EVOLUTION AND REGULARITY RESULTS FOR EPITAXIALLY STRAINED THIN FILMS AND MATERIAL VOIDS

BY

PAOLO PIOVANO

DISSERTATION

Submitted in Partial Fulfillment of the Requirements for the Degree of

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IN

MATHEMATICS

CARNEGIE MELLON UNIVERSITY

Department of Mathematical Sciences

Pittsburgh, Pennsylvania

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“Per chi va
in direzione ostinata e contraria”

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Abstract

In this dissertation we study free boundary problems that model the evolution of interfaces in the presence of elasticity, such as thin film profiles and material void boundaries. These problems are characterized by the competition between the elastic bulk energy and the anisotropic surface energy.

First, we consider the evolution equation with curvature regularization that models the motion of a two-dimensional thin film by evaporation-condensation on a rigid substrate. The film is strained due to the mismatch between the crystalline lattices of the two materials and anisotropy is taken into account. We present the results contained in [62] where the author establishes short time existence, uniqueness and regularity of the solution using De Giorgi's minimizing movements to exploit the L^2-gradient flow structure of the equation. This seems to be the first analytical result for the evaporation-condensation case in the presence of elasticity.

Second, we consider the relaxed energy introduced in [20] that depends on admissible pairs (E, u) of sets E and functions u defined only outside of E. For dimension three this energy appears in the study of the material voids in solids, where the pairs (E, u) are interpreted as the admissible configurations that consist of void regions E in the space and of displacements u of the atoms of the crystal. We provide the precise mathematical framework that guarantees the existence of minimal energy pairs (E, u). Then, we establish that for every minimal configuration (E, u), the function u is $C^{1,\gamma}_{\text{loc}}$-regular outside an essentially closed subset of E. No hypothesis of starshapedness is assumed on the voids and all the results that are contained in [18] hold true for every dimension $d \geq 2$.

Key Words and Sentences: surface energy, elastic bulk energy, minimizing movements, evolution, gradient flow, motion by mean curvature, minimal configurations, existence, uniqueness, regularity, partial regularity, lower density bound, thin film, epitaxy, surface diffusion, evaporation-condensation, material voids, grain boundaries, anisotropy.

Contents

Acknowledgements vii

1 Introduction 1

1.1 Evolution of Epitaxially Strained Thin Films . . . 3

1.2 Material Voids in an Elastic Solid and Regularity Results for $d \geq 2$ . . . 9

2 Preliminaries 13

2.1 Continuous Functions and Lebesgue Spaces . . . 13

2.2 Sobolev Spaces . . . 14

2.3 Interpolation Inequalities . . . 15

2.4 Functions of Bounded Variation . . . 16

2.5 Sets of Finite Perimeter . . . 18

2.6 Embedding Theorems and Isoperimetric Inequalities . . . 21

2.7 Generalized Area Formula . . . 24

2.8 Approximate Continuity and Differentiability . . . 26

2.9 Fine Properties of BV Functions . . . 29

2.10 Special Functions of Bounded Variation . . . 31

2.11 Generalized Functions of Bounded Variation . . . 35

3 Evolution of Elastic Thin Films 39

3.1 Mathematical Setting . . . 40

3.2 Existence and Regularity . . . 48

3.3 Uniqueness . . . 62

4 Material Voids in an Elastic Solid and Regularity Results for $d \geq 2$ 69

4.1 Mathematical Setting for Material Voids . . . 70

4.2 Volume Penalization 75
4.3 Density Lower Bound 81

5 Future Research Projects 95

List of Symbols 97

Bibliography 101

Chapter 1

Introduction

In this dissertation we study free boundary problems that model the evolution of interfaces in the presence of elasticity, such as thin film profiles and material void boundaries. Understanding the morphological evolution of such interfaces plays a crucial role in many fields of physics, chemistry and nanotechnology, especially for the design and control of material microstructures. However, the mathematical validation of models on evolving interfaces is still incipient (see [15, 31, 47, 49, 65, 66, 67, 71, 74]). In this manuscript we extend the current state of the art adopting various analytical techniques from partial differential equations, geometric measure theory, and calculus of variations.

The physical motivation of the models that we consider can be found in [41, 45, 55], and in the references therein. These problems are characterized by the competition between the elastic bulk energy and the anisotropic surface energy. Precisely, a crystalline material that occupies an open region $U \subset \mathbb{R}^d$, $d \geq 2$, with locally Lipschitz boundary, is treated as a continuum obeying the laws of linear elasticity. Hence, denoting by u the displacement of the bulk material, $\boldsymbol{E}(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ represents the linearized strain and the bulk elastic energy takes the form

$$\int_U W(\boldsymbol{E}(u))\,\mathrm{d}z\,, \tag{1.0.1}$$

where the elastic energy density $W : \mathbb{M}^{2\times 2}_{sym} \to [0,\infty)$ is defined by

$$W(\boldsymbol{A}) := \frac{1}{2}\mathbb{C}\boldsymbol{A} : \boldsymbol{A}$$

for a positive definite fourth-order tensor $\mathbb{C}$. Furthermore, the interface ∂U is treated as an anisotropic geometrical surface. Thus, denoting by ν the unit normal vector to ∂U

that points outward from the region U, and by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional measure, the surface energy functional is defined by

$$\int_{\partial U} \psi(\nu)\, \mathrm{d}\mathcal{H}^{d-1}\,, \tag{1.0.2}$$

where $\psi : \mathbb{R}^d \setminus \{0\} \to (0, \infty)$ is a positively one-homogeneous function that is assumed to be of class C^2 away from the origin. The total energy is the result of the competition between the surface and the bulk elastic energy, and it is given by

$$\mathcal{E}(U, u) := \int_U W(\boldsymbol{E}(u))\, \mathrm{d}z + \int_{\partial U} \psi(\nu)\mathrm{d}\mathcal{H}^{d-1}\,, \tag{1.0.3}$$

for all admissible configurations (U, u).

As we will describe in detail in the following sections, the energy (1.0.3) appears in the study of epitaxially strained thin films and material voids boundaries. Indeed, in the context of thin films deposited on a flat substrate, the sets U represent the admissible regions occupied by the films and ∂U their profiles. In this case, we describe the thickness of the films by means of profile functions h. Hence, the sets U and ∂U are, respectively, the subgraphs Ω_h and the graphs Γ_h of the profile functions (see (1.1.6) below).

In the case of material voids in an elastic solid that is contained in a region Ω of the space, we denote the void regions by E. Using this notation, the sets U coincide with the admissible parts of Ω occupied by the atoms of the solid, i.e., $\Omega \setminus E$, and the sets ∂U with ∂E. The energy (1.0.3) takes the form of (1.2.1) below.

We note that in both models, the functions u stand for the admissible displacements of the atoms and satisfy appropriate Dirichlet boundary conditions. Specifically, the admissible displacements u take prescribed values at the interface between the films and the substrate, for the first model, and outside a bounded region that contains the voids, for the second model. As described below, these boundary conditions force the materials to be strained, thus generating elastic energy.

Analytical results for free boundary problems with underlying energy of the form (1.0.3) appeared only recently. A precise mathematical framework that guarantees the existence of minimal configurations of (1.0.3) is not yet available in literature for dimensions $d \geq 3$ (even for the isotropic case), but it has been provided in the context of epitaxially strained thin films for dimension two in [19, 38]. In this case, regularity, qualitative and quantitative properties of equilibrium configurations have been studied in [38, 42]. Furthermore, the existence and regularity of minimizers of (1.0.3) has been established (again for $d = 2$) in the case of starshaped material voids in elastic solids in [37]. The evolutionary counterpart of [19, 38, 42] has been developed in [39] for the surface diffusion case, and in [62] for the evaporation-condensation case.

In this dissertation, we present the results contained in [62] and in [18]. In Chapter 3 we establish short time existence, uniqueness and regularity for the solution of the evolution equation associated with the curvature regularization of the energy (1.1.6), adopting De Giorgi's minimizing movement method to exploit the L^2-gradient flow structure of the equation (see [62]). In Chapter 4 we prove regularity results in the context of material voids in elastic solids that hold true for any dimension $d \geq 2$, and without the restriction assumed in [37] that the voids are starshaped (see [18]). These results pave the road to extend the theory developed in dimension two to any dimension.

In the following two sections we introduce, respectively, the model for epitaxially strained thin films and the model for material voids in elastic solids, and the main evolution and regularity results achieved in both cases.

1.1 Evolution of Epitaxially Strained Thin Films

In Chapter 3 we study the morphologic evolution of an anisotropic epitaxial film deposited on a rigid substrate, with the film strained due to a mismatch between the crystalline lattices of the two materials. We consider the evaporation-condensation case and neglect surface diffusion, with the profile of the film being modeled as a grain-vapor interface with the vapor being considered as a reservoir that interacts with the profile of the film only through the evaporation-condensation process (see [41, Section 19]). We essentially follow the approach that is used in [39] for the surface diffusion case, and just as in [39] we restrict our attention to the two-dimensional model or, in other words, to a three-dimensional epitaxially strained film with identical vertical cross-sections.

One of the earliest theories for the evolution of an interface Γ between two phases is due to Mullins (see [60, 61]), who derived the equations that describe the planar motion of isotropic grain boundaries by evaporation-condensation and by surface diffusion. Up to a rescaling, the equations are the motion by mean curvature and the motion by surface Laplacian of mean curvature, i.e.,

$$V = k \quad \text{and} \quad V = -k_{\sigma\sigma} \text{ on } \Gamma, \tag{1.1.1}$$

respectively, where V is the normal velocity, k is the curvature of the evolving interface and $(\cdot)_\sigma$ is the tangential derivative along the interface. There is a large body of literature devoted to the study of these equations. In particular, a generalization of Mullins's models includes anisotropy (see [41, Section 19.7]). Precisely, the anisotropic surface energy functional is

$$\int_\Gamma \psi(\nu)\,\mathrm{d}\mathcal{H}^1\,, \tag{1.1.2}$$

where ν here denotes the normal vector to Γ. In particular, in [46, Section 8] and [14, 48] it is shown that the equation for the evaporation-condensation case becomes

$$\beta V = (g_{\theta\theta} + g)k - C \text{ on } \Gamma\,, \tag{1.1.3}$$

where C is a constant, the coefficient β is a material function associated with the attachment kinetics of the atoms at the interface, and g is defined by

$$g(\theta) := \psi(\cos\theta, \sin\theta) \tag{1.1.4}$$

for each angle $\theta \in [0{,}2\pi]$ that ν forms with the x-axis along Γ. We assume the kinetic coefficient to be constant and so, up to a rescaling, we take $\beta \equiv 1$.

Locally, the interface may be described as the graph of a one-dimensional function. In the context of a thin film over a flat substrate, we set the x-axis on the substrate upper boundary and describe the thickness of the film by means of a profile function $h : (0,b) \times [0,T] \to [0,\infty)$ for a positive length b and a positive time T. In this way, the graph of h represents the evolving profile Γ_h of the film. We adopt the sign convention that the normal vector ν points outward from the region Ω_h occupied by the film and k is negative when the profile is concave. Note that the normal velocity parametrized by the profile function h is given by

$$V = \frac{1}{J} h_t\,, \quad \text{where} \quad J := \sqrt{1 + |h_x|^2}\,,$$

and we denote by h_x and h_t the derivatives with respect to the first and the second component, respectively.

In [14, 46] the constant C is included in (1.1.3) to represent the difference in bulk energies between the phases. As already mentioned in [46, Remark 3.1], the theory can be extended to account for deformation (see also [41, 49]). Indeed, the inclusion of deformation is very important to model epitaxy because the difference in lattice parameters between the film and the substrate can induce large stresses in the film. In order to release the resulting elastic energy, the atoms in the film move and reorganize themselves in more convenient configurations. In analogy with [19, 38, 42] and with the surface diffusion case (see [39]), we work in the context of the elasticity theory for small deformations. Hence, fixing a time t in $[0,T]$, the bulk elastic energy is

$$\int_{\Omega_h} W(\boldsymbol{E}(u))\,\mathrm{d}z\,, \tag{1.1.5}$$

where u defined in Ω_h denotes the planar displacement of the bulk material that is assumed to be in (quasistatic) equilibrium. Therefore, the total energy of the system at time t is

$$\mathcal{F}(h,u) := \int_{\Omega_h} W(\boldsymbol{E}(u))\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu)\,\mathrm{d}\mathcal{H}^1 \tag{1.1.6}$$

that can be regarded as (1.0.3) in the context of thin films in dimension two. Furthermore, we model the mismatch of the film atoms at the interface with the substrate using the Dirichlet boundary condition $u(x,0) = (e_0 x,0)$, where the constant $e_0 > 0$ measures the misfit between the crystalline lattices. Moreover, the migration of atoms can eventually result in the formation of surface patters on the profile of the film, such as undulations, material agglomerates or isolated islands. However, these non-flat configurations have a cost in terms of surface energy which is roughly proportional to the area of the profile of the film (see (3.1.3) below). Therefore, the evolution of the film profile is the result of the competition between the bulk elastic energy and the surface energy of the film, and (1.1.3) becomes

$$V = (g_{\theta\theta} + g)k - W(\boldsymbol{E}(u)) \quad \text{on } \Gamma_h\,, \tag{1.1.7}$$

while the corresponding equation in the case of surface diffusion is

$$V = (-(g_{\theta\theta} + g)k + W(\boldsymbol{E}(u)))_{\sigma\sigma} \quad \text{on } \Gamma_h\,,$$

where $W(\boldsymbol{E}(u))$ is defined for each $t \in [0,T]$ as the trace of $W(\boldsymbol{E}(u(\cdot,t)))$ on $\Gamma_{h(\cdot,t)}$ and $u(\cdot,t)$ is the elastic equilibrium corresponding to $h(\cdot,t)$.

These evolution equations exhibit different behaviors with respect to the sign of the interfacial stiffness $f := g_{\theta\theta} + g$. In fact, the equations are parabolic on any angle interval in which f is strictly positive. In this case, (1.1.7) has been extensively studied and it behaves similarly to $V = k$ (see, e.g., [12, 13, 48]). Those angle intervals in which f is negative are relevant from the materials science viewpoint. In this range, (1.1.7) is backward parabolic and unstable and so, in order to analyze its behavior, we consider a higher order perturbation. The idea consists in allowing for a dependence on curvature of the surface energy density g in order to penalize surface patterns with large curvature, such as sharp corners (see [58, 66]). This approach was already suggested in [14] and relies on the physical argumentations of Herring (see [50, 51]). In [29], the authors choose a quadratic dependence on curvature for ψ of the form

$$\psi(\nu, k) := \psi(\nu) + \frac{\varepsilon}{2}k^2 \,, \tag{1.1.8}$$

with ε denoting a (small) positive constant (see also [47]). Hence, replacing the surface energy density in (1.1.2) with (1.1.8) and taking into account the bulk elastic energy (1.1.5), the total energy of the system at a time t in $[0, T]$ becomes

$$\mathcal{F}(h) := \int_{\Omega_h} W(\boldsymbol{E}(u_h))\,\mathrm{d}z + \int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{2}k^2\right) \mathrm{d}\mathcal{H}^1 \,, \tag{1.1.9}$$

where $u_h(\cdot, t)$ is the minimizer of the elastic energy (1.1.5) in $\Omega_{h(\cdot,t)}$ under suitable boundary and periodicity conditions. The resulting parabolic equations are

$$V = (g_{\theta\theta} + g)k - W(\boldsymbol{E}(u)) - \varepsilon\left(k_{\sigma\sigma} + \frac{1}{2}k^3\right) \quad \text{on } \Gamma_h \tag{1.1.10}$$

for the evaporation-condensation case, and

$$V = \left(-(g_{\theta\theta} + g)k + W(\boldsymbol{E}(u)) + \varepsilon\left(k_{\sigma\sigma} + \frac{1}{2}k^3\right)\right)_{\sigma\sigma} \quad \text{on } \Gamma_h \tag{1.1.11}$$

for the surface diffusion case. These equations have been already proposed in [39], where (1.1.11) has been analytically studied. To the best of our knowledge, no analytical results exist in literature for (1.1.10), unless we restrict ourselves to the case without elasticity, as in [16, 17, 23, 32, 68] (see also [12, 13]).

In this dissertation we establish short time existence, uniqueness, and regularity of spatially periodic solutions of (1.1.10). Precisely, given a time $T > 0$, we say that (h, u) is a b*-periodic configuration* in Ω_h if $h(\cdot, t)$ is b-periodic in $\mathbb{R}$ and $u(x+b, y, t) = u(x, y, t) + (e_0 b, 0)$ for each (x, y) in the subgraph of $h(\cdot, t)$ and any time $t \in [0, T]$. For an initial b-periodic profile h_0, we introduce the Cauchy problem

$$\begin{cases} \dfrac{1}{J}h_t = (g_{\theta\theta} + g)k - W(\boldsymbol{E}(u)) - \varepsilon\left(k_{\sigma\sigma} + \dfrac{1}{2}k^3\right) \text{ in } \mathbb{R} \times (0, T)\,, \\ \operatorname{div} \mathbb{C}\boldsymbol{E}(u) = 0 \text{ in } \Omega_h\,, \\ \mathbb{C}\boldsymbol{E}(u)[\nu] = 0 \text{ on } \Gamma_h \text{ and } u(x, 0, t) = (e_0\, x, 0)\,, \\ (h, u) \text{ is a } b\text{-periodic configuration in } \Omega_h\,, \\ h(\cdot, 0) = h_0\,, \end{cases} \tag{1.1.12}$$

where $W(\boldsymbol{E}(u))$ is defined for each $t \in [0,T]$ as the trace of $W(\boldsymbol{E}(u(\cdot,t)))$ on the graph of $h(\cdot,t)$. This problem is proposed in the review article [55, Section 4.2.2] to which we refer for further references. We now state the existence result that we prove in Chapter 3 (see Theorem 3.2.10).

Existence Theorem.

Let $h_0 \in H^2_{\mathrm{loc}}(\mathbb{R};(0,\infty))$ be an initial b-periodic profile. Then there exists $T_0 > 0$ such that for each $T < T_0$ the Cauchy problem (1.1.12) admits a solution (h,u) with profile $h \in L^2(0,T;H^4_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty(0,T;H^2_{\mathrm{loc}}(\mathbb{R})) \cap H^1(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))$.

This existence result appears to be the first in the presence of elasticity and without surface diffusion. Moreover, we believe that the method is so general that could be applied also to the case with surface diffusion (1.1.11) to prove a short time existence and regularity result without the use of constant speed parametrizations of the profiles. The theorem is established combining an idea of [44, Chapter 12] with the minimizing movement method introduced by De Giorgi (see [4, 10]) to exploit the fact that the equation (1.1.10) can be regarded as the gradient flow of the functional $\mathcal{F}$ with respect to the L^2-metric.

The idea of this method is based on the discretization of the time interval $[0,T]$ in $N \in \mathbb{N}$ subintervals with length τ_N, and on defining inductively the approximate solution h_N at time $i\tau_N$ by a minimum problem that depends on the approximate solution at the previous time. Precisely, we start with the initial profile $h_N(\cdot,0) := h_0$ and for each $i = 1,\dots,N$, we find $h_N(\cdot,i\tau_N)$ as the minimizer of

$$\mathcal{F}(h) + \frac{1}{2\tau_N}\mathscr{D}^2\left(h, h_N(\cdot,(i-1)\tau_N)\right) \tag{1.1.13}$$

where the function $\mathscr{D}$, that measures the L^2-distance between h and $h_N(\cdot,(i-1)\tau_N)$, is chosen so that the Euler equation of this minimum problem corresponds to a time discretization of (1.1.10) (see (3.2.23) below). Then, the discrete-time evolution h_N is defined in $[0,T]$ as the piecewise constant or linear interpolant of $\{h_N(\cdot,i\tau_N)\}$. This approach was already adopted in [3] to deal with the motion of crystalline boundaries by mean curvature. Moreover, minimizing movements have been used also more recently to study mean curvature type flows in the case without elasticity in [16, 21, 23], and for the equation (1.1.11) in [39] (see also [64] for the Hele-Shaw equation and [31]). As already observed in [22], the basic differences between the evaporation-condensation and the surface diffusion evolution equations are that the latter preserves the area underneath the film profile and it is a gradient flow of $\mathcal{F}$ with respect to another metric, the H^{-1}-distance (see also [70]).

Moreover, the method adoped provides an estimate of the $L^\infty(0,T;L^\infty(0,b))$-norm of the spacial derivative of the profile solution in terms of $\|h_0'\|_\infty$. In the following result, that is established in Theorem 3.2.11, we summarize all the regularity properties that apply to the solution of (1.1.12) given by the Existence Theorem above.

Regularity Theorem.

Let $h_0 \in H^2_{\mathrm{loc}}(\mathbb{R};(0,\infty))$ be an initial b-periodic profile and let (h,u) be the solution of (1.1.12) in $[0,T]$ given by the Existence Theorem above for $T < T_0$. Then, the profile h satisfies:

(i) $h \in C^{0,\beta}([0,T];C^{1,\alpha}([0,b]))$ *for every* $\alpha \in \left(0,\frac{1}{2}\right)$ *and* $\beta \in \left(0,\frac{1-2\alpha}{8}\right)$,

(ii) $h \in L^{\frac{12}{5}}(0,T;C^{2,1}([0,b])) \cap L^{\frac{24}{5}}(0,T;C^{1,1}([0,b]))$,

(iii) $\|h_x\|_{L^\infty(0,T;L^\infty(0,b))} \le \|h_0'\|_\infty + \sqrt{\|h_0'\|^2_\infty + 1}$.

From the Uniqueness Theorem below it follows that for each $T > 0$ the Cauchy problem (1.1.12) admits at most one solution in $[0,T]$.

Uniqueness Theorem.

Let $T > 0$ and let $h_0 \in H^2_{\mathrm{loc}}(\mathbb{R};(0,\infty))$ be an initial b-periodic profile. If (h_1,u_1) and (h_2,u_2) are two solutions of (1.1.12) in $[0,T]$ with profiles h_1 and $h_2 \in L^2(0,T;H^4_{\mathrm{loc}}(0,b)) \cap L^\infty(0,T;H^2_{\mathrm{loc}}(0,b)) \cap H^1(0,T;L^2_{\mathrm{loc}}(0,b))$, then they coincide.

Note that in the previous theorem the regularity hypothesis on the profiles h_1 and h_2 is not an artificial assumption. In fact, it is satisfied by the solution of (1.1.12) given in the Existence Theorem for $T < T_0$. Hence, this solution, found by means of minimizing movements, is the unique solution of (1.1.12) for $T < T_0$.

The study of the long time existence and the global behavior of the solution of (1.1.10), as well as the asymptotic stability, will be the subject of future work (see Chapter 5).

1.2 Material Voids in an Elastic Solid and Regularity Results for $d \geq 2$

In Chapter 4 we continue the study of surface roughening of material caused by elastic stress. The emphasis is now on providing a precise mathematical framework that guarantees the existence of minimal configurations of (1.0.3) for dimensions $d \geq 2$. We focus on models for material voids in elastic solids using the formulation introduced in [37, 65]. As in the case of epitaxially driven thin films, the morphology of void boundaries results from the competition between the elastic strain energy which tends to destabilize the interface, and the surface energy, which has a stabilizing effect (see for example [43, 65, 69, 72, 73]). Thus, the energy of the system is of the form (1.0.3) with interacting bulk and surface energies. Denoting the region in the space that contains the elastic solid by an open set Ω in $\mathbb{R}^d$, the total energy is defined by

$$\int_{\Omega \setminus E} W(\boldsymbol{E}(u))\,\mathrm{d}z + \int_{\Omega \cap \partial E} \psi(\nu)\mathrm{d}\mathcal{H}^{d-1} \tag{1.2.1}$$

on pairs (E, u) consisting of sets $E \subset \Omega$ and of functions u that represent, respectively, the voids in Ω and the displacements of the solid atoms. The energy (1.2.1) is well defined (allowing for the value $+\infty$) on sets E with locally Lipschitz boundary and functions $u \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$. Note that we formally define the functions u in the whole set Ω for technical reasons and without loss of generality since the energies that we consider account only for their values in $\Omega \setminus E$. Following the variational approach of [37], we introduce a Dirichlet boundary condition by imposing that each admissible pair (E, u) satisfies

$$u = u_0 \quad \text{in} \quad \Omega \setminus \overline{\Omega'} \quad \text{and} \quad E \subset \overline{\Omega'} \tag{1.2.2}$$

for some bounded function u_0 and some connected set $\Omega' \subset\subset \Omega$ with Lipschitz boundary, and we fix the volume of the admissible void regions by assuming that

$$|E| = \lambda \tag{1.2.3}$$

for some constant $0 < \lambda \leq |\Omega'|$.

To apply the Direct Method of the Calculus of Variations in order to establish the existence of a minimum admissible configuration, we need the functional to be lower semicontinuous with respect to an adequate topology. We consider the topology characterized by the $L^1 \times L^1$-convergence (see Chapter 2 for the definition of these notion of convergence) and since (1.2.1) is not lower semicontinuous with respect to this convergence, we need to consider its lower semicontinuous envelope (or relaxed functional).

In [20] a representation formula for the lower semicontinuous envelopes of a class of functionals slightly different from (1.2.1) has been established under the condition that ψ is convex. In particular, for $p > 1$ the representation formula applies to the relaxation of the functional $\mathcal{G} : X_{\mathrm{reg}}(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\mathcal{G}(E,u) := \int_{\Omega\setminus E} |\nabla u|^p \,\mathrm{d}x + \int_{\Omega\cap\partial E} \psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1}\,, \tag{1.2.4}$$

where

$$X_{\mathrm{reg}}(\Omega;\mathbb{R}^d) := \Big\{(E,u) \,:\, E\subset\Omega \text{ with locally Lipschitz boundary and } u\in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)\Big\}\,.$$

We observe that $\mathcal{G}$ differs from (1.2.1) only by the fact that the linear elastic bulk energy is replaced by the generalized Dirichlet functional (see [24] for the case of anti-plane shear). The relaxed functional $\overline{\mathcal{G}}$ of $\mathcal{G}$ is defined by

$$\begin{aligned}\overline{\mathcal{G}}(E,u) := \inf\Big\{ \liminf_{n\to\infty} \mathcal{G}(E_n,u_n) \,:\, \{(E_n,u_n)\}\subset X_{\mathrm{reg}}(\Omega;\mathbb{R}^d)\,,\ \chi_{E_n}\to\chi_E \text{ in } L^1(\Omega)\,,\\ \text{and } u_n\to u \text{ in } L^1(\Omega;\mathbb{R}^d)\Big\}\end{aligned}$$

for each pair (E,u) such that E is a $\mathcal{L}^d$-measurable subset of Ω and $u\in L^1(\Omega;\mathbb{R}^d)$. Let $\mathcal{P}(\Omega)$ denote the family of *sets of locally finite perimeter* and $GSBV(\Omega;\mathbb{R}^d)$ denote the space of *generalized functions of bounded variation* in Ω that take values in $\mathbb{R}^d$ (see Chapter 2). Then, by [20] the relaxed functional $\overline{\mathcal{G}}$ satisfies

$$\overline{\mathcal{G}}(E,u) = \int_{\Omega\setminus E} |\nabla^* u|^p\,\mathrm{d}x + \int_{\Omega\cap\partial^* E}\psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Omega\cap S^*_u\cap E^0} \left(\psi(\nu^*_u)+\psi(-\nu^*_u)\right)\mathrm{d}\mathcal{H}^{d-1}$$

for each pair $(E,u)\in\mathcal{P}(\Omega)\times L^1(\Omega;\mathbb{R}^d)$ such that $u\chi_{E^0}\in GSBV(\Omega;\mathbb{R}^d)$, where E^0 denotes the *external measure theoretic set* of E, while $\nabla^* u$ and S^*_u are, respectively, the gradient and the discontinuity set of u with normal ν^*_u in the sense defined in Section 2.11 (see also Remarks 4.1.1 and 4.1.3).

Another possible approach is to consider the Hausdorff convergence of sets instead of the L^1 convergence of their characteristic functions and extend to dimensions d larger than two the relaxation result contained in [37]. However, note that in [37] the admissible sets E need to be starshaped while we are not assuming this hypothesis on the void regions.

Moreover, we note that the literature does not provide any representation formula for the relaxed functional (1.2.1). Indeed, in order to find this representation formula the natural mathematical formulation is in the space SBD of *special functions of bounded deformations* (see [7]). The problem resides in the fact that the theory of SBD functions is still not well-developed as the theory of SBV functions. However, we refer the reader to the paper [27] for recent progress on SBD.

The main result achieved in Chapter 4 concerns the regularity of local minimizers (E, u) of $\overline{\mathcal{G}}$. In dimension two the regularity of minimal configurations is achieved in [37] by establishing a uniform "exterior Wulff shape condition" for the case of material voids, and in [39] by establishing the so called uniform "internal sphere condition" for the case of thin films (see [37, Theorem 6.5] and [39, Proposition 3.3], respectively). These methods are adaptations of an argument first introduced in [25] that is strongly hinged on the two dimensional geometry. Therefore, we introduce a new strategy adopting the theory developed for the Mumford-Shah functional (see [8, 9, 11, 28, 56], and see also [6] for the functional that models the mixture of two conducting materials and [5] for minimal surfaces). We present the proofs in the scalar case, i.e., for functions u with scalar values. Precisely, for every local minimal pair (E, u) that satisfies (1.2.2) and (1.2.3), we establish that the function u is $C^{1,\gamma}_{\mathrm{loc}}$-regular outside an essentially closed subset of E (see Theorem 4.3.10 and Definition 4.1.5 for the definition of local minimizer).

Regularity Theorem.

There exists $\gamma \in (0,1]$, that depends only on p and the dimension d, such that for every local minimizer (E, u) of $\overline{\mathcal{G}}$ satisfying (1.2.2) and (1.2.3), a representative of $u\chi_{E^0}$ belongs to $C^{1,\gamma}_{\mathrm{loc}}(\Omega' \setminus \overline{\Gamma}_{E,u})$, where the set

$$\Gamma_{E,u} := \partial^* E \cup \Big(S_u \cap E^0 \Big)$$

is essentially closed in Ω', i.e.,

$$\mathcal{H}^{d-1}(\Omega' \cap \overline{\Gamma}_{E,u} \setminus \Gamma_{E,u}) = 0\,. \tag{1.2.5}$$

We remark that, from the point of view of regularity, the volume constraint on the void regions introduces extra difficulties, since this implies that the only variations allowed are the ones that maintain the volume constant (see [33]). We overcome this problem in Theorem 4.2.1 by showing that every local minimizers (E,u) of $\overline{\mathcal{G}}$ satisfying (1.2.2) and (1.2.3) is also a minimizer of a suitable energy functional with a volume penalization (see [1, 33]). Then, we can easily verify that there exist a constant $\omega \geq 0$ and a radius $\varrho_0 > 0$ for (E,u) such that, for every ball $B_\varrho(x) \subset \Omega'$ with $\varrho \leq \varrho_0$, the inequality

$$\overline{\mathcal{G}}(E,u,B_\varrho(x)) \leq \overline{\mathcal{G}}(F,v,B_\varrho(x)) + \omega\varrho^d \tag{1.2.6}$$

holds for every admissible pair (F,v) with $E \triangle F \subset\subset B_\varrho(x)$ and $\{u \neq v\} \subset\subset B_\varrho(x)$, where $\overline{\mathcal{G}}(\cdot,\cdot,B_\varrho(x))$ stands for the local version of $\overline{\mathcal{G}}$ in $B_\varrho(x)$ (see (4.1.3) and Definition 4.1.7). We say that an admissible pair that satisfies (1.2.6) in an open set A is a *quasi-minimizer* of $\mathcal{G}$ in A (see Definition 4.1.7).

In view of (1.2.6), the Regularity Theorem follows by classical regularity results for minima of the generalized Dirichlet functional (see [34, 54]), and by proving that the set $\Gamma_{E,u}$ is essentially closed for every quasi-minimizer of $\mathcal{G}$ in Ω'. The latter property is established not only in Ω' but also for all the quasi-minimizers in a generic open set $A \subset \mathbb{R}^d$ following the method introduced in [28] for the Mumford-Shah functional. Precisely, the key point is to prove a uniform lower bound for the $(d-1)$-dimensional density of $\mathcal{H}^{d-1} \lfloor_{\Gamma_{E,u}}$ at the points $x \in \overline{\Gamma}_{E,u}$, and this is achieved in Theorem 4.3.8.

It seems that the method used to prove the results contained in Chapter 4 may be adapted to the case of an unbounded boundary datum u_0. In future work we plan to extend the results to the case of linear elasticity where the functional is (1.2.1) (see [27]). The regularity result obtained paves the way to address partial regularity of the boundary of the voids in elastic solids (see [8, 9, 11, 28, 56]). The author is attempting to establish that the boundary of the voids is a regular hypersurface outside a relatively closed set in Ω with negligible $(N-1)$-Hausdorff measure (see Chapter 5).

Chapter 2

Preliminaries

We begin by introducing the notation and the requisite preliminaries needed in the sequel. The results of this chapter are mainly contained in [2, 9, 36, 35, 40, 46], to which we refer for further considerations and for most of the proofs.

Let $d \in \mathbb{N}$, and let $0 \leq k \leq d$. In this dissertation, the Lebesgue outer measure in $\mathbb{R}^d$ and the k-dimensional Hausdorff measure are denoted by $\mathcal{L}^d$ and $\mathcal{H}^k$, respectively. Given a set $U \subset \mathbb{R}^d$, we denote by ∂U and $\overline{U}$, respectively, the topological boundary and closure of U. Furthermore, $\mathscr{B}(U)$ is the Borel σ-algebra of U and $\mathscr{M}(U)$ is the family of $\mathcal{L}^d$-measurable subsets of U.

Moreover, throughout this chapter Ω stands for a generic open set in $\mathbb{R}^d$.

2.1 Continuous Functions and Lebesgue Spaces

We use the sandard notation for the vector space $C^m(\Omega)$ of the real functions defined in Ω that are continuous, together with their partial derivatives up to the order $m \in \mathbb{N}_0$. We let $C(\Omega) := C^0(\Omega)$, and we define

$$C^\infty(\Omega) := \bigcap_{m=0}^{\infty} C^m(\Omega).$$

The subspaces of $C(\Omega)$, $C^m(\Omega)$ and $C^\infty(\Omega)$ consisting of all the functions with compact support are denoted by $C_c(\Omega)$, $C_c^m(\Omega)$, and $C_c^\infty(\Omega)$, respectively. For $m \in \mathbb{N}_0$ and $0 < \alpha \leq 1$, $C^{m,\alpha}(\Omega)$ is the space of real functions continuously differentiable up to the order $m \in \mathbb{N}_0$, with locally α-Hölder continuous derivatives.

Moreover, we define in the usual way the space $L^\infty(\Omega)$, and for $1 \le p < \infty$ the space $L^p(\Omega)$ of p-Lebesgue integrable functions over Ω. Consider also, for $1 \le p \le \infty$, the space $L^p_{\mathrm{loc}}(\Omega)$ of Lebesgue measurable functions that belong to $L^p(K)$ for every compact set $K \subset \Omega$. Given $M \in \mathbb{N}$ we introduce the following notations for the corresponding spaces of vector valued functions: $C(\Omega;\mathbb{R}^M)$, $C^m(\Omega;\mathbb{R}^M)$, $C^\infty(\Omega;\mathbb{R}^M)$, $C_c(\Omega;\mathbb{R}^M)$, $C_c^m(\Omega;\mathbb{R}^M)$, $C_c^\infty(\Omega;\mathbb{R}^M)$, $L^p(\Omega;\mathbb{R}^M)$ and $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$. Furthernore, we denote the norm of a function $u \in L^p(\Omega;\mathbb{R}^M)$ by

$$\|u\|_{L^p(\Omega;\mathbb{R}^M)} := \left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$, and by

$$\|u\|_{L^\infty(\Omega;\mathbb{R}^M)} := \operatorname{ess\,sup}\{|u(x)| : x \in \Omega\}$$

for $p = \infty$. Let $\|u\|_{L^p(\Omega)} := \|u\|_{L^p(\Omega;\mathbb{R})}$ and note that in the sequel, we will sometimes use the shorter notation $\|u\|_p := \|u\|_{L^p(\Omega)}$.

