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Models of $\mathbb{R}$-supercompactness

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Abstract. Under various appropriate hypotheses it is shown that there is only one determinacy model of the form $L(\mathbb{R}, \mu)$ in which μ is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. In particular, this gives a positive answer to a question asked by W.H. Woodin in 1983. It is also proven that it is relatively consistent that there are different ZF models of the form $L(\mathbb{R}, \mu)$ in which μ witnesses that ω_1 is $\mathbb{R}$-supercompact.

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Contents

CHAPTER 1

Introduction

In this thesis we deal with several set theories, most of which include ZF + DC. Here, ZF is Zermelo-Fraenkel set theory and DC is the Dependent Choice principle. One such theory is ZFC, which is ZF + AC, where AC is the Axiom of Choice. Other examples are

$$\mathrm{ZFC} + \text{ there exists a measurable cardinal,}$$

and

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}\,.$$

We start by reviewing the basic notions.

1.1. Large Cardinals and Inner Model Theory

By definition, an uncountable cardinal κ is *measurable* if and only if there is a non-principal κ-complete and normal ultrafilter over κ. In ZFC, this is equivalent to the existence of a transitive class M and an elementary embedding $j : V \to M$ with critical point κ. The proof of this equivalence uses an ultrapower construction and Łos' Theorem, which in turn uses AC. The existence of a measurable cardinal is an example of a large cardinal axiom. Another example is the existence of a supercompact cardinal. An uncountable cardinal κ is *S-supercompact* if and only if there is a κ-complete ultrafilter on $\mathcal{P}_\kappa(S)$ which is fine and normal. We say κ is *supercompact* if and only if it is S-supercompact for every non-empty set S. In ZFC, this is equivalent to, for every cardinal λ, there exists a transitive class M with ${}^\lambda M \subseteq M$ and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Again we emphasize that AC is used to prove this equivalence. Clearly, in ZFC if κ is supercompact, then κ is measurable and the set of measurable cardinals is unbounded in κ. This can be used to show that the consistency of the theory

$$\mathrm{ZFC} + \text{There is a measurable cardinal}$$

is a theorem of the theory

$$\mathrm{ZFC} + \text{There is a supercompact cardinal.}$$

In other words, the second theory has greater consistency strength than the first. It is an empirical fact that large cardinal axioms line up this way.

One of the central motivations of inner model theory is the construction of canonical inner models of ZFC for different large cardinal axioms. For example, the constructible universe, L, is the minimal transitive proper class model of ZFC. Gödel proved this fact in ZF. For any set S, we construct a transitive proper class $L[S]$ by setting $L_0[S] = \emptyset$ and $L_{\alpha+1}[S]$ to be the family of subsets of $L_\alpha[S]$ that are definable over the structure

$$(L_\alpha[S], \in, S \cap L_\alpha[S])$$

and taking unions at limits. If $\mathcal{U}$ is a normal measure on $\mathcal{P}(\kappa)$ and

$$\overline{\mathcal{U}} = \mathcal{U} \cap L[\mathcal{U}],$$

then

$$\overline{\mathcal{U}} \in L[\overline{\mathcal{U}}] = L[\mathcal{U}]$$

and

$$L[\overline{\mathcal{U}}] \models \mathrm{ZFC} + \overline{\mathcal{U}} \text{ is a normal measure on } \mathcal{P}(\kappa).$$

This is a theorem of Solovay; see [**1**]. Extending this, Kunen (cf. [**4**]) proved that for a given κ models of this form are unique (see Theorem 2). For several decades, inner model theory has strived to extend such results to more powerful large cardinals. In spite of great progress, supercompact cardinals remain beyond our reach so far in the context of ZFC.

1.2. Determinacy

If S is a set, then AD_S says that, for every game of length ω in which two players alternate choosing members of S, one or the other player has a winning strategy. The instances relevant here are AD_ω, more commonly called AD or the Axiom of Determinacy, and $\mathrm{AD}_{\mathbb{R}}$. It is an easy well known result that AC implies AD fails. In other words, $\mathrm{ZFC} + \mathrm{AD}$ is inconsistent. However, the consistency of the theory

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$$

is a theorem of the theory

$$\mathrm{ZFC} + \text{ There is a supercompact cardinal.}$$

In fact, combining results of Martin and Steel ([**5**]) and of Woodin ([**28**]) one can prove in ZFC that if there is a supercompact cardinal, then $L(\mathbb{R})$ is a model of AD. Unfortunately, the reader must distinguish between types of parentheses. Here $L(\mathbb{R})$ is constructed by setting $L_0(\mathbb{R}) = HC$ (we identify $\mathbb{R}$ with HC), $L_{\alpha+1}(\mathbb{R})$ to be the family of sets definable over $(L_\alpha(\mathbb{R}), \in)$, and taking unions at limits. Woodin reduced the hypothesis of this result to a large cardinal axiom strictly between measurability and supercompactness. In fact, he showed that the existence of a certain countable structure called $\mathcal{M}_\omega^\sharp$ suffices. Woodin also showed that the consistency of the theory

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$$

is a theorem of the theory

$$\mathrm{ZFC} + \text{There is a supercompact cardinal.}$$

1.3. Supercompactness measures under ZF+AD

One important and surprising consequence of determinacy is that ω_1 is a large cardinal. Solovay proved that, under $\mathrm{ZF} + \mathrm{AD}$, the club filter on ω_1 is a normal measure and is the unique such measure (see [**3**]). He also proved that under $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$, ω_1 is $\mathbb{R}$-supercompact as witnessed by the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (see [**12**]). We recall that C is a club subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ if there is $\pi : {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that $\sigma \in C$ if and only if σ is closed under π. We define $\mathcal{C}$ as the collection of subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$ that contain a club.

We start to discuss the theory $\mathrm{ZF} + \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact in further detail. For this we must define another kind of model which is built by a combination of two constructions. Suppose μ is a collection of subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. By $L(\mathbb{R}, \mu)$, we mean "throw in $\mathbb{R}$ at the bottom" and "use μ as a predicate". That is, define $L_0(\mathbb{R}, \mu) = \mathrm{HC}$, $L_{\alpha+1}(\mathbb{R}, \mu)$ to be the collection of sets definable over the structure

$$(L_\alpha(\mathbb{R}, \mu), \in, \mu \cap L_\alpha(\mathbb{R}, \mu))$$

and take unions at limits. Notice that μ might not belong to $L(\mathbb{R}, \mu)$ but $\mu \cap L(\mathbb{R}, \mu)$ does and

$$L(\mathbb{R}, \mu) = L(\mathbb{R}, \mu \cap L(\mathbb{R}, \mu))$$

We usually think of $L(\mathbb{R}, \mu)$ as a structure in which the extra symbol $\dot{\mu}$ is interpreted as $\mu \cap L(\mathbb{R}, \mu)$. It is immediate from Solovay's theorem about $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$ and other well-known facts that assuming $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$, $L(\mathbb{R}, \mathcal{C})$ is a model of the theory

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \omega_1 \text{ is } \mathbb{R}\text{-supercompact}$$

where the $\mathbb{R}$-supercompactness is witnessed by $\dot{\mu}^{L(\mathbb{R},\mathcal{C})} = \mathcal{C} \cap L(\mathbb{R}, \mathcal{C})$. It is natural to ask whether under ZFC one can also build canonical models of this theory. Towards this we have the following, which is proved in Chapter 3.

Theorem 1 (Rodríguez, Trang). *Assume* ZFC *and suppose* $\mathcal{M}^\sharp_{\omega^2}$ *exists. Let* $\mathcal{C}$ *be the club filter on* $\mathcal{P}_{\omega_1}(\mathbb{R})$. *Then* $L(\mathbb{R}, \mathcal{C})$ *models*

$$\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \mathcal{C} \cap L(\mathbb{R}, \mathcal{C}) \textit{ is an } \mathbb{R}\textit{-supercompactness measure.}$$

Moreover, if $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ *is such that*

$$L(\mathbb{R}, \mu) \models \mathrm{AD} + \dot{\mu} \textit{ is a } \mathbb{R}\textit{-supercompactness measure,}$$

then $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$.

The meaning of $\mathcal{M}^\sharp_{\omega^2}$ and the sense in which it is iterable will be discussed in Chapter 2. The large cardinal hypothesis that $\mathcal{M}^\sharp_{\omega^2}$ exists is slightly stronger than the consistency strength of the theory $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. In this context, the existence of $\mathcal{M}^\sharp_{\omega^2}$ is nearly optimal because the theories

$$\mathrm{ZF} + \mathrm{AD} + \omega_1 \text{ is } \mathbb{R}\text{-supercompact}$$

and

$$\mathrm{ZFC} + \text{there are } \omega^2\text{-many Woodin cardinals}$$

are equiconsistent (e.g. see [**25**]).

Recall Kunen's result on the uniqueness of minimal models with a measurable cardinal.

Theorem 2 (Kunen). *Assume* ZFC. *Suppose that for* $i < 2$,

$$L[\mathcal{U}_i] \models \mathrm{ZFC} + \mathcal{U}_i \textit{ is a normal measure on } \kappa.$$

Then $L[\mathcal{U}_0] = L[\mathcal{U}_1]$ *and* $\mathcal{U}_0 \cap L[\mathcal{U}_0] = \mathcal{U}_1 \cap L[\mathcal{U}_1]$.

Motivated by Theorem 2, Woodin asked the following analogous question in [**27**].

Question 3 (Woodin, 1983). Assume $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}$. Suppose that for $i < 2$,

$$L(\mathbb{R}, \mu_i) \models \mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + \dot{\mu} \text{ is an } \mathbb{R}\text{-supercompactness measure.}$$

Is is true that $L(\mathbb{R}, \mu_0) = L(\mathbb{R}, \mu_1)$?

In Chapter 4 we give a proof for the following theorem

Theorem 4 (Rodríguez, Trang). *The answer to Question 3 is yes. Moreover, if $V = L(\mathcal{P}(\mathbb{R}))$, then the unique such model is $L(\mathbb{R}, \mathcal{C})$ where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

Building on Theorems 1 and 4 and their proofs, we show in Chapter 5 how can gets rid of the hypothesis that $\mathcal{M}^{\sharp}_{\omega^2}$ exists. To obtain the following optimal result.

Theorem 5 (Rodríguez). *Assume ZFC. Then there is at most one model of the form $L(\mathbb{R}, \mu)$ that satisfies* $\mathrm{AD}+$ *$\dot{\mu}$ is an-$\mathbb{R}$-supercompactness measure.*

We also note that in the absence of $\mathcal{M}^{\sharp}_{\omega^2}$ it is relatively consistent that the unique model of the theory AD+ ω_1 is $\mathbb{R}$-supercompact is not $L(\mathbb{R}, \mathcal{C})$, we show this in Chapter 6. Finally, also in Chapter 6, we show that if we drop the hypothesis that $L(\mathbb{R}, \mu)$ satisfies AD in Theorem 5, then the conclusions can fail.

Theorem 6 (Rodríguez). *Assume* ZFC *and there exists a measurable cardinal of Mitchell order two. Then there is a proper class model of* ZFC *in which there are subsets μ and ν of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ are models of*

$$\mathrm{ZF} + \dot{\mu} \textit{ is } \mathbb{R}\textit{-supercompactness measure,}$$

and there is $A \in L(\mathbb{R})$ such that $A \in \nu \setminus \mu$.

CHAPTER 2

Preliminaries

We give a summary of theorems and results we use throughout this paper. Unless we say otherwise, our base theory will be $\mathrm{ZF}+\mathrm{DC}_{\mathbb{R}}$. We start by reviewing the general theory of AD and AD^+ models and then move into mice and genericity iterations and lastly we discuss some general theory of the models $L(\mathbb{R},\mu)$ satisfying $\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact.

2.1. The theory of AD and AD^+

Recall that AD, the axiom of determinacy proposed my Mycielski and Steinhaus in [**7**], is the statement that every game with payoff contained in ω^ω is determined. Under AD there is a complete analysis of the descriptive set theory and fine structure of $L(\mathbb{R})$ by recursion along its Wadge hierarchy. In other models determinacy alone does not yield such a detailed structural analysis. Woodin introduced AD^+ to make up for the difference. Good sources that cover this material are [**3**] and Chapter 9 of [**29**]. All theorems and definitions of this section are due to others and can be found in [**29**] unless otherwise mentioned. Before defining AD^+ the following notion get us started.

Definition 7. Let $A\subseteq\mathbb{R}$. We say A is ∞-Borel if there is a formula $\phi(x,y)$ and a set $S\subset\mathrm{ON}$ such that

$$x\in A \text{ if and only if } L[S,x]\models\phi(x,S).$$

Let us recall that Θ is the least ordinal that is not a surjective image of a function with domain $\mathbb{R}$. In other words,

$$\Theta=\{\alpha\in\mathrm{ON}\mid \text{ there is } f:\mathbb{R}\to\alpha \text{ surjective }\}.$$

Definition 8. AD^+ is the conjunction of the following two sentences:

(1) Every set of reals is ∞-Borel.
(2) Let $\lambda<\Theta$ and $\pi:\lambda^\omega\to\mathbb{R}$ be a continuous function; then $\pi^{-1}[A]$ is determined for every $A\subseteq\mathbb{R}$.

The ordinal Θ has the following approximations. For $A\subseteq\mathbb{R}$, we define

$$\theta(A)=\{\alpha\mid \text{ there is a surjection } f:\mathbb{R}\to\alpha \text{ with } f\in\mathrm{OD}_A\}$$

The Solovay sequence is defined as follows.

Definition 9. Assume AD^+. Define a sequence of ordinals $\leq\Theta$ as follows.

- $\theta_0=\theta(\emptyset)$.
- If γ is a limit ordinal, then $\theta_\gamma=\sup\{\theta_\alpha\mid\alpha<\gamma\}$.
- $\theta_{\alpha+1}=\theta(A)$ for A any set of Wadge rank θ_α.

In $L(\mathbb{R})$, the minimal model of AD^+ every set is ordinal definable from a real, hence $\Theta = \theta_0$. Adding conditions on the length of the Solovay sequence yields a hierarchy of strengthenings of AD^+. For example, $\mathrm{AD}^+ + \Theta = \theta_\omega$ implies $\mathrm{AD}_\mathbb{R}$. Another example is $\mathrm{AD}^+ + \Theta = \theta_{\omega_1}$, which implies $\mathrm{DC} + \mathrm{AD}_\mathbb{R}$.

The following ordinal is also an approximation to Θ but in a different sense.

Definition 10. $\boldsymbol{\delta^2_1}$ is the least ordinal that is not the rank of a Δ^2_1 pre-wellorder on $\mathbb{R}$, in other words

$$\boldsymbol{\delta^2_1} = \{\alpha \in \mathrm{ON} \mid \text{ there exists a } \boldsymbol{\Delta}^2_1 \text{ pre-wellorder on } \mathbb{R} \text{ of rank } \alpha\}$$

In $L(\mathbb{R})$, $\boldsymbol{\delta^2_1}$ is the least ordinal α such that $L_\alpha(\mathbb{R}) \prec_1 L(\mathbb{R})$. For this reason we call $\boldsymbol{\delta^2_1}$ the *least stable* ordinal. This is closely related to the fact, that under AD, $\Sigma_1^{L(\mathbb{R})}$ is the largest scaled point-class of $L(\mathbb{R})$ and, for every bounded formula, ϕ, if

$$L(\mathbb{R}) \models \exists A \subset \mathbb{R}\ \phi(A),$$

then $L(\mathbb{R})$ has a Suslin co-Suslin witness for ϕ. Recall that AD^+ was introduced to generalize theorems of $\mathrm{AD} + V = L(\mathbb{R})$. Here is an example.

Theorem 11. *Assume* AD^+ *and* $V = L(\mathcal{P}(\mathbb{R}))$. *Suppose that ϕ is a bounded formula such that $\phi(A)$ holds for some $A \subseteq \mathbb{R}$. Then there is a Suslin co-Suslin witness A for ϕ.*

It is an open problem whether AD implies AD^+. However, models of determinacy often come in the form of *derived models*, which we will define after the following theorem. Such models are known to satisfy AD^+.

