \rangle$ by letting $\beta^{q_{i+1}} = \alpha_{i+1}$ and, for all $k < \kappa$,

$$D^{q_{i+1}}(k, \alpha_{i+1}) = \begin{cases} \emptyset & \text{if } k < \xi + 1 \\ E_{i+1} \cup \bigcup_{\beta \in E_{i+1}} D^{q_{i+1}}(k, \beta) & \text{if } k \geq \xi + 1 \end{cases}.$$

Finally, for all $j < 2^i$ and $\ell \in \{0, 1\}$, let $(\dot{t}_{s_j \frown \langle \ell \rangle}, \dot{r}_{s_j \frown \langle \ell \rangle}, \dot{p}_{s_j \frown \langle \ell \rangle})$ be such that $q_{i+1} \Vdash$ "$(\dot{t}_{s_j \frown \langle \ell \rangle}, \dot{r}_{s_j \frown \langle \ell \rangle}, \dot{p}_{s_j \frown \langle \ell \rangle}) = (\dot{t}^*_{s_j \frown \langle \ell \rangle} \cup \{\alpha_{i+1}\}, \dot{r}^*_{s_j \frown \langle \ell \rangle}, \dot{p}^*_{s_j \frown \langle \ell \rangle})$".

Next, suppose that $i \leq \gamma$ is a limit ordinal and that $\langle q_j \mid j < i \rangle$, $\langle (\dot{t}_s, \dot{r}_s, \dot{p}_s) \mid s \in {}^{<i}2 \rangle$ and $\langle \alpha_j \mid j < i \rangle$ have been defined. Let $\xi = \sup(\{\delta \mid$ for some $s \in {}^{<i}2$, $q_{|s|} \Vdash$ "$\mathrm{otp}(\dot{t}_s) = \check{\delta}$"$\})$. Since $2^{<i} < \kappa$, we know that $\xi < \kappa$. Let $\alpha_i = \sup(\{\beta^{q_j} \mid j < i\})$ and let E_i be a club in α_i of order type cf$(\alpha_i) < \kappa$. Define q_i to be a lower bound for $\langle q_j \mid j < i \rangle$ by letting $\beta^{q_i} = \alpha_i$ and

$$D^{q_i}(k, \alpha_i) = \begin{cases} \emptyset & \text{if } k < \xi + 1 \\ E_i \cup \bigcup_{\beta \in E_i} D^{q_i}(k, \beta) & \text{if } k \geq \xi + 1 \end{cases}.$$

Finally, for all $s \in {}^i2$, let $\dot{t}_s$ be such that $q_i \Vdash$ "$\dot{t}_s = \bigcup_{j<i} \dot{t}_{s \restriction j} \cup \{\alpha_i\}$" and let $(\dot{r}_s, \dot{p}_s)$ be forced by $(q_i, \dot{t}_s)$ to be a lower bound for $\langle (\dot{r}_{s \restriction j}, \dot{p}_{s \restriction j}) \mid j < i \rangle$. This is possible, since $\mathbb{R} * j(\mathbb{Q})$ is κ-closed. This construction satisfies conditions 1-5 above and allows us to complete the proof of Claim 2.20. $\square$

But now we have shown that F, which threads $\mathcal{C}$, is in $V[G * H]$, contradicting the fact that $\mathcal{C}$ is a $\square_{\kappa, <\kappa}$-sequence in $V[G * H]$. Thus, $\square_{\kappa, <\kappa}$ fails in $V[G * H]$. $\square$

Note that, if κ is supercompact and $\lambda > \kappa$ is measurable, we can also obtain a model in which there is a transitive, normal, uniform κ-covering matrix for κ^+ but $\square_{\kappa, <\kappa}$ fails by first making the supercompactness of κ indestructible under κ-directed closed forcing and then forcing with $\mathbb{Q}$. We conjecture that we can obtain such a model for all regular, uncountable κ but do not have a proof when κ is inaccessible but not supercompact.

We now investigate counterexamples to $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ for more general shapes of covering matrices. Recall the following definitions:

Definition. *Let κ be an infinite cardinal.*

1. $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ *is a* coherent sequence *if, for all* $\alpha, \beta \in \lim(\kappa)$*,*
   a) C_α *is a club in* α*.*
   b) *If* $\alpha \in C'_\beta$*, then* $C_\alpha = C_\beta \cap \alpha$*.*
2. *Let* $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ *be a coherent sequence. If* D *is a club in* κ*, then* D *is a* thread *through* $\vec{C}$ *if, for every* $\alpha \in D'$*,* $D \cap \alpha = C_\alpha$*.*
3. $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ *is a* $\square(\kappa)$-sequence *if it is a coherent sequence that has no thread. We say that* $\square(\kappa)$ *holds if there is a* $\square(\kappa)$*-sequence.*

Let $\theta < \lambda$ be regular, infinite cardinals and suppose that $\square(\lambda)$ holds. Fix a $\square(\lambda)$-sequence $\vec{C}$. Arrange so that C_α is defined for all $\alpha < \lambda$ by letting $C_{\alpha+1} = \{\alpha\}$. The definitions of the following functions are due to Todorcevic [30]: First, define $\Lambda_\theta : [\lambda]^2 \to \lambda$ by

$$\Lambda_\theta(\alpha, \beta) = \max\{\xi \in C_\beta \cap (\alpha + 1) \mid \theta \text{ divides } \mathrm{otp}(C_\beta \cap \xi)\}$$

Next, let $\rho_\theta : [\lambda]^2 \to \theta$ be defined recursively by

$$\rho_\theta(\alpha, \beta) = \sup\{\mathrm{otp}(C_\beta \cap [\Lambda_\theta(\alpha, \beta), \alpha)), \rho_\theta(\alpha, \min(C_\beta \setminus \alpha)), \\ \rho_\theta(\xi, \alpha) \mid \xi \in C_\beta \cap [\Lambda_\theta(\alpha, \beta), \alpha)\}$$

Proofs of the following lemmas can be found in [30].

Lemma 2.22. *Let $\alpha < \beta < \gamma < \lambda$.*

1. $\rho_\theta(\alpha, \gamma) \leq \max\{\rho_\theta(\alpha, \beta), \rho_\theta(\beta, \gamma)\}$.
2. $\rho_\theta(\alpha, \beta) \leq \max\{\rho_\theta(\alpha, \gamma), \rho_\theta(\beta, \gamma)\}$.

Lemma 2.23. *Let $i < \theta$ and $\beta < \lambda$. $\{\alpha < \beta \mid \rho_\theta(\alpha, \beta) \leq i\}$ is closed.*

Define a covering matrix $\mathcal{D} = \{D(i, \beta) \mid i < \theta, \beta < \lambda\}$ by letting $D(i, \beta) = \{\alpha < \beta \mid \rho_\theta(\alpha, \beta) \leq i\}$. It is clear that $\mathcal{D}$ satisfies conditions 1 and 2 in the definition of a covering matrix. Part 1 of Lemma 2.22 implies that $\mathcal{D}$ is transitive. In fact, together with part 2 of the same lemma, it implies a stronger coherence property, namely that if $i < \theta$, $\alpha < \beta < \lambda$, and $\alpha \in D(i, \beta)$, then $D(i, \alpha) = D(i, \beta) \cap \alpha$. Lemma 2.23 implies that $\mathcal{D}$ is closed. We now show, however, that in general $\mathcal{D}$ is not uniform. First, we make the following definition:

Definition. *1. Let A be a set of ordinals and let μ be an infinite cardinal. $A^{[\mu]} = \{\alpha \in A \mid \mu$ **divides** $\mathrm{otp}(A \cap \alpha)\}$.*

2. Let $\mu < \kappa$ be infinite regular cardinals. $\vec{C}$ is a $\square^\mu(\kappa)$-sequence if $\vec{C}$ is a $\square(\kappa)$-sequence and $\{\alpha \in S^\kappa_\mu \mid C^{[\mu]}_\alpha$ is bounded below $\alpha\}$ is stationary. $\square^\mu(\kappa)$ is the statement that a $\square^\mu(\kappa)$-sequence exists.

Lemma 2.24. *Let $\beta < \lambda$ be such that $\mathrm{cf}(\beta) = \theta$ and $C^{[\theta]}_\beta$ is bounded below β. Then, for every $i < \theta$, $D(i, \beta)$ is bounded below β.*

Proof. Let $\xi = \max(C^{[\theta]}_\beta)$. Since $\xi < \beta$ and $\mathrm{cf}(\beta) = \theta$, $\mathrm{otp}(C_\beta \setminus (\xi + 1)) = \theta$. Enumerate $C_\beta \setminus (\xi + 1)$ in increasing order as $\langle \beta_i \mid i < \theta \rangle$. Now, if $i < \theta$ and $\beta_i < \alpha < \beta$, then $\Lambda_\theta(\alpha, \beta) = \xi$ and $\mathrm{otp}(C_\beta \cap [\xi, \alpha)) > i$, so $\rho_\theta(\alpha, \beta) > i$. Thus, $D(i, \beta) \subseteq (\beta_i + 1)$. ◻

Lemma 2.25. *Suppose $\vec{C}$ is a $\square^\theta(\lambda)$-sequence. Then $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ both fail.*

Proof. Suppose for sake of contradiction that $\mathrm{CP}(\mathcal{D})$ holds and is witnessed by an unbounded $T \subseteq \lambda$. Since $\vec{C}$ is a $\square^\theta(\lambda)$-sequence, we can find $\alpha \in T'$ such that $\mathrm{cf}(\alpha) = \theta$ and $C^{[\theta]}_\alpha$ is bounded below α. Let $X \in [T]^\theta$ be an unbounded subset of α. Find $i < \theta$ and $\beta < \lambda$ such that $X \subseteq D(i, \beta)$. Since $\mathcal{D}$ is closed, $\alpha \in D(i, \beta)$, so $D(i, \alpha) = D(i, \beta) \cap \alpha$ and $X \subseteq D(i, \alpha)$. This is a contradiction, as the previous lemma implies that $D(i, \alpha)$ is bounded below α. Thus, $\mathrm{CP}(\mathcal{D})$ fails. Since $\mathcal{D}$ is transitive, this means that $\mathrm{S}(\mathcal{D})$ fails as well. ◻

The question now naturally arises whether $\square^\theta(\lambda)$ is a strictly stronger assumption than $\square(\lambda)$. This and related questions are addressed in the following chapter.

CHAPTER 3

SQUARES

Definition. *Let $A \subseteq \kappa$ and let $\vec{C}$ be a $\square(\kappa)$-sequence. $\vec{C}$* avoids *A if, for every $\alpha \in \lim(\kappa)$, $C'_\alpha \cap A = \emptyset$.*

It is well known that if $\square_\kappa$ holds and $S \subseteq \kappa^+$ is stationary, then there is a stationary $T \subseteq S$ and a $\square_\kappa$-sequence that avoids T. We would like to know to what extent similar phenomena occur in connection with $\square(\kappa)$. The following proposition, whose proof we include for completeness, provides some information in this direction by showing that, if κ is regular and $S \subseteq \kappa$ is stationary, then every $\square(\kappa)$-sequence must, in a certain sense, avoid a pair of stationary subsets of S.

Proposition 3.1. *Let $\kappa > \omega_1$ be a regular cardinal, and let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ be a coherent sequence. Then the following are equivalent:*

1. *$\vec{C}$ is a $\square(\kappa)$-sequence.*
2. *For every stationary $T \subseteq \kappa$, there are stationary $S_0, S_1 \subseteq T$ such that for every $\alpha \in \lim(\kappa)$, $C'_\alpha \cap S_0 = \emptyset$ or $C'_\alpha \cap S_1 = \emptyset$.*

Proof. First, suppose that there are stationary sets S_0, $S_1 \subseteq \kappa$ such that, for every $\alpha \in \lim(\kappa)$, $C'_\alpha \cap S_0 = \emptyset$ or $C'_\alpha \cap S_1 = \emptyset$, and suppose for sake of contradiction that there is a club D in κ that threads $\vec{C}$. Since S_0 and S_1 are stationary, there are $\alpha < \beta < \gamma$ in D' such that $\alpha \in S_0$ and $\beta \in S_1$. Since $D \cap \gamma = C_\gamma$, we have $\alpha, \beta \in C'_\gamma$, contradicting the fact that $C'_\gamma \cap S_0$ or $C'_\gamma \cap S_1$ is empty. Thus, $\vec{C}$ is a $\square(\kappa)$-sequence.

Next, suppose that $\vec{C}$ is a $\square(\kappa)$-sequence and that $T \subseteq \kappa$ is stationary. There are two cases:

Case 1: There is $\alpha_0 < \kappa$ such that $\{\alpha \in T \mid C'_\alpha \cap (\alpha_0, \kappa) \neq \emptyset\}$ is nonstationary. Let $T^* = \{\alpha \in T \setminus (\alpha_0 + 1) \mid C'_\alpha \cap (\alpha_0, \kappa) = \emptyset\}$ and let $T^* = S_0 \cup S_1$ be any partition of T^* into disjoint stationary sets. Then S_0 and S_1 are as desired.

Case 2: For all $\alpha_0 < \kappa$, $\{\alpha \in T \mid C'_\alpha \cap (\alpha_0, \kappa) \neq \emptyset\}$ is stationary. For $\alpha \in \lim(\kappa)$, let $S^0_\alpha = \{\beta \in T \setminus (\alpha + 1) \mid \alpha \notin C'_\beta\}$ and let $S^1_\alpha = \{\beta \in T \setminus (\alpha + 1) \mid \alpha \in C'_\beta\}$.

Claim 3.2. *There is $\alpha \in \lim(\kappa)$ such that S^0_α and S^1_α are both stationary.*

Proof. Suppose this is not the case. Then for every $\alpha \in \lim(\kappa)$, either S^0_α or S^1_α is non-stationary. Let A be the set of $\alpha \in \lim(\kappa)$ such that S^0_α is nonstationary. We claim that A is unbounded in κ. To show this, fix $\alpha_0 < \kappa$. By Fodor's Lemma, we can fix an $\alpha \geq \alpha_0$ and a stationary $T^* \subseteq T$ such that if $\beta \in T^*$, then $\alpha = \min(C'_\beta \setminus \alpha_0)$. Then S^1_α is stationary, so S^0_α is non-stationary and hence $\alpha \in A$. Now let $\alpha < \alpha'$ be elements of A, and let D_α and $D_{\alpha'}$ be clubs in κ disjoint from S^0_α and $S^0_{\alpha'}$, respectively. Fix $\beta \in D_\alpha \cap D_{\alpha'} \cap T$. Then $C_\alpha = C_\beta \cap \alpha$ and $C_{\alpha'} = C_\beta \cap \alpha'$, so $C_\alpha = C_{\alpha'} \cap \alpha$. Thus, $D = \bigcup_{\alpha \in A} C_\alpha$ is a thread through $\vec{C}$, contradicting the fact that $\vec{C}$ is a $\square(\kappa)$-sequence. $\square$

Now let α be such that S^0_α and S^1_α are both stationary. Let $S_0 = S^0_\alpha$ and $S_1 = S^1_\alpha$. It is routine to check that S_0 and S_1 are as desired.

$\square$

The preceding observations lead us to make the following definition.

Definition. *Let κ be a regular, uncountable cardinal, and let $S \subseteq \kappa$. $\vec{C}$ is a $\square(\kappa, S)$-sequence if $\vec{C}$ is a $\square(\kappa)$-sequence and $\vec{C}$ avoids S. $\square(\kappa, S)$ is the statement that a $\square(\kappa, S)$-sequence exists.*

Recalling the definition of $\square^\mu(\kappa)$ from the previous chapter, we start our investigation of these intermediate square principles with the following simple observation.

Proposition 3.3. *Let $\mu < \kappa$ be infinite regular cardinals. The following are equivalent:*

1. $\square^\mu(\kappa)$
2. *There is a $\square(\kappa)$-sequence $\vec{C}$ such that $\{\alpha < \kappa \mid \mathrm{otp}(C_\alpha) = \mu\}$ is stationary.*

Proof. A $\square(\kappa)$-sequence as in 2. is clearly a $\square^\mu(\kappa)$ sequence. For the other direction, let $\vec{D}$ be a $\square^\mu(\kappa)$-sequence, and let $T = \{\alpha < \kappa \mid D^{[\mu]}_\alpha$ is bounded below $\alpha\}$. By definition, $T \cap S^\kappa_\mu$ is stationary in κ. Form $\vec{C}$ as follows. If $\alpha \in \lim(\kappa) \setminus T$, let $C_\alpha = D^{[\mu]}_\alpha$. If $\alpha \in T$, let $C_\alpha = D_\alpha \setminus (\max(D^{[\mu]}_\alpha) + 1)$. It is immediate that, for all $\alpha \in \lim(\kappa)$, C_α is club in α. Suppose $\alpha < \beta$ and $\alpha \in C'_\beta$. Then, by our construction, $\alpha \in D'_\beta$, so $D_\beta \cap \alpha = D_\alpha$. Also, $\alpha \in T$ if and only if $\beta \in T$, so we obtained C_α from D_α in the same way we obtained C_β from D_β, so $C_\beta \cap \alpha = C_\alpha$. We arranged so that, if $\alpha \in T \cap S^\kappa_\mu$, then $\mathrm{otp}(C_\alpha) = \mu$. Finally, there can be no thread E through $\vec{C}$, since, if there were, there would be $\alpha < \alpha'$ such that $\alpha, \alpha' \in E' \cap T \cap S^\kappa_\mu$. But then $\mathrm{otp}(E \cap \alpha) = \mu = \mathrm{otp}(E \cap \alpha')$, which is a contradiction. Thus, $\vec{C}$ is a $\square(\kappa)$-sequence as in 2. $\square$

The remainder of this chapter investigates the implications and non-implications that exist among the traditional square properties and those of the form $\square^\mu(\kappa)$ and $\square(\kappa, S)$. A diagram illustrating the complete picture when $\kappa = \omega_2$ can be found at the end of this chapter.

Proposition 3.4. *Let $\mu < \kappa$ be infinite, regular cardinals.*

1. *If $\square^\mu(\kappa)$ holds, then there is a stationary $S \subseteq S^\kappa_\mu$ such that $\square(\kappa, S)$ holds.*
2. *If there is a stationary $S \subseteq S^\kappa_\omega$ such that $\square(\kappa, S)$ holds, then $\square^\omega(\kappa)$ holds.*

Proof. First, suppose that $\vec{C}$ is a $\square^\mu(\kappa)$-sequence. Let $T = \{\alpha < \kappa \mid C^{[\mu]}_\alpha$ is bounded below $\alpha\}$. It is easily seen that the construction from the proof of Proposition 3.3 yields a $\square(\kappa, S)$-sequence, where $S = T \cap S^\kappa_\mu$.

Next, suppose that $S \subseteq S^\kappa_\omega$ is stationary and that $\vec{D}$ is a $\square(\kappa, S)$-sequence. For $\alpha < \kappa$, define C_α as follows: If $\alpha \notin S$, let $C_\alpha = D_\alpha$. If $\alpha \in S$, let C_α be any set of order type ω unbounded in α. Since $\vec{D}$ avoids S, this does not interfere with the coherence of the sequence, so $\vec{C}$ is a $\square^\omega(\kappa)$-sequence. $\square$

Proposition 3.5. *Let $\mu < \nu < \kappa$ be infinite, regular cardinals. If $\square^\nu(\kappa)$ holds, then $\square^\mu(\kappa)$ holds.*

Proof. Assume $\square^\nu(\kappa)$ holds, and fix a $\square(\kappa)$-sequence $\vec{C}$ such that $T_0 = \{\alpha < \kappa \mid \mathrm{otp}(C_\alpha) = \nu\}$ is stationary. We claim that $T_1 = \{\alpha \in S^\kappa_\mu \mid \mathrm{otp}(C_\alpha) < \nu\}$ is also stationary. To see this, let E be club in κ. Let $\beta \in E' \cap T_0$. Then $E \cap C_\beta$ is club in β. Let $\alpha \in (E \cap C_\beta)' \cap S^\kappa_\mu$. Then, since $C_\alpha = C_\beta \cap \alpha$, $\mathrm{otp}(C_\alpha) < \nu$, so $\alpha \in E \cap T_1$.

We can now apply Fodor's Lemma to $T_1 \setminus \nu$ to find a $\gamma < \nu$ and a stationary $S \subseteq T_1$ such that $\alpha \in S$ implies $\mathrm{otp}(C_\alpha) = \gamma$. Let $\langle \gamma_\xi \mid \xi < \mu \rangle$ enumerate a club in γ with each γ_ξ a limit ordinal (this will not be possible if $\mu = \omega$ and γ is not a limit of limit ordinals, but in that case $\vec{C}$ is already a $\square^\mu(\kappa)$-sequence).

We now define a $\square^{\mu}(\kappa)$-sequence $\vec{D}$ (in fact, $\vec{D}$ will also be a $\square(\kappa, S)$-sequence). First, if $\alpha \in S$, let $D_\alpha = \{\beta \in C'_\alpha \mid \text{for some } \nu < \mu,\ \text{otp}(C_\alpha \cap \beta) = \gamma_\nu\}$. Next, if $\alpha \in D'_{\alpha'}$ for some $\alpha' \in S$, let $D_\alpha = D_{\alpha'} \cap \alpha$. Note that this is well-defined. If $\alpha \in C'_{\alpha'}$ for some $\alpha' \in S$ but, for all $\beta \in S$, $\alpha \notin D'_\beta$, then let $D_\alpha = C_\alpha \setminus \max(D_{\alpha'} \cap \alpha)$. Note again that this is well-defined.

If there is $\alpha' \in S$ such that $\alpha' \in C'_\alpha$ (note that such an α' must be unique), then let $D_\alpha = C_\alpha \setminus \alpha'$. In all other cases, let $D_\alpha = C_\alpha$. It is now easy to verify that $\vec{D}$ is a $\square(\kappa)$-sequence and, since $\alpha \in S$ implies that $\text{otp}(D_\alpha) = \mu$, that it is in fact a $\square^{\mu}(\kappa)$-sequence. □

The following corollary is now immediate.

Corollary 3.6. *1. Let $\mu < \nu < \kappa$ be infinite, regular cardinals. If $\square^{\nu}(\kappa)$ holds, then there is a stationary $S \subseteq S^{\kappa}_{\mu}$ such that $\square(\kappa, S)$ holds.*

2. Let $\mu \leq \kappa$ be infinite, regular cardinals, with μ regular. If $\square_\kappa$ holds, then $\square^{\mu}(\kappa^+)$ holds.

We now show that the above implications are generally not reversible. We begin by recalling the definition of the forcing poset that adds a $\square(\kappa)$ sequence by specifying its initial segments.

Definition. *Let κ be a regular cardinal. $\mathbb{Q}(\kappa)$ is the partial order whose elements are of the form $q = \langle C^q_\alpha \mid \alpha \leq \beta^q \rangle$, where*

1. *$\beta^q < \kappa$.*
2. *For all $\alpha \leq \beta^q$, C^q_α is a club in α.*
3. *For all $\alpha < \alpha' \leq \beta^q$, if $\alpha \in C'_{\alpha'}$, then $C_\alpha = C_{\alpha'} \cap \alpha$.*

$p \leq q$ if and only if p end-extends q, i.e. $\beta^p \geq \beta^q$ and, for all $\alpha \leq \beta^q$, $C^p_\alpha = C^q_\alpha$.

Proposition 3.7. *$\mathbb{Q}(\kappa)$ is κ-strategically closed.*

Proof. We specify a winning strategy for Player II in $G_\kappa(\mathbb{Q}(\kappa))$. First, let $q_0 = \emptyset$. Let $0 < \alpha < \kappa$ be even and suppose that $\langle q_\delta \mid \delta < \alpha \rangle$ has already been played. We specify Player II's next move, q_α. Let $E_\alpha = \{\beta^{q_\delta} \mid \delta < \alpha \text{ is even}\}$ and suppose that we have satisfied the following inductive hypotheses:

1. E_α is closed below its supremum.

2. For all even ordinals $\delta < \xi < \alpha$, $\beta^{q_\delta} < \beta^{q_\xi}$ and $\beta^{q_\delta} \in (C^{q_\xi}_{\beta^{q_\xi}})'$.

First, suppose that α is a successor ordinal. Since it is even, it is in fact a double successor. Let $\alpha = \alpha' + 1 = \alpha'' + 2$. In this case, let $\beta^{q_\alpha} = \beta^{q_{\alpha'}} + \omega$. For limit ordinals $\zeta < \beta^{q_\alpha}$, let $C^{q_\alpha}_\zeta = C^{q_{\alpha'}}_\zeta$, and let

$$C^{q_\alpha}_{\beta^{q_\alpha}} = C^{q_\alpha}_{\beta^{q_{\alpha''}}} \cup \{\beta^{q_{\alpha''}}\} \cup \{\beta^{q_{\alpha'}} + n \mid n < \omega\}.$$

Next, suppose that α is a limit ordinal. Let $\beta^{q_\alpha} = \sup(\{\beta^{q_\delta} \mid \delta < \alpha\})$. For limit ordinals $\zeta < \beta^{q_\alpha}$, find $\delta < \alpha$ such that $\zeta \leq \beta^{q_\delta}$ and let $C^{q_\alpha}_\zeta = C^{q_\delta}_\zeta$. Note that this is well-defined. Let

$$C^{q_\alpha}_{\beta^{q_\alpha}} = \bigcup_{\zeta \in E_\alpha} C^{q_\alpha}_\zeta.$$

By our inductive hypotheses, this is a club in β^{q_α} and satisfies the coherence requirements.

It is clear that this procedure produces a valid condition $q_\alpha \in \mathbb{Q}(\kappa)$ that is a lower bound for $\langle q_\delta \mid \delta < \alpha \rangle$ and maintains the inductive hypotheses. Thus, $\mathbb{Q}(\kappa)$ is κ-strategically closed. $\square$

An argument similar to the proof of the previous proposition shows that, for every $\alpha < \kappa$, the set $\{q \mid \beta^q \geq \alpha\}$ is dense in $\mathbb{Q}(\kappa)$.

Corollary 3.8. *Forcing with $\mathbb{Q}(\kappa)$ preserves all cardinals $\leq \kappa$ and adds a coherent sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$. In addition, if $\kappa^{<\kappa} = \kappa$, then all cardinals are preserved.*

Proof. Let G be $\mathbb{Q}(\kappa)$-generic over V. Since $\mathbb{Q}(\kappa)$ is κ-strategically closed, it doesn't add any $< \kappa$-sequences of ordinals and hence preserves all cardinals $\leq \kappa$. Since $\{q \mid \beta^q \geq \alpha\}$ is dense in $\mathbb{Q}(\kappa)$ for every $\alpha < \kappa$, we can define $C_\alpha = C^q_\alpha$, where $q \in G$ and $\beta^q \geq \alpha$. It is clear that $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is well-defined and a coherent sequence. Finally, if $\kappa^{<\kappa}$, then $|\mathbb{Q}(\kappa)| = \kappa$. Thus, $\mathbb{Q}(\kappa)$ has the κ^+-c.c. and preserves all cardinals $\geq \kappa^+$. $\square$

Lemma 3.9. *If $\mu < \kappa$ are regular cardinals and G is $\mathbb{Q}(\kappa)$-generic over V, then the coherent sequence $\vec{C}$ added by G is a $\square^\mu(\kappa)$-sequence.*

Proof. Let $S = \{\alpha < \kappa \mid \mathrm{otp}(C_\alpha) = \mu\}$. It suffices to show that, in $V[G]$, S is stationary in κ. Note that this implies that $\vec{C}$ doesn't have a thread, since any club in κ must meet S in two points.

Work in V, let $\dot{D}$ be a $\mathbb{Q}(\kappa)$-name forced by the empty condition to be a club in κ, let $\dot{S}$ be a $\mathbb{Q}(\kappa)$-name for S, and let $q \in \mathbb{Q}(\kappa)$. We will find $p \leq q$ such that $p \Vdash \dot{D} \cap \dot{S} \neq \emptyset$.

We construct $\langle q_\alpha \mid \alpha \leq \mu \rangle$, a decreasing sequence of conditions from $\mathbb{Q}(\kappa)$ such that for every $\alpha \leq \mu$,

1. $\mathrm{otp}(C^{q_\alpha}_{\beta^{q_\alpha}}) \leq \mu$.
2. $E_\alpha = \{\beta^{q_\delta} \mid \delta < \alpha\}$ is closed below its supremum.
3. For all $\delta < \alpha$, $\beta^{q_\delta} < \beta^{q_\alpha}$ and $\beta^{q_\delta} \in (C^{q_\alpha}_{\beta^{q_\alpha}})'$.
4. If $\alpha < \mu$, then $q_{\alpha+1} \Vdash \dot{D} \cap (\beta^{q_\alpha}, \beta^{q_{\alpha+1}}) \neq \emptyset$.

To carry out this construction, we first let $\beta^{q_0} = \beta^q + \omega$. For limit ordinals $\zeta \leq \beta^q$, let $C^{q_0}_\zeta = C^q_\zeta$. Let $C^{q_0}_{\beta^{q_0}} = \{\beta^q + n \mid n < \omega\}$. Next, suppose that $\alpha = \alpha' + 1$ and that we have already constructed $\langle q_\delta \mid \delta \leq \alpha' \rangle$. Find $q^*_\alpha \leq q_{\alpha'}$ and $\xi_\alpha > \beta^{q_{\alpha'}}$ such that $q^*_\alpha \Vdash \xi_\alpha \in \dot{D}$. Find $q^{**}_\alpha \leq q^*_\alpha$ such that $\beta^{q^{**}_\alpha} \geq \xi_\alpha$. Let $\beta^{q_\alpha} = \beta^{q^{**}_\alpha} + \omega$. For limit $\zeta \leq \beta^{q^{**}_\alpha}$, let $C^{q_\alpha}_\zeta = C^{q^{**}_\alpha}_\zeta$. Finally, let $C^{q_\alpha}_{\beta^{q_\alpha}} = C^{q_\alpha}_{\beta^{q_{\alpha'}}} \cup \{\beta^{q_{\alpha'}}\} \cup \{\beta^{q^{**}_\alpha} + n \mid n < \omega\}$.

Now suppose that $\alpha < \mu$ is a limit ordinal and we have constructed $\langle q_\delta \mid \delta < \alpha \rangle$. Let $\beta^{q_\alpha} = \sup(\{\beta^{q_\delta} \mid \delta < \alpha\})$. For limit ordinals $\zeta < \beta^{q_\alpha}$, find $\delta < \alpha$ such that $\beta^{q_\delta} \geq \zeta$ and let $C^{q_\alpha}_\zeta = C^{q_\delta}_\zeta$. Let

$$C^{q_\alpha}_{\beta^{q_\alpha}} = \bigcup_{\zeta \in E_\alpha} C^{q_\alpha}_\zeta.$$

It is clear that this construction satisfies requirements 1-4 above. Let $p = q_\mu$. We have arranged so that $\mathrm{cf}(\beta^p) = \mu$ and $\mathrm{otp}(C^p_{\beta^p}) = \mu$. We have also arranged that, for every $\zeta < \beta^p$, $p \Vdash$ "$\dot{D} \cap (\zeta, \beta^p) \neq \emptyset$". Thus, since $\dot{D}$ is forced by the empty condition to be a club, $p \Vdash \beta^p \in \dot{D}$. Thus we have found our desired $p \leq q$ such that $p \Vdash$ "$\dot{D} \cap \dot{S} \neq \emptyset$". $\square$

We now introduce a forcing poset designed to add a thread of order type κ through a $\square(\kappa)$-sequence.

Definition. *Let κ be a regular cardinal and let $\vec{C}$ be a $\square(\kappa)$-sequence. $\mathbb{T}(\vec{C})$ is the partial order consisting of elements t such that:*

1. *t is a closed, bounded subset of κ.*
2. *For every $\alpha \in t'$, $t \cap \alpha = C_\alpha$.*

We denote the maximum element of a condition t by γ^t. $s \leq t$ if and only if s end-extends t, i.e. $\gamma^s \geq \gamma^t$ and $s \cap (\gamma^t + 1) = t$.

In analogy with the poset $\mathbb{T}_{\mathcal{D}}$ defined in the previous chapter, if $\vec{C}$ was added by $\mathbb{Q}(\kappa)$, then $\mathbb{T}(\vec{C})$ is quite nice.

Proposition 3.10. *Let κ be a regular cardinal and let $\vec{C}$ be a $\square(\kappa)$-sequence. For every $\alpha < \kappa$, the set $\{t \mid \gamma^t \geq \alpha\}$ is dense in $\mathbb{T}(\vec{C})$.*

Proof. If $t \in \mathbb{T}(\vec{C})$ and $\gamma^t < \alpha$, then $t \cup \{\alpha\} \in \mathbb{T}(\vec{C})$. $\square$

Proposition 3.11. *Let κ be a regular cardinal. Let $\mathbb{Q} = \mathbb{Q}(\kappa)$, $\dot{\vec{C}}$ be a $\mathbb{Q}$-name for the $\square(\kappa)$-sequence added by $\mathbb{Q}$, and $\dot{\mathbb{T}}$ be a $\mathbb{Q}$-name for $\mathbb{T}(\dot{\vec{C}})$. Then $\mathbb{Q} * \dot{\mathbb{T}}$ has a κ-closed dense subset.*

Proof. Let $\mathbb{S} = \{(q, \dot{t}) \mid q$ decides the value of $\dot{t}$ and $q \Vdash$ "$\beta^q = \gamma^t$"$\}$. We first show that $\mathbb{S}$ is dense in $\mathbb{Q} * \dot{\mathbb{T}}$. To this end, let $(q_0, \dot{t}_0) \in \mathbb{Q} * \dot{\mathbb{T}}$. Since $\mathbb{Q}$ is κ-strategically closed, $\dot{t}_0$ is forced to be in the ground model. Find $q \leq q_0$ and t^* such that $q \Vdash$ "$\dot{t}_0 = \check{t}^*$". Without loss of generality, we may assume that $\beta^q > \max(t^*)$. Let $\dot{t}$ be such that $q \Vdash$ "$\dot{t} = \dot{t}_0 \cup \{\check{\beta}^q\}$". Then $(q, \dot{t}) \leq (q_0, \dot{t}_0)$ and $(q, \dot{t}) \in \mathbb{S}$.

Next, we claim that $\mathbb{S}$ is κ-closed. Let $\alpha < \kappa$ and let $\langle (q_\delta, \dot{t}_\delta) \mid \delta < \alpha \rangle$ be a decreasing sequence of conditions from $\mathbb{S}$. Without loss of generality, we may assume that, for every $\delta < \alpha$, $\beta^{q_\delta} < \beta^q$. We will construct a lower bound $(q, \dot{t})$. Let $\beta^q = \sup(\{\beta^{q_\delta} \mid \delta < \alpha\})$. For limit $\zeta < \beta^q$, let $\delta < \alpha$ be such that $\beta^{q_\delta} \geq \zeta$ and set $C^q_\zeta = C^{q_\delta}_\zeta$. Let $X = \{\zeta \mid$ for some $\delta < \alpha, q_\delta \Vdash$ "$\check{\zeta} \in \dot{t}_\alpha$"$\}$. By our definition of $\mathbb{S}$, X is club in β^q and for every $\zeta \in X'$, $X \cap \zeta = C^q_\zeta$. Thus, we can let $C^q_{\beta^q} = X$. Finally, let $\dot{t}$ be such that $q \Vdash$ "$\dot{t} = \check{X} \cup \{\check{\beta}^q\}$". $(q, \dot{t})$ is then a lower bound of $\langle (q_\delta, \dot{t}_\delta) \mid \delta < \alpha \rangle$ in $\mathbb{S}$. $\square$

A key point here, which will be exploited in the proof of the next theorem, is that, for an uncountable cardinal κ, one can force to add and then thread a $\square(\kappa^+)$-sequence with a two-step iteration which is κ^+-closed, whereas if one wants to add and thread, for example, a $\square_{\kappa,<\kappa}$-sequence, the best one can do is a two-step iteration which is κ-closed.

Theorem 3.12. *Suppose $\mu < \kappa$ are regular cardinals and $\lambda > \kappa$ is a measurable cardinal. Let G be $\mathrm{Coll}(\kappa, < \lambda)$-generic over V and, in $V[G]$, let H be $\mathbb{Q}(\kappa^+)$-generic over $V[G]$. Then, in $V[G * H]$, $\square^\mu(\kappa^+)$ holds and $\square_{\kappa,<\kappa}$ fails.*

Note that, in $V[G * H]$, $\kappa^{<\kappa} = \kappa$, so $\square^*_\kappa$ holds.

Proof. We have already shown that $\square^{\mu}(\kappa^+)$ holds in any extension by $\mathbb{Q}(\kappa^+)$, so it remains to show that $\square_{\kappa,<\kappa}$ fails. Let $\mathbb{Q} = \mathbb{Q}(\kappa^+)$, and let $\vec{C}$ be the $\square(\kappa^+)$-sequence added by H. In $V[G*H]$, let $\mathbb{T} = \mathbb{T}(\vec{C})$.

Fix an elementary embedding $j : V \to M$ with critical point λ. $j \restriction \mathrm{Coll}(\kappa, <\lambda) : \mathrm{Coll}(\kappa, <\lambda) \to \mathrm{Coll}(\kappa, <j(\lambda))$ is the identity map and, in $V^{\mathrm{Coll}(\kappa,<\lambda)}$, $\mathbb{Q} * \dot{\mathbb{T}}$ has a κ^+-closed dense subset which has size κ^+. Thus, we can extend j to a complete embedding of $\mathrm{Coll}(\kappa, <\lambda) * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ into $\mathrm{Coll}(\kappa, < j(\lambda))$ such that the quotient forcing, $\mathbb{R}$, is κ-closed. Then, letting I be $\mathbb{T}$-generic over $V[G*H]$ and letting J be $\mathbb{R}$-generic over $V[G*H*I]$, we can further extend j to an elementary embedding $j : V[G] \to M[G*H*I*J]$.

We would like to extend j still further to have domain $V[G*H]$. This is precisely the reason for introducing the threading poset. In $V[G*H*I*J]$, $j(\mathbb{Q})$ is the forcing poset to add a $\square(j(\lambda))$-sequence. $\langle C_\alpha \mid \alpha < \lambda \rangle$ would be a condition in $j(\mathbb{Q})$ if it had a top element. To arrange this, we define $q^* \in j(\mathbb{Q})$ by letting $\beta^{q^*} = \lambda$, $C^{q^*}_\alpha = C_\alpha$ for all $\alpha < \lambda$, and $C^{q^*}_\lambda = \bigcup I$. Since $\bigcup I$ is a thread through $\vec{C}$, q^* is a condition in $j(\mathbb{Q})$. Moreover, for every $q \in H$, $j(q) = q \le q^*$. Thus, if K is $j(\mathbb{Q})$-generic over $V[G*H*I*J]$ and $q^* \in K$, then $j[H] \subseteq K$, so we can extend j to an elementary embedding $j : V[G*H] \to M[G*H*I*J*K]$.

Now suppose for sake of contradiction that $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha \mid \alpha < \lambda \rangle$ is a $\square_{\kappa,<\kappa}$-sequence in $V[G*H]$. For each $\alpha < \lambda$, $j(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$. Let $j(\vec{\mathcal{D}}) = \langle \mathcal{D}_\alpha \mid \alpha < j(\lambda) \rangle$. $j(\vec{\mathcal{D}})$ is a $\square_{\kappa,<\kappa}$-sequence in $M[G*H*I*J*K]$. Choose $F \in \mathcal{D}_\lambda$. F is a thus a thread through $\vec{\mathcal{D}}$.

Claim 3.13. $F \in V[G*H*I*J]$.

Proof. $F \in V[G*H*I*J*K]$. However, K is generic for $j(\mathbb{Q})$, which is $j(\lambda)$-strategically closed in $V[G*H*I*J]$ and thus does not add any κ-sequences of ordinals. Thus, $F \in V[G*H*I*J]$. $\square$

Claim 3.14. $F \in V[G*H*I]$

Proof. Suppose not. Work in $V[G*H*I]$. Then there is an $\mathbb{R}$-name $\dot{F}$ such that $F = \dot{F}^J$ and $\Vdash_{\mathbb{R}}$ "$\dot{F}$ is not in the ground model".

Suppose first that κ is not strongly inaccessible. Let γ be the least cardinal such that $2^\gamma \ge \kappa$. We will construct $\langle p_s \mid s \in {}^{\le\gamma}2 \rangle$ and $\langle \alpha_\beta \mid \beta \le \gamma \rangle$ satisfying:

1. $p_{\langle\rangle} \Vdash$ "$\dot{F}$ is a thread through $\vec{\mathcal{D}}$".

2. For all $s, u \in {}^{\le\gamma}2$ such that $s \subseteq u$, we have $p_s, p_u \in \mathbb{R}$ and $p_u \le p_s$.

3. $\langle \alpha_\beta \mid \beta \leq \gamma \rangle$ is a strictly increasing, continuous sequence of ordinals less than κ^+.

4. For all $s \in {}^{<\gamma}2$, there is $\alpha < \alpha_{|s|+1}$ such that $p_{s^\frown\langle 0 \rangle}$ and $p_{s^\frown\langle 1 \rangle}$ decide the statement "$\check{\alpha} \in \dot{F}$" in opposite ways.

5. For all $\beta < \gamma$ and all $s \in {}^{\beta}2$, both $p_{s^\frown\langle 0 \rangle}$ and $p_{s^\frown\langle 1 \rangle}$ force that $\dot{F} \cap (\alpha_\beta, \alpha_{\beta+1}) \neq \emptyset$.

6. For all limit ordinals $\beta \leq \gamma$ and all $s \in {}^{\beta}2$, $p_s \Vdash$ "α_β is a limit point of $\dot{F}$" and there is $D_s \in \mathcal{D}_{\alpha_\beta}$ such that $p_s \Vdash$ "$\dot{F} \cap \alpha_\beta = D_s$".

Suppose for a moment that we have successfully constructed these sequences. Then, for all $s \in {}^{\gamma}2$, there is $D_s \in \mathcal{D}_{\alpha_\gamma}$ such that $p_s \Vdash$ "α_γ is a limit point of $\dot{F}$ and $\dot{F} \cap \alpha_\beta = D_s$". But if $s, u \in {}^{\gamma}2$ and $s \neq u$, then there is $\alpha < \alpha_\gamma$ such that p_s and p_u decide the statement "$\alpha \in \dot{F}$" in opposite ways, so $D_s \neq D_u$. But $2^\gamma \geq \kappa$, so this contradicts the fact that $|\mathcal{D}_{\alpha_\gamma}| < \kappa$.