2.2 Sobolev Spaces

Definition 2.2.1. Let $1 \le p \le \infty$. The *Sobolev space* $W^{1,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ whose distributional first-order partial derivatives belong to $L^p(\Omega)$, i.e., for all $i = 1, \dots, d$, there exists a function $v_i \in L^p(\Omega)$ such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x = -\int_\Omega v_i \phi \,\mathrm{d}x$$

for all $\phi \in C_c^\infty(\Omega)$. The function v_i is called the *weak*, or *distributional*, *partial derivative* of u with respect to x_i and it is denoted by $\frac{\partial u}{\partial x_i}$ or $\partial_i u$.

For $u \in W^{1,p}(\Omega)$ we set

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}\right).$$

In the remaining of this section, let m, $M \in \mathbb{N}$, $1 \le p \le \infty$, and for a given multi-index $\beta = (\beta_1, \dots, \beta_d) \in (\mathbb{N}_0)^d$ set

$$\partial^\beta u := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}},$$

where $|\beta| = \beta_1 + \dots + \beta_d$.

We define by induction

$$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \text{ for all } i = 1, \ldots, d \right\}$$

for $m \geq 2$, and

$$W^{m,p}_{\mathrm{loc}}(\Omega) := \{ u \in L^1_{\mathrm{loc}}(\Omega) : u \in W^{m,p}(U) \text{ for all open sets } U \subset\subset \Omega \} .$$

We denote by $W^{m,p}(\Omega;\mathbb{R}^M)$ the space of functions $u = (u_1, \ldots, u_M)$ such that $u_j \in W^{m,p}(\Omega)$ for all $j = 1, \ldots, M$, and we use the notation $H^m(\Omega;\mathbb{R}^M) = W^{m,2}(\Omega;\mathbb{R}^M)$.

We recall that $W^{m,p}(\Omega;\mathbb{R}^M)$ is a Banach space endowed with the norm defined for $u \in W^{m,p}(\Omega;\mathbb{R}^M)$ by

$$\|u\|_{W^{m,p}(\Omega;\mathbb{R}^M)} := \left(\sum_{0 \leq |\beta| \leq m} \|\partial^\beta u\|^p_{L^p(\Omega;\mathbb{R}^M)} \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$, and

$$\|u\|_{W^{m,\infty}(\Omega;\mathbb{R}^M)} := \max_{0 \leq |\beta| \leq m} \|\partial^\beta u\|_\infty$$

if $p = \infty$.

2.3 Interpolation Inequalities

In this section we present some interpolation inequalities that will be useful to establish the results of Chapter 3.

Definition 2.3.1. Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$, and let $I \subset \mathbb{R}$ be a bounded open interval with length b. The *Sobolev space* $W^{m,p}_{\#}(I)$ is the space of all functions in $W^{m,p}_{\mathrm{loc}}(\mathbb{R})$ that are b-periodic, endowed with the norm of $W^{m,p}(I)$.

The following interpolation inequalities are essentially contained in [2] and in the Appendix of [39] (see also [53]).

Theorem 2.3.2. *Let $I \subset \mathbb{R}$ be a bounded open interval. Let j, m be positive integers such that $0 \leq j < m$, and let $1 \leq p \leq q \leq \infty$ be such that $mp > 1$. Then, there exists a constant $K > 0$ such that for all $f \in W^{m,p}_{\#}(I)$*

$$\|f^{(j)}\|_{L^p(I)} \leq K \|f^{(m)}\|^{\frac{j}{m}}_{L^p(I)} \|f\|^{\frac{m-j}{m}}_{L^p(I)} . \tag{2.3.1}$$

In addition, if either f vanishes at the boundary or $\int_I f\,\mathrm{d}x = 0$, then

$$\|f\|_{L^q(I)} \leq K\|f^{(m)}\|^{\theta}_{L^p(I)}\|f\|^{1-\theta}_{L^p(I)}\,, \tag{2.3.2}$$

where $\theta := \frac{1}{m}\left(\frac{1}{p} - \frac{1}{q}\right)$.

From Theorem 2.3.2 we deduce another interpolation inequality.

Corollary 2.3.3. *Let $I \subset \mathbb{R}$ be a bounded open interval. Let j, m be positive integers such that $0 < j < m$ and let $1 \leq p \leq q \leq \infty$ be such that $(m-j)p > 1$. Then, there exists a constant $K > 0$ such that for all $f \in W^{m,p}_{\#}(I)$*

$$\|f^{(j)}\|_{L^q(I)} \leq K\|f^{(m)}\|^{\eta}_{L^p(I)}\|f\|^{1-\eta}_{L^p(I)}\,, \tag{2.3.3}$$

where $\eta := \frac{1}{m}\left(\frac{1}{p} - \frac{1}{q} + j\right)$.

Proof. Since $f^{(j)} \in W^{m-j,p}_{\#}(I)$ and $\int_I f^{(j)}\mathrm{d}x = 0$, by (2.3.2) we have

$$\|f^{(j)}\|_{L^q(I)} \leq K\|f^{(m)}\|^{\theta}_{L^p(I)}\|f^{(j)}\|^{1-\theta}_{L^p(I)}\,,$$

with $\theta := \frac{1}{m-j}\left(\frac{1}{p} - \frac{1}{q}\right)$, which, together with (2.3.1), yields (2.3.3). ☐

2.4 Functions of Bounded Variation

In this section we introduce the space of functions of bounded variation and the related properties used in the sequel.

Definition 2.4.1. The space $BV(\Omega)$ of *functions of bounded variation* in Ω is the space of all functions $u \in L^1(\Omega)$ whose distributional first-order partial derivatives are representable by finite Radon measures in Ω, i.e., for all $i = 1, \ldots, d$, there exists a finite signed measure $\mu_i : \mathscr{B}(\Omega) \to \mathbb{R}$ such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega \phi\,\mathrm{d}\mu_i$$

for all $\phi \in C^\infty_c(\Omega)$. The measure μ_i is called the *weak*, or *distributional*, *partial derivative* of u with respect to x_i and it is denoted by $D_i u$.

For $u \in BV(\Omega)$ we set $Du := (D_1 u, \ldots, D_d u)$. Furthermore, we say that $u \in BV_{\mathrm{loc}}(\Omega)$ if $u \in BV(U)$ for every open set U compactly contained in Ω.

Definition 2.4.2 (Variation)**.** Let $u \in L^1_{\mathrm{loc}}(\Omega)$. The *variation* of u in Ω is defined by

$$V(u,\Omega) := \sup\left\{ \int_\Omega u \,\mathrm{div}\varphi \,\mathrm{d}x : \varphi \in C^1_c(\Omega;\mathbb{R}^d),\ \|\varphi\|_\infty \le 1 \right\}.$$

We observe that if $u \in BV(\Omega)$ then the *total variation measure* of Du coincides with the variation of u in Ω, i.e.,

$$|Du|(\Omega) = V(u,\Omega).$$

Furthermore, $BV(\Omega)$ endowed with the norm defined for each $u \in BV(\Omega)$ by

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1} + |Du|(\Omega),$$

is a Banach space, and $W^{1,1}(\Omega) \subset BV(\Omega)$ with strict inclusion.

The following result shows that the approximability by smooth functions with gradients bounded in L^1 characterizes BV functions.

Theorem 2.4.3 (Approximation by Smooth Functions)**.** *Let $u \in L^1(\Omega)$. Then, $u \in BV(\Omega)$ if and only if there exists a sequence $\{u_n\} \subset C^\infty(\Omega)$ converging to u in $L^1(\Omega)$ and satisfying*

$$L := \lim_{n\to\infty} \int_\Omega |\nabla u_n| \,\mathrm{d}x < \infty. \tag{2.4.1}$$

Moreover, the least constant L for which (2.4.1) holds true coincides with $|Du|(\Omega)$.

Definition 2.4.4 (Weakly* Convergence in BV)**.** Let u, $u_n \in BV(\Omega)$. We say that $\{u_n\}$ *weakly* converges* in $BV(\Omega)$ to u if $\{u_n\}$ converges to u in $L^1(\Omega)$ and $\{Du_n\}$ weakly* converges (in the sense of measures) to Du in Ω, i.e.,

$$\lim_{n\to\infty} \int_\Omega \phi \,\mathrm{d}Du_n = \int_\Omega \phi \,\mathrm{d}Du$$

for every $\phi \in C_0(\Omega)$.

We now introduce the notion of extension domains Ω that is used to extend functions that are BV in Ω to functions that are BV in the whole $\mathbb{R}^d$.

Definition 2.4.5 (Extension Domains)**.** We say that an open set $\Omega \subset \mathbb{R}^d$ is an *extension domain for BV* if $\partial\Omega$ is bounded and for any open set $U \supset \overline{\Omega}$ there exists a linear and continuous *extension operator* $\mathscr{E} : BV(\Omega) \to BV(\mathbb{R}^d)$ satisfying

(i) $\mathscr{E}u(x) = 0$ for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d \setminus U$ and for every $u \in BV(\Omega)$,

(ii) $|D(\mathscr{E}u)|(\partial\Omega) = 0$ for every $u \in BV(\Omega)$,

(iii) for every $p \in [1, \infty]$ the restriction of $\mathscr{E}$ to $W^{1,p}(\Omega)$ induces a linear continuous map between this space and $W^{1,p}(\mathbb{R}^d)$.

Note that any open set Ω with compact Lipschitz boundary is an extension domain.

The following compactness property is the justification for the introduction of BV functions since it is satisfied in the BV space but not in the Sobolev space $W^{1,1}$.

Theorem 2.4.6 (Compactness in BV)**.** *Every sequence $\{u_n\} \subset BV_{\mathrm{loc}}(\Omega)$ satisfying*

$$\sup_n \left\{ |Du_n|(U) + \int_U |u_n| \,\mathrm{d}x \right\} < \infty \tag{2.4.2}$$

for each open set $U \subset\subset \Omega$ admits a subsequence $\{u_{n_k}\}$ converging in $L^1_{\mathrm{loc}}(\Omega)$ to $u \in BV_{\mathrm{loc}}(\Omega)$. If Ω is a bounded extension domain for BV and $\{u_n\}$ is bounded in $BV(\Omega)$ we can say that $u \in BV(\Omega)$ and that the subsequence weakly converges to u.*

2.5 Sets of Finite Perimeter

In this section we introduce the main properties of a particular class of BV functions, the characteristic functions of sets of finite perimeter.

Definition 2.5.1 (Sets of Finite Perimeter)**.** Given an $\mathcal{L}^d$-measurable subset E of $\mathbb{R}^d$, the *perimeter of E in Ω* is defined as the variation of χ_E in Ω, i.e.,

$$P(E, \Omega) := \sup \left\{ \int_E \mathrm{div}\varphi \,\mathrm{d}x : \varphi \in C^1_c(\Omega; \mathbb{R}^d),\ \|\varphi\|_\infty \le 1 \right\}.$$

If $P(E, \Omega) < \infty$, then we say that E is a *set of finite perimeter in Ω*.

We observe that if $|E \cap \Omega| < \infty$, then E has finite perimeter in Ω if and only if $\chi_E \in BV(\Omega)$.

Definition 2.5.2. An $\mathcal{L}^d$-measurable subset E of $\mathbb{R}^d$ is said to be a set a *set of locally finite perimeter in Ω* if $\chi_E \in BV_{\mathrm{loc}}(\Omega)$. Let $\mathscr{P}(\Omega)$ denote the collection of $\mathcal{L}^d$-measurable subsets of $\mathbb{R}^d$ of locally finite perimeter in Ω.

Definition 2.5.3 (Convergence of Sets)**.** Let $n \in \mathbb{N}$, and let E_n and E be sets with finite Lebesgue measure in Ω. If $|\Omega \cap (E_n \triangle E)|$ converges to 0 as $n \to \infty$, then we say that $\{E_n\}$ *converges to E in $L^1(\Omega)$* and we write $E_n \to E$ in $L^1(\Omega)$. If $\{E_n\}$ *converges to E in $L^1(U)$* for any open $U \subset\subset \Omega$, then we say that $\{E_n\}$ *converges to E locally in $L^1(\Omega)$* and we write that $E_n \to E$ in $L^1_{\mathrm{loc}}(\Omega)$.

Note that $E_n \to E$ in $L^1(\Omega)$ if and only if $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$. Furthermore, for measurable sets, local convergence in $L^1(\Omega)$ is equivalent to convergence in $L^1(\Omega)$ when Ω has finite measure.

A consequence of the compactness Theorem 2.4.6 is that sequences of sets with locally equibounded perimeter are relatively compact with respect to the local convergence in $L^1(\Omega)$.

Theorem 2.5.4. *Let $\{E_n\}$ be a sequence of $\mathcal{L}^d$-measurable sets satisfying*

$$\sup_n P(E_n, U) < \infty$$

for each open set $U \subset\subset \Omega$. Then there exists a subsequence $\{E_{n_k}\}$ locally converging in $L^1(\Omega)$. If $|\Omega| < \infty$ then the subsequence converges in $L^1(\Omega)$.

In the following definition we introduce another notion of boundary.

Definition 2.5.5 (Reduced Boundary)**.** Let $E \in \mathscr{P}(\Omega)$. The *reduced boundary* $\partial^* E$ is the collection of points $x \in \operatorname{supp}|D\chi_E| \cap \Omega$ such that the limit

$$\nu_E(x) := \lim_{\varrho \searrow 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))}$$

exists in $\mathbb{R}^d$ and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : \partial^* E \to \mathbb{S}^{d-1}$ is called the *generalized inner normal* to E.

We have that $\partial^* E$ is a Borel set and $\nu_E : \partial^* E \to \mathbb{S}^{d-1}$ is a Borel map.

Definition 2.5.6 (Rectifiable Sets)**.** Let $k \in [0, d]$ be an integer and let A be a $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$.

(i) We say that A is *countably k-rectifiable* if there exist countably many Lipschitz functions $f_j : \mathbb{R}^k \to \mathbb{R}^d$ such that

$$A \subset \bigcup_{j=0}^{\infty} f_j(\mathbb{R}^{d-1}) .$$

(ii) We say that A is *countably $\mathcal{H}^k$-rectifiable* if there exist countably many Lipschitz functions $f_j : \mathbb{R}^k \to \mathbb{R}^d$ such that

$$\mathcal{H}^{d-1}\left(A \setminus \bigcup_{j=0}^{\infty} f_j(\mathbb{R}^{d-1})\right) = 0 .$$

(iii) We say that A is $\mathcal{H}^k$*-rectifiable* if A is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(A) < \infty$.

For $k = 0$ countably k-rectifiable and countably $\mathcal{H}^k$-rectifiable sets correspond to finite or countable sets, while $\mathcal{H}^k$-rectifiable sets correspond to finite sets.

Theorem 2.5.7. *Let $E \in \mathscr{P}(\mathbb{R}^d)$. Then $\partial^* E$ is countably $(d-1)$-rectifiable and*

$$|D\chi_E| = \mathcal{H}^{d-1}\lfloor_{\partial^* E}\,. \tag{2.5.1}$$

In addition, for any $x_0 \in \partial^* E$ the sets $(E - x_0)/\varrho$ locally converge in $L^1(\mathbb{R}^d)$ as $\varrho \searrow 0$ to the halfspace orthogonal to $\nu_E(x_0)$ that contains $\nu_E(x_0)$, and

$$\lim_{\varrho \searrow 0} \frac{\mathcal{H}^{d-1}(\partial^* E \cap B_\varrho(x_0))}{\omega_{d-1}\varrho^{d-1}} = 1\,.$$

The following generalized Gauss-Green formula holds for sets E of finite perimeter in Ω:

$$\int_E \operatorname{div}\varphi \,\mathrm{d}x = -\int_{\partial^* E} \langle \nu_E, \varphi \rangle \,\mathrm{d}\mathcal{H}^{d-1} \tag{2.5.2}$$

for each $\varphi \in C_c^1(\Omega; \mathbb{R}^d)$.

Definition 2.5.8 (Essential Boundary)**.** Let $\theta \in [0,1]$ and E be a $\mathcal{L}^d$ -measurable subset of $\mathbb{R}^d$. Denote by E^θ the set of all points where E has density θ, i.e.

$$\left\{ x \in \mathbb{R}^d : \lim_{\varrho \searrow 0} \frac{|E \cap B_\varrho(x)|}{|B_\varrho(x)|} = \theta \right\}.$$

We call *measure theoretic interior* and *measure theoretic exterior of* E the sets E^0 and E^1, respectively, and we define the *essential boundary of* E as the set $\partial^* E := \mathbb{R}^d \setminus (E^0 \cup E^1)$ of points where the density is neither 0 nor 1.

Note that E^θ is a Borel set for every $\theta \in [0,1]$.

Theorem 2.5.9. *Let E be a set of finite perimeter in Ω. Then*

$$\partial^* E \cap \Omega \subset E^{\frac{1}{2}} \subset \partial_* E \quad \textit{and} \quad \mathcal{H}^{d-1}\left(\Omega \setminus \left(E^0 \cup \partial^* E \cup E^1\right)\right) = 0\,.$$

In addition, E has density either 0 or $\frac{1}{2}$ or 1 at $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$, and $\mathcal{H}^{d-1}$-a.e. $x \in \partial_* E \cap \Omega$ belongs to $\partial^* E$.

Due to the previous theorem, in the Gauss-Green formula (2.5.2) for sets of finite perimeter we may replace $\partial^* E$ both with $\partial_* E$ and with $E^{\frac{1}{2}}$. In addition, by Definition 2.5.1 and (2.5.1) we have

$$P(E,\Omega) = |D\chi_E|(\Omega) = \mathcal{H}^{d-1}(\Omega \cap \partial^* E) = \mathcal{H}^{d-1}(\Omega \cap \partial_* E) = \mathcal{H}^{d-1}(\Omega \cap E^{\frac{1}{2}})\,.$$

We conclude this section with a simple property that will be used in Corollary 4.3.9.

Proposition 2.5.10. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let μ be a positive Radon measure in Ω. Assume that there exist $s \in (0,\infty)$ and $B \in \mathscr{B}(\Omega)$ such that*

$$\limsup_{\varrho \searrow 0} \frac{\mu(B_\varrho(x))}{\omega_{d-1}\varrho^{d-1}} \geq s$$

for every $x \in B$. Then

$$\mu \geq s\mathcal{H}^{d-1}\lfloor_B\,.$$

2.6 Embedding Theorems and Isoperimetric Inequalities

In this section we recall higher integrability properties of BV functions and important inequalities for sets of finite perimeter.

In the sequel, given a function $u \in L^1(\Omega)$ we denote its mean value by

$$u_\Omega := \fint_\Omega u(x)\,\mathrm{d}x = C\frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x\,.$$

Theorem 2.6.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded connected extension domain for BV. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\int_\Omega |u - u_\Omega|\,\mathrm{d}x \leq C|Du|(\Omega) \tag{2.6.1}$$

for every function $u \in BV(\Omega)$.

We remark that the constant C in (2.6.1) depends only on Ω. Moreover, if we apply the previous theorem to balls $B_\varrho(x) \subset \mathbb{R}^d$, then the constant $C(B_\varrho(x))$ does not depend on the points x and a simple scaling argument shows that $C(B_\varrho(x)) = \gamma_1\varrho$, where γ_1 is the dimensional constant relative to the unit ball, i.e.,

$$\gamma_1 := C(B_1(0))\,.$$

From Theorem 2.6.1 it follows that sets of finite perimeter satisfy the isoperimetric inequality.

Theorem 2.6.2 (Isoperimetric Inequality)**.** *Let $d > 1$. If E is a set of finite perimeter in $\mathbb{R}^d$ then either E or $\mathbb{R}^d \setminus E$ has finite Lebesgue measure, and*

$$\min\left\{|E|, |\mathbb{R}^d \setminus E|\right\} \le \gamma_2 \left[P(E, \mathbb{R}^d)\right]^{\frac{d}{d-1}}$$

for some dimensional constant $\gamma_2 > 0$.

Let $1 \le p \le d$ and define p^* by

$$p^* := \begin{cases} \dfrac{dp}{d-p} & \text{if } p < d, \\ +\infty & \text{if } p = d\,. \end{cases}$$

Theorem 2.6.3. *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies $V(u, \mathbb{R}^d) < \infty$ then there exists $m \in \mathbb{R}$ such that*

$$\|u - m\|_{L^{1^*}(\mathbb{R}^d)} \le \gamma_3 V(u, \mathbb{R}^d)\,. \tag{2.6.2}$$

for a dimensional constant $\gamma_3 > 0$. If $u \in L^1(\mathbb{R}^d)$ then $m = 0$, $u \in BV(\mathbb{R}^d)$, and thus $\|u\|_{L^{1^*}} \le \gamma_3 |Du|(\mathbb{R}^d)$. In particular, the embedding $BV(\mathbb{R}^d) \hookrightarrow L^{1^*}(\mathbb{R}^d)$ is continuous.

We now state the continuous and compact embedding properties of the BV space in the Lebesgue spaces L^p.

Theorem 2.6.4 (Embedding Theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded extension domain for BV. Then the embedding $BV(\Omega) \hookrightarrow L^{1^*}(\Omega)$ is continuous and the embeddings $BV(\Omega) \hookrightarrow L^p(\Omega)$ are compact for $1 \le p < 1^*$.*

From Theorems 2.6.1 and 2.6.4 we obtain a Poincaré inequality in BV.

Proposition 2.6.5 (Poincaré Inequality in BV)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded connected extension domain for BV. Then there exists a constant $\overline{C} = \overline{C}(\Omega) > 0$ such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \le \overline{C} |Du|(\Omega)$$

for every $u \in BV(\Omega)$ and for $1 \le p \le 1^*$.

Using the previous proposition applied to balls $B_\varrho(x) \subset \mathbb{R}^d$ and a scaling argument we obtain that there exists a dimensional constant $\gamma_4 > 0$ such that

$$\|u - u_{B_\varrho(x)}\|_{L^p(B_\varrho(x))} \le \gamma_4 \varrho^{\frac{d}{p}} \frac{|Du|(B_\varrho(x))}{\varrho^{d-1}} \tag{2.6.3}$$

for every $u \in BV(B_\varrho(x))$ and for $1 \le p \le 1^*$. Moreover, from (2.6.3) with $p = 1^*$ we deduce the *relative isoperimetric inequality* for sets E of locally finite perimeter in $\mathbb{R}^d$, precisely

$$\min\left\{|B_\varrho(x) \cap E|^{\frac{d-1}{d}}, |B_\varrho(x) \setminus E|^{\frac{d-1}{d}}\right\} \le \gamma_5 P(E, B_\varrho(x)) \tag{2.6.4}$$

for a dimensional constant $\gamma_5 > 0$.

Definition 2.6.6 (Medians)**.** Let $B_\varrho(x) \subset \mathbb{R}^d$ and consider a measurable function $u : B_\varrho(x) \to \mathbb{R}$. We say that $m \in \mathbb{R}$ is a *median of* u *in* $B_\varrho(x)$ if

$$|\{u < t\}| \le \frac{|B_\varrho(x)|}{2} \quad \text{for } t < m, \quad \text{and} \quad |\{u > t\}| \le \frac{|B_\varrho(x)|}{2} \quad \text{for } t > m.$$

The existence of medians can be established by a simple continuity argument (see also Remark 2.10.5 below).

In view of (2.6.4), we obtain a local version of Theorem 2.6.3 in which $\mathbb{R}^d$ is replaced by $B_\varrho(x) \subset \mathbb{R}^d$ and the constant m (see (2.6.5)) is a median of u in $B_\varrho(x)$.

Theorem 2.6.7. *If $u \in BV(B_\varrho(x))$ and m is a median of u in $B_\varrho(x)$, then*

$$\|u - m\|_{L^p(B_\varrho(x))} \le \gamma_5 \varrho^{\frac{d}{p}} \frac{|Du|(B_\varrho(x))}{\varrho^{d-1}} \tag{2.6.5}$$

for every $1 \le p \le 1^*$.

We now state the Poincaré inequality for Sobolev spaces analogous to (2.6.3). If $1 \le p < d$ and $u \in W^{1,p}(B_\varrho(x))$, then

$$\|u - u_{B_\varrho(x)}\|_{L^q(B_\varrho(x))} \le \gamma_6 \varrho^{\frac{d}{q} - \frac{d}{p} + 1} \|\nabla u\|_{L^p(B_\varrho(x))} \tag{2.6.6}$$

for every $1 \le q \le p^*$ and for a dimensional constant $\gamma_6 > 0$.

We conclude this section introducing the Campanato Theorem [9, Theorem 7.51] that will be used in Theorem 4.3.8.

Theorem 2.6.8. *Let $u \in L^p(B_{2R}(x_0))$ for some $p \in [1, \infty)$ and $R > 0$. If for some $\alpha \in (0,1]$ and $\gamma > 0$ we have that*

$$⨍_{B_\varrho(x)} |u(y) - u_{x,\varrho}|^p \, \mathrm{d}y \le \gamma^p \left(\frac{\varrho}{R}\right)^{p\alpha}$$

for every ball $B_\varrho(x)$ *with* $\varrho \leq R$ *and* $x \in B_R(x_0)$*, then a representative of* u *is* α*-Hölder continuous in* $B_R(x_0)$*,*

$$|u(x) - u(y)| \leq c\gamma \left(\frac{|x-y|}{R} \right)^\alpha$$

for each x*,* $y \in B_R(x_0)$*, and*

$$\max_{x \in B_R(x_0)} |u(x)| \leq c\gamma + |u_{x_0,R}|\,,$$

where c *is a constant depending only on* d *and* α*.*

2.7 Generalized Area Formula

Throughout this section we let k be an integer such that $k \leq d$ and we denote by $\boldsymbol{G}_k$ the family of unoriented k-dimensional subspaces of $\mathbb{R}^d$.

Definition 2.7.1 (Approximate Tangent Space)**.** Let A be an $\mathcal{H}^k$-measurable subset of $\mathbb{R}^d$ with locally finite $\mathcal{H}^k$-measure and $x \in \mathbb{R}^d$. We say that A has *approximate tangent space* $\pi \in \boldsymbol{G}_k$ at x, and we write

$$\mathrm{Tan}^k(A, x) = \mathcal{H}^k \lfloor_\pi\,,$$

if

$$\lim_{\varrho \searrow 0} \int_{A_{x,\varrho}} \phi(y)\,\mathrm{d}\mathcal{H}^k(y) = \int_\pi \phi(y)\,\mathrm{d}\mathcal{H}^k(y)$$

for all $\phi \in C_c(\mathbb{R}^d)$, where $A_{x,\varrho} := (A - x)/\varrho$.

We identify each $\pi \in \boldsymbol{G}_k$ with the matrix $(\pi_{i,j})$ representing the orthogonal projection onto π with respect to the canonical basis e_1, e_2, $\dots$, e_d and, given a unit vector ν normal to the plane π, we define

$$\pi^\perp x := \langle x, \nu \rangle \nu$$

for each $x \in \mathbb{R}^d$. Hence, the projection πx of a point $x \in \mathbb{R}^d$ onto π is given by

$$\pi x = x - \pi^\perp x\,.$$

Remark 2.7.2 (Lipschitz k-Graphs)**.** Let $\pi \in \boldsymbol{G}_k$ and let $f : \pi \to \pi^\perp$ be a Lipschitz function. We define the set

$$P_f(x) := \{ v + d\,f_{(\pi x)}(v) : v \in \pi \}$$

for each $x \in \Gamma$ such that f is differentiable at πx. Then, the graph of f, i.e.,

$$\Gamma_f := \{x \in \mathbb{R}^d : f(\pi x) = \pi^\perp x\}$$

is countably k-rectifiable, and we have that

$$\operatorname{Tan}^k(\Gamma_f, x) = \mathcal{H}^k \lfloor_{P_f(x)}$$

for $\mathcal{H}^k$-a.e. $x \in \Gamma_f$.

We now state the locality property of approximate tangent spaces.

Proposition 2.7.3. *For $i = 1, 2$ let A_i be countably $\mathcal{H}^k$-rectifiable sets contained in $\mathbb{R}^d$. If π_i are the approximate tangent space to A_i, then*

$$\pi_1(x) = \pi_2(x)$$

for $\mathcal{H}^k$-a.e. $x \in A_1 \cap A_2$.

In view of Remark 2.7.2, the previous proposition implies that if $A \subset \mathbb{R}^d$ is a countably $\mathcal{H}^k$-rectifiable set and $\{\Gamma_{f_i}\}$ is a partition of $\mathcal{H}^k$-almost all of A into k-graphs of Lipschitz functions f_i, then

$$\operatorname{Tan}^k(A, x) := \mathcal{H}^k \lfloor_{P_{f_i}(x)}$$

for each $x \in \Gamma_{f_i}$ where $P_{f_i}(x)$ is defined.

Proposition 2.7.4. *Let $\phi : \mathbb{R}^k \to \mathbb{R}^d$ be a one-to-one Lipschitz function and let $E \subset \mathbb{R}^k$ be a $\mathcal{L}^k$-measurable set. Then the set $A = \phi(E)$ satisfies*

$$\operatorname{Tan}^k(A, x) = d\,\phi_{\phi^{-1}(x)}(\mathbb{R}^k)$$

for $\mathcal{H}^k$-a.e. $x \in A$.

Rademacher's Theorem (see for example [52, Theorem 11.49]) provides no information about the differentiability of a Lipschitz function f defined in a k-dimensional subset A of $\mathbb{R}^k$ with $k < d$ since A is $\mathcal{L}^d$-negligible. However, we can prove that a "tangential" differential does exist $\mathcal{H}^k$-almost everywhere, if A is countably $\mathcal{H}^k$-rectifiable.

Definition 2.7.5 (Tangential Differential of Lipschitz Functions)**.** Let A be a countably $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz function. We say that f is *tangentially differntiable at* $x \in A$ if the restriction of f to the affine space $x + \operatorname{Tan}^k(A, x)$ is differentiable at x. We denote the *tangential differential* that is a linear map between the spaces $\operatorname{Tan}^k(A, x)$ and $\mathbb{R}^m$ by $d^A f_x$.

We note that if f is differentiable at x, then $d^A f_x$ is the restriction of the differential $d\, f_x$ to $\mathrm{Tan}^k(A, x)$, provided that the approxiamte tangent space exists.

The following result is an extension of Rademacher's Theorem.

Theorem 2.7.6 (Tangential differentiability)**.** *Let A be a countably $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz function. Then, f is tangentially differntiable at $\mathcal{H}^k$-a.e. $x \in A$.*

Using tangential differentials and recalling the definition of k-Dimensional Jacobian of a linear map we can prove a generalized area formula.

Definition 2.7.7 (k-Dimensional Jacobian)**.** The *k-dimensional Jacobian* of a linear map $L : \mathbb{R}^k \to \mathbb{R}^d$ is denoted by $\boldsymbol{J}_k L$ and it is defined by

$$\boldsymbol{J}_k L := \sqrt{\det\left(\mathrm{L}^* \circ \mathrm{L}\right)}\,,$$

where L^* stands for the adjoint of L.

Theorem 2.7.8 (Generalized Area Formula)**.** *Let A be a countably $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz function. Then, the multiplicity function $\mathcal{H}^0\left(A \cap f^{-1}(y)\right)$ is $\mathcal{H}^k$-measurable in $\mathbb{R}^m$ and*

$$\int_{\mathbb{R}^m} \mathcal{H}^0\left(A \cap f^{-1}(y)\right) \mathrm{d}\mathcal{H}^k(y) = \int_A \boldsymbol{J}_k d^A f_x \,\mathrm{d}\mathcal{H}^k(x)\,.$$

2.8 Approximate Continuity and Differentiability

In this section we introduce the notions of approximate continuity, approximate jump points and approximate differentiability that will be used in the next section to study the fine properties of BV functions.

Definition 2.8.1 (Approximate Limit)**.** Let $u \in L^1_{\mathrm{loc}}(\Omega)$. We say that u has an *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{\varrho \searrow 0} \fint_{B_\varrho(x)} |u(y) - z|\,\mathrm{d}y = 0\,. \tag{2.8.1}$$

We call the set S_u of points at which u has no approximate limit the *approximate discontinuity set* of u. If $x \in \Omega \setminus S_u$ then the vector z, that is uniquely determined by (2.8.1), is called the approximate limit of u at x, and is denoted by $\tilde{u}(x)$. We say that u is *approximate continuous at x* if $x \in \Omega \setminus S_u$ and $\tilde{u}(x) = u(x)$, i.e. x is a Lebesgue point of u.

Observe that the essential boundary $\partial_* E$ of a $\mathcal{L}^d$ -measurable subset E of $\mathbb{R}^d$, that we introduced in Definition 2.5.8, coincides with the approximate discontinuity set S_{χ_E}.

Proposition 2.8.2 (Properties of Approximate Limits)**.** *Let* $u \in L^1_{\mathrm{loc}}(\Omega)$. *The following assertions hold true:*

(i) S_u *is a* $\mathcal{L}^d$*-negligible Borel set, and* $\tilde{u} : \Omega \setminus S_u \to \mathbb{R}$ *is a Borel function that coincides with* u *for* $\mathcal{L}^d$*-a.e.* $x \in \Omega \setminus S_u$*;*

(ii) if $f : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz map and* $v = f \circ u$*, then* $S_v \subset S_u$ *and* $\tilde{v}(x) = f(\tilde{u}(x))$ *for every* $x \in \Omega \setminus S_u$*.*

We now give a definition of approximate jump discontinuity between two values a and b along a direction ν. For this purpose we denote the two half balls contained in $B_\varrho(x)$ and determined by ν, by

$$B^+_\varrho(x, \nu) := \{y \in B_\varrho(x) : \langle y - x, \nu\rangle > 0\}$$

and

$$B^-_\varrho(x, \nu) := \{y \in B_\varrho(x) : \langle y - x, \nu\rangle < 0\} .$$

Definition 2.8.3 (Approximate Jump Points)**.** Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and let $x \in \Omega$. We say that x is an *approximate jump point of* u if there exist a, $b \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$, such that $a \neq b$ and

$$\lim_{\varrho \searrow 0} \fint_{B^+_\varrho(x,\nu)} |u(y) - a| \,\mathrm{d}y = 0 , \quad \lim_{\varrho \searrow 0} \fint_{B^-_\varrho(x,\nu)} |u(y) - b| \,\mathrm{d}y = 0 . \tag{2.8.2}$$

We denote the triplet (a, b, ν), that is uniquely determined by (2.8.2) up to a permutation of (a, b) and a change of sign of ν, by $(u^+(x), u^-(x), \nu_u(x))$. Furthermore, we call *approximate jump set of* u the set J_u of approximate jump points of u.

We say that the two triples (a, b, ν) and (a', b', ν') are equivalent if

$$\text{either} \quad (a, b, \nu) = (a', b', \nu') \quad \text{or} \quad (a, b, \nu) = (b', a', -\nu') . \tag{2.8.3}$$

Proposition 2.8.4 (Properties of the Approximate Jump Set)**.** *Let* $u \in L^1_{\mathrm{loc}}(\Omega)$. *The following assertions hold:*

(i) J_u *is a Borel subset of* S_u *and there exists a Borel function*

$$\Big(u^+(\cdot), u^-(\cdot), \nu_u(\cdot)\Big) : J_u \to \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{d-1}$$

such that (2.8.2) *is satisfied for every* $x \in J_u$*;*

(ii) if $f : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz map,* $v := f \circ u$*, and* $x \in J_u$*, then* $x \in J_v$ *if and only if* $f(u^+(x)) \neq f(u^-(x))$*. In addition, if* $x \in J_v$ *then*

$$\Big(v^+(x), v^-(x), \nu_v(x)\Big) = \Big(f(u^+(x)), f(u^-(x)), \nu_u(x)\Big) ,$$

while if $x \notin J_v$*, then* $x \notin S_v$ *and* $\tilde{v} = f(u^+(x)) = f(u^-(x))$*.*

We also introduce a notion of approximate differentiability for functions in $L^1_{\text{loc}}(\Omega)$.

Definition 2.8.5 (Approximate Differentiability)**.** Let $u \in L^1_{\text{loc}}(\Omega)$ and let $x \in \Omega \setminus S_u$. We say that u is *approximate differentiable at* x if there exists $v \in \mathbb{R}^d$ such that

$$\lim_{\varrho \searrow 0} \fint_{B^+_\varrho(x,\nu)} \frac{|u(y) - \tilde{u}(x) - v \cdot (y-x)|}{\varrho} \, \mathrm{d}y = 0 . \tag{2.8.4}$$

We denote the set of approximate differentiable points of u by $\mathcal{D}_u$. Furthermore, if u is approximate differentiable at x, then the vector v, that is uniquely determined by (2.8.4), is called *approximate differential of* u *at* x and is denoted by $\nabla u(x)$.