Theorem 12 (The Derived Model Theorem). *Suppose that δ is a limit of Woodin cardinals. Let $G \subset col(\omega, < \delta)$ be V-generic and $\mathbb{R}^*_G = \bigcup\{\mathbb{R}^{V[G\restriction\alpha]} \mid \alpha < \delta\}$. For $A \subseteq \mathbb{R}^*_G$, let $A \in \Gamma_G$ if and only if*

$$A \in V(\mathbb{R}^*_G) \text{ and } L(\mathbb{R}^*_G, A) \models \mathrm{AD}^+ .$$

Then $L(\mathbb{R}_G, \Gamma_G) \models \mathrm{AD}^+$.

We refer to $L(\mathbb{R}_G, \Gamma_G)$ as the derived model given by the generic G. Note that the theory of the derived model does not depend on G, as the forcing is homogeneous.

Here are three important theorems that relate AD, AD^+ and $\mathrm{AD}_\mathbb{R}$

Theorem 13. *Assume* AD *and let* $\Gamma = \{A \subseteq \mathbb{R} \mid L(\mathbb{R}, A) \models \mathrm{AD}^+\}$. *Then* $L(\mathbb{R}, \Gamma) \models \mathrm{AD}^+$. *Moreover, if* $\Gamma \neq \mathcal{P}(\mathbb{R})$, *then* $L(\mathbb{R}, \Gamma) \models \mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_\mathbb{R}$.

Theorem 14. *Assume* AD *and* $V = L(\mathcal{P}(\mathbb{R}))$. *Suppose* $A \subseteq \mathbb{R}$ *and* $L(\mathbb{R}, A) \neq V$. *Then* $A^\sharp$ *exists.*

Theorem 15. *Assume* AD^+ *and* $V = L(\mathcal{P}(\mathbb{R}))$. *Suppose that* ω_1 *is* $\mathbb{R}$*-supercompact. Then* $\mathrm{AD}_\mathbb{R}$ *holds.*

Theorems 13 and 14 can be used to see that $\mathrm{AD} + \neg\, \mathrm{AD}^+$ implies the consistency of $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_\mathbb{R}$. It is not known whether AD implies AD^+. The models we care most about in this thesis have the form $L(\mathbb{R}, \mu)$. This models are not of the form $L(\mathcal{P}(\mathbb{R}))$ and it turns out that $\mathrm{AD}_\mathbb{R}$ fails in these models.

2.2. Mice, genericity iterations and K

We will assume that the reader has familiarity with the basic concepts of extender models and mice. Sources for this subject are [**21**] and [**6**]. However, for the non-expert we will summarize the key parts of the theory of mice we will be using.

We start by recalling that a *premouse* $\mathcal{M}$ is a fine structural model of the form $(\mathcal{J}^{\boldsymbol{E}}_\alpha, \in, \boldsymbol{E} \restriction \alpha, E_\alpha)$, where $\boldsymbol{E}$ is a fine extender sequence. Let us fix some notation: suppose $\mathcal{M}$ is as above and $\gamma < \alpha$, we write $\mathcal{M}|\gamma$ for the structure $(\mathcal{J}^{\boldsymbol{E}\restriction\gamma}_\gamma, \in, \boldsymbol{E} \restriction \gamma, E_\gamma)$. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are two pre-mice we say that $\mathcal{M}$ *is an initial segment of* N and write $\mathcal{M} \trianglelefteq \mathcal{N}$, if there is $\gamma \leq \mathrm{ON} \cap \mathcal{N}$ such that $\mathcal{M} = \mathcal{N}|\gamma$.

If $\mathcal{M}$ is a k-sound mouse, we say that $\mathcal{M}$ is (k, β, γ) or (k, β)-iterable if player II has a winning strategy for the iteration games $\mathcal{G}_k(\mathcal{M}, \beta, \gamma)$ or $\mathcal{G}_k(\mathcal{M}, \beta)$ respectively, see sections 3 and 4 of [**21**] for a precise definition. A *mouse* is a k-sound pre-mouse that is either $(k, \omega_1 + 1)$-iterable . Whenever $k = \omega$ we will abuse notation and write $\omega_1 + 1$-iterable for $(\omega, \omega_1 + 1)$ etc. $\mathcal{M}^\sharp_{\omega^2}$ is the mouse we will be most interested in this work. We warm up for its presentation with the following definition.

Definition 16. A pre-mouse is called ω^2-*small* if whenever κ is the critical point of an extender in the sequence of $\mathcal{M}$, then

$$\mathcal{M}|\kappa \not\models \text{“There are } \omega^2 \text{ many Woodin cardinals”}.$$

Definition 17. $\mathcal{M}^\sharp_{\omega^2}$ is the unique sound, $(\omega, \omega_1, \omega_1+1)$-iterable mouse that is not ω^2-small, but all of whose initial segments are ω^2-small.

Notice that $\rho_1(\mathcal{M}^\sharp_{\omega^2}) = \omega$ and $p_1(\mathcal{M}^\sharp_{\omega^2}) = \emptyset$, hence $\mathcal{M}^\sharp_{\omega^2}$ is countable. We will see in Chapter 3 that $\mathcal{M}^\sharp_{\omega^2}$ is related to $L(\mathbb{R}, \mathcal{C})$ very much like $\mathcal{M}^\sharp_\omega$ is related to $L(\mathbb{R})$. Recall the Solovay sequence defined in section 2.1. Its length not only entails stronger versions of determinacy but also the existence of mice with certain large cardinal structure. A fact we will repeatedly use is that under AD^+ if $\Theta > \theta_0$, then there is a *non-tame mouse*, which we introduce now.

Definition 18. Let $\mathcal{M}$ be a pre-mouse. We say $\mathcal{M}$ is *tame* if for any δ such that $\mathcal{M} \models$ “δ is Woodin” and for any E_γ in the sequence of $\mathcal{M}$ with $\mathrm{crit}(E_\gamma) < \delta$, then $\gamma < \delta$.

Note that a non-tame mouse is a mouse that has a cardinal that is strong past a Woodin cardinal. Also, if a non-tame mouse exists, it is easy to see that $\mathcal{M}^\sharp_{\omega^2}$ exists.

Theorem 19 (Woodin, see [19]). *Assume* $\mathrm{AD}^+ + \Theta > \theta_0$*, then there is an* $(\omega_1 + 1)$*-iterable non-tame mouse.*

One important application of mice is that they help when analyzing certain determinacy models. For example, one of the key ingredients for the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ is that one can iterate $\mathcal{M}^\sharp_\omega$ to make any given real generic. We will also use this technique in this work.

Theorem 20 (Genericity Iterations). *Let* Σ *be an* $(\omega_1 + 1)$*-iteration strategy for a countable mouse* $\mathcal{M}$ *and* δ *be an ordinal in* $\mathcal{M}$ *such that*

$$\mathcal{M} \models \delta \textit{ is a Woodin cardinal.}$$

Then there is a Boolean algebra $\mathbb{B}^{\mathcal{M}}_\delta \in \mathcal{M}$ *such that* $\mathbb{B}^{\mathcal{M}}_\delta \subset V^{\mathcal{M}}_\delta$ *and* $\mathcal{M} \models \mathbb{B}_\delta$ *is* δ*-c.c. Moreover, for every* $x \in \mathbb{R}$*, there is a countable iteration tree* $\mathcal{T}$ *such that*

- *$\mathcal{T}$ is a play according to Σ,*
- *$\mathcal{T}$ has a final model, say $\mathcal{M}^{\mathcal{T}}_{\gamma}$,*
- *$\mathcal{T}$ is nowhere dropping and*
- *there is an $\mathcal{M}^{\mathcal{T}}_{\gamma}$-generic filter G for $\mathbb{B}^{\mathcal{M}^{\mathcal{T}}_{\gamma}}_{i^{\mathcal{T}}_{0,\gamma}(\delta)} = i^{\mathcal{T}}_{0,\gamma}\left(\mathbb{B}^{\mathcal{M}}_{\delta}\right)$ such that*

$$\mathcal{M}^{\mathcal{T}}_{\gamma}[G] = \mathcal{M}^{\mathcal{T}}_{\gamma}[x]$$

The Boolean algebra $\mathbb{B}^{\mathcal{M}}_{\delta}$ is called *the extender algebra of $\mathcal{M}$ at δ*. We will use genericity iterations and $\mathcal{M}^{\sharp}_{\omega^2}$ to compute the theory of $L(\mathbb{R}, \mathcal{C})$ in Chapter 3.

Now we discuss some basic core model theory. See [**16**] and [**2**] for a detailed treatment or [**9**] for a basic exposition. Roughly speaking the core model K (if it exists) is the maximal canonical inner model of ZFC. If $0^{\sharp}$ does not exist then $L = K$. Historically the definition of K has been given progressively with weaker anti-large cardinal hypotheses. Near the state of the art, Jensen and Steel defined K assuming "there is no proper class model with a Woodin cardinal". See their paper [**2**], their definition builds on much work by them and others as is explained in the introduction to [**2**]. We will use the phrase "K exists" to refer to any of the settings in which K can be defined and its theory works. Let us list the key properties that we will use the properties that we will use in this thesis.

First K is an extender model. Second K is absolute to generic extension. In other words, if G is set-generic for V, then $K = K^{V[G]}$. Third, the core model is inductively definable. In particular, $K \cap \mathrm{HC}$ is definable over $L_{\omega_1}(\mathbb{R})$. Fourth, the core model is maximal in the general sense that if V has certain large cardinals, then so has K (we are intentionally vague here, we will be precise when we apply this concept). Finally, although K is not absolute between inner models of ZFC we do have $K^K = K$. This will be of particular importance in Chapter 6.

2.3. The theory of $L(\mathbb{R}, \mu)$

In the present section we summarize the known theory of models of the form $L(\mathbb{R}, \mu)$ satisfying AD+ω_1 is $\mathbb{R}$-supercompact. All the results of this section can be found in [**25**] and [**29**]. First, let us recall that, by a theorem of Woodin, the consistency strength of AD+ω_1 is $\mathbb{R}$-supercompact is exactly the strength of

$$\text{ZFC} + \text{there are } \omega^2 \text{ Woodin cardinals.}$$

There is more than an equiconsistency in that there are ways of translating models from these two theories that we describe next.

First, let M be a model of ZFC with ω^2-many Woodin cardinals. Let δ^M_α be the α-th Woodin cardinal of M. Also, let

$$\lambda^M_\beta = \sup\{\delta^M_\alpha \mid \alpha \in \beta\}.$$

Suppose that G is an M-generic filter for $\mathrm{col}(\omega, < \lambda^M_{\omega^2})$. Let $\sigma_i = \bigcup_{\alpha<\omega i} \mathbb{R}^{M[G\restriction\alpha]}$ and $\mathbb{R}^* = \bigcup_{\alpha<\omega^2} \mathbb{R}^{M[G\restriction\alpha]}$. In $M[G]$, define *the tail filter*, $\mathcal{F}$, on $\mathcal{P}_{\omega_1}(\mathbb{R}^*)$ as follows: for $A \subseteq P_{\omega_1}(\mathbb{R}^*)$

$$A \in \mathcal{F} \text{ if and only if } \exists n \in \omega\, \forall m \geq n\ (\sigma_m \in A).$$

by [**25**] $L(\mathbb{R}^*, \mathcal{F}) \models \mathrm{AD}^+ + \omega_1$ is $\mathbb{R}$-supercompact. The ideas behind the proof are similar to the ideas used to prove the the derived model theorem, Theorem 12.

In fact, we will refer to $L(\mathbb{R}^*, \mathcal{F})$ as the derived model associated to M and G, although technically this is incorrect.

Towards the other direction of the equi-consistency , let $L(\mathbb{R}, \mu)$ be a model of AD in which ω_1 is $\mathbb{R}$-supercompact as witnessed by μ. Then, in a forcing extension of $L(\mathbb{R}, \mu)$ there exists a proper class inner model M of ZFC with ω^2-many Woodin cardinals. Moreover, there is G an M-generic filter for $\mathrm{col}(\omega, < \lambda^M_{\omega^2})$, such that the derived model associated to M and G is precisely $L(\mathbb{R}, \mu)$. This implies that any $L(\mathbb{R}, \mu)$ model of AD+ω_1 is $\mathbb{R}$-supercompact is actually a model of AD^+. We will use this fact repeatedly in this paper without explicitly mentioning it.

We discuss now the internal structure of $L(\mathbb{R}, \mu)$. The following approximation to Theorem 5 was proved by Woodin

Theorem 21. *Suppose $L(\mathbb{R}, \mu)$ is a model of* AD$+$ *μ is an $\mathbb{R}$-supercompactness measure. Then*

$$L(\mathbb{R}, \mu) \models \text{“}\mu \textit{ is the unique } \mathbb{R}\textit{-supercompactness measure”}$$

Suppose that $L(\mathbb{R}, \mu)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact. Theorem 21 easily implies that μ is definable in $L(\mathbb{R}, \mu)$, and hence

$$L(\mathbb{R}, \mu) \models \Theta = \theta_0$$

Combining this with Theorem 15 we have that.

Lemma 22. *Suppose that $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, Then*

$$L(\mathbb{R}, \mu) \models \text{“}\mu \textit{ is not the club filter.”}$$

Proof. Let us work in $L(\mathbb{R}, \mu)$ and for contradiction suppose that in $L(\mathbb{R}, \mu)$ μ is the club filter. Then $L(\mathcal{P}(\mathbb{R})) = L(\mathbb{R}, \mu)$, hence by Theorem 15 $L(\mathbb{R}, \mu) \models \mathrm{AD}_{\mathbb{R}}$. This contradicts the fact that $L(\mathbb{R}, \mu) \models \Theta = \theta_0$. □

Although μ cannot be seen internally by $L(\mathbb{R}, \mu)$ to be the club filter, the following theorem due to Woodin implies that these two filters agree up to $\boldsymbol{\delta^2_1}$.

Theorem 23. *Assume $L(\mathbb{R}, \mu)$ is a model of* AD$+$ *$\dot{\mu}$ is an $\mathbb{R}$-supercompactness measure, then*

- $L_{\boldsymbol{\delta^2_1}}(\mathbb{R}, \mu) \prec_1 L(\mathbb{R}, \mu)$
- $L(\mathbb{R}, \mu) \models$ “*if $A \in \mu \cap L_{\boldsymbol{\delta^2_1}}(\mathbb{R}, \mu)$, then A contains a club*”

Suppose that $L(\mathbb{R}, \mu)$ is a model of AD$+$ $\dot{\mu}$ is an $\mathbb{R}$-supercompactness measure. Recall that $\mathrm{HOD}^{L(\mathbb{R},\mu)}(\mathbb{R})$ is the smallest proper class containing $\mathrm{HOD}^{L(\mathbb{R},\mu)}$ and $\mathbb{R}$. Note that μ is definable, as it is the unique $\mathbb{R}$-supercompactness measure in $L(\mathbb{R}, \mu)$. Moreover, by the proof of Theorem 3.1 in [**25**], $\mu \in \mathrm{HOD}^{L(\mathbb{R},\mu)}(\mathbb{R})$. This implies the following key lemma.

Lemma 24. *Suppose $L(\mathbb{R}, \mu)$ is a model of* AD$+$ *$\dot{\mu}$ is an $\mathbb{R}$-supercompactness measure, then*

$$\mathrm{HOD}^{L(\mathbb{R},\mu)}(\mathbb{R}) = L(\mathbb{R}, \mu).$$

CHAPTER 3

Uniqueness under ZFC + $\mathcal{M}^\sharp_{\omega^2}$ exists

In this chapter, we proof Theorem 1. But first we prove the following proposition, which is weaker than Theorem 1. It has an extra hypothesis, namely that μ concentrates on stationary sets.

Proposition 25. Assume ZFC and suppose that $\mathcal{M}^\sharp_{\omega^2}$ exists. Then:

(1) $L(\mathbb{R}, \mathcal{C}) \models \text{AD} + \dot{\mu}$ is an $\mathbb{R}$-supercompactness measure.

(2) If $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ is such that for every $A \in \mu$, A is stationary, and

$$L(\mathbb{R}, \mu) \models \text{AD} + \dot{\mu} \text{ is a } \mathbb{R}\text{-supercompactness measure},$$

then $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$.