Now we turn our attention to the construction of such sequences. Fix $p_{\langle\rangle}$ such that $p_{\langle\rangle} \Vdash$ "$\dot{F}$ is a thread through $\vec{\mathcal{D}}$", and let $\alpha_0 = 0$. Fix $\beta < \gamma$ and suppose that $\langle p_s \mid s \in {}^{\beta}2 \rangle$ and α_β have been defined. Fix $s \in {}^{\beta}2$. Since $\Vdash_{\mathbb{R}}$ "$\dot{F}$ is unbounded in κ^+", we can find $\alpha > \alpha_\beta$ and $p'_s \leq p_s$ such that $p'_s \Vdash$ "$\check{\alpha} \in \dot{F}$". Since $\Vdash_{\mathbb{R}}$ "$\dot{F}$ is not in the ground model", we can find $\alpha_s > \alpha$ and $p_0, p_1 \leq p'_s$ such that p_0 and p_1 decide the statement "$\alpha_s \in \dot{F}$" in opposite ways. Let $p_{s^\frown\langle 0 \rangle} = p_0$ and $p_{s^\frown\langle 1 \rangle} = p_1$. Do this for all $s \in {}^{\beta}2$, and let $\alpha_{\beta+1} = \sup(\{\alpha_s + 1 \mid s \in {}^{\beta}2\})$. $2^\beta < \kappa$, so $\alpha_{\beta+1} < \kappa^+$.

If $\beta \leq \gamma$ is a limit ordinal and $\langle p_s \mid s \in {}^{<\beta}2 \rangle$ and $\langle \alpha_\delta \mid \delta < \beta \rangle$ have been constructed, let $\alpha_\beta = \sup(\{\alpha_\delta \mid \delta < \beta\})$. Fix $s \in {}^{\beta}2$. Since $\mathbb{R}$ is κ-closed, there is $p \in \mathbb{R}$ such that, for every $\delta < \beta$, $p \leq p_{s\restriction\delta}$. We have arranged that for every $\delta < \beta$ there is $\alpha > \alpha_\delta$ such that $p_{s\restriction(\delta+1)} \Vdash$ "$\check{\alpha} \in \dot{F}$". Thus, $p \Vdash$ "$\check{\alpha}_\beta$ is a limit point of $\dot{F}$". Find $p' \leq p$ and $D_s \in \mathcal{D}_{\alpha_\beta}$ such that $p' \Vdash$ "$\dot{F} \cap \check{\alpha}_\beta = \check{D}_s$". Let $p_s = p'$. Requirements 1-6 above are easily seen to be satisfied by this construction.

Now suppose that κ is strongly inaccessible. We modify the above construction slightly. By Fodor's Lemma, find $\nu < \kappa$ and a stationary $S \subseteq S^\lambda_{<\kappa}$ such that if $\alpha \in S$, then $|\mathcal{D}_\alpha| \leq \nu$. Construct $\langle p_s \mid s \in {}^{\leq\nu}2 \rangle$ and $\langle \alpha_\beta \mid \beta \leq \nu \rangle$ exactly as above. Fix a sufficiently large regular cardinal θ and let $M \prec H(\theta)$ contain all relevant information (including $\dot{F}$, $\vec{\mathcal{D}}$, $\mathbb{R}$, $\langle p_s \mid s \in {}^{\leq\nu}2 \rangle$, and $\langle \alpha_\beta \mid \beta \leq \nu \rangle$) such that $|M| = \kappa \subseteq M$ and $\lambda_M = M \cap \lambda \in S$. Fix $\langle \lambda_\eta \mid \eta < \gamma < \kappa \rangle$ increasing and cofinal in λ_M. Using the κ-closure of $\mathbb{R}$ and the fact that $\dot{F}$ is forced to be a club, find, for each $s \in {}^{\nu}2$, a decreasing sequence of conditions from $\mathbb{R}$, $\langle p_{s,\eta} \mid \eta < \gamma\rangle$ such that, for every $\eta < \gamma$, $p_{s,\eta} \in M$ and there is a ξ_η such that $\lambda_\eta < \xi_\eta < \lambda_M$ and $p_{s,\eta} \Vdash$ "$\check{\xi}_\eta \in \dot{F}$". Let p^*_s be a lower bound for $\langle p_{s,\eta} \mid \eta < \gamma\rangle$. For each $s \in {}^\nu 2$, $p^*_s \Vdash$ "$\check{\lambda}_M \in \dot{F}'$" and, for $s \neq u \in {}^\nu 2$, p^*_s and p^*_u force contradictory information about $\dot{F} \cap \lambda_M$. Since $2^\nu > \nu$, this contradicts the fact that $|\mathcal{D}_{\lambda_M}| \leq \nu$. $\Box$

Thus, $F \in V[G * H * I]$. However, $\lambda = (\kappa^+)^{V[G]}$ and, in $V[G]$, $\mathbb{Q} * \dot{\mathbb{T}}$ has a dense κ^+-closed subset. Thus, $\lambda = (\kappa^+)^{V[G*H*I]}$, contradicting the fact that F is a club in λ of order type κ. $\Box$

Next, we prove that, if $\mu < \nu \leq \kappa$, $\Box^\mu(\kappa^+)$ does not imply that there is a stationary $T \subseteq S^{\kappa^+}_\nu$ such that $\Box(\kappa^+, T)$ holds. In particular, $\Box^\mu(\kappa^+)$ does not imply $\Box^\nu(\kappa^+)$. The main idea in the argument, which comes from a modification of the proof of Theorem 18 in [6], is that, though the forcing to thread a $\Box(\kappa^+)$-sequence does not necessarily preserve stationary subsets of $S^{\kappa^+}_\nu$, the stationarity of the sets not preserved by the threading forcing is, in a sense, easy to destroy. Thus, by shooting clubs disjoint to these sets, we can arrange so that the threading forcing does in fact preserve stationary subsets of $S^{\kappa^+}_\nu$.

Theorem 3.15. *Let $\mu < \nu \leq \kappa$ be regular cardinals, and let $\lambda > \kappa$ be measurable with $2^\lambda = \lambda^+$. Then there is a forcing extension preserving all cardinals $\leq \kappa$ in which $\Box^\mu(\kappa^+)$ holds but in which, for every stationary $T \subseteq S^{\kappa^+}_\nu$, $\Box(\kappa^+, T)$ fails.*

Proof. Let the initial model be called V_0. In V_0, let $\mathbb{P} = \mathrm{Coll}(\kappa, < \lambda)$. Let $V = V_0^{\mathbb{P}}$. Work in V. Let $\mathbb{Q} = \mathbb{Q}(\kappa^+)$, and let $\dot{\vec{C}}$ be a name for the $\Box(\kappa^+)$-sequence added by $\mathbb{Q}$. In $V^{\mathbb{Q}}$, let $\mathbb{T} = \mathbb{T}(\dot{\vec{C}})$.

In $V^{\mathbb{Q}}$, we define a sequence of posets $\langle \mathbb{S}_\alpha \mid \alpha \leq \lambda^+\rangle$ by induction on α. We will show that each $\mathbb{S}_\alpha$ is λ-distributive and thus does not change any cofinalities $\leq \lambda$. For each $\beta < \lambda^+$, we will fix a $\mathbb{Q} * \mathbb{S}_\beta$-name $\dot{X}_\beta$ for a subset of S^λ_ν such that $\Vdash_{\mathbb{Q}*\mathbb{S}_\beta*\mathbb{T}}$ "$\dot{X}_\beta$ is non-stationary" and a $\mathbb{Q} * \mathbb{S}_\beta * \mathbb{T}$-name $\dot{E}_\beta$ for a club in λ such that $\Vdash_{\mathbb{Q}*\mathbb{S}_\beta*\mathbb{T}}$ "$\dot{X}_\beta \cap \dot{E}_\beta = \emptyset$". Elements of $\mathbb{S}_\alpha$ are then functions s such that:

1. $\mathrm{dom}(s) \subseteq \alpha$.
2. $|s| \leq \kappa$.
3. For every $\beta \in \mathrm{dom}(s)$, $s(\beta)$ is a closed, bounded subset of λ.
4. For every $\beta \in \mathrm{dom}(s)$, $s \restriction \beta \Vdash$ "$s(\beta) \cap \dot{X}_\beta = \emptyset$".

For $s, t \in \mathbb{S}_\alpha$, $t \leq s$ if and only if $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$ and, for every $\beta \in \mathrm{dom}(s)$, $t(\beta)$ end-extends $s(\beta)$. $\mathbb{S}_{\lambda^+}$ can be seen as a dense subset of an iteration with $\leq \kappa$-support in which the α^{th} iterand shoots a club disjoint to the interpretation of $\dot{X}_\alpha$. Thus, for each $\alpha < \lambda^+$, $\Vdash_{\mathbb{Q}*\mathbb{S}_{\lambda^+}}$ "$\dot{X}_\alpha$ is non-stationary". By a standard Δ-system argument, it is easy to see that, in $V^{\mathbb{Q}}$, $\mathbb{S}_{\lambda^+}$ has the λ^+-chain condition. Thus, since, in $V^{\mathbb{Q}}$, $2^\lambda = \lambda^+$, we can choose the sequence $\langle \dot{X}_\alpha \mid \alpha < \lambda^+ \rangle$ in such a way that, for every $\beta < \lambda^+$ and every $\mathbb{Q} * \mathbb{S}_\beta$-name $\dot{X}$ for a subset of S^λ_ν, if there is $\alpha \geq \beta$ such that $\Vdash_{\mathbb{Q}*\mathbb{S}_\alpha*\mathbb{T}}$ "$\dot{X}$ is non-stationary", then there is $\alpha^* \geq \alpha$ such that $\Vdash_{\mathbb{Q}*\mathbb{S}_{\alpha^*}}$ "$\dot{X}_{\alpha^*} = \dot{X}$". Also, again by the λ^+-chain condition of $\mathbb{S}_{\lambda^+}$, if $\dot{X}$ is a $\mathbb{Q} * \mathbb{S}_{\lambda^+}$-name for a subset of S^λ_ν and $\Vdash_{\mathbb{Q}*\mathbb{S}_{\lambda^+}*\mathbb{T}}$ "$\dot{X}$ is non-stationary", then there is $\alpha < \lambda^+$ and a $\mathbb{Q} * \mathbb{S}_\alpha$-name $\dot{Y}$ for a subset of S^λ_ν such that $\Vdash_{\mathbb{Q}*\mathbb{S}_{\lambda^+}}$ "$\dot{X} = \dot{Y}$" and $\Vdash_{\mathbb{Q}*\mathbb{S}_\alpha*\mathbb{T}}$ "$\dot{Y}$ is non-stationary". Putting this together, we have that for every $\mathbb{Q}*\mathbb{S}_{\lambda^+}$-name $\dot{X}$ for a subset of S^λ_ν, if $\Vdash_{\mathbb{Q}*\mathbb{S}_{\lambda^+}*\mathbb{T}}$ "$\dot{X}$ is non-stationary", then already $\Vdash_{\mathbb{Q}*\mathbb{S}_{\lambda^+}}$ "$\dot{X}$ is non-stationary".

Lemma 3.16. *In $V^{\mathbb{Q}}$, for every $\alpha \leq \lambda^+$, $\mathbb{S}_\alpha$ is ν-closed .*

Proof. Work in $V^{\mathbb{Q}}$. The proof is by induction on $\alpha \leq \lambda^+$. Thus, assume that, for all $\beta < \alpha$, $\mathbb{S}_\beta$ is ν-closed. Let $\langle s_\gamma \mid \gamma < \xi \rangle$ be a decreasing sequence from $\mathbb{S}_\alpha$, with $\xi < \nu$. We will define a lower bound $s \in \mathbb{S}_\alpha$ for the sequence. Let $\mathrm{dom}(s) = \bigcup_{\gamma<\xi} \mathrm{dom}(s_\gamma)$. Clearly, $|\mathrm{dom}(s)| \leq \kappa$. For $\beta \in \mathrm{dom}(s)$, let $\delta_\beta = \sup(\bigcup_{\gamma<\xi} s_\gamma(\beta))$, and let $s(\beta) = \{\delta_\beta\} \cup \bigcup_{\gamma<\xi} s_\gamma(\beta)$. It is immediate that, if $s \in \mathbb{S}_\alpha$, then it is a lower bound for $\langle s_\gamma \mid \gamma < \xi \rangle$. Thus, it remains to show that for all $\beta \in \mathrm{dom}(s)$, $s \restriction \beta \Vdash$ "$s(\beta) \cap \dot{X}_\beta = \emptyset$". But, if $\beta \in \mathrm{dom}(s)$, then, by the induction hypothesis, $\mathbb{S}_\beta$ is ν-closed. Therefore ν remains a regular cardinal in $V^{\mathbb{Q}*\mathbb{S}_\beta}$, so $s \restriction \beta \Vdash_{\mathbb{S}_\beta}$ "$\check{\delta}_\beta \notin \dot{X}_\beta$" and hence $s \restriction \beta \Vdash$ "$s(\beta) \cap \dot{X}_\beta = \emptyset$". □

By genericity, $\vec{C}$, the square sequence added by $\mathbb{Q}$, is a $\square^\mu(\lambda)$-sequence as witnessed by a stationary $S \subseteq S^\lambda_\mu$. Also, by Lemma 1.7, S remains stationary in $V^{\mathbb{Q}*\mathbb{S}_{\lambda^+}}$, so $\vec{C}$ remains a $\square^\mu(\lambda)$-sequence in $V^{\mathbb{Q}*\mathbb{S}_{\lambda^+}}$.

Lemma 3.17. *In V, for every $\alpha \leq \lambda^+$, $\mathbb{Q}*\mathbb{S}_\alpha*\mathbb{T}$ has a dense λ-closed subset.*

Proof. Work in V. For $\alpha \leq \lambda^+$, let $\mathbb{U}_\alpha$ consist of all conditions of $\mathbb{Q} * \mathbb{S}_\alpha * \mathbb{T}$ of the form $(q, \dot{s}, \dot{t})$ such that q decides the values of $\dot{s}$ and $\dot{t}$, $q \Vdash$ "$\beta^q = \gamma^t$", and, for every $\beta \in \mathrm{dom}(\dot{s})$, $(q, \dot{s} \restriction \beta, \dot{t}) \Vdash$ "$\max(\dot{s}(\beta)) \in \dot{E}_\beta$". We show by induction on α that $\mathbb{U}_\alpha$ is the desired dense λ-closed subset.

If $\alpha = 0$, this is simply Proposition 3.11. Let $\alpha = \beta + 1$. We first show that $\mathbb{U}_\alpha$ is dense. Fix $(q_0, \dot{s}_0, \dot{t}_0)$. Since $\mathbb{Q}$ is $(\kappa + 1)$-strategically closed, we may assume without loss of generality that q_0 decides the value of $\dot{s}_0$ to be s_0. Apply the induction hypothesis to get $(q_1, \dot{s}_1^*, \dot{t}_1) \leq (q_0, \check{s}_0 \restriction \beta, \dot{t}_0)$ in $\mathbb{U}_\beta$ such that there is $\gamma < \lambda$ such that $\gamma > \max(s_0(\beta))$ (we say $\max(s_0(\beta)) = 0$ if $\beta \notin \mathrm{dom}(s_0)$) and $(q_1, \dot{s}_1^*, \dot{t}_1) \Vdash$ "$\check{\gamma} \in \dot{E}_\beta$" (it must then be the case that $(q_1, \dot{s}_1^*) \Vdash$ "$\check{\gamma} \notin \dot{X}_\beta$"). Let $\dot{s}_1$ be such that $q_1 \Vdash$ "$\dot{s}_1 = \dot{s}_1^* \cup \{(\beta, s_0(\beta) \cup \{\gamma\})\}$". Then $(q_1, \dot{s}_1, \dot{t}_1)$ extends $(q_0, \dot{s}_0, \dot{t}_0)$ and is in $\mathbb{U}_\alpha$.

We now show that $\mathbb{U}_\alpha$ is λ-closed. Let $\xi < \lambda$, and let $\langle (q_\eta, \dot{s}_\eta, \dot{t}_\eta) \mid \eta < \xi \rangle$ be a decreasing sequence from $\mathbb{U}_\alpha$. By the methods of the proof of Proposition 3.11, we can find $(q, \dot{t})$ such that q decides the value of $\dot{t}$, $q \Vdash$ "$\beta^q = \gamma^t$", and, for every $\eta < \xi$, $(q, \dot{t}) \leq (q_\eta, \dot{t}_\eta)$. We now define an s so that $(q, \check{s}, \dot{t})$ is a lower bound for our sequence. Let $\mathrm{dom}(s) = \{\delta \mid$ for some $\eta < \xi, q_\eta \Vdash \delta \in \mathrm{dom}(\dot{s}_\eta)\}$. For $\delta \in \mathrm{dom}(s)$, let $r_\delta = \{\gamma \mid$ for some $\eta < \xi, q_\eta \Vdash \gamma \in \dot{s}_\eta(\delta)\}$ and let $s(\delta) = r_\delta \cup \sup(r_\delta)$. All that remains to be shown is that, for every $\delta \in \mathrm{dom}(s)$, $(q, \check{s} \restriction \delta, \dot{t}) \Vdash$ "$\max(\check{s}(\delta)) \in \dot{E}_\delta$". We show this by induction on δ. Thus, assume it holds for all ordinals less than δ in $\mathrm{dom}(s)$. Then $(q, \check{s} \restriction \delta, \dot{t}) \in \mathbb{U}_\delta$. For every $\eta < \xi$ with $\delta \in \mathrm{dom}(s_\eta)$, $(q, \dot{s}_\eta \restriction \delta, \dot{t}) \Vdash$ "$\max(\dot{s}_\eta(\delta)) \in \dot{E}_\delta$". Thus, $(q, \check{s} \restriction \delta, \dot{t}) \Vdash$ "$\dot{E}_\delta$ is unbounded in $\check{r}_\delta$". Since $\dot{E}_\delta$ is forced to be a club, $(q, \check{s} \restriction \delta, \dot{t}) \Vdash$ "$\sup(r_\delta) = \max(\check{s}(\delta)) \in \dot{E}_\delta$".

Now suppose that $\alpha \leq \lambda^+$ is a limit ordinal. To show that $\mathbb{U}_\alpha$ is dense, let $(q_0, \dot{s}_0, \dot{t}_0) \in \mathbb{Q} * \mathbb{S}_\alpha * \mathbb{T}$. Assume without loss of generality that q_0 decides the value of $\dot{s}_0$ and $\dot{t}_0$. If $\mathrm{cf}(\alpha) = \lambda$, then there is $\beta < \alpha$ such that $q_0 \Vdash$ "$\mathrm{dom}(\dot{s}_0) \subseteq \beta$". Then $(q_0, \dot{s}_0, \dot{t}_0) \in \mathbb{Q} * \mathbb{S}_\beta * \mathbb{T}$ and, by the inductive hypothesis, we can find $(q_1, \dot{s}_1, \dot{t}_1) \leq (q_0, \dot{s}_0, \dot{t}_0)$ such that $(q_1, \dot{s}_1, \dot{t}_1) \in \mathbb{U}_\beta \subseteq \mathbb{U}_\alpha$.

If $\mathrm{cf}(\alpha) < \lambda$, fix an increasing, continuous sequence $\langle \alpha_i \mid i < \xi \rangle$ cofinal in α, with $\xi \leq \kappa$. We define a sequence $\langle (q_i, \dot{s}_i, \dot{t}_i) \mid 1 \leq i < \xi \rangle$ such that, for all $1 \leq i < j < \xi$,

- $(q_i, \dot{s}_i, \dot{t}_i) \in \mathbb{U}_{\alpha_i}$.
- $(q_i, \dot{s}_i, \dot{t}_i) \leq (q_0, \dot{s}_0 \restriction \alpha_i, \dot{t}_0)$.
- $(q_j, \dot{s}_j, \dot{t}_j) \leq (q_i, \dot{s}_i, \dot{t}_i)$.

If $i = i' + 1$, define $(q_i, \dot{s}_i, \dot{t}_i)$ as follows. Let $\dot{s}_i^*$ be such that $q_{i'} \Vdash$ "$\dot{s}_i^* = \dot{s}_{i'} \cup \dot{s}_0 \restriction [\alpha_{i'}, \alpha_i)$". By the inductive hypothesis, find $(q_i, \dot{s}_i, \dot{t}_i) \in \mathbb{U}_{\alpha_i}$ such that $(q_i, \dot{s}_i, \dot{t}_i) \leq (q_{i'}, \dot{s}_i^*, \dot{t}_{i'})$. If i is a limit ordinal, then, by the inductive hypothesis, $\mathbb{U}_{\alpha_i}$ is λ-closed, so we can find $(q_i, \dot{s}_i, \dot{t}_i) \in \mathbb{U}_{\alpha_i}$ that is a lower bound for $\langle (q_j, \dot{s}_j, \dot{t}_j) \mid j < i \rangle$.

We now define $(q, \dot{s}, \dot{t}) \in \mathbb{U}_\alpha$ which is a lower bound for $\langle (q_i, \dot{s}_i, \dot{t}_i) \mid i < \xi \rangle$. First, by previous arguments, find $(q, \dot{t}) \in \mathbb{U}_0$ which is a lower bound for

$\langle (q_i, \dot{t}_i) \mid i < \xi \rangle$. Next, let $X = \{\beta \mid \text{for some } i < \xi, q_i \Vdash \text{“}\check{\beta} \in \mathrm{dom}(\dot{s}_i)\text{”}\}$. If $\beta \in X$, let $\dot{s}^*(\beta)$ be such that $q \Vdash \text{“}\dot{s}^*(\beta) = \bigcup_{i<\xi} \dot{s}_i(\beta)\text{”}$. Let $\dot{s}$ be such that $q \Vdash \text{“}\mathrm{dom}(\dot{s}) = X$ and, if $\beta \in X$, then $\dot{s}(\beta) = \dot{s}^*(\beta) \cup \{\sup(\dot{s}^*(\beta)\}\text{”}$. It is routine to check that $(q, \dot{s}, \dot{t}) \in \mathbb{U}_\alpha$ and $(q, \dot{s}, \dot{t}) \leq (q_0, \dot{s}_0, \dot{t}_0)$.

The proof that $\mathbb{U}_\alpha$ is λ-closed is the same as in the successor case. $\square$

It follows that, for all $\alpha \leq \lambda^+$, $\mathbb{S}_\alpha$ is λ-distributive and thus preserves all cardinals and cofinalities. It remains to show that, in $V^{\mathbb{Q}*\mathbb{S}_{\lambda^+}}$, $\square(\lambda, T)$ fails for every stationary $T \subseteq S^\lambda_\nu$.

Fix $j : V_0 \to M$ with $\mathrm{crit}(j) = \lambda$. Let G be $\mathbb{P}$-generic over V_0, let H be $\mathbb{Q}$-generic over $V_0[G] = V$, let I be $\mathbb{S}_{\lambda^+}$-generic over $V[H]$, and let J be $\mathbb{T}$-generic over $V[H * I]$. Since, in V, $\mathbb{Q} * \mathbb{S}_{\lambda^+} * \mathbb{T}$ has a λ-closed dense subset and has size λ^+, by Fact 1.5 we can extend the identity map $i : \mathbb{P} \to j(\mathbb{P})$ to a complete embedding $i^* : \mathbb{P} * \mathbb{Q} * \mathbb{S}_{\lambda^+} * \mathbb{T} \to j(\mathbb{P})$ such that the quotient forcing $\mathbb{R} = j(\mathbb{P})/i^*[\mathbb{P} * \mathbb{Q} * \mathbb{S}_{\lambda^+} * \mathbb{T}]$ is κ-closed. Thus, letting K be $\mathbb{R}$-generic over $V[H * I * J]$, we can lift j to an elementary embedding $j : V \to M[G * H * I * J * K]$.

Suppose now for sake of contradiction that, in $V[H * I]$, $T \subseteq S^\lambda_\nu$ is stationary and $\vec{D} = \langle D_\alpha \mid \alpha < \lambda \rangle$ is a $\square(\lambda, T)$-sequence. For $\xi < \lambda^+$, let $I_\xi = I \cap \mathbb{S}_\xi$. Each I_ξ is then $\mathbb{S}_\xi$-generic over $V[H]$. Since $\mathbb{S}_{\lambda^+}$ has the λ^+-c.c., we can fix $\xi^* < \lambda^+$ such that $T, \vec{D} \in V[H * I_{\xi^*}]$.

We would like to lift j further to have domain $V[H * I_{\xi^*}]$. To do this, we define a master condition $(q^*, \check{s}^*, \check{t}^*) \in j(\mathbb{Q} * \mathbb{S}_{\xi^*} * \mathbb{T})$. q^* is defined exactly as in the proof of Theorem 3.12. Let E be the club added by J, and let $t^* = E \cup \{\lambda\}$. Then $(q^*, \check{t}^*) \in j(\mathbb{Q} * \mathbb{T})$. Let s be the generic object added by I_{ξ^*}. s is thus a function with domain ξ^*, where, for each $\alpha < \xi^*$, $s(\alpha)$ is a club in λ. Let s^* be such that $\mathrm{dom}(s^*) = j[\xi^*]$ and, for each $\alpha < \xi^*$, $s^*(j(\alpha)) = s(\alpha) \cup \{\lambda\}$. It is clear that, if $(q^*, \check{s}^*) \in j(\mathbb{Q} * \mathbb{S}_{\xi^*})$, then it is a lower bound for $j[H * I_{\xi^*}]$. Thus, all that needs to be checked is that, for every $\alpha < \xi^*$, $(q^*, \check{s}^* \restriction j(\alpha)) \Vdash \text{“}\check{s}^*(j(\alpha)) \cap j(\dot{X}_\alpha) = \emptyset\text{”}$. We show this by induction on α. It suffices to show that $(q^*, \check{s}^* \restriction j(\alpha)) \Vdash \text{“}\check{\lambda} \notin j(\dot{X}_\alpha)\text{”}$. Suppose for sake of contradiction that there is $(q', \dot{s}') \leq (q^*, \check{s}^* \restriction j(\alpha))$ such that $(q', \dot{s}') \Vdash \text{“}\check{\lambda} \in j(\dot{X}_\alpha)\text{”}$. Recall that $\dot{E}_\alpha$ is a $\mathbb{Q} * \mathbb{S}_\alpha * \mathbb{T}$-name for a club in λ. $(q', \dot{s}', \check{t}^*)$ is a lower bound for $H * I \restriction \alpha * J$, so, for every $\beta < \lambda$, $(q', \dot{s}', \check{t}^*) \Vdash \text{“}\check{\beta} \in j(\dot{E}_\alpha)\text{”}$ if and only if $\beta \in E_\alpha$. Thus, $(q', \dot{s}', \check{t}^*) \Vdash \text{“}\lambda$ is a limit point of $j(\dot{E}_\alpha)\text{”}$, so $(q', \dot{s}', \check{t}^*) \Vdash \text{“}\lambda \notin j(\dot{X}_\alpha)\text{”}$. This is a contradiction.

Thus, $(q^*, \check{s}^*)$ is a lower bound for $j[H * I_{\xi^*}]$ in $j(\mathbb{Q} * \mathbb{S}_{\xi^*})$, so, if we let $H^+ * I^+_{\xi^*}$ be $j(\mathbb{Q}*\mathbb{S}_{\xi^*})$-generic over $M[G*H*I*J*K]$ with $(q^*, \check{s}^*) \in H^+ * I^+_{\xi^*}$, then we can lift j to an elementary embedding $j : V[H * I_{\xi^*}] \to M[G * H * I * J * K * H^+ * I^+_{\xi^*}]$.

Now $\vec{D}$ and T are in the domain of j, $j(\vec{D}) = \langle D_\alpha \mid \alpha < j(\lambda)\rangle$ is a $\square(j(\lambda), j(T))$-sequence, and $j(T) \cap \lambda = T$. Thus, D_λ is a thread through $\vec{D}$ and avoids T. Since $j(\mathbb{Q} * \mathbb{S}_{\xi^*})$ is $j(\lambda)$-distributive, $H^+ * I^+_{\xi^*}$ could not have added D_λ, so $D_\lambda \in V[H * I * J * K]$. Also, since K is generic for κ-closed forcing, the argument from Theorem 3.12 shows that it can not add a thread through $\vec{D}$, so $D_\lambda \in V[H * I * J]$ and witnesses that T is not stationary in $V[H * I * J]$. But we arranged our iteration S_{λ^+} so that this implies that T is already non-stationary in $V[H*I]$. This is a contradiction, so $\square(\lambda, T)$ fails in $V[H * I]$. $\square$

Finally, we show that the existence of a stationary $S \subset S^{\kappa^+}_{\kappa}$ such that $\square(\kappa^+, S)$ holds does not imply the existence of a stationary $T \subset S^{\kappa^+}_{<\kappa}$ for which $\square(\kappa^+, T)$ holds. In [18], Harrington and Shelah show that, after collapsing a Mahlo cardinal to be κ^+, one can iteratively force to shoot clubs disjoint to non-reflecting subsets of $S^{\kappa^+}_{<\kappa}$, thus obtaining a model in which every stationary subset of $S^{\kappa^+}_{<\kappa}$ reflects. We will work in L and carry out the forcing iteration used by Harrington and Shelah, arguing that our desired conclusion holds in the final model. We use the following theorem of Jensen about squares in L ([9], Chapter VII).

Theorem 3.18. *Suppose $V = L$. Let λ be an inaccessible cardinal which is not weakly compact, and let $S \subseteq \lambda$ be stationary. Then there is a stationary $S' \subseteq S$ such that $\square(\lambda, S')$ holds.*

Theorem 3.19. *Suppose $V = L$, κ is a regular, uncountable cardinal, and $\lambda > \kappa$ is the least Mahlo cardinal greater than κ. Then there is a forcing extension in which $\lambda = \kappa^+$, there is a stationary $S \subseteq S^\lambda_\kappa$ such that $\square(\lambda, S)$ holds, and all stationary subsets of $S^\lambda_{<\kappa}$ reflect (and hence $\square(\lambda, T)$ fails for every stationary $T \subseteq S^\lambda_{<\kappa}$).*

Proof. By Theorem 3.18, fix a stationary $S \subseteq \lambda$ consisting of inaccessible cardinals and a $\square(\lambda, S)$-sequence, $\vec{C}$. Let $\mathbb{P} = \mathrm{Coll}(\kappa, <\lambda)$. In $V^{\mathbb{P}}$, we define an iteration $\langle \mathbb{Q}_\alpha \mid \alpha \leq \lambda^+\rangle$, which will shoot clubs disjoint to non-reflecting sets of ordinals, by induction on α. For each $\alpha < \lambda^+$, we will fix a $\mathbb{Q}_\alpha$-name $\dot{X}_\alpha$ such that $\Vdash_{\mathbb{Q}_\alpha}$ "$\dot{X}_\alpha \subseteq S^\lambda_{<\kappa}$ and $\dot{X}_\alpha$ does not reflect at any ordinal of uncountable cofinality". Elements of $\mathbb{Q}_\alpha$ are functions q such that:

1. $\mathrm{dom}(q) \subseteq \alpha$.

2. $|q| \leq \kappa$.

3. For every $\beta \in \mathrm{dom}(q)$, $q(\beta)$ is a closed, bounded subset of λ.

4. For every $\beta \in \mathrm{dom}(q)$, $q \restriction \beta \Vdash$ "$q(\beta) \cap \dot{X}_\beta = \emptyset$".

For $p, q \in \mathbb{Q}_\alpha$, $q \leq p$ if and only if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and, for every $\beta \in \mathrm{dom}(p)$, $q(\beta)$ end-extends $p(\beta)$. An easy Δ-system argument shows that $\mathbb{Q}_{\lambda^+}$ has the λ^+-c.c. Thus, with a suitable choice of the names $\dot{X}_\alpha$, we can arrange that, in $V^{\mathbb{P} * \mathbb{Q}_{\lambda^+}}$, every stationary subset of $S^\lambda_{<\kappa}$ reflects.

Back in V, fix a sufficiently large, regular cardinal θ (in particular, $\theta > \lambda^+$). Let $\mathcal{N}$ be the set of N such that $\mathbb{P} \in N$, $N \preceq (H(\theta), \in, \lhd)$ (where $\lhd$ is a well-ordering of $H(\theta)$), $\lambda_N := N \cap \lambda$ is an inaccessible cardinal, $|N| = \lambda_N$, and $N^{<\lambda_N} \subseteq N$. For $N \in \mathcal{N}$, let $\pi_N : N \to \bar{N}$ be the transitive collapse. If $x \in N$, let $x_N = \pi_N(x)$.

Lemma 3.20. *For every $x \in H(\theta)$, there is $N \in \mathcal{N}$ such that $x \in N$.*

Proof. Fix $x \in H(\theta)$. We find the desired $N \in \mathcal{N}$ by building an increasing, continuous chain $\langle N_\alpha \mid \alpha < \lambda \rangle$ such that, for each $\alpha < \lambda$,

1. $x, \lambda \in N_\alpha$

2. $N_\alpha \prec H(\theta)$.

3. $|N_\alpha| < \lambda$.

4. $\mathcal{P}(N_\alpha) \subseteq N_{\alpha+1}$.

5. $\sup(N_\alpha \cap \lambda) \subseteq N_{\alpha+1}$.

Let E be the set of $\alpha < \lambda$ such that $N_\alpha \cap \lambda = \alpha = |N_\alpha|$. E is a club in λ, so, since λ is Mahlo, there is $\alpha^* \in E$ such that α^* is inaccessible. N_{α^*} is then in $\mathcal{N}$. $\square$

If $N \in \mathcal{N}$, then, since N is closed under $< \lambda_N$-sequences, $\mathbb{P}_N = \mathbb{P} \cap N = \mathrm{Coll}(\kappa, < \lambda_N)$, so $N^{\mathbb{P}} \prec H(\theta)^{\mathbb{P}}$. Also, since $\mathbb{P}_N$ has the λ_N-c.c., $\bar{N}^{\mathbb{P}_N} \cong N^{\mathbb{P}}$ is closed under $< \lambda_N$-sequences from $V^{\mathbb{P}_N}$.

The following Lemma, which will show that $\mathbb{Q}_{\lambda^+}$ is λ-distributive, is proven in [18].

Lemma 3.21. *Let $\alpha < \lambda^+$.*

1. *For all $N \in \mathcal{N}$ such that $\alpha \in N$, in $V^{\mathbb{P}_N}$, $(\mathbb{Q}_\alpha)_N$ has a λ_N-closed dense subset.*

2. *In $V^{\mathbb{P}}$, $\mathbb{Q}_\alpha$ is λ-distributive.*

3. *For all $N \in \mathcal{N}$ such that $\alpha \in N$, in $V^{\mathbb{P}_N * (\mathbb{Q}_\beta)_N}$, $(X_\beta)_N$ is not stationary in λ_N.*

By the chain condition, every $< \lambda$-sequence in $V^{\mathbb{P}*\mathbb{Q}_{\lambda^+}}$ appears in $V^{\mathbb{P}*\mathbb{Q}_\alpha}$ for some $\alpha < \lambda^+$, so we have that, in $V^{\mathbb{P}}$, $\mathbb{Q}_{\lambda^+}$ is λ-distributive and thus preserves all cardinals $\leq \lambda$.

Lemma 3.22. *In $V^{\mathbb{P}*\mathbb{Q}_{\lambda^+}}$, there is no stationary $T \subseteq S^\lambda_{<\kappa}$ such that $\square(\lambda, T)$ holds.*

Proof. Suppose there is such a T, and let $\vec{D}$ be a $\square(\lambda, T)$-sequence. Then, for each $\alpha < \lambda$ of uncountable cofinality, D'_α witnesses that $T \cap \alpha$ is non-stationary, so T does not reflect. However, in $V^{\mathbb{P}*\mathbb{Q}_{\lambda^+}}$, every stationary subset of $S^\lambda_{<\kappa}$ reflects. Contradiction. $\square$

In $V^{\mathbb{P}*\mathbb{Q}_{\lambda^+}}$, $\vec{C}$ is clearly still a coherent sequence avoiding S and $S \subseteq S^\lambda_\kappa$. Thus, the following lemma suffices to prove the theorem.

Lemma 3.23. *S is stationary in $V^{\mathbb{P}*\mathbb{Q}_{\lambda^+}}$.*

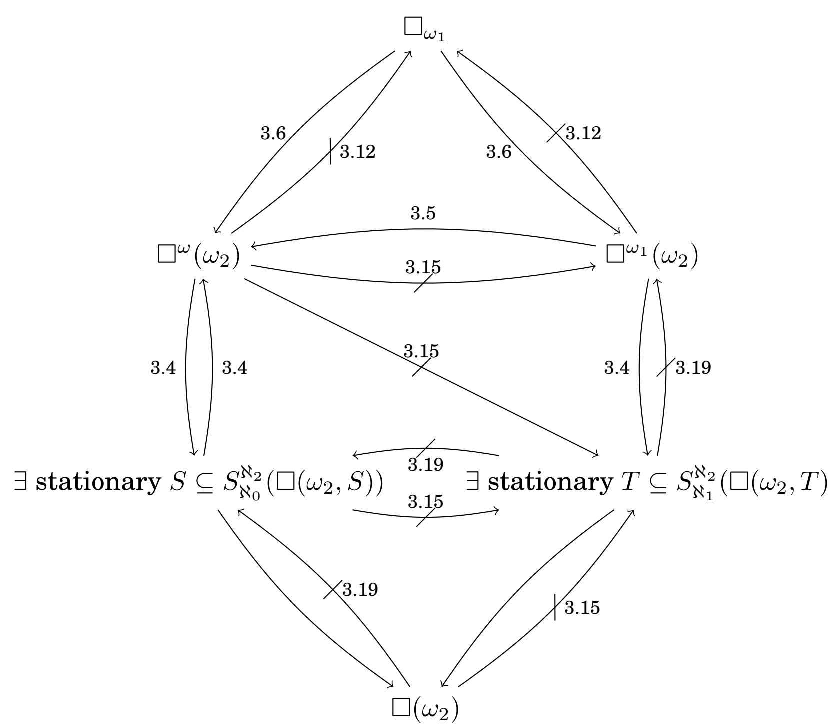
Proof. Let $E \in V^{\mathbb{P}*\mathbb{Q}_{\lambda^+}}$ be a club in λ. By the chain condition, there is $\alpha < \lambda^+$ such that $E \in V^{\mathbb{P}*\mathbb{Q}_\alpha}$. Thus, it suffices to show that S remains stationary in $V^{\mathbb{P}*\mathbb{Q}_\alpha}$ for every $\alpha < \lambda^+$.

To this end, fix $\alpha < \lambda^+$, $(p, \dot{q}) \in \mathbb{P} * \mathbb{Q}_\alpha$, and $\dot{E}$, a $\mathbb{P} * \mathbb{Q}_\alpha$-name for a club in λ. By the argument from the proof of Lemma 3.20, we can find $N \in \mathcal{N}$ such that $\{(p, \dot{q}), \dot{E}, \alpha\} \subseteq N$ and $\lambda_N \in S$. Let G be $\mathbb{P}$-generic over V with $p \in G$, and let G_N be the restriction of G to $\mathrm{Coll}(\kappa, < \lambda_N)$. Since $\mathbb{P}$ has the λ-c.c., S is still stationary in $V[G]$. Let q be the interpretation of $\dot{q}$ in $V[G]$, and reinterpret $\dot{E}$ in $V[G]$ as a $\mathbb{Q}_\alpha$-name. Also, we can extend π_N to an isomorphism of $N[G]$ and $\bar{N}[G_N]$. Enumerate the dense open sets of $(\mathbb{Q}_\alpha)_N$ lying in $\bar{N}[G_N]$ as $\langle D_\xi \mid \xi < \lambda_N \rangle$. By Lemma 3.21(1) and the fact that $\bar{N}[G_N]$ is closed under $< \lambda_N$-sequences, we can find a decreasing sequence $\langle q_\xi \mid \xi < \lambda_N \rangle$ of conditions from the λ_N-closed dense subset of $(\mathbb{Q}_\alpha)_N$ such that $q_0 = q$ and, for all $\xi < \lambda_N$, $q_{\xi+1} \in D_\xi \cap \bar{N}[G_N]$. We define q^* to be a lower bound for $\langle \pi_N^{-1}(q_\xi) \mid \xi < \lambda_N \rangle$ in $\mathbb{Q}_\alpha$ by letting $\mathrm{dom}(q^*) = N \cap \alpha$ and, for each $\beta \in \mathrm{dom}(q^*)$,

$$q^*(\beta) = \bigcup_{\xi < \lambda_N} q_\xi(\pi_N(\beta)) \cup \{\lambda_N\}.$$

$\mathrm{cf}(\lambda_N) = \kappa$ in $V^{\mathbb{P}*\mathbb{Q}_\beta}$, so λ_N is forced not to be in $\dot{X}_\beta$ and thus $q^* \in \mathbb{Q}_\alpha$. For every $\gamma < \lambda_N$, there is $\xi < \lambda_N$ and $\delta \in (\gamma, \lambda_N)$ such that $q_\xi \Vdash_{(\mathbb{Q}_\alpha)_N}$ "$\check{\delta} \in \pi(\dot{E})$". Thus, $\pi_N^{-1}(q_\xi) \Vdash_{\mathbb{Q}_\alpha}$ "$\check{\delta} \in \dot{E}$". so $q^* \Vdash_{\mathbb{Q}_\alpha}$ "$\dot{E}$ is unbounded in $\check{\lambda}_N$". Since $\dot{E}$ is a name for a club, $q^* \Vdash_{\mathbb{Q}_\alpha}$ "$\lambda_N \in \dot{E} \cap \check{S}$", so S is stationary in $V^{\mathbb{P}*\mathbb{Q}_\alpha}$. $\square$

We conclude with a diagram illustrating the situation at ω_2, where we now have a complete picture. Arrows correspond to implications, and struck-out arrows to non-implications. Numbers labeling the arrows refer to the lemmas or theorems from which they follow.



CHAPTER 4

SCALES

This chapter builds on work inspired by two questions in singular cardinal combinatorics. Woodin asked whether the failure of the Singular Cardinals Hypothesis at κ implies that $\square^*_\kappa$ holds, and Cummings, Foreman, and Magidor [6] asked whether the existence of a very good scale of length κ^+ implies that $\square^*_\kappa$ holds. Gitik and Sharon, in [15], answer both of these questions by producing, starting with a supercompact cardinal, a model in which there is a singular, strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^{++}$, AP_κ fails (and hence $\square^*_\kappa$ fails), and there is an $A \subseteq \kappa$ that carries a very good scale.