Proposition 2.8.6 (Properties of the Approximate Differentiability Set)**.** *Let* $u \in L^1_{\text{loc}}(\Omega)$*. The following assertions hold true:*

(i) the set $\mathcal{D}_u \in \Omega \setminus S_u$ *of points where* u *is approximately differentiable is a Borel subset of* Ω*, and* $\nabla u : \mathcal{D}_u \to \mathbb{R}^d$ *is a Borel map;*

(ii) if $x \in \mathcal{D}_u$ *and* $f : \mathbb{R} \to \mathbb{R}$ *is a function with linear growth at infinity and differentiable at* $\tilde{u}(x)$*, then* $v = f \circ u$ *is approximately differentiable at* x*, and*

$$\nabla v(x) = \nabla f(\tilde{u}(x)) \nabla u(x) .$$

Proposition 2.8.7 (Locality Properties)**.** *Let* $u, v \in L^1_{\text{loc}}(\Omega)$*. The following hold:*

(i) $\tilde{u}(x) = \tilde{v}(x)$ *for every point* $x \in \Omega \setminus (S_u \cup S_v)$ *of density* 1 *of* $\{u = v\}$*, and hence for* $\mathcal{L}^d$*-a.e.* $x \in \{u = v\}$*;*

(ii) if $x \in J_u \cap J_v$ *and* $\{u = v\}$ *has density* 1 *at* x*, then* $(u^+(x), u^-(x), \nu_u(x))$ *is equivalent to* $(v^+(x), v^-(x), \nu_v(x))$ *(see* (2.8.3)*);*

(iii) $\nabla u(x) = \nabla v(x)$ *for every* $x \in \mathcal{D}_u \cap \mathcal{D}_v$ *of density* 1 *of* $\{u = v\}$*, and hence for* $\mathcal{L}^d$*-a.e.* $x \in \{u = v\} \cap \mathcal{D}_u \cap \mathcal{D}_v$*.*

2.9 Fine Properties of BV Functions

In this section we recall the approximate continuity and differentiability properties of a generic function $u \in BV(\Omega)$ and we analyse the decomposition of its distributional derivative Du in different terms.

The following result asserts that the mean value of $|u|^{1^*}$ on balls $B_\varrho(x)$ is uniformly bounded as $\varrho \searrow 0$ for $\mathcal{H}^{d-1}$-a.e. point x.

Lemma 2.9.1. *If $u \in BV(\Omega)$ then the set*

$$I := \left\{ x \in \Omega : \limsup_{\varrho \searrow 0} \fint_{B_\varrho(x)} |u(y)|^{1^*} \, \mathrm{d}y = \infty \right\}$$

is $\mathcal{H}^{d-1}$-negligible.

We now compare $|Du|$ with $\mathcal{H}^{d-1}$, and note that $|Du|$ vanishes on any $\mathcal{H}^{d-1}$-negligible set.

Lemma 2.9.2. *Let $u \in BV(\Omega)$. Then*

$$|Du| \geq |u^+ - u^-| \mathcal{H}^{d-1} \lfloor_{J_u} .$$

Moreover, for every Borel set $B \subset \Omega$ the following two assertions hold true:

(i) if $\mathcal{H}^{d-1}(B) = 0$, then $|Du|(B) = 0$;

(ii) if $\mathcal{H}^{d-1}(B) < \infty$ and $B \cap S_u = \varnothing$, then $|Du|(B) = 0$.

If $u \in BV(\Omega)$ then by the Radon-Nikodým Theorem we have

$$Du = D^a u + D^s u ,$$

where $D^a u$ and $D^s u$ are the absolutely continuous and singular part with respect to $\mathcal{L}^d$, respectively. We may further decompose the singular part $D^s u$.

Definition 2.9.3 (Jump and Cantor Parts)**.** If $u \in BV(\Omega)$ then the measures

$$D^j u := D^s u \lfloor_{J_u} \quad \text{and} \quad D^c u := D^s u \lfloor_{(\Omega \setminus S_u)} ,$$

are called *jump part* of the derivative and *Cantor part* of the derivative, respectively.

In analogy with what was established in Theorems 2.5.7 and 2.5.9 for the essential boundary $\partial_* E$ of a set E of finite perimeter in Ω, the following result asserts that the discontinuity set S_u is $\mathcal{H}^{d-1}$-rectifiable, and $\mathcal{H}^{d-1}$-almost every point in S_u is an approximate jump point.

Theorem 2.9.4 (Federer-Vol'pert)**.** *For every $u \in BV(\Omega)$ the discontinuity set S_u is countably $\mathcal{H}^{d-1}$-rectifiable and $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$. Moreover,*

$$Du\lfloor_{J_u} = (u^+ - u^-)\nu_u \mathcal{H}^{d-1}\lfloor_{J_u} .$$

Since by Theorems 2.9.2 and 2.9.4 Du vanishes on the $S_u \setminus J_u$, from Definition 2.9.3 we obtain that Du may be decomposed as

$$Du = D^a u + D^j u + D^c u . \tag{2.9.1}$$

The following theorem states that $D^a u = \nabla u \mathcal{L}^d$, where ∇u is the approximate differential of u (see Definition 2.8.5).

Theorem 2.9.5 (Calderón-Zygmund)**.** *If $u \in BV(\Omega)$ then u is approximately differentiable at $\mathcal{L}^d$-a.e. point in Ω, and the approximate differential ∇u is the density of the absolutely continuous part of Du with respect to $\mathcal{L}^d$.*

Moreover, given a function $u \in BV(\Omega)$ we have that $D^j u = Du\lfloor_{J_u}$, and for every $B \in \mathscr{B}(\Omega)$

$$D^j u(B) = \int_{B \cap J_u} (u^+(x) - u^-(x))\nu_u(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) . \tag{2.9.2}$$

In the following proposition we state the main properties of the three components of Du.

Theorem 2.9.6 (Properties of $D^a u$, $D^j u$, and $D^c u$)**.** *Let $u \in BV(\Omega)$. The following hold:*

(i) $D^a u = Du\lfloor_{(\Omega \setminus \Upsilon_u)}$ *and* $D^s u = Du\lfloor_{\Upsilon_u}$ *where* Υ_u *is defined by*

$$\Upsilon_u := \left\{ x \in \Omega : \lim_{\varrho \searrow 0} \frac{|Du|(B_\varrho(x))}{\varrho^d} = \infty \right\} .$$

Moreover, if $B \in \mathbb{R}$ is $\mathcal{H}^1$-negligible Borel set, then ∇u vanishes $\mathcal{L}^d$-a.e. on $u^{-1}(B)$.

(ii) Let $\Theta_u \subset \Upsilon_u$ be the set defined by

$$\Theta_u := \left\{ x \in \Omega : \lim_{\varrho \searrow 0} \frac{|Du|(B_\varrho(x))}{\varrho^{d-1}} > 0 \right\} .$$

Then $J_u \subset \Theta_u$, $\mathcal{H}^{d-1}(\Theta_u \setminus J_u) = 0$, and $D^j u = Du\lfloor_{\Theta_u}$. More generally, $D^j u = Du\lfloor_{\Xi}$ for every Borel set Ξ containing J_u and σ-finite with respect to $\mathcal{H}^{d-1}$.

(iii) $D^c u = Du\lfloor_{(\Upsilon_u \setminus \Theta_u)}$, $D^c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^{d-1}$ and on sets of the form $\tilde{u}^{-1}(B)$ with $B \subset \mathbb{R}$, $\mathcal{L}^1(B) = 0$.

Remark 2.9.7. Let $u \in BV(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function satisfying $f(0) = 0$ if $|\Omega| = \infty$. By Rademacher Theorem (see for example [52, Theorem 11.49]) we have that f is differentiable at $\mathcal{L}^d$-a.e. point of $\mathbb{R}$. Furthermore, by Theorem 2.9.6 (i) we have that $\nabla u = 0$ $\mathcal{L}^d$-a.e. in the set where $f'(u)$ is not defined. Therefore, the map $w = f'(u)\nabla u$ is a well-defined map if we assume that $w(x) = 0$ or, equivalently, that $f'(u(x))$ is an arbitrary real number, when f is not differentiable at $u(x)$. Similarly, by Theorem 2.9.6 (iii), $f'(\tilde{u})D^c u$ is a well-defined measure since $f'(\tilde{u})$ is undefined at most on a $|D^c u|$-negligible set.

In view of the previous remark, the following chain rule holds for BV functions.

Theorem 2.9.8 (Chain Rule in BV)**.** *Let $u \in BV(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function satisfying $f(0) = 0$ if $|\Omega| = \infty$. Then $v = f \circ u \in BV(\Omega)$ and*

$$Dv = f'(u)\nabla u \mathcal{L}^d + \Big(f(u^+) - f(u^-) \Big) \nu_u \mathcal{H}^{d-1}\lfloor_{J_u} + f'(\tilde{u}) D^c u \,.$$

2.10 Special Functions of Bounded Variation

In this section we define the space of special functions of bounded variation and we describe the main properties of these functions.

Definition 2.10.1. We say that $u \in BV(\Omega)$ is a *special function of bounded variation* in Ω, and we write $u \in SBV(\Omega)$, if the Cantor part of its distributional derivative is zero.

Note that $SBV(\Omega)$ is a proper subspace of $BV(\Omega)$ and that, from (2.9.1), Theorems 2.9.4 and 2.9.5, it follows that

$$Du = D^a u + D^j u = \nabla u \mathcal{L}^d + (u^+(x) - u^-(x))\nu_u(x)\mathcal{H}^{d-1}\lfloor_{J_u} \tag{2.10.1}$$

for every $u \in SBV(\Omega)$. Furthermore, $W^{1,1}(\Omega) \subset SBV(\Omega)$, and by (2.10.1) we have that $u \in W^{1,1}(\Omega)$ if and only if $\mathcal{H}^{d-1}(S_u) = 0$. Thus, also the inclusion of $W^{1,1}(\Omega)$ in $SBV(\Omega)$ is strict.

Theorem 2.10.2 (Closure in SBV)**.** *Let $\xi : [0,\infty) \to [0,\infty]$ and $\vartheta : (0,\infty) \to (0,\infty]$ be lower semicontinuous increasing functions such that*

$$\lim_{s\to\infty} \frac{\xi(s)}{s} = \infty \quad \textit{and} \quad \lim_{s\to 0} \frac{\vartheta(s)}{s} = \infty\,. \tag{2.10.2}$$

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set and, let $\{u_n\} \subset SBV(\Omega)$ be such that

$$\sup_n \left\{ \int_\Omega \xi\left(|\nabla u_n|\right)\,\mathrm{d}x + \int_{J_{u_n}} \vartheta\left(|u_n^+ - u_n^-|\right)\,\mathrm{d}\mathcal{H}^{d-1} \right\} < \infty\,. \tag{2.10.3}$$

If $\{u_n\}$ weakly converges in $BV(\Omega)$ to a function u, then $u \in SBV(\Omega)$, the approximate gradients ∇u_n weakly converge to ∇u in $L^1(\Omega;\mathbb{R}^d)$, and $D^j u_n$ weakly* converge to $D^j u$ in Ω. In addition, we have that*

$$\int_\Omega \xi\left(|\nabla u|\right)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega \xi\left(|\nabla u_n|\right)\,\mathrm{d}x \quad \textit{if } \xi \textit{ is convex}$$

and

$$\int_{J_u} \vartheta\left(|u^+ - u^-|\right)\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{J_{u_n}} \vartheta\left(|u_n^+ - u_n^-|\right)\,\mathrm{d}\mathcal{H}^{d-1} \quad \textit{if } \vartheta \textit{ is concave.}$$

By the previous theorem and the Compactness Theorem 2.4.6 we may ensure the existence of convergent subsequences under an extra equiboundedness assumption.

Theorem 2.10.3 (Compactness in SBV)**.** *Let ξ, ϑ and Ω as in Theorem 2.10.2. Let $\{u_n\} \subset SBV(\Omega)$ satisfy (2.10.3), and assume, in addition, that*

$$\sup_n \|u_n\|_\infty < \infty\,. \tag{2.10.4}$$

Then there exists a subsequence $\{u_{n_k}\}$ weakly converging in $BV(\Omega)$ to $u \in SBV(\Omega)$.*

The uniform L^∞-bound (2.10.4) is necessary in Theorem 2.10.3 to estimate $D^j u_n$, and in turn, to have compactness in the BV weak*-topology, according to the Compactness Theorem 2.4.6. Indeed, without this hypothesis the limit functions need not be of bounded variation in Ω (unless $d = 1$ and $\vartheta \equiv 1$). The generalized (special) function of bounded variation that we present in the following section have been introduced exactly to overcome this difficulty.

Given an SBV function u defined on a ball B_r, we introduce a specific truncation of u that is characterized by the fact that the truncation levels depend on the size of $\mathcal{H}^{d-1}(S_u \cap B_r)$.

Definition 2.10.4 (Truncations)**.** Let $r > 0$ and let $u : B_r \to \mathbb{R}$ be a measurable function. Recall that we denote by γ_5 the dimensional constant in the relative isoperimetric inequality (see Formula (2.6.4)). If

$$s_u := \Big(2\gamma_5 \mathcal{H}^{d-1}(S_u \cap B_r)\Big)^{\frac{d}{d-1}} < \frac{|B_r|}{2}, \tag{2.10.5}$$

then we define $u_*(s,r) := \inf\{t \in [-\infty, \infty] : |\{y \in B_r : u(y) < t\}| \geq s\}$ for every $s \in [0, |B_r|]$, and introduce the *truncation of* u defined by

$$\overline{u} := \tau^-(u,r) \vee u \wedge \tau^+(u,r) \tag{2.10.6}$$

where

$$\begin{cases} \tau^-(u,r) := u_*(s_u, r) \\ \tau^+(u,r) := u_*(|B_r| - s_u, r)\,. \end{cases} \tag{2.10.7}$$

Remark 2.10.5. Let $r > 0$ and let $u : B_r \to \mathbb{R}$ be a measurable function. It can be shown that $u_*(|B_r|/2, r)$ is a median of u in B_r, and that, by (2.10.5), $\tau^-(u,r) \leq m \leq \tau^+(u,r)$ for any other median m of u in B_r (see Definition 2.6.6). Also, since

$$\{u \neq \overline{u}\} = \{u > \tau^+(u,r)\} \cup \{\, u < \tau^-(u,r)\}\,,$$

by (2.10.7) we obtain that

$$|\{u \neq \overline{u}\}| \leq 2\left(2\gamma_5 \mathcal{H}^{d-1}(S_u \cap B_r)\right)^{\frac{d}{d-1}}\,. \tag{2.10.8}$$

We now state the Poincaré inequality for functions in $SBV(B_r)$ introduced in [28]. In view of Definition 2.10.4 and Remark 2.10.5, this inequality reduces to the Poincaré inequality for Sobolev functions when $\mathcal{H}^{d-1}(S_u \cap B_r) = 0$.

Theorem 2.10.6 (Poincaré Inequality in SBV)**.** *Let $r > 0$, $1 \leq p < d$, and let $u \in SBV(B_r)$. If (2.10.5) holds, then the truncation $\overline{u}$ of u defined by (2.10.6) satisfies*

$$|D\overline{u}|(B_r) \leq 2\int_{B_r} |\nabla u|\,\mathrm{d}x$$

and

$$\left(\int_{B_r} |\overline{u} - m|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \leq \frac{2\gamma_5 p(d-1)}{d-p}\left(\int_{B_r} |\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}$$

for every median m of u in B_r.

The following result is a consequence of the Poincaré's inequality in SBV and of the Compactness Theorem 2.10.3.

Proposition 2.10.7. *Let $r > 0$ and consider a sequence $\{w_n\} \subset SBV(B_r)$ such that*

$$\sup_{n\in\mathbb{N}} \int_{B_r} |\nabla w_n|^p \,\mathrm{d}x < \infty \quad \textit{and} \quad \lim_{n\to\infty} \mathcal{H}^{d-1}(S_{w_n}) = 0\,.$$

Furthermore, for each $n \in \mathbb{N}$ consider a median m_n of w_n in B_r and the truncation $\overline{w}_n$ defined in (2.10.6). Then, there exists a subsequence $\{\overline{w}_{n_k}\}$ and a function $w \in W^{1,p}(B_r)$ such that

$$\begin{aligned} &(\overline{w}_{n_k} - m_{n_k}) \to w \quad \textit{in } L^p(B_r),\\ &(w_{n_k} - m_{n_k}) \to w \quad \mathcal{L}^d\textit{-a.e. in } B_r \textit{ as } k \to \infty, \end{aligned}$$

and

$$\int_{B_r} |\nabla w|^p \,\mathrm{d}x \le \liminf_{k\to\infty} \int_{B_r} |\nabla \overline{w}_{n_k}|^p \,\mathrm{d}x\,.$$

See Theorem 7.5 and Remark 7.6 in [9] for a proof of the previous result. Another application of the Poincaré inequality for SBV functions is the following theorem that provides a sufficient condition for the existence of the approximate limit at a given point.

Theorem 2.10.8. *Let $p,\ q > 1$, $w \in SBV_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$. If*

$$\lim_{\varrho\searrow 0} \varrho^{1-d} \left[\int_{B_\varrho(x)} |\nabla w|^p \,\mathrm{d}y\,, + \mathcal{H}^{d-1}\left(S_w \cap B_\varrho(x)\right) \right] = 0$$

and

$$\limsup_{\varrho\searrow 0} \fint_{B_\varrho(x)} |u(y)|^q \,\mathrm{d}y < \infty\,,$$

then $x \notin S_w$.

We conclude this section defining

$$SBV^p(\Omega) := \{u \in SBV(\Omega) : \nabla u \in L^p(\Omega) \text{ and } \mathcal{H}^{d-1}(S_u \cap \Omega) < \infty\} \tag{2.10.9}$$

for $p > 1$.

2.11 Generalized Functions of Bounded Variation

In this section we introduce the space of generalized functions of bounded variations and their main properties. These functions can appear as limit of sequences of functions of bounded variation.

Definition 2.11.1. We say that $u : \Omega \to \mathbb{R}^M$ is a *a generalized function of bounded variation* in Ω, and we write $u \in GBV(\Omega; \mathbb{R}^M)$, if for every $\phi \in C^1(\mathbb{R}^d)$ with the support of $\nabla\phi$ compact, the composition $\phi \circ u$ belongs to $BV_{\mathrm{loc}}(\Omega)$. Furthermore, we say that $u : \Omega \to \mathbb{R}^M$ is a *generalized special function of bounded variation* in Ω, and we write $u \in GSBV(\Omega; \mathbb{R}^M)$, if for every ϕ as above the composition $\phi \circ u$ belongs to $SBV_{\mathrm{loc}}(\Omega)$.

We denote as it is usual $GBV(\Omega) = GBV(\Omega; \mathbb{R})$ and $GSBV(\Omega) = GSBV(\Omega; \mathbb{R})$.

Remark 2.11.2. If $M = 1$ then the previous definition can be rephrased by saying that $u \in GBV(\Omega)$ if the truncated functions

$$u^\tau := (-\tau) \vee u \wedge \tau \tag{2.11.1}$$

belong to $BV_{\mathrm{loc}}(\Omega)$ for any $\tau > 0$, and that $u \in GSBV(\Omega)$ if $u^\tau \in SBV_{\mathrm{loc}}(\Omega)$ for any $\tau > 0$.

Note that the space $[GSBV(\Omega)]^M$ is strictly contained in $GBV(\Omega; \mathbb{R}^M)$ for $M > 1$.

The notions of approximate continuity, jump points, and differentiability introduced in Section 2.8 do not apply to generalized functions of bounded variation since GBV functions are not necessarily locally integrable. We use a weaker notion of approximate limit in order to introduce the analogous of S_u, J_u, and ∇u.

Definition 2.11.3 (Approximate Limit)**.** Let $u : \Omega \to \mathbb{R}$ be a Borel function and let $x \in \overline{\Omega}$ be a point where the lower density of Ω is strictly positive. We define the *upper* and *lower weak approximate limits of u at x* by, respectively,

$$u^\vee(x) := \inf\left\{ s \in \overline{\mathbb{R}} : \lim_{\varrho \searrow 0} \frac{|\{u > s\} \cap B_\varrho(x)|}{\varrho^d} = 0 \right\},$$

and

$$u^\wedge(x) := \sup\left\{ s \in \overline{\mathbb{R}} : \lim_{\varrho \searrow 0} \frac{|\{u < s\} \cap B_\varrho(x)|}{\varrho^d} = 0 \right\}.$$

If $u^\vee(x) = u^\wedge(x)$ then their common value is called the *weak approximate limit of u at x* and it is denoted by $\tilde{u}_*(x)$. We also set $S_u^* := \{x \in \Omega : u^\wedge(x) < u^\vee(x)\}$.

We note that the notions of approximate limit and weak approximate limit are all equivalent for functions in $L^\infty_{\mathrm{loc}}(\Omega)$ (see Proposition 3.65 and Remark 4.29 in in [9] for further considerations). Moreover, we have that

$$GSBV(\Omega) \cap L^\infty(\Omega) = SBV_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)\,. \tag{2.11.2}$$

Definition 2.11.4 (Weak Approximate Jump Points)**.** Let $u : \Omega \to \mathbb{R}$ be a Borel function. We say that $x \in \Omega$ is a *weak approximate jump point of* u, and we write $x \in J_u^*$, if there exist $a,\, b \in \overline{\mathbb{R}}$ with $a > b$, and a unit vector $\nu \in \mathbb{R}^d$, such that, setting

$$H^+ := \{y \in \Omega : \langle y - x, \nu\rangle > 0\} \quad \text{and} \quad H^- := \{y \in \Omega : \langle y - x, \nu\rangle < 0\}\,,$$

the weak approximate limit of the restiction of u to H^+ is a and the weak approximate limit of the restiction of u to H^- is b. If $x \in J_u^*$ then $a = u^\vee(x)$ and $b = u^\wedge(x)$. The vector ν, uniquely determined by this condition, will be denoted by $\nu_u^*(x)$.

We notice that the direction of $\nu_u^*(x)$ is uniquely determined by the previous definition, and if the values a and b are finite, then they are characterized by the following conditions:

$$\lim_{\varrho \searrow 0} \frac{|\{y \in \Omega \cap B_\varrho^+(x) : |u(y) - a| > \epsilon\}|}{\varrho^d} = 0\,,$$

and

$$\lim_{\varrho \searrow 0} \frac{|\{y \in \Omega \cap B_\varrho^-(x) : |u(y) - b| > \epsilon\}|}{\varrho^d} = 0$$

for all $\epsilon > 0$, where $B_\varrho^+(x) := B_\varrho(x) \cap H^+$ and $B_\varrho^-(x) := B_\varrho(x) \cap H^-$.

Definition 2.11.5 (Weakly Approximate Differentiability)**.** Let $u : \Omega \to \mathbb{R}$ be a Borel function and let $x \in \Omega \setminus S_u^*$. We say that u is *weakly approximately differentiable at* x if $\tilde{u}_*(x) \in \mathbb{R}$ and if there exists a linear map $L : \mathbb{R}^d \to \mathbb{R}$ such that, for every $\epsilon > 0$, the set

$$\left\{ y \in \Omega \setminus \{x\} : \frac{|u(y) - \tilde{u}_*(x) - L(y - x)|}{|y - x|} > \epsilon \right\}$$

has density 0 at x. In this case, we set $\nabla^* u(x) = L$.

Remark 2.11.6. Given two Borel functions u and v, if x is a point of density 1 for $\{u = v\}$, then u is weakly approximately continuous at x if and only if the same applies to v. In this case, the weak approximate limits coincide, and either both u and v are weakly differentiable (with $\nabla^* u(x) = \nabla^* v(x)$) or neither one is. Analogously, $x \in J_u^*$ if and only if $x \in J_v^*$, and if x is a weak approximate jump point of u and v then

$$\left(u^\vee(x), u^\wedge(x), \nu_u^*(x)\right) = \left(v^\vee(x), v^\wedge(x), \nu_v^*(x)\right)\,.$$

We now define the Cantor part of the derivative of a function in $GBV(\Omega)$.

Definition 2.11.7. Let $u \in GBV(\Omega)$. We define the *Cantor part of the derivative of* u for every measurable set $E \subset \Omega$ by

$$|D^c u|(E) := \sup \left\{ \sum_{\tau>0} |D^c u^\tau|(E_\tau) : E_\tau \subset \Omega \text{ pairwise disjoint measurable sets, } E = \bigcup_{\tau>0} E_\tau \right\},$$

where the truncations u^τ are defined in (2.11.1).

The following theorem asserts that the structure of the generalized derivative of a $GSBV$ function is similar to that of a BV functions.

Theorem 2.11.8 (Fine Properties of GBV Functions)**.** *Let $u \in GBV(\Omega)$, let $\tau \geq 0$, and recall (2.11.1). The following hold:*

(i) S_u^* *is countably* $\mathcal{H}^{d-1}$*-rectifiable,* $\mathcal{H}^{d-1}(S_u^* \setminus J_u^*) = 0$, $S_u^* = \bigcup_{\tau>0} S_{u^\tau}$,

$$u^\vee(x) = \lim_{\tau \nearrow \infty} (u^\tau)^\vee(x), \quad \text{and} \quad u^\wedge(x) = \lim_{\tau \nearrow \infty} (u^\tau)^\wedge(x);$$

(ii) u *is weakly approximately differentiable* $\mathcal{L}^d$*-a.e. in* Ω*, and*

$$\nabla^* u(x) = \nabla u^\tau(x)$$

for $\mathcal{L}^d$*-a.e.* $x \in \{|u| \leq \tau\}$*;*

(iii) $\{u > s\}$ *has finite perimeter in* Ω *for* $\mathcal{L}^d$*-a.e.* $s \in \mathbb{R}$*, and*

$$\int_{-\infty}^{+\infty} P(\{u > s\}, B)\, \mathrm{d}s = \int_B |\nabla^* u|\, \mathrm{d}x + \int_{J_u^* \cap B} \left(u^+ - u^-\right) \mathrm{d}\mathcal{H}^{d-1} + |D^c u|(B)$$

for every Borel set $B \subset \Omega$*.*

The following theorem asserts that if (2.10.4) is not verified in the Compactness Theorem in SBV 2.10.3 then the compactness holds in $GSBV$ instead of SBV.

Theorem 2.11.9 (Compactness in GSBV)**.** *Let $\xi : [0, \infty) \to [0, \infty]$ and $\vartheta : (0, \infty) \to (0, \infty]$ be two lower semicontinuous increasing functions verifying (2.10.2), and let $g : [0, \infty) \to [0, \infty]$ be increasing, with $g(s) \to \infty$ as $s \to \infty$. Let $\{u_n\} \subset GSBV(\Omega)$ be such that*

$$\sup_n \left\{ \int_\Omega [\xi(|\nabla^* u_n|) + g(|u_n|)]\, \mathrm{d}x + \int_{J_{u_n}^*} \vartheta\left(u_n^\vee - u_n^\wedge\right) \mathrm{d}\mathcal{H}^{d-1} \right\} < \infty. \tag{2.11.3}$$

Then there exist a subsequence $\{u_{n_k}\}$ *and a function* $u \in GSBV(\Omega)$ *such that* $u_{n_k} \to u$ $\mathcal{L}^d$*-a.e. in* Ω *and* $\nabla^* u_{n_k} \rightharpoonup \nabla^* u$ *weakly in* $L^1(U; \mathbb{R}^d)$ *for every open set* $A \subset\subset \Omega$*. Moreover,*

$$\int_\Omega \xi\left(|\nabla^* u|\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega \xi\left(|\nabla^* u_n|\right) \mathrm{d}x \quad \textit{if } \xi \textit{ is convex,}$$

and

$$\int_{J_u^*} \vartheta\left(|u^+ - u^-|\right) \mathrm{d}\mathcal{H}^{d-1} \leq \liminf_{n\to\infty} \int_{J_{u_n}^*} \vartheta\left(|u_n^+ - u_n^-|\right) \mathrm{d}\mathcal{H}^{d-1} \quad \textit{if } \vartheta \textit{ is concave.}$$

We end this chapter by introducing

$$GSBV^p(B) := \{u \in GSBV(B) : \nabla u \in L^p(B) \text{ and } \mathcal{H}^{d-1}(S_u \cap B) < \infty\}, \tag{2.11.4}$$

and by observing that from (2.10.9) and (2.11.2) it follows that

$$GSBV^p(\Omega) \cap L^\infty(\Omega) = SBV^p_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega).$$

Chapter 3

Evolution of Elastic Thin Films

In this chapter we prove the results introduced in Section 1.1 to which we refer the reader for the introductory explanation of the model under consideration and for the presentation of its physical motivation. We proceed as follows. In Section 3.1 we introduce the incremental minimum problem (1.1.13) choosing the appropriate function $\mathscr{D}$ (see the penalization term (3.1.12)), and we prove the existence of the discrete-time evolutions. Since in the evaporation-condensation case there are no constraints on the area of Ω_h, we proceed in a different way with respect to [39]. In fact, following an argument in [44, Chapter 12], we find h_N among functions with spatial derivative uniformly bounded by some constant $r > 0$. In particular, we start considering admissible profile functions in $H^2_{\mathrm{loc}}(\mathbb{R}; [0, \infty))$.

In Section 3.2 we prove that for each T and r, the corresponding discrete-time evolutions h_N converge to a function h in $C^{0,\beta}([0,T]; C^{1,\alpha}([0,b]))$ for every $\alpha \in \left(0, \frac{1}{2}\right)$ and $\beta \in \left(0, \frac{1-2\alpha}{8}\right)$. Furthermore, since we prove that $\{h_N\}$ is equicontinuous in time with respect to the $C^{1,\alpha}$-metric, we are allowed to select a time T_0 small enough and r_0 such that h_N is a weak solution of the time discretization of (1.1.10) for each $T < T_0$ (see (3.2.23)). Then, using the time discretization of (1.1.10) to estimate higher order derivatives of h_N, we prove that $h \in L^2(0,T; H^4_{\mathrm{loc}}(\mathbb{R})) \cap H^1(0,T; L^2_{\mathrm{loc}}(\mathbb{R}))$. Finally, in Theorem 3.2.10 we prove that h is a solution of (1.1.12), and in Theorem 3.2.11 we state the regularity properties satisfied by h.

Finally, in Section 3.3 we prove that the solution found with the minimizing movement method is the unique solution of (1.1.12) in $[0,T]$ with $T < T_0$. Since (1.1.10) does not necessarily preserve the area underneath the profile of the film, the proof is more involved than the one in [39] for the case with surface diffusion.

3.1 Mathematical Setting

In this section we introduce the precise mathematical formulation of the problem. Following the literature (see [19, 39]), we consider periodic conditions on the evolving profile and on the corresponding elastic displacement. Given a constant $b > 0$, we denote by $H^m_\#(0,b)$, for $m = 0,1,\dots$, the space of all functions in $H^m_{\mathrm{loc}}(\mathbb{R})$ that are b-periodic, endowed with the norm in $H^m(0,b)$. The class of admissible profile functions is

$$AP := \left\{ h : \mathbb{R} \to [0,\infty) : h \in H^2_\#(0,b) \right\}$$

for a positive constant b. Furthermore, given $h \in AP$,

$$\Gamma_h := \{z = (x,h(x)) : 0 < x < b\} \quad \text{and} \quad \Omega_h := \{z = (x,y) : 0 < x < b, 0 < y < h(x)\}$$

denote, respectively, the profile and the reference configuration of the film with respect to the interval $(0,b)$, while the corresponding sets on all the domain $\mathbb{R}$ are denoted by $\Gamma^\#_h$ and $\Omega^\#_h$. Moreover, the class of admissible planar displacements is

$$\begin{aligned} AD_h := \{u : \Omega^\#_h \to \mathbb{R}^2 : u \in H^1(\Omega_h;\mathbb{R}^2)\,,\ u(\cdot,0) = (e_0\,\cdot\,,0) \text{ in the sense of traces,} \\ \text{and } u(x+b,y) = u(x,y) + (e_0 b,0) \text{ for a.e. } (x,y) \in \Omega^\#_h\}\,, \end{aligned} \tag{3.1.1}$$

where the constant $e_0 > 0$ represents the mismatch between the lattices of the film and the substrate. Consequently, the functional space of admissible configurations is

$$X_{e_0} := \{(h,u) : h \in AP, u \in AD_h\}\ .$$

As in [39], we define the surface energy density $g : [0,2\pi] \to (0,\infty)$ by

$$g(\theta) := \psi(\cos\theta, \sin\theta)\,, \tag{3.1.2}$$

where $\psi : \mathbb{R}^2 \setminus \{0\} \to (0,\infty)$ is a positively one-homogeneous function of class C^2 away from the origin. Note that these are the only hypotheses assumed on ψ throughout the chapter. From these assumptions it follows that there exist two positive constants M_1 and M_2 such that

$$M_1|\xi| \le \psi(\xi) \le M_2|\xi| \tag{3.1.3}$$

for each $\xi \in \mathbb{R}^2$.

We recall that $W : \mathbb{M}^{2\times 2}_{sym} \to [0,\infty)$ is defined by

$$W(A) := \frac{1}{2}\mathbb{C}A : A\,,$$

with $\mathbb{C}$ a constant positive definite fourth-order tensor, and thus the total energy functional (1.1.9) becomes

$$\mathcal{F}(h,u) := \int_{\Omega_h} W(\boldsymbol{E}(u))\,\mathrm{d}z + \int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{2}k^2\right)\mathrm{d}\mathcal{H}^1 \tag{3.1.4}$$

for each $(h,u) \in X_{e_0}$, where $\boldsymbol{E}(u) := \frac{1}{2}(\nabla u + \nabla^T u)$, ν is the outer normal vector to Ω_h, k is the curvature of Γ_h, and ε is a (small) positive constant. In particular, given $h \in AP$, we have that

$$k = \left(\frac{h'}{\sqrt{1+(h')^2}}\right)' \quad \text{and} \quad \nu = \frac{(-h',1)}{\sqrt{1+(h')^2}}\,.$$

In the following result we establish a Korn-type inequality for subgraphs of Lipschitz functions.

Lemma 3.1.1. *Let $h : [0,b] \to [-L,L]$ be a Lipschitz function with $\operatorname{Lip} h \leq L$ for some $L > 0$ and consider $U_h := \{z = (x,y) : 0 < x < b, -L(1+3b) < y < h(x)\}$. If $1 < p < \infty$, then there exists a constant $C = C(p,b,L) > 0$ such that*

$$\int_{U_h} |u|^p\,\mathrm{d}z + \int_{U_h} |\nabla u|^p\,\mathrm{d}z \leq C\int_{U_h} |\boldsymbol{E}(u)|^p\,\mathrm{d}z\,, \tag{3.1.5}$$

for all $u \in W^{1,p}(U_h;\mathbb{R}^2)$ with $u(\cdot,-L(1+3b)) = 0$ (in the sense of traces).