Proposition 25 is proved in section 3.1. To read Section 3.1 the reader should be familiar with the technique of iterating mice to make reals generic as summarized in Chapter 2.

In Section 3.2, we prove Theorem 1. For this, we use the HOD analysis of the models $L(\mathbb{R}, \mu)$ that satisfied $\text{AD} + \omega_1$ is $\mathbb{R}$-supercompact, to show that on a Turing cone of reals x,

$$\text{HOD}_x^{L(\mathbb{R},\mathcal{C})} = \text{HOD}_x^{L(\mathbb{R},\mu)}.$$

See [**17**] for the HOD analysis in $L(\mathbb{R})$; other good sources on this subject are [**23**] and [**25**]. We recall here that the meaning of ordinal definability in $L(\mathbb{R}, \mu)$ is different from the usual notion in that the language for the definitions includes the predicate $\dot{\mu}$ which is interpreted as $\mu \cap L(\mathbb{R}, \mu)$.

3.1. The ZFC case under the stationarity assumption

In this section, we prove Proposition 25. Assume its hypotheses. Recall that $\mathcal{M}^\sharp_{\omega^2}$ is the unique, active, sound mouse projecting to ω, with ω^2-many Woodin cardinals all whose initial segments are ω^2-small. See Chapter 2 for its definition. Part of what it means to be a mouse is that $\mathcal{M}^\sharp_{\omega^2}$ has an $(\omega, \omega_1, \omega_1 + 1)$-iteration strategy, which happens to be unique; we call it Σ. We now make an additional assumption about Σ that we will eliminate when we finish the proof of Proposition 25 at the end of this section. Let $\kappa = (2^{\mathfrak{c}})^+$. Assume that Σ is coded by a κ-universally Baire set of reals. In other words, there are trees T and U such that:

- $p[T]$ codes Σ and $p[U] = \mathbb{R} \setminus p[T]$.
- If $\mathbb{P} \in V_\kappa$, then $V^{\mathbb{P}}$ satisfies that $p[T]$ codes an $(\omega, \omega_1, \omega_1 + 1)$-iteration strategy on $\mathcal{M}^\sharp_{\omega^2}$ and $p[U] = \mathbb{R} \setminus p[T]$.

Fix such trees T and U. We will abuse notation by using Σ to refer to the strategy coded by $p[T]$ in any small generic extension of V. If P is a countable Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, then Σ can also be considered an $(\omega, \omega_1, \omega_1 + 1)$-iteration strategy on P. If

Q is a Σ-iterate of such a P and there is no dropping in model on the branch from P to Q, then we write $P \overset{\Sigma}{\to} Q$.

Recall the construction at the beginning of Section 2.3. Eventually, we will find a Σ-iterate, M and an M-generic filter G on $\mathrm{col}(\omega, <\lambda^M_{\omega^2})$ such that the associated $\mathbb{R}^*$ is $\mathbb{R}$ and the corresponding tail filter contains the club filter, $\mathcal{C}$. Towards this, the following gets us started.

Lemma 26. *Suppose that γ is a cardinal such that $\gamma \geq 2^{\mathfrak{c}^+}$. Let X_0 and X_1 be countable elementary substructures of H_γ such that $\mathbb{R} \cap X_0 \in X_1$ and $T, U \in X_0$. Then there is an iteration tree $\mathcal{T}$ on $\mathcal{M}^\sharp_{\omega^2}$ of successor length $\zeta + 1$ such that $\mathcal{T} \restriction \alpha \in X_0$ for all $\alpha < \zeta$ and $\mathcal{T} \in X_1$, and there exists $G \in X_1$ such that G is $M^{\mathcal{T}}_\zeta$-generic on $\mathrm{col}(\omega, <\lambda^{M^{\mathcal{T}}_\zeta}_\omega)$ and the associated set of symmetric reals is $\mathbb{R} \cap X_0$.*

Proof. Given the assumptions above note that $\mathbb{R} \cap X_0 \in X_1$, so there is $\langle x_i \mid i \in \omega \rangle$ an enumeration of $\mathbb{R} \cap X_0$ in X_1. By Theorem 20 there is $\mathcal{T}_0$, an iteration tree on $\mathcal{M}^\sharp_{\omega^2}$ according to Σ, with last model P_0, such that $i : \mathcal{M}^\sharp_{\omega^2} \to P_0$ exists and x_0 is generic for $\mathbb{B}^{P_0}_{\delta_0}$, the extender algebra at $\delta_0^{P_0}$. Note $\mathcal{M}^\sharp_{\omega^2} \in X_0$ and has a unique strategy, hence $\mathcal{T}_0$ belongs to X_0 and is countable there. We continue iterating $P_0 \to P_1$ in the interval (δ_0, δ_1), say via $\mathcal{T}_1$, to make the next real x_1, generic for the extender algebra at $\delta_1^{P_1}$. Note that in this case both x_0 and x_1 are set generic over P_1 for posets in $V^{P_1}_{\lambda^{P_1}_\omega}$. Continuing in this fashion we get Σ-iteration trees $\mathcal{T}_n$ with branch embeddings $P_{n-1} \to P_n$ such that x_n is P_n-generic for the extender algebra at $\delta_n^{P_n}$. Also every x_i for $i < n$ is set generic over P_n.

In X_1, define $\mathcal{T}$ to be the concatenation of the $\mathcal{T}_n$. Now $\mathcal{T}$ has a unique cofinal branch b. Let $P = M^{\mathcal{T}}_b$.

Claim: There is a P-generic filter G for $\mathrm{col}(\omega, <\lambda^P_\omega)$ in X_1 such that the associated set of symmetric reals is $\mathbb{R} \cap X_0$.

Proof of Claim. Let $\sigma = \mathbb{R} \cap X_0$ and $\lambda = \lambda^P_\omega$. By construction the following hold:

(1) For every $x \in \sigma$ there is a poset $\mathbb{P} \in V^P_\lambda$ such that x is P-generic for $\mathbb{P}$.
(2) $\lambda = \sup\{\omega_1^{P[x]} \mid x \in \sigma\}$.
(3) $P \models$ "λ is a strong limit cardinal".

Define a poset $\mathbb{T}$ in $P(\sigma)$ as follows.

- g is a condition in $\mathbb{T}$ if there is $\alpha < \lambda$ and $x \in \sigma$ such that g is P-generic for $\mathrm{col}(\omega, <\alpha)$ and $g \in P[x]$.
- $g_0 \leq_{\mathbb{T}} g_1$ if $g_0 \supseteq g_1$.

Suppose that H is $P(\sigma)$-generic for $\mathbb{T}$ and let $H = \bigcup G$. We claim that G is as wanted. We left to the reader to check that H is indeed P-generic for $\mathrm{col}(\omega, <\lambda)$. Moreover note that by construction

$$\bigcup_{\alpha<\lambda} \mathbb{R}^{P[H\restriction\alpha]} \subseteq \sigma$$

Finally, if $x \in \sigma$ let $D_x = \{g \in \mathbb{T} \mid x \in P[g]\}$. We show that D_x is dense in $\mathbb{T}$. For this let $g \in \mathbb{T}$, so there is $y \in \sigma$ such that $g \in P[y]$ and g is P-generic for some small collapse. By (1) there are $\alpha < \lambda$ and $\mathbb{P} \in V^{P[y]}_\alpha$ such that x is $P[y]$-generic for $\mathbb{P}$. Also as λ is a strong limit in P, by (2), there is $z \in \sigma$ such that $V^{P[y]}_{\alpha+1}$ is countable in $P[z]$. We may assume also without loss that x and y are in $P[z]$. So, there is $\tilde{g} \subset \mathrm{col}(\omega, <\alpha)$ in $P[z]$ extending g such that $x \in P[\tilde{g}]$ and $\tilde{g}$ is P-generic. Hence D_x is dense, and so

$$\bigcup_{\alpha<\lambda} \mathbb{R}^{P[H\upharpoonright\alpha]} = \sigma$$

as wanted. □

□

Note that in the proof of the Lemma 26 we used the interval $(\lambda_0^M, \lambda_\omega^M)$, where $M = \mathcal{M}^\sharp_{\omega^2}$ but we could have used any Σ-iterate M of $\mathcal{M}^\sharp_{\omega^2}$ with $M \in X_0$ and any interval $(\lambda^M_{\omega i}, \lambda^M_{\omega(i+1)})$ to obtain the same result.

Lemma 27. *In $V^{col(\omega,2^{\mathfrak{c}})}$, there is a Σ -iterate P of $\mathcal{M}^\sharp_{\omega^2}$ and a P-generic filter G for $col(\omega, <\lambda^P_{\omega^2})$ such that if $\mathcal{F}$ is the associated tail filter, then $\mathcal{C}^V$ is contained in $\mathcal{F}$.*

Proof. In the statement of the lemma, we are writing $\mathcal{C}^V$ for the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ as computed in V. In $V^{\mathrm{col}(\omega,(2^{\mathfrak{c}}))}$, we let $\langle X_i | i \in \omega\rangle$ be a chain of countable elementary substructures of $H^V_{\kappa^+}$ such that $\bigcup_{i\in\omega} X_i \supseteq \mathcal{P}(\mathbb{R})^V$ and if $\sigma_i = X_i \cap \mathbb{R}$, then $\sigma_i \in X_{i+1}$ and σ_i is countable in V. We may assume that $\mathcal{M}^\sharp_{\omega^2}$, T and U are in X_0. We construct an iteration of the form $\mathcal{M}^\sharp_{\omega^2} \to P_0 \to P_1 \to \cdots \to P_i \to P_{i+1} \cdots \to P$ by recursion using Lemma 26 so that the iteration $P_{i-1} \to P_i$ is done in the interval $(\lambda_{\omega(i-1)}, \lambda_{\omega i})$ and makes σ_{i-1} the set of symmetric reals associated to a P_i-generic on $\mathrm{col}(\omega, <\lambda^{P_i}_{\omega i})$. Let P be the direct limit of the P_i, by a similar argument given in the claim of Lemma 26, there is a P-generic filter G for $\mathrm{col}(\omega, <\lambda^P_{\omega^2})$ such that $\sigma_i = \bigcup_{\alpha<\omega i} \mathbb{R}^{P[G\upharpoonright\alpha]}$. Note that the set of symmetric reals associated to G and P is $\mathbb{R}^V$. Let $\mathcal{F}$ be the corresponding tail filter. Consider any $A \in \mathcal{C}^V$. Let $\pi \in V$ be such that $\pi : \mathbb{R}^{<\omega} \to \mathbb{R}$ and its closure points belong to A. Then there is an $n \in \omega$ such that $\pi \in X_n$. So for all $m \geq n$, $\pi \in X_m$ and σ_m is closed under π, thus $A \in \mathcal{F}$. □

The two key facts in the proof of Lemma 27 are that if A is an element of $\mathcal{C}^V$, then there is an $i \in \omega$ such that $A \in X_i$, and that every X_i is closed under Σ. This motivates the following definition.

Definition 28. Suppose N is a set model of some set theory, such that $\mathcal{P}(\mathbb{R})^N$ is countable. Then we say $\langle X_i \,|\, i \in \omega\rangle$ is a *good resolution* of N if for all $i \in \omega$, we have $X_i \prec N$ and $\mathbb{R} \cap X_i \in X_{i+1}$, and $\bigcup_{i\in\omega} X_i \supset \mathcal{P}(\mathbb{R})^N$.

Note that in the proof of Lemma 27 instead of $H_{(2^{\mathfrak{c}})^+}$ we could have used any N that is a model of enough set theory and $\mathcal{P}(\mathbb{R})^N$ is countable in $V^{\mathrm{col}(\omega,2^{\mathfrak{c}})}$. We give an example of such a situation in the following lemma.

Lemma 29. *Suppose that A is stationary in $\mathcal{P}_{\omega_1}(\mathbb{R})$. Then it is forced by $col(\omega, 2^{\mathfrak{c}})$ that there exist a Σ-iterate P of $\mathcal{M}^\sharp_{\omega^2}$, and P-generic filter G for $col(\omega, <\lambda^P)$ such that A belongs to the tail filter associated to G and P.*

Proof. By homogeneity it is enough to find a generic filter for $\mathrm{col}(\omega, 2^{\mathfrak{c}})$ with the desired property. Consider $\mathbb{P}_A$, the forcing poset whose conditions are countable, closed, increasing sequences from A. In other, words $p = \langle \sigma_\alpha \mid \alpha < \beta \rangle$ is a condition in $\mathbb{P}_A$ if

- for all $\alpha < \beta$, we have that σ_α belongs to A,
- for every α and α' in β if $\alpha < \alpha'$ then $\sigma_\alpha \subseteq \sigma_{\alpha'}$, and
- if $\alpha < \beta$ is a limit ordinal, then $\sigma_\alpha = \bigcup_{i \in \alpha} \sigma_i$.

We say $p \leq q$ if p end-extends q. It is easy to see that this poset shoots a club through A. Also, the usual argument will show that this forcing is (ω_1, ∞)-distributive, so in particular it does not add any new reals. Let h be V-generic for $\mathbb{P}_A$. Then $\mathbb{R}^{V[h]} = \mathbb{R}$ and, as the forcing has size continuum, we have that $2^{\mathfrak{c}}$ is the same ordinal in V and $V[h]$. Applying Lemma 27 in $V[h]$, if G' is $V[h]$-generic for $\mathrm{col}(\omega, 2^{\mathfrak{c}})$, then in $V[h][G']$ the conclusion of the lemma holds as $A \in \mathcal{C}^{V[h]}$. Finally, note that there is a V-generic filter G for $\mathrm{col}(\omega, 2^{\mathfrak{c}})$ such that $V[G] = V[h][G']$. $\square$

Suppose that in $V^{\mathrm{col}(\omega, 2^{\mathfrak{c}})}$ there are two Σ-iterates P and Q of $\mathcal{M}^\sharp_{\omega^2}$ and generic filters G and H for $\mathrm{col}(\omega, < \lambda^P_{\omega^2})$ and $\mathrm{col}(\omega, < \lambda^Q_{\omega^2})$ respectively such that the set of symmetric reals of $P[G]$ and $Q[H]$ is precisely $\mathbb{R}^V$. Let $\mathcal{E}$ and $\mathcal{F}$ be the tail filters associated to P, G and Q, H respectively. We will show that if this is the case then $L(\mathbb{R}, \mathcal{E}) = L(\mathbb{R}, \mathcal{F})$.

Lemma 30. *In $V^{col(\omega, 2^{\mathfrak{c}})}$, let N_1 and N_2 be transitive sets containing T and U that model a reasonable amount of* ZFC *such that $\mathbb{R}^{N_i} = \mathbb{R}^V$ for $i = 1, 2$. Let $\langle X^1_i | i \in \omega \rangle$ and $\langle X^2_i | i \in \omega \rangle$ be good resolutions of N_1 and N_2 respectively and $\mathcal{F}^1$ and $\mathcal{F}^2$ be the the associated tail filters. Then $L(\mathbb{R}, \mathcal{F}^1) = L(\mathbb{R}, \mathcal{F}^2)$.*

In practice N_1 would be $H^V_{\kappa^+}$ and N_2 would be $H^{V[h]}_{\kappa^+}$ V-generic for some filter h which is V-generic for small forcing.

Proof. Let $\sigma^1_j = X^1_j \cap \mathbb{R}$ and similarly $\sigma^2_j = X^2_j \cap \mathbb{R}$. Iterate $\mathcal{M}^\sharp_{\omega^2}$ inductively as follows. Let $\sigma_0 = \sigma^1_0$ and let $\mathcal{M}^\sharp_{\omega^2} \to P_0$ be the iteration to make σ_0 generic on the first ω-many Woodins. Note that σ_0 can be coded as a single real, so there is i_1 such that $\sigma_0 \in X^2_{i_1}$ and thus the iteration $M^\sharp_\omega \to P_0$ is actually in $X^2_{i_1}$. Define $\sigma_1 = \sigma^2_{i_1}$ and iterate $P_0 \to P_1$ on the second ω-many Woodins to make σ_1 generic. There is i_2 such that $\sigma_1 \in X^1_{i_2}$. Let $\sigma_2 = \sigma^1_{i_2}$ and continue the iteration in this fashion. We get an iteration $\mathcal{M}^\sharp_{\omega^2} \to P_0 \to P_1 \cdots \to P_i \to P_{i+1} \to \cdots \to P$ and a P-generic filter G for $\mathrm{col}(\omega, < \lambda^P_{\omega^2})$ such that $\sigma_i = \bigcup_{\alpha < \omega i} \mathbb{R}^{P[G \restriction \alpha]}$. Let $\mathcal{F}$ be the associated tail filter. Note also that for any $i \in \omega$ there are $j > i$ and $k > i$, and natural numbers m and n such that $\sigma^1_j = \sigma_m$ and $\sigma^2_k = \sigma_n$.
Claim: $L(\mathbb{R}, \mathcal{F}^1) = L(\mathbb{R}, \mathcal{F}) = L(\mathbb{R}, \mathcal{F}^2)$.