Cummings and Foreman, in [5], show that, in the model of [15], there is a $B \subseteq \kappa$ that carries a bad scale, thus providing another proof of the failure of AP_κ. Cummings and Foreman go on to raise a number of other questions, three of which we address in this chapter:

1. Do there exist any other interesting scales in the model of [15]?
2. Into which case of Shelah's Trichotomy Theorem do the bad points of the bad scale in [5] fall?
3. When the first PCF generator exists, does it have a maximal (modulo bounded subsets of κ) subset which carries a good scale?

Remark We assume throughout this chapter that, if A is a set of regular cardinals and we are considering $\prod A$, then A is progressive, i.e. $|A| < \min(A)$.

4.1 Diagonal supercompact Prikry forcing

We review here some key facts from Gitik and Sharon's construction in [15]. At the heart of their argument is a diagonal version of supercompact Prikry forcing.

Let κ be a supercompact cardinal, let $\mu = \kappa^{+\omega+1}$, and let U be a normal, fine ultrafilter over $\mathcal{P}_\kappa(\mu)$. For $n < \omega$, let U_n be the projection of U on $\mathcal{P}_\kappa(\kappa^{+n})$, i.e. $X \in U_n$ if and only if $\{y \in \mathcal{P}(\kappa^{+\omega+1}) \mid y \cap \kappa^{+n} \in X\} \in U$. Note that each U_n is a normal, fine ultrafilter over $\mathcal{P}_\kappa(\kappa^{+n})$ concentrating on the set $X_n = \{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid \kappa_x := x \cap \kappa$ is an inaccessible cardinal and, for all $i \leq n, \mathrm{otp}(x \cap \kappa^{+i}) = \kappa_x^{+i}\}$.

We are now ready to define the diagonal supercompact Prikry forcing, $\mathbb{Q}$. Conditions of $\mathbb{Q}$ are of the form $q = \langle x_0^q, x_1^q, \ldots, x_{n-1}^q, A_n^q, A_{n+1}^q, \ldots \rangle$, where

1. For all $i < n$, $x_i^q \in X_i$.
2. For all $i \geq n$, $A_i^q \in U_i$.
3. For all $i < j < n$, $x_i^q \prec x_{i+1}^q$.
4. For all $i < n$, $j \geq n$, and $y \in A_j^q$, $x_i^q \prec y$.

n is called the *length* of q and is denoted $\mathrm{lh}(q)$. $\langle x_0^q, \ldots, x_{n-1}^q \rangle$ is called the *lower part* of q, and $\langle A_n^q, A_{n+1}^q, \ldots \rangle$ is called the *upper part* of q. If $s = \langle x_0, \ldots, x_{n-1} \rangle$ is a lower part and $t = \langle A_n, A_{n+1}, \ldots \rangle$ is an upper part, then $s^\frown t$ denotes the condition $\langle x_0, \ldots, x_{n-1}, B_n, B_{n+1} \ldots \rangle$, where $B_m = \{y \in A_m \mid x_{n-1} \prec y\}$. If s is a lower part of length n, let $s^\frown \mathbb{1}$ denote the condition $s^\frown \langle X_n, X_{n+1}, \ldots \rangle$. If $s = \langle x_0, \ldots, x_{n-1} \rangle$ is a lower part and q is a condition of the form $\langle B_0, B_1, \ldots \rangle$ such that, for every $i < n$, $x_i \in B_i$, then $s^\frown q$ denotes the condition $\langle x_0, \ldots, x_{n-1}, B'_n, B'_{n+1} \ldots \rangle$, where, for $i \geq n$, $B'_i = \{y \in B_i \mid x_{n-1} \prec y\}$.

For $p, q \in \mathbb{Q}$, $p \leq q$ if and only if

1. $\mathrm{lh}(p) \geq \mathrm{lh}(q)$.
2. For all $i < \mathrm{lh}(q)$, $x_i^p = x_i^q$.
3. For all i such that $\mathrm{lh}(q) \leq i < \mathrm{lh}(p)$, $x_i^p \in A_i^q$.
4. For all $i \geq \mathrm{lh}(p)$, $A_i^p \subseteq A_i^q$.

We say p is a *direct extension of* q, and write $p \leq^* q$, if $p \leq q$ and $\mathrm{lh}(p) = \mathrm{lh}(q)$.

We now summarize some relevant facts about $\mathbb{Q}$. The reader is referred to [15] for proofs.

- (Diagonal intersection) Suppose that, for every lower part s, A_s is an upper part such that $s^\frown A_s \in \mathbb{Q}$. Then there is a sequence $\langle B_n \mid n < \omega \rangle$ such that, for every n, $B_n \in U_n$ and, for every lower part s of length n, every extension of $s^\frown \langle B_i \mid i \geq n \rangle$ is compatible with $s^\frown A_s$.

- (Prikry property) Let $q \in \mathbb{Q}$ and let ϕ be a statement in the forcing language. Then there is $p \leq^* q$ such that $p \parallel \phi$. In particular, $\mathbb{Q}$ adds no new bounded subsets of κ.

- The generic object added by $\mathbb{Q}$ is an ω-sequence $\langle x_n \mid n < \omega \rangle$, where, for all $n < \omega$, $x_n \in X_n$ and $x_n \prec x_{n+1}$. Letting $\kappa_n = \kappa_{x_n}$, $\langle \kappa_n \mid n < \omega \rangle$ is cofinal in κ, so $\mathrm{cf}(\kappa)^{V[G]} = \omega$. $\bigcup_{n<\omega} x_n = \kappa^{+\omega}$, so, in $V[G]$, for all $i \leq \omega$, κ^{+i} is an ordinal of cofinality ω and size κ.

- Any two conditions with the same lower part are compatible. In particular, since there are only $\kappa^{+\omega}$-many lower parts, $\mathbb{Q}$ satisfies the $\kappa^{+\omega+1}$-c.c. Thus, $\kappa^{+\omega+1} = \mu$ is preserved in the extension, and $(\kappa^+)^{V[G]} = \mu$.

- Let $\langle A_n \mid n < \omega \rangle$ be such that, for each $n < \omega$, $A_n \in U_n$. Then there is n^* such that, for all $n \geq n^*$, $x_n \in A_n$.

- If $A \in V[G]$ is a set of ordinals such that $\mathrm{otp}(A) = \nu$, where $\omega < \nu = \mathrm{cf}^V(\nu) < \kappa$, then there is an unbounded subset B of A such that $B \in V$.

4.2 Scales in the Gitik-Sharon model

In [15], Gitik and Sharon obtain their desired model by starting with a supercompact cardinal, κ, performing an Easton-support iteration to make $2^\kappa = \kappa^{+\omega+2}$ while preserving the supercompactness of κ, and, in the resulting model, forcing with $\mathbb{Q}$. They then show that, in the final model, there is a very good scale in $\prod_{n<\omega} \kappa_n^{+\omega+1}$. We show that, with a bit more care in preparing the ground model, we can arrange so that there are other scales with many very good points.

Let κ be supercompact, and suppose that GCH holds. Let $\mu = \kappa^{+\omega+1}$, let U be a supercompactness measure on $\mathcal{P}_\kappa(\mu)$, and let $j : V \to M \cong Ult(V, U)$. If λ is a regular cardinal, let $\mathbb{A}(\lambda)$ denote the full-support product of $\mathrm{Add}(\lambda^{+n}, \lambda^{+\omega+2})$ for $n < \omega$, where $\mathrm{Add}(\lambda^{+n}, \lambda^{+\omega+2})$ is the poset whose conditions are functions f such that $\mathrm{dom}(f) \subseteq \lambda^{+\omega+2}$, $|\mathrm{dom}(f)| < \lambda^{+n}$, and, for every $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a partial function from λ^{+n} to λ^{+n} of size less than λ^{+n}. If $p = \langle p_n \mid n < \omega \rangle$ is a condition in $\mathbb{A}(\lambda)$ and $\alpha < \lambda^{+\omega+2}$, denote by $p \restriction \alpha$ the condition $\langle p_n \restriction \alpha \mid n < \omega \rangle$, and let $\mathbb{A}(\lambda) \restriction \alpha = \{p \restriction \alpha \mid p \in \mathbb{A}(\lambda)\}$. Let $\mathbb{P}$ denote the iteration with backward Easton support of $\mathbb{A}(\lambda)$

for inaccessible $\lambda \leq \kappa$. For each λ, let $\mathbb{P}_{<\lambda}$ denote the iteration below λ and let $\mathbb{P}_\lambda$ be $\mathbb{P}_{<\lambda} * \mathbb{A}(\lambda)$. Let G be $\mathbb{P}$-generic over V.

Lemma 4.1. *In $V[G]$, we can extend j to j^*, a μ-supercompactness embedding with domain $V[G]$, such that for every $n < \omega$ and every $\beta < j(\kappa^{+n})$, there is $g^n_\beta : \kappa^{+n} \to \kappa^{+n}$ such that $j^*(g^n_\beta)(\sup(j``\kappa^{+n})) = \beta$.*

Proof. First note that, in V, for every $\alpha \leq \mu^+$, $|j(\alpha)| \leq |{}^{\mathcal{P}_\kappa(\mu)}\alpha| \leq \mu^+$, and, since ${}^\mu M \subseteq M$, $\operatorname{cf}(j(\lambda)) = \mu^+$ for every regular λ with $\kappa \leq \lambda \leq \mu^+$.. Thus, the number of antichains of $j(\mathbb{P}_{<\kappa})/G$ in $M[G]$ is μ^+ and, since ${}^\mu M[G] \subseteq M[G]$ and $j(\mathbb{P}_{<\kappa})/G$ is μ^+-closed, we can find $H \in V[G]$ that is $j(\mathbb{P}_{<\kappa})/G$-generic over $M[G]$. Let $G_\kappa = \langle f^n_\alpha \mid n < \omega,\ \alpha < \mu^+\rangle$ be the generic object for $\mathbb{A}(\kappa)$ added by G. For $n < \omega$, let $\langle \delta^n_\alpha \mid \alpha < \mu^+\rangle$ enumerate $j(\kappa^{+n})$.

As before, we can find, in $V[G]$, an I^* which is $j(\mathbb{A}(\kappa)) = \mathbb{A}(j(\kappa))^{M[G*H]}$-generic over $M[G*H]$. For $\alpha < j(\mu^+)$, let $I^* \restriction \alpha = \{p \restriction \alpha \mid p \in I^*\}$. Note that $I^* \restriction \alpha$ is $j(\mathbb{A}(\kappa)) \restriction \alpha$-generic over $M[G*H]$. For each $\alpha < j(\mu^+)$, let $I \restriction \alpha$ be formed by minimally adjusting $I^* \restriction \alpha$ so that, for every $p \in I$, $n < \omega$, and $\eta < \mu^+$, if $j(\eta) < \alpha$ and $j(\eta) \in \operatorname{dom}(p_n)$, then $p_n(j(\eta))$ is compatible with $j``f^n_\eta$ and $p_n(j(\eta))(\sup(j``\kappa^{+n})) = \delta^n_\eta$. Since $j``\mu^+$ is cofinal in $j(\mu^+)$, the number of changes to each condition is at most μ, so each adjusted p is itself in $M[G*H]$ and $I \restriction \alpha$ is $j(\mathbb{A}(\kappa)) \restriction \alpha$-generic over $M[G*H]$. Let

$$I = \bigcup_{\alpha < j(\mu^+)} I \restriction \alpha.$$

By chain condition, every maximal antichain of $j(\mathbb{A}(\kappa))$ is a subset of $j(\mathbb{A}(\kappa)) \restriction \alpha$ for some $\alpha < j(\mu^+)$, so I is $j(\mathbb{A}(\kappa))$-generic over $M[G*H]$. Now $j``G \subseteq G*H*I$, so we can lift $j$ to $j^*$ with domain $V[G]$ and $j(G) = G*H*I$. By construction, for every $n < \omega$ and $\alpha < \mu^+$, $j^*(f^n_\alpha)(\sup(j``\kappa^{+n})) = \delta^n_\alpha$, so, for $\beta < j(\kappa^{+n})$, letting $g^n_\beta = f^n_\alpha$, where $\delta^n_\alpha = \beta$, gives j^* the desired properties. □

Let U^* be the measure on $\mathcal{P}_\kappa(\mu)$ derived from j^*, and, for $n < \omega$, let U^*_n be the projection of U^* onto $\mathcal{P}_\kappa(\kappa^{+n})$ and $j^*_{U^*_n}$ be the embedding derived from U^*_n. Note that $U^*_n = \{X \subseteq \mathcal{P}_\kappa(\kappa^{+n}) \mid j``\kappa^{+n} \in j^*(X)\}$. Also note that, for all $n < \omega$, the functions $\langle g^n_\alpha \mid \alpha < j(\kappa^{+n})\rangle$ witness that $j^*_n(\kappa^{+n}) = j(\kappa^{+n})$. Let $\mathbb{Q}$ be the diagonal supercompact Prikry forcing defined using the U^*_n's. Let $H = \langle x_n \mid n < \omega\rangle$ be $\mathbb{Q}$-generic over $V[G]$, and let $\kappa_n = x_n \cap \kappa$.

Theorem 4.2. *In $V[G * H]$, there is a scale in*

$$\prod_{\substack{n<\omega \\ i\leq n}} \kappa_{n+1}^{+i}$$

of length μ^+ such that every $\alpha < \mu^+$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good.

Proof. For each $n < \omega$, fix an increasing, continuous sequence of ordinals $\langle \alpha_\zeta^n \mid \zeta < \mu^+ \rangle$ cofinal in $j(\kappa^{+n})$. For all $\zeta < \mu^+$, $n < \omega$, and $i \leq n$, let $f_\zeta(n,i) = g^i_{\alpha^i_\zeta}(\sup(x_n \cap \kappa^{+i}))$.

Claim 4.3. *Let $\zeta < \mu^+$. There is $n_\zeta < \omega$ such that for all $n \geq n_\zeta$ and all $i \leq n$, $f_\zeta(n,i) < \sup(x_{n+1} \cap \kappa^{+i})$.*

Proof. For $n < \omega$, let $A_{n+1} = \{x \in \mathcal{P}_\kappa(\kappa^{+n+1}) \mid$ for all $i \leq n$ and all $y \in \mathcal{P}_\kappa(\kappa^{+n})$ with $y \prec x$, $g^i_{\alpha^i_\zeta}(\sup(y \cap \kappa^{+i})) < \sup(x \cap \kappa^{+i})\}$. Now suppose that $\bar{y} \in \mathcal{P}_{j(\kappa)}(j(\kappa^{+n}))$ is such that $\bar{y} \prec j``\kappa^{+n+1}$. Then, since $j``\kappa^{+n+1} \cap j(\kappa) = \kappa$, $\mathrm{otp}(\bar{y}) < \kappa$ and, since $\bar{y} \subseteq j``\kappa^{+n+1}$, if $y \in \mathcal{P}_\kappa(\kappa^{+n})$ is the inverse image of $\bar{y}$ under $j$, then $j^*(y) = \bar{y}$, so, for all $i \leq n$, $j^*(g^i_{\alpha^i_\zeta})(\sup(\bar{y} \cap j(\kappa^{+i}))) = j^*(g^i_{\alpha^i_\zeta}(\sup(y \cap \kappa^{+i}))) < \sup(j``\kappa^{+i}) = \sup(j``\kappa^{+n+1} \cap j(\kappa^{+i}))$. Thus, $j``\kappa^{+n+1} \in j^*(A_{n+1})$, so $A_{n+1} \in U^*_{n+1}$. By genericity, there is $n_\zeta < \omega$ such that $x_{n+1} \in A_{n+1}$ for all $n \geq n_\zeta$. The claim follows. □

Thus, by adjusting each f_ζ on only finitely many coordinates, we may assume that, for all $\zeta < \mu^+$,

$$f_\zeta \in \prod_{\substack{n<\omega \\ i\leq n}} \sup(x_{n+1} \cap \kappa^{+i}).$$

Claim 4.4. *For all $\zeta < \zeta' < \mu^+$, $f_\zeta <^* f_{\zeta'}$.*

Proof. Fix $\zeta < \zeta' < \mu^+$. For all $n < \omega$ and $i \leq n$, we know that $\alpha^i_\zeta = j^*(g^i_{\alpha^i_\zeta})(\sup(j``\kappa^{+n} \cap j(\kappa^{+i}))) < j^*(g^i_{\alpha^i_{\zeta'}})(\sup(j``\kappa^{+n} \cap j(\kappa^{+i}))) = \alpha^i_{\zeta'}$. Thus, the set $B_n = \{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid$ for all $i \leq n$, $g^i_{\alpha^i_\zeta}(\sup(x \cap \kappa^{+i})) < g^i_{\alpha^i_{\zeta'}}(\sup(x \cap \kappa^{+i}))\}$ is in U^*_n. By genericity, $x_n \in B_n$ for large enough $n < \omega$, so, for large enough n, for all $i \leq n$, $f_\zeta(n,i) < f_{\zeta'}(n,i)$. □

Lemma 4.5. *In $V[G * H]$, let*

$$h \in \prod_{\substack{n<\omega \\ i\leq n}} \sup(x_{n+1} \cap \kappa^{+i}).$$

Then there is $\langle H_{n,i} \mid n < \omega,\ i \leq n \rangle \in V[G]$ such that $H_{n,i} : \mathcal{P}_\kappa(\kappa^{+n}) \to \kappa^{+i}$ and, for large enough n, for all $i \leq n$, $h(n,i) < H_{n,i}(x_n)$.

Proof. Let h be as in the statement of the lemma, and let $\dot{h}$ be a $\mathbb{Q}$-name for h. We may assume that, in fact,

$$h \in \prod_{\substack{n<\omega \\ i\leq n}} x_{n+1} \cap \kappa^{+i}$$

by considering instead h', where $h'(n,i) = \min(x_{n+1} \setminus h(n,i))$.

We show that, for every $q \in \mathbb{Q}$, there is $p \leq^* q$ forcing the desired conclusion. We assume for simplicity that q is the trivial condition and that

$$q \Vdash \text{“}\dot{h} \in \prod_{\substack{n<\omega \\ i\leq n}} \dot{x}_{n+1} \cap \kappa^{+i}\text{”}.$$

A tedious but straightforward adaptation of our proof gives the general case.

Work in $V[G]$. If s is a lower part of length $n+2$ with maximum element x^s_{n+1}, then, for every $i \leq n$, $s^\frown \mathbb{1} \Vdash$ “$\dot{h}(n,i) \in x^s_{n+1}$”. Since $|x^s_{n+1}| < \kappa$ and $(\mathbb{Q}, \leq^*)$ is κ-closed, repeated application of the Prikry property yields an upper part A_s and ordinals $\langle \alpha_{s,i} \mid i \leq n\rangle$ such that, for all $i \leq n$, $s^\frown A_s \Vdash$ “$\dot{h}(n,i) = \alpha_{s,i}$”. By taking a diagonal intersection, we obtain a condition $q' = \langle B_0, B_1, \ldots\rangle$ such that for every lower part s of length $n+2$ compatible with q' and every $i \leq n$, $s^\frown q' \Vdash$ “$\dot{h}(n,i) = \alpha_{s,i}$”.

Now suppose t is a lower part of length $n+1$ compatible with q', and let $i \leq n$. Consider the regressive function with domain B_{n+1} which takes x and returns $\alpha_{t^\frown\langle x\rangle,i}$. By Fodor's lemma, this function is constant on a measure-one set $B_{t,i}$. Let $B_t = \bigcap_{i\leq n} B_{t,i}$. By taking the diagonal intersections of the B_t's, we obtain a condition $p = \langle C_0, C_1, \ldots\rangle$ such that for every lower part t of length $n+1$ compatible with p and for every $i \leq n$, there is $\beta_{t,i}$ such that $t^\frown p \Vdash$ “$\dot{h}(n,i) = \beta_{t,i}$”.

Now, for $n < \omega$, $i \leq n$, and $x \in \mathcal{P}_\kappa(\kappa^{+n})$, let $H_{n,i}(x) = \sup(\{\beta_{t,i} + 1 \mid t$ is a lower part of length $n+1$ with top element $x\})$. Since there are fewer than κ-many such lower parts, it is clear that $H_{n,i} : \mathcal{P}_\kappa(\kappa^{+n}) \to \kappa^{+i}$ and, for all $n < \omega$ and $i \leq n$, $p \Vdash$ “$\dot{h}(n,i) < H_{n,i}(\dot{x}_n)$”. $\square$

Claim 4.6. *$\vec{f} = \langle f_\zeta \mid \zeta < \mu^+\rangle$ is cofinal in*

$$\prod_{\substack{n<\omega \\ i\leq n}} \sup(x_{n+1} \cap \kappa^{+i}).$$

Proof. Let

$$h \in \prod_{\substack{n<\omega \\ i\leq n}} \sup(x_{n+1} \cap \kappa^{+i}).$$

Find $\langle H_{n,i} \mid n < \omega,\ i \leq n\rangle \in V[G]$ as in the previous lemma. For each n and i, $[H_{n,i}]_{U_n} < j(\kappa^{+i})$. For $i < \omega$, let $\alpha_i = \sup(\{[H_{n,i}]_{U_n} + 1 \mid n \geq i\})$. For every $i < \omega$, $\alpha_i < j(\kappa^{+i})$. Find $\zeta < \mu^+$ such that, for all $i < \omega$, $\alpha_i < \alpha^i_\zeta$. For all $n < \omega$ and $i \leq n$, $[H_{n,i}]_{U_n} < \alpha^i_\zeta = j^*(g^i_{\alpha^i_\zeta})(\sup(j``\kappa^{+i}))$. Thus, the set of $x \in \mathcal{P}_\kappa(\kappa^{+n})$ such that $H_{n,i}(x) < g^i_{\alpha^i_\zeta}(\sup(x \cap \kappa^{+i}))$ is in U_n. By genericity, for large enough n and all $i \leq n$, $h(n,i) < H_{n,i}(x_n) < g^i_{\alpha^i_\zeta}(\sup(x_n \cap \kappa^{+i})) = f_\zeta(n,i)$, so $\vec{f}$ is in fact cofinal in the desired product. $\square$

Claim 4.7. *Suppose $\alpha < \mu^+$ and $\omega < \mathrm{cf}(\alpha) < \kappa$ (in $V[G * H]$). Then α is very good for $\vec{f}$.*

Proof. Since $\omega < \mathrm{cf}(\alpha) < \kappa$ in $V[G*H]$, the same is true in $V[G]$. Let $C \in V[G]$ be a club in α with $\mathrm{otp}(C) = \mathrm{cf}(\alpha)$. Let $n < \omega$, $i \leq n$, and $\zeta < \zeta'$ with $\zeta, \zeta' \in C$. Then it is easy to see that $A_{n,i,\zeta,\zeta'} := \{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid g^i_{\alpha^i_\zeta}(\sup(x \cap \kappa^{+i})) < g^i_{\alpha^i_{\zeta'}}(\sup(x \cap \kappa^{+i}))\}$ is in U^*_n. Since $|C| < \kappa$ and U^*_n is κ-complete, we get that

$$A_n = \bigcap_{\substack{i \leq n \\ \zeta < \zeta' \in C}} A_{n,i,\zeta,\zeta'} \in U^*_n.$$

Thus, for large enough n, $x_n \in A_n$, so C witnesses that α is very good. $\square$

We now have a scale with all of the desired properties, except it lives in the wrong product. Notice, though, that for every $n < \omega$ and $i \leq n$, we have arranged that $\mathrm{cf}(\sup(x_{n+1} \cap \kappa^{+i})) = \kappa^{+i}_{n+1}$. Thus, the following general lemma allows us to collapse $\vec{f}$ to a scale of the same length, with the same very good points, in

$$\prod_{\substack{n<\omega \\ i \leq n}} \kappa^{+i}_{n+1}.$$

Lemma 4.8. *Let $\langle \gamma_n \mid n < \omega\rangle$ be a sequence of ordinals of uncountable cofinality. For each $n < \omega$, let $\kappa_n = \mathrm{cf}(\gamma_n)$. Assume that $\langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence, and let $\kappa = \sup(\{\kappa_n \mid n < \omega\})$. Suppose μ is a regular cardinal, $\mu > \kappa$, and $\vec{f}$ is a scale of length μ in*

$$\prod_{n<\omega} \gamma_n$$

such that, on a club $C \subseteq \mu$, every $\alpha < \mu$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good. Then there is a scale $\vec{g}$ of length μ in

$$\prod_{n<\omega} \kappa_n$$

such that, on the club C, every $\alpha < \mu$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good.

Proof. For each $n < \omega$, let $\langle \xi^n_\eta \mid \eta < \kappa_n \rangle$ be increasing, continuous, and cofinal in γ_n. Define $\vec{g} = \langle g_\alpha \mid \alpha < \mu \rangle$ by $g_\alpha(n) = \min(\{\eta \mid f_\alpha(n) \leq \xi^n_\eta\})$. Since $\vec{f}$ is $<^*$-increasing, it is easily seen that $\vec{g}$ is $\leq^*$-increasing and that there is a club in μ on which $\vec{g}$ is $<^*$-increasing. By thinning out to this club, we may assume that $\vec{g}$ is $<^*$-increasing. Also, since $\vec{f}$ is cofinal in

$$\left(\prod_{n<\omega} \gamma_n, <^* \right),$$

it follows that $\vec{g}$ is cofinal in

$$\left(\prod_{n<\omega} \kappa_n, <^* \right).$$

Thus, it remains to check that $\vec{g}$ has many very good points.

Let $\alpha \in C$, and let $D \subseteq \alpha$ be a club in α witnessing that α is very good for $\vec{f}$. Thus, there is $n^* < \omega$ such that, for every $n* \leq n < \omega$ and every $\beta < \beta' \in D$, $f_\beta(n) < f_{\beta'}(n)$, which implies that $g_\beta(n) \leq g_{\beta'}(n)$.

For each $n \geq n^*$, we attempt to construct a club $D_n \subseteq D$ such that for all $\beta < \beta' \in D_n$ and all $m \geq n$, $g_\beta(m) < g_{\beta'}(m)$. If this construction succeeds for any value of n, then that n and D_n will witness that α is very good for $\vec{g}$. To carry out the construction, we will recursively construct D_n, which will be enumerated in increasing fashion as $\{\delta^n_i \mid i < \mathrm{cf}(\alpha)\}$. We let $\delta^n_0 = \min(D)$. If $i < \mathrm{cf}(\alpha)$ is a limit ordinal and the construction has succeeded so far, then $\delta^n_i = \sup(\{\delta^n_j \mid j < i\})$ is easily seen to satisfy our demands. Finally, given δ^n_i, try to find δ^n_{i+1} such that, for all $m \geq n$, $g_{\delta^n_i}(m) < g_{\delta^n_{i+1}}(m)$. If such a δ^n_{i+1} does not exist, then the construction halts, and we set $\delta^n = \delta^n_i$. Note that, since the construction failed at stage i, there is some $m_n \geq n$ such that, for all $\delta \in D \setminus \delta^n$, we have $g_\delta(m_n) = g_{\delta^n}(m_n)$.

Now suppose for sake of contradiction that this construction fails for all $n \geq n^*$. Let $\delta^* = \sup(\{\delta^n \mid n < \omega\})$. Since $\mathrm{cf}(\alpha) > \omega$, we have $\delta^* < \alpha$. Also, for all $\delta \in D \setminus \delta^*$ and all $n \geq n^*$, we have $g_\delta(m_n) = g_{\delta^*}(m_n)$, contradicting the fact that $\vec{g}$ is $<^*$-increasing.

$\square$

Theorem 4.9. *Suppose $\sigma \in {}^\omega\omega$ and, for all $n < \omega$, $\sigma(n) \geq n$. Then, in $V[G * H]$, there is a bad scale of length μ in*

$$\prod_{n<\omega} \kappa_n^{+\sigma(n)}.$$

Proof. Since, in $V[G]$, κ is supercompact, there is a scale $\vec{g} = \langle g_\alpha \mid \alpha < \mu \rangle$ in

$$\prod_{n<\omega} \kappa^{+\sigma(n)}$$

with stationarily many bad points of cofinality $< \kappa$ (see [7] for a proof). We have arranged with our preparatory forcing that, for every $n < \omega$, $j_n^*(\kappa) = j^*(\kappa)$. For each $n < \omega$ and each $\eta < \kappa^{+\sigma(n)}$, let $F_\eta^n : \mathcal{P}_\kappa(\kappa^{+n}) \to \kappa$ be such that $[F_\eta^n]_{U_n^*} = \eta$. We may assume that, for all $x \in \mathcal{P}_\kappa(\kappa^{+n}),\ F_\eta^n(x) < \kappa_x^{+\sigma(n)}$. We now define $\langle f_\alpha \mid \alpha < \mu \rangle$ in

$$\prod_{n<\omega} \kappa_n^{+\sigma(n)}$$

by letting $f_\alpha(n) = F_{g_\alpha(n)}^n(x_n)$.

Claim 4.10. *If $\alpha < \alpha' < \kappa^{+\omega+1}$, then $f_\alpha <^* f_{\alpha'}$.*

Proof. Since $\vec{g}$ is a scale, there is n^* such that, for all $n \geq n^*$, $g_\alpha(n) < g_{\alpha'}(n)$. Thus, for every $n \geq n^*$, $\{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid F_{g_\alpha(n)}^n(x) < F_{g_{\alpha'}(n)}^n(x)\} \in U_n^*$, so, by genericity, for large enough n, $f_\alpha(n) < f_{\alpha'}(n)$. $\square$

Lemma 4.11. *In $V[G * H]$, let*

$$h \in \prod_{n<\omega} \kappa_n^{+\sigma(n)}.$$

Then there is $\langle H_n \mid n < \omega \rangle \in V[G]$ such that $\mathrm{dom}(H_n) = \mathcal{P}_\kappa(\kappa^{+n})$, $H_n(x) < \kappa_x^{+\sigma(n)}$ for all $x \in \mathcal{P}_\kappa(\kappa^{+n})$, and, for large enough n, $h(n) < H_n(x_n)$.

Proof. Let h be as in the statement of the lemma, and let $\dot{h} \in V[G]$ be a $\mathbb{Q}$-name for h. Let $q \in \mathbb{Q}$. We show that there is $p \leq^* q$ forcing the desired conclusion. As in the proof of Lemma 4.5, we assume that q is the trivial condition and that

$$q \Vdash \text{“}\dot{h} \in \prod_{n<\omega} \dot{\kappa}_n^{+\sigma(n)}\text{”}.$$

Work in $V[G]$. If s is a lower part of length $n+1$ with maximum element x_n^s, then $s^\frown \mathbb{1} \Vdash$ “$\dot{h}(n) < \kappa_{x_n^s}^{+\sigma(n)} < \kappa$”. Thus, by the Prikry property and the κ-completeness of the measures, there is an upper part A_s and an ordinal

$\alpha_s < \kappa_{x_n^s}^{+\sigma(n)}$ such that $s^\frown A_s \Vdash$ "$\dot{h}(n) = \alpha_s$". By taking a diagonal intersection, we obtain a condition $q' = \langle B_0, B_1, \ldots\rangle$ such that, for every lower part s of length $n+1$ compatible with q', $s^\frown q' \Vdash$ "$\dot{h}(n) = \alpha_s$".

For $x \in \mathcal{P}_\kappa(\kappa^{+n})$, let $H_n(x) = \sup(\{\alpha_s \mid s$ is a lower part of length $n+1$ with top element $x\})$. Note that, if $m < n, y \in \mathcal{P}_\kappa(\kappa^{+m})$, and $y \prec x$, then $y \subseteq x \cap \kappa^{+m}$. Since $|x \cap \kappa^{+m}| = \kappa_x^{+m}$, there are fewer than $\kappa_x^{+n} \leq \kappa_x^{+\sigma(n)}$-many lower parts of length $n+1$ with top element x, so $H_n(x) < \kappa_x^{+\sigma(n)}$. Moreover, it is clear that, for every $n < \omega$, $q' \Vdash$ "$\dot{h}(n) < H_n(\dot{x}_n)$". $\square$

Claim 4.12. *$\vec{f} = \langle f_\alpha \mid \alpha < \mu\rangle$ is cofinal in*

$$\prod_{n<\omega} \kappa_n^{+\sigma(n)}$$

Proof. Let

$$h \in \prod_{n<\omega} \kappa_n^{+\sigma(n)}$$

and let $\langle H_n \mid n < \omega\rangle \in V[G]$ be as given by the previous lemma. For each $n < \omega$, $[H_n]_{U_n^*} < \kappa^{+\sigma(n)}$, so we can find $\alpha < \mu$ and $n^* < \omega$ such that for all $n \geq n^*$, $[H_n]_{U_n^*} < g_\alpha(n)$. Then, for all $n \geq n^*$, $\{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid H_n(x) < F^n_{g_\alpha(n)}(x)\} \in U_n^*$. Thus, by genericity, for large enough n, $h(n) < H_n(x_n) < f_\alpha(n)$. $\square$

Claim 4.13. *If $\alpha < \mu$ is good for $\vec{f}$, then it is good for $\vec{g}$ as well.*

Proof. Let α be good for $\vec{f}$. $\omega < \mathrm{cf}(\alpha) < \kappa$, and this is true in $V[G]$ as well. Since every unbounded subset of α in $V[G * H]$ contains an unbounded subset in $V[G]$, we can choose $A \in V[G]$ unbounded in α and $n^* < \omega$ witnessing that α is good for $\vec{f}$. Moreover, we may assume that $\mathrm{otp}(A) = \mathrm{cf}(A)$. Let $q = \langle x_0, x_1, \ldots, x_{n-1}, A_n, A_{n+1}, \ldots\rangle$ force that A and n^* witness the goodness of α. It must be the case that for every $m \geq n, n^*$, $\{x \in \mathcal{P}_\kappa(\kappa^{+m}) \mid \langle F^m_{g_\beta(m)}(x) \mid \beta \in A\rangle$ is strictly increasing$\} \in U_m^*$, since otherwise we could find $p \leq q$ forcing that $\langle f_\beta(m) \mid \beta \in A\rangle$ is not strictly increasing. Thus, for all $m \geq n, n^*$ and $\beta, \gamma \in A$ with $\beta < \gamma$, $g_\beta(m) = [F^m_{g_\beta(m)}]_{U_m^*} < [F^m_{g_\gamma(m)}]_{U_m^*} = g_\gamma(m)$. Thus, A witnesses that α is good for $\vec{g}$. $\square$

We know that, in $V[G]$, there is a stationary set of $\alpha < \mu$ with $\omega < \alpha < \kappa$ such that α is bad for $\vec{g}$. Since $\mathbb{Q}$ has the μ-c.c., this set remains stationary in $V[G * H]$. Thus, $\vec{f}$ is a bad scale in $V[G * H]$. $\square$

In [15], Gitik and Sharon show that, in $V[G * H]$, there is a very good scale in $\prod \kappa_n^{+\omega+1}$ of length μ and a scale in $\prod \kappa_n^{+\omega+2}$ of length μ^+ such that every $\alpha < \mu^+$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good. We now show that, above this, there is no essentially new behavior.

Theorem 4.14. *In $V[G]$, let $\sigma : \omega \to \kappa$ be such that, for all $n < \omega$, $\sigma(n) \geq \omega + 1$. Then, in $V[G * H]$, there is a scale of length μ^+ in $\prod \kappa_n^{+\sigma(n)+1}$ such that every $\alpha < \mu^+$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good.*

Proof. First note that, by genericity, for large enough n, $\kappa_n^{+\sigma(n)+1} < \kappa_{n+1}$. Thus, we may assume without loss of generality that this is true for all $n < \omega$. Work in $V[G]$. For $n < \omega$, let $\eta_n = (\kappa^{+\sigma(n)+1})^M$. $\eta_n < j(\kappa)$ and $|j(\kappa)| = \mu^+$, so, since M is closed under μ-sequences of ordinals, $|\eta_n| = \mathrm{cf}(\eta_n) = \mu^+$. Let $\langle \alpha^n_\zeta \mid \zeta < \mu^+ \rangle$ be increasing, continuous, and cofinal in η_n.

Recall, letting $n = 0$ in Lemma 4.1, that for all $\alpha < j(\kappa)$ (so certainly for all $\alpha < \eta_n$), there is $g_\alpha : \kappa \to \kappa$ such that $j^*(g_\alpha)(\kappa) = \alpha$. Now, moving to $V[G * H]$, define $\vec{f} = \langle f_\zeta \mid \zeta < \mu^+ \rangle$ by letting $f_\zeta(n) = g_{\alpha^n_\zeta}(\kappa_n)$. The proofs of the following claims are only minor modifications of the proofs of the analogous claims from Theorem 4.2 and are thus omitted.

Claim 4.15. *For all $\zeta < \mu^+$, for all large enough $n < \omega$, $f_\zeta(n) < \kappa_n^{+\sigma(n)+1}$.*

Claim 4.16. *In $V[G * H]$, let*

$$h \in \prod_{n<\omega} \kappa_n^{+\sigma(n)+1}.$$

Then there is $\langle H_n \mid n < \omega \rangle \in V[G]$ such that $\mathrm{dom}(H_n) = \mathcal{P}_\kappa(\kappa^{+n})$, $H_n(x) < \kappa_x^{+\sigma(n)}$ for all $x \in \mathcal{P}_\kappa(\kappa^{+n})$, and, for large enough n, $h(n) < H_n(x_n)$.

Claim 4.17. *$\vec{f}$ is a scale in $\prod \kappa_n^{+\sigma(n)+1}$.*

Claim 4.18. *If $\alpha < \mu^+$ and $\omega < \mathrm{cf}(\alpha) < \kappa$, then α is very good for $\vec{f}$.*

□

We now take a step back momentarily to survey the landscape. Things become a bit clearer if, in $V[G * H]$, we force with $\mathrm{Coll}(\mu, \mu^+)$, producing a generic object I. Since this forcing is so highly closed, all relevant scales in $V[G*H]$ remain scales in $V[G*H*I]$, and the goodness or badness of points of uncountable cofinality is preserved. The only thing that is changed is that, in $V[G * H * I]$, all relevant scales have length $\mu = \kappa^+$. Moreover, we have a very detailed picture of which scales are good and which are bad. Let $\sigma \in V[G]$ with $\sigma : \omega \to \kappa$ and, for all $n < \omega$, either $\sigma(n) = 0$ or $\sigma(n)$ is a successor ordinal, and consider a scale $\vec{f}$ of length μ in $\prod \kappa_n^{+\sigma(n)}$. First consider the case $\sigma : \omega \to \omega$. If, for large enough n, $\sigma(n) < n$, then $\vec{f}$ is a good scale, and in fact there is a very good scale in the same product. On the other hand, if $\sigma(n) \geq n$ for infinitely many n, then $\vec{f}$ is bad. Thus, the diagonal sequence $\langle \kappa_n^{+n} \mid n < \omega \rangle$ is a dividing line between goodness and badness in the finite successors of the κ_n's. If, alternatively, $\sigma(n) > \omega$ for all sufficiently large n, then $\vec{f}$ is once again a good scale, and there is a very good scale in the same product.

4.3 Very weak square in the Gitik-Sharon model

We take a brief moment to note that, though AP_κ necessarily fails in the forcing extension by $\mathbb{Q}$, the weaker Very Weak Square principle may hold. We first recall the following definition from [13].

Definition. *Let λ be a singular cardinal. A Very Weak Square sequence at λ is a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that, for a club of $\alpha < \lambda^+$,*

- *C_α is an unbounded subset of α.*
- *For all bounded $x \in [C_\alpha]^{<\omega_1}$, there is $\beta < \alpha$ such that $x = C_\beta$.*

Note that we may assume in the above definition that, for the relevant club of $\alpha < \lambda^+$, $\mathrm{otp}(C_\alpha) = \mathrm{cf}(\alpha)$.

The existence of a Very Weak Square sequence at λ follows from AP_λ, but the converse is not true. In fact, in [13], Foreman and Magidor prove that the existence of a Very Weak Square sequence at every singular cardinal is consistent with the existence of a supercompact cardinal. Also, note that a Very Weak Square sequence at λ is preserved by any countably-closed forcing which also preserves λ and λ^+. In particular, our preparation forcing $\mathbb{P}$ preserves Very Weak Square sequences. Thus, we may assume that, prior to forcing with $\mathbb{Q}$, there is a Very Weak Square sequence at $\kappa^{+\omega}$.

Let V denote the model over which we will force with $\mathbb{Q}$, and suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \mu \rangle$ is a Very Weak Square sequence in V, where $\mu = \kappa^{+\omega+1}$. Assume additionally that there is a club $E \subseteq \mu$ such that, for all $\alpha \in E$, $\mathrm{otp}(C_\alpha) = \mathrm{cf}(\alpha)$ and, for all bounded $x \in [C_\alpha]^{<\omega_1}$, there is $\beta < \alpha$ such that $x = C_\beta$. Let G be $\mathbb{Q}$-generic over V. In $V[G]$, form $\vec{D} = \langle D_\alpha \mid \alpha < \mu \rangle$ as follows.

- If $\alpha \notin E$ or $\alpha \in E$ and $\mathrm{cf}^V(\alpha) < \kappa$, let $D_\alpha = C_\alpha$.

- If $\alpha \in E$ and $\mathrm{cf}^V(\alpha) \geq \kappa$, let $D_\alpha \subseteq C_\alpha$ be an ω-sequence cofinal in α.

Now, using the fact that forcing with $\mathbb{Q}$ does not add any bounded subsets of κ, it is easy to verify that $\vec{D}$ is a Very Weak Square sequence at κ.

4.4 Classifying bad points

We now turn our attention to Cummings and Foreman's second question from [5]. We first recall some relevant definitions and the Trichotomy Theorem, due to Shelah [28].

Definition. *Let X be a set, let I be an ideal on X, and let $f, g : X \to ON$. Then $f <_I g$ if $\{x \in X \mid g(x) \leq f(x)\} \in I$. $\leq_I$, $=_I$, $>_I$, and $\geq_I$ are defined analogously. If D is the dual filter to I, then $<_D$ is the same as $<_I$.*

Thus, if X is a set of ordinals and I is the ideal of bounded subsets of X, then $<_I$ is the same as $<^*$.