Proof. Fix a ball B contained in $(0,b)\times(-L(1+3b),-L(1+2b))$. Since U_h is an open bounded domain starshaped with respect to B, by a classical version of Korn's inequality (see [59, 63]) there exists a constant $C_1 = C_1(p,b,L) > 0$ such that

$$\int_{U_h} |\nabla u|^p\,\mathrm{d}z \leq C_1\left(\int_{U_h} |u|^p\,\mathrm{d}z + \int_{U_h} |\boldsymbol{E}(u)|^p\,\mathrm{d}z\right) \tag{3.1.6}$$

for all $u \in W^{1,p}(U_h;\mathbb{R}^2)$. Thus, it is enough to prove that

$$\int_{U_h} |u|^p\,\mathrm{d}z \leq C_2\int_{U_h} |\boldsymbol{E}(u)|^p\,\mathrm{d}z \tag{3.1.7}$$

for some constant $C_2 = C_2(p,b,L) > 0$. By contradiction, assume that there exists a sequence $\{h_n\}$ as in the statement and a sequence $\{u_n\} \subset W^{1,p}(U_{h_n};\mathbb{R}^2)$ of functions with $u_n(\cdot,-L(1+3b)) = 0$ (in the sense of traces) such that

$$\int_{U_{h_n}} |u_n|^p \,\mathrm{d}z > n \int_{U_{h_n}} |\boldsymbol{E}(u_n)|^p \,\mathrm{d}z\,.$$

By the Ascoli-Arzelà Theorem, since $\{h_n\}$ is bounded in $C^{0,1}([0,b])$ by L, up to a subsequence (not relabeled), it converges uniformly to a Lipschitz function $\bar{h}$ with $\operatorname{Lip}\bar{h} \leq L$. Furthermore, for every $n \in \mathbb{N}$, the function

$$v_n := \frac{u_n}{\|u_n\|_{L^p(U_{h_n})}}$$

satisfies

$$\int_{U_{h_n}} |v_n|^p \,\mathrm{d}z = 1\,, \qquad \int_{U_{h_n}} |\boldsymbol{E}(v_n)|^p \,\mathrm{d}z \to 0 \text{ as } n \to \infty\,, \tag{3.1.8}$$

and its trace on the segment $(0,b) \times \{-L(1+3b)\}$ is equal to zero. Hence,

$$\sup_n \int_{U_{h_n}} |\nabla v_n|^p \,\mathrm{d}z < +\infty$$

by (3.1.6), and since U_{h_n} has Lipschitz boundary we can extend the functions v_n to the rectangle $R_L := (0,b) \times (L(1+3b), -L(1+3b))$ in such a way that $\{v_n\}$ is bounded in $W^{1,p}(R_L;\mathbb{R}^2)$ with null trace on $(0,b) \times \{-L(1+3b)\}$. Thus, up to a subsequence (not relabeled), $\{v_n\}$ converges weakly in $W^{1,p}(R_L;\mathbb{R}^2)$ to some function v. Note that (3.1.8) implies that

$$\int_{U_{\bar{h}}} |v|^p \,\mathrm{d}z = 1\,, \tag{3.1.9}$$

since $\{v_n \chi_{U_{h_n}}\}$ converges to $v\chi_{U_{\bar{h}}}$ in $L^p(R_L;\mathbb{R}^2)$ by the Lebesgue Dominated Theorem and the uniqueness of the limit. Moreover, v has trace zero on the segment $(0,b)\times\{-L(1+3b)\}$ (see [52]), and $\{\boldsymbol{E}(v_n)\}$ converges weakly to $\boldsymbol{E}(v)$ in $L^p(R_L;\mathbb{R}^2)$. Thus, in view of the uniform convergence of $\{h_n\}$ to $\bar{h}$ and by the Lebesgue Monotone Convergence Theorem, we have

$$\int_{U_{\bar{h}}} |\boldsymbol{E}(v)|^p \,\mathrm{d}z \leq \liminf_{n\to\infty} \int_{U_{h_n}} |\boldsymbol{E}(v_n)|^p \,\mathrm{d}z = 0\,,$$

and so $\boldsymbol{E}(v) \equiv 0$ $\mathcal{L}^2$-a.e in $U_{\bar{h}}$. Since $U_{\bar{h}}$ is connected, this yields that $v(z) = a + Az$ for some $a \in \mathbb{R}^2$ and some skew-symmetric matrix $A \in \mathbb{M}^{2\times 2}$. Thus, since v is continuous, $v(\cdot,-L(1+3b)) = 0$ in $(0,b)$ and so $a = 0$ and $A = 0$. We have reached a contradiction with (3.1.9). □

Consider a non-identically zero profile $h \in AP$ and introduce the elastic energy

$$\int_{\Omega_h} W(\boldsymbol{E}(v))\,\mathrm{d}z \tag{3.1.10}$$

defined for each $v \in AD_h$. By Lemma 3.1.1 there exists a minimizer of (3.1.10) in AD_h that is unique due to the Dirichlet condition.

Definition 3.1.2. Given $h \in AP$ with $h \not\equiv 0$, we say that $u \in AD_h$ is the elastic equilibrium corresponding to h if u minimizes (3.1.10) among all $v \in AD_h$. Moreover, $(h_0, u_0) \in X_{e_0}$ is said to be an initial configuration if $h_0 \not\equiv 0$ and u_0 is the elastic equilibrium corresponding to h_0.

Consider an initial configuration $(h_0, u_0) \in X_{e_0}$, fix $r > \|h_0'\|_\infty$, $T > 0$, $N \in \mathbb{N}$, and set

$$\tau_N := T/N\,.$$

We now introduce the iterative minimization process used to define the discrete-time evolutions.

The incremental minimum problem. Set $(h^r_{0,N}, u^r_{0,N}) := (h_0, u_0)$, and for $i = 1, \dots, N$, define inductively $(h^r_{i,N}, u^r_{i,N})$ as a solution of the following minimum problem:

$$\min\Big\{G_{i,N}(h,u) : (h,u) \in X_{e_0} \text{ and } \|h'\|_\infty \le r\Big\}\,. \tag{$M^r_{i,N}$}$$

The functional $G_{i,N}$ is given by

$$G_{i,N}(h,u) := \mathcal{F}(h,u) + P_{i,N}(h)\,, \tag{3.1.11}$$

with the penalization term $P_{i,N}$ defined by

$$P_{i,N}(h) := \frac{1}{2\tau_N}\int_{\Gamma_{h^r_{i-1,N}}} \left(\frac{h - h^r_{i-1,N}}{J^r_{i-1,N}}\right)^2 \mathrm{d}\mathcal{H}^1 = \frac{1}{2\tau_N}\int_0^b \frac{(h - h^r_{i-1,N})^2}{J^r_{i-1,N}}\mathrm{d}x\,, \tag{3.1.12}$$

where $J^r_{i-1,N} := \sqrt{1 + ((h^r_{i-1,N})')^2}\,$.

The incremental minimum problem is well defined. In fact, for each $i = 1, \dots, N$, we can recursively find a solution of the minimum problem $(M^r_{i,N})$ as it is established by the following result.

Theorem 3.1.3. *Let* $(h_0, u_0) \in X_{e_0}$ *be an initial configuration and let* $r > \|h_0'\|_\infty$, $T > 0$ *and* $N \in \mathbb{N}$. *Then, for* $i = 1, \dots, N$, *the minimum problem* $(M^r_{i,N})$ *admits a solution* $(h^r_{i,N}, u^r_{i,N}) \in X_{e_0}$ *with* $\|(h^r_{i,N})'\|_\infty \le r$.

Proof. Fix $i = 1, \dots, N$, and if $i > 1$, consider a solution $(h^r_{j,N}, u^r_{j,N})$ of $(M^r_{j,N})$ for each $j = 1, \dots, i-1$. We want to find a solution of $(M^r_{i,N})$. First observe that by (3.1.11), (3.1.12), and by the minimality of $(h^r_{j,N}, u^r_{j,N})$, we have

$$\mathcal{F}(h^r_{j,N}, u^r_{j,N}) \le G_{j,N}(h^r_{j,N}, u^r_{j,N}) \le G_{j,N}(h^r_{j-1,N}, u^r_{j-1,N}) = \mathcal{F}(h^r_{j-1,N}, u^r_{j-1,N}),$$

and so

$$0 \le \inf_{(h,u)\in X_{e_0}} G_{i,N}(h,u) \le G_{i,N}(h^r_{i-1,N}, u^r_{i-1,N}) = \mathcal{F}(h^r_{i-1,N}, u^r_{i-1,N}) \le \cdots \le \mathcal{F}(h_0, u_0).$$

Therefore, we are allowed to select a minimizing sequence $\{(h_n, u_n)\} \subset X_{e_0}$ for $(M^r_{i,N})$ such that $\|h_n'\|_\infty \le r$ for each n and $\sup_n G_{i,N}(h_n, u_n) < \infty$.

Since $\sup_n P_{i,N}(h_n, u_n) < \infty$ and $J^r_{i-1,N} \le \sqrt{1+r^2}$, we have that $\{h_n\}$ is bounded in $L^2(0,b)$ (by a constant depending on r). Furthermore, $\{h_n\}$ is bounded in $H^2(0,b)$ since $\|h_n'\|_\infty \le r$ and

$$\frac{\varepsilon}{2(1+r^2)^{\frac{5}{2}}}\|h_n''\|^2_{L^2([0,b])} \le \frac{\varepsilon}{2}\int_0^b \frac{(h_n'')^2}{(1+(h_n')^2)^{\frac{5}{2}}}\mathrm{d}x = \frac{\varepsilon}{2}\int_{\Gamma_{h_n}} k^2 \mathrm{d}\mathcal{H}^1 < \infty. \tag{3.1.13}$$

Thus, there exists $h \in AP$ with $\|h'\|_\infty \le r$ such that, up to a subsequence (not relabeled), $h_n \rightharpoonup h$ in $H^2(0,b)$ and $h_n \to h$ in $W^{1,\infty}(0,b)$. Using Fatou's Lemma, we conclude that

$$P_{i,N}(h) \le \liminf_{n\to\infty} P_{i,N}(h_n), \tag{3.1.14}$$

and in view of the continuity of ψ, we have

$$\int_{\Gamma_h} \psi(\nu)\,\mathrm{d}\mathcal{H}^1 = \int_0^b \psi(-h',1)\,\mathrm{d}x \le \liminf_{n\to\infty}\int_0^b \psi(-h_n',1)\,\mathrm{d}x = \liminf_{n\to\infty}\int_{\Gamma_{h_n}} \psi(\nu)\,\mathrm{d}\mathcal{H}^1, \tag{3.1.15}$$

where in the first and last equality we used the fact that ψ is positively one-homogeneous. Furthermore, since $(1+(\cdot)^2)^{-\frac{5}{4}}$ is uniformly continuous on $[-r,r]$, the sequence $\{(1+(h_n')^2)^{-\frac{5}{4}}\}$ converges uniformly to $(1+(h')^2)^{-\frac{5}{4}}$, and so

$$\frac{h_n''}{(1+(h_n')^2)^{\frac{5}{4}}} \rightharpoonup \frac{h''}{(1+(h')^2)^{\frac{5}{4}}} \text{ in } L^2(0,b),$$

due to the weak convergence of $\{h_n''\}$ in $L^2(0,b)$. Thus, we have

$$
\begin{aligned}
\int_{\Gamma_h} k^2 \mathrm{d}\mathcal{H}^1 &= \int_0^b \frac{(h'')^2}{(1+(h')^2)^{\frac{5}{2}}}\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \int_0^b \frac{(h_n'')^2}{(1+(h_n')^2)^{\frac{5}{2}}}\mathrm{d}x = \liminf_{n\to\infty} \int_{\Gamma_{h_n}} k^2 \mathrm{d}\mathcal{H}^1 \,.
\end{aligned}
\tag{3.1.16}
$$

In order to prove that the sequence $\{u_n\}$ is bounded in an appropriate space, we need to apply Lemma 3.1.1 in the Appendix. For this purpose, we consider a constant

$$
L \geq \sup_n \|h_n\|_{C^1([0,b])} \,,
$$

we define a set $U := (0,b)\times(0,-L(1+3b))$, and we choose $w \in H^1(U;\mathbb{R}^2)$ with null trace on $(0,b)\times\{-L(1+3b)\}$ and trace equal to $(e_0\cdot,0)$ on $(0,b)\times\{0\}$ such that

$$
\|w\|_{H^1(U;\mathbb{R}^2)} \leq C\|Tr(w)\|_{H^{\frac{1}{2}}(\partial U)}
\tag{3.1.17}
$$

for some constant $C>0$ (see [52]), where $Tr(\cdot)$ is the trace operator. We may now extend each u_n to $U_{h_n} := \{z=(x,y): 0<x<b, -L(1+3b)<y<h_n(x)\}$ with w, without relabeling it. Applying Lemma 3.1.1 to each U_{h_n}, we obtain

$$
\int_{U_{h_n}} |u_n|^2 \,\mathrm{d}z + \int_{U_{h_n}} |\nabla u_n|^2 \,\mathrm{d}z \leq C\left(\int_{\Omega_{h_n}} |\boldsymbol{E}(u_n)|^2 \,\mathrm{d}z + \|Tr(w)\|^2_{H^{\frac{1}{2}}(\partial U)}\right)
$$

for some constant $C>0$ depending only on L. Therefore, since $\sup_n \int_{\Omega_{h_n}} |\boldsymbol{E}(u_n)|^2 \mathrm{d}z < \infty$, we have that $\|u_n\|_{H^1(U_{h_n};\mathbb{R}^2)}$ are equibounded. Proceeding now as in Lemma 3.1.1, since each U_{h_n} has Lipschitz boundary, we extend u_n to the rectangle $R_L := (0,b)\times(-L(1+3b), L(1+3b))$ and we obtain that, up to a subsequence (not relabeled), $\{u_n\}$ converges weakly in $H^1(R_L;\mathbb{R}^2)$ to some function u with trace equal to $(e_0\cdot,0)$ on $(0,b)\times\{0\}$ (see [52]). Furthermore, we extend u to $\Omega_h^\#$ by defining $u(x+b,y) := u(x,y) + (e_0 b,0)$ for every $(x,y)\in\Omega_h^\#\setminus\Omega_h$, so that $(h,u)\in X_{e_0}$.

Finally, since $\{\boldsymbol{E}(u_n)\}$ weakly converges to $\boldsymbol{E}(u)$ in $L^2(R_L;\mathbb{R}^2)$ and $\{h_n\}$ convergences uniformly to h, we conclude that

$$
\int_{\Omega_h} W(\boldsymbol{E}(u))\,\mathrm{d}z \leq \liminf_{n\to\infty} \int_{\Omega_{h_n}} W(\boldsymbol{E}(u_n))\,\mathrm{d}z \,,
\tag{3.1.18}
$$

which, together with (3.1.14), (3.1.15) and (3.1.16), implies that (h,u) is a minimizer of $(M^r_{i,N})$. $\square$

Remark 3.1.4. Let $f \in H^{\frac{1}{2}}(0,b)$. The previous theorem still holds true if we replace the Dirichlet boundary condition $u(\cdot,0) = (e_0 \cdot ,0)$ in (3.1.1) with the more general condition $u(\cdot,0) = (f(\cdot),0)$. Precisely, let $h_0 \in H^2(0,b)$ be an initial profile and let $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. Then, for $i = 1, \dots, N$, the functional (3.1.11) admits a minimizer in

$$X_f^r := \{(u,h) : h \in H^2(0,b) \text{ with } \|h'\|_\infty \le r\,,\; u \in H^1(\Omega_h;\mathbb{R}^2) \text{ with } u(\cdot,0) = (f(\cdot),0)\}\,.$$

In fact, this result follows from the same arguments used in the previous proof with the only difference that we need now to select the function $w \in H^1(U;\mathbb{R}^2)$ in (3.1.17) with null trace on $(0,b) \times \{-L(1+3b)\}$ and trace equal to $(f(\cdot),0)$ on $(0,b) \times \{0\}$. We choose such a function w by extending f to $\mathbb{R}$ by [30, Theorem 5.4], using the surjectivity of the trace operator from $H^1(\mathbb{R}^2_-)$ to $H^{\frac{1}{2}}(\mathbb{R})$ (see [52]), and finally truncating near $\mathbb{R} \times \{-L(1+3b)\}$ with a cut-off function.

In view of Theorem 3.1.3 we may define the notion of discrete-time evolution of (1.1.10).

Definition 3.1.5. Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration and let $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. For $i = 1, \dots, N$, consider a solution $h^r_{i,N}$ to $(M^r_{i,N})$ given by Theorem 3.1.3. The piecewise linear interpolation $h^r_N : \mathbb{R} \times [0,T] \to [0,\infty)$ of the functions $h^r_{i,N}$, namely the function defined by

$$h^r_N(x,t) := h^r_{i-1,N}(x) + \frac{1}{\tau_N}(t-(i-1)\tau_N)(h^r_{i,N}(x) - h^r_{i-1,N}(x)) \tag{3.1.19}$$

if $(x,t) \in \mathbb{R} \times [(i-1)\tau_N, i\tau_N]$, for $i = 1, \dots, N$, is said to be a discrete-time evolution of (1.1.10). In addition, for each $t \in [0,T]$ we denote by $u^r_N(\cdot,t)$ the elastic equilibrium corresponding to $h^r_N(\cdot,t)$.

We observe that, by Theorem 3.1.3, if $(h_0, u_0) \in X_{e_0}$ is an initial configuration, $r > \|h_0'\|_\infty$ and $T > 0$, then for each $N \in \mathbb{N}$ there exists a discrete-time evolution h^r_N of (1.1.10) and we have that $h^r_N(\cdot,t) \in AP$ and $\left\|\dfrac{\partial h^r_N}{\partial x}(\cdot,t)\right\|_\infty \le r$ for all t in $[0,T]$.

Remark 3.1.6. In what follows, given a regular height function $h : \mathbb{R} \times [0,T] \to [0,\infty)$, h_x and h_t stand for the derivatives with respect to the space and the time, respectively. Moreover, for each $t \in [0,T]$, given a regular function $u(\cdot,t) : \Omega^\#_{h(\cdot,t)} \to \mathbb{R}^2$, we denote by $\nabla u(\cdot,t)$ the gradient of u with respect to the spatial coordinates and by $\boldsymbol{E}(u)(\cdot,t) := \frac{1}{2}(\nabla u(\cdot,t) + \nabla^T u(\cdot,t))$ its symmetric part. Furthermore, $\boldsymbol{E}(u)(\cdot,h(\cdot,t)) : \mathbb{R} \to \mathbb{M}^{2\times 2}_{sym}$ is the trace of $\boldsymbol{E}(u)(\cdot,t)$ on $\Gamma^\#_{h(\cdot,t)}$.

If we use the parametrization with the height function, then the curvature, the normal velocity of the evolving profile Γ_h, and the outward normal vector ν to Ω_h at the point $(\cdot, h(\cdot))$ are given, respectively, by

$$k = \left(\frac{h_x}{\sqrt{1+|h_x|^2}}\right)_x, \quad V = \frac{1}{J}h_t, \quad \text{and} \quad \nu = \frac{1}{J}(-h_x,1) .$$

Also, we have that $(\cdot)_\sigma = \frac{1}{J}(\cdot)_x$.

We now introduce the notion of a solution of (1.1.12) in the interval of time $[0,T]$.

Definition 3.1.7. Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration. A solution of (1.1.12) in $[0,T]$ with initial configuration (h_0,u_0) is a function $h \in L^2(0,T;H^4_\#(0,b)) \cap H^1(0,T;L^2_\#(0,b))$ that satisfies $h(\cdot,0) = h_0(\cdot)$ in $[0,b]$, and

$$\frac{1}{J}h_t = -\varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} - \frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x + \partial_{11}\psi(-h_x,1)h_{xx} - W \tag{3.1.20}$$

in $(0,b)\times(0,T]$, where $J := \sqrt{1+|h_x|^2}$, $\partial_{11}\psi$ denotes the second derivative of ψ with respect to the first component, $W(\cdot,t) := W(\boldsymbol{E}(u)(\cdot,h(\cdot,t)))$ and $u(\cdot,t)$ is the elastic equilibrium corresponding to $h(\cdot,t)$ for each $t \in [0,T]$.

Note that (3.1.20) is (1.1.10) using the parametrization with the height function. The following two lemmas provide the identities used to derive (3.1.20).

Lemma 3.1.8. *Let g be the function introduced in (3.1.2). Then,*

$$g(\theta) + g_{\theta\theta}(\theta) = \frac{\partial_{11}\psi(\cos\theta,\sin\theta)}{\sin^2\theta}$$

for every $\theta \in (0,2\pi)\setminus\{\pi\}$.

Lemma 3.1.9. *The curvature regularization term satisfies the identity*

$$k_{\sigma\sigma} + \frac{1}{2}k^3 = \left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x$$

for h sufficiently smooth.

3.2 Existence and Regularity

In this section we establish the existence of a solution of (1.1.12) in the sense of the Definition 3.1.7 for short time intervals and we study its regularity (see Theorems 3.2.10 and 3.2.11). First, we consider an initial configuration $(h_0, u_0) \in X_{e_0}$ and we prove that, if $\{h_N^r\}$ is a sequence of discrete-time evolutions for $r > \|h_0'\|_\infty$ and $T > 0$ (see Definition 3.1.5), then, up to a subsequence (not relabeled), it converges to some function h^r as $N \to \infty$. Next, we select a time T_0 small enough and r_0 appropriate to have that $\|(h_{i,N}^{r_0})'\|_\infty < r_0$ for each $T < T_0$, $N \in \mathbb{N}$, and $i = 1, \dots, N$. For $T < T_0$ the profile function $h_{i,N}^{r_0}$ satisfies the Euler-Lagrange equation (3.2.23) corresponding to the minimum problem $(M_{i,N}^{r_0})$. Finally, using the estimates provided by (3.2.23), we prove that h^{r_0} is a solution of (1.1.12) on $[0,T]$ for $T < T_0$.

We begin by showing that the discrete-time evolutions h_N^r introduced in Definition 3.1.5 are uniformly bounded in $L^\infty(0,T;H^2(0,b)) \cap H^1(0,T;L^2(0,b))$. In the following, we pay attention to the dependence on r of the constants involved in the estimates used to select T_0 in Corollary 3.2.3.

Theorem 3.2.1. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration and let $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. For $i = 1, \dots, N$, consider a solution $h_{i,N}^r$ to $(M_{i,N}^r)$ given by Theorem 3.1.3 and the related discrete-time evolution introduced in Definition 3.1.5. Then,*

$$\int_0^T \int_0^b \left| \frac{\partial h_N^r}{\partial t}(\cdot, t) \right|^2 \mathrm{d}x\, \mathrm{d}t \le C_0(r) \quad \text{and} \quad \sup_i \|h_{i,N}^r\|_{H^2(0,b)} \le \sqrt{C_0(r)T} + C_1(r), \tag{3.2.1}$$

where $C_0(r), C_1(r) > 0$ are constants that depend only on r.

Therefore, up to a subsequence,

$$h_N^r \rightharpoonup h^r \text{ in } L^2(0,T;H^2(0,b)) \quad \text{and} \quad h_N^r \rightharpoonup h^r \text{ in } H^1(0,T;L^2(0,b)) \tag{3.2.2}$$

as $N \to \infty$, for some function $h^r \in L^2(0,T;H^2(0,b)) \cap H^1(0,T;L^2(0,b))$. Moreover, for every $\gamma \in \left(0, \frac{1}{2}\right)$ we have

$$h_N^r \to h^r \text{ in } C^{0,\gamma}([0,T];L^2(0,b)) \quad \text{as} \quad N \to \infty, \tag{3.2.3}$$

$h^r \in L^\infty(0,T;H^2(0,b))$, $h^r(\cdot,t) \in AP$, and $\left\| \frac{\partial h^r}{\partial x}(\cdot,t) \right\|_\infty \le r$ for every t in $[0,T]$.

Proof. Fix $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. For simplicity, in this proof, we disregard the dependence on r in the notation of $h_{i,N}^r$ and h_N^r. For each $i = 1, \dots, N$, we have that

$$G_{i,N}(h_{i,N}, u_{i,N}) \le G_{i,N}(h_{i-1,N}, u_{i-1,N}) = \mathcal{F}(h_{i-1,N}, u_{i-1,N}) \tag{3.2.4}$$

by (3.1.11), (3.1.12) and the minimality of $(h_{i,N}, u_{i,N})$. Thus, $P_{i,N}(h_{i,N}) \le \mathcal{F}(h_{i-1,N}, u_{i-1,N}) - \mathcal{F}(h_{i,N}, u_{i,N})$ and so,

$$\frac{1}{2\tau_N\sqrt{1+r^2}} \int_0^b (h_{i,N} - h_{i-1,N})^2 \mathrm{d}x \le \mathcal{F}(h_{i-1,N}, u_{i-1,N}) - \mathcal{F}(h_{i,N}, u_{i,N}) \,.$$

Recalling (3.1.19) and summing over $i = 1, \dots, N$, since $\mathcal{F} \ge 0$ we obtain

$$\frac{1}{2\sqrt{1+r^2}} \int_0^T \int_0^b \left| \frac{\partial h_N}{\partial t}(x,t) \right|^2 \mathrm{d}x \, \mathrm{d}t \le \mathcal{F}(h_0, u_0) \,,$$

i.e. the first estimate in (3.2.1) with $C_0(r) := 2\sqrt{1+r^2}\mathcal{F}(h_0, u_0)$. Now, since $h_N(x, \cdot)$ is absolutely continuous on $[0, T]$, for all $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, using Hölder's inequality and Fubini's Theorem, we have

$$\begin{aligned} \|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{L^2(0,b)} &\le \left(\int_0^b \left(\int_{t_1}^{t_2} \frac{\partial h_N}{\partial t}(x,t) \mathrm{d}t \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\ &\le \left(\int_{t_1}^{t_2} \left\| \frac{\partial h_N}{\partial t}(\cdot, t) \right\|^2_{L^2(0,b)} \mathrm{d}t \right)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \,. \end{aligned}$$

Therefore, from the first estimate in (3.2.1) we obtain

$$\|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{L^2(0,b)} \le \sqrt{C_0(r)}(t_2 - t_1)^{\frac{1}{2}} \tag{3.2.5}$$

and, in particular, selecting $t_1 = 0$ and $t_2 = i\tau_N$, since $h_N(\cdot, 0) = h_0(\cdot)$ and $h_N(\cdot, i\tau_N) = h_{i,N}(\cdot)$, (3.2.5) implies that $\|h_{i,N}\|_{L^2(0,b)} \le \sqrt{C_0(r)}\sqrt{T} + \|h_0\|_{L^2([0,b])}$. Furthermore, from (3.2.4) we observe that $\mathcal{F}(h_{i,N}, u_{i,N}) \le \mathcal{F}(h_{i-1,N}, u_{i-1,N})$ for each $i = 1, \dots, N$, and so,

$$\frac{\varepsilon}{2(1+r^2)^{\frac{5}{2}}} \|(h_{i,N})''\|^2_{L^2([0,b])} \le \frac{\varepsilon}{2} \int_{\Gamma_{h^r_{i,N}}} k^2 \, \mathrm{d}\mathcal{H}^1 \le \mathcal{F}(h_{i,N}, u_{i,N}) \le \dots \le \mathcal{F}(h_0, u_0) \,.$$

where we have used the fact that $\|h'_{i,N}\|_\infty \le r$. Thus,

$$\|h''_{i,N}\|_{L^2(0,b)} \le C_2(r) \tag{3.2.6}$$

for $C_2(r) := \sqrt{\frac{2}{\varepsilon}\mathcal{F}(h_0, u_0)}(1+r^2)^{\frac{5}{4}}$, and the second estimate in (3.2.1) follows.

Therefore, since

$$\sup_{t \in [0,T]} \|h_N(\cdot, t)\|_{H^2(0,b)} \le \sqrt{C_0(r)T} + C_1(r), \tag{3.2.7}$$

up to a subsequence (not relabeled), $h_N \rightharpoonup h$ in $L^2(0,T;H^2(0,b))$ for some function h. On the other hand, the first estimate in (3.2.1) implies that, up to a further subsequence (not relabeled), $\left\{\dfrac{\partial h_N}{\partial t}\right\}$ converges weakly in $L^2(0,T;L^2(0,b))$, and we deduce that $\dfrac{\partial h}{\partial t} \in L^2(0,T;L^2(0,b))$, i.e., $h \in H^1(0,T;L^2(0,b))$. Finally, note that (3.2.5) togheter with Ascoli-Arzelà Theorem (see e.g. [10, Proposition 3.3.1]), implies (3.2.3). Thus, since by (3.2.7) for each t in $[0,T]$, we can find a sequence $\{h_{N_k}(\cdot,t)\}$ that converges in $W^{1,\infty}(0,b)$, by the uniqueness of the limit we have that $h(\cdot,t) \in AP$ and $\left\|\frac{\partial h}{\partial x}(\cdot,t)\right\|_\infty \leq r$. □

From now on, we denote by $\{h_N^r\}$ and h^r, respectively, a subsequence and a limit function provided by Theorem 3.2.1. In the next result we improve the convergence of $\{h_N^r\}$ to h^r.

Theorem 3.2.2. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration. For $r > \|h_0'\|_\infty$, $T > 0$, we have that $h^r \in C^{0,\beta}([0,T];C^{1,\alpha}([0,b]))$ and*

$$h_N^r \to h^r \text{ in } C^{0,\beta}([0,T];C^{1,\alpha}([0,b])) \quad \text{as} \quad N \to \infty \tag{3.2.8}$$

for every $\alpha \in \left(0,\frac{1}{2}\right)$ and $\beta \in \left(0,\frac{1-2\alpha}{8}\right)$. Furthermore, $h^r(\cdot,t) \to h_0$ in $C^{1,\alpha}([0,b])$ as $t \to 0^+$.

Proof. Fix $r > \|h_0'\|_\infty$, $T > 0$ and $N \in \mathbb{N}$. In this proof, we disregard again the dependence on r in the notation of $h_{i,N}^r$ and h_N^r. Since for each t_1,t_2 in $[0,T]$, with $t_1 < t_2$, the function $g := h_N(\cdot,t_2) - h_N(\cdot,t_1)$ is b-periodic, by the interpolation inequality (2.3.3), we have that

$$\|g'\|_\infty \leq K\|g''\|_{L^2(0,b)}^{\frac{3}{4}}\|g\|_{L^2(0,b)}^{\frac{1}{4}} \tag{3.2.9}$$

for some constant $K > 0$, and since $\|g''\|_{L^2(0,b)} \leq 2\sup_{i,N}\|h_{i,N}''\|_{L^2(0,b)}$, we obtain

$$\|g'\|_\infty \leq K(2C_2(r))^{\frac{3}{4}}\|g\|_{L^2(0,b)}^{\frac{1}{4}}$$

where we used (3.2.6). Thus, by (3.2.5) we find that

$$\left\|\frac{\partial h_N}{\partial x}(\cdot,t_2) - \frac{\partial h_N}{\partial x}(\cdot,t_1)\right\|_\infty \leq C_3(r)(t_2-t_1)^{\frac{1}{8}}, \tag{3.2.10}$$

for $C_3(r) := 2^{\frac{3}{4}}KC_2^{\frac{3}{4}}(r)C_0^{\frac{1}{8}}(r) > 0$.

Furthermore, by the Mean Value Theorem there exists $\bar{x} \in [0,b]$ such that

$$g(\bar{x}) = \frac{1}{b}\int_0^b g(x)\,\mathrm{d}x\,,$$

and so

$$|g(x)| \leq \mid g(x) - g(\bar{x})| + |g(\bar{x})| \leq b\|g'\|_\infty + \frac{1}{\sqrt{b}}\|g\|_{L^2(0,b)}\,,$$

for each $x \in [0,b]$. Therefore, by (3.2.5) and (3.2.10), we obtain

$$\|h_N(\cdot,t_2) - h_N(\cdot,t_1)\|_\infty \leq C_3(r)b(t_2-t_1)^{\frac{1}{8}} + \sqrt{\frac{C_0(r)}{b}}(t_2-t_1)^{\frac{1}{2}}\,. \tag{3.2.11}$$

Moreover, for every $\alpha \in (0,\frac{1}{2})$, if $|\cdot|_\alpha$ denotes the α-Hölder seminorm, we have

$$|g'|_\alpha := \sup\left\{\frac{|g'(x)-g'(y)|}{|x-y|^\alpha} : x,y \in [0,b], x \neq y\right\} \leq |g'|_{\frac{1}{2}}^{2\alpha}\left(2\|g'\|_\infty\right)^{1-2\alpha}\,. \tag{3.2.12}$$

Since (3.2.7) implies that

$$\left|\frac{\partial h_N}{\partial x}(\cdot,t_2) - \frac{\partial h_N}{\partial x}(\cdot,t_1)\right|_{\frac{1}{2}} \leq 2K_M\left(\sqrt{C_0(r)T} + C_1(r)\right)$$

where K_M is the constant of the Morrey's inequality (see [2, 52]), by (3.2.10) and (3.2.12) we deduce that

$$\left|\frac{\partial h_N}{\partial x}(\cdot,t_2) - \frac{\partial h_N}{\partial x}(\cdot,t_1)\right|_\alpha \leq C_4(r,\alpha,T)(t_2-t_1)^{\frac{1-2\alpha}{8}}\,, \tag{3.2.13}$$

for $C_4(r,\alpha,T) := 2K_M^{2\alpha}\left(\sqrt{C_0(r)T} + C_1(r)\right)^{2\alpha}(C_3(r))^{1-2\alpha} > 0$.

Therefore, it follows from (3.2.10), (3.2.11), and (3.2.13), that for every $\alpha \in (0,\frac{1}{2})$, h_N is uniformly equicontinuous with respect to the $C^{1,\alpha}([0,b])$-norm topology and that

$$\|h_N(\cdot,t_2) - h_N(\cdot,t_1)\|_{C^{1,\alpha}([0,b])} \leq C(r,\alpha,T)(t_2-t_1)^{\frac{1-2\alpha}{8}}\,, \tag{3.2.14}$$

for some $C(r,\alpha,T) > 0$. In particular, we find (3.2.8) applying Ascoli-Arzelà Theorem (see e.g. [10, Proposition 3.3.1]). Finally, since $\|h_N(\cdot,t) - h_N(\cdot,t_1)\|_{C^{1,\alpha}([0,b])} \to 0$ as $t \to t_1$, we conclude the proof choosing $t_1 = 0$. □

It follows from the previous theorem, that we can select r_0 and a small time T_1 (the largest one with respect to the estimate (3.2.10)) so that $\left\|\frac{\partial h_N^{r_0}}{\partial x}\right\|_{L^\infty([0,b]\times[0,T])} < r_0$ for every $T < T_1$ and $N \in \mathbb{N}$.

Corollary 3.2.3. *Let* $(h_0, u_0) \in X_{e_0}$ *be an initial configuration, and set*

$$r_0 := \|h_0'\|_\infty + \sqrt{\|h_0'\|_\infty^2 + 1} \quad \text{and} \quad T_1 := \frac{(1 + \|h_0'\|_\infty^2)^4}{\sigma_0(\varepsilon)(1 + r_0^2)^8}\,, \tag{3.2.15}$$

where $\sigma_0(\varepsilon) := 2^{10} K^8 \varepsilon^{-3} F^4(h_0, u_0)$ *and* K *is the interpolation constant in* (3.2.9). *Then, for* $T < T_1$ *we have that* $\sup\limits_{i,N} \|(h_{i,N}^{r_0})'\|_\infty < r_0$.

Proof. We recall that the constant in (3.2.10) is $C_3(r) := K(2C_2(r))^{\frac{3}{4}} C_0^{\frac{1}{8}}(r)$, where K is the interpolation constant in (3.2.9), $C_0(r) := 2\sqrt{1 + r^2}\mathcal{F}(h_0, u_0)$ and $C_2(r) := \sqrt{\frac{2}{\varepsilon}\mathcal{F}(h_0, u_0)}(1 + r^2)^{\frac{5}{4}}$. Hence, $C_3(r) = \sigma_0^{\frac{1}{8}}(\varepsilon)(1 + r^2)$. Therefore, choosing $t_1 = 0$ and $t_2 = i\tau_N$ in (3.2.10) we find that

$$\|(h_{i,N}^r)'\|_\infty \leq (1 + r^2)(\sigma_0(\varepsilon)T)^{\frac{1}{8}} + \|h_0'\|_\infty\,,$$

for $N \in \mathbb{N}$ and $i = 1, \ldots, N$. Thus, if $r > \|h_0'\|_\infty$ then it follows that $\sup\limits_{i,N} \|(h_{i,N}^r)'\|_\infty < r$ for every $T < T_1(r)$, where

$$T_1(r) := \frac{(r - \|h_0'\|_\infty)^8}{\sigma_0(\varepsilon)(1 + r^2)^8}\,. \tag{3.2.16}$$

Choose $r_0 := \|h_0'\|_\infty + \sqrt{\|h_0'\|_\infty^2 + 1}$ to maximize $T_1(r)$ and let $T_1 := T_1(r_0)$. □

Remark 3.2.4. If $h_0 > 0$ then there exists a time $T_2 = T_2(h_0) > 0$ such that $h_N^{r_0} > 0$ in $[0, b] \times [0, T]$ for every $T < T_2$. Indeed, by (3.2.11) with $t_1 = 0$ and $t_2 = t$ we have that

$$h_N^{r_0}(x, t) \geq h_0(x) - C_3(r_0)bt^{\frac{1}{8}} - \sqrt{\frac{C_0(r_0)}{b}}t^{\frac{1}{2}} \geq \min_{x \in [0,b]} h_0(x) - C_3(r_0)bT^{\frac{1}{8}} - \sqrt{\frac{C_0(r_0)}{b}}T^{\frac{1}{2}}$$

for every $(x, t) \in [0, b] \times [0, T]$.