Proof of the Claim. We have that $\mathcal{F}^1$ is an ultrafilter relative to sets in $L(\mathbb{R}, \mathcal{F}^1)$, and similarly $\mathcal{F}$ is an ultrafilter in $L(\mathbb{R}, \mathcal{F})$. Now by an induction on $\alpha \in \mathrm{ON}$, we see that $L_\alpha(\mathbb{R}, \mathcal{F}^1) = L_\alpha(\mathbb{R}, \mathcal{F})$ and $\mathcal{F} \cap L_\alpha(\mathbb{R}, \mathcal{F}^1) = \mathcal{F} \cap L_\alpha(\mathbb{R}, \mathcal{F})$, which would give the desired claim. Limit stages are clear. Now if the induction hypotheses hold at α, it is clear that $L_{\alpha+1}(\mathbb{R}, \mathcal{F}^1) = L_{\alpha+1}(\mathbb{R}, \mathcal{F})$. Given $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$, a set in $L_{\alpha+1}(\mathbb{R}, \mathcal{F}^1) \cap \mathcal{F}^1$, we have that either A or its complement is in

$\mathcal{F}$. But A contains a tail of the σ_i^1, hence by construction its complement cannot contain a tail of σ_i, which means $A \in \mathcal{F}$. The other direction is similar, so the induction hypotheses hold at $\alpha + 1$. $\square$

Clearly the claim completes the proof of the Lemma 30. $\square$

For simplicity we will refer to the unique model in $V^{\mathrm{col}(\omega, 2^{\mathfrak{c}})}$ coming from constructions a la Lemma 27 as $L(\mathbb{R}, \mathcal{F})$. Note that by the homogeneity of the collapse, $L(\mathbb{R}, \mathcal{F})$ is definable from T and U in V, as is $\mathcal{F} \cap V$. We refer to $\mathcal{F} \cap V$ as $\mathcal{F}$ when there is no ambiguity.

Lemma 31. *Let $L(\mathbb{R}, \mu) \models \mathrm{AD} +$ $\dot{\mu}$ is a $\mathbb{R}$-supercompactness measure, and suppose that μ contains only stationary sets. Then $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F})$.*

Proof. First we will show again inductively that $L_\alpha(\mathbb{R}, \mathcal{F}) = L_\alpha(\mathbb{R}, \mu)$ and $\mathcal{F} \cap L_\alpha(\mathbb{R}, \mathcal{F}) = \mu \cap L_\alpha(\mathbb{R}, \mathcal{F})$. As in the proof of the claim in the last theorem, we only need to take care of the successor steps. Now given $A \in \mathcal{F} \cap L_{\alpha+1}(\mathbb{R}, \mathcal{F})$, by induction either A or its complement is in μ. For contradiction suppose $A \notin \mu$. Then $A^c \in \mu$, so A^c is stationary, applying Lemmas 29 and 30 giving $A^c \in \mathcal{F}$, which is a contradiction. $\square$

Lemma 32. *$\mathcal{C} \cap L(\mathbb{R}, \mathcal{F}) = \mathcal{F} \cap L(\mathbb{R}, \mathcal{F})$.*

Proof. Otherwise by Lemma 27, there is $A \in \mathcal{F} \cap L(\mathbb{R}, \mathcal{F})$ that does not contain a club, which means that A^c is stationary so by Lemma 29 and 30 we have $A^c \in \mathcal{F}$, which gives a contradiction. $\square$

To summarize we have seen that $L(\mathbb{R}, \mathcal{C})$ is the unique model of AD + ω_1 is $\mathbb{R}$-supercompact under the hypotheses of Proposition 25 and the additional assumption that Σ is $(2^{\mathfrak{c}})^+$-universally Baire. Our final step is to eliminate this extra assumption.

Assume that $\mathcal{M}_{\omega^2}^{\sharp}$ exists and Σ is an $(\omega, \omega_1, \omega_1 + 1)$-iteration strategy but not necessarily universally Baire. Suppose that μ is as in the statement of the proposition. Let γ be such that V_γ reflects enough set theory, and let $N \prec V_\gamma$ be countable such that Σ and μ are in N. Let H be the transitive collapse of N and $\pi : H \to N$ be the uncollapsing map. Define $\bar{\mu} = \pi^{-1}(\mu)$ and $\overline{\Sigma} = \pi^{-1}(\Sigma)$.

Let us review how the universal Baireness of Σ was used in the proofs of the earlier lemmas. The key points are that Σ canonically extends to a strategy in $V^{\mathrm{col}(\omega, 2^{\mathfrak{c}})}$ and $\mathcal{P}(\mathbb{R})^V$ is countable in $V^{\mathrm{col}(\omega, 2^{\mathfrak{c}})}$. The relationship between H and V is similar enough to the relationship between V and $V^{\mathrm{col}(\omega, 2^{\mathfrak{c}})}$ to obtain the following in V without assuming Σ universally Baire. There is a countable iterate P of $\mathcal{M}_{\omega^2}^{\sharp}$ and a P-generic filter K for $\mathrm{col}(\omega, < \lambda_{\omega^2}^P)$ such that $\mathbb{R}^H$ is the set of symmetric reals of $P[K]$. Moreover, if $\overline{\mathcal{F}}$ is the associated tail filter, then $\overline{\mathcal{F}} \cap H$ belongs to H and in H, $L(\mathbb{R}^H, \overline{\mathcal{F}}) = L(\mathbb{R}^H, \overline{\mu}) = L(\mathbb{R}^H, \mathcal{C}^H)$, and the three filters are the same on the common model. By elementarity and the choice of γ, $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{C})$ and the two filters agree on the common model. This completes the proof of Proposition 25.

3.2. The general ZFC case

Assume $\mathcal{M}_{\omega^2}^{\sharp}$ exists. In the last section we saw that if μ consists only of stationary sets and $L(\mathbb{R}, \mu)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact, then $\mu \cap L(\mathbb{R}, \mu) = \mathcal{C} \cap L(\mathbb{R}, \mu)$. Let us give an example that illustrates there is more to do.

Let $\mathcal{S}^V$ be the collection of stationary subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$ in V. By Proposition 25 we have that $L(\mathbb{R}, \mathcal{S}^V)$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact. Let $A \subset \mathcal{P}_{\omega_1}(\mathbb{R})$ be a stationary set whose complement is also stationary and let h be a V-generic filter for the poset that shoots a club through A^c (as in the proof of Theorem 29). Applying Proposition 25 in $V[h]$, $L(\mathbb{R}, \mathcal{S}^{V[h]})$ is the unique model of AD + ω_1 is $\mathbb{R}$-supercompact. We would like to conclude that $L(\mathbb{R}, \mathcal{S}^V) = L(\mathbb{R}, \mathcal{S}^{V[h]})$ but it does not follow from Proposition 25 applied in $V[h]$ because $A \in \mathcal{S}^V$ but A is nonstationary.

Notice that the proof given in the last section relies heavily on the fact that if $A \in \mu$, then one can shoot a club through A without adding reals. Without this available to us we need a different idea. We use Woodin's Analysis of HOD in order to prove Theorem 1. The HOD Analysis for structures of the form $L(\mathbb{R}, \mu)$ was done in [**25**], however we will use a variant closer to the exposition of [**17**]. We start by doing the analysis for $L(\mathbb{R}, \mathcal{C})$ and then generalize to $L(\mathbb{R}, \mu)$. We first give some useful definitions and lemmas. We will work, as in the last section, with $\mathcal{M}^{\sharp}_{\omega^2}$ and its strategy Σ, as well as with trees T and U that witness that Σ is $(2^{\mathfrak{c}})^+$-universally Baire. Ultimately, the universally Baire assumption on Σ will be eliminated using the same ideas from last section.

3.2.1. $\mathcal{P}(\mathbb{R})$ in Models of AD + ω_1 is $\mathbb{R}$ -supercompact

We start by analyzing $\mathcal{P}(\mathbb{R})$ in "minimal" AD models of ω_1 is $\mathbb{R}$-supercompact. the following terminology will get us started.

Definition 33. Given μ, a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$, we use the following notation:

- $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$
- $\boldsymbol{\delta}^2_1(\mu) = \boldsymbol{\delta}^{2\,L(\mathbb{R},\mu)}_1$
- $\Theta(\mu) = \Theta^{L(\mathbb{R},\mu)}$

The following lemma says that the power sets of the reals of such models line up with that of $L(\mathbb{R}, \mathcal{C})$.

Lemma 34. *Suppose that $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ is such that $L(\mathbb{R}, \mu) \models$ AD + ω_1 is $\mathbb{R}$ -supercompact. Then either $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\mathcal{C}(\mathbb{R})$ or $\mathcal{P}_\mathcal{C}(\mathbb{R}) \subseteq \mathcal{P}_\mu(\mathbb{R})$.*

Proof. Suppose neither $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\mathcal{C}(\mathbb{R})$ nor $\mathcal{P}_\mathcal{C}(\mathbb{R}) \subseteq \mathcal{P}_\mu(\mathbb{R})$. Let $\Gamma = \mathcal{P}_\mathcal{C}(\mathbb{R}) \cap \mathcal{P}_\mu(\mathbb{R})$. By Theorem 3.7.1 of [**26**] $L(\mathbb{R}, \Gamma) \models \mathrm{AD}_\mathbb{R}$. Hence by a theorem of Solovay mentioned in the introduction, if ν is the club filter defined in $L(\mathbb{R}, \Gamma)$, then $L(\mathbb{R}, \nu) \models$ AD + ω_1 is $\mathbb{R}$-supercompact. Moreover, we have that ν is a subset of $\mathcal{C}$, so by an induction in the constructive hierarchy (like the one in the proof of Lemma 30) we have $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mathcal{C})$, which readily gives a contradiction. □

We will need the notion of the *envelope* of a point-class. For a complete exposition of this subject the reader may consult Chapter 3 of [**26**]. We will mostly be interested in envelopes of point-classes of the form $\Sigma_1^{Lp(\mathbb{R})|\gamma}$ where Lp is the *lower part* operator (see Chapter 3 of [**10**]). We recall the definitions below.

Definition 35. For a set X we have the following.

- Given a mouse $\mathcal{M}$ on X we say that $\mathcal{M}$ is *countably iterable* if for any $\bar{\mathcal{M}}$ countable and elementary embeddable into $\mathcal{M}$, we have that $\bar{\mathcal{M}}$ is $\omega_1 + 1$ iterable.

- $Lp(X)$ is the union of all countably iterable and sound X-mice that project to X.

Definition 36. Suppose that γ is an admissible ordinal of $Lp(\mathbb{R})$. Let $\Gamma = \Sigma_1^{Lp(\mathbb{R})|\gamma}$. For $A \subseteq \mathbb{R}$

- We say $A \in \mathrm{OD}^{<\gamma}$ if there is $\alpha < \gamma$ such that A is $\mathrm{OD}^{Lp(\mathbb{R})|\alpha}$.
- We say $A \in Env(\Gamma)$ if for every $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ there is $A' \in \mathrm{OD}^{<\gamma}$ such that $A \cap \sigma = A' \cap \sigma$.

We also note that the definition of the envelope can be relativized to any real x. Recall that $\mathbf{Env}(\Gamma)$, the boldface envelope, is $\bigcup_{x\in\mathbb{R}} Env(\Gamma(x))$. The notion of the envelope is particularly useful when analyzing the Σ_1-gaps and the pattern of scales in the structure $Lp(\mathbb{R})$ (see [**14**], [**20**] and [**11**]).

We turn now to prove that for any μ such that $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, we have that $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\mathcal{C}(\mathbb{R})$.

Lemma 37. *Suppose that μ is a subset of $\mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu)$ satisfies $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then $L(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\mu)$ have the same sets of reals.*

Proof. For contradiction suppose that this is not the case. Without loss of generality we may assume that μ and $\mathcal{C}$ measure some subset of $\mathcal{P}_{\omega_1}(\mathbb{R})$ differently, as otherwise the lemma would follow trivially. By Lemma 34 we have the following two cases.

Case 1: $\mathcal{P}_\mu(\mathbb{R})$ is strictly contained in $\mathcal{P}_\mathcal{C}(\mathbb{R})$.
In this case without loss we will assume that μ is such that $\mathcal{P}_\mu(\mathbb{R})$ is minimal. In other words, given any other $\nu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\nu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, then $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\nu(\mathbb{R})$.

By $(\mathbb{R},\mu)^\sharp$, we mean the theory of the reals and indiscernibles of $L(\mathbb{R},\mu)$ in a language with predicates for membership and μ and constant symbols $\dot{x}$ for each real x. Let B belong to $\mathcal{P}_\mathcal{C}(\mathbb{R})$ but not to $\mathcal{P}_\mu(\mathbb{R})$. Then $(\mathbb{R},\mu)^\sharp = \oplus_{n\in\omega}\mathcal{T}_n^\mu$, where each $\mathcal{T}_n^\mu$ is Wadge reducible to B. Since there is a real x that codes all these reductions, $(\mathbb{R},\mu)^\sharp \in L(\mathbb{R},\mathcal{C})$. Also, recall that $L_{\boldsymbol{\delta}_1^2(\mathcal{C})}(\mathbb{R},\mathcal{C}) \prec_1 L(\mathbb{R},\mathcal{C})$ (see Chapter 2), hence there is such a sharp in $L_{\boldsymbol{\delta}_1^2(\mathcal{C})}(\mathbb{R},\mathcal{C})$. Let $\bar{\mu}$ be such that $(\mathbb{R},\bar{\mu})^\sharp \in L_{\boldsymbol{\delta}_1^2(\mathcal{C})}(\mathbb{R},\mathcal{C})$ and $L(\mathbb{R},\bar{\mu}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Claim: In $L(\mathbb{R},\mathcal{C})$, $\mathcal{M}_{\omega^2}^\sharp$ exists and is ω_1+1-iterable.

Proof of the claim. Let us work in $L(\mathbb{R},\mathcal{C})$. First, by results of [**25**] we have that $\mathcal{P}_C(\mathbb{R}) \subseteq Lp(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ and $\mathcal{P}_{\bar{\mu}}(\mathbb{R}) \subseteq Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}$. Note that if M is an $\mathbb{R}$-mouse in $L(\mathbb{R},\bar{\mu})$ projecting to $\mathbb{R}$, there is a set of reals in $\mathcal{P}_{\bar{\mu}}(\mathbb{R})$ coding it. Thus $M \in L_{\boldsymbol{\delta}_1^2(\mathcal{C})}(\mathbb{R},\mathcal{C})$. Also, if M is countably iterable in $L(\mathbb{R},\bar{\mu})$, by definition, if $\bar{M}$ is a countable hull of M it is iterable in $L(\mathbb{R},\bar{\mu})$. As $\mathbb{R} \subset L(\mathbb{R},\mathcal{C})$ any such $\bar{M}$ is ω_1-iterable in $L(\mathbb{R},\mathcal{C})$. But ω_1 is measurable in $L(\mathbb{R},\mathcal{C})$ hence $\bar{M}$ is iterable in $L(\mathbb{R},\mathcal{C})$. This gives us that:

$$Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})} \lhd (Lp(\mathbb{R})|\boldsymbol{\delta}_1^2(\mathcal{C}))^{L(\mathbb{R},\mathcal{C})}.$$

This implies that $\boldsymbol{\delta}_1^2(\bar{\mu})$ starts a Σ_1-gap in $Lp(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$. Let

$$\Gamma = \Sigma_1^{Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}}.$$

We claim that $\mathbf{Env}(\Gamma) = P_{\bar{\mu}}(\mathbb{R})$, where the envelope is as defined in $L(\mathbb{R},\mathcal{C})$.