Definition. *Let I be an ideal on X, β an ordinal, and $\vec{f} = \langle f_\alpha \mid \alpha < \beta\rangle$ a $<_I$-increasing sequence of functions in ${}^X ON$. $g \in {}^X ON$ is an* exact upper bound *(or* eub*) for $\vec{f}$ if the following hold:*

1. *For all $\alpha < \beta$, $f_\alpha <_I g$.*
2. *For all $h \in {}^X ON$ such that $h <_I g$, there is $\alpha < \beta$ such that $h <_I f_\alpha$.*

We note that, easily, if $\vec{f}$ is a $<_I$-increasing sequence of functions and g and h are both eubs for $\vec{f}$, then $g =_I h$. The following is a standard alternate characterization of good points in scales.

Proposition 4.19. *Let κ be singular, let $A \subseteq \kappa$ be a cofinal set of regular cardinals of order type $\mathrm{cf}(\kappa)$, and let $\vec{f} = \langle f_\alpha \mid \alpha < \mu\rangle$ be a scale in $\prod A$. Let $\beta < \mu$ be such that $\mathrm{cf}(\kappa) < \mathrm{cf}(\beta) < \kappa$. Then the following are equivalent:*

1. *β is good for $\vec{f}$.*
2. *$\langle f_\alpha \mid \alpha < \beta\rangle$ has an eub, g, such that, for all $i \in A$, $\mathrm{cf}(g(i)) = \mathrm{cf}(\beta)$.*
3. *There is a $<$-increasing sequence of functions $\langle h_\xi \mid \xi < \mathrm{cf}(\beta)\rangle$ that is cofinally interleaved with $\vec{f} \restriction \beta$, i.e. for every $\alpha < \beta$ there is $\xi < \mathrm{cf}(\beta)$ such that $f_\alpha <^* h_\xi$ and, for every $\xi < \mathrm{cf}(\beta)$, there is $\alpha < \beta$ such that $h_\xi <^* f_\alpha$. In this case, the function $i \mapsto \sup(\{h_\xi(i) \mid \xi < \mathrm{cf}(\beta)\})$ is an eub for $\vec{f} \restriction \beta$.*

Theorem 4.20. *(Trichotomy) Suppose I is an ideal on X, $|X|^+ < \lambda = \mathrm{cf}(\lambda)$, and $\langle f_\alpha \mid \alpha < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^X ON$. Then one of the following holds:*

1. *(Good) $\langle f_\alpha \mid \alpha < \lambda \rangle$ has an eub, g, such that, for all $x \in X$, $\mathrm{cf}(g(x)) > |X|$.*
2. *(Bad) There is an ultrafilter U on X extending the dual filter to I and a sequence $\langle S_x \mid x \in X \rangle$ such that $|S_x| \leq |X|$ for all $x \in X$ and, for all $\alpha < \lambda$, there are $h \in \prod_{x \in X} S_x$ and $\beta < \lambda$ such that $f_\alpha <_U h <_U f_\beta$.*
3. *(Ugly) There is a function $h \in {}^X ON$ such that the sequence of sets $\langle \{x \mid f_\alpha(x) < h(x)\} \mid \alpha < \lambda \rangle$ does not stabilize modulo I.*

We note that the above terminology is slightly misleading. For β to be a good point in a scale, for example, requires more than $\langle f_\alpha \mid \alpha < \beta \rangle$ falling into the Good case of the Trichotomy Theorem. It also requires that the eub have uniform cofinality equal to $\mathrm{cf}(\beta)$. Also, β being a bad point in a scale does not imply that $\langle f_\alpha \mid \alpha < \beta \rangle$ falls into the Bad case of the Trichotomy Theorem.

We now answer Cummings and Foreman's question asking into which case of the Trichotomy Theorem the bad points in the Gitik-Sharon model fall. We also answer the analogous question for some other models in which bad scales exist, showing that, in the standard models in which bad scales exist at relatively small cardinals, there is considerable diversity of behavior at the bad points. We first recall the following fact from [7].

Fact 4.21. *Suppose κ is a supercompact cardinal, $\sigma \in {}^\omega \omega$ is a function such that, for all $n < \omega$, $\sigma(n) \geq n$, and $\langle f_\alpha \mid \alpha < \kappa^{+\omega+1} \rangle$ is a scale in $\prod_{n<\omega} \kappa^{+\sigma(n)}$. Then there is an inaccessible cardinal $\delta < \kappa$ such that, for stationarily many $\beta \in \kappa^{+\omega+1} \cap \mathrm{cof}(\delta^{+\omega+1})$, $\langle f_\alpha \mid \alpha < \beta \rangle$ has an eub, g, such that, for all $n < \omega$, $\mathrm{cf}(g(n)) = \delta^{+\sigma(n)}$.*

The content of the next theorem is that these eubs of non-uniform cofinality get transferred down to the bad scales defined in extensions by diagonal supercompact Prikry forcing. Note that the proof of the existence of bad scales in Theorem 4.9 did not rely on our preparatory forcing $\mathbb{P}$, so we dispense with it here.

Theorem 4.22. *Let κ be supercompact, let $\mu = \kappa^{+\omega+1}$, and let $\sigma \in {}^\omega \omega$ be such that, for all $n < \omega$, $\sigma(n) \geq n$. Let $\mathbb{Q}$ be diagonal supercompact Prikry* *forcing at κ defined from $\langle U_n \mid n < \omega \rangle$, where U_n is a measure on $\mathcal{P}_\kappa(\kappa^{+n})$. In $V^{\mathbb{Q}}$, let $\vec{f}$ be the bad scale in $\prod_{n<\omega} \kappa_n^{+\sigma(n)}$ defined as in Theorem 4.9. Then there is an inaccessible cardinal $\delta < \kappa$ such that, for stationarily many $\beta \in \mu \cap \mathrm{cof}(\delta^{+\omega+1})$, $\langle f_\alpha \mid \alpha < \beta \rangle$ has an eub, g, such that, for all $n < \omega$, $\mathrm{cf}(g(n)) = \delta^{+\sigma(n)}$.*

Proof. Let G be $\mathbb{Q}$-generic over V, and let $\langle x_n \mid x < \omega \rangle$ be the associated generic sequence. $\vec{f} = \langle f_\alpha \mid \alpha < \mu \rangle$ is the bad scale in $\prod_{n<\omega} \kappa_n^{+\sigma(n)}$ defined as in the proof of Theorem 4.9, i.e. $f_\alpha(n) = F^n_{g_\alpha(n)}(x_n)$, where $\vec{g} = \langle g_\alpha \mid \alpha < \mu \rangle \in V$ is a scale in $\prod_{n<\omega} \kappa^{+\sigma(n)}$ and, for each $n < \omega$ and $\eta < \kappa^{+\sigma(n)}$, $F^n_\eta : \mathcal{P}_\kappa(\kappa^{+n}) \to \kappa$ is such that $[F^n_\eta]_{U_n} = \eta$.

By Fact 4.21, there is an inaccessible $\delta < \kappa$ and a stationary $S \subseteq \mu \cap \mathrm{cof}(\delta^{+\omega+1})$ such that, for all $\beta \in S$, $\langle g_\alpha \mid \alpha < \beta \rangle$ has an eub, g, such that, for all $n < \omega$, $\mathrm{cf}(g(n)) = \delta^{+\sigma(n)}$. Without loss of generality, $\vec{g}$ is a continuous scale and this eub is in fact g_β.

Since $\mathbb{Q}$ has the μ-c.c. and preserves the inaccessibility of δ and the regularity of $\delta^{+\omega+1}$, S remains a stationary subset of $\mu \cap \mathrm{cof}(\delta^{+\omega+1})$ in $V[G]$. Thus, we will be done if we show that, for all $\beta \in S$, $\langle f_\alpha \mid \alpha < \beta \rangle$ has an eub g such that, for all $n < \omega$, $\mathrm{cf}(g(n)) = \delta^{+\sigma(n)}$. In fact, we claim that this eub is, up to finite adjustments, f_β.

Claim 4.23. *Let $\beta \in S$. Then, for sufficiently large $n < \omega$, $\mathrm{cf}(f_\beta(n)) = \delta^{+\sigma(n)}$*

Proof. For all n, $\mathrm{cf}(g_\beta(n)) = \delta^{+\sigma(n)}$. Thus, since $\delta^{+\sigma(n)} < \kappa$ and $[F^n_{g(\beta)}]_{U_n} = g(\beta)$, we know that, for all $n < \omega$ and almost all $x \in X_n$, $\mathrm{cf}(F^n_{g_\beta(n)}(x)) = \delta^{+\sigma(n)}$. Therefore, by genericity of $\langle x_n \mid n < \omega \rangle$, $\mathrm{cf}(f_\beta(n)) = \mathrm{cf}(F^n_{g_\beta(n)}(x_n)) = \delta^{+\sigma(n)}$ for all sufficiently large $n < \omega$. $\square$

Claim 4.24. *Let $\beta \in S$. Then f_β is an eub for $\langle f_\alpha \mid \alpha < \beta \rangle$.*

Proof. In $V[G]$, fix $h \in \prod_{n<\omega} \kappa_n^{+\sigma(n)}$ such that $h < f_\beta$. We want to find $\alpha < \beta$ such that $h <^* f_\alpha$. We first define some auxiliary functions.

In V, for each $n < \omega$, let $\langle \eta^n_\xi \mid \xi < \delta^{+\sigma(n)} \rangle$ be an increasing sequence of ordinals cofinal in $g_\beta(n)$. In $V[G]$, for $\tau \in \prod_{n<\omega} \delta^{+\sigma(n)}$, define f_τ by letting, for all $n < \omega$, $f_\tau(n) = F^n_{\eta^n_{\tau(n)}}(x_n)$.

Since, for all $n < \omega$, $\langle \eta^n_\xi \mid \xi < \delta^{+\sigma(n)} \rangle$ is increasing and cofinal in $g_\beta(n)$ and since $\delta^{+\sigma(n)} < \kappa$, we have, by the κ-completeness of the measures, that, for all $n < \omega$, the set of $x \in X_n$ such that $\langle F^n_{\eta^n_\xi}(x) \mid \xi < \delta^{+\sigma(n)} \rangle$ is increasing and cofinal in $F^n_{g_\beta(n)}(x)$ is in U_n. Thus, for sufficiently large $n < \omega$, $\langle F^n_{\eta^n_\xi}(x_n) \mid \xi < \delta^{+\sigma(n)} \rangle$ is increasing and cofinal in $F^n_{g_\beta(n)}(x_n) = f_\beta(n)$.

Thus, we can find $\tau \in \prod_{n<\omega} \delta^{+\sigma(n)}$ such that, for large enough $n < \omega$, $h(n) < F^n_{\eta^n_{\tau(n)}}(x_n)$, i.e. $h <^* f_\tau$.

Since $\mathbb{Q}$ does not add any bounded subsets of κ, we actually have $\tau \in V$. For all $n < \omega$, $\eta^n_{\tau(n)} < g_\beta(n)$, so, since g_β is an eub for $\langle g_\alpha \mid \alpha < \beta \rangle$, there is $\alpha < \beta$ such that, for large enough $n < \omega$, $\eta^n_{\tau(n)} < g_\alpha(n)$. Thus, we know by genericity that, again for sufficiently large $n < \omega$, $F^n_{\eta^n_{\tau(n)}}(x_n) < F^n_{g_\alpha(n)}(x_n)$, i.e. $f_\tau <^* f_\alpha$. So $h <^* f_\tau <^* f_\alpha$, and we have have shown that f_β is in fact an eub for $\langle f_\alpha \mid \alpha < \beta \rangle$ and hence proven the theorem. □

Thus, the bad scales in the Gitik-Sharon model have stationarily many points which lie in the Good case of the Trichotomy Theorem but are nonetheless bad, since the eubs at these points have non-uniform cofinality.

We now turn our attention to a bad scale isolated by Cummings, Foreman, and Magidor in [7]. Let κ be a supercompact cardinal and let $\vec{f} = \langle f_\alpha \mid \alpha < \kappa^{+\omega+1} \rangle$ be a continuous scale in $\prod_{n<\omega} \kappa^{+n}$. Let δ be as given in Fact 4.21 and let $G_0 \times G_1$ be $\mathrm{Coll}(\omega, \delta^{+\omega}) \times \mathrm{Coll}(\delta^{+\omega+2}, <\kappa)$-generic over V. Cummings, Foreman, and Magidor show that, in $V[G_0 \times G_1]$, $\vec{f}$ remains a scale, now living in $\prod_{n<\omega} \aleph_{n+3}$, and that the stationary set S of bad points in V of cofinality $\delta^{+\omega+1}$ is a stationary set of bad points in $V[G_0 \times G_1]$ of cofinality ω_1. The Trichotomy Theorem does not apply to points of cofinality ω_1 in increasing sequences of countable reduced products (see [22] for a counterexample), but we show that the points in S nonetheless fall into the Ugly case of the Trichotomy Theorem.

Theorem 4.25. *In $V[G_0 \times G_1]$, if $\beta \in S$, then there is $h < f_\beta$ such that the sequence of sets $\langle \{n \mid f_\alpha(n) < h(n)\} \mid \alpha < \beta \rangle$ does not stabilize modulo bounded sets.*

Proof. Let $\beta \in S$. We must produce an h such that, for every $\alpha < \beta$, there is $\alpha' < \beta$ such that $\{n \mid f_\alpha(n) < h(n) < f_{\alpha'}(n)\}$ is infinite. We will actually show that such an h exists in $V[G_0]$. First, for concreteness, we remark that we are thinking of conditions in $\mathrm{Coll}(\omega, \delta^{+\omega})$ as finite partial functions from ω into $\delta^{+\omega}$. Let g be the generic surjection from ω onto $\delta^{+\omega}$ added by G_0. In V, we know that, for every $n < \omega$, $\mathrm{cf}(f_\beta(n)) = \delta^{+n}$. For each $n < \omega$, let $\langle \eta^n_\xi \mid \xi < \delta^{+n} \rangle$ be increasing and cofinal in $f_\beta(n)$. In $V[G_0]$, define h as follows: if n is such that $g(n) < \delta^{+n}$, then let $h(n) = \eta^n_{g(n)}$. If $g(n) > \delta^{+n}$, let $h(n) = 0$.

Clearly, $h < f_\beta$. Let $\alpha < \beta$. In V, there is $\tau \in \prod \delta^{+n}$ such that, for sufficiently large n, $f_\alpha(n) < \eta^n_{\tau(n)}$. Also, since f_β is an eub, there is $\alpha' < \beta$

such that, again, for sufficiently large n, $\eta^n_{\tau(n)} < f_{\alpha'}(n)$. An easy density argument shows that, for infinitely many $n < \omega$, $g(n) = \tau(n)$ and thus $h(n) = \eta^n_{\tau(n)}$. Therefore, for infinitely many $n < \omega$, $f_\alpha(n) < h(n) < f_{\alpha'}(n)$. $\square$

We end this section by briefly remarking on two other models in which bad scales exist. A result of Magidor [8] shows that, if Martin's Maximum holds, then any scale of length $\aleph_{\omega+1}$ in $\prod_A \aleph_n$, where $A \subseteq \omega$, is bad. Foreman and Magidor, in [13], show that the same conclusion follows from the Chang's Conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. The proofs of these results immediately yield that, in both cases, such a scale has stationarily many points of cofinality ω_1 that fall into the Bad case of the Trichotomy Theorem.

4.5 Down to $\aleph_{\omega^2}$

In [15], Gitik and Sharon show how to modify their diagonal supercompact Prikry forcing to arrange that κ, which is supercompact in V, becomes $\aleph_{\omega^2}$ in the forcing extension, SCH and approachability both fail at $\aleph_{\omega^2}$, and there is $A \subseteq \aleph_{\omega^2}$ that carries a very good scale. We start this section by reviewing their construction, being slightly more careful with our preparation of the ground model so that our results from Section 4.2 carry down.

In V, let $j : V \to M$ be the elementary embedding derived from U, a supercompactness measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$. Let $\mathbb{P}$ be the backward Easton-support iteration from Section 4.2, and let $j^* : V[G] \to M[G * H * I]$ be as in Lemma 4.1. Let U^* be the measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$ derived from j^* and, for $n < \omega$, let U^*_n be the projection of U^* onto $\mathcal{P}_\kappa(\kappa^{+n})$. Let $i^*_n : V[G] \to N_n$ be the elementary embedding derived from the ultrapower of $V[G]$ by U^*_n and let $k_n : N_n \to M[G{*}H{*}I]$ be the factor map. The functions $\{g^n_\beta \mid \beta < j(\kappa^{+n})\}$ witness that $i^*_n(\kappa^{+n}) = j(\kappa^{+n})$, so $\mathrm{crit}(k_n) > j(\kappa^{+n})$.

As before, we can find, in $V[G]$, an H^* that is $\mathrm{Coll}(\kappa^{+\omega+2}, < j(\kappa))^{M[G*H*I]}$-generic over $M[G{*}H{*}I]$. For all $n < \omega$, since $\mathrm{crit}(k_n) > j(\kappa)$ and $\mathrm{Coll}(\kappa^{+\omega+2}, < i^*_n(\kappa))^{N_n}$ has the $i^*_n(\kappa)$-c.c., the filter generated by $k_n^{-1}[H^*]$, which we will call H_n, is $\mathrm{Coll}(\kappa^{+\omega+2}, < i^*_n(\kappa))^{N_n}$-generic over N_n.

In $V[G]$, let $\delta < \kappa$ be an inaccessible cardinal such that, for every $\sigma \in {}^\omega\omega$ such that $\sigma(n) \geq n$ for all $n < \omega$, there is a scale $\vec{f}$ in $\prod_{n<\omega} \kappa^{+\sigma(n)}$ of length $\kappa^{+\omega+1}$ such that there are stationarily many $\beta < \kappa^{+\omega+1}$ of cofinality $\delta^{+\omega+1}$ such that $\vec{f} \restriction \beta$ has an eub, g, such that, for all $n < \omega$, $\mathrm{cf}(g(n)) = \delta^{+\sigma(n)}$. Such a δ exists by the proof of Fact 4.21, which can be found in [7].

We now define a version of the diagonal supercompact Prikry forcing with interleaved collapses. Conditions in this poset, which we again call $\mathbb{Q}$, are of the form

$$q = \langle c^q, x_0^q, f_0^q, x_1^q, f_1^q, \ldots, x_{n-1}^q, f_{n-1}^q, A_n^q, F_n^q, A_{n+1}^q, F_{n+1}^q, \ldots \rangle$$

such that the following conditions hold:

1. $\langle x_0^q, x_1^q, \ldots, x_{n-1}^q, A_n^q, A_{n+1}^q \ldots \rangle$ is a condition in the diagonal supercompact Prikry forcing defined using the U_n^*'s.
2. If $n = 0$, $c^q \in \mathrm{Coll}(\omega, < \delta) \times \mathrm{Coll}(\delta^{+\omega+2}, < \kappa)$.
3. If $n \geq 1$, $c^q \in \mathrm{Coll}(\omega, < \delta) \times \mathrm{Coll}(\delta^{+\omega+2}, < \kappa_{x_0^q})$.
4. For all $i < n - 1$, $f_i^q \in \mathrm{Coll}(\kappa_{x_i^q}^{+\omega+2}, < \kappa_{x_{i+1}^q})$.
5. $f_{n-1}^q \in \mathrm{Coll}(\kappa_{x_{n-1}^q}^{+\omega+2}, < \kappa)$.
6. For all $\ell \geq n$, F_ℓ^q is a function with domain A_ℓ^q such that $F_\ell^q(x) \in \mathrm{Coll}(\kappa_x^{+\omega+2}, < \kappa)$ for all $x \in A_\ell^q$ and $i_\ell^*(F_\ell^q)(i_\ell``\kappa^{+\ell}) \in H_\ell$.

As before, n is the length of q, denoted $\mathrm{lh}(q)$. If $p, q \in \mathbb{Q}$, where $\mathrm{lh}(p) = n$ and $\mathrm{lh}(q) = m$, then $p \leq q$ if and only if:

1. $n \geq m$ and $\langle x_0^p, \ldots, x_{n-1}^p, A_n^p, \ldots \rangle \leq \langle x_0^q, \ldots, x_{m-1}^q, A_m^q, \ldots \rangle$ in the standard supercompact diagonal Prikry poset.
2. $c^p \leq c^q$.
3. For all $i \leq m - 1$, $f_i^p \leq f_i^q$.
4. For all i such that $m \leq i \leq n - 1$, $f_i^p \leq F_i^q(x_i^p)$.
5. For all $i \geq n$ and all $x \in A_i^p$, $F_i^p(x) \leq F_i^q(x)$.

If J is $\mathbb{Q}$-generic over $V[G]$ and $\langle x_n \mid n < \omega \rangle$ is the associated Prikry sequence, then, for each $n < \omega$, letting $\kappa_n = \kappa_{x_n}$, we have $\kappa_n = (\aleph_{\omega\cdot(n+1)+3})^{V[G*J]}$, $\kappa = (\aleph_{\omega^2})^{V[G*H]}$, and $(\kappa^{+\omega+1})^{V[G]} = (\aleph_{\omega^2+1})^{V[G*J]}$.

The results about scales transfer down in a straightforward manner. Namely, in $V[G * J]$, we have the following situation:

- $\prod_{n<\omega} \aleph_{\omega\cdot n+1}$ carries a very good scale of length $\aleph_{\omega^2+1}$.
- $\prod_{n<\omega} \aleph_{\omega\cdot n+2}$ carries a scale of length $\aleph_{\omega^2+2}$ such that, for every $\beta < \aleph_{\omega^2+2}$ such that $\omega_1 \leq \mathrm{cf}(\beta) < \aleph_{\omega^2}$, β is very good for the scale.

- If $\sigma \in {}^\omega\omega$ is such that, for every $n < \omega$, $\sigma(n) < n$, then $\prod_{n<\omega} \aleph_{\omega\cdot(n+1)+\sigma(n)+3}$ carries a scale of length $\aleph_{\omega^2+2}$ such that, as in the previous item, all relevant points are very good.

- If $\sigma \in {}^\omega\omega$ is such that, for all $n < \omega$, $\sigma(n) \geq n$, then $\prod_{n<\omega} \aleph_{\omega\cdot(n+1)+\sigma(n)+3}$ carries a bad scale of length $\aleph_{\omega^2+1}$.

Also, just as in Section 5, the bad scales in $\prod_{n<\omega} \aleph_{\omega\cdot(n+1)+\sigma(n)+2}$ in $V[G * J]$ have stationarily many points of cofinality $\aleph_{\omega+1}$ where there are eubs g such that, for all $n < \omega$, $\mathrm{cf}(g(n)) = \aleph_{\sigma(n)+1}$. Finally, as before, if we force over $V[G * J]$ with $\mathrm{Coll}(\aleph_{\omega^2+1}, \aleph_{\omega^2+2})$, then all scales in $V[G * J]$ remain scales (now all of length $\aleph_{\omega^2+1}$) in the further extension. The bad scales remain bad, the very good scales remain good, and all scales which were of length $\aleph_{\omega^2+2}$ in $V[G * J]$ are also very good in the further extension.

4.6 The good ideal

We now turn to the third question raised by Cummings and Foreman in [5]. We first give some background.

Let κ be a singular cardinal, and let A be a progressive set of regular cardinals cofinal in κ. Let I be the collection of $B \subseteq A$ such that either B is bounded or $\prod B$ carries a scale of length κ^+. I is easily seen to be an ideal, and one of the seminal results of PCF Theory is the fact [28] that I is singly generated, i.e. there is $B^* \subseteq A$ such that, for all $B \in I$, $B \subseteq^* B^*$, where $\subseteq^*$ denotes inclusion modulo bounded sets. Thus, if κ is a singular cardinal that is not a cardinal fixed point, then there is a largest subset of the regular cardinals below κ, modulo bounded sets, which carries a scale of length κ^+. Such a set is called the first PCF generator.

Cummings and Foreman asked whether, when the first PCF generator exists, there is also a maximal set, again modulo bounded sets, which carries a good scale. This is the case in the models obtained above by diagonal supercompact Prikry forcing, which were also the first known models in which this set is nontrivial and different from the first PCF generator itself, i.e. in which certain sets carry good scales and others carry bad scales. For example, in our final model in Section 6, after forcing with $\mathrm{Coll}(\aleph_{\omega^2+1}, \aleph_{\omega^2+2})$, the largest subset of regular cardinals below $\aleph_{\omega^2}$ that carries a good scale is, modulo bounded subsets, $\{\aleph_{\omega\cdot n+m} \mid n \leq \omega, m < n+2\}$. Does such a set always exist? In this section, we will only be concerned with singular cardinals of countable cofinality and with scales in

$\prod A$, where $\mathrm{otp}(A) = \omega$. With this in mind, we make the following definition.

Definition. *Suppose κ is a singular cardinal of countable cofinality. Then $I_{gd}[\kappa]$ is the collection of $A \subseteq \kappa$ such that A is a set of regular cardinals and either A is finite or $\mathrm{otp}(A) = \omega$ and $\prod A$ carries a good scale of length κ^+.*

It is easily seen that $I_{gd}[\kappa]$ is an ideal. A question related to that of Cummings and Foreman is whether or not $I_{gd}[\kappa]$ is a P-ideal, i.e. whether or not, given $\langle A_n \mid n < \omega \rangle$ such that, for all $n < \omega$, $A_n \in I_{gd}[\kappa]$, there exists $A \in I_{gd}[\kappa]$ such that, for all $n < \omega$, $A_n \subseteq^* A$. We do not answer this question, but we provide some partial results. In the first part of this section, we analyze individual good and bad points in scales, proving that there are consistently local obstacles to proving that $I_{gd}[\kappa]$ is a P-ideal (though we do not know whether this pathological behavior can consistently occur simultaneously at enough points to provide an actual counterexample to $I_{gd}[\kappa]$ being a P-ideal). In the second part of the section, we show that, after forcing with a finite-support iteration of Hechler forcing of length ω_1, $I_{gd}[\kappa]$ is necessarily a P-ideal.

Let $\langle A_n \mid n < \omega \rangle$ be a sequence of elements of $I_{gd}[\kappa]$ and suppose, without loss of generality, that each A_n is infinite. Then, by results of Shelah, there is $A \subseteq \kappa$ of order type ω such that $\prod A$ carries a scale of length κ^+ and, for all $n < \omega$, $A_n \subseteq^* A$. Again without loss of generality, by adjusting the A_n's if necessary, we may assume that $A = \bigcup A_n$ and, for all $n < \omega$, $A_n \subseteq A_{n+1}$.

Let $\vec{f} = \langle f_\alpha \mid \alpha < \kappa^+ \rangle$ be a scale in $\prod A$. For $B \subseteq A$, let $\vec{f}^B$ denote $\langle f_\alpha \restriction B \mid \alpha < \kappa^+ \rangle$. If B is infinite, then $\vec{f}^B$ is a scale in $\prod B$. Also, since, for each $n < \omega$, $\prod A_n$ carries a good scale, we know that $\vec{f}^{A_n}$ is itself a good scale. Thus, for each $n < \omega$, there is a club $C_n \subseteq \kappa^+$ such that for every $\beta \in C_n$ of uncountable cofinality, β is good for $\vec{f}^{A_n}$. Intersecting the clubs, there is a club $C \subseteq \kappa^+$ such that, for every $n < \omega$ and every $\beta \in C$ of uncountable cofinality, β is good for $\vec{f}^{A_n}$.

We want to know if there is $B \subseteq A$ such that $\prod B$ carries a good scale and, for all $n < \omega$, $A_n \subseteq^* B$. We first take a local view and focus on individual good points. Suppose that $\beta \in C$ has uncountable cofinality. For $n < \omega$, let $g_n \in \prod A_n$ be an eub for $\vec{f}^{A_n} \restriction \beta$ such that, for all $i \in A_n$, $\mathrm{cf}(g_n(i)) = \mathrm{cf}(\beta)$. By uniqueness of eubs, if $m < n$, then $g_n \restriction A_m =^* g_m$. Define $g \in {}^A\mathrm{On}$ by letting $g \restriction A_0 = g_0$ and, for all $n < \omega$, $g \restriction (A_{n+1} \setminus A_n) = g_{n+1} \restriction (A_{n+1} \setminus A_n)$. Then we have that $g \restriction A_n =^* g_n$ for all $n < \omega$.

By uniqueness of eubs, if $B^* \subseteq A$, $A_n \subseteq^* B^*$ for all $n < \omega$, and g^* is an eub for $\vec{f}^{B^*} \restriction \beta$, then, for all $n < \omega$, $g^* \restriction (A_n \cap B^*) =^* g_n \restriction (A_n \cap B^*)$. Thus, if there is such a B^* and g^*, then there is a $B \subseteq A$ such that $A_n \subseteq^* B$ for all $n < \omega$ and $g \restriction B$ is an eub for $\vec{f}^B \restriction \beta$. We now turn to the question of when such a B exists.

We note that a subset of A that almost contains all of the A_n's can be specified by an element of ${}^\omega\omega$. Namely, if each A_n is enumerated in increasing order as $\langle i^n_k \mid k < \omega \rangle$ and $\sigma \in {}^\omega\omega$, let

$$B_\sigma = \bigcup_{n<\omega} A_n \setminus i^n_{\sigma(n)}.$$

Each B_σ is a subset of A that almost contains all of the A_n's, and each subset of A that almost contains all of the the A_n's contains a set of the form B_σ.

With this in mind, it is not surprising that cardinal characteristics of the continuum have an impact on the situation. Recall that $\mathfrak{b}$, or the *bounding number*, is the size of the smallest family of functions that is unbounded in $({}^\omega\omega, <^*)$. $\mathfrak{d}$, or the *dominating number*, is the size of the smallest family of functions that is cofinal in $({}^\omega\omega, <^*)$.

For simplicity, when considering questions about the local behavior of scales, we will without loss of generality think of the A_n's as being subsets of ω and consider increasing sequences of functions in ${}^A\mathrm{On}$.

Lemma 4.26. *Let λ be a regular, uncountable cardinal. Let $\langle A_n \mid n < \omega \rangle$ be such that each A_n is an infinite subset of ω and, for all $m < n$, $A_m \subseteq A_n$. Let $A = \bigcup_{n<\omega} A_n$. Let $\vec{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ be a $<^*$-increasing sequence of functions in ${}^A\mathrm{On}$ such that, for every $n < \omega$, $\vec{f}^{A_n} = \langle f_\alpha \restriction A_n \mid \alpha < \lambda \rangle$ has an eub of uniform cofinality λ. If either $\lambda < \mathfrak{b}$ or $\lambda > \mathfrak{d}$, then there is $B \subseteq A$ such that, for all n, $A_n \subseteq^* B$ and $\vec{f}^B$ has an eub of uniform cofinality λ.*

Proof. For $n < \omega$, let g_n be an eub of uniform cofinality λ for $\vec{f}^{A_n}$. Define $g \in {}^A\mathrm{On}$ as before so that, for all $i \in A$, $\mathrm{cf}(g(i)) = \lambda$ and, for all $n < \omega$, $g \restriction A_n =^* g_n$.

For each $i \in A$, let $\langle \gamma^i_\xi \mid \xi < \lambda \rangle$ be increasing and cofinal in $g(i)$ and, for $\xi < \lambda$, define $h_\xi \in {}^A\mathrm{On}$ by $h_\xi(i) = \gamma^i_\xi$.

We first consider the case $\lambda < \mathfrak{b}$. For each $\alpha < \lambda$ and each $n < \omega$, there is $\xi < \lambda$ such that $f_\alpha \restriction A_n <^* h_\xi \restriction A_n$. Thus, there is $\xi_\alpha < \lambda$ such that, for all $n < \omega$, $f_\alpha \restriction A_n <^* h_{\xi_\alpha} \restriction A_n$. Define $\sigma_\alpha \in {}^\omega\omega$ by letting $\sigma_\alpha(n)$ be the least i such that $f_\alpha \restriction (A_n \setminus i) < h_{\xi_\alpha} \restriction (A_n \setminus i)$.

Similarly, for each $\xi < \lambda$ and each $n < \omega$, there is $\alpha < \lambda$ such that $h_\xi \restriction A_n <^* f_\alpha \restriction A_n$, so there is $\alpha_\xi < \lambda$ such that, for all $n < \omega$, $h_\xi \restriction A_n <^* f_{\alpha_\xi} \restriction A_n$. Define $\tau_\xi \in {}^\omega\omega$ by letting $\tau_\xi(n)$ be the least i such that $h_\xi \restriction (A_n \setminus i) < f_{\alpha_\xi} \restriction (A_n \setminus i)$.

Since $\lambda < \mathfrak{b}$, we can find $\sigma \in {}^\omega\omega$ such that for all $\alpha, \xi < \lambda$, we have $\sigma_\alpha, \tau_\xi <^* \sigma$. Let $B = B_\sigma$. We claim that this B is as desired. To see this, let $\alpha < \lambda$. $f_\alpha \restriction B_{\sigma_\alpha} < h_{\xi_\alpha} \restriction B_{\sigma_\alpha}$. But, since $\sigma_\alpha <^* \sigma$, we know that $B \subseteq^* B_{\sigma_\alpha}$, so $f_\alpha \restriction B <^* h_{\xi_\alpha} \restriction B$. By the same argument, for each $\xi < \lambda$, $h_\xi \restriction B <^* f_{\alpha_\xi} \restriction B$. Thus, $\langle h_\xi \restriction B \mid \xi < \lambda \rangle$ is cofinally interleaved with $\vec{f}^B$, so, by Proposition 4.19, $g \restriction B$ is an eub for $\vec{f}^B$.

Now suppose that $\lambda > \mathfrak{d}$. Let $\mathcal{F}$ be a family of functions cofinal in ${}^\omega\omega$ such that $|\mathcal{F}| = \mathfrak{d}$. As above, for each $\alpha < \lambda$, we can find $\xi_\alpha < \lambda$ and $\sigma_\alpha \in {}^\omega\omega$ such that $f_\alpha \restriction B_{\sigma_\alpha} <^* h_{\xi_\alpha} \restriction B_{\sigma_\alpha}$, but we now also require that $\sigma_\alpha \in \mathcal{F}$. This is possible because $\mathcal{F}$ is cofinal in ${}^\omega\omega$. Since $\lambda > \mathfrak{d}$, there is $\sigma^* \in \mathcal{F}$ such that $\sigma_\alpha = \sigma^*$ for cofinally many $\alpha < \lambda$. But, since $\vec{f}$ is $<^*$-increasing, it is actually the case that, for every $\alpha < \lambda$, there is $\xi^*_\alpha < \lambda$ such that $f_\alpha \restriction B_{\sigma^*} <^* h_{\xi^*_\alpha} \restriction B_{\sigma^*}$.

Similarly, we can find $\tau^* \in \mathcal{F}$ such that, for every $\xi < \lambda$, there is $\alpha^*_\xi < \lambda$ such that $h_\xi \restriction B_{\tau^*} <^* f_{\alpha^*_\xi} \restriction B_{\tau^*}$. Let $\sigma > \sigma^*, \tau^*$, and let $B = B_\sigma$. It is easily seen as before that $\langle h_\xi \restriction B \mid \xi < \lambda \rangle$ is cofinally interleaved with $\vec{f}^B$, so $g \restriction B$ is an eub for $\vec{f}^B$. $\square$

Now suppose that $\mathfrak{b} \leq \lambda \leq \mathfrak{d}$. Let $\langle A_n \mid n < \omega \rangle$ be a $\subseteq$-increasing sequence of infinite subsets of ω, let $A = \bigcup A_n$, and let $\vec{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ be a $<^*$-increasing sequence of functions in ${}^A\mathrm{On}$ such that, for each $n < \omega$, $\vec{f}^{A_n}$ has an eub, g_n, of uniform cofinality λ. Let $g \in {}^A\mathrm{On}$ be of uniform cofinality λ such that, for all $n < \omega$, $g \restriction A_n =^* g_n$. We would like to understand how g can fail to be an eub.

For each $i \in A$, let $\langle \gamma^i_\xi \mid \xi < \lambda \rangle$ be increasing and cofinal in $g(i)$ and, for $\xi < \lambda$, let $h_\xi \in {}^A\mathrm{On}$ be defined by $h_\xi(i) = \gamma^i_\xi$. For $\sigma \in {}^\omega\omega$, the statement that $g \restriction B_\sigma$ is an eub for $\vec{f}^{B_\sigma}$ is equivalent to the statement that $\langle h_\xi \restriction B_\sigma \mid \xi < \lambda \rangle$ is cofinally interleaved in $\vec{f}^{B_\sigma}$. There are two ways in which this could fail to happen:

1. There is $\alpha < \lambda$ such that, for every $\xi < \lambda$, $f_\alpha \restriction B_\sigma \not<^* h_\xi \restriction B_\sigma$.
2. There is $\xi < \lambda$ such that, for every $\alpha < \lambda$, $h_\xi \restriction B_\sigma \not<^* f_\alpha \restriction B_\sigma$.

Thus, if $g \restriction B_\sigma$ fails to be an eub for $\vec{f}^{B_\sigma}$ for all $\sigma \in {}^\omega\omega$, one of the above two situations must occur for a $<^*$-cofinal set of ${}^\omega\omega$, so in fact one of them must occur for all $\sigma \in {}^\omega\omega$.

We first concentrate on case 1.

Lemma 4.27. *Let $\sigma \in {}^\omega\omega$. Suppose $\alpha < \lambda$ is such that, for every $\xi < \lambda$, $f_\alpha \restriction B_\sigma \not<^* h_\xi \restriction B_\sigma$. Then $f_\alpha \restriction B_\sigma \not<^* g \restriction B_\sigma$.*

Proof. Suppose for sake of contradiction that $f_\alpha \restriction B_\sigma <^* g \restriction B_\sigma$. Let $k < \omega$ be such that $f_\alpha(i) < g(i)$ for all $i \in B_\sigma \setminus k$. For $i \in B_\sigma \setminus k$, let $\xi_i < \lambda$ be such that $f_\alpha(i) < h_{\xi_i}(i)$. Let $\xi = \sup_{i \in B_\sigma \setminus k} \xi_i$. Then $f_\alpha(i) < h_\xi(i)$ for all $i \in B_\sigma \setminus k$, so $f_\alpha \restriction B_\sigma <^* h_\xi \restriction B_\sigma$. □

Thus, we are in case 1 if and only if $g \restriction B_\sigma$ is not in fact an upper bound for $\vec{f}^{B_\sigma}$. We now show that, if $\lambda = \mathfrak{b} = \omega_1$, this can hold simultaneously for all $\sigma \in {}^\omega\omega$. Without loss of generality, in what follows we assume that $A = \omega$.

Lemma 4.28. *Suppose $\mathfrak{b} = \omega_1$. Let $\langle A_n \mid n < \omega \rangle$ be such that, for each n, $A_n \subseteq A_{n+1}$, $A_{n+1} \setminus A_n$ is infinite, and $\bigcup_{n<\omega} A_n = \omega$. There is a sequence of functions $\vec{f} = \langle f_\alpha \mid \alpha < \omega_1 \rangle$, $<^*$-increasing in ${}^\omega\mathrm{On}$, such that, for every $n < \omega$, $\vec{f}^{A_n}$ has an eub, g_n, of uniform cofinality ω_1 but, letting g be such that $g \restriction A_n =^* g_n$ for every $n < \omega$, for every $\sigma \in {}^\omega\omega$, $g \restriction B_\sigma$ is not an upper bound for $\vec{f}^{B_\sigma}$.*

Proof. Fix a $<^*$-increasing, unbounded sequence $\langle \sigma_\alpha \mid \alpha < \omega_1 \rangle$ in ${}^\omega\omega$. We first construct a useful sequence of subsets of ω.

Claim 4.29. *There is a sequence $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ such that, for every $\alpha < \beta < \omega_1$,*

1. *$X_\alpha \subseteq \omega$ and $X_\alpha \subseteq^* X_\beta$.*
2. *For all $n < \omega$, $X_\alpha \cap A_n$ is finite.*
3. *For all $n < \omega$ $X_{\alpha+1} \cap (A_{n+1} \setminus (A_n \cup \sigma_\alpha(n+1)))$ is nonempty.*

Proof. We construct $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ by recursion on α. Let $X_0 = \emptyset$. Given X_α, let $X_{\alpha+1} = X_\alpha \cup \{\min(A_{n+1} \setminus (A_n \cup \sigma_\alpha(n+1))) \mid n < \omega\}$. Requirement 1 is clearly satisfied. For each $n < \omega$, $|X_{\alpha+1} \cap A_n| \leq |X_\alpha \cap A_n| + n$, so requirement 2 is satisfied as well. Finally, since, for all n, $A_{n+1} \setminus A_n$ is infinite, $X_{\alpha+1}$ contains an element of $A_{n+1} \setminus (A_n \cup \sigma_\alpha(n+1))$, so requirement 3 is satisfied.

If $\beta < \omega_1$ is a limit ordinal, then we just need X_β to satisfy requirements 1 and 2. Fix a bijection $\tau_\beta : \omega \to \beta$, and let

$$X_\beta = \bigcup_{n<\omega} (X_{\tau_\beta(n)} \setminus \bigcup_{k<n} A_k).$$

For $\alpha < \beta$, if $\alpha = \tau_\beta(n)$, then, since $X_\alpha \cap A_k$ is finite for all k, $X_\alpha \subseteq^* X_\alpha \setminus (\bigcup_{k<n} A_k) \subseteq X_\beta$, so $X_\alpha \subseteq^* X_\beta$. Also, for all $n < \omega$,

$$X_\beta \cap A_n \subseteq (\bigcup_{k\leq n} X_{\tau_\beta(k)}) \cap A_n,$$

which is finite. Thus, X_β satisfies 1 and 2 as desired, completing the construction. $\square$

We now use the sequence $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ to construct the desired sequence of functions $\vec{f}$. For $\alpha < \omega_1$ and $k < \omega$, let

$$f_\alpha(k) = \begin{cases} \alpha & \text{if } k \notin X_\alpha \\ \omega_1 + \alpha & \text{if } k \in X_\alpha \end{cases}$$

Claim 4.30. *$\vec{f}$ is $<^*$-increasing.*

Proof. Let $\alpha < \beta < \omega_1$. Then $f_\beta(k) \leq f_\alpha(k)$ if and only if $k \in X_\alpha \setminus X_\beta$. But $X_\alpha \subseteq^* X_\beta$, so this only happens for finitely many values of k. Thus, $f_\alpha <^* f_\beta$. $\square$

For an ordinal γ, let c_γ denote the constant function with domain ω taking value γ everywhere.