Define

$$T_0 := \min\{T_1, T_2\}\,, \tag{3.2.17}$$

and note that Theorems 3.2.1 and 3.2.2 hold true for r_0 and every $T < T_0$. In the rest of the chapter we assume that $T < T_0$ and, to simplify the notation, we denote $h := h^{r_0}$, $h_N := h_N^{r_0}$, $h_{i,N} := h_{i,N}^{r_0}$, $J_{i,N}^{r_0} := J_{i,N}$, $u_N := u_N^{r_0}$ and $u_{i,N} := u_{i,N}^{r_0}$ for all $N \in \mathbb{N}$ and $i = 1, \ldots, N$.

Moreover, for technical reasons, in the sequel we use the piecewise constant interpolations of $\{J_{i,N}\}$, and $\{V_{i,N}\}$, where $V_{i,N}$ is defined by

$$V_{i,N}(x) := \frac{1}{\tau_N} \frac{h_{i,N}(x) - h_{i-1,N}(x)}{J_{i-1,N}(x)}$$

for every $x \in \mathbb{R}$, $i = 1, \ldots, N$ and $N \in \mathbb{N}$. We will also use the piecewise constant interpolations for $\{u_{i,N}\}$ and $\{h_{i,N}\}$, in place of the piecewise linear interpolations introduced in (3.1.19).

Definition 3.2.5. Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration, and for $N \in \mathbb{N}$ and $i = 1, \ldots, N$, consider $I_{i,N} := ((i-1)\tau_N, i\tau_N]$. Define $\tilde{u}_N(z,0) := u_0$ for all $z \in \Omega_{h_0}$ and

$$\tilde{u}_N(z,t) := u_{i,N}(z) \quad \text{for all} \quad z \in \Omega_{h_{i,N}} \quad \text{if} \quad t \in I_{i,N}\,. \tag{3.2.18}$$

Analogously, define $\tilde{h}_N$ and $V_N : \mathbb{R} \times (0,T] \to [0,\infty)$ by, respectively,

$$\tilde{h}_N(\cdot,t) := h_{i,N} \quad \text{and} \quad V_N(\cdot,t) := V_{i,N} \quad \text{if} \quad t \in I_{i,N}\,.$$

In addition, set $\tilde{J}_N := \sqrt{1 + \left(\dfrac{\partial \tilde{h}_N}{\partial x}\right)^2}$.

Remark 3.2.6. Fix $T < T_0$. In view of Theorem 3.2.2, we deduce the following convergence results for $\{\tilde{h}_N\}$, $\{\tilde{J}_N\}$ and $\{V_N\}$.

(i) For $\alpha \in (0, \frac{1}{2})$,

$$\tilde{h}_N \to h \text{ in } L^\infty(0,T; C^{1,\alpha}([0,b]))\,, \tag{3.2.19}$$

as $N \to \infty$. This can be easily verified using the equicontinuity of the sequence $\{h_N\}$ with respect to the $C^{1,\alpha}([0,b])$-norm topology (see (3.2.14)).

(ii) It follows from (i) that $\tilde{J}_N \to J := \sqrt{1 + |h_x|^2}$ in $L^\infty(0,T; C([0,b]))$.

(iii) Furthermore,

$$V_N \rightharpoonup V := \frac{1}{J} h_t \text{ in } L^2(0,T; L^2(0,b))\,. \tag{3.2.20}$$

Indeed, from Definition 3.1.5 we have that for all $t \in ((i-1)\tau_N, i\tau_N)$, $x \in \mathbb{R}$,

$$V_N(x,t) = \frac{1}{J_{i-1,N}(x)} \frac{\partial h_N}{\partial t}(x,t)\,.$$

Hence, (3.2.20) follows from (ii) and the fact that $\dfrac{\partial h_N}{\partial t} \rightharpoonup \dfrac{\partial h}{\partial t}$ in $L^2(0,T; L^2(0,b))$ by the second assertion in (3.2.2).

For the convergence of $\{u_N\}$ and $\{\tilde{u}_N\}$, we follow the last part of the proof of [39, Theorem 3.4]. We recall the following result established in [39, Lemma 6.10] using standard elliptic estimates (see [42, Proposition 8.9]) and we use the notation introduced in Remark 3.1.6.

Lemma 3.2.7. *Let $M > 0$ and $c_0 > 0$. Consider h_1, $h_2 \in H^2_\#(0,b)$ with $h_i \geq c_0$ and $\|h_i\|_{H^2_\#(0,b)} \leq M$ for $i = 1{,}2$, and let u_1 and u_2 the corresponding elastic equilibrium in Ω_{h_1} and Ω_{h_2}, respectively. Then, for every $\alpha \in (0, \frac{1}{2}]$*

$$\|\boldsymbol{E}(u_1)(\cdot, h_1(\cdot)) - \boldsymbol{E}(u_2)(\cdot, h_2(\cdot))\|_{C^{1,\alpha}([0,b])} \leq C\|h_1 - h_2\|_{C^{1,\alpha}([0,b])}$$

for some constant $C > 0$ depending only on M, c_0 and α.

In the remainder of the chapter, we assume that the initial profile is strictly positive, i.e.,

$$h_0 > 0\,. \tag{3.2.21}$$

The following theorem is a consequence of [42, Proposition 8.9] and Lemma 3.2.7.

Theorem 3.2.8. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration with $h_0 > 0$, and let $T < T_0$. Then*

(i) there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $i = 0, \dots, N$,

$$\|\nabla u_{i,N}\|_{C^{0,\frac{1}{2}}(\overline{\Omega}_{h_{i,N}};\mathbb{M}^{2\times 2})} \leq C\,,$$

(ii) $\boldsymbol{E}(u_N)(\cdot, h_N) \to \boldsymbol{E}(u)(\cdot, h)$ in $C^{0,\beta}([0,T]; C^{1,\alpha}([0,b]))$,

(iii) $\boldsymbol{E}(\tilde{u}_N)(\cdot, \tilde{h}_N) \to \boldsymbol{E}(u)(\cdot, h)$ in $L^\infty(0,T; C^{1,\alpha}([0,b]))$,

for every $\alpha \in \left(0, \frac{1}{2}\right)$ and $\beta \in \left(0, \frac{1-2\alpha}{8}\right)$, where $u(\cdot, t)$ is the elastic equilibrium corresponding to $h(\cdot, t)$.

Proof. Recall that by Remark 3.2.4 we have $h_N, \tilde{h}_N > 0$ in $[0,b] \times [0,T]$. Using standard elliptic estimates (see [42, Proposition 8.9]), for all $N \in \mathbb{N}$ and $i = 0, \dots, N$, we may bound the norm of $\nabla u_{i,N}$ in $C^{0,\frac{1}{2}}(\overline{\Omega}_{h_{i,N}}; \mathbb{M}^{2\times 2})$ by a constant that depends only on the $C^{1,\frac{1}{2}}[0,b]$-norm of $h_{i,N}$ (and the fourth order tensor $\mathbb{C}$). Thus, the first assertion follows from the second estimate in (3.2.1).

In view of Lemma 3.2.7 and the second estimate in (3.2.1), the second and third assertions are implied by (3.2.8) and (3.2.19), respectively. $\square$

To simplify the notation, we define the function W_N in $[0,b]\times(0,T]$ by $W_N(\cdot,t) := W_{i,N}$ for each $N \in \mathbb{N}$ and $t \in I_{i,N}$, where

$$W_{i,N}(x) := W\left(\boldsymbol{E}(u_{i,N})(x, h_{i,N}(x))\right) ,$$

for each $i = 1, \dots, N$ and $x \in [0,b]$. Consider also, the function defined by $W(\cdot,t) := W\left(\boldsymbol{E}(u)(\cdot, h(\cdot,t))\right)$ in $[0,b]$ for each $t \in (0,T]$.

Theorem 3.2.9. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration that satisfies (3.2.21) and let $T < T_0$. Then*

(i) there exists a constant $C > 0$ such that for each $N \in \mathbb{N}$ we have

$$\int_0^T \int_0^b \left| \frac{\partial^4 \tilde{h}_N(x,t)}{\partial x^4} \right|^2 \mathrm{d}x\, \mathrm{d}t \leq C\,; \tag{3.2.22}$$

(ii) $h \in L^2(0,T;H^4(0,b))$ and $\tilde{h}_N \rightharpoonup h$ in $L^2(0,T;H^4(0,b))$.

Proof. By Corollary 3.2.3, for all $N \in \mathbb{N}$ and $i = 1, \dots, N$, $h_{i,N}$ satisfies the Euler-Lagrange equation

$$\int_0^b \left[\varepsilon \frac{h''_{i,N}}{J^5_{i,N}} \varphi'' - \frac{5\varepsilon}{2} \frac{(h''_{i,N})^2}{J^7_{i,N}} h'_{i,N} \varphi' - \partial_1 \psi(-h'_{i,N},1)\varphi' \right] \mathrm{d}x + \int_0^b (W_{i,N} + V_{i,N})\, \varphi\, \mathrm{d}x = 0 \tag{3.2.23}$$

for all $\varphi \in AP$, where $\partial_1\psi$ is the partial derivative of ψ with respect to the first component and $W_{i,N}(x)$ is a continuous function in $[0,b]$ by Theorem 3.2.8. In particular, for all $N \in \mathbb{N}$, $i = 1, \dots, N$, and $\varphi \in C^2_c(0,b)$, we have that

$$\int_0^b f_{i,N} \varphi'' \mathrm{d}x = 0\,,$$

where the function $f_{i,N}$, defined by

$$f_{i,N}(x) := \varepsilon \frac{h''_{i,N}}{J^5_{i,N}} + \int_0^x \left(\frac{5\varepsilon}{2} \frac{(h''_{i,N})^2}{J^7_{i,N}} h'_{i,N} + \partial_1\psi(-h'_{i,N},1) \right) \mathrm{d}r + \int_0^x \int_0^r (W_{i,N} + V_{i,N})\, \mathrm{d}\zeta\, \mathrm{d}r\,,$$

for $x \in [0,b]$, belongs to $L^2(0,b)$. Therefore, we conclude that

$$f_{i,N}(x) = c_{i,N} x + d_{i,N} \tag{3.2.24}$$

for every $x \in [0,b]$ and some constants $c_{i,N}$ and $d_{i,N}$. Now, solving (3.2.24) for $h''_{i,N}$, we obtain

$$
\begin{aligned}
h''_{i,N} = \frac{J^5_{i,N}}{\varepsilon}\Bigg[-\int_0^x \left(\frac{5\varepsilon}{2}\frac{(h''_{i,N})^2}{J^7_{i,N}}h'_{i,N} + \partial_1\psi(-h'_{i,N},1)\right) \mathrm{d}r \\
-\int_0^x\int_0^r (W_{i,N}+V_{i,N})\, \mathrm{d}\zeta\, \mathrm{d}r + c_{i,N}x + d_{i,N}\Bigg],
\end{aligned} \tag{3.2.25}
$$

from which we conclude that $h''_{i,N}$ is absolutely continuous on $[0,b]$, and so it is b-periodic (since $h_{i,N}$ is b-periodic). Furthermore, differentiating both side of (3.2.24) and solving the resulting equation for $h'''_{i,N}$, we obtain

$$
h'''_{i,N} = \frac{5}{2}\frac{(h''_{i,N})^2}{J^2_{i,N}}h'_{i,N} + \frac{J^5_{i,N}}{\varepsilon}\left(-\partial_1\psi(-h'_{i,N},1) - \int_0^x (W_{i,N}+V_{i,N})\, \mathrm{d}r + c_{i,N}\right). \tag{3.2.26}
$$

Hence, $h'''_{i,N}$ is also absolutely continuous on $[0,b]$, and so it is b-periodic. Differentiating (3.2.24) once more and solving the resulting equation for $h^{(\mathrm{iv})}_{i,N}$, we obtain

$$
\begin{aligned}
h^{(\mathrm{iv})}_{i,N} = 10\,\frac{h'''_{i,N}h''_{i,N}h'_{i,N}}{J^2_{i,N}} + \frac{5}{2}\frac{(h''_{i,N})^3}{J^2_{i,N}} - \frac{35}{2}\frac{(h''_{i,N})^3(h'_{i,N})^2}{J^4_{i,N}} + \\
+ \frac{J^5_{i,N}h''_{i,N}}{\varepsilon}\partial_{11}\psi(-h'_{i,N},1) - \frac{J^5_{i,N}}{\varepsilon}(W_{i,N}+V_{i,N}).
\end{aligned}
$$

Thus, since ψ is of class C^2 away from the origin, $h_{i,N} \in C^4([0,b])$, and so $h_{i,N} \in H^4_{\#}(0,b)$ with $h^{(\mathrm{iv})}_{i,N}$ b-periodic. Furthermore, by Theorems 3.2.1 and 3.2.8, we have

$$
\begin{aligned}
\int_0^b |h^{(\mathrm{iv})}_{i,N}|^2\, \mathrm{d}x &\leq C\int_0^b \left(1 + |h''_{i,N}|^6 + |h'''_{i,N}|^2|h''_{i,N}|^2 + V^2_{i,N}\right) \mathrm{d}x \\
&\leq C\int_0^b |h''_{i,N}|^6\, \mathrm{d}x + C\int_0^b |h'''_{i,N}|^3\, \mathrm{d}x + C\int_0^b \left(1+V^2_{i,N}\right) \mathrm{d}x,
\end{aligned}
$$

where in the last inequality we used Young's inequality. Now we apply (2.3.2) and (2.3.3) to $h''_{i,N}$ to estimate $\|h''_{i,N}\|_{L^6(0,b)}$ and $\|h'''_{i,N}\|_{L^3(0,b)}$, respectively. It follows that

$$
\begin{aligned}
\|h^{(\mathrm{iv})}_{i,N}\|^2_{L^2} &\leq C\|h''_{i,N}\|^5_{L^2}\|h^{(\mathrm{iv})}_{i,N}\|_{L^2} + C\|h''_{i,N}\|^{\frac{5}{4}}_{L^2}\|h^{(\mathrm{iv})}_{i,N}\|^{\frac{7}{4}}_{L^2} + C\int_0^b \left(1+V^2_{i,N}\right) \mathrm{d}x \\
&\leq \gamma\|h^{(\mathrm{iv})}_{i,N}\|^2_{L^2(0,b)} + C_\gamma\int_0^b \left(1+V^2_{i,N}\right) \mathrm{d}x,
\end{aligned} \tag{3.2.27}
$$

where in the last inequality we used Young's inequality with an arbitrary $\gamma > 0$ and (3.2.1) to estimate $\|h''_{i,N}\|_{L^2}$. Choosing $\gamma < 1$ in (3.2.27), multiplying for $\frac{T}{N}$, and summing over all $i = 1, \dots, N$, we obtain

$$\sum_{i=1}^{N} \frac{T}{N} \int_0^b |h^{(\mathrm{iv})}_{i,N}|^2 \, \mathrm{d}x \leq C \int_0^T \int_0^b \left(1 + V_N^2\right) \, \mathrm{d}x \, \mathrm{d}t \, .$$

Hence, recalling the definition of $\tilde{h}_N$ since V_N is bounded in $L^2(0, T; L^2(0, b))$ by (3.2.20) we obtain (i).

We now prove the second assertion. We start by considering $M > N$, $i = 1, \dots, N$ and $j = 1, \dots, M$. Subtracting to (3.2.23) the Euler-Lagrange equation satisfied by $h_{j,M}$, and considering the test function $\varphi = h_{i,N} - h_{j,M}$, we obtain

$$\begin{aligned}
\int_0^b \left(\frac{h''_{i,N}}{J^5_{i,N}} - \frac{h''_{j,M}}{J^5_{j,M}}\right)(h''_{i,N} - h''_{j,M}) \, \mathrm{d}x &= \frac{5}{2} \int_0^b \left(\frac{(h''_{i,N})^2}{J^7_{i,N}} h'_{i,N} - \frac{(h''_{j,M})^2}{J^7_{j,M}} h'_{j,M}\right)(h'_{i,N} - h'_{j,M}) \, \mathrm{d}x \\
&\quad + \frac{1}{\varepsilon} \int_0^b \big(\partial_1 \psi(-h'_{i,N}, 1) - \partial_1 \psi(-h'_{j,M}, 1)\big)(h'_{i,N} - h'_{j,M}) \, \mathrm{d}x \\
&\quad - \frac{1}{\varepsilon} \int_0^b (W_{i,N} - W_{j,M})(h_{i,N} - h_{j,M}) \, \mathrm{d}x \\
&\quad - \frac{1}{\varepsilon} \int_0^b (V_{i,N} - V_{j,M})(h_{i,N} - h_{j,M}) \, \mathrm{d}x \, .
\end{aligned} \tag{3.2.28}$$

Fix $\eta > 0$ and recall the notation $I_{i,N} = ((i-1)\tau_N, i\tau_N]$ and $I_{j,M} = ((j-1)\tau_N, j\tau_N]$. Since $\tilde{h}_N \to h$ in $L^\infty(0, T; C^1([0, b]))$, for N and M sufficiently large and for every i and j such that $|I_{i,N} \cap I_{j,M}| \neq 0$, we have that $\|h_{i,N} - h_{j,M}\|_{C^1([0,b])} \leq \eta$. We claim that

$$\int_0^b |h''_{i,N} - h''_{j,M}|^2 \, dx \leq C\eta \int_0^b (1 + |V_{i,N}| + |V_{j,M}|) \, \mathrm{d}x \tag{3.2.29}$$

for some constant $C > 0$. Indeed, the left-hand side of (3.2.28) satisfies

$$\begin{aligned}
&\left| \int_0^b \left(\frac{h''_{i,N}}{J^5_{i,N}} - \frac{h''_{j,M}}{J^5_{j,M}}\right)(h''_{i,N} - h''_{j,M}) \, \mathrm{d}x \right| \\
&\qquad \geq \int_0^b \frac{|h''_{i,N} - h''_{j,M}|^2}{J^5_{i,N}} \, \mathrm{d}x - \left| \int_0^b h''_{j,M} \left(\frac{1}{J^5_{j,M}} - \frac{1}{J^5_{i,N}}\right)(h''_{i,N} - h''_{j,M}) \, \mathrm{d}x \right| \\
&\qquad \geq C \int_0^b |h''_{i,N} - h''_{j,M}|^2 \, \mathrm{d}x - \int_0^b \left| \frac{1}{J^5_{j,M}} - \frac{1}{J^5_{i,N}} \right| |h''_{j,M}|(|h''_{i,N}| + |h''_{j,M}|) \, \mathrm{d}x \\
&\qquad \geq C \int_0^b |h''_{i,N} - h''_{j,M}|^2 \, \mathrm{d}x - C\eta
\end{aligned}$$

where we used the Lipschitz continuity of the function $s \mapsto (1+s^2)^{-\frac{5}{2}}$ on $[0, r_0]$, $J_{i,N} \leq \sqrt{1+r_0^2}$, and (3.2.6). Thus, the claim follows from the fact that the absolute value of the right-hand side may be estimated from above by $C\eta$ for some constant $C > 0$, since $h_{i,N}$, $h_{j,M} \leq r_0$, (3.2.6), $\partial_1 \psi$ is continuous away from the origin, and in view of assertion (iii) of Theorem 3.2.8.

Furthermore, integrating (3.2.29) over $I_{i,N} \cap I_{j,M}$, we have that for N and M sufficiently large,

$$\int_{I_{i,N}\cap I_{j,M}} \int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta \int_{I_{i,N}\cap I_{j,M}} \int_0^b (1 + |V_{i,N}| + |V_{j,M}|)\,\mathrm{d}x\,\mathrm{d}t$$

for each i and j such that $|I_{i,N} \cap I_{j,M}| \neq 0$. Now, we first fix $i = 1, \dots, N$, and sum the previous estimate with respect to every j such that $|I_{i,N} \cap I_{j,M}| \neq 0$ to obtain

$$\int_{I_{i,N}} \int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta \int_{I_{i,N}} \int_0^b (1 + |V_N| + |V_M|)\,\mathrm{d}x\,\mathrm{d}t\,,$$

and then we sum over i, so that (3.2.20) implies

$$\int_0^T \int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta \tag{3.2.30}$$

for M, N sufficiently large and some constant $C > 0$.

Moreover, by (2.3.1),

$$\int_0^b \left| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(x,t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(x,t) \right|^2 \mathrm{d}x \leq C \left(\int_0^b \left| \frac{\partial^4 \tilde{h}_N}{\partial x^4}(x,t) - \frac{\partial^4 \tilde{h}_M}{\partial x^4}(x,t) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_0^b \left| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(x,t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(x,t) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Finally, we integrate with respect to t and use Hölder's inequality, the first assertion and (3.2.30) to deduce that

$$\int_0^T \int_0^b \left| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(x,t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(x,t) \right|^2 \mathrm{d}x\,\mathrm{d}t \leq C\eta^{\frac{1}{2}}\,, \tag{3.2.31}$$

for N and M sufficiently large. Thus, by (3.2.30) and (3.2.31), $\left\{\frac{\partial^2 \tilde{h}_N}{\partial x^2}\right\}$ is a Cauchy sequence in $L^2(0,T;H^1(0,b))$ and, since by Theorem 3.2.1 and (3.2.19) $\tilde{h}_N \rightharpoonup h$ in $L^2(0,T;H^2(0,b))$, we have that $\tilde{h}_N \to h$ in $L^2(0,T;H^3(0,b))$. Hence, in view of (i) we obtain that $\tilde{h}_N \rightharpoonup h$ in $L^2(0,T;H^4(0,b))$. $\square$

Note that $h \in L^2(0,T;H^4_\#(0,b)) \cap H^1(0,T;L^2_\#(0,b))$ and recall Definition 3.1.7. In the following theorem, we prove the existence of a solution of (1.1.12) in $[0,T]$ for $T < T_0$.

Theorem 3.2.10. *Let $(h_0,u_0) \in X_{e_0}$ be an initial configuration such that $h_0 > 0$, and let $T_0 > 0$ be as defined in (3.2.17). Then the Cauchy problem (1.1.12) admits a solution in $[0,T]$ for each $T < T_0$ in the sense of Definition 3.1.7.*

Proof. Fix $\varphi \in C_c^\infty((0,b)\times(0,T))$. It follows from (3.2.23) that for all $N \in \mathbb{N}$,

$$\int_0^b \left[\varepsilon \frac{(\tilde{h}_N)_{xx}}{\tilde{J}_N^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{(\tilde{h}_N)_{xx}^2}{\tilde{J}_N^7}(\tilde{h}_N)_x\varphi_x - \partial_1\psi(-(\tilde{h}_N)_x,1)\varphi_x + W_N\varphi\right]\mathrm{d}x = -\int_0^b V_N\varphi\,\mathrm{d}x$$

in $(0,T]$. Integrating over $(0,T]$, we obtain

$$\int_0^T A_N\,\mathrm{d}t = -\int_0^T\int_0^b V_N\varphi\,\mathrm{d}x\mathrm{d}t\,, \tag{3.2.32}$$

where

$$A_N := \int_0^b \left[\varepsilon \frac{(\tilde{h}_N)_{xx}}{\tilde{J}_N^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{(\tilde{h}_N)_{xx}^2}{\tilde{J}_N^7}(\tilde{h}_N)_x\varphi_x - \partial_1\psi(-(\tilde{h}_N)_x,1)\varphi_x + W_N\varphi\right]\mathrm{d}x$$

in $(0,T]$. By Lebesgue Dominated Convergence Theorem, $\{A_N\}$ converges to

$$A := \int_0^b \left[\varepsilon \frac{h_{xx}}{J^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{h_{xx}^2}{J^7}h_x\varphi_x - \partial_1\psi(-h_x,1)\varphi_x + W\varphi\right]\mathrm{d}x$$

in $L^1(0,T)$. Indeed, we have that

$$|A_N| \le C\|\varphi\|_{C^2((0,b)\times(0,T))}\int_0^b \left[|(\tilde{h}_N)_{xx}| + |(\tilde{h}_N)_{xx}|^2 + W_N\right]\mathrm{d}x$$

in $(0,T]$ for some constant $C > 0$, since $(\tilde{h}_N)_x$ is uniformly bounded in $[0,b]\times(0,T]$, $\partial_1\psi$ is continuous away from the origin, and $\tilde{J}_N \ge 1$. Thus, by (3.2.1) and assertion (i) of Theorem 3.2.8, A_N is uniformly bounded in $(0,T]$. Moreover, $A_N \to A$ $\mathcal{L}^1$-a.e. in $(0,T)$ because $\partial_1\psi$ is continuous away from the origin, $W_N(\cdot,t) \to W(\cdot,t)$ in $C([0,b])$ by Theorem 3.2.8, and $\tilde{h}_N(\cdot,t) \to h(\cdot,t)$ in $C^2([0,b])$ by Theorem 3.2.9.

Therefore, since $A_N \to A$ in $L^1(0,T)$ and also by (3.2.20), we obtain that

$$\int_0^T \int_0^b \left[\varepsilon \frac{h_{xx}}{J^5}\varphi_{xx} - \frac{5\varepsilon}{2}\frac{h_{xx}^2}{J^7} h_x \varphi_x - \partial_1\psi(-h_x,1)\varphi_x + W\varphi\right] \mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_0^b V\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Integrating by parts, we have

$$\int_0^T \int_0^b f\varphi\,\mathrm{d}x\,\mathrm{d}t = 0\,, \tag{3.2.33}$$

where the function f defined in $[0,b]\times(0,T)$ by

$$f := \varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x + (\partial_1\psi(-h_x,1))_x + W + V\,,$$

belongs to $L^2(0,T;L^2(0,b))$. Indeed, since h_x is uniformly bounded in $[0,b]\times[0,T]$, $J \geq 1$, and $\partial_{11}\psi$ is continuous away from the origin, we have

$$\begin{aligned}
\int_0^T \int_0^b |f|^2\,\mathrm{d}x\,\mathrm{d}t &\leq C\int_0^T \int_0^b \left[|h_{xxxx}|^2 + |h_{xxx}|^2|h_{xx}|^2 + |h_{xx}|^6 + |h_{xx}|^2 + W^2 + |V|^2\right] \mathrm{d}x\,\mathrm{d}t \\
&\leq C\int_0^T \int_0^b \left[1 + |h_{xxx}|^2|h_{xx}|^2 + |h_{xx}|^6\right] \mathrm{d}x\,\mathrm{d}t \\
&\leq C\int_0^T \int_0^b \left[1 + |h_{xxx}|^3 + |h_{xx}|^6\right] \mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for some constant $C > 0$, where in the second inequality we used the fact that h belongs to $L^2(0,T_0;H^4(0,b))$, (3.2.20) and Theorem 3.2.8, while the last one follows from Young's inequality. Moreover, since $h_{xx}(\cdot,t) \in H^2_\#(0,b)$ for $\mathcal{L}^1$-a.e. t in $[0,T_0]$, we may use the interpolation results (2.3.2) and (2.3.3) to estimate $\|h_{xxx}(\cdot,t)\|_{L^3(0,b)}$ and $\|h_{xx}(\cdot,t)\|_{L^6(0,b)}$, respectively, as done in (3.2.27), and then applying again Young's inequality, we obtain

$$\int_0^T \int_0^b |f|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\left[1 + \int_0^T \int_0^b |h_{xxxx}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \left(\int_0^b |h_{xx}|^2\,\mathrm{d}x\right)^5 \mathrm{d}t\right].$$

Note that since $h \in L^2(0,T;H^4(0,b)) \cap L^\infty(0,T;H^2(0,b))$, the right-hand side of the previous inequality is bounded.

By the arbitrariness of φ and the density of $C_c^\infty((0,b)\times(0,T))$ in $L^2((0,b)\times(0,T))$, we deduce from (3.2.33) that $f \equiv 0$. Thus, h satisfies

$$V = -\varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} - \frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x - (\partial_1\psi(-h_x,1))_x - W\,,$$

which is (3.1.20). □

The following regularity result applies to the solution h of (1.1.12) for $T < T_0$.

Theorem 3.2.11. *Let* $(h_0, u_0) \in X_{e_0}$ *be an initial configuration such that* $h_0 > 0$ *and let* $T < T_0$*. Then, the solution* h *of* (1.1.12) *in* $[0, T]$ *given in Theorem 3.2.10, satisfies:*

(i) $h \in L^2(0, T; H^4_\#(0, b)) \cap L^\infty(0, T; H^2_\#(0, b)) \cap H^1(0, T; L^2_\#(0, b))$,

(ii) $h \in C^{0,\beta}([0, T]; C^{1,\alpha}([0, b]))$ *for every* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* $\beta \in \left(0, \frac{1-2\alpha}{8}\right)$,

(iii) $\|h_x\|_{L^\infty(0,T;L^\infty(0,b))} \le \|h_0'\|_\infty + \sqrt{\|h_0'\|^2_\infty + 1}$,

(iv) $h \in L^{\frac{12}{5}}(0, T; C^{2,1}_\#([0, b])) \cap L^{\frac{24}{5}}(0, T; C^{1,1}_\#([0, b]))$.

Proof. Properties (i)-(iii) have been established in Theorems 3.2.1, 3.2.2, 3.2.9, and Corollary 3.2.3. In order to prove (iv), we fix $N, M \in \mathbb{N}$ and we follow [39, Corollary 3.7]. By (2.3.3), we have

$$\begin{aligned}
&\Big\| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(\cdot, t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(\cdot, t) \Big\|_\infty \\
&\quad \le C \left(\int_0^b \Big| \frac{\partial^4 \tilde{h}_N}{\partial x^4}(x, t) - \frac{\partial^4 \tilde{h}_M}{\partial x^4}(x, t) \Big|^2 \mathrm{d}x \right)^{\frac{5}{12}} \left(\int_0^b \Big| \frac{\partial \tilde{h}_N}{\partial x}(x, t) - \frac{\partial \tilde{h}_M}{\partial x}(x, t) \Big|^2 \mathrm{d}x \right)^{\frac{1}{12}}
\end{aligned}$$

$\mathcal{L}^1$-a.e. in $[0, T]$. Raising both sides to the power $\frac{12}{5}$, integrating over $[0, T]$ and recalling (3.2.22), we obtain

$$\int_0^T \left\| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(\cdot, t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(\cdot, t) \right\|_\infty^{\frac{12}{5}} \mathrm{d}t \le C \sup_{t \in [0,T]} \left\| \frac{\partial \tilde{h}_N}{\partial x}(\cdot, t) - \frac{\partial \tilde{h}_M}{\partial x}(\cdot, t) \right\|_\infty^{\frac{2}{5}}.$$

Then, by (3.2.19) we have that $\tilde{h}_N \to h$ in $L^{\frac{12}{5}}(0, T; C^{2,1}_\#([0, b]))$ and $h \in L^{\frac{12}{5}}(0, T; C^{2,1}_\#([0, b]))$. Furthermore, by (2.3.1), we have

$$\left\| \frac{\partial^2 \tilde{h}_N}{\partial x^2}(\cdot, t) - \frac{\partial^2 \tilde{h}_M}{\partial x^2}(\cdot, t) \right\|_\infty \le C \left\| \frac{\partial^3 \tilde{h}_N}{\partial x^3}(\cdot, t) - \frac{\partial^3 \tilde{h}_M}{\partial x^3}(\cdot, t) \right\|_\infty^{\frac{1}{2}} \left\| \frac{\partial \tilde{h}_N}{\partial x}(\cdot, t) - \frac{\partial \tilde{h}_M}{\partial x}(\cdot, t) \right\|_\infty^{\frac{1}{2}}$$

$\mathcal{L}^1$-a.e. in $[0, T]$. Thus, raising both sides to the power $\frac{24}{5}$, we proceed as before to conclude that $\tilde{h}_N \to h$ in $L^{\frac{24}{5}}(0, T; C^{1,1}_\#([0, b]))$ and $h \in L^{\frac{24}{5}}(0, T; C^{1,1}_\#([0, b]))$.

$\square$

3.3 Uniqueness

From Theorem 3.3.1 below, it follows that the solution provided by Theorem 3.2.10 is the unique solution of (1.1.12) in $[0,T]$ for $T < T_0$. Since (3.1.20) does not necessarily preserve the area underneath the profile of the film, the proof is more involved than the one in [39] for the case with surface diffusion.

Theorem 3.3.1. *Let $(h_0, u_0) \in X_{e_0}$ be an initial configuration such that $h_0 > 0$, and let $T > 0$. If h_1, $h_2 \in L^2(0,T; H^4_\#(0,b)) \cap L^\infty(0,T; H^2_\#(0,b)) \cap H^1(0,T; L^2_\#(0,b))$ are two solutions of (1.1.12) in $[0,T]$ with initial configuration (h_0, u_0) (see Definition 3.1.7), then $h_1 = h_2$.*

Proof. For simplicity of notation, in this proof, we denote by $(\cdot)'$ the differentiation with respect to x. Consider a constant $M > 0$ such that

$$\|h_i\|_{L^\infty(0,T;H^2_\#(0,b))} \leq M \tag{3.3.1}$$

for $i = 1,2$. We want to apply Gronwall's Lemma to the function

$$t \mapsto H(t) := \int_0^b |h_2 - h_1|^2 \,\mathrm{d}x + \int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x \,.$$

We claim that $H \in W^{1,1}(0,T)$, and that there exists a constant $C > 0$, that depends only on M, such that

$$\frac{\partial H}{\partial t}(t) \leq C G(t) H(t) \tag{3.3.2}$$

for almost every $t \in (0,T)$, where

$$G(t) := 1 + \|h_1^{(\mathrm{iv})}(\cdot,t)\|^2_{L^2} + \|h_2^{(\mathrm{iv})}(\cdot,t)\|^2_{L^2} \,.$$

We proceed in four steps. In the sequel of this proof, constants denoted by the same symbol may change from formula to formula.

Step 1: We begin by proving that $H \in W^{1,1}(0,T)$, and that for almost every $t \in (0,T)$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2 - h_1|^2 \,\mathrm{d}x = \int_0^b \left(\frac{\partial h_2}{\partial t} - \frac{\partial h_1}{\partial t}\right)(h_2 - h_1)\,\mathrm{d}x \,, \tag{3.3.3}$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x = -\int_0^b \left(\frac{\partial h_2}{\partial t} - \frac{\partial h_1}{\partial t}\right)(h_2'' - h_1'')\,\mathrm{d}x \,. \tag{3.3.4}$$

To this purpose, we mollify the b-periodic function $\bar{h}$ defined in $\mathbb{R} \times (-T,2T)$ by

$$\bar{h}(\cdot,t) := \begin{cases} (h_2 - h_1)(\cdot,t) & \text{if } t \in [0,T], \\ (h_2 - h_1)(\cdot,-t) & \text{if } t \in (-T,0), \\ (h_2 - h_1)(\cdot,2T-t) & \text{if } t \in (T,2T). \end{cases}$$

For each $\epsilon > 0$ small enough, the mollification $\bar{h}_\epsilon$ is defined and smooth in $\mathbb{R} \times [0,T]$ and so, it satisfies

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^b |\bar{h}_\epsilon|^2 \,\mathrm{d}x = \int_0^b \frac{\partial \bar{h}_\epsilon}{\partial t}\bar{h}_\epsilon \,\mathrm{d}x \quad \text{and} \quad \frac{1}{2}\frac{\partial}{\partial t}\int_0^b |\bar{h}'_\epsilon|^2 \,\mathrm{d}x = -\int_0^b \frac{\partial \bar{h}_\epsilon}{\partial t}\bar{h}''_\epsilon \,\mathrm{d}x \tag{3.3.5}$$

in $[0,T]$, where we used the fact that $\bar{h}_\epsilon(\cdot,t)$ is b-periodic for each $t \in [0,T]$. Furthermore, $\bar{h}_\epsilon \to \bar{h}$ in $H^1((0,b)\times(0,T))$ since $\bar{h} \in H^1((-b,2b)\times(-T,2T))$, and $\bar{h}''_\epsilon \to \bar{h}''$ in $L^2((0,b)\times(0,T))$ since $\bar{h}'' \in L^2((-b,2b)\times(-T,2T))$ (see [52]). Therefore, by (3.3.5) we obtain that $\int_0^b |\bar{h}|^2 \,\mathrm{d}x$ and $\int_0^b |\bar{h}'|^2 \,\mathrm{d}x$ are weakly differentiable in the sense of distributions in $(0,T)$ and satisfy (3.3.3) and (3.3.4), respectively.