For this note that by results of [**20**], we have $\mathbf{Env}(\Gamma) = P(\mathbb{R})^{Lp(\mathbb{R})|\gamma}$, where γ is the largest ordinal such that $Lp(\mathbb{R})|\boldsymbol{\delta^2_1}(\bar{\mu}) \prec_1 Lp(\mathbb{R})|\gamma$. Note that $\gamma \geq \Theta^{L(\mathbb{R},\bar{\mu})}$, and for all we know this inequality could be strict. However, since $[\boldsymbol{\delta^2_1}(\bar{\mu}), \gamma]$ is a Σ_1-gap, we have that $Lp(\mathbb{R})|(\gamma+1)$ is the first initial segment of $L(\mathbb{R})^{L(\mathbb{R},\mathcal{C})}$ that has a subset of the reals not in $Lp(\mathbb{R})^{L(\mathbb{R},\bar{\mu})}$, in fact $(\mathbb{R}, \bar{\mu})^\sharp \in Lp(\mathbb{R})|(\gamma+1)$. Thus $\mathcal{P}_{\bar{\mu}}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap Lp(\mathbb{R})|\gamma$ and so $\mathbf{Env}(\Gamma) = \mathcal{P}_{\bar{\mu}}(\mathbb{R})$, as wanted.

Let $\vec{B}$ be a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Since $\vec{B}$ is countable, there exists a real x such that each element in $\vec{B}$ is $\mathrm{OD}_x^{L(\mathbb{R},\bar{\mu})}$. As a result of the analysis of $\mathrm{HOD}^{L(\mathbb{R},\bar{\mu})}$ (done in Section 3 of [**25**]) for every real y there is a set extender model $\mathcal{M}_y$ with the following properties.

- $\mathcal{M}_y \models$ "there are exactly ω^2-many Woodin cardinals."
- $V_{\Theta(\bar{\mu})}^{\mathrm{HOD}_y^{L(\mathbb{R},\bar{\mu})}} = \mathcal{M}_y|\Theta(\bar{\mu})$ and $\mathcal{M}_y \models$ "$\Theta(\bar{\mu})$ is the first Woodin cardinal".
- For every $\mathrm{OD}_y^{L(\mathbb{R},\mu)}$ set of reals A and any Woodin cardinal δ in $\mathcal{M}_y$, there is a term $\tau_{A,\delta}$ that captures[1] A at δ.

We point out here that in section 3.2.2 we develop a variant of the HOD analysis. Our Theorem 61 will also imply the existence of the extender models $\mathcal{M}_y$ (the proof of Theorem 61 does not depend on this lemma).

Let $\mathcal{M} = \mathcal{M}_x$. Then $\mathcal{M}$ has ω^2-many Woodin cardinals and terms capturing every B in $\vec{B}$ at every Woodin cardinal δ of $\mathcal{M}$. Let $\tau_{B,\delta}$ be the standard term witnessing this. Let us define $\mathcal{N} = Hull^{\mathcal{M}}(\{\tau_{B,\delta} \mid B \in \vec{B} \text{ and } \delta \text{ Woodin in } \mathcal{M}\})$ (sketch: the Woodin cardinals of $\mathcal{N}$ remain Woodin in $L[N] \subseteq L(\mathbb{R}, \bar{\mu})$ and, in $L(\mathbb{R}, \mathcal{C})$ we have $(\mathbb{R}, \bar{\mu})^\sharp$.). Hence $\mathcal{N}$ is an x-mouse that captures all the elements of a self-justifying system. Thus, by a theorem of Woodin, the strategy that picks branches that are realizable into $\mathcal{M}$ and moves these term relations correctly is an iteration strategy for $\mathcal{N}$ (see [**10**]). In other words, $\mathcal{N}$ is ω_1-iterable in $L(\mathbb{R}, \mathcal{C})$, and hence $\mathcal{N}^\sharp$ exists and is ω_1-iterable in $L(\mathbb{R}, \mathcal{C})$. Therefore $\mathcal{M}^\sharp_{\omega^2}$ exists and it is ω_1 (and hence $\omega_1 + 1$) iterable.

□

We claim that if ν is the club filter in $L(\mathbb{R}, \mathcal{C})$, then $L(\mathbb{R}, \nu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. We cannot apply Proposition 25 directly but we can work our way into a situation where the proof can be adapted. For this, let α be such that $L_\alpha(\mathbb{R}, \mathcal{C})$ reflects enough set theory. By results of [**25**] we have that DC holds in $L(\mathbb{R}, \mathcal{C})$. Therefore, there is a countable set N, such that $N \prec L_\alpha(\mathbb{R}, \mathcal{C})$ and $\mathcal{M}^\sharp_{\omega^2}$ and its unique strategy are in N. Let $\bar{N}$ be the transitive collapse of N. Then the proof of Proposition 25 implies that $\bar{N}$ believes that "if $\bar{\nu}$ is the club filter, then $L(\mathbb{R}, \bar{\nu}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact". By elementarily and the choice of α, we get that $L(\mathbb{R}, \mathcal{C})$ believes this as well. Also, $\nu \subseteq \mathcal{C}$ and by an induction on the constructive hierarchy, as in the proof of Lemma 30, we get $L(\mathbb{R}, \nu) = L(\mathbb{R}, \mathcal{C})$. This contradicts Theorem 22.

Case 2: $\mathcal{P}_{\mathcal{C}}(\mathbb{R})$ is strictly contained in $\mathcal{P}_\mu(\mathbb{R})$.
Using the same argument as in Case 1, we have that $\mathcal{P}_{\mathcal{C}}(\mathbb{R})$ is contained in $L_{\boldsymbol{\delta^2_1}(\mu)}(\mathbb{R}, \mu)$. Recall that $\mu \cap L_{\boldsymbol{\delta^2_1}(\mu)}(\mathbb{R}, \mu)$ is a subset of the club filter of $L(\mathbb{R}, \mu)$. So if $A \in \mathcal{P}_{\mathcal{C}}(\mathbb{R}) \cap \mu$

[1] We say that τ captures A at δ in $\mathcal{M}$ if for any $\mathcal{M}$-generic $H \subset \mathrm{col}(\omega, \delta)$, $A \cap \mathbb{R}^{\mathcal{M}[H]} = \tau[H]$.

then A contains a club in V. Hence $\mathcal{C} \cap L(\mathbb{R}, \mathcal{C}) \subseteq \mu$. By an induction on the construction hierarchy we have $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$, a contradiction. $\square$

Note that the proof of Lemma 37 implies that any two models, $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ that satisfy $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact would have the same Θ and so they would also share the same $\boldsymbol{\delta^2_1}$. This justifies referring to $\Theta(\mu)$ and $\boldsymbol{\delta^2_1}(\mu)$ simply as Θ and $\boldsymbol{\delta^2_1}$ respectively.

Also, in the proof of Lemma 37, we used that $L(\mathbb{R}, \mu)$ cannot have an $\omega_1 + 1$ iteration strategy for $\mathcal{M}^\sharp_{\omega^2}$. The careful reader might note that more is true: if $L(\mathbb{R}, \mu)$ is a model of AD+ ω_1 is $\mathbb{R}$-supercompact, then it cannot have an iterable proper class model with ω^2 many Woodin cardinals that is also iterable. Also, recall Theorem 19, which implies that if $\mathrm{AD}^+ + \Theta > \theta_0$ holds then there is a non-tame mouse. These two facts combined yield:

Lemma 38. *Suppose that $L(\mathbb{R}, \mu)$ is a model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then $L(\mathbb{R}, \mu)$ does not contain a proper class model N of ZF such that $\mathbb{R} \subset N$ and $N \models \Theta > \theta_0$.*

Proof. Recall that given the hypotheses $L(\mathbb{R}, \mu) \models \mathrm{AD}^+$ and so $N \models AD^+$. If $N \models \Theta > \theta_0$, then we have that N has a non-tame mouse and hence $\mathcal{M}^\sharp_{\omega^2}$ exists and is $\omega_1 + 1$-iterable in N. But $\mathbb{R} \subset N$, hence $\mathcal{M}^\sharp_{\omega^2}$ exists and is $\omega_1 + 1$ iterable in $L(\mathbb{R}, \mu)$, which is a contradiction. $\square$

We finish this section mentioning a useful corollary to Lemma 38.

Corollary 39. *Suppose that $L(\mathbb{R}, \mu)$ is a model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then $L(\mathcal{P}_\mu(\mathbb{R})) \models \mathrm{AD}^+ + \Theta = \theta_0$.*

3.2.2. A HOD Analysis for $L(\mathbb{R}, \mathcal{C})$

We start the outline of the HOD analysis by adapting the standard notions. This means, we will define, in V, a directed system whose limit agrees with a rank initial segment $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$, understand what the rest of $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$ looks like and define in $L(\mathbb{R}, \mathcal{C})$ a corresponding covering system. Then we will generalize these results to models $L(\mathbb{R}, \mu)$ of $\mathrm{ZF} + \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Definition 40. We say P is a δ_0-*bounded* Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$ if there is an iteration tree $\mathcal{T}$ on $\mathcal{M}^\sharp_{\omega^2}$ built according to Σ, such that,

- P is the last model of $\mathcal{T}$,
- all extenders used in $\mathcal{T}$ have critical point below the image of $\delta_0^{\mathcal{M}^\sharp_{\omega^2}}$, and
- there is no drop in model on the branch leading to P so that there is an embedding $i : \mathcal{M}^\sharp_{\omega^2} \to P$ given by $\mathcal{T}$.

Let

$$\mathcal{D}^+ = \{P \mid P \text{ is a } \delta_0\text{-bounded iterate of } \mathcal{M}^\sharp_{\omega^2}\}.$$

For P and Q in $\mathcal{D}^+$, say $P \preceq^+ Q$ if P iterates to Q via Σ in a δ_0-bounded way, in which case we let $\pi_{P,Q}$ be the corresponding embedding given by Σ. By the Dodd-Jensen property of Σ, $\pi_{P,Q}$ is well defined. The Dodd-Jensen property also guarantees that $(\mathcal{D}^+, \preceq^+, \pi_{Q,P})$ is a directed system. Take the direct limit of $(\mathcal{D}^+, \preceq^+, \pi_{P,Q})$ and iterate the away the sharp ON-many times to obtain a proper class model M^+_∞. Also, let $\pi_{Q,\infty}$ be the natural map from Q to M^+_∞.

We will eventually prove that, $L[M^+_\infty, \Sigma \restriction X] = \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$, for some set of iteration trees X in $L(\mathbb{R},\mathcal{C})$. Motivated by the work of Steel and Woodin for $L(\mathbb{R})$ the next step is to adapt the definition of suitability. From now on, we work in $L(\mathbb{R},\mathcal{C})$.

Definition 41. Let $\alpha < \omega^2$. Let P be a premouse. We say P is α-*suitable* if there exist a sequence $\langle \delta_i^P \rangle_{i<\alpha}$ in P such that

(1) For every cut-point η of P, $Lp(P|\eta) \trianglelefteq P$.
(2) If $\eta < o(P)$, but is not a Woodin cardinal of P, then $Lp(P|\eta) \models$ "η is not Woodin".
(3) If $\lambda = \sup_{i\in\alpha} \delta_i^P$, then $o(P) = \sup_{n\in\omega} (\lambda^{+n})^P$.

We will say P is *suitable* if there is $\alpha < \omega^2$ such that P is α-suitable and define $\alpha(P) = \alpha$. It is an easy consequence of mouse capturing and the definition of suitability that if P is suitable, ξ a P-cardinal and $A \subseteq \xi$ is such that A is $\mathrm{OD}_P^{L(\mathbb{R},\mathcal{C})}$, then $A \in P$. On the other hand if $A \in P$, then A is $\mathrm{OD}_P^{L(\mathbb{R},\mathcal{C})}$. We will use this fact repeatedly without explicitly mentioning it.

Definition 42. Let $\mathcal{T}$ be normal tree on a suitable mouse P. We say $\mathcal{T}$ is *guided* if and only if for all limit $\eta < \mathrm{lh}(\mathcal{T})$, we have that $Q([0,\eta)_T, \mathcal{T} \restriction \eta)$ exists and is an initial segment of $Lp(\mathcal{M}(\mathcal{T} \restriction \eta))$. We say that $\mathcal{T}$ is *maximal* if $Lp(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin; otherwise we say $\mathcal{T}$ is *short*.

Notice that if $\mathcal{T}$ has successor length, then it is short.

Definition 43. Consider an α-suitable premouse P and $A \subseteq \mathbb{R}$. Let η be a cardinal of P. We say that P captures A at η if there is a $\mathrm{col}(\omega,\eta)$ name τ, such that whenever g is P-generic on $\mathrm{col}(\omega,\eta)$, we have $\tau[g] \cap \mathbb{R} = A \cap \mathbb{R}$. We say that P captures A if for every $i < \alpha(P)$, P captures A at δ_i^P.

Note that given $A \subset \mathbb{R}$ and a suitable P that captures A at δ_i^P, say via τ, there is a standard term that witnesses the capturing, following [**17**], we give its definition

$$\tau^P_{A,i} = \{(p,\sigma) \mid \sigma \text{ is a name for a real and } p \Vdash^{\mathrm{col}(\omega,\delta_i^P)} \sigma \in \tau\}.$$

Our next step is to define a notion of iterability that is strong enough so that one can compare suitable mice. Note the connection with [**17**], where the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ used a system of suitable mice with only finitely many Woodin cardinals. In our situation, however, suitable mice are allowed to have fewer than ω^2 many Woodin cardinals. That is why we need a stronger form of iterability that we describe below.

We will define a slight modification of Definition 1.8 from [**15**]. A suitable P is said to be *weakly* (ω,ω^2)-iterable* if player II has a winning strategy for the game $\mathcal{WG}^*(P,\omega^2)$ in which I and II alternate moves for ω^2 many rounds as follows. The game starts by letting $P_0 = P$. At round α, player I plays a countable normal, guided, putative iteration tree $\mathcal{T}_\alpha$ on P_α. At that point player II has two options. The first option is only available if $\mathcal{T}_\alpha$ has a wellfounded final model; then II may accept I's move in which case we set $P_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}_{lh(\mathcal{T}_\alpha)-1}$. The second option is for player II to play a maximal wellfounded branch b_α on $\mathcal{T}_\alpha$, such that, if $\mathcal{T}_\alpha$ is short then $Q(b_\alpha, \mathcal{T}_\alpha)$ exists and is an initial segment of $Lp(\mathcal{M}(\mathcal{T}_\alpha))$. The game continues by setting $P_{\alpha+1} = \mathcal{M}^{\mathcal{T}_\alpha}_{b_\alpha}$. There are additional requirements for both players at limit rounds. Namely, if I and II have played for all $\beta < \gamma$ and γ is a limit ordinal then:

- If there is $i < \alpha(P)$ such that for infinitely many $\beta < \gamma$, we have that $\mathcal{T}_\beta$ is a tree based on $P_\beta | \delta_i^{P_\beta}$ then I loses.
- The direct limit of P_β for $\beta < \gamma$ is wellfounded. Otherwise II loses.

After the ω^2 rounds have been played, the only condition for II is that the direct limit along the main branch is well-founded. We illustrate the weak* game, $\mathcal{WG}^*(P, \omega^2)$ game as follows:

$$\begin{array}{l|llllllll}
\text{Player} & 0 & & 1 & & \dots & & \omega & & \dots \\
\hline
\text{I} & \mathcal{T}_0 \text{ on } P_0 & & \mathcal{T}_1 \text{ on } \mathcal{M}_{b_0}^{\mathcal{T}_0} & & & \left.\right\} P_\omega & \mathcal{T}_\omega & & \\
\text{II} & & b_0 & & b_1 & & & & b_\omega &
\end{array}$$

Note that if P and Q are suitable premice such that II has a winning strategies τ_P and τ_Q for $\mathcal{WG}^*(P, \omega^2)$ and $\mathcal{WG}^*(Q, \omega^2)$ respectively. Then one can form guided iteration trees $\mathcal{T}_P$ and $\mathcal{T}_Q$ using the extenders that cause the "least" disagreement, and using τ_P and τ_Q when a maximal tree arises in this comparison. Since each P and Q have $< \omega^2$-many Woodin cardinals, this comparison succeeds. Note also that the end model of this comparison is still weakly* (ω, ω^2)-iterable.