Claim 4.31. *For every $n < \omega$, $c_{\omega_1} \restriction A_n$ is an eub for $\vec{f}^{A_n}$.*

Proof. For every $\alpha < \omega_1$, $X_\alpha \cap A_n$ is finite, so $f_\alpha \restriction A_n =^* c_\alpha \restriction A_n$. Since $c_{\omega_1} \restriction A_n$ is an eub for $\langle c_\alpha \restriction A_n \mid \alpha < \omega_1 \rangle$, it is also an eub for $\vec{f}^{A_n}$. $\square$

Claim 4.32. *For every $\sigma \in {}^\omega\omega$, $c_{\omega_1} \restriction B_\sigma$ is not an upper bound for $\vec{f}^{B_\sigma}$.*

Proof. Fix $\sigma \in {}^\omega\omega$, and find $\alpha < \omega_1$ such that $\sigma_\alpha \not<^* \sigma$. Then $Y = \{n \mid \sigma(n+1) \leq \sigma_\alpha(n+1)\}$ is infinite and, for every $n \in Y$, $X_{\alpha+1}$ contains an element of $A_{n+1} \setminus (A_n \cup \sigma_\alpha(n+1))$. Thus, $X_{\alpha+1} \cap B_\sigma$ is infinite. Since, for all $k \in X_{\alpha+1}$, $f_{\alpha+1}(k) > \omega_1$, we have $f_{\alpha+1} \restriction B_\sigma \not<^* c_{\omega_1} \restriction B_\sigma$, so $c_{\omega_1} \restriction B_\sigma$ is not an upper bound for $\vec{f}^{B_\sigma}$. $\square$

Thus, for all $\sigma \in {}^\omega\omega$, $\vec{f}^{B_\sigma}$ has no eub. $\square$

We now turn our attention to case 2 and show that it too is consistently possible.

Lemma 4.33. *Suppose $\mathfrak{d} = \omega_1$. Let $\langle A_n \mid n < \omega \rangle$ be such that, for each n, $A_n \subseteq A_{n+1}$, $A_{n+1} \setminus A_n$ is infinite, and $\bigcup_{n<\omega} A_n = \omega$. There is a sequence of functions $\vec{f} = \langle f_\alpha \mid \alpha < \omega_1 \rangle$, $<^*$-increasing in ${}^\omega\mathrm{On}$ such that, for every $n < \omega$, $\vec{f}^{A_n}$ has an eub but, defining g and $\langle h_\xi \mid \xi < \omega_1 \rangle$ as above, for every $\sigma \in {}^\omega\omega$, there is $\xi_\sigma < \omega_1$ such that, for every $\alpha < \omega_1$, $h_{\xi_\sigma} \restriction B_\sigma \not<^* f_\alpha \restriction B_\sigma$.*

Proof. Fix a $<^*$-increasing sequence $\langle \sigma_\alpha \mid \alpha < \omega_1 \rangle$ cofinal in ${}^\omega\omega$. For $\alpha < \omega_1$, let B_α denote B_{σ_α}.

Claim 4.34. *There are subsets of ω, $\langle X^\beta_\alpha \mid \alpha < \beta < \omega_1 \rangle$, such that, letting $X^\beta = \bigcup_{\alpha<\beta} X^\beta_\alpha$,*

1. *For all $\beta < \omega_1$ and all $n < \omega$, $X^\beta \cap A_n$ is finite.*
2. *For all $\beta < \omega_1$ and all $\alpha < \alpha' < \beta$, $X^\beta_\alpha \subseteq X^\beta_{\alpha'}$.*
3. *For all $\beta < \omega_1$ and all $\alpha < \beta$, $X^\beta_\alpha \cap B_\alpha$ is infinite.*
4. *For all $\beta < \beta' < \omega_1$, $\bigcup_{\alpha<\beta} (X^\beta_\alpha \setminus X^{\beta'}_\alpha)$ is finite.*

Proof. We construct $\langle X^\beta_\alpha \mid \alpha < \beta < \omega_1 \rangle$ by recursion on β. First, let $X^1_0 = \{\min(A_{n+1} \setminus (A_n \cup \sigma_0(n+1))) \mid n < \omega\}$.

Next, suppose $\beta' < \omega_1$ and we have constructed $\langle X^\beta_\alpha \mid \alpha < \beta \leq \beta' \rangle$ satisfying requirements 1-4 above. For $\alpha < \beta'$, let $X^{\beta'+1}_\alpha = X^{\beta'}_\alpha$, and let $X^{\beta'+1}_{\beta'} = X^{\beta'} \cup \{\min(A_{n+1} \setminus (A_n \cup \sigma_{\beta'}(n+1))) \mid n < \omega\}$. It is immediate by the inductive hypothesis that this still satisfies the requirements.

Finally, suppose $\beta' < \omega_1$ is a limit ordinal and that we have constructed $\langle X^\beta_\alpha \mid \alpha < \beta < \beta' \rangle$. Fix a bijection $\tau : \omega \to \beta'$. For $\alpha < \beta'$, denote $\tau^{-1}(\alpha)$ as i_α. Note that, for a fixed α, $\langle X^\beta_\alpha \mid \alpha < \beta < \beta' \rangle$ is $\subseteq^*$-decreasing, and each X^β_α has an infinite intersection with B_α, so any finite subsequence has an infinite intersection with B_α. For all $\alpha < \beta'$, we will first define $Y_\alpha = \{y^\alpha_m \mid m < \omega\}$ by recursion on m. We will also define an increasing sequence of natural numbers, $\langle n^\alpha_m \mid m < \omega \rangle$.

Fix $\alpha < \beta'$. Let n^α_0 be the least n such that

$$\bigcap_{\substack{j \leq i_\alpha \\ \tau(j) > \alpha}} X^{\tau(j)}_\alpha \cap (A_n \setminus \sigma_\alpha(n))$$

is nonempty, and let y_0^α be an element of this intersection. Given y_m^α and n_m^α, let n_{m+1}^α be the least $n > n_m^\alpha$ such that

$$\bigcap_{\substack{j \le i_\alpha + m \\ \tau(j) > \alpha}} X_\alpha^{\tau(j)} \cap (A_n \setminus (\sigma_\alpha(n) \cup (y_m^\alpha + 1)))$$

is nonempty, and let y_{m+1}^α be an element of this intersection. Note that we can always find such an n_{m+1}^α, since

$$\bigcap_{\substack{j \le i_\alpha + m \\ \tau(j) > \alpha}} X_\alpha^{\tau(j)} \cap B_\alpha$$

is infinite.

If $\alpha < \beta < \beta'$ and $i_\beta \le i_\alpha + m$, $y_k^\alpha \in X_\alpha^\beta$ for all $k > m$, so $Y_\alpha \subseteq^* X_\alpha^\beta$ and, if $i_\beta < i_\alpha$, we actually have $Y_\alpha \subseteq X_\alpha^\beta$. From this, it follows that, for all $n < \omega$, $Y_\alpha \cap A_n$ is finite. Also, by construction, $Y_\alpha \cap B_\alpha$ is infinite. Now let

$$Y_\alpha^* = Y_\alpha \setminus \bigcup_{n < i_\alpha} A_n.$$

Note that Y_α^* has all of the properties of Y_α mentioned above. Finally, let

$$X_\alpha^{\beta'} = \bigcup_{\gamma \le \alpha} Y_\gamma^*.$$

We claim that this construction has succeeded. To see that requirement 1 is satisfied, note that for all $n < \omega$, $\{\alpha < \beta' \mid Y_\alpha^* \cap A_n \neq \emptyset\} \subseteq \tau``(n+1)$, which is finite, and, for each $\alpha < \beta'$, $Y_\alpha^* \cap A_n$ is finite, so $X^{\beta'} \cap A_n$ is the finite union of finite sets and hence finite.

Requirements 2 and 3 are obviously satisfied by the construction. To see 4, fix $\beta < \beta'$. If $\alpha < \beta$ is such that $i_\alpha > i_\beta$, then $Y_\alpha^* \subseteq X_\alpha^\beta$, so

$$\bigcup_{\alpha < \beta} (X_\alpha^\beta \setminus X_\alpha^{\beta'}) \subseteq \bigcup_{\substack{\alpha < \beta \\ \imath_\alpha < i_\beta}} (Y_\alpha^* \setminus X_\alpha^\beta),$$

which is finite. □

We are now ready to define $\vec{f} = \langle f_\beta \mid \beta < \omega_1 \rangle$. First, if $\beta < \omega_1$ and $k \in X^\beta$, let α_k^β be the least $\alpha < \beta$ such that $k \in X_\alpha^\beta$. For $\beta < \omega_1$ and $k < \omega$, let

$$f_\beta(k) = \begin{cases} \omega_1 \cdot \beta & \text{if } k \notin X^\beta \\ \omega_1 \cdot \alpha_k^\beta + \beta & \text{if } k \in X^\beta \end{cases}$$

Claim 4.35. *$\vec{f}$ is $<^*$-increasing.*

Proof. Let $\beta < \beta' < \omega_1$. If $k < \omega$, the only way we can have $f_{\beta'}(k) \leq f_\beta(k)$ is if there is $\alpha < \beta$ such that $k \in X^\beta_\alpha \setminus X^{\beta'}_\alpha$. By construction, there are only finitely many such values of k. $\square$

Claim 4.36. *For every $n < \omega$, $c_{\omega_1^2} \restriction A_n$ is an eub for $\vec{f}^{A_n}$.*

Proof. For each $\beta < \omega_1$ and $n < \omega$, $X^\beta \cap A_n$ is finite, so $f_\beta \restriction A_n =^* c_{\omega_1 \cdot \beta} \restriction A_n$. Since $c_{\omega_1^2} \restriction A_n$ is an eub for $\langle c_{\omega_1 \cdot \beta} \restriction A_n \mid \beta < \omega_1 \rangle$, it is also an eub for $\vec{f}^{A_n}$. $\square$

Claim 4.37. *For all $\alpha, \beta < \omega_1$, $c_{\omega_1 \cdot (\alpha+1)} \restriction B_\alpha \not<^* f_\beta \restriction B_\alpha$.*

Proof. If $\beta \leq \alpha$, then $f_\beta(k) < \omega_1 \cdot (\alpha + 1)$ for all $k < \omega_1$, so $f_\beta < c_{\omega_1 \cdot (\alpha+1)}$. If $\beta > \alpha$, then $X^\beta_\alpha \cap B_{\sigma_\alpha}$ is infinite and, for all $k \in X^\beta_\alpha$, $f_\beta(k) < \omega_1 \cdot (\alpha + 1)$. Thus, $c_{\omega_1 \cdot (\alpha+1)} \restriction B_{\sigma_\alpha} \not<^* f_\beta \restriction B_\alpha$. $\square$

By the above claims, we can let $g = c_{\omega_1^2}$ and, for $\xi < \omega_1$, $h_\xi = c_{\omega_1 \cdot \xi}$. Let $\sigma \in {}^\omega\omega$. Since $\langle \sigma_\alpha \mid \alpha < \omega_1 \rangle$ is dominating, we can find $\alpha < \omega_1$ such that $\sigma <^* \sigma_\alpha$. Then $B_\sigma \subseteq^* B_\alpha$, so, by the last claim, for all $\beta < \omega_1$, $h_{\alpha+1} \restriction B_\sigma \not<^* f_\beta \restriction B_\sigma$. $\square$

Thus, at individual points of scales we can have a situation in which β is good for $\vec{f}^{A_n}$ for every $n < \omega$ but fails to be good for $\vec{f}^{B_\sigma}$ for every $\sigma \in {}^\omega\omega$. It remains open whether this can happen simultaneously at stationarily many points, thus providing a counterexample to $I_{gd}[\kappa]$ being a P-ideal. In fact, only something slightly weaker needs to happen to provide a counterexample, namely, for every $\sigma \in {}^\omega\omega$, there are stationarily many $\beta \in \kappa^+ \cap \mathrm{cof}(\geq \omega_1)$ such that β is good for every $\vec{f}^{A_n}$ but fails to be good for $\vec{f}^{B_\sigma}$.

We now show that, starting in any ground model, a very mild forcing, namely a finite support iteration of Hechler forcing, forces $I_{gd}[\kappa]$ to be a P-ideal for every singular κ of countable cofinality. This further suggests that the question under consideration perhaps has more to do with the structure of ${}^\omega\omega$ than with PCF-theoretic behavior at higher cardinals.

We first recall the Hechler forcing notion. Conditions in the forcing poset are of the form $p = (s^p, f^p)$, where $s^p \in {}^{n^p}\omega$ for some $n^p < \omega$ and $f^p \in {}^\omega\omega$. $q \leq p$ if and only if

1. $n^q \geq n^p$

2. $s^q \restriction n^p = s^p$

3. For every $k \in [n^p, n^q)$, $s^q(k) > f^p(k)$.

4. $f^q > f^p$.

Hechler forcing adds a dominating real, i.e. a $\sigma^* \in {}^\omega\omega$ such that, for every $\sigma \in ({}^\omega\omega)^V$, $\sigma <^* \sigma^*$.

We will need the following fact from [1].

Fact 4.38. *Let X be a set, let I be an ideal on X, and let $\vec{f} = \langle f_\alpha \mid \alpha < \zeta \rangle$ be a $<_I$-increasing sequence of functions in ${}^X\mathrm{On}$. Suppose that λ is such that $|X| < \lambda = \mathrm{cf}(\lambda) < \mathrm{cf}(\zeta)$. Then the following are equivalent.*

1. *There are stationarily many $\beta \in \zeta \cap \mathrm{cof}(\lambda)$ such that $\vec{f} \restriction \beta$ has an eub of uniform cofinality λ.*

2. *$\vec{f}$ has an eub, g, such that, for all $x \in X$, $\mathrm{cf}(g(x)) \geq \lambda$.*

Lemma 4.39. *Let κ be a singular cardinal of countable cofinality. For each $n < \omega$, let A_n be an increasing ω-sequence of regular cardinals, cofinal in κ, such that $\prod A_n$ carries a good scale. Let $\mathbb{P}$ be Hechler forcing. Then, in $V^{\mathbb{P}}$, there is an ω-sequence $B \subset \kappa$ such that, for all $n < \omega$, $A_n \subseteq^* B$ and $\prod B$ carries a good scale.*

Proof. In V, let B_0 be an ω-sequence, cofinal in κ, such that $A_n \subseteq^* B_0$ for all $n < \omega$ and $\prod B_0$ carries a scale. Without loss of generality, we may assume that $B_0 = \bigcup_{n<\omega} A_n$ and that all elements of B_0 are uncountable. Also, for each $n < \omega$, enumerate A_n in increasing order by $\langle i^n_k \mid k < \omega \rangle$. Let $\vec{f} = \langle f_\alpha \mid \alpha < \kappa^+ \rangle$ be a scale in $\prod B_0$. Then, for each $n < \omega$, $\vec{f}^{A_n}$ is a scale in $\prod A_n$ and, since each $\prod A_n$ carries a good scale, each $\vec{f}^{A_n}$ is good. Let C_n be club in κ^+ such that, if $\beta \in C_n \cap \mathrm{cof}(\geq \omega_1)$, then β is good for $\vec{f}^{A_n}$, and let $C = \bigcap_{n<\omega} C_n$.

Let $\beta \in C \cap \mathrm{cof}(\geq \omega_1)$ and, for each $n < \omega$, let $g^\beta_n \in \prod A_n$ be an eub for $\vec{f}^{A_n} \restriction \beta$ such that $\mathrm{cf}(g^\beta_n(i)) = \mathrm{cf}(\beta)$ for all $i \in A_n$. Let $g_\beta \in \prod B_0$ be such that, for all $i \in B_0$, $\mathrm{cf}(g_\beta(i)) = \mathrm{cf}(\beta)$ and, for all $n < \omega$, $g_\beta \restriction A_n =^* g^\beta_n$.

For each $i \in B_0$, let $\langle \beta^i_\xi \mid \xi < \mathrm{cf}(\beta) \rangle$ be increasing and cofinal in $g_\beta(i)$. For $\xi < \mathrm{cf}(\beta)$, let $h^\beta_\xi \in \prod B_0$ be defined by $h^\beta_\xi(i) = \beta^i_\xi$. For $\xi < \xi' < \mathrm{cf}(\beta)$, we have $h^\beta_\xi < h^\beta_{\xi'} < g_\beta$.

For $\xi < \mathrm{cf}(\beta)$ and $n < \omega$, let $\alpha^n_\xi < \beta$ be such that $h^\beta_\xi \restriction A_n <^* f_{\alpha^n_\xi} \restriction A_n$. Such an ordinal exists, because $h_\xi < g_\beta$ and $g_\beta \restriction A_n$ is an eub for $\langle f_\alpha \restriction A_n \mid$

$\alpha < \beta\rangle$. Let $a_\xi = \sup(\{a^n_\xi \mid n < \omega\})$. Define a function $\sigma^\beta_\xi \in {}^\omega\omega$ by letting $\sigma^\beta_\xi(n)$ be the least j such that $h^\beta_\xi(i^n_k) < f_{\alpha_\xi}(i^n_k)$ for all $k \geq j$.

For $\alpha < \beta$ and $n < \omega$, $f_\alpha \restriction A_n <^* g_\beta \restriction A_n$, so, since $\mathrm{cf}(\beta) > \omega$, there is $\xi^n_\alpha < \mathrm{cf}(\beta)$ such that $f_\alpha \restriction A_n <^* h^\beta_{\xi^n_\alpha} \restriction A_n$. Let $\xi_\alpha = \sup(\{\xi^n_\alpha \mid n < \omega\})$. Define a function $\tau^\beta_\alpha \in {}^\omega\omega$ by letting $\tau^\beta_\alpha(n)$ be the least j such that $f_\alpha(i^n_k) < h^\beta_{\xi_\alpha}(i^n_k)$ for all $k \geq j$.

Now let G be $\mathbb{P}$-generic over V. Since $\mathbb{P}$ has the c.c.c., all cardinalities and cofinalities are preserved by $\mathbb{P}$.

Claim 4.40. *In $V[G]$, $\vec{f}$ is still a scale in $\prod B_0$.*

Proof. $\vec{f}$ is clearly still $<^*$ increasing, so it remains to check that it is cofinal in $\prod B_0$. To this end, let $\dot{h}$ be a $\mathbb{P}$ name for a member of $\prod B_0$. For $i \in B_0$, let $X_i = \{\delta \mid$ for some $p \in \mathbb{P}$, $p \Vdash$ "$\dot{h}(\check{i}) = \check{\delta}$"$\}$. By the c.c.c., each X_i is countable, so we may define a function $h^* \in \prod B_0$ in V by $h^*(i) = \sup(X_i)$. Then there is $\alpha < \kappa^+$ such that $h^* <^* f_\alpha$, so $\Vdash$ "$\dot{h} <^* \check{f}_\alpha$". □

In $V[G]$, let $\sigma \in {}^\omega\omega$ be the real added by G. In particular, σ dominates all reals in V. Let $B = \bigcup_{n<\omega}(A_n \setminus i^n_{\sigma(n)})$. Clearly, B is an ω-sequence cofinal in κ such that, for all $n < \omega$, $A_n \subseteq^* B$.

Claim 4.41. *For all $\beta \in C \cap \mathrm{cof}(\geq \omega_1)$, β is good for $\vec{f}^B$.*

Proof. Fix $\beta \in C \cap \mathrm{cof}(\geq \omega_1)$. It suffices to show that $\langle h^\beta_\xi \restriction B \mid \xi < \mathrm{cf}(\beta)\rangle$ is cofinally interleaved with $\langle f_\alpha \restriction B \mid \alpha < \beta\rangle$. Fix $\alpha < \beta$, and let ξ be the ξ_α used above in the definition of τ^β_α. $\tau^\beta_\alpha <^* \sigma$, so there is $m < \omega$ such that $\tau^\beta_\alpha(n) < \sigma(n)$ for all $n \geq m$. Thus, by the definition of τ^β_α, we know that for all $n \geq m$, $f_\alpha \restriction \bigcup_{n \geq m}(A_n \setminus i^n_{\sigma(n)}) < h^\beta_\xi \restriction \bigcup_{n\geq m}(A_n \setminus i^n_{\sigma(n)})$. Also, for each $n < m$, $f_\alpha \restriction A_n <^* h^\beta_\xi \restriction A_n$. Putting this together, we get $f_\alpha \restriction B <^* h^\beta_\xi \restriction B$. An identical argument shows that, for all $\xi < \mathrm{cf}(\beta)$, there is $\alpha < \beta$ such that $h^\beta_\xi \restriction B <^* f_\alpha \restriction B$. □

Thus, $\vec{f}^B$ is a good scale in $\prod B$, and the proof is complete. □

Theorem 4.42. *Let $\langle \mathbb{P}_\gamma \mid \gamma \leq \omega_1\rangle$ be a finite-support iteration of Hechler forcing. Then, in $V^{\mathbb{P}_{\omega_1}}$, for every singular cardinal κ of countable cofinality, $I_{gd}[\kappa]$ is a P-ideal.*

Proof. Let $\mathbb{P} = \mathbb{P}_{\omega_1}$. Since every pair of conditions p and q in the Hechler poset such that $s^p = s^q$ are compatible, Hechler forcing is σ-centered. Thus, since $\mathbb{P}$ is a finite-support iteration of length ω_1 of σ-centered forcings, $\mathbb{P}$ is itself σ-centered, and hence ω_1-Knaster.

Let G be $\mathbb{P}$-generic over V. For $\eta < \gamma \leq \omega_1$, let $\mathbb{P}_\gamma = \mathbb{P}_\eta * \mathbb{P}_{\eta\gamma}$, and let G_η and $G_{\eta\gamma}$ be the generic filters induced by G on $\mathbb{P}_\eta$ and $\mathbb{P}_{\eta\gamma}$, respectively.

Let κ be a singular cardinal of countable cofinality, and, in $V[G]$, let A be an ω-sequence cofinal in κ such that $\prod A$ carries a good scale. By chain condition, there is $\gamma < \omega_1$ such that $A \in V[G_\gamma]$.

Claim 4.43. *$\prod A$ carries a good scale in $V[G_\gamma]$.*

Proof. Work in $V[G_\gamma]$. Let $\dot{\vec{f}} = \langle \dot{f}_\alpha \mid \alpha < \kappa^+ \rangle$ be a $\mathbb{P}_{\gamma\omega_1}$-name such that $\Vdash_{\mathbb{P}_{\gamma\omega_1}}$ "$\dot{\vec{f}}$ is a good scale in $\prod \check{A}$". For $\alpha < \kappa^+$ and $i \in A$, let $Y_{\alpha,i} = \{\nu \mid$ for some $p \in \mathbb{P}_{\gamma\omega_1}$, $p \Vdash$ "$\dot{f}_\alpha(i) = \check{\nu}$"$\}$. Since $\mathbb{P}_{\gamma\omega_1}$ has the c.c.c., each $Y_{\alpha,i}$ is countable.

Now define $\vec{g} = \langle g_\alpha \mid \alpha < \kappa^+ \rangle$ by recursion on α as follows. Let g_0 be an arbitrary function in $\prod A$. Given g_α, let $g_{\alpha+1} \in \prod A$ be such that $g_{\alpha+1} > g_\alpha$ and, for all $i \in A$, $g_{\alpha+1}(i) > \sup(Y_{\alpha,i})$. If $\alpha < \kappa^+$ is a limit ordinal, let $g_\alpha \in \prod A$ be such that $g_\beta <^* g_\alpha$ for all $\beta < \alpha$. This is easily accomplished by a standard diagonalization argument.

It is clear that $\vec{g}$ is $<^*$-increasing. To see that it is cofinal in $\prod A$, let $h \in \prod A$ be given. Since $\dot{\vec{f}}$ is forced to be a scale in $\prod A$ and by the c.c.c., there is $\alpha < \kappa^+$ such that $\Vdash_{\mathbb{P}_{\gamma\omega_1}}$ "$\check{h} <^* \dot{f}_\alpha$". But $\Vdash_{\mathbb{P}_{\gamma\omega_1}}$ "$\dot{f}_\alpha < \check{g}_{\alpha+1}$", so $h <^* g_{\alpha+1}$. Thus, $\vec{g}$ is a scale in $\prod A$ and, by previous arguments, it remains a scale in $V[G]$. Since $\prod A$ carries a good scale in $V[G]$, $\vec{g}$ must be good in $V[G]$.

Subclaim 4.44. *If $\beta \in \kappa^+ \cap \mathrm{cof}(\geq \omega_1)$ is good for $\vec{g}$ in $V[G]$, then it is good in $V[G_\gamma]$.*

Proof. We proceed by induction on β and split into two cases depending on the cofinality of β. First, suppose $\mathrm{cf}(\beta) = \omega_1$. In $V[G]$, there is $i \in A$ and $\langle \beta_\xi \mid \xi < \omega_1 \rangle$ witnessing that β is good, i.e.

- $\langle \beta_\xi \mid \xi < \omega_1 \rangle$ is increasing and cofinal in β.
- For all $\xi < \xi' < \omega_1$ and all $j \in A \setminus i$, $g_\xi(j) < g_{\xi'}(j)$.

In $V[G_\gamma]$, find $p \in \mathbb{P}_{\gamma\omega_1}$ such that p forces β to be good for $\vec{g}$, and let $\dot{i}$ and $\langle \dot{\beta}_\xi \mid \xi < \omega_1 \rangle$ be names forced by p to witness that β is good. First, find $p^* \leq p$ and $i^* \in A$ such that $p^* \Vdash$ "$\dot{i} = i^*$". For $\xi < \omega_1$, find $p_\xi \leq p^*$ and β^*_ξ such that p_ξ "$\Vdash \dot{\beta}_\xi = \beta^*_\xi$". Since $\mathbb{P}_{\gamma\omega_1}$ is ω_1-Knaster, there is an unbounded $X \subseteq \omega_1$ such that if $\xi, \xi' \in X$, then p_ξ and $p_{\xi'}$ are compatible. Then i^* and $\langle \beta^*_\xi \mid \xi \in X \rangle$ witness that β is good for $\vec{g}$ in $V[G_\gamma]$.

Now suppose $\mathrm{cf}(\beta) > \omega_1$. In $V[G]$, there is a club C in β of order type $\mathrm{cf}(\beta)$ such that if $\alpha \in C \cap \mathrm{cof}(\geq \omega_1)$, then α is good for $\vec{g}$. By chain condition, there is a club $D \subseteq C$ such that $D \in V[G_\gamma]$, and, by induction, every $\alpha \in D \cap \mathrm{cof}(\geq \omega_1)$ is good for $\vec{g}$ in $V[G_\gamma]$. Thus, by Fact 4.38, in $V[G_\gamma]$, $\langle g_\alpha \mid \alpha < \beta \rangle$ has an eub, h, such that, for all $i \in A$, $\mathrm{cf}(h(i)) > \omega$. But then, by chain condition, h remains an eub in $V[G]$, where β is good for $\vec{g}$. Thus, $\mathrm{cf}(h(i)) = \mathrm{cf}(\beta)$ for all but finitely many $i \in A$, so β is good for $\vec{g}$ in $V[G_\gamma]$. □

Since $\vec{g}$ is a good scale in $V[G]$, there is a club $E \subseteq \kappa^+$ such that for all $\beta \in E \cap \mathrm{cof}(\geq \omega_1)$, β is good for $\vec{g}$. By chain condition, there is a club $E' \subseteq E$ such that $E' \in V[G_\gamma]$. The previous subclaim implies that for all $\beta \in E' \cap \mathrm{cof}(\geq \omega_1)$, β is good for $\vec{g}$ in $V[G_\gamma]$. Thus, $\vec{g}$ is a good scale in $V[G_\gamma]$. □

We finally show that $I_{gd}[\kappa]$ is a P-ideal in $V[G]$. So, in $V[G]$, let $\langle A_n \mid n < \omega \rangle$ be such that, for all $n < \omega$, $A_n \in I_{gd}[\kappa]$. We can assume that none of the A_n's is finite. By the previous claim, there is $\gamma < \omega_1$ such that, for all $n < \omega$, $A_n \in V[G_\gamma]$ and $\prod A_n$ carries a good scale in $V[G_\gamma]$. Then, by Lemma 4.39, in $V[G_{\gamma+1}]$ there is $B \in I_{gd}[\mu]$ such that, for all $n < \omega$, $A_n \subseteq^* B$. Arguments as before show the good scale in $\prod B$ in $V[G_\gamma]$ remains a good scale in $V[G]$, so $B \in I_{gd}[\kappa]$ in $V[G]$ as well. Thus, $I_{gd}[\kappa]$ is a P-ideal in $V[G]$. □

CHAPTER 5

BOUNDED STATIONARY REFLECTION

5.1 Preliminaries on stationary reflection

Definition. *Let κ be a regular cardinal, and let $S \subseteq \kappa$ be stationary. If $\alpha < \kappa$, then S* reflects at α *if $\operatorname{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. S* reflects *if there is $\alpha < \kappa$ such that S reflects at α. $\operatorname{Refl}(\kappa)$ is the statement that every stationary subset of κ reflects.*

The extent of stationary reflection is a topic of considerable interest in set theory, particularly regarding the investigation of the tension existing between incompactness phenomena and canonical inner models on one hand and large cardinals and reflection principles on the other. Quoting two basic results in this vein, it is an easy consequence of Π^1_1-indescribability that, if κ is weakly compact, then $\operatorname{Refl}(\kappa)$ holds, while Jensen [20] showed that, if $V = L$ and κ is a regular, uncountable cardinal, then $\operatorname{Refl}(\kappa)$ holds if and only if κ is weakly compact. Also note that, if $\kappa = \lambda^+$ and λ is a regular cardinal, then $\operatorname{Refl}(\kappa)$ cannot hold, since S^κ_λ is a non-reflecting stationary subset of κ.

We will be concerned with stationary reflection at successors of singular cardinals. The following fundamental result, due to Solovay [29] serves as a template for many of the proofs in this area.

Proposition 5.1. *Suppose $\langle \kappa_i \mid i < \omega \rangle$ is an increasing sequence of supercompact cardinals, and let $\kappa_\omega = \sup(\{\kappa_i \mid i < \omega\})$. Then $\operatorname{Refl}(\kappa_\omega^+)$ holds.*

Proof. Let $S \subset \kappa_\omega^+$ be stationary. By shrinking S if necessary, we may assume that there is $\lambda < \kappa_\omega$ such that $S \subseteq S^{\kappa_\omega^+}_\lambda$. Let $i^* < \omega$ be such that $\lambda < \kappa_{i^*}$, and let $j : V \to M$ be an elementary embedding witnessing that κ_{i^*} is κ_ω^+-supercompact. In M, $j(S)$ is a stationary subset of $S^{j(\kappa_\omega^+)}_\lambda$. Let $\eta = \sup(j``\kappa_\omega^+)$. $\eta < j(\kappa_\omega^+)$, and we claim that, in M, $j(S) \cap \eta$ is stationary in η.

Suppose this is not the case, and let $C \in M$ be a club in η such that $C \cap j(S) = \emptyset$. Since $j``\kappa_\omega^+$ is a $< \kappa_{i^*}$-closed, unbounded subset of $\eta$, $C \cap j``\kappa_\omega^+$ is a $< \kappa_{i^*}$-closed, unbounded subset of η that is disjoint from $j(S)$. Let $D = j^{-1}``(C \cap j``\kappa_\omega^+)$. D is a $< \kappa_{i^*}$-closed, unbounded subset of κ_ω^+ that is disjoint from S. But this is a contradiction, since S is a stationary subset of $S^{\kappa_\omega^+}_\lambda$ and $\lambda < \kappa_{i^*}$.

Thus, $M \models$ "$j(S)$ reflects at η", so, by elementarity, there is $\beta < \kappa_\omega^+$ such that $V \models$ "S reflects at β". □

Magidor, in [24], brings this result down to smaller cardinals by showing that, assuming the existence of ω-many supercompact cardinals, it is consistent that $\mathrm{Refl}(\aleph_{\omega+1})$ holds. In [28], Shelah produces, starting from a proper class of supercompact cardinals, a model in which, among other things, $\mathrm{Refl}(\mu^+)$ holds for every singular cardinal μ. In fact, he shows that, under more stringent large cardinal assumptions, such a model can contain a cardinal κ which is κ^{+n}-supercompact for every $n < \omega$. On the other hand, he proves in the same paper that if there is a cardinal κ that is $\kappa^{+\omega+1}$-supercompact, then there is a singular cardinal μ such that $\mathrm{Refl}(\mu^+)$ fails. In [4], Chayut presents a simpler argument that one can force stationary reflection at the successor of every singular cardinal.

In this chapter, we investigate questions about the cofinality of ordinals at which stationary sets reflect. These questions are of interest in, for example, the study of square bracket partition relations, where Eisworth has shown [12] that, if μ is a singular cardinal and $\mu^+ \to [\mu^+]^2_{\mu^+}$, then, for every stationary set $S \subseteq \mu^+$ and every regular $\lambda < \mu$, there is $\beta \in S^{\mu^+}_{>\lambda}$ such that S reflects at β. Eisworth [10] raised the natural question as to whether this is always the case assuming $\mathrm{Refl}(\mu^+)$ holds. With this in mind, we make the following definition.

Definition. *Let μ be a singular cardinal.* Bounded stationary reflection *holds at μ^+ if $\mathrm{Refl}(\mu^+)$ holds but there is a stationary $S \subseteq \mu^+$ and a $\lambda < \mu$ such that S does not reflect at any ordinal in $S^{\mu^+}_{\geq\lambda}$.*

An easy argument shows that bounded stationary reflection cannot hold at $\aleph_{\omega+1}$.

Proposition 5.2. *Suppose $\mathrm{Refl}(\aleph_{\omega+1})$ holds. Then, for every $n < \omega$, every stationary subset of $\aleph_{\omega+1}$ reflects to an ordinal β such that $\mathrm{cf}(\beta) \geq \aleph_n$.*

Proof. For a stationary $T \subseteq \aleph_{\omega+1}$, let $T' = \{\beta < \aleph_{\omega+1} \mid T \text{ reflects at } \beta\}$. It is immediate that, for every stationary $T \subseteq \aleph_{\omega+1}$, T' is stationary. For, if not, let C be club in $\aleph_{\omega+1}$ such that $C \cap T' = \emptyset$. Then $C \cap T$ is a stationary subset of $\aleph_{\omega+1}$ that does not reflect, contradicting our hypotheses.

Now let $S \subseteq \aleph_{\omega+1}$ be stationary and let $n < \omega$. We will show that S reflects at an ordinal of cofinality at least $\aleph_n$. Define sequences $\langle S_k \mid k < \omega \rangle$ and $\langle i_k \mid k < \omega \rangle$ as follows. Find $i_0 < \omega$ such that $S \cap \mathrm{cof}(\aleph_{i_0})$ is stationary, and let $S_0 = S \cap \mathrm{cof}(\aleph_{i_0})$. Given S_k and i_k, find $i_{k+1} < \omega$ such that $S'_k \cap \mathrm{cof}(\aleph_{i_{k+1}})$ is stationary, and let $S_{k+1} = S'_k \cap \mathrm{cof}(\aleph_{i_{k+1}})$. Each S_k is a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_{i_k})$ and, since such a set can only reflect at ordinals of cofinality greater than $\aleph_{i_k}$, it follows that $\langle i_k \mid k < \omega \rangle$ is a strictly increasing sequence.

Claim 5.3. *For every $k < \omega$ and $\beta < \aleph_{\omega+1}$, if S_k reflects at β, then S reflects at β.*

Proof. We proceed by induction on k. $k = 0$ is trivial, since $S_0 \subseteq S$. Now suppose we have proven the claim for k and that S_{k+1} reflects at β. Let C be club in β, and let $\alpha \in C' \cap S_{k+1}$. Then S_k reflects at α, so, since $C \cap \alpha$ is club in α, $C \cap S_k \neq \emptyset$. Thus, S_k reflects at β and, by induction, S does as well. □

Now find $i < \omega$ such that $i_k \geq n$. Then S_k reflects at an ordinal β, and, by our choice of k, $\mathrm{cf}(\beta) \geq \aleph_n$. By the claim, S also reflects at β, and we are done. □

In this chapter, we show that the situation is different at larger cardinals. Starting from sufficiently many supercompact cardinals, we produce a model in which bounded stationary reflection holds at the successors of many singular cardinals.

5.2 Forcing constructions

In this section, we introduce some forcing posets which will be used in the proof of our theorem.

We first define a forcing notion to shoot a club through the set of approachable points of the successor of a singular cardinal. This forcing poset is a key component of Chayut's proof of the consistency of stationary reflection in [4], and we will use it in a similar way here. Suppose μ is singular, $\lambda = \mu^+$, and $\lambda^{<\lambda} = \lambda$. Let $\vec{a}$ be an enumeration of the bounded subsets of λ in order type λ, and let S be the set of ordinals that are approachable with respect to $\vec{a}$. Define $\mathbb{Q}_{\vec{a}}$ to the be the poset consisting of closed, bounded subsets of S, where, if $p, q \in \mathbb{Q}_{\vec{a}}$, then $p \leq q$ if and only if p is an end-extension of q.

Lemma 5.4. *$\mathbb{Q}_{\vec{a}}$ is strongly $< \lambda$-strategically closed.*

Proof. Let $\beta < \lambda$. We describe a winning strategy for Player II in $G^*_\beta(\mathbb{Q}_{\vec{a}})$. Fix a large, regular cardinal θ. In the course of the game, as the plays $\langle q_\alpha \mid \alpha < \beta \rangle$ are made, we will be defining a continuous, internally-approachable chain $\langle M_\alpha \mid \alpha < \beta \rangle$ of elementary submodels of $H(\theta)$ subject to the following conditions:

- $\mathbb{Q}_{\vec{a}} \in M_0$, $\beta \subset M_0$, and, for every limit $\alpha < \beta$, there is a club $C_\alpha \subseteq \alpha$ of order type $\mathrm{cf}(\alpha)$ such that $C_\alpha \in M_0$.
- For all $\alpha < \beta$, $|M_\alpha| < \lambda$ and $M_\alpha \cap \lambda \in \lambda$.
- For all $\alpha < \beta$, $\langle q_\gamma \mid \gamma \leq \alpha \rangle \in M_{\alpha+1}$.

If α is an even successor ordinal and $\langle q_\gamma \mid \gamma < \alpha \rangle$ has been played, then Player II plays q_α such that

- $q_\alpha \in M_\alpha$.
- $q_\alpha \leq q_{\alpha-1}$.
- $\max(q_\alpha) > M_{\alpha-1} \cap \lambda$.

This is possible, since S is stationary in λ and M_α contains all relevant information needed to find such a q_α.

Now suppose that $\alpha < \beta$ is a limit ordinal and that $\langle q_\gamma \mid \gamma < \alpha \rangle$ has been played. We show that $\bigcup_{\gamma<\alpha} q_\gamma \cup \{(M_\alpha \cap \lambda)\}$ is a valid condition, thus completing the proof. It suffices to show that $M_\alpha \cap \lambda$ is approachable with respect to $\vec{a}$. To this end, consider $D = \{M_\gamma \cap \lambda \mid \gamma \in C_\alpha\}$. D is a club in $M_\alpha \cap \lambda$ of order type $\mathrm{cf}(\alpha) = \mathrm{cf}(M_\alpha \cap \lambda)$. Also, every initial segment of D is in M_α, since every initial segment can be calculated from C_α and a sufficient initial segment of $\langle M_\gamma \mid \gamma < \alpha \rangle$, which are in M_α. But $M_\alpha \models$ "$\vec{a}$ is an enumeration of the bounded subsets of λ", so, for every $\eta < M_\alpha \cap \lambda$, there is $\xi < M_\alpha \cap \lambda$ such that $D \cap \eta = a_\xi$. Thus, D witnesses that $M_\alpha \cap \lambda$ is approachable with respect to $\vec{a}$. □

We now introduce a forcing notion to add a particular type of stationary set to the successor of a singular cardinal. Let $\mu > \aleph_\omega$ be a singular cardinal, let $\lambda = \mu^+$, and let $\kappa < \mu$ be a regular uncountable cardinal. $\mathbb{S}_{\lambda,\kappa}$ is a forcing poset whose conditions are of the form $p = (s^p, \gamma^p)$, where s^p is a bounded subset of S^λ_ω such that, for all $\beta \in S^\lambda_{\geq\kappa}$, $s^p \cap \beta$ is not stationary in β and γ^p is an ordinal such that for all $\alpha \in s^p$, $\alpha < \gamma^p < \lambda$. If $p, q \in \mathbb{S}_{\lambda,\kappa}$, then

$p \leq q$ if and only if s^p end-extends s^q and $s^p \setminus s^q \subset [\gamma^q, \lambda)$. It is immediate that forcing with $\mathbb{S}_{\lambda,\kappa}$ adds a stationary subset of S^λ_ω that does not reflect at any $\beta \in S^\lambda_{\geq\kappa}$ and that $\mathbb{S}_{\lambda,\kappa}$ is κ-closed. We will be interested mostly in forcings of the form $\mathbb{S}_{\lambda,\aleph_{\omega+1}}$, which we will denote simply as $\mathbb{S}_\lambda$.

Lemma 5.5. *$\mathbb{S}_{\lambda,\kappa}$ is $< \lambda$-strategically closed.*

Proof. Let $\beta < \lambda$. We describe a winning strategy for Player II in $G_\beta(\mathbb{S}_{\lambda,\kappa})$. In the course of the game, as the conditions $\langle p_\alpha \mid \alpha < \beta \rangle$ are being played, we define closed, bounded subsets $\langle C_\alpha \mid \alpha < \beta \text{ is even} \rangle$ of λ, ensuring that

- For all even $\alpha < \alpha' < \beta$, $C_{\alpha'}$ end-extends C_α.
- For all even $\alpha < \beta$ and all $\gamma \leq \alpha$, $C_\alpha \cap s^{p_\gamma} = \emptyset$.
- For all even $\alpha < \beta$, $\gamma^{p_\alpha} = \max(C_\alpha) + 1$.