Step 2: Inserting (3.1.20) for h_1 and h_2 in (3.3.3), integrating by parts, and using the periodicity of $h_1(\cdot,t)$ and $h_2(\cdot,t)$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2 - h_1|^2 \,\mathrm{d}x = &-\varepsilon \int_0^b \left[\frac{h''_2}{J_2^5}(J_2(h_2-h_1))'' - \frac{h''_1}{J_1^5}(J_1(h_2-h_1))''\right] \mathrm{d}x \\ &+ \frac{5\varepsilon}{2}\int_0^b \left[\frac{(h''_2)^2 h'_2}{J_2^7}(J_2(h_2-h_1))' - \frac{(h''_1)^2 h'_1}{J_1^7}(J_1(h_2-h_1))'\right] \mathrm{d}x \\ &+ \int_0^b \partial_1\psi(-h'_2,1)(J_2(h_2-h_1))' - \partial_1\psi(-h'_1,1)(J_1(h_2-h_1))' \,\mathrm{d}x \\ &- \int_0^b (W_2J_2 - W_1J_1)(h_2-h_1)\,\mathrm{d}x =: I_1 + I_2 + I_3 + I_4\,, \end{aligned} \tag{3.3.6}$$

where J_i and W_i refer to the function h_i for $i = 1,2$. In the sequel of this step, we estimate the integrals on the right-hand side of the previous equality.

First, we consider I_1 and I_2 and observe that

$$
\begin{aligned}
I_1 + I_2 + \varepsilon \int_0^b \frac{|h_2'' - h_1''|^2}{J_2^4}\,\mathrm{d}x =& \varepsilon \int_0^b h_1'' \left(\frac{1}{J_2^4} - \frac{1}{J_1^4}\right)(h_2'' - h_1'')\,\mathrm{d}x \\
&+ \frac{3\varepsilon}{2}\int_0^b \left(\frac{(h_2'')^2 h_2'}{J_2^6} - \frac{(h_1'')^2 h_1'}{J_1^6}\right)(h_2' - h_1')\,\mathrm{d}x \\
&+ \frac{5\varepsilon}{2}\int_0^b \left(\frac{(h_2''')^3 (h_2')^2}{J_2^8} - \frac{(h_1''')^3 (h_1')^2}{J_1^8}\right)(h_2 - h_1)\,\mathrm{d}x \\
- \int_0^b \Bigg(\frac{(h_2'')^3 + h_2''' h_2'' h_2' + h_2''' h_2'' (h_2')^3}{J_2^8} &- \frac{(h_1'')^3 + h_1''' h_1'' h_1' + h_1''' h_1'' (h_1')^3}{J_1^8}\Bigg)(h_2 - h_1)\,\mathrm{d}x\,.
\end{aligned}
$$

In view of (3.3.1), h_1' and h_2' are uniformly bounded and so there exists a constant $C_\varepsilon > 0$ that depends on M such that

$$
\inf_{(0,b)\times(0,T)} \frac{\varepsilon}{J_2^4} \geq C_\varepsilon\,. \tag{3.3.7}
$$

Thus, since for $n \in \mathbb{N}$ the function $s \mapsto (1+s^2)^{-\frac{n}{2}}$ is locally Lipschitz continuous and we have

$$
|(h_2'')^n - (h_1'')^n| \leq (\|h_1''\|_\infty^{n-1} + \|h_2''\|_\infty^{n-1})|h_2'' - h_1''| \tag{3.3.8}
$$

in $(0,b)\times(0,T)$, we obtain

$$
\begin{aligned}
I_1 + I_2 + C_\varepsilon \int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x \leq C\Bigg[&\int_0^b |h_1''||h_2'' - h_1''||h_2' - h_1'|\,\mathrm{d}x + \int_0^b |h_2''|^2 |h_2' - h_1'|^2\,\mathrm{d}x \\
&+ (\|h_1''\|_\infty + \|h_2''\|_\infty)\int_0^b |h_2'' - h_1''||h_2' - h_1'|\,\mathrm{d}x + \int_0^b (|h_2''||h_2' - h_1'|)(|h_2''|^2|h_2 - h_1|)\,\mathrm{d}x \\
&+ (\|h_1''\|_\infty^2 + \|h_2''\|_\infty^2)\int_0^b |h_2'' - h_1''||h_2 - h_1|\,\mathrm{d}x + \int_0^b (|h_2''||h_2' - h_1'|)(|h_2'''||h_2 - h_1|)\,\mathrm{d}x \\
&+ \int_0^b |h_2''||h_2''' - h_1'''||h_2 - h_1|\,\mathrm{d}x + \int_0^b |h_1'''||h_2'' - h_1''||h_2 - h_1|\,\mathrm{d}x\Bigg]\,.
\end{aligned}
$$

We now apply Young's inequality to each integral on the right-hand side of the previous inequality. Precisely, for the integrals that present the term $|h_2'' - h_1''|$ or $|h_2''' - h_1'''|$, we use Young's inequality with a parameter $\eta > 0$. In this way, we have that

$$
\begin{aligned}
I_1 + I_2 + C_\varepsilon \int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x \leq& \eta \int_0^b |h_2''' - h_1'''|^2\,\mathrm{d}x + \eta \int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x \\
&+ C_\eta \left(\|h_1''\|_\infty^2 + \|h_2''\|_\infty^2\right)\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x \\
+ C_\eta(\|h_2''\|_\infty^2 + \|h_1''\|_\infty^4 + \|h_2''\|_\infty^4 &+ \|h_1'''\|_\infty^2 + \|h_2'''\|_\infty^2)\int_0^b |h_2 - h_1|^2\,\mathrm{d}x\,.
\end{aligned} \tag{3.3.9}
$$

Next, we estimate I_3 from above. As before, we begin by observing that

$$I_3 = \int_0^b (\partial_1\psi(-h_2',1)J_2 - \partial_1\psi(-h_1',1)J_1)(h_2' - h_1')\,\mathrm{d}x + \int_0^b (\partial_1\psi(-h_2',1)\frac{h_2''h_2'}{J_2} - \partial_1\psi(-h_1',1)\frac{h_1''h_1'}{J_1})(h_2 - h_1)\,\mathrm{d}x\,.$$

Then, using the fact that the function $s \mapsto \partial_1\psi(s,1)$ is locally Lipschitz continuous, and again invoking the fact that h_1' and h_2' are uniformly bounded, we have

$$\begin{aligned} I_3 &\le C\Big[\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + \int_0^b |h_2''||h_2' - h_1'||h_2 - h_1|\,\mathrm{d}x + \int_0^b |h_2'' - h_1''||h_2 - h_1|\,\mathrm{d}x\Big] \\ &\le \eta\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x + C\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta(1 + \|h_2''\|_\infty^2)\int_0^b |h_2 - h_1|^2\,\mathrm{d}x\,. \end{aligned} \tag{3.3.10}$$

Now, we consider I_4. Observe that by Lemma 3.2.7 and by the definition of W, there exists a constant C, that depends on M, such that $\|W_i\|_{L^\infty((0,b)\times(0,T))} \le C$ for $i = 1,2$, and

$$\int_0^b |W_2 - W_1|^2\,\mathrm{d}x \le C\|h_1 - h_2\|_{H^2}^2 \le C\int_0^b |h_2 - h_1|^2\,\mathrm{d}x + C\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x \tag{3.3.11}$$

in $(0,T)$, where in the last estimate we applied Poincaré inequality. Therefore, since the function $s \mapsto (1+s^2)^{\frac{1}{2}}$ is locally Lipschitz continuous, W_i and h_i' are uniformly bounded for $i = 1,2$, we have

$$\begin{aligned} I_4 &:= -\int_0^b (W_2J_2 - W_1J_1)(h_2 - h_1)\,\mathrm{d}x \\ &\le C\int_0^b |W_2 - W_1||h_2 - h_1|\,\mathrm{d}x + C\int_0^b |h_2' - h_1'||h_2 - h_1|\,\mathrm{d}x \\ &\le \eta\int_0^b |W_2 - W_1|^2\,\mathrm{d}x + C\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta\int_0^b |h_2 - h_1|^2\,\mathrm{d}x \\ &\le \eta\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x + C\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta\int_0^b |h_2 - h_1|^2\,\mathrm{d}x\,, \end{aligned} \tag{3.3.12}$$

where in the second inequality we used Young's inequality (with and without a small parameter $\eta > 0$), while in the last we used (3.3.11).

Finally, combining (3.3.9), (3.3.10) and (3.3.12) with (3.3.6), we obtain that

$$\begin{aligned} \frac{\partial}{\partial t}\int_0^b |h_2 - h_1|^2\,\mathrm{d}x + C_\varepsilon\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x &\le \eta\int_0^b |h_2''' - h_1'''|^2\,\mathrm{d}x + \eta\int_0^b |h_2'' - h_1''|^2\,\mathrm{d}x \\ &\quad + C_\eta\left(1 + \|h_1''\|_\infty^2 + \|h_2''\|_\infty^2\right)\int_0^b |h_2' - h_1'|^2\,\mathrm{d}x + C_\eta(1 + D)\int_0^b |h_2 - h_1|^2\,\mathrm{d}x\,, \end{aligned} \tag{3.3.13}$$

for a small $\eta > 0$ and for a function D defined in (0,T) by

$$D(t) := \sum_{i=1,2} \left(\|h_i''(\cdot,t)\|_\infty^2 + \|h_i''(\cdot,t)\|_\infty^4 + \|h_i'''(\cdot,t)\|_\infty^2 \right) . \tag{3.3.14}$$

Step 3: We now insert (3.1.20) for h_1 and h_2 in (3.3.4). Since

$$\left(\frac{h_i''}{J_i^5}\right)'' = \left(\frac{h_i'''}{J_i^5}\right)' - 5\left(\frac{(h_i'')^2 h_i'}{J_i^7}\right)'$$

for $i = 1{,}2$, integrating by parts and using the periodicity of $h_1(\cdot,t)$ and $h_2(\cdot,t)$, we have that

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x = &-\int_0^b \left[\varepsilon \frac{h_2'''}{J_2^5} - \frac{5\varepsilon}{2}\frac{(h_2'')^2 h_2'}{J_2^7} + \partial_1\psi(-h_2',1)\right](J_2(h_2''-h_1''))'\,\mathrm{d}x \\ &+\int_0^b \left[\varepsilon \frac{h_1'''}{J_1^5} - \frac{5\varepsilon}{2}\frac{(h_1'')^2 h_1'}{J_1^7} + \partial_1\psi(-h_1',1)\right](J_1(h_2''-h_1''))'\,\mathrm{d}x \\ &+\int_0^b (W_2J_2 - W_1J_1)(h_2''-h_1'')\,\mathrm{d}x := \bar I_1 + \bar I_2 + \bar I_3 \,. \end{aligned} \tag{3.3.15}$$

Proceeding analogously to the second step, we estimate the integrals on the right-hand side of the previous equality.

First, we observe that

$$\begin{aligned} \bar I_1 + \bar I_2 + \varepsilon\int_0^b \frac{|h_2'''-h_1'''|^2}{J_2^4}\,\mathrm{d}x = &-\varepsilon\int_0^b h_1'''\left(\frac{1}{J_2^4} - \frac{1}{J_1^4}\right)(h_2'''-h_1''')\,\mathrm{d}x \\ &-\varepsilon\int_0^b \left(\frac{h_2'''h_2''h_2'}{J_2^6} - \frac{h_1'''h_1''h_1'}{J_1^6}\right)(h_2''-h_1'')\,\mathrm{d}x \\ &+\frac{5\varepsilon}{2}\int_0^b \left(\frac{(h_2'')^2h_2'}{J_2^6} - \frac{(h_1'')^2h_1'}{J_1^6}\right)(h_2'''-h_1''')\,\mathrm{d}x \\ &+\frac{5\varepsilon}{2}\int_0^b \left(\frac{(h_2'')^3(h_2')^2}{J_2^8} - \frac{(h_1'')^3(h_1')^2}{J_1^8}\right)(h_2''-h_1'')\,\mathrm{d}x \\ &-\int_0^b (\partial_1\psi(-h_2',1)J_2 - \partial_1\psi(-h_1',1)J_1)(h_2'''-h_1''')\,\mathrm{d}x \\ &-\int_0^b (\partial_1\psi(-h_2',1)\frac{h_2''h_2'}{J_2} - \partial_1\psi(-h_1',1)\frac{h_1''h_1'}{J_1})(h_2''-h_1'')\,\mathrm{d}x \,. \end{aligned}$$

Thus, recalling (3.3.7) and using as before the facts that h_1' and h_2' are uniformly bounded, that for $n \in \mathbb{N}$, (3.3.8) holds, and that the functions $s \mapsto (1+s^2)^{-\frac{n}{2}}$ and $s \mapsto \partial_1\psi(s,1)$

are locally Lipschitz continuous, we obtain

$$
\begin{aligned}
\bar I_1+\bar I_2 + C_\varepsilon \int_0^b |h_2''' - h_1'''|^2 \,\mathrm{d}x \leq C\Bigg[&\int_0^b |h_1'''||h_2''' - h_1'''||h_2' - h_1'| \,\mathrm{d}x \\
&+ \int_0^b (|h_2''||h_2'' - h_1''|)(|h_2'''||h_2' - h_1'|) \,\mathrm{d}x + \int_0^b |h_2''||h_2''' - h_1'''||h_2'' - h_1''| \,\mathrm{d}x + \int_0^b |h_1'''||h_2'' - h_1''|^2 \,\mathrm{d}x \\
&+ \int_0^b |h_2''|^2|h_2''' - h_1'''||h_2' - h_1'| \,\mathrm{d}x + (\|h_1''\|_\infty + \|h_2''\|_\infty) \int_0^b |h_2''' - h_1'''||h_2'' - h_1''| \,\mathrm{d}x \\
&+ \int_0^b (|h_2''||h_2'' - h_1''|)(|h_2''|^2|h_2' - h_1'|) \,\mathrm{d}x + (\|h_1''\|_\infty^2 + \|h_2''\|_\infty^2) \int_0^b |h_2'' - h_1''|^2 \,\mathrm{d}x \\
&+ \int_0^b |h_2''' - h_1'''||h_2' - h_1'| \,\mathrm{d}x + \int_0^b |h_2''||h_2'' - h_1''||h_2' - h_1'| \,\mathrm{d}x + \int_0^b |h_2'' - h_1''|^2 \,\mathrm{d}x\Bigg].
\end{aligned}
$$

We then apply Young's inequality to each integral on the right-hand side of the previous inequality. Precisely, for the integrals that present the term $|h_2''' - h_1'''|$ we apply Young's inequality with a parameter $\eta > 0$. In this way, we have

$$
\begin{aligned}
\bar I_1 + \bar I_2 + C_\varepsilon \int_0^b |h_2''' - h_1'''|^2 \,\mathrm{d}x \leq{}& \eta \int_0^b |h_2''' - h_1'''|^2 \,\mathrm{d}x \\
&+ C_\eta \left(1 + \|h_1''\|_\infty^2 + \|h_2''\|_\infty^2 + \|h_1'''\|_\infty\right) \int_0^b |h_2'' - h_1''|^2 \,\mathrm{d}x \\
&+ C_\eta (1 + \|h_2''\|_\infty^2 + \|h_2''\|_\infty^4 + \|h_1'''\|_\infty^2 + \|h_2'''\|_\infty^2) \int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x\,.
\end{aligned} \tag{3.3.16}
$$

Next, we estimate $\bar I_3$ from above. From the facts that the function $s \mapsto (1+s^2)^{\frac12}$ is locally Lipschitz continuous, that W_i and h_i' are uniformly bounded for $i = 1{,}2$ and (3.3.11), it follows that

$$
\begin{aligned}
\bar I_3 &\leq C \int_0^b |W_2 - W_1||h_2'' - h_1''| \,\mathrm{d}x + C \int_0^b |h_2'' - h_1''||h_2' - h_1'| \,\mathrm{d}x \\
&\leq C \int_0^b |h_2'' - h_1''|^2 \,\mathrm{d}x + C \int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x + C \int_0^b |h_2 - h_1|^2 \,\mathrm{d}x\,,
\end{aligned} \tag{3.3.17}
$$

where we used Young's inequality and (3.3.11).

Now, since

$$
\|h_2'' - h_1''\|_{L^2} \leq C \|h_2''' - h_1'''\|_{L^2}^{\frac12} \|h_2' - h_1'\|_{L^2}^{\frac12}
$$

by (2.3.1) applied to $h_2' - h_1'$ with $j = 1$ and $m = 2$, we observe that

$$
\begin{aligned}
&C_\eta(1 + \|h_1''\|_\infty^2 + \|h_2''\|_\infty^2 + \|h_1'''\|_\infty) \int_0^b |h_2'' - h_1''|^2 \,\mathrm{d}x \\
&\qquad \leq C_\eta(1 + \|h_1''\|_\infty^2 + \|h_2''\|_\infty^2 + \|h_1'''\|_\infty)\|h_2''' - h_1'''\|_{L^2}\|h_2' - h_1'\|_{L^2} \\
&\qquad \leq \eta \int_0^b |h_2''' - h_1'''|^2 + C_\eta \left(1 + \|h_1''\|_\infty^4 + \|h_2''\|_\infty^4 + \|h_1'''\|_\infty^2\right) \int_0^b |h_2' - h_1'|^2
\end{aligned} \tag{3.3.18}
$$

where, in the last inequality, we used again Young's inequality for $\eta > 0$.

Finally, by (3.3.15), (3.3.16), (3.3.17) and (3.3.18), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t} \int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x + C_\varepsilon \int_0^b |h_2''' - h_1'''|^2 \,\mathrm{d}x \leq \\
&\quad \leq \eta \int_0^b |h_2''' - h_1'''|^2 \,\mathrm{d}x + C_\eta \,(1 + D) \int_0^b |h_2' - h_1'|^2 \,\mathrm{d}x + C \int_0^b |h_2 - h_1|^2 \,\mathrm{d}x \,,
\end{aligned} \tag{3.3.19}
$$

where D is the function defined in $(0, T)$ by (3.3.14).

Step 4: Adding (3.3.13) and (3.3.19), and choosing η small enough, we deduce that

$$
\frac{\partial H}{\partial t}(t) \leq C(1 + D(t))H(t) \,, \tag{3.3.20}
$$

for some costant $C > 0$ and for each $t \in (0, T)$. We note that, for each $t \in (0, T)$ and for $i = 1{,}2$, by (2.3.2) with $m = 2$, $p = 2$, and $q = \infty$ applied to $h_i''(\cdot, t)$, we have

$$
\|h_i''(\cdot, t)\|_\infty \leq C\|h_i^{(\mathrm{iv})}(\cdot, t)\|_{L^2(0,b)}^{\frac{1}{4}} \|h_i''(\cdot, t)\|_{L^2(0,b)}^{\frac{3}{4}} \leq CM^{\frac{3}{4}}\|h_i^{(\mathrm{iv})}(\cdot, t)\|_{L^2(0,b)}^{\frac{1}{4}} \,,
$$

and by (2.3.3) with $m = 2$, $j = 1$, $p = 2$, and $q = \infty$ again applied to $h_i''(\cdot, t)$, we have

$$
\|h_i'''(\cdot, t)\|_\infty \leq C\|h_i^{(\mathrm{iv})}(\cdot, t)\|_{L^2(0,b)}^{\frac{3}{4}} \|h_i''(\cdot, t)\|_{L^2(0,b)}^{\frac{1}{4}} \leq CM^{\frac{1}{4}}\|h_i^{(\mathrm{iv})}(\cdot, t)\|_{L^2(0,b)}^{\frac{3}{4}} \,.
$$

Therefore, we may find a constant $C > 0$ that depends only on M such that $D(t) \leq CG(t)$, and so (3.3.2) follows from (3.3.20). In view of the fact that $G \in L^1(0, T)$, we may apply Gronwall's Lemma to obtain that H satisfies

$$
H(t) \leq H(0) \exp\left(\int_0^t G(s) \,\mathrm{d}s\right)
$$

for every $t \in [0, T]$. Since $H(0) = 0$, this concludes the proof.

□

Chapter 4

Material Voids in an Elastic Solid and Regularity Results for $d \geq 2$

In this chapter we prove the results about the existence and regularity of the minimal configurations in dimensions $d \geq 2$ that we presented in Section 1.2 with reference to the applications to material voids in an elastic solid. We refer the reader to Section 1.2 for the introduction to the model and for its physical motivation. We proceed as follows. In Section 4.1 we present the mathematical setting in the scalar case using a formulation of the model consistent with [37, 65]. As described in the Introduction, we present the relaxation result contained in [20], we introduce the notion of volume-constrained local minimizer and the notion of quasi-minimizer of $\overline{\mathcal{G}}$. Then, we study the compactness property of sequences of admissible pairs with equibounded energies, and we analyse the scaling properties of the functional $\overline{\mathcal{G}}$ and of its quasi-minimizers.

In Section 4.2 we prove that local minimizers are also quasi-minimizers of $\overline{\mathcal{G}}$. This is a consequence of Proposition 4.2.1 in which we show that every local minimizer of $\overline{\mathcal{G}}$ is also a free minimizer of a new functional obtained from $\overline{\mathcal{G}}$ by adding a suitable penalization term.

In Section 4.3 we establish the lower density bound for every quasi-minimizer of $\overline{\mathcal{G}}$ (see Theorem 4.3.8). This result follows from a blow-up argument that, in view of the scaling properties proved in Section 4.1, provides an estimation of the decay of $\overline{\mathcal{G}}$ in small balls (see the Decay Lemma 4.3.6). A first consequence of the lower density bound is that the set

$$\Gamma_{E,u} := \partial^* E \cup \left(S_u \cap E^0\right) ,$$

is essentially closed in Ω for every quasi-minimizer (E, u). Therefore, by Section 4.2 we obtain that $\Gamma_{E,u}$ is essentially closed also for the volume-constrained local minimizer of $\overline{\mathcal{G}}$. Finally, the Regularity Theorem presented in Section 1.2 follows from the classical regularity results for minima of the generalized Dirichlet functional with exponent $p > 1$ (see [34] for the case with $p > 2$ and [54] for the case $2 \geq p > 1$).

4.1 Mathematical Setting for Material Voids

Let $d \geq 2$, $p > 1$, and let $\Omega \subset \mathbb{R}^d$ be an open set. We define the space of pairs $X_{\mathrm{reg}}(\Omega)$ by

$$X_{\mathrm{reg}}(\Omega) := \Big\{(E,u) \in \mathscr{P}(\Omega) \times L^1(\Omega) : \, u \in W^{1,p}(\Omega) \text{ and } \partial E \text{ is locally Lipschitz}\Big\} ,$$

and the functional $\mathcal{G} : \mathscr{M}(\Omega) \times L^1(\Omega) \to [0, +\infty]$ by

$$\mathcal{G}(E,u) := \begin{cases} \displaystyle\int_{\Omega \setminus E} |\nabla u|^p \, \mathrm{d}x + \int_{\Omega \cap \partial E} \psi(\nu_E) \, \mathrm{d}\mathcal{H}^{d-1} & \text{if } (E,u) \in X_{\mathrm{reg}}(\Omega), \\ \\ +\infty & \text{otherwise,} \end{cases} \tag{4.1.1}$$

where ν_E denotes the interior normal to E. As defined in the Introduction,

$$\psi : \mathbb{R}^2 \setminus \{0\} \to (0, \infty)$$

stands for a positively one-homogeneous function of class C^2 away from the origin. We recall that from these assumptions it follows that

$$M_1|\xi| \leq \psi(\xi) \leq M_2|\xi| \tag{4.1.2}$$

for each $\xi \in \mathbb{R}^d$ and some positive constants M_1 and M_2 (see (3.1.3)). Since the functional $\mathcal{G}$ is not lower semicontinuous with respect to the convergence in $L^1(\Omega) \times L^1(\Omega)$, we consider its lower semicontinuous envelope $\overline{\mathcal{G}}$ with respect to the same topology, that is defined by

$$\overline{\mathcal{G}}(E,u) := \inf\Big\{\mathcal{G}(E_n, u_n) : \, \{(E_n,u_n)\} \subset \mathscr{M}(\Omega) \times L^1(\Omega) \,, \quad E_n \to E \quad \text{in} \quad L^1(\Omega) \,, \quad \text{and} \quad u_n \to u \quad \text{in} \quad L^1(\Omega)\Big\}$$

for each $\mathscr{M}(\Omega) \times L^1(\Omega)$. In order to introduce the integral representation of $\overline{\mathcal{G}}$, we define the space of admissible pairs $X(\Omega)$ by

$$X(\Omega) := \Big\{(E,u) \in \mathscr{P}(\Omega) \times L^1(\Omega) :\, u\chi_{E^0} \in GSBV(\Omega)\Big\}\,.$$

Remark 4.1.1. Let $(E,u) \in X(\Omega)$ and define $w := u\chi_{E^0}$. Since each $x \in E^0$ is a point of density 1 for $\{u = w\}$, we have that u is weakly approximately continuous and differentiable in E^0 at the same points of w by Remark 2.11.6. Precisely, we have that $S^*_u \cap E^0 = S^*_w \cap E^0$, $J^*_u \cap E^0 = J^*_w \cap E^0$, $\nu^*_u(x) = \nu^*_w(x)$ for each weak approximate jump point $x \in J^*_u \cap E^0$, and $\nabla^* u = \nabla^* w$ $\mathcal{L}^d$-a.e. in $\Omega \setminus E$.

In view of Remark 4.1.1, we present the following relaxation result that has been established in [20].

Theorem 4.1.2. *Assume that ψ is convex. Then, the lower semicontinuous envelope of $\mathcal{G}$ with respect to the $L^1(\Omega) \times L^1(\Omega)$ topology is the functional $\overline{\mathcal{G}} : \mathscr{M}(\Omega) \times L^1(\Omega) \to [0,+\infty]$ defined by*

$$\overline{\mathcal{G}}(E,u) = \begin{cases} \displaystyle\int_{\Omega\setminus E} |\nabla^* u|^p \,\mathrm{d}x + \int_{\Omega\cap\partial^* E} \psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Omega\cap S^*_u\cap E^0} \big(\psi(\nu^*_u) + \psi(-\nu^*_u)\big)\,\mathrm{d}\mathcal{H}^{d-1} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } (E,u) \in X(\Omega), \\ +\infty & \textit{otherwise.} \end{cases}$$

Remark 4.1.3. We observe that if $\overline{\mathcal{G}}(E,u) < \infty$, then E is a set of finite perimeter in Ω, and $u\chi_{E^0} \in GSBV^p(\Omega)$.

Furthermore, for every $(E,u) \in X(\Omega)$, Borel set $B \subset \Omega$, and constant $c > 0$, we use the notation

$$\overline{\mathcal{G}}(E,u,c,B) := \int_{B\setminus E} |\nabla^* u|^p \,\mathrm{d}x + c\int_{B\cap\partial^* E} \psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + c\int_{B\cap S^*_u\cap E^0} \big(\psi(\nu^*_u) + \psi(-\nu^*_u)\big)\,\mathrm{d}\mathcal{H}^{d-1}\,,$$

and

$$\overline{\mathcal{G}}(E,u,B) := \overline{\mathcal{G}}(E,u,1,B)\,. \tag{4.1.3}$$

In this chapter we are mainly interested in the regularity properties of the pairs (E,u) that (locally) minimize the functional $\overline{\mathcal{G}}$ under a volume constraint on the sets E and such that the displacements u take prescribed values outside a bounded region in Ω. Hence, we introduce a Dirichlet boundary condition by assuming that $\Omega' \subset\subset \Omega$ is a bounded, open set with Lipschitz boundary and $u_0 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and we set

$$X_{u_0}(\Omega, \Omega') := \left\{(E, u) \in X(\Omega) : E \subset \overline{\Omega'},\ u = u_0 \text{ a.e. in } \Omega \setminus \overline{\Omega'}\right\}.$$

Moreover, given $0 < \lambda < |\Omega'|$, we impose a volume constraint defining

$$X_{u_0,\lambda}(\Omega, \Omega') := \left\{(E, u) \in X_{u_0}(\Omega, \Omega') : |E| = \lambda\right\}.$$

In the following proposition we show that sequences of pairs with equibounded energies are compact in a suitable topology.

Proposition 4.1.4. *Let $(E_n, u_n) \in X_{u_0}(\Omega, \Omega')$ be such that*

$$\sup_n \overline{\mathcal{G}}(E_n, u_n) < +\infty \quad \textit{and} \quad \sup_n \|u_n\|_{L^\infty(\Omega)} < +\infty.$$

Then there exist $(E, u) \in X_{u_0}(\Omega, \Omega')$ and a subsequence (E_{n_k}, u_{n_k}) such that $E_{n_k} \to E$ in $L^1(\Omega)$, $u_{n_k}\chi_{E^0_{n_k}} \to u$ in $L^1(\Omega)$.

Proof. From the uniform bound on the energies and from (4.1.2) we have that the sets E_n have equibounded perimeters, hence by Theorem 2.5.4 up to a subsequence (not relabeled) they converge in $L^1(\Omega)$ to a finite perimeter set $E \in \mathscr{P}(\Omega)$, $E \subset \overline{\Omega'}$. In addition, the functions $v_n := u_n\chi_{E^0_n}$ belong to $GSBV(\Omega) \cap L^\infty(\Omega)$ and they coincide with a $W^{1,p}$-function in $\Omega \setminus \overline{\Omega'}$, hence we deduce that $v_n \in SBV(\Omega)$. Since by assumption

$$\sup_n \left\{\int_\Omega |\nabla v_n|^p \,\mathrm{d}x + \mathcal{H}^{d-1}(S_{v_n}) + \|v_n\|_{L^\infty(\Omega)}\right\} < +\infty,$$

by the Compactness Theorem 2.10.3 in SBV we have that up to a further subsequence (not relabeled) $v_n \to u$ in $L^1(\Omega)$, with $u \in SBV(\Omega)$. Finally, we clearly have $u = 0$ on E and $u = u_0$ in $\Omega \setminus \overline{\Omega'}$, from which it follows that $(E, u) \in X_{u_0}(\Omega, \Omega')$. □

We remark that, as the functional $\overline{\mathcal{G}}$ is lower semicontinuous with respect to the convergence stated in the previous proposition by the results contained in [20], the minimum problem

$$\text{(P)} \qquad \min\left\{\overline{\mathcal{G}}(E, u) : (E, u) \in X_{u_0,\lambda}(\Omega, \Omega')\right\}$$

always admits a solution by the Direct Method of the Calculus of Variation (note that by a truncation argument we can always assume that a minimizing sequence is bounded in L^∞, so that the assumptions of Proposition 4.1.4 are satisfied).

In the following definition we introduce the notion of local minimizer of $\overline{\mathcal{G}}$ corresponding to the previous minimum problem.

Definition 4.1.5. Given $\delta > 0$, we say that a pair $(E, u) \in X_{u_0,\lambda}(\Omega, \Omega')$ is a δ-*local minimizer (in its volume class) of* $\overline{\mathcal{G}}$ if $\overline{\mathcal{G}}(E, u) < \infty$ and

$$\overline{\mathcal{G}}(E, u) \leq \overline{\mathcal{G}}(F, v) \tag{4.1.4}$$

for every $(F, v) \in X_{u_0,\lambda}(\Omega, \Omega')$ such that $|E \triangle F| \leq \delta$. Furthermore, we say that a δ-local minimizer (E, u) is *isolated* if (4.1.4) holds with the strict inequality whenever $|E \triangle F| > 0$.

We now state the quasi-minimality property introduced in (1.2.6) that we will prove it applies to δ-local minimizer of $\overline{\mathcal{G}}$.

Definition 4.1.6. Let $A \subset \mathbb{R}^d$ be an open set and $c > 0$ be a constant. Given $(E, u) \in X(A)$ such that $\overline{\mathcal{G}}(E, u, c, A) < \infty$, we define the *deviation from minimality* $\mathrm{Dev}(E, u, c, A)$ of (E, u) in A (with respect to c) as the smallest $\theta \in [0, \infty]$ such that

$$\overline{\mathcal{G}}(E, u, c, A) \leq \overline{\mathcal{G}}(F, v, c, A) + \theta$$

for every $(F, v) \in X(A)$ such that $E \triangle F \subset\subset A$ and $\{u \neq v\} \subset\subset A$. We write

$$\mathrm{Dev}(E, u, A) := \mathrm{Dev}(E, u, 1, A)\,.$$

Definition 4.1.7. Let $A \subset \mathbb{R}^d$ be an open set. We say that a pair $(E, u) \in X(A)$ is a *quasi-minimizer* of $\overline{\mathcal{G}}$ in A if $\overline{\mathcal{G}}(E, u, A) < \infty$ and there exist a constant $\omega \geq 0$ and a radius $\varrho_0 > 0$ such that for all balls $B_\varrho(x) \subset A$ with $\varrho \leq \varrho_0$ we have that $\mathrm{Dev}(E, u, B_\varrho(x)) \leq \omega \varrho^d$, i.e.,

$$\overline{\mathcal{G}}(E, u, B_\varrho(x)) \leq \overline{\mathcal{G}}(F, v, B_\varrho(x)) + \omega \varrho^d$$

for every $(F, v) \in X(B_\varrho(x))$ such that $E \triangle F \subset\subset B_\varrho(x)$ and $\{u \neq v\} \subset\subset B_\varrho(x)$. We write $(E, u) \in \mathcal{M}_\omega(A)$.

We conclude this section by presenting some scaling properties of the functional $\overline{\mathcal{G}}$ and of the deviation from minimality for admissible pairs (E, u).

Definition 4.1.8. Let $z \in \Omega$ and $\varrho > 0$ be such that $B_\varrho(z) \subset \Omega$ and consider the map $\gamma = \gamma_{z,\varrho}$ defined by

$$\gamma(x) := \frac{x - z}{\varrho}$$

for every $x \in \Omega$. Given a set $S \subset \Omega$ and a function v defined in Ω, we define the *rescaled set* $S_{z,\varrho}$ and the *rescaled function* $v_{z,\varrho}$ *with respect to* z *and* ϱ by, respectively,

$$S_{z,\varrho} := \gamma(S) \quad \text{and} \quad v_{z,\varrho}(y) := \varrho^{\frac{1-p}{p}} v(\gamma^{-1}(y)) = \varrho^{\frac{1-p}{p}} v(z + \varrho y) \tag{4.1.5}$$

for every $y \in \Omega_{z,\varrho}$. Moreover, given a Radon measure μ on $(\Omega, \mathscr{B}(\Omega))$, we define the *push-forward measure* $\gamma_\sharp\mu$ *of* μ by

$$\gamma_\sharp\mu(B) := \mu\left(\gamma^{-1}(B)\right)$$

for every $B \in \mathscr{B}(\Omega_{z,\varrho})$.

The proof of the Decay Lemma 4.3.6 of Section 4.3 is based on a typical blow-up argument for which we need the following remark.

Remark 4.1.9. Let $(E, u) \in X(\Omega)$ be such that $w := u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$. Let $z \in \Omega$ and $\varrho > 0$ be such that $B_\varrho(z) \subset \Omega$. With the notation introduced in (4.1.5), consider $\Omega_{z,\varrho}$, $E_{z,\varrho}$, $u_{z,\varrho}$, and $w_{z,\varrho}$. The following assertions hold.