Recall that a *stack* $\vec{\mathcal{T}}$ on a premouse P is a pair consisting of a sequence of iteration trees $\langle \mathcal{T}_i \,|\, i < \gamma \rangle$ and a sequence of premice $\langle P_i \,|\, i \leq \gamma \rangle$ such that

- $P_0 = P$,
- for every $i < \gamma$, $\mathcal{T}_i$ is an iteration tree of successor length on P_i and with last model P_{i+1}, and
- for every limit ordinal $\beta < \gamma$, P_β is the direct limit of $\langle P_i \,|\, i < \beta \rangle$ and the tree embeddings.

Note that $\langle P_i \,|\, i \leq \gamma \rangle$ is determined by the sequence $\langle \mathcal{T}_i \,|\, i < \gamma \rangle$. Also for a stack $\vec{\mathcal{T}}$ on P, we define $\mathcal{M}_\infty^{\vec{\mathcal{T}}} = P_\gamma$ and $i_\infty^{\vec{\mathcal{T}}} : P \to \mathcal{M}_\infty^{\vec{\mathcal{T}}}$, the natural embedding associated to this stack (if it exists). Notice that in $\mathcal{WG}^*(P, \omega^2)$ players I and II collaborate to form a stack on P.

If P is a suitable mouse capturing some $A \subseteq \mathbb{R}$ it will be desirable that "good" iterations of P maintain the suitability condition and move the terms capturing A correctly.

Definition 44 (A-iterations). For a suitable P that captures a set of reals A we define the following.

(1) We say P is A-iterable if II has a winning strategy for the game $\mathcal{WG}^*(P, \omega^2)$ such that whenever $\vec{\mathcal{T}}$ is a stack given by a game according to the strategy and $i_\infty^{\vec{\mathcal{T}}} : P \to \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ exists, then $\mathcal{M}_\infty^{\vec{\mathcal{T}}}$ is suitable and for any $i < \alpha(P)$ we have that $i_\infty^{\vec{\mathcal{T}}}(\tau_{A,i}^P) = \tau_{A,i}^{\mathcal{M}_\infty^{\vec{\mathcal{T}}}}$. We will call such a strategy an A-*strategy* for P.

(2) An A-*iteration* of P is a stack $\vec{\mathcal{T}}$ on P given by a run in $\mathcal{WG}^*(P, \omega^2)$ according to an A-strategy.

(3) We will say Q is an A-*iterate* of P if there is an A-iteration $\vec{\mathcal{T}}$ on P such that $Q = \mathcal{M}_\infty^{\vec{\mathcal{T}}}$ and $i_\infty^{\vec{\mathcal{T}}}$ exists.

(4) For $\mathcal{T}$, a normal guided tree on P of successor length $\eta+1$, such that $\mathcal{T}\restriction\eta$ is maximal, we let $\mathcal{T}^- = \mathcal{T}\restriction\eta$. In other words $\mathcal{T}^-$ is $\mathcal{T}$ without the last branch.

Also, given a a finite sequence $\vec{A}$ of sets of reals, we say P is $\vec{A}$-*iterable* in case there exists a winning strategy in $\mathcal{WG}^*(P,\omega^2)$ that simultaneously witnesses A-iterability for every A in the sequence $\vec{A}$.

So far we do not know whether there are A-iterable suitable mice but we prove next that these exist when $\mathcal{M}^\sharp_{\omega^2}$ is present. The following pair of lemmas are adaptations of results in Chapter 3 of [**17**].

Lemma 45. *Suppose $A\subseteq\mathbb{R}$ is definable in $L(\mathbb{R},\mathcal{C})$ from indiscernibles and assume that $\tilde{N}$ is a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, such that $i:\mathcal{M}^\sharp_{\omega^2}\to N$ (given by Σ) exists. Then any suitable initial segment of $\tilde{N}$ is A-iterable.*

The idea in the proof is the following. Note that if $\tilde{N}$ is a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, by Lemma 27, given a Woodin cardinal δ of $\tilde{N}$, we can iterate $\tilde{N}\overset{\Sigma}{\to}K$ above δ to make $L(\mathbb{R},\mathcal{C})$ realizable as the derived model associated to K and some K-generic filter. Hence, one can define truth in $L(\mathbb{R},\mathcal{C})$ in K using the homogeneity of the collapse. We show the details below.

Proof. Let $\tilde{N}$ as above and suppose that $A\subseteq\mathbb{R}$ is definable in $L(\mathbb{R},\mathcal{C})$ from indiscernibles. Let φ be a formula such that for any increasing sequence of indiscernibles $c_0<c_1<\cdots<c_n$ for $L(\mathbb{R},\mathcal{C})$

$$x\in A \Leftrightarrow L(\mathbb{R},\mathcal{C})\models\varphi(c_0,\ldots,c_{n-1},x).$$

Let Q be a suitable initial segment of $\tilde{N}$, and $\alpha=\alpha(Q)$. Let us define N to be the proper class model resulting when iterating the last extender of $\tilde{N}$ ON-many times. Let $\beta<\alpha$ and define τ as $(p,x)\in\tau$ if

$$p\Vdash_N^{\mathrm{col}(\omega,<\delta^N_\beta)}\Vdash_N^{\mathrm{col}(\omega,<\lambda^N_{\omega^2})} L(\dot{\mathbb{R}},\dot{\mathcal{F}})\models\varphi(\check{c_0},\check{c}_1,\ldots,\check{c}_{n-1},x),$$

where $\dot{\mathbb{R}}$ is the standard name for the symmetric reals under $\mathrm{col}(\omega,<\lambda^N_{\omega^2})$ and $\dot{\mathcal{F}}$ is the name for the tail filter associated to this forcing as defined in Chapter 2. Now by suitability of Q we have that $\tau\in Q$.

Let us see first that τ captures A at δ^Q_β. For this let G be Q-generic for $\mathrm{col}(\omega,\delta^Q_\beta)$. Note that by suitability G is also N-generic. Now, we can use Σ to iterate N, above δ^Q_β, in the fashion of Lemma 27 to get an embedding $j:N\to M$ and an $M[G]$-generic filter H such that $G*H$ is M-generic for $\mathrm{col}(\omega,\lambda^M_{\omega^2})$, and $j(\dot{\mathcal{F}})[G*H]=\mathcal{C}$ and $j(\dot{\mathbb{R}})[G*H]=\mathbb{R}$. With no loss we may assume that the indiscernibles are fixed by j. This implies that if $(p,x)\in\tau$ and $p\in G$, then

$$M[G*H]\models L(\mathbb{R},\mathcal{C})\models\varphi(c_0,\ldots,c_{n-1},x[G])$$

In other words if $x[G]\in\tau[G]\Rightarrow x[G]\in A$. Conversely if $x\in A\cap Q[G]$, then by homogeneity of the second forcing over N we have that

$$\Vdash_N^{\mathrm{col}(\omega,<\lambda^N_{\omega^2})} L(\dot{\mathbb{R}},\dot{\mathcal{F}})\models\varphi(\check{c_0},\check{c}_1,\ldots,\check{c}_{n-1},(\check{x[G]})),$$

so there is a condition $p\in G$ such that

$$p\Vdash_N^{\mathrm{col}(\omega,<\delta^N_\beta)}\Vdash_N^{\mathrm{col}(\omega,<\lambda^N_{\omega^2})} L(\dot{\mathbb{R}},\dot{\mathcal{F}})\models\varphi(\check{c_0},\check{c}_1,\ldots,\check{c}_{n-1},x).$$

In other words $A \cap Q[G] \subseteq \tau[G]$, and so $\tau[G] = A \cap Q[G]$.

By the way τ is defined it is easy to see that Σ moves τ correctly and so it is an A-iteration strategy. $\square$

Corollary 46. *Suppose A is an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals. Then for every $\alpha < \omega^2$ there is an α-suitable P that is A-iterable.*

PROOF. Suppose that there is a counterexample A. By minimizing the ordinals from which A is defined we may assume that A is actually definable in $L(\mathbb{R}, \mathcal{C})$. Let $\alpha < \omega^2$ and Q is the suitable initial segment of $\mathcal{M}^\sharp_{\omega^2}$ with $\alpha(Q) = \alpha$. By Lemma 46 Q is A-iterable, a contradiction. $\square$

The point in the proof of Corollary 46 is that given a counterexample in $L(\mathbb{R}, \mathcal{C})$ to a statement of the form "for all $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ sets of reals" then one can, by minimizing the counterexample, find a definable one. However it is the case that, usually, the suitable initial segments of $\mathcal{M}^\sharp_{\omega^2}$ witness that there are no definable counterexamples. The same argument essentially gives the following lemma.

Lemma 47 (Comparison). *Suppose that P is A-iterable and Q is B-iterable. Then there is an $A \oplus B$-iterable suitable mouse R, an A-iteration from P to a suitable initial segment of R and a B iteration from Q to suitable initial segment of R.*

It is clear that our notion of A-iterability is downwards absolute to $L(\mathbb{R}, \mathcal{C})$. The next step is to define a covering system using pairs (P, A), where A is an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals and P is an A-iterable mouse. However, it could be the case that for such a P, there are two different A-iterations $\pi : P \to Q$ and $\sigma : P \to Q$, and this would be a clear problem in building a directed limit. For this reason we need to work with relevant hulls and a stronger notion of iterability. We define below these concepts.

Definition 48. For an A-iterable mouse P, we let

(1) $P^- = P|(\delta_0^{+\omega})^P$
(2) $\gamma^P_{A,i} = \sup(Hull^P(\tau^P_{A,i}) \cap \delta^P_0)$.
(3) $\gamma^P_A = \sup_{i \in \alpha(P)} \gamma^P_{A,i}$.
(4) $\xi^P_A = \gamma^{P^-}_A$.
(5) $H(P, A) = Hull^P(\xi^P_A \cup \{\tau^P_{A,i} \mid i < \alpha(P)\})$

Note that if P is a suitable A-iterable mouse, then $P|(\delta_0^{+\omega})^P$ is 1-suitable and A-iterable .

Using the usual "zipper argument" (see [**13**] or [**18**]) we get the following lemma.

Lemma 49. *Let $\mathcal{T}$ be a tree of limit length on P, a suitable pre-mouse. Suppose further that there are branches b and c such that $\mathcal{T}^\frown b$ and $\mathcal{T}^\frown c$ are A-iterations and $\mathcal{M}^\mathcal{T}_b$ and $\mathcal{M}^\mathcal{T}_c$ are A-iterable. Then $i^\mathcal{T}_b \restriction \gamma^P_A = i^\mathcal{T}_c \restriction \gamma^P_A$ and so $i^\mathcal{T}_b \restriction H(P, A) = i^\mathcal{T}_c \restriction H(P, A)$.*

A delicate point here is that if P is an A-iterable mouse we could potentially have two A-iterations associated to two different trees on P leading to the same end model Q, so Lemma 49 would not apply. Hence we define the notion of strong iterability in the natural way and prove the existence of strongly iterable mice.

Definition 50. For $A \subseteq \mathbb{R}$ and a suitable A-iterable mouse P, we say P is *strongly A-iterable*, if whenever $i : P \to Q$ and $j : P \to Q$ are two A-iterations, then $i \restriction H(P, A) = j \restriction H(P, A)$.

Note again that when proving that for any $A \subseteq \mathbb{R}$ which is $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ there is a strongly A-iterable mouse it is sufficient to prove that for any definable set A there is a strongly A-iterable mouse. The following lemma, in contrast to most of what we have discussed so far, is not an "easy" generalization of the HOD Analysis in $L(\mathbb{R})$. The reason of this is the extra complexity in the iteration games considered. We give a detailed proof for the existence of strongly A-iterable mice.

Lemma 51. *Let A be an $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ set of reals and let P be A-iterable. Then there is an A-iterate of P that is strongly A-iterable.*

Proof. By the discussion above, we may without loss assume that A is definable in $L(\mathbb{R}, \mathcal{C})$. Given an A-iterable mouse P, by comparison we can A-iterate P to Q, an initial segment of a correct iterate of $\mathcal{M}^\sharp_{\omega^2}$. We claim that Q is as wanted.

Suppose $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ are A-iteration stacks on Q with the same last model R. We want to show that the embeddings given by $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ agree on $H(Q, A)$. We will actually show that both embeddings agree with embeddings given by Σ on $H(Q, A)$. Here we have to be an extra bit more careful than in the analogous situation of $L(\mathbb{R})$, because our iteration games can have more rounds and at limit stages it is not straightforward how to proceed, we will show next the details of how to overcome this difficulty.

We look inductively at the trees in the stack $\vec{\mathcal{T}} = \langle \mathcal{T}_i \mid i \in \alpha \rangle$. Let Q_i (for $i \in \alpha$) be the model starting round i in the weak* game. We will construct trees $\mathcal{S}_i$ inductively such that $\vec{\mathcal{S}} = \langle \mathcal{S} \mid i \in \alpha \rangle$ is according to Σ and has the property that the embedding given by $\vec{\mathcal{S}}$ agrees with $i^{\vec{\mathcal{T}}}$ on γ^Q_A. We will assume with no loss of generality that every tree $\mathcal{T}_i$ for $i \in \alpha$ is based on a window of the form $(\delta^{Q_i}_{k_i}, \delta^{Q_i}_{k_i+1})$.

Start with $\mathcal{T}_0$. Let us define $\mathcal{S}_0$ as follows. First suppose that $\mathcal{T}_0$ is based on ${Q_0}^-$. If it is according to Σ we let $\mathcal{S}_0 = \mathcal{T}_0$. Otherwise $\mathcal{T}_0$ is a maximal tree with a last branch b. Recall that $\mathcal{T}_0^-$ denotes the maximal part of $\mathcal{T}_0$. Let c be the branch given by Σ through $\mathcal{T}_0^-$ and, note that, by Lemma 45, c respects A. Let $\mathcal{S}_0$ be $\mathcal{T}_0^- {}^\frown c$. Also, by Lemma 49, we have that $i^{\mathcal{T}_0}$ and $i^{\mathcal{S}_0}$ agree on ξ^Q_A. Recall that Q_1 is the last model of $\mathcal{T}_0$, and let $\bar{Q}_1$ be the last model of $\mathcal{S}_0$, hence by fullness and maximality of $\mathcal{T}_0^-$, we get $Q_1^- = \bar{Q}_1^-$.

If $\mathcal{T}_0$ is above δ_0 then let $\mathcal{S}_0 = \emptyset$, $i^{\mathcal{S}_0} = \mathrm{id}$ and $\bar{Q}_1 = Q_0$. Here we get also get trivially that $i^{\mathcal{T}_0}$ and $i^{\mathcal{S}_0}$ agree on ξ^Q_A and $Q_1^- = \bar{Q}_1^-$.

Let us consider then $\mathcal{T}_1$. If it is based on ${Q_1}^-$ we can regard it as a tree on $\bar{Q}_1$ and then we can again use Σ to get $\mathcal{S}_1$ on $\bar{Q}_1$ such that $i^{\mathcal{T}_1}$ and $i^{\mathcal{S}_1}$ agree on $\xi^{Q_1}_A = \xi^{\bar{Q}_1}_A$. Again, by fullness we get that if $\bar{Q}_2$ is the last model of $\mathcal{S}_1$, then $Q_2^- = \bar{Q}_2^-$.

Otherwise we just let $\mathcal{S}_1 = \emptyset$ and the desired agreement is maintained so far.

Note that by the rules of the weak* game one has that $\mathcal{T}_i$ can be based on ${Q_i}^-$ only for finitely many $i \in \omega$. Hence $\bar{Q}_\omega$ agrees with Q_ω up to their common 1-suitable initial segment and the embedding on the $\mathcal{T}$-side agrees with the one given by the $\mathcal{S}$-side up to ξ^Q_A.