Suppose that $\alpha < \beta$ is an even successor ordinal and $\langle p_\gamma \mid \gamma < \alpha \rangle$ has been played. Player II finds $\eta_\alpha > \gamma^{p_{\alpha-1}}$, lets $C_\alpha = C_{\alpha-2} \cup \{\eta_\alpha\}$, and plays $p_\alpha = (s^{p_{\alpha-1}}, \eta_\alpha + 1)$. If $\alpha < \beta$ is a limit ordinal, let $\eta_\alpha = \sup(\{\max(C_\gamma) \mid \gamma < \alpha\})$. Player II lets $C_\alpha = \bigcup_{\gamma<\alpha} C_\gamma \cup \{\eta_\alpha\}$ and plays $p_\alpha = \left(\bigcup_{\gamma<\alpha} s^{p_\gamma}, \eta_\alpha + 1\right)$. $p_\alpha \in \mathbb{S}_{\lambda,\kappa}$, since $\bigcup_{\gamma<\alpha} C_\gamma$ witnesses that $s_\alpha \cap \eta_\alpha$ is not stationary in η_α. $\square$

5.3 The main theorem

Theorem 5.6. *Suppose there is a proper class of supercompact cardinals and GCH holds. Then there is a forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$ that is not a cardinal fixed point, every stationary subset of μ^+ reflects but there is a stationary $S \subseteq S^{\mu^+}_\omega$ such that S does not reflect at any ordinal $\beta \in S^{\mu^+}_{\geq\aleph_{\omega+1}}$.*

Proof. Let $\langle \kappa_i \mid i \in \mathrm{On} \rangle$ be an increasing, continuous sequence of cardinals such that

- $\kappa_0 = \omega$.
- If κ_i is singular, then $\kappa_{i+1} = \kappa_i^+$.
- If κ_i is regular, then κ_{i+1} is supercompact.

We may assume without loss of generality that, if i is a limit ordinal, then κ_i is singular, for, if this is not the case, then we may work not in V but in $V_{\kappa_{i^*}}$, where i^* is the least limit ordinal i for which κ_i is regular.

We now define a class forcing $\langle \mathbb{P}_i \mid i \in \mathrm{On} \rangle$ such that:

- If i is a limit ordinal, then $\mathbb{P}_i$ is the inverse (i.e. full support) limit of $\langle \mathbb{P}_j \mid j < i \rangle$.
- If i is a successor ordinal, then $\mathbb{P}_{i+1} = \mathbb{P}_i * \mathrm{Coll}(\kappa_i, < \kappa_{i+1})$.
- If $i > \omega$ is a limit ordinal and $i < \kappa_i$, then let $\dot{\vec{a}}$ be a $\mathbb{P}_i$-name for an enumeration of the bounded subsets of κ_{i+1} in order type κ_{i+1} and let $\mathbb{P}_{i+1} = \mathbb{P}_i * \dot{\mathbb{Q}}_{\dot{\vec{a}}} * \dot{\mathbb{S}}_{\kappa_{i+1}}$.
- If i is a limit ordinal and $i = \kappa_i$, then $\mathbb{P}_{i+1} = \mathbb{P}_i * \{\mathbb{1}\}$, where $\{\mathbb{1}\}$ is trivial forcing.

For ordinals $i < k$, let $\dot{\mathbb{P}}_{i,k}$ be such that $\mathbb{P}_k = \mathbb{P}_i * \dot{\mathbb{P}}_{i,k}$. We think of conditions in $\mathbb{P}_i$ as being functions p with domain i. If $\ell < i$ is a limit ordinal, $\mathbb{P}_{\ell,\ell+1}$ is of the form $\dot{\mathbb{Q}}_{\dot{\vec{a}}} * \dot{\mathbb{S}}_{\kappa_{\ell+1}}$, and $p \in \mathbb{P}_i$, then $p(\ell)$ is thought of as a pair $(p(\ell)_0, p(\ell)_1)$ in the natural way. Note that, by Lemmas 5.4 and 5.5 and Fact 1.6, for all ordinals $i < k$, in $V^{\mathbb{P}_i}$ we have that $\dot{\mathbb{P}}_{i,k}$ is $< \kappa_i$-strategically closed and hence does not add any new sequences of ordinals of length less than κ_i. Thus, for $i < k$, $(H(\kappa_i))^{V^{\mathbb{P}_k}} = (H(\kappa_i))^{V^{\mathbb{P}_i}}$, so $V^{\mathbb{P}} = \bigcup_{i \in \mathrm{On}} V^{\mathbb{P}_i}$ is a model of ZFC.

We now show, by induction on ordinals i, that all of the κ_i's remain cardinals in $V^{\mathbb{P}}$. It is clear that there can be no other cardinals in $V^{\mathbb{P}}$ and so, when we have shown this, it will follow that $\kappa_i = (\aleph_i)^{V^{\mathbb{P}}}$. $\kappa_0 = \omega$, so there is nothing to worry about here. We first consider cardinals κ_{i+1}, where i is a successor ordinal or 0. In this case, since $|\mathbb{P}_i| < \kappa_{i+1}$, κ_{i+1} remains supercompact in $V^{\mathbb{P}_i}$. Since $\mathbb{P}_{i+1} = \mathbb{P}_i * \mathrm{Coll}(\kappa_i, < \kappa_{i+1})$, $\kappa_{i+1} = (\kappa_i^+)^{\mathbb{P}_{i+1}}$. Finally, since for all $k > i+1$, $\mathbb{P}_{i+1,k}$ is $< \kappa_{i+1}$-strategically closed, κ_{i+1} remains a cardinal in $V^{\mathbb{P}_k}$ for all k and hence in $V^{\mathbb{P}}$.

If i is a limit ordinal, then, by the previous paragraph, κ_i is, in $V^{\mathbb{P}}$, the limit of cardinals and hence a cardinal. Finally, consider κ_{i+1}, where i is a limit ordinal. Since, for all $k > i+1$, $\mathbb{P}_{i+1,k}$ is $< \kappa_{i+1}$-strategically closed, it suffices to show that κ_{i+1} remains a cardinal in $V^{\mathbb{P}_{i+1}}$. Suppose this is not the case. Then there is $i_0 < i$ such that $(\mathrm{cf}(\kappa_{i+1}))^{V^{\mathbb{P}_{i+1}}} = \kappa_{i_0}$. Since $\mathbb{P}_{i_0+1,i+1}$ is $\kappa_{i_0}+1$-strategically closed, it must be the case that $(\mathrm{cf}(\kappa_{i+1}))^{V^{\mathbb{P}_{i_0+1}}} = \kappa_{i_0}$. But $|\mathbb{P}_{i_0+1}| < \kappa_i$, so κ_{i+1} remains a regular cardinal in $V^{\mathbb{P}_{i_0+1}}$. This is a contradiction, so in fact κ_{i+1} remains a cardinal in $V^{\mathbb{P}_{i+1}}$.

It remains to show that, in $V^{\mathbb{P}}$, for every limit ordinal $i > \omega$ that is not a cardinal fixed point, every stationary subset of κ_{i+1} reflects but there is a stationary $S_i \subseteq S^{\kappa_{i+1}}_{\omega}$ that does not reflect at any ordinal in $S^{\kappa_{i+1}}_{\geq \aleph_{\omega+1}}$. We prove the latter statement first, since it is almost immediate. Let $i > \omega$

be such an ordinal, and let S_i be the stationary set added by $\mathbb{S}_{\kappa_{i+1}}$. By the definition of $\mathbb{S}_{\kappa_{i+1}}$, in $V^{\mathbb{P}_{i+1}}$, S_i is a stationary subset of $S^{\kappa_{i+1}}_{\omega}$ that does not reflect at any ordinal in $S^{\kappa_{i+1}}_{\geq\aleph_{\omega+1}}$. The fact that S_i does not reflect at any such ordinal is clearly preserved in any further forcing extension. Also, since AP_{κ_i} holds in $V^{\mathbb{P}_{i+1}}$ and, for all $j > i+1$, $\mathbb{P}_{i+1,j}$ is countably closed, S_i remains stationary in every $V^{\mathbb{P}_j}$ and hence in $V^{\mathbb{P}}$. Thus, S_i is as desired.

The following lemma, which will be useful in proving that all stationary subsets of κ_{i+1} reflect, comes from [4]. We thank Menachem Magidor and Yair Chayut for communicating it to us.

Lemma 5.7. *Let $i \in \mathrm{On}$, and let X_i be the set of limit ordinals $k < i$ such that $\omega < k < \kappa_k$. Then the set D_i of $p \in \mathbb{P}_i$ such that there is a function $g \in \prod_{k\in X_i} \kappa_{k+1}$ such that $g \in V$ and, for all $k \in X_i$, $p \restriction k \Vdash_{\mathbb{P}_k}$ "$\max(p(k)_0) < g(k)$" and $p \restriction k^\frown p(k)_0 \Vdash_{\mathbb{P}_k * \mathbb{Q}_{\bar{a}}}$ "$\gamma^{p(k)_1} < g(k)$" is dense in $\mathbb{P}_i$.*

Proof. We proceed by induction on i. First, suppose $i = k+1$. If k is a successor ordinal or k is limit and $k = \kappa_k$, then the conclusion is trivial, since $k \notin X_i$ in this case. Thus, assume k is a limit ordinal and $k < \kappa_k$. Let $p \in \mathbb{P}_i$. Find $p' \leq p \restriction k^\frown p(k)_0$ and $\eta < \kappa_{i+1}$ such that $p' \Vdash$ "$\gamma^{p(k)_1} < \eta$". Find $p'' \leq p' \restriction k$ and $\xi < \kappa_{i+1}$ such that $p'' \Vdash$ "$\max(p'(k)_0) < \xi$". Finally, by the inductive hypothesis, find $p^* \leq p''$ such that $p^* \in D_k$ as witnessed by $h \in \prod_{\ell\in X_k} \kappa_{\ell+1}$. Form $\bar{p} \in \mathbb{P}_i$ by letting $\bar{p} = p^{*\frown} p'(k)_0{}^\frown p(k)_1$, and let $g \in \prod_{\ell\in X_i} \kappa_{\ell+1}$ by letting $g(\ell) = h(\ell)$ for $\ell < k$ and $g(k) = \max(\{\eta, \xi\})$. Now $\bar{p} \leq p$, and $\bar{p} \in D_i$, as witnessed by g.

Now suppose i is a limit ordinal, and let $p \in \mathbb{P}_i$. Recall that κ_i is singular and that $\mathrm{cf}(\kappa_i) = \mathrm{cf}(i)$. Let $\langle \ell_\alpha \mid \alpha < \mathrm{cf}(i)\rangle$ be increasing and cofinal in i, with ℓ_0 a successor ordinal and $\kappa_{\ell_0} > \mathrm{cf}(i)$. Move to $V^{\mathbb{P}_{\ell_0}}$, assuming that $p \restriction \ell_0$ is in the generic subset of $\mathbb{P}_{\ell_0}$. We can apply the inductive hypothesis to $\mathbb{P}_{\ell_0,\ell_\alpha}$ for $\alpha < \mathrm{cf}(i)$ to get dense sets D_{ℓ_0,ℓ_α}. For $\ell^* > \ell_0$, define X_{ℓ_0,ℓ^*} to be the set of limit ordinals $k \in (\ell_0, \ell^*)$ such that $k < \kappa_k$. Note that $\mathbb{P}_{\ell_0,i}$ is $\mathrm{cf}(i)+1$-strategically closed. We will play the first $\mathrm{cf}(i)$-many moves of the game $G_{\mathrm{cf}(i)+1}(\mathbb{P}_{\ell_0,i})$ to produce $\langle p_\alpha \mid \alpha < \mathrm{cf}(i)\rangle$. Player II will play her winning strategy, and Player I will play $p \restriction [\ell_0, i)$ on her first move and, on her ξ^{th} move, will play q such that $q \restriction [\ell_0, \ell_\alpha) \in D_{\ell_0,\ell_\alpha}$ as witnessed by $h_\alpha \in \prod_{k\in X_{\ell_0,\ell_\alpha}} \kappa_{k+1}$. $\langle p_\alpha \mid \alpha < \mathrm{cf}(i)\rangle$ has a lower bound p^*, and moreover, defining $h^* \in \prod_{k \in X_{\ell_0,i}} \kappa_{k+1}$ by $h^*(k) = \sup(\{h_\alpha(k) \mid \alpha < \mathrm{cf}(i)\})$, we can assure that $p^* \in D_{\ell_0,i}$ as witnessed by h^*.

Now move back to V, and let $\dot{h}$ be a $\mathbb{P}_{\ell_0}$-name for h^*. Since $\mathbb{P}_{\ell_0}$ satisfies the $\kappa_{\ell_0}^+$-c.c., we can find $h \in V$ such that $\Vdash_{\mathbb{P}_{\ell_0}}$ "$\dot{h} < h$". Thus, there is $q \leq p \restriction \ell_0$ such that $q \Vdash_{\mathbb{P}_{\ell_0}}$ "$p^* \in D_{\ell_0,i}$ as witnessed by h". Find $q' \leq q$ and h' such that $q' \in D_{\ell_0}$ as witnessed by h'. Finally, let $\bar{p} = q' {}^\frown p^*$ and $g = h' {}^\frown h$. Then $\bar{p} \leq p$ and $\bar{p} \in \mathbb{P}_i$, as witnessed by g. □

Fix a limit ordinal $i > \omega$ such that i is not a cardinal fixed point in $V^{\mathbb{P}}$. We will show that every stationary subset of κ_{i+1} reflects. Since every subset of κ_{i+1} in $V^{\mathbb{P}}$ appears in $V^{\mathbb{P}_{i+2}}$, it suffices to prove stationary reflection in $V^{\mathbb{P}_{i+2}}$. Let G be $\mathbb{P}_{i+2}$-generic over V. For $i_0 < i+2$, let G_{i_0} be the $\mathbb{P}_{i_0}$-generic filter induced by G. Let $T \subseteq \kappa_{i+1}$ be stationary in $V[G]$. Without loss of generality, there is a successor ordinal $k < i$ such that $T \subseteq S^{\kappa_{i+1}}_{\kappa_k}$. We will show that T reflects. The proof breaks into two cases.

Case 1: $k < \omega$**.**

Let $k^* < \omega$ be such that $k < k^*$, and let $i^* = k^* + 1$. In $V[G_{k^*}]$, κ_{i^*} is supercompact. Let $j : V[G_{k^*}] \to M[G_{k^*}]$ witness that κ_{i^*} is κ_{i+2}-supercompact. $j(\mathbb{P}_{k^*,i^*}) = \mathrm{Coll}(\kappa_{k^*}, < j(\kappa_{i^*}))$ and $\mathbb{P}_{i^*,i+2}$ is strongly κ_{k^*}-strategically closed, so, by Fact 1.5, $j(\mathbb{P}_{k^*,i^*}) \cong \mathbb{P}_{k^*,i+2} * \mathbb{R}$, where $\mathbb{R}$ is κ_{k^*}-closed. Thus, letting H be $\mathbb{R}$-generic over $V[G]$, we can extend j to $j : V[G_{i^*}] \to M[G * H]$

We would like to extend j further to have domain $V[G]$. Let G^{i^*} be the $\mathbb{P}_{i^*,i+2}$-generic filter over $V[G_{i^*}]$ induced by G. We recursively build a condition $p^* \in j(\mathbb{P}_{i^*,i+2})$ such that $p^* \leq j(p)$ for all $p \in G^{i^*}$. Conditions in $\mathbb{P}_{i^*,i+2}$ can be seen as functions with domain $[i^*, i+2)$, so conditions in $j(\mathbb{P}_{i^*,i+2})$ can be thought of as functions with domain $[i^*, j(i+2))$. We recursively define $p^*(\alpha)$ for $\alpha \in [i^*, j(i+2))$. Suppose α is a successor ordinal and we have defined $p^* \restriction [i^*, \alpha)$ such that $p^* \restriction [i^*, \alpha) \leq j(p) \restriction [i^*, \alpha)$ for all $p \in G$. The forcing at coordinate α in $j(\mathbb{P}_{i^*,i+2})$ is a Levy collapse that is $j(\kappa_{i^*})$-directed closed, and, in $M[G * H]$, $|G^{i^*}| = \kappa_{k^*}$, so there is $\dot{q}$ such that $p^* \restriction [i^*, \alpha) \Vdash$ "$\dot{q} \leq j(p)(\alpha)$" for every $p \in G^{i^*}$. Let $p^* \restriction \alpha + 1 = p^* \restriction \alpha {}^\frown \dot{q}$.

Now suppose α is a limit ordinal. We would like to thank Menachem Magidor and Yair Chayut for conveying the following argument. By Lemma 5.7, $\Vdash_{j(\mathbb{P}_{i^*})}$ "$\sup(\{j(p)(\alpha)_0 \mid p \in G^{i^*}\}) = \sup(\{j(g)(\alpha) \mid g \in \prod_{\ell \leq i} \kappa_{\ell+1} \cap V\})$". Let this common supremum be denoted by γ, and note that $\mathrm{cf}^V(\gamma) \leq \kappa_{i+1}$.

We would like to let $p^*(\alpha)_0$ be forced by $p^* \restriction \alpha$ to be equal to $\bigcup_{p \in G^{i^*}} j(p)(\alpha)_0 \cup \{\gamma\}$. Let $A = \langle \vec{a}_\ell \mid \ell \in X_{i^*,i+2} \rangle$ be the sequence of names for enumerations of bounded subsets of $\kappa_{\ell+1}$ to be used in the forcings to shoot clubs through the sets of approachable sets. We must show that γ is approachable with respect to $j(A)(\alpha)$. In $M[G * H]$, κ_{i+1} has been collapsed to have size κ_{k^*}, so we can find a sequence $\langle g_\delta \mid \delta < \kappa_{k^*} \rangle$ of elements of $\prod_{\ell \leq i} \kappa_{\ell+1}$ such that:

- For all $\eta < \delta < \kappa_{k^*}$ and $i^* < \ell \leq i$, $g_\eta(\ell) < g_\delta(\ell)$.
- For all $\delta < \kappa_{k^*}$, and $i^* < \ell \leq i$ with $\ell \in X_{i^*,i+2}$, the sequence $\langle g_\eta(\ell) \mid \eta < \delta \rangle$ is enumerated in $A(\ell)$ before $g_\delta(\ell)$.
- For every $g \in \prod_{\ell \leq i} \kappa_{\ell+1}$, there is $\delta < \kappa_{k^*}$ such that, for every $i^* < \ell \leq i$, $g(\ell) < g_\delta(\ell)$.

By the closure properties of M, $\langle j(g_\delta)(\alpha) \mid \delta < \kappa_{k^*} \rangle \in M[G * H]$, and this sequence is easily seen to witness the approachability of γ with respect to $j(A)(\alpha)$, so our definition of $p^*(\alpha)_0$ is valid. Finally, we define $p^*(\alpha)_1$ by noting that the second component of coordinate α in $j(\mathbb{P}_{i^*,i+2})$ is $j(\kappa_{\omega+1}) > \kappa_{i+2}$-directed closed, so we can define $p^*(\alpha)_1$ to be a name for a lower bound for $\{j(p)(\alpha)_1 \mid p \in G^{i^*}\}$.

We now have successfully completed the construction of p^*. Let I be $j(\mathbb{P}_{i^*,i+2})$-generic over $M[G*H]$ such that $p^* \in I$, and extend j to $j : V[G] \to M[G * H * I]$.

Now suppose, for sake of contradiction, that T does not reflect in $V[G]$. Then, in $M[G * H * I]$, $j(T)$ is a stationary subset of $S^{j(\kappa_{i+1})}_{\kappa_k}$ that does not reflect. In particular, letting $\eta = \sup(j``\kappa_{i+1})$, $j(T)$ does not reflect at $\eta$. Let $D$ be a club in $\eta$ disjoint from $j(T)$. Since $j``\kappa_{i+1}$ is $< \kappa_{i^*}$-closed and unbounded in η, $D \cap j``\kappa_{i+1}$ is $< \kappa_{i^*}$-closed and unbounded in $\eta$ and is disjoint from $j(T)$. Thus, $E = j^{-1}``(D \cap j``\kappa_{i+1})$ is a $< \kappa_{i^*}$-closed, unbounded subset of κ_{i+1} that is disjoint from T. $E \in V[G * H * I]$. However, as I is generic for $j(\kappa_{i^*})$-strategically closed forcing and $j(\kappa_{i^*}) > \kappa_{i+1}$, it must be the case that $E \in V[G*H]$. Thus, since $\kappa_k < \kappa_{i^*}$ and $T \subseteq S^{\kappa_{i+1}}_{\kappa_k}$, E witnesses that T is not stationary in $V[G * H]$. However, since T is stationary in $V[G]$, $AP_{\kappa_{i+1}}$ holds in $V[G]$, and H is generic for κ_{k^*}-closed forcing, Fact 1.7 implies that T remains stationary in $V[G * H]$. This is a contradiction, so T does reflect in $V[G]$.

Case 2: $k > \omega$**.** Let k^* be a successor ordinal such that $k < k^* < i$ and $i < \kappa_{k^*}$ (note that we can do this because $i < \kappa_i$), and let $i^* = k^* + 1$. Let X

be the set of limit ordinals $\ell \in (i^*, i]$ such that $\ell < \kappa_\ell$. In $V[G]$, for $\ell \in X$, let S_ℓ be the stationary subset of $S^{\kappa_{\ell+1}}_\omega$ added by $\mathbb{S}_{\kappa_{\ell+1}}$. We would like to repeat the argument for Case 1, but now the stationary sets S_ℓ are an obstacle to lifting the relevant elementary embedding. Thus, we force to destroy the stationarity of the sets S_ℓ for $\ell \in X$. To this end, let $\mathbb{T}$ be the forcing poset whose conditions are functions t such that $\mathrm{dom}(t) = X$ and, for all $\ell \in X$, $t(\ell)$ is a closed, bounded subset of $\kappa_{\ell+1}$ such that $t(\ell) \cap S_\ell = \emptyset$. If $s, t \in \mathbb{T}$, then $s \leq t$ if and only if, for every $\ell \in X$, $s(\ell)$ end-extends $t(\ell)$. Let K be $\mathbb{T}$-generic over $V[G]$.

Claim 5.8. *In $V[G_{i^*}]$, there is a dense subset of $\mathbb{P}_{i^*,i+2} * \mathbb{T}$ that is strongly κ_{k^*}-strategically closed.*

Proof. Let $\mathbb{U}$ be the set of $(p, \dot{t}) \in \mathbb{P}_{i^*,i+2} * \mathbb{T}$ such that there is $g \in \prod_{\ell \in X} \kappa_{\ell+1}$ such that, for all $\ell \in X$, $p \restriction \ell^\frown p(\ell)_0 \Vdash_{\mathbb{P}_{i^*,\ell} * \mathbb{Q}_{\vec{a}}}$ "$\gamma^{p(\ell)_1} = g(\ell)+1$" and $p \Vdash_{\mathbb{P}_{i^*,i+2}}$ "$\max(\dot{t}(\ell)) = g(\ell)$".

We first show that $\mathbb{U}$ is dense in $\mathbb{P}_{i^*,i+2} * \mathbb{T}$. Given $(p_0, \dot{t}_0) \in \mathbb{P}_{i^*,i+2} * \mathbb{T}$, find $p_1 \leq p_0$ such that there is $h_0 \in \prod_{\ell \leq i} \kappa_{\ell+1}$ such that, for every $\ell \in X$, $p_1 \Vdash_{\mathbb{P}_{i^*,i+2}}$ "$\max(\dot{t}_0(\ell)) < h_0(\ell)$". Then, find $p_2 \leq p_1$ such that, as in Lemma 5.7, $p_2 \in D_{i^*,i+2}$, as witnessed by $h_1 \in \prod_{\ell \leq i} \kappa_{\ell+1}$, where $h_1 > h_0$. Now let $(p, \dot{t}) \leq (p_2, \dot{t}_0)$ be such that, for all $\ell \in X$, $p \restriction \ell^\frown p(\ell)_0 \Vdash_{\mathbb{P}_{i^*,\ell} * \mathbb{Q}_{\vec{a}}}$ "$\gamma^{p(\ell)_1} = h_1(\ell) + 1$" and $p \Vdash_{\mathbb{P}_{i^*,i+2}}$ "$\max(\dot{t}(\ell)) = h(\ell)$". Then $(p, \dot{t}) \in \mathbb{U}$ and $(p, \dot{t}) \leq (p_0, \dot{t}_0)$, so $\mathbb{U}$ is dense in $\mathbb{P}_{i^*,i+1} * \mathbb{T}$.

We now show that $\mathbb{U}$ is strongly κ_{k^*}-strategically closed. We thus describe a winning strategy for Player II in the game $G^*_{\kappa_{k^*}}(\mathbb{U})$. Suppose that $\beta < \kappa_{k^*}$ is an even successor ordinal and $\langle (p_\alpha, \dot{t}_\alpha) \mid \alpha < \beta \rangle$ has been played. All of the forcing iterands in $\mathbb{P}_{i^*,i+2}$ are already known to be strongly κ_{k^*}-strategically closed except for those of the form $\mathbb{S}_{\ell+1}$ so, on all of the other coordinates, Player II plays at stage β according to her winning strategy. To finish, at all $\ell \in X$, she simply lets $p_\beta(\ell)_1 = p_{\beta-1}(\ell)_1$ and lets $\dot{t}_\beta = \dot{t}_{\beta-1}$. This is easily seen to describe a condition in $\mathbb{U}$ extending $(p_{\beta-1}, \dot{t}_{\beta-1})$. If $\beta < \kappa_{k^*}$ is a limit ordinal and $\langle (p_\alpha, \dot{t}_\alpha) \mid \alpha < \beta \rangle$ has been played, we need to exhibit a greatest lower bound, $(p_\beta, \dot{t}_\beta)$, for this sequence of conditions. We define $p_\beta(\ell)$ recursively for $\ell \in [i^*, i+2)$. Let $p_\beta(\ell)$ (or $p_\beta(\ell)_0$, if $\ell \in X$) be such that $p_\beta \restriction \ell$ forces that $p_\beta(\ell)$ is the greatest lower bound of $\langle p_\alpha(\ell) \mid \alpha < \beta \rangle$. Such a greatest lower bound exists because Player II has been playing according to her winning strategy on coordinate ℓ. For $\ell \in X$, we have two cases for $p_\beta(\ell)_1$. If $\langle \gamma^{p_\alpha(\ell)_1} \mid \alpha < \beta \rangle$ is eventually equal to some ordinal $\eta + 1$, then let $p_\beta(\ell)_1$ be forced by $p_\beta \restriction \ell^\frown p_\beta(\ell)_0$ to be equal to $(\bigcup_{\alpha<\beta} s^{p_\alpha(\ell)_1}, \eta + 1)$. If $\langle \gamma^{p_\alpha(\ell)_1} \mid \alpha < \beta \rangle$ is not eventually constant, let $\eta = \sup(\{\gamma^{p_\alpha(\ell)_1} \mid \alpha < \beta\})$ and again let $p_\beta(\ell)_1$ be forced by $p_\beta \restriction \ell^\frown p_\beta(\ell)_0$ to be equal to $(\bigcup_{\alpha<\beta} s^{p_\alpha(\ell)_1}, \eta + 1)$. Finally, let $\dot{t}_\beta$ be such that, for all $\ell \in X$, letting η be as in the definition of $p_\beta(\ell)_1$, $p_\beta \Vdash_{\mathbb{P}_{i^*,i+2}}$ "$\dot{t}_\beta(\ell) = \bigcup_{\alpha<\beta} \dot{t}_\alpha(\ell) \cup \{\eta\}$". It is easily seen that $(p_\beta, \dot{t}_\beta) \in \mathbb{U}$ and is a greatest lower bound of $\langle (p_\alpha, \dot{t}_\alpha) \mid \alpha < \beta \rangle$. □

As in Case 1, κ_{i^*} is still supercompact in $V[G_{k^*}]$, so let $j : V[G_{k^*}] \to M[G_{k^*}]$ witness that κ_{i^*} is κ_{i+2}-supercompact. Since $\mathbb{P}_{i^*,i+2} * \mathbb{T}$ has a dense, strongly κ_{k^*}-strategically closed subset, we can use Fact 1.5 to lift j to $j : V[G_{i^*}] \to M[G * K * H]$, where H is generic over $V[G * K]$ for κ_{k^*}-closed forcing. We now define a master condition $p^* \in j(\mathbb{P}_{i^*,i+2})$ such that $p^* \leq j(p)$ for every $p \in G^{i^*}$. This is done exactly as in Case 1 except for the following: if $\ell \in X$, let $\eta_\ell = j``\kappa_{\ell+1}$. Then $p^*(\ell)_1$ is defined so that it is forced by $p^* \restriction \ell^\frown p^*(\ell)_0$ to be equal to $(\bigcup_{p\in G^{i^*}} s^{p(\ell)_1}, \eta + 1)$. Note that it is forced that $\sup(s^{p^*(\ell)_1}) = \eta_\ell$ and $p^*(\ell)_1 \in j(\mathbb{S}_{\kappa_{\ell+1}})$, since $\bigcup_{t\in K} j(t)(\ell)$ is forced to be a club in η disjoint from $s^{p^*(\ell)_1}$. Also, since $i < \mathrm{crit}(j)$, elements of $j(\mathbb{P}_{i^*,i+2})$ can also be thought of as functions with domain $[i^*, i + 2)$, so this finishes the definition of p^*.

Thus, letting I be $j(\mathbb{P}_{i^*,i+2})$-generic over $V[G * K * H]$ with $p^* \in I$, we can lift our embedding to $j : V[G] \to M[G * K * H * I]$. If T does not reflect in $V[G]$, then, as before, we can find a club E in κ_{i+1} such that $E \in V[G*K]$ and $E \cap T = \emptyset$. Thus, we will reach a contradiction and finish the proof if we demonstrate the following claim.

Claim 5.9. *T remains stationary in $V[G * K]$.*

Proof. Work in $V[G]$. Let $t_0 \in \mathbb{T}$, and let $\dot{D}$ be a $\mathbb{T}$-name for a club in κ_{i+1}. We will find $t \leq t_0$ such that $t \Vdash_{\mathbb{T}}$ "$\dot{D} \cap \check{T} \neq \emptyset$". Let θ be a sufficiently large regular cardinal. Since AP_{κ_i} holds (it was forced by $\mathbb{P}_{i,i+1}$), we can find an internally approachable continuous chain of elementary substructures of $H(\theta)$, $\langle M_\xi \mid \xi < \kappa_k \rangle$ such that:

- $\mathbb{T}, \{\kappa_\ell \mid \ell \leq i + 1\}, t_0, \dot{D} \in M_0$.
- $\kappa_k^+ \subset M_0$.
- For all $\xi < \kappa_k$, $|M_\xi| < \kappa_{k^*}$.

- Letting $M = \bigcup_{\xi<\kappa_k} M_\xi$, $\sup(M \cap \kappa_{i+1}) \in T$.

For all $\ell \in X$, let $\lambda_\ell = \sup(M \cap \kappa_{\ell+1})$. For each $\ell \in X$, $\mathrm{cf}(\lambda_\ell) = \kappa_k$ and $\langle \sup(M_\xi \cap \kappa_{\ell+1}) \mid \xi < \kappa_k \rangle$ enumerates a club C^0_ℓ in λ_ℓ, all of whose initial segments are in M. In fact, since $\kappa_k^+ \subset M$ every bounded subset of C^0_ℓ is in M. Also, for all $\ell \in X$, since $k > \omega$, S_ℓ does not reflect at λ_ℓ, so there is a club C^1_ℓ in λ_ℓ such that $C^1_\ell \cap S_\ell = \emptyset$. For all $\ell \in X$, let $C_\ell = C^0_\ell \cap C^1_\ell$. By the preceding, we have that C_ℓ is a club in $\kappa_{\ell+1}$ disjoint from S_ℓ, all of whose initial segments are in M. In fact, again since $\kappa_k^+ \subset M$, any sequence $\langle C'_\ell \mid \ell \in X \rangle$ such that each C'_ℓ is an initial segment of C_ℓ is in M. For each $\ell \in X$, let $\langle \eta^\ell_\alpha \mid \alpha < \kappa_k \rangle$ be an increasing enumeration of C_ℓ.

We now construct a descending sequence $\langle t_\alpha \mid \alpha < \kappa_k \rangle$ of conditions in $\mathbb{T} \cap M$. In fact, any initial segment of the construction can be computed inside M, so any initial segment of the sequence of conditions will also be in M.

t_0 has already been given. Suppose $t_\alpha \in M$ is given. We will construct $t_{\alpha+1}$. First, let t'_α be the $<_\theta$-least condition in $\mathbb{T}$ such that $t'_\alpha \leq t_\alpha$ and there is $\gamma > \eta^i_\alpha$ such that $t'_\alpha \Vdash_{\mathbb{T}}$ "$\check{\gamma} \in \dot{D}$". Then, let $t_{\alpha+1}$ be the $<_\theta$-least condition such that $t_{\alpha+1} \leq t'_\alpha$ and, for every $\ell \in X$, there is $\beta > \alpha$ such that $\max(t_{\alpha+1}(\ell)) = \eta^\ell_\beta$. Note that the calculation of $t_{\alpha+1}$ only requires sufficiently long initial segments of the clubs C_ℓ, so $t_{\alpha+1} \in M$.

If $\beta < \kappa_k$ is a limit ordinal and $\langle t_\alpha \mid \alpha < \beta \rangle$ has been constructed, then, for each $\ell \in X$, let $\delta_\ell = \sup(\bigcup_{\alpha<\beta} t_\alpha(\ell))$ and define t_β by letting $t_\beta(\ell) = \bigcup_{\alpha<\beta} t_\alpha(\ell) \cup \{\delta_\ell\}$. t_β is a valid condition in $\mathbb{T}$ since, for each $\ell \in X$, δ_ℓ is in C_ℓ and hence not in S_ℓ. Also, the calculation of t_β only requires $\langle t_\alpha \mid \alpha < \beta \rangle$, calculation of which itself only requires sufficiently long initial segments of the C_ℓ's, so $t_\beta \in M$.

Finally, define $t \in \mathbb{T}$ by, for each $\ell \in X$, letting $t(\ell) = \bigcup_{\alpha<\kappa_k} t_\alpha(\ell) \cup \{\lambda_\ell\}$. Each λ_ℓ has cofinality $\kappa_k > \omega$, so $\lambda_\ell \notin S_\ell$ and thus t is in fact in $\mathbb{T}$. Also, $t \leq t_\alpha$ for all $\alpha < \kappa_k$, so $t \Vdash_{\mathbb{T}}$ "$\check{\lambda}_i \in \dot{D}$", so, in particular, $t \Vdash_{\mathbb{T}}$ "$\dot{D} \cap \check{T} \neq \emptyset$". $\square$

It remains open whether we can find a model in which bounded stationary reflection holds at the successor of every singular cardinal greater $\aleph_\omega$. The difficulty in dealing with successors of cardinal fixed points lies in the fact that, if we are unable to use an elementary embedding with critical point in the interval (i, κ_i), then our proof of Claim 5.9 does not work. By suitably varying the cofinalities of the points at which the stationary sets S_ℓ are allowed to reflect, we can obtain a model in which bounded stationary reflection holds at the successors of all singular cardinals greater than $\aleph_\omega$ which are not limits of cardinal fixed points, but it seems that this approach cannot be extended to attain a truly global result.

CHAPTER 6

RADIN FORCING

Gitik, Kanovei, and Koepke, in [17], prove the following theorem, completely characterizing the intermediate models of Prikry-generic forcing extensions:

Theorem 6.1. *Suppose $\mathbb{P}$ is Prikry forcing. Let G be $\mathbb{P}$-generic over V, let g be the associated ω-sequence, and let X be a set of ordinals in $V[G]$. Then there is some $d \subseteq g$ such that $X \equiv_V d$.*

In this chapter, we extend their analysis to analyze intermediate models of generic extensions by Radin forcing at a large cardinal κ using measure sequences of length less than κ.

6.1 The forcing

In this section we recall the basic facts about Radin forcing, beginning with the following definition.

Definition. *A coherent sequence of measures is a sequence $\vec{U} = \langle U(\beta, i) \mid \beta < \mathrm{lh}(\vec{U}),\ i < o^{\vec{U}}(\beta)\rangle$ such that, for all $\beta < \mathrm{lh}(\vec{U})$ and $i < o^{\vec{U}}(\beta)$,*

1. *$U(\beta, i)$ is a normal ultrafilter over β.*
2. *If $j_i^\beta : V \to M_i^\beta \cong \mathrm{Ult}(V, U(\beta, i))$ is the canonical embedding, then $j_i^\beta(\vec{U}) \restriction \beta = \vec{U} \restriction (\beta, i)$, where $\vec{U} \restriction \beta = \langle U(\beta', i') \mid \beta' < \beta,\ i' < o^{\vec{U}}(\beta')\rangle$ and $\vec{U} \restriction (\beta, i) = \langle U(\beta', i') \mid \beta' < \beta,\ i' < o^{\vec{U}}(\beta')$ or $\beta' = \beta,\ i' < i\rangle$.*

Suppose $\vec{U}$ is a coherent sequence of measures, $\mathrm{lh}(\vec{U}) = \kappa + 1$ for some cardinal κ, and $o^{\vec{U}}(\kappa) > 0$. Our version of Radin forcing associated with $\vec{U}$, which we denote $\mathbb{P}_{\vec{U}}$, has conditions of the form $p = \langle (\alpha_k, A_k) \mid k \leq m\rangle$, where

- $m < \omega$.
- $\langle \alpha_k \mid k \leq m\rangle$ is an increasing sequence of ordinals.
- $\alpha_m = \kappa$ and $A_m \in \bigcap_{i < o^{\vec{U}}(\kappa)} U(\kappa, i)$.

- For $k < m$, if $o^{\vec{U}}(\alpha_k) = 0$, then $A_k = \emptyset$, and if $o^{\vec{U}}(\alpha_k) > 0$, then $A_k \in \bigcap_{i<o^{\vec{U}}(\alpha_k)} U(\alpha_k, i)$. Moreover, if $j < k \leq m$, then $A_k \cap (\alpha_j + 1) = \emptyset$.

- For all $k \leq m$ and $\beta \in A_k$, $o^{\vec{U}}(\beta) < o^{\vec{U}}(\alpha_k)$.

Remark The last requirement in the above definition is not standard. We introduce it because it will simplify future notation and does not change any key properties of the forcing.

If $p = \langle(\alpha_k, A_k) \mid k \leq m\rangle$ is a condition, we say that $m = \mathrm{lh}(p)$, $\langle(\alpha_k, A_k) \mid k < m\rangle$ is the *lower part of* p, and (κ, A_m) is the *upper part of* p. Let $a^p = \{\alpha \mid \text{for some } k < m, \alpha = \alpha_k\}$. If $n < m$ and $o^{\vec{U}}(\alpha_n) > 0$, then $p^{\leq n} = \langle(\alpha_k, A_k) \mid k \leq n\rangle$ and $p^{>n} = \langle(\alpha_k, A_k) \mid n < k \leq m\rangle$. Note that $p^{\leq n} \in \mathbb{P}_{\vec{U}\restriction(\alpha_n+1)}$ and $p^{>n} \in \mathbb{P}_{\vec{U}}$.

If $p = \langle(\alpha_j, A_j) \mid j \leq m\rangle$ and $q = \langle(\beta_k, B_k) \mid k \leq n\rangle$ are in $\mathbb{P}_{\vec{U}}$, then $q \leq p$ if and only if

- $n \geq m$.

- For every $j \leq m$, there is $k \leq n$ such that $\alpha_j = \beta_k$ and $B_k \subseteq A_j$.

- If $k < n$ is such that there is no $j < m$ with $\alpha_j = \beta_k$ then, letting $i \leq m$ be least such that $\beta_k < \alpha_i$, we have $\beta_k \in A_i$ and $B_k \subseteq A_i$.

We say q is a *direct extension* of p and write $q \leq^* p$ if $q \leq p$ and $n = m$.

For the rest of the chapter, we will assume for simplicity that $o^{\vec{U}}(\kappa) = \lambda < \kappa$ and that, for all $p \in \mathbb{P}_{\vec{U}}$, $a^p \subseteq \kappa \setminus (\lambda + 1)$. We now list some of the basic properties of the Radin forcing $\mathbb{P}_{\vec{U}}$. See [16] for proofs.

- If p and q are conditions with the same lower part, then p and q are compatible. Thus, since there are only κ-many lower parts, $\mathbb{P}_{\vec{U}}$ has the κ^+-c.c.

- $\mathbb{P}_{\vec{U}}$ has the Prikry property, i.e. for every sentence ϕ in the forcing language and every $p \in \mathbb{P}_{\vec{U}}$, there is $q \leq^* p$ such that $p \parallel \phi$.

- If $p = \langle(\alpha_k, A_k) \mid k \leq m\rangle$ is in $\mathbb{P}_{\vec{U}}$ and $n < m$ is such that $o^{\vec{U}}(\alpha_n) > 0$, then $\mathbb{P}_{\vec{U}}/p \cong \mathbb{P}_{\vec{U}\restriction(\alpha_n+1)}/p^{\leq n} \times \mathbb{P}_{\vec{U}}/p^{>n}$.

- Let G be $\mathbb{P}_{\vec{U}}$-generic over V, and let $g = \{\alpha \mid \text{for some } p \in G, \alpha \in a^p\}$. Then g is a club in κ and has order type $\omega^{o^{\vec{U}}(\kappa)}$, and $V[G] = V[g]$. In a slight abuse of notation, we will alternately think of this g both as a subset of κ and as a function, with domain $\omega^{o^{\vec{U}}(\kappa)}$, enumerating this subset in increasing fashion.

If $V[G]$ is a set-generic forcing extension of V and X is a set of ordinals in $V[G]$, then

$$V[X] = \bigcup_{\substack{z \subseteq \mathrm{On} \\ z \in V}} L[z, X]$$

is the smallest inner model W of ZFC such that $V \subseteq W \subseteq V[G]$ and $X \in W$. If X and Y are sets of ordinals in $V[G]$, we say $X \leq_V Y$ if $X \in V[Y]$ and $X \equiv_V Y$ if $X \leq_V Y$ and $Y \leq_V X$, i.e. $V[X] = V[Y]$.

6.2 Preliminaries on indiscernibles

In this section, we recast the results from [17] about sets of indiscernibles in a form appropriate for Radin forcing.

As above, let $\vec{U} = \langle U(\beta, i) \mid \beta \leq \kappa, i < o^{\vec{U}}(\beta)\rangle$ be a coherent sequence of measures. For notational simplicity, we denote $o^{\vec{U}}(\beta)$ by $o(\beta)$. For $\beta \leq \kappa$, let $\vec{U}_\beta = \langle U(\beta, i) \mid i < o(\beta)\rangle$.