(i) $(E_{z,\varrho}, u_{z,\varrho}) \in X(\Omega_{z,\varrho})$ and $w_{z,\varrho} = u_{z,\varrho}\chi_{E^0_{z,\varrho}} \in SBV_{\mathrm{loc}}(\Omega_{z,\varrho})$.
In fact, observe that $\partial_* E_{z,\varrho} = (\partial_* E)_{z,\varrho}$, $E^0_{z,\varrho} = (E^0)_{z,\varrho}$, and by the definitions of distributional derivative and push-forward, we have that

$$D\left(\chi_{E_{z,\varrho}}\right) = \varrho^{1-d}\gamma_\sharp\left(D\chi_E\right) \tag{4.1.6}$$

(see [9, Remark 3.18]). Thus, $E_{z,\varrho}$ is a set of locally finite perimeter.
Furthermore, $u_{z,\varrho} \in L^\infty(\Omega_{z,\varrho})$, $S_{w_{z,\varrho}} = (S_w)_{z,\varrho}$, and using again the definitions of distributional derivative and push-forward, we obtain

$$Dw_{z,\varrho} = \varrho^{1-d}\gamma_\sharp(\varrho^{\frac{1-p}{p}} Dw)$$

(see [9, Remark 3.18]). Hence, $w_{z,\varrho} \in BV_{\mathrm{loc}}(\Omega_{z,\varrho})$ and so, by [9, Proposition 3.92] we have that

$$\begin{aligned} D^a w_{z,\varrho} &= \nabla w_{z,\varrho}\mathcal{L}^d = \varrho^{\frac{1}{p}}\nabla w \circ \gamma^{-1}\mathcal{L}^d\,, \\ D^j w_{z,\varrho} &= \left[(w_{z,\varrho})^+ - (w_{z,\varrho})^-\right]\nu_w \circ \gamma^{-1}\,\mathcal{H}^{d-1}_{|\Omega_{z,\varrho}\cap(S_w)_{z,\varrho}}\,, \\ D^c w_{z,\varrho} &= 0\,. \end{aligned} \tag{4.1.7}$$

(ii) Let $A \subset \Omega$ be an open set. By (4.1.6) we have

$$\int_{A_{z,\varrho}\cap\partial^* E_{z,\varrho}} \psi(\nu_{E_{z,\varrho}})\,\mathrm{d}\mathcal{H}^{d-1} = \varrho^{1-d}\int_{\gamma(A)} \psi(\nu_E)\circ\gamma^{-1}\,\mathrm{d}\gamma_\sharp\,(D\chi_E)$$
$$= \varrho^{1-d}\int_{A\cap\partial^* E} \psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1}\,.$$

Also, in view of Remark 4.1.1 and (4.1.7), we have that

$$\int_{A_{z,\varrho}\setminus E_{z,\varrho}} |\nabla u_{z,\varrho}|^p\,\mathrm{d}y = \varrho\int_{\gamma(A)} |\nabla w(z+\varrho y)|^p\,\mathrm{d}y = \varrho^{1-d}\int_{A\setminus E} |\nabla u|^p\,\mathrm{d}x$$

since $\nabla u_{z,\varrho}(y) = \nabla w_{z,\varrho}(y) = \varrho^{\frac{1}{p}}\nabla w\circ\gamma^{-1}(y)$ for $\mathcal{L}^d$-a.e. $y \in \Omega_{z,\varrho}\setminus E_{z,\varrho}$, and

$$\int_{A_{z,\varrho}\cap S_{u_{z,\varrho}}\cap E^0_{z,\varrho}} [\psi(\nu_{u_{z,\varrho}}) + \psi(-\nu_{u_{z,\varrho}})]\,\mathrm{d}\mathcal{H}^{d-1}$$
$$= \varrho^{1-d}\int_{\gamma(A\cap S_w\cap E^0)} [\psi(\nu_u)+\psi(-\nu_u)]\circ\gamma^{-1}\,\mathrm{d}\mathcal{H}^{d-1}$$
$$= \varrho^{1-d}\int_{A\cap S_u\cap E^0} [\psi(\nu_u)+\psi(-\nu_u)]\,\mathrm{d}\mathcal{H}^{d-1}\,.$$

Therefore, for every $c > 0$ we have that

$$\overline{\mathcal{G}}(E_{z,\varrho}, u_{z,\varrho}, c, A_{z,\varrho}) = \varrho^{1-d}\overline{\mathcal{G}}(E, u, c, A)\,,$$

and

$$\mathrm{Dev}(E_{z,\varrho}, u_{z,\varrho}, c, A_{z,\varrho}) = \varrho^{1-d}\mathrm{Dev}(E, u, c, A)\,.$$

(iii) If $(E, u) \in \mathcal{M}_\omega(A)$ for some constant $\omega \geq 0$ and some open set $A \subset \Omega$, then $(E_{z,\varrho}, u_{z,\varrho}) \in \mathcal{M}_{\omega\varrho}(A_{z,\varrho})$.

4.2 Volume Penalization

As explained in Section 1.2, in studying the regularity properties of local minimizers of the functional $\overline{\mathcal{G}}$, one initially seeks to get rid of the volume constraint in order to gain more freedom in the admissible variations. This is acquired by showing that every volume-constrained local minimizer of $\overline{\mathcal{G}}$ is also a free minimizer of a new functional obtained from $\overline{\mathcal{G}}$ by adding a suitable penalization term.

Proposition 4.2.1. *Assume that Ω' is connected, and consider a δ-local minimizer $(E,u) \in X_{u_0,\lambda}(\Omega,\Omega')$ of $\overline{\mathcal{G}}$ (see Definition 4.1.5). Then, there exists $\beta_0 > 0$ such that (E,u) is a solution of the minimum problem*

$$\min\left\{\overline{\mathcal{G}}(F,v) + \beta\big|\lambda - |F|\big| : \ (F,v) \in X_{u_0}(\Omega,\Omega'),\ |E \triangle F| \leq \frac{\delta}{2}\right\} \tag{4.2.1}$$

for all $\beta \geq \beta_0$.

Proof. We follow the argument in [33, Section 2] (see also [1, Proposition 2.7]). Observe that, by the Direct Method of the Calculus of Variations, for every $\beta > 0$ the problem (4.2.1) admits a solution, which we denote by (E_β, u_β), and in addition

$$\overline{\mathcal{G}}(E_\beta,u_\beta) \leq \overline{\mathcal{G}}(E_\beta,u_\beta) + \beta\big|\lambda - |E_\beta|\big| \leq \overline{\mathcal{G}}(E,u). \tag{4.2.2}$$

We shall prove that, if β is sufficiently large, each minimizer (E_β,u_β) satisfies the volume constraint $|E_\beta| = \lambda$, so that by the local minimality of (E,u) and by (4.2.2) the result will follow.

Assume by contradiction that there exists a sequence $\beta_n \to +\infty$ such that $|E_{\beta_n}| \neq \lambda$ for every $n \in \mathbb{N}$. Without loss of generality we may assume that $|E_{\beta_n}| < \lambda$ for every n (the proof for the case in which $|E_{\beta_n}| > \lambda$ is similar). In order to simplify the notation, we set $(E_n,u_n) := (E_{\beta_n},u_{\beta_n})$. Our aim is to construct suitable competitors $(F_n,v_n) \in X_{u_0,\lambda}(\Omega,\Omega')$, with $|F_n \triangle E| \leq \delta$, such that $\overline{\mathcal{G}}(F_n,v_n) < \overline{\mathcal{G}}(E,u)$, thus contradicting the local minimality of (E,u).

Step 1: Since, by (4.2.2), the sets E_n have equibounded perimeters and $|E_n| \to \lambda$, up to a subsequence (not relabeled) $E_n \to F$ in $L^1(\Omega)$ for some set F of finite perimeter in Ω with $|F| = \lambda$. Since F has finite perimeter in Ω, $0 < |F| < |\Omega'|$ and Ω' is connected, there exists a point $x_0 \in \partial^* F \cap \Omega'$. By Theorem 2.5.7 the translated and rescaled sets $F_{x_0,r} := \frac{1}{r}(F - x_0)$ converge in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ to the half space $H := \{z \cdot \nu_F(x_0) > 0\}$ as $r \to 0$ where ν_F is the generalized inner normal to F at x_0 (see Definition 2.5.5). Hence, given $\varepsilon > 0$, there exists $r > 0$ such that, setting $x_r := x_0 - r\nu_F(x_0)/2$, we have $B_r(x_r) \subset \Omega'$ and

$$|F \cap B_{r/2}(x_r)| < \varepsilon r^d, \qquad |F \cap B_r(x_r)| > \frac{\omega_d r^d}{2^{d+2}},$$

and assuming for simplicity that $x_r = 0$, the convergence of E_n to F implies that

$$|E_n \cap B_{r/2}| < \varepsilon r^d, \qquad |E_n \cap B_r| > \frac{\omega_d r^d}{2^{d+2}} \tag{4.2.3}$$

for all n sufficiently large. For a sequence of constants $0 < \sigma_n < \frac{1}{2^d}$ to be chosen, we consider the sequence of bi-Lipschitz maps defined by

$$\Phi_n(x) := \begin{cases} x - \sigma_n(2^d - 1)x & \text{if } |x| < \frac{r}{2}, \\ x + \sigma_n\Big(1 - \frac{r^d}{|x|^d}\Big)x & \text{if } \frac{r}{2} \le |x| < r, \\ x & \text{if } |x| \ge r. \end{cases} \tag{4.2.4}$$

In the sequel $J\Phi_n$ stands for the jacobian of Φ_n, while $J_{d-1}\Phi_{n,x}$ denotes the $(d-1)$-dimensional jacobian of the tangential differential of Φ_n at $x \in \partial^* E_n$, i.e.,

$$J_{d-1}\Phi_{n,x} := \boldsymbol{J}_{d-1} d^{\partial^* E_n}(\Phi_n)_x$$

(see Definitions 2.7.5 and 2.7.7). In [33] the following estimates are established:

$$\|\nabla\Phi_n^{-1}(\Phi_n(x))\| \le \big(1 - (2^d - 1)\sigma_n\big)^{-1} \qquad \text{for every } x \in B_r \setminus B_{r/2}, \tag{4.2.5}$$

$$1 + C_1\sigma_n \le J\Phi_n(x) \le 1 + 2^d d\sigma_n \qquad \text{for every } x \in B_r \setminus B_{r/2}, \tag{4.2.6}$$

$$J_{d-1}\Phi_{n,x} \le 1 + \sigma_n + 2^d(d-1)\sigma_n \qquad \text{for every } x \in \partial^* E_n \cap (B_r \setminus B_{r/2}), \tag{4.2.7}$$

where C_1 is a dimensional constant. We define $F_n := \Phi_n(E_n)$, $v_n := u_n \circ \Phi_n^{-1}$, so that $(F_n, v_n) \in X_{u_0}(\Omega, \Omega')$ (we have not altered the boundary datum, since Φ_n coincides with the identity outside B_r). We first note that by (4.2.3), (4.2.4), and (4.2.6) we have

$$\begin{aligned} |F_n| - |E_n| &= \int_{B_r \cap E_n} (J\Phi_n - 1)\,\mathrm{d}x \\ &\ge C_1\sigma_n |E_n \cap (B_r \setminus B_{r/2})| + \Big[\big(1 - \sigma_n(2^d - 1)\big)^d - 1\Big]|E_n \cap B_{r/2}| \\ &\ge C_1\sigma_n\Big(\frac{\omega_d r^d}{2^{d+2}} - \varepsilon r^d\Big) - \sigma_n(2^d - 1)d\varepsilon r^d \ge C_2\sigma_n r^d\,, \end{aligned} \tag{4.2.8}$$

where in the last inequality we have chosen ε sufficiently small (independently of n). Hence, we can choose σ_n so that $|F_n| = \lambda$ for every n. In particular, note that this implies that $\sigma_n \to 0$.

Given a function $f \in C^1(\Omega)$, by (4.2.4) we observe that

$$\begin{aligned} \int_\Omega |f(\Phi_n^{-1}(x)) - f(x)|\,\mathrm{d}x &\le \int_\Omega \int_0^1 |\nabla f(tx + (1-t)\Phi_n^{-1}(x))||\Phi_n^{-1}(x) - x|\,\mathrm{d}t\,\mathrm{d}x \\ &\le c\sigma_n \int_0^1 \int_{B_r} |\nabla f(tx + (1-t)\Phi_n^{-1}(x))|\,\mathrm{d}x\,\mathrm{d}t \le C_3\sigma_n \int_{B_r} |\nabla f(y)|\,\mathrm{d}y\,, \end{aligned}$$

where the last inequality is obtained by a change of variables. Thus, by approximating χ_{E_n} with a sequence of functions $f_k^{(n)} \in C^1(\Omega)$ according to Theorem 2.4.3, we deduce that

$$\begin{aligned}|F_n \triangle E_n| &= \int_\Omega |\chi_{E_n} \circ \Phi_n^{-1} - \chi_{E_n}| \,\mathrm{d}x = \lim_k \int_\Omega |f_k^{(n)} \circ \Phi_n^{-1} - f_k^{(n)}| \,\mathrm{d}x \\ &\leq \lim_k C_3 \sigma_n \int_{B_r} |\nabla f_k^{(n)}| \,\mathrm{d}x = C_3 \sigma_n P(E_n, B_r).\end{aligned}$$

Therefore, if n is sufficiently large, since $\sigma_n \to 0$ and $P(E_n, B_r)$ are equibounded, we deduce that $|F_n \triangle E_n| \leq \frac{\delta}{2}$ and hence $|F_n \triangle E| \leq \delta$.

Step 2: Since, by the previous step, $|F_n| = \lambda$ and $|F_n \triangle E| \leq \delta$ for every n, in order to get a contradiction we shall prove that $\overline{\mathcal{G}}(F_n, v_n) < \overline{\mathcal{G}}(E, u)$. From the minimality of (E_n, u_n) and (4.2.8) it follows that

$$\begin{aligned}\overline{\mathcal{G}}(E,u) - \overline{\mathcal{G}}(F_n, v_n) &\geq \overline{\mathcal{G}}(E_n, u_n) + \beta_n \big|\lambda - |E_n|\big| - \overline{\mathcal{G}}(F_n, v_n) \\ &\geq \overline{\mathcal{G}}(E_n, u_n) - \overline{\mathcal{G}}(F_n, v_n) + \beta_n \sigma_n C_2 r^d.\end{aligned} \tag{4.2.9}$$

We now estimate the term

$$I_{1,n} := \int_{\Omega \setminus E_n} |\nabla^* u_n|^p \,\mathrm{d}x - \int_{\Omega \setminus F_n} |\nabla^* v_n|^p \,\mathrm{d}x$$

that by a change of variables satisfies

$$I_{1,n} = \int_{B_r \setminus E_n} \Big[|\nabla^* u_n(x)|^p - |\nabla^* u_n(x) \nabla \Phi_n^{-1}(\Phi_n(x))|^p J\Phi_n(x) \Big] \,\mathrm{d}x\,.$$

Splitting the previous integral in $B_{r/2} \setminus E_n$ and in $(B_r \setminus B_{r/2}) \setminus E_n$, we observe that there exists a constant c depending only on d and p for which, using (4.2.4),

$$\begin{aligned}&\int_{B_{r/2} \setminus E_n} \Big[|\nabla^* u_n(x)|^p - |\nabla^* u_n(x) \nabla \Phi_n^{-1}(\Phi_n(x))|^p J\Phi_n(x) \Big] \,\mathrm{d}x \\ &\qquad = \int_{B_{r/2} \setminus E_n} |\nabla^* u_n|^p \Big[1 - \big(1 - \sigma_n(2^d - 1)\big)^{d-p} \Big] \,\mathrm{d}x \geq -c\sigma_n \int_{B_{r/2} \setminus E_n} |\nabla^* u_n|^p \,\mathrm{d}x\end{aligned}$$

and, using (4.2.5) and (4.2.6),

$$\begin{aligned}&\int_{(B_r \setminus B_{r/2}) \setminus E_n} \Big[|\nabla^* u_n(x)|^p - |\nabla^* u_n(x) \nabla \Phi_n^{-1}(\Phi_n(x))|^p J\Phi_n(x) \Big] \,\mathrm{d}x \\ &\qquad \geq \int_{(B_r \setminus B_{r/2}) \setminus E_n} |\nabla^* u_n|^p \Big[1 - \big(1 - (2^d - 1)\sigma_n\big)^{-p} (1 + 2^d d \sigma_n) \Big] \,\mathrm{d}x \\ &\qquad \geq -c\sigma_n \int_{(B_r \setminus B_{r/2}) \setminus E_n} |\nabla^* u_n|^p \,\mathrm{d}x\,.\end{aligned}$$

Hence, we obtain that

$$I_{1,n} \geq -C_4\sigma_n \int_{B_r\setminus E_n} |\nabla^* u_n|^p \,\mathrm{d}x \geq -C_4\sigma_n \overline{\mathcal{G}}(E,u)\,. \tag{4.2.10}$$

We estimate the term

$$I_{2,n} := \int_{\Omega\cap\partial^* E_n} \psi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1} - \int_{\Omega\cap\partial^* F_n} \psi(\nu_{F_n})\,\mathrm{d}\mathcal{H}^{d-1}$$

by means of the Generalized Area Formula (see Theorem 2.7.8) obtaining that

$$\begin{aligned} I_{2,n} &= \int_{\overline{B}_r\cap\partial^* E_n} \Big(\psi(\nu_{E_n}) - \psi(\nu_{F_n}\circ\Phi_n) J_{d-1}\Phi_{n,x}\Big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\geq \int_{\overline{B}_r\cap\partial^* E_n} \psi(\nu_{F_n}\circ\Phi_n)(1 - J_{d-1}\Phi_{n,x})\,\mathrm{d}\mathcal{H}^{d-1} \\ &\qquad + \int_{\overline{B}_r\cap\partial^* E_n} \nabla\psi(\nu_{F_n}\circ\Phi_n)\cdot(\nu_{E_n} - \nu_{F_n}\circ\Phi_n)\,\mathrm{d}\mathcal{H}^{d-1} \\ &=: I^a_{2,n} + I^b_{2,n}\,, \end{aligned} \tag{4.2.11}$$

where in the last inequality we used the convexity of ψ. We now proceed as before splitting in $B_{r/2}\cap\partial^* E_n$ and in $(\overline{B}_r\setminus B_{r/2})\cap\partial^* E_n$ both the integrals $I^a_{2,n}$ and $I^b_{2,n}$. Regarding $I^a_{2,n}$, we first observe that

$$\int_{B_{r/2}\cap\partial^* E_n} \psi(\nu_{F_n}\circ\Phi_n)(1 - J_{d-1}\Phi_{n,x})\,\mathrm{d}\mathcal{H}^{d-1} \geq 0 \tag{4.2.12}$$

since Φ_n is a contraction in $B_{r/2}$ and $J_{d-1}\Phi_{n,x} < 1$. Then, from (4.2.7) it follows that

$$\begin{aligned} &\int_{(\overline{B}_r\setminus B_{r/2})\cap\partial^* E_n} \psi(\nu_{F_n}\circ\Phi_n)(1 - J_{d-1}\Phi_{n,x})\,\mathrm{d}\mathcal{H}^{d-1} \\ &\qquad \geq \int_{(\overline{B}_r\setminus B_{r/2})\cap\partial^* E_n} \psi(\nu_{F_n}\circ\Phi_n)(-\sigma_n - 2^d(d-1)\sigma_n)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\qquad \geq -\sigma_n 2^d d\frac{M_2}{M_1}\int_{(\overline{B}_r\setminus B_{r/2})\cap\partial^* E_n} \psi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1}\,, \end{aligned} \tag{4.2.13}$$

where M_1 and M_2 are the constants appearing in (4.1.2). To estimate $I^b_{2,n}$ we observe that $\nu_{F_n}\circ\Phi_n = \nu_{E_n}$ in $B_{r/2}$ by the definition of Φ_n, and so,

$$\int_{B_{r/2}\cap\partial^* E_n} \nabla\psi(\nu_{F_n}\circ\Phi_n)\cdot(\nu_{E_n} - \nu_{F_n}\circ\Phi_n)\,\mathrm{d}\mathcal{H}^{d-1} = 0\,. \tag{4.2.14}$$

Then, since in $\overline{B}_r \setminus B_{r/2}$ by the definition of Φ_n and (4.2.5) we have that

$$\begin{aligned}|\nu_{E_n} - \nu_{F_n} \circ \Phi_n| &= \left| \nu_{E_n} - \frac{\nu_{E_n}(\nabla \Phi_n)^{-1}}{|\nu_{E_n}(\nabla \Phi_n)^{-1}|} \right| \\ &\leq |\nu_{E_n} - \nu_{E_n}(\nabla \Phi_n)^{-1}| + \left| \nu_{E_n}(\nabla \Phi_n)^{-1} - \frac{\nu_{E_n}(\nabla \Phi_n)^{-1}}{|\nu_{E_n}(\nabla \Phi_n)^{-1}|} \right| \\ &\leq 2|\nu_{E_n} - \nu_{E_n}(\nabla \Phi_n)^{-1}| \leq 2|\nu_{E_n}(\nabla \Phi_n^{-1} \circ \Phi_n)||\nabla \Phi_n - I| \\ &\leq 2\big(1 - (2^d - 1)\sigma_n\big)^{-1}|\nabla \Phi_n - I| \leq \bar{c}\sigma_n \,,\end{aligned}$$

where $\bar{c}$ is a dimensional constant, we obtain

$$\begin{aligned}&\int_{(\overline{B}_r \setminus B_{r/2}) \cap \partial^* E_n} \nabla \psi(\nu_{F_n} \circ \Phi_n) \cdot (\nu_{E_n} - \nu_{F_n} \circ \Phi_n) \, \mathrm{d}\mathcal{H}^{d-1} \\ &\qquad \geq -c\sigma_n \frac{L}{M_1} \int_{(\overline{B}_r \setminus B_{r/2}) \cap \partial^* E_n} \psi(\nu_{E_n}) \, \mathrm{d}\mathcal{H}^{d-1} \,,\end{aligned} \tag{4.2.15}$$

where $L := \|\nabla \psi\|_\infty$. By (4.2.11), (4.2.12), (4.2.13), (4.2.14), and (4.2.15) we deduce that

$$I_{2,n} \geq -C_5 \sigma_n \int_{\overline{B}_r \cap \partial^* E_n} \psi(\nu_{E_n}) \, \mathrm{d}\mathcal{H}^{d-1} \geq -C_5 \sigma_n \overline{\mathcal{G}}(E, u) \,. \tag{4.2.16}$$

A totally similar argument leads to the following estimate:

$$\begin{aligned}&\int_{\Omega \cap S^*_{u_n} \cap E_n^{(0)}} (\psi(\nu^*_{u_n}) + \psi(-\nu^*_{u_n})) \, \mathrm{d}\mathcal{H}^{d-1} - \int_{\Omega \cap S^*_{v_n} \cap F_n^{(0)}} (\psi(\nu^*_{v_n}) + \psi(-\nu^*_{v_n})) \, \mathrm{d}\mathcal{H}^{d-1} \\ &\qquad \geq -C_6 \sigma_n \overline{\mathcal{G}}(E, u) \,.\end{aligned} \tag{4.2.17}$$

Collecting (4.2.10), (4.2.16), (4.2.17) and recalling (4.2.9) we finally deduce that

$$\overline{\mathcal{G}}(E, u) - \overline{\mathcal{G}}(F_n, v_n) \geq (C_2 \beta_n r^d - (C_4 + C_5 + C_6)\overline{\mathcal{G}}(E, u))\sigma_n > 0$$

for n large enough, which is the desired contradiction. □

An immediate consequence of Proposition 4.2.1 is that local minimizers of $\overline{\mathcal{G}}$ are also quasi-minimizers.

Corollary 4.2.2. *Assume that Ω' is connected, and let $(E, u) \in X_{u_0,\lambda}(\Omega, \Omega')$ be a δ-local minimizer of $\overline{\mathcal{G}}$. Then (E, u) is a* quasi-minimizer *of $\overline{\mathcal{G}}$ in Ω' and, in particular, $(E, u) \in \mathcal{M}_{\beta_0 \omega_d}(\Omega')$, where β_0 is given by Proposition 4.2.1. Furthermore, we have that $u\chi_{E^0} \in SBV(\Omega) \cap L^\infty(\Omega)$.*

Proof. We begin by establishing that (E, u) is a quasi-minimizer of $\overline{\mathcal{G}}$ in Ω'. Fix $\varrho_0 > 0$ such that $\omega_d \varrho_0^d \leq \frac{\delta}{2}$ and consider any $\varrho \leq \varrho_0$, $B_\varrho(x) \subset \Omega'$, and $(F, v) \in X(\Omega')$ such that $(E \triangle F) \cup \{u \neq v\} \subset\subset B_\varrho(x)$. We clearly have that $(F, v) \in X_{u_0}(\Omega, \Omega')$ and $|E \triangle F| \leq \omega_d \varrho^d \leq \frac{\delta}{2}$. Thus, it follows from Proposition 4.2.1 that

$$\overline{\mathcal{G}}(E, u, B_\varrho(x)) \leq \overline{\mathcal{G}}(F, v, B_\varrho(x)) + \beta_0 ||E| - |F|| \leq \overline{\mathcal{G}}(F, v, B_\varrho(x)) + \beta_0 \omega_d \varrho^d ,$$

showing that $\mathrm{Dev}(E, u, B_\varrho(x)) \leq \beta_0 \omega_d \varrho^d$. Then, by a truncation argument we deduce that $u\chi_{E^0} \in L^\infty(\Omega)$ since $u_0 \in L^\infty(\Omega)$. Recalling (2.11.2), this concludes the proof since u coincides with a $W^{1,p}$-function in $\Omega \setminus \overline{\Omega'}$. □

4.3 Density Lower Bound

In this section we prove the density lower bound for quasi-minimizers $(E, u) \in \mathcal{M}_\omega(\Omega)$ with $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$ (see Theorem 4.3.8) and study some of its consequences. We begin by establishing an upper bound for the energy that follows from a simple comparison argument.

Lemma 4.3.1 (energy upper bound)**.** *If $(E, u) \in \mathcal{M}_\omega(\Omega)$ then*

$$\overline{\mathcal{G}}(E, u, B_\varrho(x)) \leq M_2 d \omega_d \varrho^{d-1} + \omega \varrho^d \tag{4.3.1}$$

for every ball $B_\varrho(x) \subset \Omega$ with $\varrho \leq \varrho_0$, where M_2 and ϱ_0 are given by (4.1.2) and Definition 4.1.7, respectively.

Proof. Consider $\varrho' \leq \varrho \leq \varrho_0$ and $F := E \cup \overline{B_{\varrho'}(x)}$. Since $E \triangle F \subset\subset B_\varrho(x)$, the quasi-minimality of (E, u) implies that

$$\overline{\mathcal{G}}(E, u, B_\varrho(x)) \leq \overline{\mathcal{G}}(E \cup \overline{B_{\varrho'}(x)}, u, B_\varrho(x)) + \omega \varrho^d .$$

Hence, we have that

$$\begin{aligned} \overline{\mathcal{G}}(E, u, \overline{B_{\varrho'}(x)}) &\leq \overline{\mathcal{G}}(E \cup \overline{B_{\varrho'}(x)}, u, \overline{B_{\varrho'}(x)}) + \omega \varrho^d \\ &= \int_{\overline{B_{\varrho'}(x)} \cap \partial^* F} \psi(\nu_F) \, \mathrm{d}\mathcal{H}^{d-1} + \omega \varrho^d \leq M_2 d \omega_d (\varrho')^{d-1} + \omega \varrho^d , \end{aligned}$$

and we obtain (4.3.1) letting $\varrho' \nearrow \varrho$. □

The following proposition will be used in Theorem 4.3.5.

Proposition 4.3.2. *Let $r > 0$. Consider a sequence of constants $c_n > 0$ and a sequence of sets F_n of finite perimeter in B_r such that*

$$\lim_{n\to\infty} \mathcal{H}^{d-1}(\partial^* F_n \cap B_r) = 0\,, \tag{4.3.2}$$

$$c_n \to c_\infty \in [0,\infty]\,,$$

$$\sup_{n\to\infty} c_n \mathcal{H}^{d-1}(\partial^* F_n \cap B_r) < \infty\,, \tag{4.3.3}$$

and set $G_n^\varrho := F_n \cap (B_r \setminus B_\varrho)$ and $H_n^\varrho := F_n \cup B_\varrho$ for $0 < \varrho < r$. Then, G_n^ϱ and H_n^ϱ are sets of finite perimeter in B_r, and either

$$\lim_{n\to\infty} c_n \mathcal{H}^{d-1}(\partial^* G_n^\varrho \cap \partial B_\varrho) = 0 \quad \textit{for } \mathcal{L}^1\textit{-a.e. } \varrho \in (0,r)\,, \tag{4.3.4}$$

or

$$\lim_{n\to\infty} c_n \mathcal{H}^{d-1}(\partial^* H_n^\varrho \cap \partial B_\varrho) = 0 \quad \textit{for } \mathcal{L}^1\textit{-a.e. } \varrho \in (0,r)\,. \tag{4.3.5}$$

Proof. We begin by observing that if $c_\infty < +\infty$, then it follows immediately from (4.3.2) that

$$\lim_{n\to\infty} c_n \left(\mathcal{H}^{d-1}(\partial^* F_n \cap B_r)\right)^{1^*} = 0\,, \tag{4.3.6}$$

while if $c_\infty = +\infty$ then the same holds true by (4.3.3) since

$$c_n \left(\mathcal{H}^{d-1}(\partial^* F_n \cap B_r)\right)^{1^*} = \left(c_n \mathcal{H}^{d-1}(\partial^* F_n \cap B_r)\right)^{1^*} c_n^{-\frac{1}{d-1}}\,.$$

Note also that

$$\mathcal{H}^{d-1}(\partial^* F_n \cap \partial B_\varrho) = 0 \tag{4.3.7}$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0,r)$. Moreover, by the relative isoperimetric inequality (2.6.4) and (4.3.2), it follows that $\{F_n\}$ converges in measure in B_r to a set F that is either $F = \varnothing$ or $F = B_1$.

We distinguish the two cases and begin by proving that if $F = \varnothing$ then (4.3.4) holds. Since

$$\partial^* G_n^\varrho \cap B_r \subset \left(F_n^1 \cap \partial B_\varrho\right) \cup \left(\partial^* F_n \cap \left(B_r \setminus \overline{B}_\varrho\right)\right) \cup \left(\partial^* F_n \cap \partial B_\varrho\right)\,,$$

we have that G_n^ϱ is a set of finite perimeter in B_r, and from (4.3.7) it follows that

$$\mathcal{H}^{d-1}(\partial^* G_n^\varrho \cap \partial B_\varrho) \leq \mathcal{H}^{d-1}\left(F_n^1 \cap \partial B_\varrho\right) \tag{4.3.8}$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0,r)$. Furthermore, the relative isoperimetric inequality implies that

$$c_n \int_0^r \mathcal{H}^{d-1}\left(F_n^1 \cap \partial B_\varrho\right) \mathrm{d}\varrho = c_n |F_n \cap B_r| \leq c_n \left(\mathcal{H}^{d-1}\left(\partial^* F_n \cap B_r\right)\right)^{1^*} ,$$

and so, by (4.3.6) and (4.3.8), up to a subsequence (not relabeled), we have that

$$\lim_{n\to\infty} c_n \mathcal{H}^{d-1}\left(\partial^* G_n^\varrho \cap \partial B_\varrho\right) = 0$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0,r)$.

If $F = B_1$ then we may proceed in a similar way with respect to the previous case and prove (4.3.5). In fact, since

$$\partial^* H_n^\varrho \cap B_r \subset \left(F_n^0 \cap \partial B_\varrho\right) \cup \left(\partial^* F_n \cap \left(B_r \setminus \overline{B}_\varrho\right)\right) \cup \left(\partial^* F_n \cap \partial B_\varrho\right) ,$$

F_n^ϱ is a set of finite perimeter in B_r, and by (4.3.7) we have that

$$\mathcal{H}^{d-1}\left(\partial^* H_n^\varrho \cap \partial B_\varrho\right) \leq \mathcal{H}^{d-1}\left(F_n^0 \cap \partial B_\varrho\right) \tag{4.3.9}$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0,r)$. Applying the relative isoperimetric inequality (2.6.4) we obtain

$$c_n \int_0^r \mathcal{H}^{d-1}\left(F_n^0 \cap \partial B_\varrho\right) \mathrm{d}\varrho = c_n |B_r \setminus F_n| \leq c_n \left(\mathcal{H}^{d-1}\left(\partial^* F_n \cap B_r\right)\right)^{1^*}$$

which, together with (4.3.6) and (4.3.9), implies (4.3.5). □

We now define the notion of local minimizer of the generalized Dirichlet functional

$$v \mapsto \mathcal{D}(v, \Omega) := \int_\Omega |\nabla v|^p \,\mathrm{d}x . \tag{4.3.10}$$

Definition 4.3.3. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *local minimizer* of the generalized Dirichlet functional in Ω if for every $U \subset\subset \Omega$ and every $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\{u \neq v\} \subset\subset U$ we have

$$\int_U |\nabla u|^p \,\mathrm{d}x \leq \int_U |\nabla v|^p \,\mathrm{d}x .$$

The following result that applies to local minimizers of the generalized Dirichlet functional is established in [57] (see also [9, Theorem 7.12]).

Theorem 4.3.4. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the generalized Dirichlet functional $\mathcal{D}(\cdot, \Omega)$. Then u is locally Lipschitz in Ω and there exists $C_0(p,d) > 0$ such that*

$$\sup_{y \in B_{\varrho/2}(x)} |\nabla u|^p \leq C_0 \fint_{B_\varrho(x)} |\nabla u|^p \,\mathrm{d}y$$

for each ball $B_\varrho(x) \subset \Omega$.

Using Propositions 2.10.7 and 4.3.2 we establish the following result that describes the limit behavior of a sequence $\{(F_n, v_n)\}$ when the deviations from minimality (see Definition 4.1.6), $\mathcal{H}^{d-1}(\partial^* F_n)$, and $\mathcal{H}^{d-1}(S_{v_n} \cap F_n^0)$ tends to zero.

Theorem 4.3.5. *Let $r > 0$ and let $\{(F_n, v_n)\}$ be a sequence of pairs such that F_n are sets of finite perimeter in B_r, $v_n \in L^1(B_r)$, and $w_n := v_n \chi_{F_n^0} \in SBV(B_r)$. For each $n \in \mathbb{N}$ consider a median m_n of w_n in B_r and a constant $c_n > 0$. If*

(a) $$\lim_{n\to\infty} \mathcal{H}^{d-1}(\partial^* F_n \cap B_r) = 0\,,$$

(b) $$\lim_{n\to\infty} \mathcal{H}^{d-1}(S_{v_n} \cap F_n^0 \cap B_r) = 0\,,$$

(c) $$\sup_{n\in\mathbb{N}} \overline{\mathcal{G}}(v_n, F_n, c_n, B_r) < \infty\,,$$

(d) $$\lim_{n\to\infty} \mathrm{Dev}(v_n, F_n, c_n, B_r) = 0\,,$$

and

(e) $$\lim_{n\to\infty} (w_n - m_n) = w \in W^{1,p}(B_r) \quad \mathcal{L}^d\text{-a.e. in } B_r\,,$$

then w is a local minimizer of $\mathcal{D}(\cdot, B_r)$, and

$$\lim_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) = \int_{B_\varrho} |\nabla w|^p \,\mathrm{d}x \tag{4.3.11}$$

for every $\varrho \in (0, r)$.

Proof. **Step 1:** Since (c) implies that the increasing functions $\varrho \mapsto \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho)$ are equibounded, by [52, Lemma 2.37] there exists an increasing function $\alpha : (0, r) \to \mathbb{R}$ such that, up to a subsequence (not relabeled),

$$\alpha(\varrho) := \lim_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) \tag{4.3.12}$$

for all $\varrho \in (0, r)$ and $c_\infty := \lim_{n\to\infty} c_n \in [0, \infty]$. In this step we prove that

$$\lim_{n\to\infty} \overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) = \alpha(\varrho) \tag{4.3.13}$$

$$\lim_{n\to\infty} \mathrm{Dev}(F_n, \overline{w}_n, c_n, B_\varrho) = 0 \tag{4.3.14}$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0,r)$, where the notation $\overline{w}_n$ was introduced in (2.10.6).