We proceed inductively in this fashion. At successors simply use Σ if the tree is based below the least Woodin cardinal, and otherwise define the corresponding tree in the $\mathcal{S}$-side as empty.

After α-many steps in this induction we will have that $\bar{Q}_\alpha$ is a Σ-iterate of Q. Let $\bar{\sigma}$ be the branch embedding. Then we have that $\bar{Q}_\alpha$ agrees with $Q_\alpha = R$ up to their common 1-suitable initial segment, and that $\bar{\sigma} \restriction \xi_A^Q = i^{\vec{\mathcal{T}}} \restriction \xi_A^Q$.

Similarly for $\vec{\mathcal{U}}$ one can get the analogous construction. So, we get that $i^{\vec{\mathcal{U}}}$ agrees with $\sigma' : Q \to Q'_\alpha$, an embedding given by Σ. Furthermore R, $\bar{Q}_\alpha$ and Q'_α agree up to their 1-suitable initial segment, and so since δ_0^R is a cut-point of both $\bar{Q}_\alpha$ and Q'_α by the Dodd Jensen property of Σ we can conclude that $\bar{\sigma}$ and σ' agree up to $\delta_0^{Q_0}$, and so $i^{\vec{\mathcal{T}}}$ and $i^{\vec{\mathcal{U}}}$ agree up to ξ_A^Q. Hence $i^{\vec{\mathcal{T}}} \restriction H(Q, A) = i^{\vec{\mathcal{U}}} \restriction H(Q, A)$ as wanted. □

Our covering system in $L(\mathbb{R}, \mathcal{C})$ will be

$$\mathcal{D} = \{H(P, \vec{A}) \mid P \text{ is strongly } \vec{A}\text{-iterable and } \vec{A} \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}\}.$$

Also we let $(P, \vec{A}) \preceq (Q, \vec{B})$ if Q is an A-iterate of P and $\vec{A} \subseteq \vec{B}$. We let $\sigma_{(P,\vec{A}),(Q,\vec{B})}$ be the unique embedding from $H(P, \vec{A})$ to $H(Q, \vec{B})$ given by an (any) $\vec{A}$-iteration from P to Q. The following results show that the suitable initial segments of correct iterates of $\mathcal{M}^\sharp_{\omega^2}$ together with the theories of indiscieribles for $L(\mathbb{R}, \mathcal{C})$ are in some sense "dense" in $\mathcal{D}$.

Let

$$M_\infty = \lim(\mathcal{D}, \preceq, \sigma_{(P,A),(Q,B)})$$

and let us define $\sigma_{(P,A),\infty}$ the natural embedding from $H(P, A)$ to this direct limit.

Let $\mathcal{T}_n^{\mathcal{C}}$ be the theory of n-many indiscernibles with real parameters of $L(\mathbb{R}, \mathcal{C})$ (coded as a subset of $\mathbb{R}$). Lemma 5.6 and Lemma 5.9 in [**17**] give the following results, we omit their proofs as they are word by word the same, except that we use $\mathcal{M}^\sharp_{\omega^2}$ and $L(\mathbb{R}, \mathcal{C})$ instead of $\mathcal{M}_\omega$ and $L(\mathbb{R})$ (the key, again, is that one can realize $L(\mathbb{R}, \mathcal{C})$ as the derived model of an iterate of $\mathcal{M}^\sharp_{\omega^2}$).

Lemma 52. *Suppose that P is a suitable initial segment of a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, then $\delta_0^P = \sup\{\xi^P_{\mathcal{T}_n^{\mathcal{C}}} \mid n \in \omega\}$.*

Lemma 53. *Assume A is $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$, and P is a strongly A-iterable suitable mouse. Then there is R, a suitable initial segment of a Σ-iterate of $\mathcal{M}^\sharp_{\omega^2}$, and a natural number n such that $(P, A) \preceq (R, A \oplus \mathcal{T}_n^{\mathcal{C}})$ and moreover $H(R, A \oplus \mathcal{T}_n^{\mathcal{C}}) = H(R, \mathcal{T}_n^{\mathcal{C}})$.*

Let us pause for a moment and discuss the general $L(\mathbb{R}, \mu)$ case. The lemma above will also be valid in this context by an application of Σ_1-reflection.

Lemma 54. *Suppose $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact, and let A be $\mathrm{OD}^{L(\mathbb{R},\mu)}$. Given P a strongly A-iterable suitable mouse and B an $\mathrm{OD}^{L(\mathbb{R},\mu)}$ set of reals, with $A \leq_W B$, there is R a suitable and $A \oplus B$-iterable mouse, such that $(P, A) \preceq (R, A \oplus B)$ and moreover $H(R, A \oplus B) = H(R, B)$.*

Proof. Otherwise fix A and B a counterexample to the statement. Fix γ large enough such that $L_\gamma(\mathbb{R}, \mu) \models \mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$ and A and B are ordinal definable over $L_\gamma(\mathbb{R}, \mu)$, but $L_\gamma(\mathbb{R}, \mu)$ has no R and A-iteration of P witnessing the conclusion of the Lemma. This Σ_1 statement about γ can then be reflected below $\boldsymbol{\delta}^2_1$. Hence there is such a $\gamma < \boldsymbol{\delta}^2_1$. But then $L_\gamma(\mathbb{R}, \mathcal{C}) = L_\gamma(\mathbb{R}, \mu)$ since below $\boldsymbol{\delta}^2_1$ both μ and $\mathcal{C}$

are just the club filter. We get then that there are A and B counterexamples of the statement in $L_\gamma(\mathbb{R},\mathcal{C})$ (and moreover OD in this structure). But then we can get the desired R and A-iteration in $L(\mathbb{R},\mathcal{C})$ and, by closure of γ, an A-iteration of P leading to R can be computed in $L_\gamma(\mathbb{R},\mu)$, so A and B cannot be the counterexample of $L_\gamma(\mathbb{R},\mathcal{C})$, contradiction. □

The lemmas above allow us to compute the direct limit of $\mathcal{D}$ just by looking at suitable initial segments of Σ-iterates of $\mathcal{M}^\sharp_{\omega^2}$ with corresponding theories of indiscernibles. We have the following agreement.

Theorem 55. $M_\infty = M^+_\infty|\lambda^{M^+_\infty}_{\omega^2}$

PROOF. We define a map $i : M_\infty \to M^+_\infty$ that is surjective below $\lambda^{M^+_\infty}_{\omega^2}$ and respects the membership relation as follows. For $x \in M_\infty$ there is a natural n and a suitable initial segment of a correct iterate of $\mathcal{M}^\sharp_{\omega^2}$, say P, such that x is in the range of $\sigma_{(P,A\oplus\mathcal{T}^{\mathcal{C}}_n),\infty}$, and there is $z \in H(P,\mathcal{T}^{\mathcal{C}}_n)$ such that $\sigma_{(P,A\oplus\mathcal{T}^{\mathcal{C}}_n),\infty}(z) = x$. Now we have an iteration $\mathcal{M}^\sharp_{\omega^2} \xrightarrow{\Sigma} \mathcal{N}$ such that P is a suitable initial segment of $\mathcal{N}$. Note that $\mathcal{N}$ might not be a δ_0-bounded iterate of $\mathcal{M}^\sharp_{\omega^2}$. We can however split the iteration from $\mathcal{M}^\sharp_{\omega^2}$ to $\mathcal{N}$ in a δ_0-bounded part and the rest. Namely, there is $\mathcal{N}^*$ such that $\mathcal{M}^\sharp_{\omega^2} \xrightarrow{\Sigma} \mathcal{N}^*$ into a δ_0-bounded way, and $\mathcal{N}^* \xrightarrow{\Sigma} \mathcal{N}$. Note that this second iteration does not move $\delta_0^{\mathcal{N}^*}$ (because all its extenders have critical point above the first Woodin cardinal). This implies that z is in the range of $\pi_{\mathcal{N}^*,\mathcal{N}}$, let $\bar{z}$ be its pre-image. Then we define $i(x) = \pi_{\mathcal{N}^*,\infty}(\bar{z})$, it is routine to show that i is well defined (see Theorem 5.10. of [**17**]). Now Lemma 53 gives us the surjectivity as follows: Let $x \in M^+_\infty|\lambda^{M^+_\infty}_{\omega^2}$ so there is $z \in \mathcal{N}$ a correct iterate of $\mathcal{M}^\sharp_{\omega^2}$ such that $\pi_{\mathcal{N},\infty}(z) = x$. Let P be a suitable initial segment of $\mathcal{N}$ such that $z \in P$. Because $\mathcal{N}$ is an δ_0 bounded iterate of $\mathcal{M}^\sharp_{\omega^2}$ we have that z is definable from ordinals less than $\delta_0^{\mathcal{N}}$ and indiscernibles, but this is easily computable from $\mathcal{T}^{\mathcal{C}}_n$ for a suitable n (again this follows essentially by Corollary 5.7 of [**17**]). Because $\xi^P_{\mathcal{T}^{\mathcal{C}}_n}$ is unbounded in $\delta_0^{\mathcal{N}}$ we conclude that $z \in H(P,\mathcal{T}^{\mathcal{C}}_n)$ for a sufficiently large n. This readily implies x is in the range of i as wanted. □

Let us work for a moment in $V^{\mathrm{col}(\omega,\mathbb{R})}$. Here we have that M^+_∞ is a countable Σ-iterate of $\mathcal{M}_{\omega^2}$. Also if G is M^+_∞-generic for $\mathrm{col}(\omega, < \lambda^{M_\infty}_{\omega^2})$, and $\mathbb{R}^*$ and $\mathcal{F}$ are the symmetric reals and associated tail filter, then $L(\mathbb{R}^*,\mathcal{F})$ is model of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Following the notation and the content of Chapter 6 from [**17**], for every n we can define, $\mathcal{T}^{\mathcal{C}^*}_n$, an $\mathrm{OD}^{L(\mathbb{R}^*,\mathcal{F})}$ set, by pieces as follows. For $(P,\mathcal{T}^{\mathcal{C}}_n)$, an element of $\mathcal{D}$, and for $i < o(P)$ let

$$\tau^*_{\mathcal{T}^{\mathcal{C}}_n,i} = \sigma_{(P,\mathcal{T}^{\mathcal{C}}_n),\infty}(\tau^P_{\mathcal{T}^{\mathcal{C}}_n,i})$$

and

$$\mathcal{T}^{\mathcal{C}^*}_n = \bigcup_{i\in\omega^2} \tau^*_{\mathcal{T}^{\mathcal{C}}_n,i}[G \upharpoonright \delta_i^{M_\infty}].$$

We also have that any suitable initial segment of M_∞ is strongly $\mathcal{T}^{\mathcal{C}^*}_n$-iterable (in $V^{\mathrm{col}(\omega,\mathbb{R})}$ as witnessed by Σ and in $L(\mathbb{R}^*,\mathcal{F})$ by absoluteness). Similarly we define A^*, an ordinal definable in $L(\mathbb{R}^*,\mathcal{F})$ set of reals, for each A in $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$. Recall that M^-_∞ is the 1-suitable initial segment of M_∞. We summarize the discussion above in the following lemmas.

Lemma 56. *For any set of reals A which is OD in $L(\mathbb{R},\mathcal{C})$ we have that A^* is OD in $L(\mathbb{R}^*,\mathcal{F})$. Moreover for any such A, M_∞^- is strongly A^*-iterable in the sense of $L(\mathbb{R}^*,\mathcal{F})$.*

Proof. This follows exactly as in the case of $L(\mathbb{R})$ so we omit details. These proofs can be essentially be found in Chapter 6: Claims 1,2 and 3 of [**17**]. □

Recall that X is the set of finite full stacks on M_∞^- in $M_\infty|(\lambda_{\omega^2}^{M_\infty})$. We then have that when computing the correct branches through $\vec{\mathcal{T}}$ it is enough to choose the unique branch that moves all the terms for A^* correctly. That is to say

Lemma 57. *Suppose $\mathcal{T} \in L(\mathbb{R}^*,\mathcal{F})$ is a guided maximal tree on M_∞^- as in the sense of $L(\mathbb{R}^*,\mathcal{F})$. Then $\Sigma(\mathcal{T}) = b$ if and only if $\mathcal{T}^\frown b$ is an A^*-iteration for all A in $\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$.*

Proof. This is claim 4 of Chapter 6 in [**17**]. Here we use Lemma 53 instead of Lemma 5.8 of [**17**], everything else follows word by word. □

From this and the homogeneity of the collapse it follows that $L[M_\infty, \Sigma \restriction X] \subseteq \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$. Also, note that if M_∞^* is the direct limit defined in $L(\mathbb{R}^*,\mathcal{F})$, then there is an embedding $\sigma : M_\infty^- \to M_\infty^*$, where $\sigma = \displaystyle\bigcup_{A\in\mathrm{OD}^{L(\mathbb{R},\mathcal{C})}} \sigma_{(M_\infty^-,A^*),\infty}$.

Lemma 58. *Let σ be the embedding above. Then $\sigma \in L[M_\infty, \Sigma \restriction X]$.*

Proof. Note that although σ is not in $L(\mathbb{R}^*,\mathcal{F})$; for any $A \in \mathrm{OD}^{L(\mathbb{R},\mathcal{C})}$ the partial embedding $\sigma_{(M^-,A^*),\infty}$ is in $L(\mathbb{R}^*,\mathcal{F})$ as it is one of the embeddings of the directed system $\mathcal{D}$ (as computed in $L(\mathbb{R}^*,\mathcal{F})$). Hence, it is enough to show that the iterates of M_∞^- given by Σ and trees in X are cofinal in the $\emptyset$-iterates of M_∞^- in $L(\mathbb{R}^*,\mathcal{F})$. For this, let $Q \in L(\mathbb{R}^*,\mathcal{F})$ be a $\emptyset$-iterate of M_∞^-. Then there is $p \in \mathrm{col}(\omega, <\lambda_{\omega^2}^{M_\infty})$ and a name $\dot{q}$ for Q such that

$$p \Vdash_{M_\infty} L(\dot{\mathbb{R}},\dot{\mathcal{F}}) \models \dot{q} \text{ is an } \emptyset\text{-iterate of } M_\infty^-$$

Let $\delta_i^{M_\infty}$ be such that p is an element of $\mathrm{col}(\omega,\delta_i^{M_\infty})$. Using the definability of forcing note that one can define in M_∞ an iteration tree that compares M_∞^- simultaneously with any $\dot{q}[H]$ (such that H is M_∞-generic with $p \in H$). This implies that there is a $\emptyset$-iterate of Q in $L[M_\infty, \Sigma \restriction X]$ as wanted. □

Recall that $\delta_0^{M_\infty}$ is the smallest Woodin cardinal of M_∞. We have the following

Lemma 59. $\delta_0^{M_\infty} = \Theta$

Proof. We first show that $\delta_0^{M_\infty} \leq \Theta$. Fix $\alpha < \delta_0^{M_\infty}$. Then there is a $(P,A) \in \mathcal{D}$ and an $\bar{\alpha} < \delta_0^P$ such that $\pi_{(P,A),\infty)}(\bar{\alpha}) = \alpha$. Consider

$$\tilde{A} = \{(x, H(Q,A)) \,|\, (P,A) \prec (Q,A) \text{ and } x \in H(Q,A)\}$$

Note that $\tilde{A}$ can be coded as a subset of the reals. For $(x, H(Q,A)) \in \tilde{A}$ let $f(x,H(Q,A)) = \pi_{(Q,A),\infty}(x)$. Then we have that $\alpha \subseteq \mathrm{range}(f)$, and f is a function on $\mathbb{R}$. This implies $\Theta > \alpha$.