For now, let us focus on $\vec{U}_\kappa$. Let $\lambda = o(\kappa)$ and, for $i < \lambda$, let $U_i = U(\kappa, i)$. Recall our assumption that $\lambda < \kappa$. For $i < \lambda$, let $A^*_i = \{\beta < \kappa \mid o(\beta) = i\}$. Note that, for all $i < \lambda$, $A^*_i \in U_i$.

Let $n < \omega$ and $b \in \lambda^n$. For a set $A \subset \kappa$, let $[A]^b$ denote the set of increasing n-tuples x from A such that, for all $k < n$, $x(k) \in U_{b(k)}$. We indicate that $x \in [\kappa]^b$ by writing $\mathrm{tp}(x) = b$.

Proposition 6.2. *Suppose $B \subseteq [\kappa]^{<\omega}$. Then there is $A \in \bigcap \vec{U}_\kappa$ such that, for all $b \in \lambda^{<\omega}$, either*

1. *For all $x \in [A]^b$, $x \in B$ or*
2. *For all $x \in [A]^b$, $x \notin B$.*

Proof. We show, by induction on $n < \omega$ simultaneously for all $B \subseteq [\kappa]^{<\omega}$, that for all $b \in \lambda^n$, there is $A_b \in \bigcap \vec{U}_\kappa$ such that either for all $x \in [A_b]^b$, $x \in B$ or for all $x \in [A_b]^b$, $x \notin B$. Then $A = \bigcap_{b \in \lambda^{<\omega}} A_b$ will witness the truth of the proposition.

First, let $b \in \lambda^1$. Let $X_b = \{\alpha \mid \{\alpha\} \in B \text{ and } \alpha \in A_{b(0)}\}$. If $X_b \in U_{b(0)}$, then let $A_b = X_b \cup \bigcup_{k \in \lambda \setminus \{i\}} A_k$. If $X_b \notin U_{b(0)}$, let $A_b = (A_{b(0)} \setminus X_b) \cup \bigcup_{k \in \lambda \setminus \{i\}} A_k$. In either case, it is easily verified that A_b is as desired.

Now suppose that $b \in \lambda^{n+1}$. Let $a = b \restriction n$, and let $B_a = \{x \restriction n \mid x \in [\kappa]^\beta \cap B\}$. Then $B_a \subseteq [\kappa]^a$. By the induction hypothesis, we may find $C_a \in \bigcap \vec{U}_\kappa$ witnessing the desired conclusion for a and B_a. First, suppose that, for all $x \in [C_a]^a$, $x \notin B_a$. Then we also have that, for all $x \in [C_a]^b$, $x \notin B$, so we can let $A_b = C_a$. Now suppose that, for all $x \in [C_a]^a$, $x \in B_a$. For each $x \in [C_a]^a$, let $Y_x = \{\alpha \mid x^\frown(n, \alpha) \in B\}$. Let $B_a^* = \{x \in B_a \mid Y_x \in U_{b(n)}\}$. Applying the induction hypothesis again, we may find $C_a^* \subseteq C_a$ witnessing the desired conclusion for a and B_a^*.

First suppose that, for all $x \in [C_a^*]^a$, $x \in B_b$. Let $Y^* = \{\alpha \in A_{b(n)} \mid$ for all $x \in [C_a^*]^a$ such that $\max(\operatorname{ran}(x)) < \alpha$, $\alpha \in Y_x\}$. Since $Y_x \in U_{b(n)}$ for all $x \in [C_a^*]^a$, we also have $Y^* \in U_{b(n)}$. Now let $A_b = Y^* \cup \bigcup_{k \in \lambda \setminus \{b(n)\}}(C_a^* \cap A_k)$. It is easily verified that, if $x \in [A_b]^b$, then $x \in B$.

Finally, suppose that, for all $x \in [C_a^*]^a$, $x \notin B_b$. The argument is exactly as in the previous case, letting $Y^* = \{\alpha \in A_{b(n)} \mid$ for all $x \in [C_a^*]^a$ such that $\max(\operatorname{ran}(x)) < \alpha$, $\alpha \notin Y_x\}$. $\square$

Proposition 6.3. *Suppose $B \subseteq [\kappa]^{<\omega}$. Then there is $A \in \bigcap \vec{U}_\kappa$ such that for every $x \in [\kappa]^{<\omega}$, every $b \in \lambda^{<\omega}$, and any $y_0, y_1 \in [A \setminus (\max(x)+1)]^b$, $x^\frown y_0 \in B$ if and only if $x^\frown y_1 \in B$.*

Proof. For $x \in [\kappa]^{<\omega}$, let $B_x = \{y \in [\kappa]^{<\omega} \mid \max(x) < \min y$ and $x^\frown y \in B\}$. For each $\alpha < \kappa$, repeatedly use Proposition 6.2 to find $A_\alpha \in \bigcap \vec{U}_\kappa$ such that for every $x \in [\alpha+1]^{<\omega}$, every $b \in \lambda^{<\omega}$, and every $y_0, y_1 \in [A_\alpha]^b$, $y_0 \in B_x$ if and only if $y_1 \in B_x$. Without loss of generality, we may assume that $A_\alpha \subset \kappa \setminus (\alpha+1)$. Let $A = \Delta_{\alpha<\kappa} A_\alpha$. It is routine to verify that $A \in \bigcap \vec{U}_\kappa$ and that A has the desired properties. $\square$

Lemma 6.4. *Suppose F is a function with domain $[\kappa]^{<\omega}$. For every $n < \omega$, $b \in \lambda^n$, and $x \in [\kappa]^{<\omega}$, there are sets $J_b(x) \in \bigcap \vec{U}_\kappa$ and $\operatorname{bas}_b(x) \subseteq n$ such that for all $y_0, y_1 \in [J_b(x)]^b$ with $\max(x) < \min(y_0), \min(y_1)$, we have $F(x^\frown y_0) = F(x^\frown y_1)$ if and only if $y_0 \restriction \operatorname{bas}_b(x) = y_1 \restriction \operatorname{bas}_b(x)$.*

Proof. Let θ be a sufficiently large regular cardinal, and let $\mathcal{F}_0$ be the collection of all sets definable in $H(\theta)$ with $F, \vec{U}_\kappa$, and ordinals less than λ as the only parameters. Since $|\mathcal{F}_0| < \kappa$, we can repeatedly apply Proposition 6.3 to obtain a set $I_0 \in \vec{U}_\kappa$ such that, for every $B \in \mathcal{F}_0$, $b \in \lambda^{<\omega}$, and $x \in [\kappa]^{<\omega}$, for every $y_0, y_1 \in [I_0]^b$ such that $\max(x) < \min(y_0), \min(y_1)$, we have $x^\frown y_0 \in B$ if and only if $x^\frown y_1 \in B$. Let $\mathcal{F}$ be the collection of all sets definable in $H(\theta)$ with I_0 as the only parameter, and again apply Proposition 6.3 to find $I \subseteq I_0$, a set of indiscernibles for $\mathcal{F}$. The point is that for every $\eta < \lambda$ and every $\alpha \in I \cap A_\eta$, α is a limit point of $I_0 \cap A_\eta$.

We proceed by induction on n, simultaneously for all $x \in [\kappa]^{<\omega}$. First, suppose $n = 1$ and let $b \in \lambda^n$, $x \in [\kappa]^{<\omega}$. Consider the set $B = \{z^\frown\langle\alpha,\beta\rangle \mid z \in [\kappa]^{<\omega}, \max(z) < \alpha, \beta < \kappa$, and $F(z^\frown\langle\alpha\rangle) = F(z^\frown\langle\beta\rangle\}$. $B \in \mathcal{F}_0$. Thus, by our choice of I, $F(x^\frown\langle\alpha\rangle) = F(x^\frown\langle\beta\rangle)$ holds for a particular pair of $\alpha < \beta$ from $(I \cap A_{b(0)}) \setminus (\max(x)+1)$ if and only if it holds for all such pairs. We can thus let $J_b(x) = I$ and, choosing $\alpha < \beta$ from $(I \cap A_{b(0)}) \setminus (\max(x)+1)$, let $\mathrm{bas}_b(x) = \emptyset$ if $F(x^\frown\langle\alpha\rangle) = F(x^\frown\langle\beta\rangle)$ and $\mathrm{bas}_b(x) = \{0\}$ if $F(x^\frown\langle\alpha\rangle) \neq F(x^\frown\langle\beta\rangle)$.

Now suppose $n = m+1$. Let $b \in \lambda^n$ and $x \in [\kappa]^{<\omega}$. Define $b' \in \lambda^m$ by letting $b'(k) = b(k+1)$. If $\alpha \in A_{b(0)} \setminus (\max(x)+1)$, then we can apply the inductive hypothesis to obtain $J_{b'}(x^\frown\langle\alpha\rangle)$ and $bas_{b'}(x^\frown\langle\alpha\rangle)$. Find $A \subseteq A_{b(0)} \setminus (\max(x)+1)$ and $s \subseteq m$ such that $A \in U_{b(0)}$ and, for all $\alpha \in A$, $\mathrm{bas}_{b'}(x^\frown\langle\alpha\rangle) = s$. Let $J = (I \cap A \cap \Delta_{\alpha \in A}(J_{b'}(x^\frown\langle\alpha\rangle)) \cap \lim(J_{b'}(x^\frown\langle\alpha\rangle))) \cup \bigcup_{\eta \in (\lambda \setminus \{b(0)\})} I \cap A_\eta \cap \Delta_{\alpha \in A}(J_{b'}(x^\frown\langle\alpha\rangle) \cap \lim(J_{b'}(x^\frown\langle\alpha\rangle)))$. We claim that we can let $J_b(x) = J$. We thus need to define $\mathrm{bas}_b(x)$ and show that this definition works. To this end, let $\alpha < \beta \in J \cap A_{b(0)}$ and let $y \in [J]^{b'}$ be such that $\beta < \min(y)$. There are two cases.

Case 1: $F(x^\frown\langle\alpha\rangle^\frown y) = F(x^\frown\langle\beta\rangle^\frown y)$**.** In this case, let $\mathrm{bas}_b(x) = \{1+k \mid k \in s\}$. To check that this works, let $y_0, y_1 \in [J]^b$. Let $\xi = y_0(0)$ and $\eta = y_1(0)$. Suppose without loss of generality that $\xi \leq \eta$. Define $y_1' \in [J]^b$ by letting $y_1'(0) = \xi$ and $y_1'(k) = y_1(k)$ for all $k > 0$. Then, by indiscernibility and the case assumption, we have that $F(x^\frown y_1) = F(x^\frown y_1')$. Moreover, by the construction of J, $F(x^\frown y_1') = F(x^\frown y_0)$ if and only if $y_1' \restriction \{1+k \mid k \in s\} = y_0 \restriction \{1+k \mid k \in s\}$ if and only if $y_1 \restriction \mathrm{bas}_b(x) = y_0 \restriction \mathrm{bas}_b(x)$.

Case 2: $F(x^\frown\langle\alpha\rangle^\frown y) \neq F(x^\frown\langle\beta\rangle^\frown y)$**.** In this case, let $\mathrm{bas}_b(x) = \{0\} \cup \{1+k \mid k \in s\}$. Let $y_0, y_1 \in [J]^b$. If $y_0 \restriction \mathrm{bas}_b(x) = y_1 \restriction \mathrm{bas}_b(x)$, then it is routine to check that $F(x^\frown y_0) = F(x^\frown y_1)$. Thus, suppose that $y_0 \restriction \mathrm{bas}_b(x) \neq y_1 \restriction \mathrm{bas}_b(x)$ but, for sake of contradiction, $F(x^\frown y_0) = F(x^\frown y_1)$. There are two subcases here.

Subcase 1: $y_0(0) \neq y_1(0)$ **and** $y_0 \restriction [1,n) = y_1 \restriction [1,n)$**.** Suppose without loss of generality that $y_0(0) < y_1(0)$. Let $y_1' \in [J]^b$ be defined by letting $y_1'(0) = y_0(0)$ and $y_1'(k) = y_1(k)$ for $k > 0$. By the case assumption and indiscernibility, $F(x^\frown y_1') \neq F(x^\frown y_1)$. However, y_1' is equal to y_0, so $F(x^\frown y_1') = F(x^\frown y_0) = F(x^\frown y_1)$ by assumption. This is a contradiction.

Subcase 2: There is $k^* \in s$ **such that** $y_0(k^*+1) \neq y_1(k^*+1)$**.** Let the *order structure* of y_0 and y_1 refer to the set of $\leq$-equations describing the order relations in the set $\{y_0(k) \mid k < n\} \cup \{y_1(k) \mid k < n\}$. Since $F(x^\frown y_0) = F(x^\frown y_1)$ we have that, by indiscernibility, $F(x^\frown \bar{y}_0) = F(x^\frown \bar{y}_1)$ for any $\bar{y}_0$, $\bar{y}_1$ in $[I]^b$ having the same order structure as y_0 and y_1. By indiscernibility and the construction of J, we can adjust y_0 on coordinate k^* to form $\bar{y}_0$ such that $\bar{y}_0(k) = y_0 k$ for all $k \neq k^*$, $\bar{y}_0(k^*) \in J_{b'}(x^\frown \langle y_0(0)\rangle)$, and $\bar{y}_0$ and y_1 have the same order structure as y_0 and y_1. By the definition of $J_{b'}(x^\frown \langle y_0(0)\rangle$, $F(x^\frown y_0) \neq F(x^\frown \bar{y}_0)$. However, by order structure considerations, $F(x^\frown \bar{y}_0) = F(x^\frown y_1) = F(x^\frown y_0)$. This provides a contradiction and finishes the proof of the lemma. $\square$

6.3 Coding fresh subsets

For the rest of this chapter, we will be working to prove the following theorem.

Theorem 6.5. *Suppose κ is a cardinal, $\vec{U}$ is a coherent sequence of measures, $\mathrm{lh}(\vec{U}) = \kappa + 1$, and $0 < o^{\vec{U}}(\kappa) < \kappa$. Let G be $\mathbb{P}_{\vec{U}}$-generic over V, and let g be the associated club in κ. Suppose $X \in V[G]$ is a set of ordinals. Then there is $d \subseteq g$ such that $d \equiv_V X$.*

Note that every model of ZFC, W, with $V \subseteq W \subseteq V[G]$, is equal to $V[X]$ for some set of ordinals $X \in V[G]$. Thus, this theorem will characterize all intermediate models of $V[G]$.

The proof will proceed by induction on κ. We will first prove the result for fresh subsets of κ. Recall the following definition.

Definition. *Suppose W is an outer model of V, $\beta \in \mathrm{On}$, $A \subseteq \beta$, and $A \in W$. A is a* fresh subset of β *if, for all $\alpha < \beta$, $A \cap \alpha \in V$.*

Let $n < \omega$, and let $b \in \lambda^n$. Then there is $f_b : n \to \omega^\lambda$ such that, whenever $p \in \mathbb{P}_{\vec{U}}$ is such that $a^p \in [\kappa]^b$, for all $k < n$, if $\dot{g}$ is the canonical name for the generic club in κ added by $\mathbb{P}_{\vec{U}}$, $p \Vdash$ "α_k is the $f_b(k)^{th}$ element of $\dot{g}$". If $a \in \lambda^m$ and $b \in \lambda^n$ are such that $\mathrm{ran}(f_a) \subseteq \mathrm{ran}(f_b)$, denote this by $a \sqsubseteq b$ and let $f_{a,b} : m \to n$ be the unique function such that $f_a = f_b \circ f_{a,b}$. If $x \in [\kappa]^b$ and $a \sqsubseteq b$, let $(x \restriction a) \in [\kappa]^a$ be defined by $(x \restriction a)(k) = x(f_{a,b}(k))$.

Let G be $\mathbb{P}_{\vec{U}}$-generic over V, and suppose $X \in V[G]$ is a fresh subset of κ. Let g be the generic club in κ of order type ω^λ defined from G. We want to show that there is $d \subseteq g$ such that $d \equiv_V X$. Without loss of generality, we may assume that we are working below the top condition, $\langle(\kappa, \kappa)\rangle$, i.e.

that there is a $\mathbb{P}_{\vec{U}}$-name $\dot{X}$ for X that is forced by the top condition to be a fresh subset of κ and that we are trying to find a condition q such that, for some $C \subseteq \omega^\lambda$ $q \Vdash$ "$\dot{X} \equiv_V \dot{g} \restriction \check{C}$". To see this, suppose we are working instead below an arbitrary condition $p = \langle(\alpha_i, A_i) \mid i \leq m\rangle$. First, if, for all $i < m$, we have $o(\alpha_i) = 0$, then the arguments below work as stated, using $\langle \alpha_i \mid i < m\rangle$ as the stem x in Propositions 6.2 and 6.3 and Lemma 6.4. If not, then let $i < m$ be largest such that $o(\alpha_i) > 0$. Then $\mathbb{P}_{\vec{U}} \cong \mathbb{P}_{\vec{U}\restriction \alpha_i+1}/p^{\leq i} \times \mathbb{P}_{\vec{U}}/p^{>i}$, and an application of the inductive hypothesis, the arguments below, and the results of Section 6.5 yield the desired proof.

We may also assume that $\dot{X}$ is a name such that, for all $\alpha < \kappa$, if $o(\alpha) > 0$ and $p \in \mathbb{P}_{\vec{U}}$ is such that $\alpha \in a^p$, then there is a $\mathbb{P}_{\vec{U}\restriction(\alpha+1)}$-name $\dot{Y}$ such that $p \Vdash$ "$\dot{X} \cap \check{\alpha} = \dot{Y}$". This follows from an application of the Prikry property and the basic fact about Radin forcing that, if g is a Radin-generic club, α is a limit point of g, and $Y \in V[g]$ is a subset of α, then $Y \in V[g \cap \alpha]$.

Define a function F with domain $[\kappa]^{<\omega}$ by letting $F(x) = \{(\alpha, Y) \mid$ there is $p \in \mathbb{P}_{\vec{U}}$ such that $a^p = x$ and $p \Vdash$ "$\dot{X} \cap \check{\alpha} = \check{Y}$"$\}$. Note that, if p and q are such that $a^p = a^q$, then p and q are compatible, so, for every $x \in [\kappa]^{<\omega}$ and $\alpha < \kappa$, there is at most one Y such that $(\alpha, Y) \in F(x)$. If $\beta < \kappa$, let $F(x) \restriction \beta$ denote $\{(\alpha, Y) \in F(x) \mid \alpha < \beta\}$. Apply Lemma 6.4 to F to attain sets $J_b(\emptyset)$ and $\mathrm{bas}_b(\emptyset)$ for $b \in \lambda^{<\omega}$. Let $J = \bigcap_{b \in \lambda^{<\omega}} J_b(\emptyset)$, and denote $\mathrm{bas}_b(\emptyset)$ by bas_b.

Proposition 6.6. *Let $n < \omega$, $a \in \lambda^n$, and $b \in \lambda^{n+1}$ be such that $a \sqsubseteq b$. Suppose that $i \leq n$ is such that $i \notin \mathrm{ran}(f_{a,b})$. Suppose $k \in \mathrm{bas}_a$. Then one of the following holds:*

1. $f_{a,b}(k) \in \mathrm{bas}_b$.
2. $k = i$ *and* $i \in \mathrm{bas}_b$.

Proof. Suppose for sake of contradiction that neither alternative holds. Let $m = f_{a,b}(k)$. Let $x, y \in [J]^a$ be such that $x(\ell) = y(\ell)$ for all $\ell \neq k$ and $x(k) \neq y(k)$. Since $k \in \mathrm{bas}_a$, $F(x) \neq F(y)$. Let $(\alpha, Y) \in F(x) \setminus F(y)$. Let $p = \langle(\alpha_i, A_i) \mid i \leq n\rangle$ and $q = \langle(\beta_i, B_i) \mid i \leq n\rangle$ be such that $a^p = x$, $a^q = y$, $p \Vdash$ "$\dot{X} \cap \check{\alpha} = \check{Y}$", and $q \Vdash$ "$\dot{X} \cap \check{\alpha} \neq \check{Y}$". Note that we can find such a q by the Prikry property.

We are assuming that item 2 in the statement of the proposition fails. Let us first suppose that it fails because $k \neq i$. In this case, $\alpha_i = \beta_i$, so we may extend p and q to $\bar{p}$ and $\bar{q}$ so that $\bar{x} = a^{\bar{p}}$ and $\bar{y} = a^{\bar{q}}$ are in $[J]^b$ and

$\bar{x}(i) = \bar{y}(i)$. Then $F(\bar{x}) = F(\bar{y})$ and $(\alpha, Y) \in F(\bar{x})$, so there is a condition q^* with $a^{q^*} = \bar{y}$ such that $q^* \Vdash$ "$\dot{X} \cap \check{\alpha} = \check{Y}$". $a^{q^*} = a^{\bar{q}}$, so q^* and $\bar{q}$ are compatible, but $\bar{q} \Vdash$ "$\dot{X} \cap \check{\alpha} \neq \check{Y}$". Contradiction.

Now suppose that item 2 fails because $i \notin \mathrm{bas}_b$. In this case, we may extend p and q to $\bar{p}$ and $\bar{q}$ with $\bar{x}$ and $\bar{y}$, and we have that $\bar{x} \restriction \mathrm{bas}_b = \bar{y} \restriction \mathrm{bas}_b$ regardless of whether or not $\bar{x}(i) = \bar{y}(i)$. The rest of the argument is as in the previous paragraph. $\square$

Let θ be a sufficiently large, regular cardinal, and let $\mathcal{F}$ be the collection of sets definable in $H(\theta)$ using only $\mathbb{P}_{\vec{U}}, \dot{X}, J, \{\mathrm{bas}_b \mid b \in \lambda^{<\omega}\}$, and ordinals less than λ as parameters. Since $|\mathcal{F}| < \kappa$, we can find $I \in \bigcap \vec{U}_\kappa$ that is indiscernible for all sets in $\mathcal{F}$ in the sense of Proposition 6.3.

Let $Z = \{\eta < \omega^\lambda \mid$ for some $b^* \in \lambda^{<\omega}$, for all $b \in \lambda^{<\omega}$ such that $b^* \sqsubseteq b$, we have $\eta \in f_b$"$\mathrm{bas}_b\}$. Let $\bar{Z}$ denote the closure of Z under the order topology. Let $d = g \restriction Z$ and $\bar{d} = g \restriction \bar{Z}$. Since g is a club, $d \equiv_V \bar{d}$. Let $\dot{d}$ denote the canonical name for d.

Theorem 6.7. $\langle(\kappa, I)\rangle \Vdash$ "$\dot{X} \equiv_V \dot{d}$".

Before we prove the theorem, we need some notation.

Definition. *Suppose $A \in \bigcup \vec{U}_\kappa$ and $p = \langle(\alpha_k, A_k) \mid k \leq m\rangle \leq \langle(\kappa, A)\rangle$. Then $a^p {}^\frown A = \langle(\alpha_k, B_k) \mid k \leq m\rangle$, where $B_k = \emptyset$ if $o(\alpha_k) = 0$ and $B_k = (\bigcup_{\eta < o(\alpha_k)} A \cap A^*_\eta) \cap (\alpha_{k-1}, \alpha_k)$ (where $\alpha_{-1} = 0$).*

Note that $a^p {}^\frown A$ is the greatest element $q \leq \langle(\kappa, A)\rangle$ such that $a^q = a^p$.

Definition. *If $\alpha < \kappa$, then $\alpha^\dagger = \min(I \setminus (\alpha + 1))$.*

Lemma 6.8. *Suppose $p \leq \langle(\kappa, I)\rangle$, $\alpha = \max(a^p)$, $o(\alpha) > 0$, and there is Y such that $p \Vdash$ "$\dot{X} \cap \alpha^\dagger = \check{Y}$". Then $a^p {}^\frown I \Vdash$ "$\dot{X} \cap \alpha^\dagger = \check{Y}$".*

Proof. Suppose not, and let $\bar{q} \leq a^p {}^\frown I$ be such that, for some $\bar{Y} \neq Y$, $\bar{q} \Vdash$ "$\dot{X} \cap \alpha^\dagger = \check{\bar{Y}}$". Without loss of generality, we may assume that $\max(a^{\bar{q}}) = \alpha$. Let $q \leq p$ such that the order structure of a^q and a^p is the same as that of $a^{\bar{q}}$ and a^p. Then, by indiscernibility and the fact that $F(a^q) \cap (\alpha^\dagger + 1) = F(a^p) \cap (\alpha^\dagger + 1)$, it must be the case that $F(a^{\bar{q}}) \cap (\alpha^\dagger + 1) = F(a^p) \cap (\alpha^\dagger + 1)$. But $(\alpha^\dagger, \bar{Y}) \in F(a^{\bar{q}}) \cap (\alpha^\dagger + 1)$, while $(\alpha^\dagger, Y) \in F(a^p) \cap (\alpha^\dagger + 1)$. This is a contradiction. $\square$

Lemma 6.9. *For every $b \in \lambda^{<\omega}$, there is $b' \sqsupseteq b$ such that $f_{b'}$"$\mathrm{bas}_{b'} \subseteq \bar{Z}$.*

Proof. If $\eta < \omega^\lambda$ is such that $\eta \notin \bar{Z}$, let $\delta_\eta = \max(\bar{Z} \cap \eta)$.

Claim 6.10. *Suppose* $\eta < \omega^\lambda$. *If* $\eta \notin \bar{Z}$ $n < \omega$, $b \in \lambda^n$, *and there are* $\ell < m < n$ *such that* $f_b(m) = \eta$ *and* $f_b(\ell) \in (\delta_\eta, \eta)$. *Then* $m \notin \mathrm{bas}_b$.

Proof. We proceed by induction on η. Let $\eta < \omega^\lambda$ be such that $\eta \notin \bar{Z}$ and suppose we have proven the claim for all $\eta' < \eta$. Let $n < \omega$ and $b \in \lambda^n$ be such that there are $\ell < m < n$ with $f_b(m) = \eta$ and $f_b(\ell) \in (\delta_\eta, \eta)$, and suppose for sake of contradiction that $m \in \mathrm{bas}_b$. By Proposition 6.6 and the fact that $\eta \notin Z$, there is $b' \sqsupseteq b$ such that $f_{b,b'}(m) \notin \mathrm{bas}_{b'}$ but there is k such that $f_{b,b'}(\ell) < k < f_{b,b'}(m)$ and $k \in \mathrm{bas}_{b'}$. Let $\xi = f_{b'}(k)$ and note that $\delta_\eta < \xi < \eta$. Thus, $\xi \notin \bar{Z}$ and $\delta_\xi = \delta_\eta$. But $f_{b'}(f_{b,b'}(\ell)) \in (\delta_\xi, \xi)$, contradicting the inductive hypothesis. $\square$

Now fix $b \in \lambda^{<\omega}$. We will find $b' \sqsupseteq b$ such that $f_{b'}$"$\mathrm{bas}_{b'} \subseteq \bar{Z}$. We will do this by recursively defining a sequence $\langle b_i \mid i < \omega\rangle$ such that if $i < j$, then $b_i \sqsubseteq b_j$ and $b_i \in \lambda^{n_i}$. If during the recursive construction we reach i such that f_{b_i}"$\mathrm{bas}_{b_i} \subseteq \bar{Z}$, then the process halts and we let $b' = b_i$. For each $i < \omega$ such that b_i is not as desired, we assign an ordinal ν_i, where ν_i is the largest ν such that, for some $m < n_i$, we have $b_i(m) = \nu$, $f_{b_i}(m) \notin Z$, and there is no $k < m$ such that $f_{b_i}(k) \in (\delta_{f_{b_i}(m)}, f_{b_i}(m))$.

If b_i has been constructed and the process has not halted, we let $b_{i+1} \sqsupseteq b_i$ be such that, for every $m < n_i$ with $b_i(m) = \nu_i$, there is $k < n_{i+1}$ such that $f_{b_{i+1}}(k) \in (\delta_{f_{b_i}(m)}, f_{b_i}(m))$. We moreover require that, for every $k \in n_{i+1} \setminus \mathrm{ran}(f_{b_i,b_{i+1}})$, $b_{i+1}(k) < \nu_i$. This ensures that, if the process does not halt at stage $i+1$, then $\nu_{i+1} < \nu_i$. Now suppose that the process never stops. Then $\langle \nu_i \mid i < \omega\rangle$ forms a strictly decreasing sequence of ordinals, which is a contradiction. Thus, there is $i < \omega$ such that f_{b_i}"$\mathrm{bas}_{b_i} \subseteq \bar{Z}$. $\square$

Lemma 6.11. $\langle(\kappa, I)\rangle \Vdash$ "$\dot{X} \leq_V \dot{\bar{d}}$".

Proof. Let G be $\mathbb{P}_{\vec{U}}$-generic over V with $\langle(\kappa, I)\rangle \in G$. We will show how to recover X given $\bar{d}$. If $b \in \lambda^{<\omega}$ and $x \in [\kappa]^b$, we say x is *compatible with* $\bar{d}$ if, for every $\eta \in \mathrm{ran}(f_b) \cap \bar{Z}$, we have, letting $k_\eta = f_b^{-1}(\eta)$, $x(k_\eta) = \bar{d}(\eta)$. Let $[\bar{d}]^b$ denote the set of $x \in [I]^b$ such that x is compatible with $\bar{d}$. Let $X_b = \bigcup_{x \in [\bar{d}]^b} F(x)$, and let $X' = \{(\alpha, Y) \mid$ for some $b^* \in \lambda^{<\omega}$, for all $b \in \lambda^{<\omega}$ such that $b^* \sqsubseteq b$, we have $(\alpha, Y) \in X_b\}$. Clearly, $X' \in V[\bar{d}]$. We claim that, for all $\alpha < \kappa$, $X \cap \alpha = Y$ if and only if $(\alpha, Y) \in X'$.

First, suppose that $X \cap \alpha = Y$. Let $p^* \in G$ be such that $p^* \Vdash$ "$\dot{X} \cap \check{\alpha} = \check{Y}$". Let $b^* = \mathrm{tp}(a^{p^*})$. We claim that b^* witnesses that $(\alpha, Y) \in X'$. To see this, let $b \sqsupseteq b^*$. Let $p \leq p^*$ be such that $p \in G$ and $\mathrm{tp}(p) = b$. Then $(\alpha, Y) \in F(a^p) \subseteq X_b$, so $(\alpha, Y) \in X'$.

Next, suppose that $(\alpha, Y) \in X'$ and suppose for sake of contradiction that $X \cap \alpha = Y' \neq Y$. Fix b^* witnessing that $(\alpha, Y) \in X'$, and let $p \in G$ be such that $p \Vdash$ "$\dot{X} \cap \check{\alpha} = \check{Y}'$". Without loss of generality, $b = \mathrm{tp}(a^p) \sqsupseteq b^*$. By Lemma 6.9, there is $b' \sqsupseteq b$ such that $f_{b'}$"$\mathrm{bas}_{b'} \subseteq \bar{Z}$. $(\alpha, Y) \in X_{b'}$, so there is $x \in [I]^{b'}$ compatible with $\bar{d}$ such that $(\alpha, Y) \in F(x)$. Let $q \leq p$ be such that $q \in G$ and $\mathrm{tp}(a^q) = b'$. Then $(\alpha, Y) \notin F(a^q)$, but, since x is compatible with $\bar{d}$ and $f_{b'}$"$\mathrm{bas}_{b'} \subseteq \bar{Z}$, we have $a^q \restriction \mathrm{bas}_{b'} = x \restriction \mathrm{bas}_{b'}$. This is a contradiction. Thus, for all $\alpha < \kappa$, $X \cap \alpha = Y$ if and only if $(\alpha, Y) \in X'$, so $X \in V[d]$. $\square$

Lemma 6.12. $\langle(\kappa, I)\rangle \Vdash$ "$\dot{d} \leq_V \dot{X}$".

Proof. Let G be $\mathbb{P}_{\vec{U}}$-generic over V with $\langle(\kappa, I)\rangle \in G$. We want to recover d from X. Let W be the set of $x \in [I]^{<\omega}$ such that $o(\max(x)) > 0$ and $(\max(x)^\dagger, X \cap \max(x)^\dagger) \in F(x)$. Clearly, $W \in V[X]$. Also note that, for every $b \in \lambda^{<\omega}$, there is $x \in W$ such that $b \sqsubseteq \mathrm{tp}(x)$. To see this, let $b \in \lambda^{<\omega}$ and find $p \in G$ such that $b \sqsubseteq \mathrm{tp}(a^p)$. Without loss of generality, $o(\max(a^p)) > 0$. Find $q \leq p$ with $q \in G$ such that $\max(a^q) = \max(a^p)$ and q decides the value of $\dot{X} \cap \max(a^q)^\dagger$. Then a^q is as desired.

Claim 6.13. *Suppose $b \in \lambda^{<\omega}$, $x, y \in [I]^b$, and $x, y \in W$. Then $F(x) = F(y)$.*

Proof. First, suppose $\max(x) = \max(y) = \alpha$ for some $\alpha < \kappa$. By assumption, $F(x) \cap \alpha^\dagger = F(y) \cap \alpha^\dagger$, so, by indiscernibility, $F(x) \cap \beta = F(y) \cap \beta$ for all $\beta \in I \cap A^*_{o(\alpha^\dagger)}$. Thus, $F(x) = F(y)$.

Therefore, we may assume that $\max(x) < \max(y)$. Let $\alpha = \max(x)$ and $\beta = \max(y)$, and let $n < \omega$ be such that $b \in \lambda^n$. We also may assume that $\mathrm{bas}_b \neq \emptyset$, since otherwise $F(x) = F(y)$ would trivially be true. Note that, since $x, y \in W$ and $\alpha < \beta$, it must be the case that $F(x) \cap \alpha^\dagger = F(y) \cap \alpha^\dagger$. The proof now splits into two cases.

Case 1: $n - 1 \notin \mathrm{bas}_b$**.** Let $z \in [I]^b$ be such that $z(n-1) = \alpha$ and, for all other $k < n$, $z(k) = x(k)$. Then $z \restriction \mathrm{bas}_b = x \restriction \mathrm{bas}_b$, so $F(z) = F(x)$. Let $\gamma \in (I \cap A^*_{b(n-1)}) \setminus (\beta + 1)$. Let y' be defined by $y'(n-1) = \gamma$ and $y'(k) = y(k)$ for all other $k < n$, and let z' be similarly defined in relation to z. Then $F(x) = F(z')$ and $F(y) = F(y')$. We also know that $F(z) \cap \alpha^\dagger = F(y) \cap \alpha^\dagger$, so, by indiscernibility, $F(z') \cap \beta^\dagger = F(y') \cap \beta^\dagger$. Thus, by the first paragraph of this proof, $F(z) = F(y)$, so $F(x) = F(y)$.

Case 2: $n - 1 \in \mathrm{bas}_b$**.**

Subcase 2a: $\mathrm{bas}_b = \{n-1\}$**.** By suitably restricing our measure one set, we can in fact ensure that every indiscernible is in fact a limit point of $(I \cap A^*_\eta)$ for every $\eta < \lambda$. Thus, by shifting around the elements of x

and y, we can assume that $\alpha^\dagger < \min(y)$. Now let $z \in [I]^b$ be such that $\beta^\dagger < \min(z)$. Since $F(y) \cap \alpha^\dagger = F(x) \cap \alpha^\dagger$, we have, by indiscernibility, $F(z) \cap \beta^\dagger = F(x) \cap \beta^\dagger$ and $F(z) \cap \beta^\dagger = F(y) \cap \beta^\dagger$. Thus, $F(x) \cap \beta^\dagger = F(y) \cap \beta^\dagger$, so, again by the first paragraph of this proof, $F(x) = F(y)$.

Subcase 2b: bas_b **contains more than one element.** Let $u = x \restriction \mathrm{bas}_b$ and $v = y \restriction \mathrm{bas}_b$. There are now two sub-subcases to finish the proof of the claim.

Sub-subcase 2bi: $u \restriction (\mathrm{bas}_b \setminus \{n-1\}) = v \restriction (\mathrm{bas}_b \setminus \{n-1\})$**.** Form z as before by letting $z(n-1) = \beta$ and $z(k) = x(k)$ for all other $k < n$. Then $z \restriction \mathrm{bas}_b = y \restriction \mathrm{bas}_b$, so $F(z) = F(y)$, so $F(z) \cap \alpha^\dagger = F(x) \cap \alpha^\dagger$. Let $w = x \restriction (n-1)$. We may assume that $\alpha^\dagger < \beta$, so, since $F(w^\frown\langle n-1, \alpha\rangle) \cap \alpha^\dagger = F(w^\frown\langle n-1, \beta\rangle) \cap \alpha^\dagger$, so, if $\gamma < \nu$ are such that $\beta < \gamma$, $\gamma \in I \cap A^*_0$, and $\nu \in I \cap A^*_{b(n-1)}$, then, by indiscernibility, $F(w^\frown\langle n-1, \alpha\rangle) \cap \gamma = F(w^\frown\langle n-1, \nu\rangle) \cap \gamma = F(w^\frown(n-1, \beta\rangle) \cap \gamma$, so $F(x) = F(z)$ and therefore $F(x) = F(y)$.

Sub-subcase 2bii: $u \restriction (\mathrm{bas}_b \setminus \{n-1\}) \neq v \restriction (\mathrm{bas}_b \setminus \{n-1\})$**.** Find $\gamma \in \mathrm{ran}(u) \setminus \mathrm{ran}(v)$ with $\gamma < \alpha$. Let k be such that $x(k) = \gamma$. Let $\delta = \min((I \cap A^*_{b(k)}) \setminus (\gamma+1))$. We may assume that $\delta < x(k+1)$, $\delta \notin \mathrm{ran}(y)$, and, if $\gamma = y(i)$ for some $i < n$, then $\delta < y(i+1)$. Let x' be such that $x'(k) = \delta$ and $x'(i) = x(i)$ for all other $i < n$. If $y(i) = \gamma$ for some $i < n$, let y' be such that $y'(i) = \delta$ and $y'(\ell) = y(\ell)$ for all other $\ell < n$. If $\gamma \notin \mathrm{ran}(y)$, let $y' = y$. Note that $x' \restriction \mathrm{bas}_b \neq x \restriction \mathrm{bas}_b$, so $F(x) \neq F(x')$. In particular, $F(x) \cap \alpha^\dagger \neq F(x') \cap \alpha^\dagger$. However, $y' \restriction \mathrm{bas}_b = y \restriction \mathrm{bas}_b$, so $F(y) = F(y')$. Also, the order structure of x, y, and $\alpha^\dagger$ is the same as that of x', y', and $\alpha^\dagger$, so $F(x) \cap \alpha^\dagger = F(y) \cap \alpha^\dagger$ and $F(x') \cap \alpha^\dagger = F(y') \cap \alpha^\dagger$ should hold or fail simultaneously. This is a contradiction, since the first equality holds by assumption and the second fails. This finishes Case 2 and hence the proof of Claim 6.13. □

We now show how to recover d from W. Let $\eta \in Z$, and let b^* be such that, for all $b \sqsupseteq b^*$, $\eta \in f_b$"bas_b. Find $x \in W$ such that $\mathrm{tp}(x) = b \sqsupseteq b^*$. By the preceding claim and the remark immediately preceding the claim, $x(f_b^{-1}(\eta))$ is independent of our choice of x and is equal to $d(\eta)$. This finishes the proof of the Lemma and hence of Theorem 6.7. □

6.4 Subsets of larger cardinals

The proof of the following theorem is a straightforward adaptation of the analogous theorem from [17]. We provide a sketch for completeness.

Theorem 6.14. *Suppose that G is $\mathbb{P}_{\vec{U}}$-generic over V and, for every $X \in V[G]$ such that $X \subseteq \kappa$, there is $d \subseteq g$ such that $X \equiv_V d$. Then for every set of ordinals $Y \in V[G]$, there is $d \subseteq g$ such that $Y \equiv_V d$.*

Proof. We proceed by induction on μ, where μ is the least cardinal such that $Y \subseteq \mu$. We are assuming the theorem holds if $\mu \leq \kappa$.

First, suppose $\delta = \mathrm{cf}(\mu) \leq \kappa$. Let $\langle \mu_\xi \mid \xi < \delta \rangle$ be an increasing sequence of cardinals cofinal in μ. By the inductive hypothesis, we have that, in $V[Y]$, for every $\xi < \delta$, there is $Y_\xi \subseteq \kappa$ such that $(X \cap \mu_\xi) \equiv_V Y_\xi$. Thus, there are $z_\xi \in V$ and ordinals α_ξ and β_ξ such that Y_ξ is the α_ξ-th element in the canonical well-ordering of $L[p_\xi, Y \cap \mu_\xi]$ and $Y \cap \mu_\xi$ is the β_ξ-th element in the canonical well-ordering of $L[p_\xi, Y_\xi]$. Since $\mathbb{P}_{\vec{U}}$ has the κ^+-c.c. and $\delta \leq \kappa$, there are sets A, B, and P in V, all of cardinality κ, such that, for every $\xi < \delta$, there are α_ξ, β_ξ, and p_ξ as above in A, B, and P, respectively. The map $\xi \mapsto (a_\xi, b_\xi, p_\xi)$ can then be coded as a subset X of κ. It is then clear that $X \in V[Y]$ and $Y \in V[X]$.

Now suppose $\delta = \mathrm{cf}(\mu) > \kappa$. By the inductive hypothesis, we have that, in $V[G]$, for every $\xi < \mu$, there is $d_\xi \subseteq g$ such that $(Y \cap \xi) \equiv_V d_\xi$. Without loss of generality, we may assume that each d_ξ is closed and that, if $\xi < \xi' < \mu$, then $d_\xi \subseteq d_{\xi'}$. Since $\delta > \kappa$, there is a closed $d \subseteq g$ such that, for all sufficiently large $\xi < \lambda$, $(Y \cap \xi) \equiv_V d$. Clearly, $d \in V[Y]$. We claim that $Y \in V[d]$.

Let $\mathbb{P}_{\vec{U}}/d$ be the quotient forcing. We have the following lemma. The proof is essentially the same as that of the corresponding lemma in [17], so we only give a sketch.

Lemma 6.15. *In $V[G]$, $\mathbb{P}_{\vec{U}}/d$ has the κ^+-c.c.*

Proof. First suppose that d is bounded below κ. Then, for some $\alpha < \kappa$, $\mathbb{P}_{\vec{U}}/d = (\mathbb{P}_{\vec{U} \restriction \alpha}/d) \times (\mathbb{P}_{\vec{U}}/\langle (\kappa, \kappa \setminus \alpha) \rangle)$. Since the first poset in this product is small and the second has the κ^+-c.c., $\mathbb{P}_{\vec{U}}/d$ itself has the κ^+-c.c.