We begin by observing that

$$\lim_{n\to\infty} \mathcal{H}^{d-1}(S_{w_n} \cap B_r) = 0 \tag{4.3.15}$$

by (a) and (b), and that

$$\sup_{n\in\mathbb{N}} \int_{B_r} |\nabla w_n|^p \, \mathrm{d}x < \infty$$

by (c). Thus, from Proposition 2.10.7 it follows that

$$(\overline{w}_n - m_n) \to w \quad \text{in } L^p(B_r), \tag{4.3.16}$$

and

$$\int_{B_\varrho} |\nabla w|^p \, \mathrm{d}x \le \liminf_{n\to\infty} \int_{B_\varrho} |\nabla \overline{w}_n|^p \, \mathrm{d}x \tag{4.3.17}$$

for every $\varrho \in (0,r]$. Moreover, if $c_\infty < +\infty$ then it follows immediately from (4.3.15) that

$$\lim_{n\to\infty} c_n \left(2\gamma_5 \mathcal{H}^{d-1}(s_{w_n} \cap B_r)\right)^{1^*} = 0\,,$$

while if $c_\infty = +\infty$ then the same holds by (c), since

$$c_n \left(\mathcal{H}^{d-1}(s_{w_n} \cap B_r)\right)^{1^*} = \left(c_n \mathcal{H}^{d-1}(s_{w_n} \cap B_r)\right)^{1^*} c_n^{-\frac{1}{d-1}} \le \left(\frac{1}{M_1}\overline{\mathcal{G}}(v_n, F_n, c_n, B_r)\right)^{1^*} c_n^{-\frac{1}{d-1}}.$$

Therefore, in view of the fact that

$$c_n \int_0^r \mathcal{H}^{d-1}\left(\{\tilde{w}_n \neq \tilde{\overline{w}}_n\} \cap \partial B_\varrho\right) \mathrm{d}\varrho = c_n |\{w_n \neq \overline{w}_n\} \cap B_r\}| \le 2c_n \left(2\gamma_5 \mathcal{H}^{d-1}(s_{w_n} \cap B_r)\right)^{1^*}\,,$$

where we used (2.10.8), we conclude that, up to another subsequence (not relabeled), we have

$$\lim_{n\to\infty} c_n \mathcal{H}^{d-1}\left(\{\tilde{w}_n \neq \tilde{\overline{w}}_n\} \cap \partial B_\varrho\right) = 0 \tag{4.3.18}$$

for $\mathcal{L}^1$-a.e. $\varrho \in (0,r)$.

By (2.10.6) and by comparing the energies of (F_n, v_n) and $(F_n, v_n \chi_{B_r \setminus B_\varrho} + \overline{w}_n \chi_{B_\varrho}) \in X(B_r)$, we obtain that

$$\begin{aligned}\overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) &\le \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho)\\ &\le \overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) + 2M_2 c_n \mathcal{H}^{d-1}(\{\tilde{w}_n \neq \tilde{\overline{w}}_n\} \cap F_n^0 \cap \partial B_\varrho) + \theta_n\end{aligned}$$

where $\theta_n := \mathrm{Dev}(F_n, v_n, c_n, B_r)$. Therefore, by (d), (4.3.12) and (4.3.18) we obtain (4.3.13).

In order to prove (4.3.14), let $(G, z) \in X(B_\varrho)$ be such that $F_n \triangle G \subset\subset B_\varrho$ and $\{\overline{w}_n \neq z\} \subset\subset B_\varrho$. Consider $z' := z\chi_{B_\varrho} + v_n \chi_{B_r \setminus B_\varrho}$ and observe that by the definition of θ_n (see also Definition 4.1.6) we have

$$\begin{aligned}\overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) &\leq \overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) + \overline{\mathcal{G}}(G, z', c_n, B_r) - \overline{\mathcal{G}}(F_n, v_n, c_n, B_r) + \theta_n \\ &\leq \overline{\mathcal{G}}(G, z, c_n, B_\varrho) + [\overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) - \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) \\ &\qquad + 2M_2 c_n \mathcal{H}^{d-1}(\{\tilde{w}_n \neq \tilde{\overline{w}}_n\} \cap F_n^0 \cap \partial B_\varrho) + \theta_n],\end{aligned}$$

and so, by Definition 4.1.6 we deduce that

$$\begin{aligned}\mathrm{Dev}(F_n, \overline{w}_n, c_n, B_\varrho) &\leq \overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_\varrho) - \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) \\ &\qquad + 2M_2 c_n \mathcal{H}^{d-1}(\{\tilde{w}_n \neq \tilde{\overline{w}}_n\} \cap F_n^0 \cap \partial B_\varrho) + \theta_n.\end{aligned}$$

We have that (4.3.14) follows now by (d), (4.3.12), (4.3.18), and (4.3.13).

Step 2: In this step we prove that w is a local minimum of $\mathcal{D}(\cdot, B_r)$ that satisfies (4.3.11). To this end, we consider $v \in W^{1,p}_{\mathrm{loc}}(B_r)$ such that $\{w \neq v\} \subset\subset B_r$, and choose $\varrho < \varrho\,' < r$ such that (4.3.13) and (4.3.14) hold, $\{w \neq v\} \subset\subset B_\varrho$, and α is continuous at ϱ'. Moreover, let $\varphi \in C_c^\infty(B_{\varrho'})$ be such that $\varphi \equiv 1$ in $\overline{B}_\varrho$, $0 \leq \varphi \leq 1$, and $|\nabla\varphi| \leq 2/(\varrho' - \varrho)$.

By (a) and (c) we apply Proposition 4.3.2 to obtain that

$$\lim_{n\to\infty} c_n \mathcal{H}^{d-1}\left(\partial^* \tilde{F}_n^\varrho \cap \partial B_\varrho\right) = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } \varrho \in (0, r) \tag{4.3.19}$$

where $\tilde{F}_n^\varrho$ is either $F_n \cap (B_r \setminus B_\varrho)$ or $F_n \cup B_\varrho$. By comparing the energies of $(F_n, \overline{w}_n)$ and $(\tilde{F}_n^\varrho, w_n') \in X(B_r)$, where

$$w_n' := \varphi(v + m_n) + (1 - \varphi)\overline{w}_n,$$

we deduce that

$$\begin{aligned}\overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_{\varrho'}) &\leq \overline{\mathcal{G}}(\tilde{F}_n^\varrho, w_n', c_n, B_{\varrho'}) + \mathrm{Dev}(F_n, \overline{w}_n, c_n, B_{\varrho'}) \\ &\leq \int_{B_\varrho \setminus \tilde{F}_n^\varrho} |\nabla v|^p \,\mathrm{d}x + \overline{\mathcal{G}}(F_n, w_n', c_n, B_{\varrho'} \setminus \overline{B}_\varrho) + c_n \int_{\partial B_\varrho \cap \partial^* \tilde{F}_n^\varrho} \psi(\nu_{\tilde{F}_n^\varrho}) \,\mathrm{d}\mathcal{H}^{d-1} \\ &\qquad + \mathrm{Dev}(F_n, \overline{w}_n, c_n, B_{\varrho'}) \\ &\leq \int_{B_\varrho} |\nabla v|^p \,\mathrm{d}x + \overline{\mathcal{G}}(F_n, w_n', c_n, B_{\varrho'} \setminus \overline{B}_\varrho) + M_2 c_n \mathcal{H}^{d-1}\left(\partial^* \tilde{F}_n^\varrho \cap \partial B_\varrho\right) \\ &\qquad + \mathrm{Dev}(F_n, \overline{w}_n, c_n, B_{\varrho'}),\end{aligned} \tag{4.3.20}$$

where in the second inequality we used the facts that $B_\varrho \cap \partial^* \tilde{F}_n^\varrho = \varnothing$ and that $S_{w'_n} \cap B_r \subset S_{\overline{w}_n} \cap (B_r \setminus \overline{B}_\varrho)$. Then, we observe that there exists a constant $c_p > 0$, that depends only on p, such that

$$\begin{aligned} \overline{\mathcal{G}}(F_n, w'_n, c_n, B_{\varrho'} \setminus \overline{B}_\varrho) \le c_p \Bigg[& \overline{\mathcal{G}}(F_n, \overline{w}_n, c_n, B_{\varrho'} \setminus B_\varrho) + \int_{B_{\varrho'} \setminus B_\varrho} |\nabla v|^p \,\mathrm{d}x \\ & + \frac{1}{(\varrho' - \varrho)^p} \int_{B_{\varrho'} \setminus B_\varrho} |\overline{w}_n - m_n - v|^p \,\mathrm{d}x \Bigg] . \end{aligned} \tag{4.3.21}$$

Therefore, by (4.3.20), (4.3.21), and passing to the limit as $n \to \infty$, we have that

$$\alpha(\varrho') \le \int_{B_\varrho} |\nabla v|^p \,\mathrm{d}x + c_p \int_{B_{\varrho'} \setminus B_\varrho} |\nabla v|^p \,\mathrm{d}x + c_p \left[\alpha(\varrho') - \alpha(\varrho)\right] + \frac{1}{(\varrho' - \varrho)^p} \int_{B_{\varrho'} \setminus B_\varrho} |w - v|^p \,\mathrm{d}x$$

by (4.3.13), (4.3.14), (4.3.16), and (4.3.19). Since α is continuous at ϱ' and $w = v$ in $B_{\varrho'} \setminus B_\varrho$ if ϱ is close enough to ϱ', if we let $\varrho \nearrow \varrho'$ in the previous inequality, we obtain that

$$\alpha(\varrho') \le \int_{B_{\varrho'}} |\nabla v|^p \,\mathrm{d}x \,. \tag{4.3.22}$$

From (4.3.22) with $v = w$, and (4.3.17) it follows that

$$\alpha(\varrho') = \int_{B_{\varrho'}} |\nabla w|^p \,\mathrm{d}x \,,$$

and so, using again (4.3.22), we deduce that w is a local minimum of $\mathcal{D}(\cdot, B_r)$ that satisfies (4.3.11) for every $\varrho \in (0, r)$ at which α is continuous.

Finally, fix any point $\varrho \in (0, r)$ and consider $\varrho' \in (\varrho, r)$ at which α is continuous. We note that

$$\limsup_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) \le \lim_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_{\varrho'}) = \int_{B_{\varrho'}} |\nabla w|^p \,\mathrm{d}x \,.$$

Letting $\varrho' \searrow \varrho$, we obtain

$$\limsup_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) \le \int_{B_\varrho} |\nabla w|^p \,\mathrm{d}x$$

which, together with (4.3.17), concludes the proof. $\square$

We now establish an estimate of the decay of $\overline{\mathcal{G}}$ in small balls that follows from Theorems 4.3.5 and 4.3.4.

Lemma 4.3.6 (Decay). *There exists a constant $C_1 = C_1(d,p) > 0$ with the property that for every $0 < \tau < 1$ we can find $\varepsilon(\tau), \theta(\tau) > 0$ such that if $(E,u) \in X(\Omega)$, $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$, and $B_\varrho(x) \subset\subset \Omega$ satisfy*

$$\mathcal{H}^{d-1}(\partial^* E \cap B_\varrho(x)) + \mathcal{H}^{d-1}(S_u \cap E^0 \cap B_\varrho(x)) \leq \varepsilon \varrho^{d-1}$$

$$\text{and} \quad \mathrm{Dev}(E,u,B_\varrho(x)) \leq \theta \overline{\mathcal{G}}(E,u,B_\varrho(x)),$$

then

$$\overline{\mathcal{G}}(E,u,B_{\tau\varrho}(x)) \leq C_1 \tau^d \overline{\mathcal{G}}(E,u,B_\varrho(x))\,.$$

Proof. We fix $0 < \tau < \frac{1}{2}$ and prove the decay property for $C_1 > C_0$ where C_0 is the constant given by Theorem 4.3.4 in the Appendix. Proceeding by contradiction we choose sequences $\{\varepsilon_n\}$ and $\{\theta_n\}$ such that $\varepsilon_n\,, \theta_n \to 0$, and for every $n \in \mathbb{N}$ we consider a pair $(E_n, u_n) \in X(\Omega)$, with $u_n \chi_{E_n^0} \in SBV_{\mathrm{loc}}(\Omega)$, and a ball $B_{\varrho_n}(x_n) \subset\subset \Omega$, that satisfy

$$\mathcal{H}^{d-1}(\partial_* E_n \cap B_{\varrho_n}(x_n)) + \mathcal{H}^{d-1}(S_{u_n} \cap E_n^0 \cap B_{\varrho_n}(x_n)) = \varepsilon_n \varrho_n^{d-1}\,,$$

$$\mathrm{Dev}(E_n,u_n,B_{\varrho_n}(x_n)) = \theta_n \overline{\mathcal{G}}(E_n,u_n,B_{\varrho_n}(x_n)\,,$$

and

$$\overline{\mathcal{G}}(E_n,u_n,B_{\tau\varrho_n}(x_n)) > C_1 \tau^d \overline{\mathcal{G}}(E_n,u_n,B_{\varrho_n}(x_n)\,.$$

Let $c_n := \varrho_n^{d-1}[\overline{\mathcal{G}}(E_n,u_n,B_{\varrho_n}(x_n))]^{-1}$ and define (F_n, v_n) by

$$F_n := E_{x_n,\varrho_n} \quad \text{and} \quad v_n := c_n^{\frac{1}{p}} u_{x_n,\varrho_n}\,,$$

where $(E_{x_n,\varrho_n}, u_{x_n,\varrho_n})$ is the rescaled pair with respect to x_n and ϱ_n (see (4.1.5)). By Remark 4.1.9 we have that $(F_n,v_n) \in X(\Omega_{x_n,\varrho_n})$, $w_n := v_n \chi_{F_n^0} \in SBV_{\mathrm{loc}}(\Omega_{x_n,\varrho_n})$,

$$\begin{aligned}
&\mathcal{H}^{d-1}(\partial^* F_n \cap B_1) + \mathcal{H}^{d-1}(S_{v_n} \cap F_n^0 \cap B_1) = \varepsilon_n\,,\\
&\mathrm{Dev}(F_n,v_n,c_n,B_1) = \theta_n\,,\\
&\overline{\mathcal{G}}(F_n,v_n,c_n,B_1) = 1\,,
\end{aligned}$$

and

$$\overline{\mathcal{G}}(F_n,v_n,c_n,B_\tau) > C_1 \tau^d\,. \tag{4.3.23}$$

Thus, it follows that $w_n \in SBV(B_1)$,

$$\int_{B_1} |\nabla w_n|^p \,\mathrm{d}y \leq 1 \quad \text{and} \quad \mathcal{H}^{d-1}(S_{w_n} \cap B_1) \leq \varepsilon_n \to 0\,,$$

and so, by Proposition 2.10.7 there exists a function $w \in W^{1,p}(B_1)$ such that, up to a subsequence (not relabeled), we have

$$\begin{aligned} &(\overline{w}_n - m_n) \to w \quad \text{in } L^p(B_1), \\ &(w_n - m_n) \to w \quad \mathcal{L}^d\text{-a.e. in } B_1 \text{ as } n \to \infty, \\ \text{and} \quad &\int_{B_1} |\nabla w|^p \, \mathrm{d}y \le \liminf_{n\to\infty} \int_{B_1} |\nabla \overline{w}_n|^p \, \mathrm{d}y \le 1 \,, \end{aligned} \tag{4.3.24}$$

where $\overline{w}_n$ was defined in (2.10.6).

Also, $\{F_n\}$ is a sequence of sets of finite perimeter in B_1 such that $\mathcal{H}^{d-1}(\partial^* F_n \cap B_1) \to 0$ as $n \to \infty$, and so the relative isoperimetric inequality (2.6.4) implies that $\{F_n\}$ converges in measure in B_1 to a set F that is either $F = \varnothing$ or $F = B_1$. Note also that $w = 0$ for $\mathcal{L}^d$-a.e in F.

By Theorem 4.3.5, we conclude that w is a local minimizer of

$$v \mapsto \int_{B_1} |\nabla v|^p \, \mathrm{d}y$$

in $W^{1,p}(B_1)$, and that

$$\lim_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_\varrho) = \int_{B_\varrho} |\nabla w|^p \, \mathrm{d}y \tag{4.3.25}$$

for every $\varrho \in (0,1)$. Hence, by Theorem 4.3.4 we deduce that w is locally Lipschitz in B_1, and that there exists $C_0(p,d) > 0$ such that

$$\sup_{y \in B_{1/2}} |\nabla w(y)|^p \le C_0 \fint_{B_1} |\nabla w|^p \, \mathrm{d}y \,. \tag{4.3.26}$$

Therefore, by (4.3.24), (4.3.25), and (4.3.26)

$$\begin{aligned} \lim_{n\to\infty} \overline{\mathcal{G}}(F_n, v_n, c_n, B_\tau) = \int_{B_\tau} |\nabla w|^p \, \mathrm{d}y &\le \omega_d \tau^d \sup_{y\in B_{1/2}} |\nabla w|^p \\ &\le C_0 \omega_d \tau^d \fint_{B_1} |\nabla w|^p \, \mathrm{d}y \le C_0 \tau^d. \end{aligned}$$

This contradicts (4.3.23).

In the case $\frac{1}{2} < \tau < 1$, the decay property follows immediately for $C_1 \ge 2^d$. □

The following remark will be used to prove the density lower bound.

Remark 4.3.7. Let $(E,u) \in X(\Omega)$ and $x \in \Omega$. If $\overline{\mathcal{G}}(E,u,B_r(x)) = o(r^{d-1})$ as $r \searrow 0$, then $x \notin \partial^* E$. Indeed, for every $y \in \partial^* E$ we have that

$$\lim_{\varrho \searrow 0} \frac{\mathcal{H}^{d-1}\left(\partial^* E \cap B_\varrho(y)\right)}{\omega_{d-1}\varrho^{d-1}} = 1$$

by Theorem 2.5.7.

In view of the Decay Lemma 4.3.6, we are now able to prove the density lower bound for quasi-minimizers $(E,u) \in \mathcal{M}_\omega(\Omega)$ with $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$.

Theorem 4.3.8 (Density Lower Bound)**.** *There exist two positive constants θ_0 and r_0 depending only on d, p, M_1, and M_2, such that for every quasi-minimizer $(E,u) \in \mathcal{M}_\omega(\Omega)$ with $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$,*

$$\mathcal{H}^{d-1}(\Gamma_{E,u} \cap B_\varrho(x)) > \theta_0 \varrho^{d-1} \tag{4.3.27}$$

for every ball $B_\varrho(x) \subset\subset \Omega$ with center $x \in \overline{\Gamma}_{E,u}$ and radius $\varrho \leq \varrho_\omega := \min\{\varrho_0, \frac{r_0}{\omega}\}$, where

$$\Gamma_{E,u} := \left[\partial^* E \cup \left(S_u \cap E^0\right)\right] \cap \Omega \tag{4.3.28}$$

and ϱ_0 is given by Definition 4.1.7.

Proof. Fix τ, $\sigma \in (0,1)$ such that $C_1\tau^d \leq \tau^{d-\frac{1}{2}}$ and $C_1\sigma(d\omega_d M_2 + 1) < \varepsilon(\tau)M_1$, respectively, and define

$$\theta_0 := \varepsilon(\sigma) \quad \text{and} \quad r_0 := \min\{1, M_1\varepsilon(\tau)\tau^d\theta(\tau), \varepsilon(\tau)\sigma^{d-1}\theta(\sigma)M_1\}$$

where C_1, $\varepsilon(\cdot)$, $\theta(\cdot)$ are given by the Decay Lemma 4.3.6. Consider a quasi-minimizer $(E,u) \in \mathcal{M}_\omega(\Omega)$ such that $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$.

Step 1: In this step we prove that if

$$\mathcal{H}^{d-1}(\Gamma_{E,u} \cap B_\varrho(x)) \leq \theta_0 \varrho^{d-1} \tag{4.3.29}$$

for some ball $B_\varrho(x) \subset\subset \Omega$ with $\varrho \leq \varrho_\omega := \min\{\varrho_0, \frac{r_0}{\omega}\}$, then

$$\overline{\mathcal{G}}(E,u,B_r(x)) = o(r^{d-1}) \quad \text{as } r \searrow 0. \tag{4.3.30}$$

We observe that (4.3.30) follows immediately from the following claim:

$$\overline{\mathcal{G}}(E,u,B_{\sigma\tau^n\varrho}(x)) \leq M_1\varepsilon(\tau)\tau^{\frac{n}{2}}\left(\sigma\tau^n\varrho\right)^{d-1} \quad \text{for each } n \in \mathbb{N}. \tag{4.3.31}$$

In order to prove (4.3.31), we proceed by induction on n. We show that (4.3.31) holds for $n=0$. If

$$\mathrm{Dev}(E,u,B_\varrho(x)) \le \theta(\sigma)\overline{\mathcal{G}}(E,u,B_\varrho(x))\,, \tag{4.3.32}$$

then we may apply the Decay Lemma 4.3.6 and so, also by the energy upper bound established in Lemma 4.3.1 and the choice of σ, we obtain that

$$\begin{aligned}\overline{\mathcal{G}}(E,u,B_{\sigma\varrho}(x)) &\le C_1\sigma^d\overline{\mathcal{G}}(E,u,B_\varrho(x)) \le C_1\sigma^d\left(M_2 d\omega_d\varrho^{d-1}+\omega\varrho^d\right)\\ &\le (\sigma\varrho)^{d-1}C_1\sigma\left(M_2 d\omega_d+1\right) \le M_1\varepsilon(\tau)\,(\sigma\varrho)^{d-1}\,.\end{aligned}$$

If (4.3.32) fails to hold, then from the quasi-minimality of (E,u) it follows that

$$\begin{aligned}\overline{\mathcal{G}}(E,u,B_{\sigma\varrho}(x)) \le \overline{\mathcal{G}}(E,u,B_\varrho(x)) &\le \frac{1}{\theta(\sigma)}\mathrm{Dev}(E,u,B_\varrho(x))\\ &\le \frac{\omega\varrho^d}{\theta(\sigma)} \le M_1\varepsilon(\tau)\,(\sigma\varrho)^{d-1}\,,\end{aligned}$$

where we used that $\varrho\le\varrho_\omega$.

Now we prove that if (4.3.31) is true for a given $n\ge 0$, then it holds also for $n+1$. As before, we distinguish two cases. If we have

$$\mathrm{Dev}(E,u,B_{\sigma\tau^n\varrho}(x)) \le \theta(\tau)\overline{\mathcal{G}}(E,u,B_{\sigma\tau^n\varrho}(x))\,, \tag{4.3.33}$$

then again by the Decay Lemma 4.3.6, we obtain

$$\overline{\mathcal{G}}(E,u,B_{\sigma\tau^{n+1}\varrho}(x)) \le C_1\tau^d\overline{\mathcal{G}}(E,u,B_{\sigma\tau^n\varrho}(x)) \le C_1\tau^d M_1\varepsilon(\tau)\tau^{\frac{n}{2}}\left(\sigma\tau^n\varrho\right)^{d-1}$$

and this implies (4.3.31) for $n+1$ due to the choice of τ. If (4.3.33) does not hold, then we obtain

$$\begin{aligned}\overline{\mathcal{G}}(E,u,B_{\sigma\tau^{n+1}\varrho}(x)) \le \overline{\mathcal{G}}(E,u,B_{\sigma\tau^n\varrho}(x)) &\le \frac{1}{\theta(\sigma)}\mathrm{Dev}(E,u,B_{\sigma\tau^n\varrho}(x))\\ &\le \frac{\omega\left(\sigma\tau^n\varrho\right)^d}{\theta(\tau)} \le M_1\varepsilon(\tau)\tau^{\frac{n+1}{2}}\left(\sigma\tau^{n+1}\varrho\right)^{d-1}\,,\end{aligned}$$

where, as before, we used the fact that $\varrho\le\varrho_\omega$. Therefore, claim (4.3.31) holds.

Step 2: We begin by observing that if $x\in\partial^*E$, then (4.3.27) holds for every ball $B_\varrho(x)\subset\subset\Omega$ with $\varrho\le\varrho_\omega$. In fact, otherwise we find a ball $B_\varrho(x)\subset\subset\Omega$ with $\varrho\le\varrho_\omega$ and such that

$$\mathcal{H}^{d-1}(\Gamma_{E,u}\cap B_\varrho(x)) \le \theta_0\varrho^{d-1}\,.$$

and by the first step and Remark 4.3.7 we have a contradiction. Furthermore, we observe that, by a density argument, (4.3.27) holds also for all balls $B_\varrho(x) \subset\subset \Omega$ with radius $\varrho \leq \varrho_\omega$ and centered at every $x \in \overline{\partial^* E}$.

Let now $w := u\chi_{E^0}$ and

$$I := \left\{ x \in \Omega : \limsup_{\varrho \searrow 0} \fint_{B_\varrho(x)} |w(y)|^{1^*} \, \mathrm{d}y = \infty \right\}.$$

We claim that if $x \in \overline{\Gamma_{E,u} \setminus I}$, then (4.3.27) holds for all balls $B_\varrho(x) \subset\subset \Omega$ with $\varrho \leq \varrho_\omega$. To prove this claim, let $x \in \Gamma_{E,u}$ and consider a ball $B_\varrho(x) \subset\subset \Omega$ with $\varrho \leq \varrho_\omega$ and such that

$$\mathcal{H}^{d-1}(\Gamma_{E,u} \cap B_\varrho(x)) \leq \theta_0 \varrho^{d-1}.$$

As before, from the first step and Remark 4.3.7 it follows that $x \in S_u \cap E^0$. Furthermore, again by the first step and using Theorem 2.10.8 applied to $w := u\chi_{E^0}$ with $q = 1^*$, we obtain that $x \in I$. Therefore, (4.3.27) holds for balls centered at any $x \in \Gamma_{E,u} \setminus I$ and by a density argument the claim follows.

To conclude the proof, we need to establish that

$$\overline{\Gamma_{E,u} \setminus I} \cup \overline{\partial^* E} = \overline{\Gamma_{E,u}}. \tag{4.3.34}$$

Let $x \notin \overline{\Gamma_{E,u} \setminus I} \cup \overline{\partial^* E}$. Since Lemma 2.9.1 implies that I is $\mathcal{H}^{d-1}$-negligible, we may find a bounded neighborhood U of x such that $\mathcal{H}^{d-1}(U \cap \Gamma_{E,u}) = 0$. Furthermore, from Remark 4.1.3 it follows that $w \in SBV^p(U)$ and since $\mathcal{H}^{d-1}(U \cap S_w) = 0$, we have that $w \in W^{1,p}(U)$. Thus, we may apply the Poincaré inequality for Sobolev functions (see (2.6.6)) and, by the energy upper bound, we obtain

$$\int_{B_\varrho(x)} \left| w(y) - \fint_{B_\varrho(x)} w(z) \, \mathrm{d}z \right|^p \mathrm{d}y \leq \gamma_6^p \varrho^p \int_{B_\varrho(x)} |\nabla w(y)|^p \, \mathrm{d}y \leq K \varrho^{p+d-1}.$$

Therefore, by Theorem 2.6.8 we have that (a representative of) w is Hölder continuous in U, and so we deduce that $x \notin \overline{S}_w$. Thus, (4.3.34) follows and this concludes the proof. □

A first consequence of the density lower bound is that for every quasi-minimizer (E, u) in Ω with $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$ the set $\Gamma_{E,u}$ is essentially closed, and so, by Corollary 4.2.2 we have the same property also for every δ-local minimizer of $\overline{\mathcal{G}}$.

Corollary 4.3.9. *If $(E, u) \in \mathcal{M}_\omega(\Omega)$ with $u\chi_{E^0} \in SBV_{\mathrm{loc}}(\Omega)$, then*

$$\mathcal{H}^{d-1}\left(\left(\overline{\Gamma}_{E,u} \cap \Omega\right) \setminus \Gamma_{E,u}\right) = 0, \tag{4.3.35}$$

where $\Gamma_{E,u}$ is the set defined in (4.3.28).

Proof. For every $x \in \overline{\Gamma}_{E,u} \cap \Omega$ by Theorem 4.3.8 we have that

$$\liminf_{\varrho \searrow 0} \frac{\mathcal{H}^{d-1}\left(\Gamma_{E,u} \cap B_\varrho(x)\right)}{\omega_{d-1}\varrho^{d-1}} \geq \frac{\theta_0}{\omega_{d-1}}\,,$$

and so by Proposition 2.5.10 we obtain

$$\mathcal{H}^{d-1}_{\Gamma_{E,u}} \geq \frac{\theta_0}{\omega_{d-1}} \mathcal{H}^{d-1}_{\overline{\Gamma}_{E,u}\cap\Omega}\,,$$

since $\Gamma_{E,u}$ is an $\mathcal{H}^{d-1}$-measurable set with $\mathcal{H}^{d-1}(\Gamma_{E,u}) < \infty$. Therefore,

$$\mathcal{H}^{d-1}_{\Gamma_{E,u}}\left(\left(\overline{\Gamma}_{E,u} \cap \Omega\right) \setminus \Gamma_{E,u}\right) \geq \frac{\theta_0}{\omega_{d-1}} \mathcal{H}^{d-1}\left(\left(\overline{\Gamma}_{E,u} \cap \Omega\right) \setminus \Gamma_{E,u}\right)$$

and this concludes the proof. $\square$

Theorem 4.3.10. *Let $1 < p < \infty$, $\delta > 0$, and Ω' be connected. Then, for every δ-local minimizer $(E,u) \in X_{u_0,\lambda}(\Omega,\Omega')$ of $\overline{\mathcal{G}}$, we have that*

$$\mathcal{H}^{d-1}(\Omega' \cap \overline{\Gamma}_{E,u} \setminus \Gamma_{E,u}) = 0\,, \tag{4.3.36}$$

and a representative of $u\chi_{E^0}$ is in $C^{1,\gamma}_{\mathrm{loc}}(\Omega' \setminus \overline{\Gamma}_{E,u})$ for a $\gamma \in (0,1]$ that depends only on p and the dimension d.

Proof. Since by Corollary 4.2.2, (E,u) is a quasi-minimizer of $\overline{\mathcal{G}}$ in Ω' and $u\chi_{E^0} \in SBV(\Omega')$, (4.3.36) follows directly from (4.3.35).

Let $\overline{B_\varrho(x)} \subset \Omega' \setminus \overline{\Gamma}_{E,u}$ and observe that either $|B_\varrho(x) \setminus E| = 0$ or $|B_\varrho(x) \cap E| = 0$ by the relative isoperimetric inequality (2.6.4). If $|B_\varrho(x) \setminus E| = 0$ we have that $u\chi_{E^0} \equiv 0$ for $\mathcal{L}^d$-a.e. in $B_\varrho(x)$. Thus, without lost of generality we may assume that $|B_\varrho(x) \cap E| = 0$. Due to the fact that (E,u) is a δ-local minimizer of $\overline{\mathcal{G}}$ we have that

$$\overline{\mathcal{G}}(E,u,\Omega') \leq \overline{\mathcal{G}}(E,u+\varphi,\Omega')$$

for every $\varphi \in C^\infty_c(B_\varrho(x))$, and so,

$$\int_{B_\varrho(x)} |\nabla u|^p \,\mathrm{d}x \leq \int_{B_\varrho(x)} |\nabla u + \nabla\varphi|^p \,\mathrm{d}x\,,$$

since $S_{u+\varphi} = S_u$ and $\operatorname{supp}\varphi \subset\subset B_\varrho(x)$. Hence, $u \in W^{1,p}(B_\varrho(x))$ minimizes the generalized Dirichlet functional among the functions $v \in u + W^{1,p}_0(B_\varrho(x))$, and

$$\int_{B_\varrho(x)} |\nabla u|^{p-2}\left(\nabla u \cdot \nabla\varphi\right) \mathrm{d}x = 0$$

for all $\varphi \in C_c^\infty(B_\varrho(x))$. Therefore, by [34] for the case $p > 2$ and by [54] for $1 < p \leq 2$ there exists $\gamma \in (0,1]$ that depends only on d and p such that $u \in C^{1,\gamma}_{\mathrm{loc}}(B_\varrho(x))$ and this concludes the proof. $\square$

Chapter 5

Future Research Projects

The results established in Chapter 3 lead to investigate new aspects of the evolution of interfaces, including:

- Study long time existence and global regularity, as well as asymptotic stability of the solution of (1.1.12).
- Extend the analysis performed in the case of evolving graphs to the case of evolving curves in the plane.
- As explained in Section 1.1, the plausibility of the regularization (1.1.8) for rounding corners is clear. However, the literature does not provide a concrete description of how solutions of (1.1.10) relate to (1.1.7). Therefore, the study of the solutions of (1.1.10) in the limit as $\varepsilon \to 0$ plays a key role.

Future research objectives related to the topics of Chapter 4 include the following projects:

- Establish the partial regularity of the boundary of material voids in elastic solids.
- Study the evolution problem in dimension $d \geq 2$.
- Carry out the same program undertaken for material voids in the case of epitaxially strained thin films (see [26]).

List of Symbols

General Notations

Let d and k be positive integers, and let E and F be sets in a topological space.

$\mathbb{N}$, $\mathbb{R}$	set of positive integers and real numbers, respectively.
$\mathbb{N}_0$	set containing the positive integers and zero.
$\overline{\mathbb{R}}$	Extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$.
$\mathbb{R}^d$, $\mathbb{S}^{d-1}$	Euclidean d-dimensional space and its unit sphere.
$\mathcal{L}^d$	Lebesgue outer measure in $\mathbb{R}^d$.
$\mathcal{H}^k$	k-dimensional Hausdorff measure.
$\mathscr{B}(\Omega)$	Family of Borel subsets of the open set $\Omega \subset \mathbb{R}^d$.
$\mathscr{M}(\Omega)$	Family of Lebesgue measurable subsets of the open set $\Omega \subset \mathbb{R}^d$.
$\mathscr{P}(\Omega)$	Collection of sets $E \in \mathscr{M}(\mathbb{R}^d)$ with locally finite perimeter in $\Omega \subset \mathbb{R}^d$.
$B_\varrho(x)$, B_ϱ	Open balls in $\mathbb{R}^d$ with radius $\varrho > 0$, and center $x \in \mathbb{R}^d$ and 0, respectively.
ω_d	$\mathcal{L}^d$-measure of B_1 in $\mathbb{R}^d$.
$a \wedge b$, $a \vee b$	Minimum and maximum of two scalars a and b.
$\subset$	Inclusion, not necessarily strict.
$E \subset\subset F$	Compact inclusion: $\overline{E} \subset F$, $\overline{E}$ compact.
$\triangle$	Symmetric difference.
$\overline{E}$	Topological closure of the set E.
∂E	Topological boundary of the set E.
$\partial^* E$	Reduced boundary of the set $E \in \mathscr{P}(\mathbb{R}^d)$.
$\partial_* E$	Essential boundary of the set $E \in \mathscr{M}(\mathbb{R}^d)$.

Function Spaces

Let $d,\ M \in \mathbb{N}$, let $m \in \mathbb{N}_0$, let $0 < \alpha \leq 1$, let $1 \leq p \leq \infty$, let $\Omega \subset \mathbb{R}^d$ be an open set, and let $I \subset \mathbb{R}$ be a bounded open interval.

$C^m(\Omega)$	Space of real functions that are continuous together with their partial derivatives up to the order m 13
$C^\infty(\Omega)$	Space of real functions that are infinitely differentiable 13
$C^m_c(\Omega)$, $C^\infty_c(\Omega)$	Subspaces of $C^m(\Omega)$ and $C^\infty(\Omega)$, respectively, consisting of all the functions with compact support 13
$C^{m,\alpha}(\Omega)$	Space of real functions continuously differentiable up to the order m, with locally α-Hölder continuous derivatives 13
$L^p(\Omega)$	Lebesgue space of p-Lebesgue integrable functions in Ω for $1 < p < \infty$ and essentially bounded functions for $p = \infty$ 14
$W^{m,p}(\Omega;\mathbb{R}^M)$	Sobolev spaces, $H^m(\Omega;\mathbb{R}^M) := W^{m,2}(\Omega;\mathbb{R}^M)$ 15
$W^{m,p}_{\#}(I)$	Space of all functions in $W^{m,p}_{\mathrm{loc}}(\mathbb{R})$ that are $\lvert I\rvert$-periodic, endowed with the norm of $W^{m,p}(I)$ 15
$BV(\Omega)$	Space of functions of bounded variation 16
$SBV(\Omega)$	Space of special functions of bounded variation 31
$G(S)BV(\Omega)$	Spaces of generalized (special) functions of bounded variation 35

Functions of (Generalized) Bounded Variation

S_u	Approximate discontinuity set of u 26
$\tilde{u}(x)$	Approximate limit of u at x 26

J_u	Approximate jump set of u	27
$\nu_u(x)$	Approximate unit normal to the jump set at the point $x \in J_u$	27
$u^+(x)\,,\ u^-(x)$	Approximate limits of u at a point $x \in J_u$	27
∇u	Approximate differential of u	28
S_u^*	Weak approximate discontinuity set of u	35
$\tilde{u}_*(x)$	Weak approximate limit of u at x	35
J_u^*	Weak approximate jump set of u	36
$\nu_u^*(x)$	Weak approximate unit normal to J_u^* at the point $x \in J_u^*$	36
$\nabla^* u$	Weak approximate differential of u	36

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