Now suppose d is unbounded in g. Think of g as an increasing function from ω^λ to κ, and let $C \subseteq \omega^\lambda$ be a club such that $d = g \restriction C$. Let $\mathbb{Q}$ be the set of $p \in \mathbb{P}_{\vec{U}}$ such that, letting $p = \langle (\alpha_k, A_k) \mid k \leq m \rangle$ and $b = \mathrm{tp}(a^p)$, we have:

- For all $k < m$ such that $f_b(k) \in C$, $\alpha_k = g(f_b(k))$.
- $f_b(m-1) \in C$.
- For all $k < m$ such that $f_b(k) \in C'$, it is the case that either $k = 0$ and $g \cap \alpha_k \subseteq A_k$ or $k > 0$, $f_b(k-1) \in C$, and $g \cap (\alpha_{k-1}, \alpha_k) \subseteq A_k$.

- $g \cap (\alpha_{m-1}, \kappa) \subseteq A_m$.
- For all $k < m$ such that $f_b(k) \notin C'$, it is the case that either $k = 0$ or $k > 0$ and $C \cap (f_b(k-1), f_b(k)) = \emptyset$.

Intuitively, the elements of $\mathbb{Q}$ are the conditions $p \in \mathbb{P}_{\vec{U}}$ for which it is clear that there is a $\mathbb{P}_{\vec{U}}$-generic filter, H, with associated club h, such that $p \in H$ and $h \restriction C = d$. It is easily verified as in [17] that $\mathbb{Q}$ is a dense subset of $\mathbb{P}_{\vec{U}}/d$ and that if $p, q \in \mathbb{Q}$ are such that $a^p = a^q$, then a common lower bound in $\mathbb{Q}$ can be found by intersecting the measure-one sets. Thus, $\mathbb{P}_{\vec{U}}/d$ has the κ^+-c.c. in $V[G]$ (and hence in $V[d]$ as well). $\square$

Now $Y \in V[G] = V[d][G]$, and $V[d][G]$ is an extension of $V[d]$ by κ^+-c.c. forcing. Let $\dot{Y} \in V[d]$ be a $\mathbb{P}_{\vec{U}}/d$-name for Y. We know that, for every $\xi < \mu$, $Y \cap \xi \in V[d]$. Let Q_ξ be the set of all conditions $q \in \mathbb{P}_{\vec{U}}/d$ such that $q \Vdash$ "$\dot{Y} \cap \check{\xi} = \check{y}_\xi$", where $y_\xi = Y \cap \xi$. Each Q_ξ is nonempty and, if $\xi < \xi'$, then $Q_{\xi'} \subseteq Q_\xi$. First, suppose there is $q \in \bigcap_{\xi < \mu} Q_\xi$. Then, for every $\xi < \mu$, $q \Vdash$ "$\dot{Y} \cap \check{\xi} = \check{y}_\xi$", so $Y \in V[d]$.

Thus, suppose that $\bigcap_{\xi < \mu} Q_\xi = \emptyset$. Then $Z = \{\xi < \mu \mid Q_\xi \setminus Q_{\xi+1} \neq \emptyset$ is unbounded in μ. For each $\xi \in Z$, let $q_\xi \in Q_\xi \setminus Q_{\xi+1}$. Since $q_\xi \nVdash$ "$\dot{Y} \cap \check{\xi} + 1 = \check{y}_{\xi+1}$", we can find $p_\xi \leq q_\xi$ such that $p_\xi \Vdash$ "$\dot{Y} \cap \check{\xi} + 1 \neq \check{y}_{\xi+1}$". Then $\{p_\xi \mid \xi \in Z\} \in V[G]$ is an antichain in $\mathbb{P}_{\vec{U}}/d$. Since $|Z| > \kappa$, this is a contradiction. $\square$

6.5 Products of Radin forcing

Let $\vec{U}_0$ and $\vec{U}_1$ be coherent sequences of measures with $\mathrm{lh}(\vec{U}_0) = \kappa_0$ and $\mathrm{lh}(\vec{U}_1) = \kappa_1$, where $\kappa_0 < \kappa_1$. Suppose we have proven that, for $i < 2$, if G_i is $\mathbb{P}_{\vec{U}_i}$-generic over V and g_i is the associated generic club, then, for every set of ordinals $X \in V[G_i]$, there is $d_i \subseteq g_i$ such that $X \equiv_V d_i$. Let $\mathbb{P} = \mathbb{P}_{\vec{U}_0}$ and $\mathbb{Q} = \{q \in \mathbb{P}_{\vec{U}_1} \mid a^q \subseteq \kappa_1 \setminus (\kappa_0 + 1)\}$.

Let G be $\mathbb{P}$-generic over V. Then, for every β such that $\kappa_0 < \mu \leq \kappa_1$ and every $i < o^{\vec{U}_1}(\beta)$, define $U_1^*(\beta, i)$ in $V[G]$ to be the collection of sets $A \subseteq \beta$ such that there is $B \in U_1(\beta, i)$ such that $B \subseteq A$. Then $U_1^*(\beta, i)$ is a β-complete normal ultrafilter over β extending $U(\beta, i)$. Let $\vec{U}_1^* = \langle U_1^*(\beta, i) \mid \kappa_0 < \beta \leq \kappa_1$ and $i < o^{\vec{U}_1}(\beta)\rangle$. Then a club $h \subseteq \kappa_1$ generates a $\mathbb{Q}$-generic filter over V if and only if generates a $\mathbb{P}_{\vec{U}_1^*}$-generic filter over $V[G]$.

Similarly, let H be $\mathbb{Q}$-generic over V. Since forcing with $\mathbb{Q}$ does not change $\mathcal{P}(\kappa_0)$, $\mathbb{P}$ remains Radin forcing in $V[H]$, and G is $\mathbb{P}$-generic over V if and only if it is $\mathbb{P}$-generic over $V[H]$.

Now let G be $\mathbb{P}$-generic over V and let H be $\mathbb{Q}$-generic over V, with g and h the respective clubs. Let X be a set of ordinals in $V[g \times h]$. By assumption, there is $d_h \subseteq h$ such that $X \equiv_{V[G]} d_h$. There is then a $d_g \subseteq g$ such that $X \equiv_{V[d_h]} d_g$. Going another step, there is $d_h^* \subseteq d_h$ such that $X \equiv_{V[d_g]} d_h^*$. We claim that we can in fact let $d_h^* = d_h$. Suppose not, so that $d_h \setminus d_h^*$ is infinite. Then, since $X \in V[g \times d_h^*]$ and $d_h \in V[g \times X]$, we have $d_h \in V[g \times d_h^*]$, which contradicts the assumption that $d_h \setminus d_h^*$ is infinite. Thus, from any two of d_g, d_h, and X, we can recover the third. This shows that $X \in V[d_g \cup d_h]$. We now show that $d_g \cup d_h \in V[X]$.

Suppose g' and h' are two Radin-generic clubs such that $X \in V[g' \times h']$. Then, as above, there are $d_{g'}$ and $d_{h'}$ such that any of X, $d_{g'}$, and $d_{h'}$ can be recovered from the other two. Thus, any of d_g, d_h, $d_{g'}$, and $d_{h'}$ can be recovered from the other three. Therefore, $d_g =^* d_{g'}$ and $d_h =^* d_{h'}$, so $d_g \cup d_h \in V[g' \times h']$. Thus, $d_g \cup d_h$ is in every $(\mathbb{P} \times \mathbb{Q})/X$-generic extension of $V[X]$. Thus, by the product lemma, $d_g \cup d_h \in V[X]$, so we in fact have $X \equiv_V d_g \cup d_h$.

6.6 Coding arbitrary subsets of κ

Let $\vec{U}$ be a coherent sequence of measures with $\mathrm{lh}(\vec{U}) = \kappa + 1$ and $o(\kappa) = \lambda < \kappa$. Let $\mathbb{P} = \mathbb{P}_{\vec{U}}$, let G be $\mathbb{P}$-generic over V, let g be the club in κ derived from G, and let $X \in V[G]$ be a subset of κ. We would like to show that there is $d \subseteq g$ such that $X \equiv_V d$. We proceed by induction on λ, with the base case, $\lambda = 1$, being covered in [17]. The proof breaks into two cases.

Case 1: There is $\beta < \kappa$ such that, for all $\alpha < \kappa$, $X \cap \alpha \in V[g \restriction \beta]$.

The desired conclusion in this case will follow from the results of the previous three sections. Let γ be a limit point of g such that $\gamma > \beta$, and let $p = \langle (\alpha_k, A_k) \mid k \leq m \rangle \in G$ with $\gamma = \alpha_k$ for some $k \leq m$. Below p, $\mathbb{P}$ can be thought of as a product $\mathbb{P}_{\vec{U} \restriction (\gamma+1)}/p^{\leq k} \times \mathbb{P}/p^{>k}$. By the inductive hypothesis and Theorem 6.7 respectively, there are $d_0 \subseteq g \restriction \gamma$ and $d_1 \subseteq g \restriction [\gamma, \kappa)$ such that any of d_0, d_1, and X is recoverable from the other two. Then, by the arguments regarding products of Radin forcings from the previous section, $X \equiv_V (d_0 \cup d_1)$.

Case 2: For all $\beta < \kappa$, there is $\alpha < \kappa$ such that $X \cap \alpha \notin V[g \restriction \beta]$.

This case requires some more work. First, let us fix a $\mathbb{P}$-name $\dot{X}$ for X. We may assume that $\dot{X}$ has the property that, for every $\beta < \kappa$ with $o(\beta) > 0$, for every $p \in \mathbb{P}$ with $\beta \in a^p$, there is a $\mathbb{P}_{\vec{U} \restriction (\beta+1)}$-name $\dot{X}_{\beta,p}$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{X}_{\beta,p} = \dot{X} \cap \beta$".

Let $b = \{\gamma < \kappa \mid o(\gamma) > 0$ and γ is singular in $V[X]\}$. It is immediate that $b \subseteq g$ and $b \in V[X]$. b is unbounded in κ, since, if there were $\delta < \kappa$ such that $b \subseteq \delta$, then an easy argument shows that every initial segment of X lies in $V[g \restriction \delta]$.

For every $\gamma \in b$, let $d_\gamma \subseteq g \restriction \gamma$ be such that $d_\gamma \equiv_V X \restriction \gamma$. We may assume that, if $\gamma < \gamma'$, then $d_\gamma \subseteq d_{\gamma'}$ and that, since $\mathcal{P}(\lambda)^V = \mathcal{P}(\lambda)^{V[G]}$, that the sequence $\langle d_\gamma \mid \gamma \in b\rangle \in V[X]$. Thus, letting $d^* = \bigcup_{\gamma \in b} d_\gamma$, we have that d^* is an unbounded subset of g (and in fact the limit points of d^* form an unbounded subset of g), $d^* \in V[X]$, and, for every $\alpha < \kappa$, $X \cap \alpha \in V[d^*]$.

Let $\dot{d}^*$ be a $\mathbb{P}$-name for d^*. Let $c \subseteq \omega^\lambda$ be such that $d^* = g \restriction c$. $c \in V$, and, by the remarks at the beginning of Section 6.3, we may assume without loss of generality that $\Vdash_{\mathbb{P}}$ "$\dot{d}^* = \dot{g} \restriction \check{c}$".

For every $\eta \in c'$, it is forced that there is a set of ordinals $z \in V$ such that $\dot{X} \cap \dot{g}(\eta) \in L[z, \dot{d}^* \restriction (c \cap \dot{g}(\eta))]$. Let χ be a sufficiently large regular cardinal such that, for every $\eta \in c'$ it is forced that there is such a set z in $H(\chi)$, and let $<_\chi$ be a well-ordering of $H(\chi)$.

Define a function F with domain $[\kappa]^{<\omega}$ by letting $F(x) = (\max(x), \alpha, z)$ if the following hold:

- If $\mathrm{tp}(x) = b \in \lambda^n$, then $f_b(n-1) \in c'$.
- α is an ordinal.
- $z \in V$ is a set of ordinals.
- There is $p \in \mathbb{P}$ such that $a^p = x$ and $p \Vdash$ "$\check{z}$ is the $<_\chi$ -least set of ordinals such that $\dot{X} \cap \max(x) \in L[z, \dot{d}^* \restriction (c \cap \max(x))]$".
- $p \Vdash$ "$\dot{X} \cap \max(x)$ is the α^{th} element in the canonical well-ordering of $L[z, \dot{d}^* \restriction (c \cap \max(x))]$".

Let $F(x) = \emptyset$ otherwise. As before, apply Lemma 6.4 to F to attain sets $J_b(\emptyset)$ and $\mathrm{bas}_b(\emptyset)$ for $b \in \lambda^{<\omega}$. Let $J = \bigcap_{b \in \lambda^{<\omega}} J_b(\emptyset)$ and, for $b \in \lambda^{<\omega}$, let bas_b denote $\mathrm{bas}_b(\emptyset)$. The following is proven in the same way as Proposition 6.6.

Proposition 6.16. *Let $n < \omega$, $a \in \lambda^n$, and $b \in \lambda^{n+1}$ be such that $a \sqsubseteq b$ and $f_a(n-1) = f_b(n) \in c'$. Suppose that $i < n$ is such that $i \notin \mathrm{ran}(f_{a,b})$ and that $k \in \mathrm{bas}_a$. Then one of the following holds:*

1. $f_{a,b}(k) \in \mathrm{bas}_b$.
2. $k = i$ *and* $i \in \mathrm{bas}_b$.

Let θ be a sufficiently large regular cardinal, and let $\mathcal{F}$ be the collection of sets in $H(\theta)$ definable using only $\mathbb{P}$, $\dot{X}$, J, $\{\mathrm{bas}_b \mid b \in \lambda^{<\omega}\}$, c, and ordinals less than λ as parameters. Apply Proposition 6.3 to find $I \subseteq J$ such that $I \in \bigcap \vec{U}_\kappa$ and I is indiscernible for all sets in $\mathcal{F}$.

For $\eta \in c'$, let B_η be the set of $b \in \lambda^{<\omega}$ such that $f_b(|x|-1) = \eta$. Let $c_\eta = \{\xi < \eta \mid$there is $b^* \in B_\eta$ such that, for all $b \in B_\eta$ such that $b^* \sqsubseteq b$, we have $\xi \in f_b$"$\mathrm{bas}_b\}$. Let $c^* = c \cup \bigcup_{\eta \in c'} c_\eta$, and let $d = g \restriction c^*$, and let $\dot{d}$ be the canonical name for d.

Lemma 6.17. $\langle(\kappa, I)\rangle \Vdash$ "$\dot{X} \equiv_V \dot{d}$".

Proof. Let G be $\mathbb{P}$-generic over V with $\langle(\kappa, I)\rangle \in G$. We first show how to recover X from d. We have already shown that all initial segments of X are in $V[d]$. We now show how, given $\eta \in c'$, to recover $X \cap d(\eta)$.

For $b \in B_\eta$, let $X_b = \{F(x) \mid x \in [d]^b\}$, and let $X'_\eta = \{(d(\eta), \alpha, z) \mid$ for some $b^* \in B_\eta$, for all $b \in B_\eta$ such that $b^* \sqsubseteq b$, we have $(d(\eta), \alpha, z) \in X_b\}$.

We first show that there is a $(d(\eta), \alpha, z) \in X'_\eta$ such that $X \cap d(\eta)$ is the α^{th} element in the canonical well-ordering of $L[z, d \restriction (c \cap d(\eta))]$. To this end, let $p \in G$ be such that $\mathrm{tp}(a^p) \in B_\eta$ and such that there are α and z such that p witnesses that $(d(\eta), \alpha, z) \in F(a^p)$. Then X is in fact the α^{th} element in the canonical well-ordering of $L[z, d \restriction (c \cap d(\eta))]$, and we claim that $\mathrm{tp}(a^p)$ witnesses that $(d(\eta), \alpha, z) \in X'_\eta$. To see this, let $b \in B_\eta$ be such that $b \sqsupseteq \mathrm{tp}(a^p)$, and let $q \in G$ be such that $q \leq p$ and $\mathrm{tp}(a^q) = b$. Then a^q witnesses that $(d(\eta), \alpha, z) \in X_b$.

Next, suppose that $(d(\eta), \alpha, z) \in X'$ and suppose for sake of contradiction that $X \cap d(\eta)$ is not the α^{th} element in the canonical well-ordering of $L[z, d \restriction (c \cap d(\eta))]$. Fix b^* witnessing that $(d(\eta), \alpha, z) \in X'$, and let $p \in G$ force that $X \cap \eta$ is not the α^{th} element in the canonical well-ordering of $L[z, d \restriction (c \cap d(\eta))]$. Without loss of generality, $b = \mathrm{tp}(a^p) \sqsupseteq b^*$. As in Lemma 6.9, we can find $b' \sqsupseteq b$ such that $f_{b'}$"$\mathrm{bas}_{b'} \subseteq c^*$. Find $x \in [I]^{b'}$ such that $F(x) = (d(\eta), \alpha, z)$. Also, let $q \leq p$ be such that $q \in G$ and $\mathrm{tp}(a^q) = b'$. Then $a^q \restriction \mathrm{bas}_{b'} = x \restriction \mathrm{bas}_{b'}$, but $F(a^q) \neq (d(\eta), \alpha, z)$. This is a contradiction.

Therefore, given d, we can recover X by, for each $\eta \in c'$, finding a $(d(\eta), \alpha, z) \in X'_\eta$ and letting $X \cap d(\eta)$ equal the α^{th} element in the canonical well-ordering of $L[z, d \restriction (c \cap d(\eta))]$.

We now show how to recover d from X. Since we have already shown that $d^* = d \restriction c \in V[X]$, it suffices to recover d from X and d^*. For $\eta \in c'$, let z_η be the $<_\chi$-least set of ordinals such that $X \cap d(\eta) \in L[z, d \restriction (c \cap$

$d(\eta))]$, and let α_η be such that X is the α_η^{th} element in the canonical well-ordering of $L[z, d \upharpoonright (c \cap d(\eta))]$. Let $W_\eta = \{x \in \kappa^{<\omega} \mid \mathrm{tp}(x) \in B_\eta$ and $F(x) = (d(\eta), \alpha_\eta, z_\eta)\}$.

Let $p \in G$ be such that $\max(a^p) = d(\eta)$ and p decides the values of z_η and α_η. Let $b' = \mathrm{tp}(a^p)$. Then, for every $b \in B_\eta$ with $b \sqsupseteq b'$, by considering $q \leq p$ with $\mathrm{tp}(a^q) = b'$, it follows that $g \upharpoonright \mathrm{ran}(f_b) \in W_\eta$. Moreover, it is immediate from the definitions that, if $x, x' \in W_\eta$ and $\mathrm{tp}(x) = b = \mathrm{tp}(x')$, then $x \upharpoonright \mathrm{bas}_b = x' \upharpoonright \mathrm{bas}_b$. Thus, given $\xi \in c_\eta$, we can determine $d(\xi)$ by finding $b^* \in B_\eta$ such that $b^* \sqsupseteq b'$ such that, for all $b \in B_\eta$ with $b \sqsupseteq b^*$, $\xi \in f_b``\mathrm{bas}_b$, finding $x \in W_\eta$ such that $\mathrm{tp}(x) = b$, and letting $d(\xi) = x(f_b^{-1}(\xi))$. This allows us to recover d from X and d^*, thus finishing the proof.

□

Putting all of this together, we can now obtain a complete proof of Theorem 6.5.

Moreover, following Gitik, Kanovei, and Koepke, we can characterize the structure of the collection of intermediate submodels of $V[G]$. It should be clear that, if $c, d \subseteq g$ are such that the closures of c and d differ by only finitely many elements, then $V[c] = V[d]$. In fact, this is the only case in which $V[c] = V[d]$.

Theorem 6.18. *Suppose that G is $\mathbb{P}_{\vec{U}}$-generic over V and g is the club in κ derived from G. Suppose $c, d \subseteq g$ are closed and $c \setminus d$ is infinite. Then $c \notin V[d]$.*

Proof. Think of g as being an increasing function from ω^λ to κ. Let $C, D \subseteq \omega^\lambda$ be such that $c = g \upharpoonright C$ and $d = g \upharpoonright D$. Notice that $C, D \in V$. Suppose for sake of contradiction that $c \in V[d]$. Then there an ordinal α and a set $z \in V$ such that $g \upharpoonright C$ is the α^{th} element in the canonical well-ordering of $L[z, g \upharpoonright D]$. Let $p = \langle (\alpha_k, A_k) \mid k \leq m \rangle \in G$ force this to be true. Let η be the least element of $C \setminus D$ not in the range of $f_{\mathrm{tp}(a^p)}$. By extending p if necessary, we may assume that there is $b \sqsupseteq \mathrm{tp}(a^p)$ such that $|b| = |\mathrm{tp}(a^p)| + 1 = m + 1$, $\eta = f_b(k^*)$ for some $k^* \leq m$, and $D \cap (f_b(k^* - 1), \eta) = \emptyset$ (where we define $f_b(-1) = 0$). For $i < 2$, we may extend p to $q_i = \langle (\alpha_0, A_0), \ldots, (\alpha_{k^*-1}, A_{k^*-1}), (\beta_i, B_i), (\alpha_k^*, A_k^* \setminus (\beta_i + 1)), \ldots, (\alpha_m, A_m) \rangle$ with $\mathrm{tp}(a^{q_i}) = b$ such that $\beta_0 \neq \beta_1$. Suppose $q_0 \in G$ Let H be a generic filter containing q_1, and let h be the club derived from H. Let $g^* = (g \cap [0, \alpha_{k^*-1})] \cup (h \cap (\alpha_{k^*-1}, \beta_1]) \cup (g \cap (\beta_1, \kappa))$. Then g^* generates a $\mathbb{P}_{\vec{U}}$-generic filter G^* such that $p \in G^*$. Also, $g^* \upharpoonright D = g \upharpoonright D$, so $g^* \upharpoonright C = g \upharpoonright C$, since the sets are defined in the same way from $g \restriction D$ and parameters in V. However, $\beta_0 = g(\eta)$, $\beta_1 = g^*(\eta)$, and $\eta \in C$. This is a contradiction. Thus, $c \notin V[d]$. $\square$

CHAPTER 7

FORCING ITERATIONS WITH UNCOUNTABLE SUPPORT

Iterated forcing with uncountable support is much less well understood than forcing with finite or countable support. In particular, a satisfactory generalization of the notion of properness to higher cardinals has remained elusive. Trying to generalize properness in the most straightforward way, we arrive at the following definitions. For concreteness, we will work throughout this chapter at ω_1. All results generalize in a straightforward way to larger uncountable regular cardinals.

Definition. *Let θ be a sufficiently large, regular cardinal, and let $\mathbb{P} \in H(\theta)$ be a poset. We say $N \prec H(\theta)$ is* relevant for $\mathbb{P}$ *if:*

- $|N| = \aleph_1$.
- ${}^{\omega}N \subseteq N$.
- $N = \bigcup_{\alpha<\omega_1} N_\alpha$*, where $\langle N_\alpha \mid \alpha < \omega_1\rangle$ is an internally approachable chain of countable elementary substructures of $H(\theta)$.*

Definition. *Let $\mathbb{P}$ be a poset, and let N be relevant for $\mathbb{P}$. $q \in \mathbb{P}$ is $(N, \mathbb{P})$-*generic *if, for all dense, open sets D of $\mathbb{P}$ such that $D \in N$, $q \Vdash$ "$\dot{G}_{\mathbb{P}} \cap D \cap N \neq \emptyset$".*

Definition. $\mathbb{P}$ *is* ω_1-proper *if, for all sufficiently large, regular θ, for all $N \prec H(\theta)$ relevant for $\mathbb{P}$, and for all $p \in \mathbb{P} \cap N$, there is $q \leq p$ such that q is $(N, \mathbb{P})$-generic.*

In analogy with the classical notion of properness, ω_1-proper forcing posets preserve ω_2, and both ω_2-c.c. and ω_2-closed forcings are ω_1-proper. However, iterations of ω_1-proper forcings with ω_1-size supports are not in general ω_1-proper and can in fact collapse ω_2 (see [27]). In this chapter, we discuss the iterability of versions of forcings to add reals at higher cardinals.

Kanamori, in [21], introduces a generalization of Sacks forcing for uncountable cardinals and proves that, assuming $\diamondsuit$, the iteration of ω_1-Sacks forcing with ω_1-size supports preserves ω_2. We adapt his argument to show that, assuming CH, the iteration of ω_1-Cohen forcing with ω_1-sized supports is ω_1-proper.

Recall that ω_1-Cohen forcing, which we denote as $\mathbb{P}$, has conditions that are elements of ${}^{<\omega_1}2$, ordered by end-extension. We will think of conditions in an iteration of length γ with ω_1-support as being functions whose domains are subsets of γ of size $\leq \omega_1$;. We will need the following Lemma.

Lemma 7.1. *Let γ be an ordinal, and let $\mathbb{P}_\gamma$ be an iteration of ω_1-Cohen forcing of length γ with ω_1-support. Let $p \in \mathbb{P}_\gamma$, and let F be a countable subset of* $\mathrm{dom}(p)$. *There is $q \leq p$ such that, for every $\beta \in F$, there is $\alpha_\beta < \omega_1$ and $s_\beta : \alpha_\beta \to 2$ such that $q \restriction \beta \Vdash$ "$q(\beta) = s_\beta$".*

Proof. The proof is by induction on γ. The lemma is trivially true for $\gamma = 1$. Suppose $\gamma = \eta + 1$. We may assume $\eta \in F$. First, extend $p \restriction \eta$ to $r \in \mathbb{P}_\eta$ such that, for some $\alpha_\eta < \omega_1$ and $s_\eta : \alpha_\eta \to 2$, $r \Vdash$ "$p(\eta) = s_\eta$". Then, using the inductive hypothesis, extend r to t such that, for all $\beta \in F \cap \eta$, there is $\alpha_\beta < \omega_1$ and $s_\beta : \alpha_\beta \to 2$ such that $t \restriction \beta \Vdash$ "$t(\beta) = s_\beta$". Then $t^\frown p(\eta)$ is as desired.

Now suppose γ is a limit ordinal of countable cofinality. Let $\langle \gamma_n \mid n < \omega \rangle$ be an increasing sequence of ordinals cofinal in γ with $\gamma_0 = 0$. We build a sequence $\langle p_n \mid n < \omega \rangle$ such that $p_n \in \mathbb{P}_{\gamma_n}$ and $\langle p_n{}^\frown p \restriction [\gamma_n, \gamma) \mid n < \omega \rangle$ is decreasing in $\mathbb{P}_\gamma$. We ensure that, for every $n < \omega$ and every $\beta \in F \cap \gamma_n$, there is $\alpha_{\beta,n} < \omega_1$ and $s_{\beta,n} : \alpha_{\beta,n} \to 2$ such that $p_n \restriction \beta \Vdash$ "$p_n(\beta) = s_{\beta,n}$". This is easily achieved by the inductive hypothesis. Now let q be the greatest lower bound of the sequence $\langle p_n{}^\frown p \restriction [\gamma_n, \gamma) \mid n < \omega \rangle$. Letting $\alpha_\beta = \sup(\{\alpha_{\beta,n} \mid n < \omega\})$ and $s_\beta = \bigcup_{n<\omega} s_{\beta,n}$ for all $\beta \in F$, it is easily seen that q is as desired.

Finally, suppose γ is a limit ordinal of uncountable cofinality. Then there is $\eta < \gamma$ such that $F \subseteq \eta$. We can then finish by applying the inductive hypothesis to $p \restriction \eta$. $\square$

Theorem 7.2. *Assume CH. Let γ be an ordinal, and let $\mathbb{P} = \mathbb{P}_\gamma$ be an iteration of ω_1-Cohen forcing of length γ with ω_1-support. Then $\mathbb{P}_\gamma$ is ω_1-proper.*

Proof. Assume $\diamondsuit$ holds in V. (Since $\diamondsuit$ is added by ω_1-Cohen forcing, this is not really an additional assumption). Let $\bar{A} = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence guessing subsets of $\omega_1 \times \omega_1$, i.e., for each $\alpha < \omega_1$, $A_\alpha \subseteq (\alpha \times \alpha)$, and for all $X \subseteq (\omega_1 \times \omega_1)$, there are stationarily many $\alpha < \omega_1$ such that $X \cap (\alpha \times \alpha) = A_\alpha$.

Definition. *If $p \in \mathbb{P}$ and F is a countable subset of* $\mathrm{dom}(p)$*, we say $q \leq_F p$ if $q \leq p$ and, for all $\beta \in F$, $q(\beta) = p(\beta)$.*

Let θ be a sufficiently large, regular cardinal, and let $N \prec H(\theta)$ be relevant for $\mathbb{P}$. Let $p \in \mathbb{P} \cap N$. We will find $q \leq p$ such that q is $(N, \mathbb{P})$-generic.

Let $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ enumerate all dense open subsets of $\mathbb{P}$ that lie in N. We will build a decreasing sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ of conditions in $\mathbb{P} \cap N$. We will also beforehand fix a bookkeeping device that will give us a sequence of countable sets $\langle F_\alpha \mid \alpha < \omega_1 \rangle$, functions $\langle g_\alpha \mid \alpha < \omega_1 \rangle$, and ordinals $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ such that:

- $F_\alpha \subseteq \mathrm{dom}(p_\alpha)$.
- $g_\alpha : F_\alpha \to \eta_\alpha$ is a bijection and $\eta_\alpha \geq \alpha$.
- If $\alpha < \beta$, then $F_\alpha \subseteq F_\beta$ and $g_\alpha \subseteq g_\beta$.
- If β is a limit ordinal, then $F_\beta = \bigcup_{\alpha < \beta} F_\alpha$.
- $\bigcup_{\alpha < \omega_1} F_\alpha = \bigcup_{\alpha < \omega_1} \mathrm{dom}(p_\alpha)$.

In our construction, we will ensure that, if $\alpha < \beta < \omega_1$, then $p_\beta \leq_{F_\alpha} p_\alpha$. This will allow us to find a lower bound for the sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$.

Let $p_0 = p$. If $\beta < \omega_1$ is a limit ordinal, let p_β be the greatest lower bound of $\langle p_\alpha \mid \alpha < \beta \rangle$. Now suppose p_α has been defined. Assume that $\eta_\alpha = \alpha$. (This happens for a club of α. If it is not the case, then let $p_{\alpha+1} = p_\alpha$.) Now define a function $\sigma_\alpha : F_\alpha \to {}^{\alpha}2$ as follows: if $\beta \in F_\alpha$ and $\delta < \alpha$, then let

$$(\sigma_\alpha(\beta))(\delta) = \begin{cases} 1 & \text{if } (g_\alpha(\beta), \delta) \in A_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Now ask whether there is $r \leq p_\alpha$ such that:

- $r \in \bigcap_{\beta < \alpha} D_\beta$.
- For all $\beta \in F_\alpha$, $r \restriction \beta \Vdash$ "$r(\beta) = \sigma_\alpha(\beta)$".

Let r_α be such an r if it exists (if not, just let $p_{\alpha+1} = p_\alpha$). Note that, by elementarity, exploiting the fact that N is closed under countable sequences, we may assume that $r_\alpha \in N$. We now define $p_{\alpha+1}$ to resemble r_α as closely as possible while requiring that $p_{\alpha+1} \leq_{F_\alpha} p_\alpha$. Namely, we let $\mathrm{dom}(p_{\alpha+1}) = \mathrm{dom}(r_\alpha)$. For $\beta \in F_\alpha$, $p_{\alpha+1}(\beta) = p_\alpha(\beta)$. If $\beta \in \mathrm{dom}(p_\alpha) \setminus F_\alpha$, then $p_{\alpha+1}(\beta)$ is a name such that $r_\alpha \upharpoonright \beta \Vdash$ "$p_{\alpha+1}(\beta) = r_\alpha(\beta)$" and, if $c \leq p_\alpha \upharpoonright \beta$ is incompatible with $r_\alpha \upharpoonright \beta$, $c \Vdash$ "$p_{\alpha+1}(\beta) = p_\alpha(\beta)$". If $\beta \in \mathrm{dom}(r_\alpha) \setminus \mathrm{dom}(p_\alpha)$, then $p_{\alpha+1}(\beta)$ is a name such that $r_\alpha \upharpoonright \beta \Vdash$ "$p_{\alpha+1}(\beta) = r_\alpha(\beta)$" and, if $c \leq p_\alpha \upharpoonright \beta$ is incompatible with $r_\alpha \upharpoonright \beta$, $c \Vdash$ "$p_{\alpha+1}(\beta) = \emptyset$". It is clear that $p_{\alpha+1} \leq_{F_\alpha} p_\alpha$ and, since everything needed to define $p_{\alpha+1}$ is in N, we may assume $p_{\alpha+1} \in N$.

Let $g = \bigcup_{\alpha<\omega_1} g_\alpha$. Then g is a bijection from $\bigcup_{\alpha<\omega_1} F_\alpha$ to ω_1. Let q be a lower bound for $\langle p_\alpha \mid \alpha < \omega_1\rangle$. We claim that q is $(N, \mathbb{P})$-generic. To prove this, fix $t \leq q$ and $\xi < \omega_1$. We show that t is compatible with an element of $N \cap D_\xi$. To do this, we will construct sequences $\langle t_\alpha \mid \alpha < \omega_1\rangle$, $\langle \rho_\alpha \mid \alpha < \omega_1\rangle$ and $\langle \tau_\alpha \mid \alpha < \omega_1\rangle$ such that:

- $\langle t_\alpha \mid \alpha < \omega_1\rangle$ is a decreasing sequence of conditions from $\mathbb{P}_\gamma$ and t_α is a greatest lower bound of $\langle t_\beta \mid \beta < \alpha\rangle$ if α is a limit ordinal.
- $\langle \rho_\alpha \mid \alpha < \omega_1\rangle$ is an increasing, continuous sequence of countable ordinals, with $\rho_\alpha \geq \alpha$.
- For $\alpha < \alpha' < \omega_1$, $\tau_\alpha : F_\alpha \to {}^{\rho_\alpha}2$ and, for $\beta \in F_\alpha$, $\tau_\alpha(\beta) \subseteq \tau_{\alpha'}(\beta)$.
- For $\alpha < \omega_1$ and $\beta \in F_\alpha$, $t_\alpha \upharpoonright \beta \Vdash$ "$t_\alpha(\beta) = \tau_\alpha(\beta)$".
- For $\alpha < \omega_1$, $t_\alpha \in \bigcap_{\beta<\alpha} D_\beta$.

Let $t_0 = t$. If α is a limit ordinal, it is clear how to proceed. Suppose $\alpha = \zeta + 1$ and t_ζ, ρ_ζ, and τ_ζ have been defined. Let $t^*_\alpha \leq t$ be such that $t^* \in D_\alpha$. Apply Lemma 7.1 to t^*_α and F_α to get $t'_\alpha \leq t^*_\alpha$, $\{\rho_{\alpha,\beta} \mid \beta \in F_\alpha\}$ and $\{s_\beta \mid \beta \in F_\alpha\}$ such that, for all $\beta \in F_\alpha$, $s_\beta : \rho_{\alpha,\beta} \to 2$ and $t'_\alpha \upharpoonright \beta \Vdash$ "$t'_\alpha(\beta) = s_\beta$". We can then find $\rho_\alpha \geq \alpha$ greater than all of the $\rho_{\alpha,\beta}$'s. and arbitrarily extend all of the s_β's to be functions $\tau_\alpha(\beta)$ in ${}^{\rho_\alpha}2$. We can then define $t_{\alpha+1} \leq t'_\alpha$ as desired.

At the end of this construction, let $X = \{(g(\beta), \delta) \mid \beta \in \bigcup_{\alpha<\omega_1} F_\alpha,\ \delta < \omega_1$, and, for all α such that $\beta \in F_\alpha$ and $\delta < \rho_\alpha$, $(\tau_\alpha(\beta))(\delta) = 1\}$. $X \subseteq \omega_1 \times \omega_1$. Note that the set of limit ordinals $\alpha < \omega_1$ such that $\eta_\alpha = \rho_\alpha = \alpha$ is club. Let $\alpha > \xi$ in this club be such that $X \cap (\alpha \times \alpha) = A_\alpha$. Working through the definitions, this implies that $\tau_\alpha = \sigma_\alpha$. Thus, $t_\alpha \leq q \leq p_\alpha$, $t_\alpha \in \bigcap_{\beta<\alpha} D_\beta$ and, for all $\beta \in F_\alpha$, $t_\alpha \upharpoonright \beta \Vdash$ "$t_\alpha(\beta) = \sigma_\alpha(\beta)$", so, in our construction of $p_{\alpha+1}$, we answered our question positively and thus were in the non-trivial case. Now, noting that $t_\alpha \leq p_{\alpha+1}$, it is easily verified that, in fact, $t_\alpha \leq r_\alpha$ (simply check by induction on $\beta \in \mathrm{dom}(r_\alpha) = \mathrm{dom}(p_{\alpha+1})$ that $t_\alpha \upharpoonright \beta \Vdash$ "$t_\alpha(\beta) \leq r_\alpha(\beta)$"). But $r_\alpha \in N \cap \bigcap_{\beta<\alpha} D_\beta$, so $r_\alpha \in N \cap D_\xi$, so we have demonstrated that t is compatible with an element of $N \cap D_\xi$, thus completing the proof. $\square$

We can also show, following [21], that, assuming some cardinal arithmetic, sufficiently short iterations have the ω_3-c.c. and thus preserve all cardinals. Before we give the proof, we introduce the following bit of notation.

Definition. *Let $\mathbb{P}_\gamma$ be an iteration of ω_1-Cohen forcing of length γ with ω_1-supports. Let $p \in \mathbb{P}_\gamma$, let $F \subseteq \mathrm{dom}(p)$ be countable, let $\alpha < \omega_1$, and suppose $\sigma : F \to {}^{\alpha}2$. Then $p|\sigma$ is a function with domain $\mathrm{dom}(p)$ such that, if $\beta \notin F$, then $(p|\sigma)(\beta) = p(\beta)$ and, if $\beta \in F$, then $(p|\sigma) \restriction \beta \Vdash$ "$p|\sigma(\beta) = \sigma(\beta)$".*

Note that $p|\sigma \in \mathbb{P}_\gamma$ and $p|\sigma \leq p$ if and only if, for all $\beta \in F$, $(p|\sigma) \restriction \beta \Vdash$ "$p(\beta) \subseteq \sigma(\beta)$".

Lemma 7.3. *Suppose $\diamondsuit$ holds and $\gamma < \omega_3$. Then $\mathbb{P}_\gamma$ has the $(2^{\omega_1})^+$-c.c.*

Proof. Let $\mathbb{U}$ be the set of $p \in \mathbb{P}_\gamma$ such that there are $S, \langle F_\alpha \mid \alpha \in S\rangle$, and $\langle \sigma_\alpha \mid \alpha \in S\rangle$ such that:

1. $S_\alpha \subseteq \omega_1$ is stationary.
2. For all $\alpha \in S$, $|F_\alpha| = \aleph_0$.
3. For all $\alpha < \alpha' \in S$, $F_\alpha \subseteq F_{\alpha'}$, and $\bigcup_{\alpha \in S} F_\alpha = \mathrm{dom}(p)$.
4. For all $\alpha \in S$, $\sigma_\alpha : F_\alpha \to {}^{\alpha}2$, $p|\sigma_\alpha \in \mathbb{P}_\gamma$, and $p|\sigma_\alpha \leq p$.
5. For all $q \leq p$, there are stationarily many $\alpha \in S$ such that q and $p|\sigma_\alpha$ are compatible.

By the proof of Theorem 7.2, $\mathbb{U}$ is dense in $\mathbb{P}_\gamma$ (the condition q defined in the proof is in $\mathbb{U}$). Also, the number of choices for $S, \langle F_\alpha \mid \alpha \in S\rangle$, and $\langle \sigma_\alpha \mid \alpha \in S\rangle$ in the definition of $\mathbb{U}$ is 2^{ω_1}. Thus, it will be enough to show that, if p and q are in $\mathbb{U}$ as witnessed by the same $S, \langle F_\alpha \mid \alpha \in S\rangle$, and $\langle \sigma_\alpha \mid \alpha \in S\rangle$, then p and q are compatible.

Suppose not, and fix a counterexample p and q. Since p and q have the same $\langle F_\alpha \mid \alpha \in S\rangle$, $\mathrm{dom}(p) = \mathrm{dom}(q)$. Since p and q are incompatible, there must be $\beta \in \mathrm{dom}(p)$ and $r \leq p \restriction \beta, q \restriction \beta$ such that $r \Vdash$ "$p(\beta)$ and $q(\beta)$ are incompatible", since otherwise it would be simple to recursively construct a common extension of p and q. Fix such a β and r, and suppose moreover that there are $\delta_0, \delta_1 < \omega_1$, $\tau_0 \in {}^{\delta_0}2$, $\tau_1 \in {}^{\delta_1}2$, and $\delta^* < \min(\{\delta_0, \delta_1\})$ such that $r \Vdash$ "$p(\beta) = \tau_0$ and $q(\beta) = \tau_1$" and $\tau_0(\delta^*) \neq \tau_1(\delta^*)$.

Let $t = r^\frown p \restriction (\mathrm{dom}(p) \setminus \beta)$. $t \leq p$, so we can find $\alpha \in S$ such that $\alpha > \max(\{\delta_0, \delta_1\})$, $\beta \in F_\alpha$, and t is compatible with $p|\sigma_\alpha$. Let $u \leq t, p|\sigma_\alpha$. Then $\tau_0 \subseteq \sigma_\alpha(\beta)$ and $u \restriction \beta \leq (q|\sigma_\alpha) \restriction \beta$, so $u \restriction \beta \Vdash$ "$q(\beta) \subseteq \sigma_\alpha(\beta)$". But $u \restriction \beta \leq r$ and $r \Vdash$ "$q(\beta) = \tau_1 \not\subseteq \sigma_\alpha(\beta)$". This is a contradiction. $\square$

This immediately yields the following corollary, using a standard Δ-system argument for the case $\gamma = \omega_3$.

Corollary 7.4. *Suppose $\diamondsuit$ holds and $2^{\omega_1} = \omega_2$. Then, if $\gamma \leq \omega_3$, $\mathbb{P}_\gamma$ has the ω_3-c.c.*

We remark that essentially the same arguments yield the same results about properness and chain condition for the generalization of Hechler forcing to ω_1. In future work, we would like to fit all of these examples into a general framework that would serve as a sort of analog to Axiom A at higher cardinals.

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