Now we show that $\Theta \leq \delta_0^{M_\infty}$. Let $\alpha < \Theta$. Then there is a pre-wellorder on $\mathbb{R}$ of rank α. So, there is a formula φ and a sequence of ordinals s such that $\alpha = \{\beta \,|\, L(\mathbb{R},\mathcal{C}) \models$ "β is the unique ordinal such that $\varphi(s,x,\beta)$" for some $x \in \mathbb{R}\}$. Let us work in $V^{\mathrm{col}(\omega,(2^{\mathfrak{c}})^+)}$. We will use the construction done in Lemma 27, to get sequences $\langle M_i \,|\, i \in \omega\rangle$ and $\langle M_i^\omega \,|\, i \in \omega\rangle$ satisfying the following:

(1) $M_i \in \mathcal{D}^+$ and for any $M \in D^+$ there is a $k \in \omega$ such that $M \prec^+ M_k$,
(2) $M_i \prec^+ M_k$ for any two natural numbers $i < k$. In this case we will write $\pi^+_{i,k} : M_i \to M_k$ for the embedding given by Σ.
(3) For any $x \in \mathbb{R}^V$, x is M_k-generic for $\mathbb{B}^{M_k}_{\delta_0^{M_k}}$ for infinitely many k.
(4) M_i^ω is a Σ-iterate of M_i and there are embeddings (given by Σ)

$$\pi_i^\omega : M_i \to M_i^\omega$$

(5) There are embeddings $\pi^\omega_{i,k} : M_i^\omega \to M_k^\omega$, given by Σ, for any $i < k$, such that

$$\pi_k^\omega \circ \pi^+_{i,k} = \pi_i^\omega \circ \pi^\omega_{i,k}$$

(6) For any $i \in \omega$ there is an M_i^ω-generic filter for the collapse up to the supremum of the Woodins of M_i^ω, say G_i, such that $L(\mathbb{R}, \mathcal{C})$ is realized as the derived model given by G_i over M_i (see Section 2.3).

To construct such sequences let us fix a big enough γ and let $\langle X_i | i \in \omega\rangle$ be an elementary chain of substructures of H_γ^V, such that $P(\mathbb{R})^V \subseteq \bigcup_{i\in\omega} X_i$ and $\sigma_i = X_i \cap \mathbb{R} \in X_{i+1}$ and it is countable there. Let $\langle x_i \mid i \in \omega\rangle$ be an enumeration of $\mathbb{R}^V$ such that $x_i \in X_i$.

Let $\langle M_i \mid i \in \omega\rangle$ be such that it satisfies conditions (1), (2) and (3) above (note that this is possible in $V^{\mathrm{col}(\omega,(2^{\mathfrak{c}})^+)}$, where $\mathbb{R}^V$ is countable).

We construct M_i^ω inductively as follows (see the diagram below). By Lemma 26 there is a tree $\mathcal{T}_0^0$ on M_0 (based on the window $(\delta_0^{M_0}, \lambda_\omega^{M_0})$), with last model M_0^1, such that σ_0 is realized as the symmetric reals for an M_0^1-generic filter for $\mathrm{col}(\omega, < \lambda_\omega^{M_0^1})$. We let $\pi_0^1 : M_0 \to M_0^1$ be the iteration embedding.

Note, that there is an i such that $\mathcal{T}_0^0$ and M_1 are elements of X_i .Let i_1^0 be the least such i. Hence in $X_{i_1^0}$ we can copy $\mathcal{T}_0^0$ using $\pi^+_{0,1}$ to get the tree $\pi^+_{0,1}\mathcal{T}_0^0$ on M_1. Use Lemma 26 again to get a tree on the last model of $\pi_{0,1}\mathcal{T}_0^0$, with last model M_1^1, such that σ_0 the symmetric reals of an M_1^1 generic filter for $\mathrm{col}(\omega, < \lambda_\omega^{M_1^1})$. Let $\mathcal{T}_1^0$ be the concatenation of $\pi^+_{0,1}\mathcal{T}_0^0$ with the latter tree and $\pi_1^1 : M_1 \to M_1^1$ the tree embedding. By copying, we have also an embedding $\pi^1_{0,1} : M_0^1 \to M_1^1$ such that $\pi^1_{0,1} \circ \pi^1_0 = \pi^1_1 \circ \pi^+_{0,1}$ (note that we can do all this construction in $X_{i_1^0}$).

Note that there is $i_2^0 > i_1^0$ such that $\mathcal{T}_1^0$ and M_2 are in $X_{i_2^0}$. Using the copying construction together with Lemma 26 we get a tree $\mathcal{T}_2^0$ on M_2 with last model M_2^1 such that σ_0 can be realized as the symmetric reals for an M_2^1-generic filter for the collapse up to $\lambda_\omega^{M_2^2}$, together with embeddings satisfying $\pi_2^1 \circ \pi^+_{1,2} = \pi^1_{1,2} \circ \pi^1_1$ (where $\pi^1_{1,2} : M_1^1 \to M_2^1$ and $\pi_2^1 : M_2 \to M_2^1$ are the embeddings coming from the copying construction and iteration tree respectively).

We continue in this fashion inductively. So, for $i \in \omega$, we get M_i^1 an iterate of M_i, such that σ_0 can be realized as the symmetric reals of an M_i^1-generic filter for $\mathrm{col}(\omega, < \lambda_\omega^{M_i^1})$. We get also embeddings $\pi^1_{i,i+1}$ and π^1_i satisfying $\pi^1_{i,i+1} \circ \pi^1_i = \pi^1_{i+1} \circ \pi^+_{i,i+1}$.

Let $\mathcal{T}_0^1$ be the tree on M_0^1, with last model M_0^2, such that σ_1 are the symmetric reals for an M_0^2-generic filter for the collapse up to $\lambda_{2\omega}^{M_0^2}$. We can then copy this tree using $\pi^1_{0,1}$ and then apply Lemma 26 to get a tree $\mathcal{T}_1^1$ on M_1^1 with last model M_1^2, such that $\sigma_{i_1^0}$ can be realized as the symmetric reals associated to a M_1^2-generic filter for the collapse over $\mathrm{col}(\omega, <\lambda_{2\omega}^{M_1^2})$.

It is clear how to proceed inductively in this fashion. So for $i<\omega$ let M_i^ω be the direct limit of the models M_i^k and $\pi_i^\omega : M_i \to M_i^\omega$ the direct limit embedding. Note that by construction we get also embeddings $\pi_{i,i+1}^\omega : M_i^\omega \to M_{i+1}^\omega$. We define M_∞^ω to be the direct limit of the models M_i^ω and embeddings $\pi_{i,i+1}^\omega$. We call $\pi_{i,\infty}^\omega$ the direct limit embedding from M_i^ω into M_∞^ω (see the diagram below). Note that M_∞^ω embeds naturally into a Σ-iterate of M_∞ so it is wellfounded.

Now, by construction we have that $\mathrm{crit}(\pi_i^\infty) > \delta_0^{M_i}$. Hence $\pi_{i,i+1}^+ \restriction M_i^- = \pi_{i,i+1}^\omega \restriction M_i^-$.

Now let us go back to the proof. For $\beta<\alpha$ there is an $x\in\mathbb{R}$ such that

$$L(\mathbb{R},\mathcal{C}) \models \text{“}\beta \text{ is the unique ordinal such that } \varphi(s,x,\beta)\text{”}$$

Let $i\in\omega$ be large enough so that for all $k\geq i$ we have that $\pi_{k,k+1}^\omega(s)=s$ and $\pi_{k,k+1}^\omega(\beta)=\beta$. For $k\geq i$ let B_k be the set of ordinals defined by:

$$\eta \in B_k \text{ if, there is } p \in \mathbb{B}_{\delta_0}^{M_k^\omega},$$

such that

$$p \Vdash^{\mathbb{B}_{\delta_0}^{M_k^\omega}} \Vdash^{\mathrm{col}(\omega,<\lambda_{\omega^2}^{M_k^\omega})} L(\dot{\mathbb{R}},\dot{\mathcal{F}}) \models \text{“}\check{\eta} \text{ is the unique ordinal such that } \varphi(s,\dot{x},\check{\eta})\text{”}$$

Here $\dot{x}$ is the name for the generic object associated to the extender algebra $\mathbb{B}_{\delta_0}^{M_k^\omega}$, $\dot{\mathbb{R}}$ and $\dot{\mathcal{F}}$ are the symmetric reals and the tail filter associated to the collapse respectively. By construction we have that $\mathbb{R}$ and $\mathcal{C}$ can be realized as $\dot{\mathbb{R}}[G]$ and $\dot{\mathcal{F}}[G]$, for some M_k^ω-generic G.

$$\begin{tikzcd}
M_0 \arrow[r, "\pi_{1,0}^+"] \arrow[d, "\pi_0^1"] \arrow[dddd, bend right, "\pi_0^\infty"'] & M_1 \arrow[r, "\pi_{1,2}^+"] \arrow[d, "\pi_1^1"] & M_2 \arrow[r] \arrow[d, "\pi_2^1"] & \cdots \arrow[r] & M_i \arrow[r] \arrow[d, "\pi_i^1"] & \cdots \arrow[r] & M_\infty \arrow[d] \\
M_0^1 \arrow[r, "\pi_{0,1}^1"] \arrow[d, "\pi_0^2"] & M_1^1 \arrow[r, "\pi_{1,2}^1"] \arrow[d, "\pi_1^2"] & M_2^1 \arrow[r] \arrow[d, "\pi_2^2"] & \cdots \arrow[r] & M_i^1 \arrow[r] \arrow[d, "\pi_i^2"] & \cdots \arrow[r] & M_\infty^1 \arrow[d] \\
M_0^2 \arrow[r, "\pi_{0,1}^2"] \arrow[d] & M_1^2 \arrow[r, "\pi_{1,2}^2"] \arrow[d] & M_2^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & M_i^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & M_\infty^2 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \cdots & \vdots \arrow[d] & \vdots & \vdots \arrow[d] \\
M_0^\omega \arrow[r, "\pi_{0,1}^\omega"] \arrow[rrrrrr, bend right, "\pi_{0,\infty}^\omega"'] & M_1^\omega \arrow[r, "\pi_{1,2}^\omega"] & M_2^\omega \arrow[r] & \cdots \arrow[r] & M_i^\omega \arrow[r] & \cdots \arrow[r] & M_\infty^\omega
\end{tikzcd}$$

Intuitively B_k is the set of "all possible values for β". Note that as $\mathbb{B}_{\delta_0}^{M_k}$ is $\delta_0^{M_k}$-c.c. we have that B_k has order type less than $\delta_0^{M_k}$ and if x is generic for this extender algebra, we have that $\beta\in B_k$. For β and k as above define $\gamma(\beta,k)$ to be the unique γ such that β is the γ-th element of B_k and define $\xi(\beta)=\pi_{k,\infty}^+(\gamma(\beta,k))$

Claim 1: $\beta \mapsto \xi(\beta)$ is well defined (i.e. independent of k)

PROOF OF CLAIM 1. Let M_{k_0} and M_{k_1} be two candidates for the definition of $\xi(\beta)$. Without loss we may assume $M_{k_0} \prec^+ M_{k_1}$, also by elementarity and the fact that β gets fixed by $\pi^+_{k_0,k_1}$ we have that $\pi^+_{k_0,k_1}(\gamma(\beta,k_0))$ is the unique γ such that β is the γ-th element of B_{k_1}, that is

$$\pi^+_{k_0,k_1}(\gamma(\beta,k_0)) = \gamma(\beta,k_1)$$

Which in turn implies that $\xi(\beta)$ does not depend on k.

□

To finish note that for $\beta_0 < \beta_1 < \alpha$ we have $\xi(\beta_0) < \xi(\beta_1) < \delta_0^{M_\infty}$, this implies $\alpha \le \delta_0^{M_\infty}$. Since α was an arbitrary ordinal below Θ we conclude $\Theta \le \delta_0^{M_\infty}$. □

Lemma 60. *Suppose that N_0 and N_1 are countable Σ-iterates of $\mathcal{M}^{\sharp}_{\omega^2}$, G_i is N_i-generic for $col(\omega, \lambda^{N_i}_{\omega^2})$ and $L(\mathbb{R}_i, \mathcal{F}_i)$ are the associated derived models in the sense of 2.3.*

Then given $x \in \mathbb{R}_0 \cap \mathbb{R}_1$ we have

$$\langle L(\mathbb{R}_0,\mathcal{F}_0), x, T^0_n\rangle \equiv \langle L(\mathbb{R}_1,\mathcal{F}_1), x, T^1_n\rangle,$$

where T^i_n is the theory of n indiscernibles for $L(\mathbb{R}_i, \mathcal{F}_i)$.

PROOF. Fix x as in the hypotheses. Then there exist $k \in \omega^2$ such that for $i = 0, 1$ we have $x \in \mathbb{R}^{N_i[G_i \restriction \delta^{N_i}_k]}$. Fix $c_0 < c_1 < \cdots < c_{n-1}$ indiscernibles for $L(\mathbb{R}_0, \mathcal{F}_0)$, $L(\mathbb{R}_1, \mathcal{F}_1)$ and $L(\mathbb{R}, \mathcal{C})$. Let φ be a formula and assume $L(\mathbb{R}_0, \mathcal{F}_0) \models \varphi(x, c_0, \ldots, c_{n-1})$. We will see that $L(\mathbb{R}, \mathcal{C})$ satisfies the same formula. By homogeneity of the collapse we have that

$$\Vdash^{\mathrm{col}(\omega,<\lambda^{N_0}_{\omega^2})}_{N_0[G\restriction\delta^{N_0}_k]} L(\dot{\mathbb{R}}, \dot{\mathcal{F}}) \models \varphi(\check{x}, \check{c}_0, \ldots, \check{c}_{n-1})$$

Now in $V^{\mathrm{col}(\omega,(2^{\mathfrak{c}})^+)}$ we can iterate N_0 above $\delta^{N_0}_k$ to realize $L(\mathbb{R}, \mathcal{C})$ as a derived model (see Lemma 45). Picking $c_0, \ldots c_n$ large enough we get that

$$L(\mathbb{R}, \mathcal{C}) \models \varphi(x, c_0, \ldots, c_{n-1})$$

By symmetry of the argument we cannot have $L(\mathbb{R}_1, \mathcal{F}_1) \models \neg\varphi(x, c_0, \ldots, c_{n-1})$, which completes the proof. □

We have the following HOD analysis result.

Theorem 61. *Suppose $\mathcal{M}^{\sharp}_{\omega^2}$ exists and its iteration strategy is $(2^{\mathfrak{c}})^+$-universally Baire. Then the following are the same model.*

(1) $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$
(2) $L[M_\infty, \Sigma \restriction X]$
(3) $L[M_\infty, \sigma]$

PROOF. Note that by the discussion after Lemma 57 we have that $L[M_\infty, \Sigma \restriction X] \subseteq \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}$. By Lemma 58 we have that $L[M_\infty, \sigma] \subseteq L[M_\infty, \Sigma \restriction X]$. So, it is enough to show that $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})} \subseteq L[M_\infty, \sigma]$. By Theorem 3.1 in [**25**] we have that $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})} = L[B]$ for some $B \subset \Theta^{L(\mathbb{R},\mathcal{C})}$ definable in $L(\mathbb{R}, \mathcal{C})$. Let us fix φ a formula such that

$$L(\mathbb{R}, \mathcal{C}) \models \varphi(\beta) \text{ if and only if, } \beta \in B$$

We will show that $B \in L[M_\infty, \sigma]$. Let $\beta < \Theta$, then by lemmas 52, 53 and 59 there exists P a suitable initial segment of a Σ-iterate of $\mathcal{M}^{\sharp}_{\omega^2}$ such that $\alpha(P) = 1$ and for some natural number n, there is $\bar{\beta} \in H(P, T_n)$ such that $\pi_{(P,T_n),\infty}(\bar{\beta}) = \beta$. Let us define $\psi(u, v, w)$ to be the formula

$$\text{“}(u, v) \in \mathcal{D} \text{ and } w \in u \text{ and } \varphi(\pi_{(u,v),\infty}(w))\text{”}$$

We have

$$\begin{aligned} L(\mathbb{R}, \mathcal{C}) \models \varphi(\beta) \text{ iff } & L(\mathbb{R}, \mathcal{C}) \models \psi(P, T_n, \bar{\beta}) \\ \text{iff } & L(\mathbb{R}^*, \mathcal{F}) \models \psi(P, T_n^*, \bar{\beta}), \text{ by Lemma 60} \\ \text{iff } & L(\mathbb{R}^*, \mathcal{F}) \models \psi(M_\infty^-, T_n^*, \beta), \text{ factoring through } M_\infty^- \\ \text{iff } & L(\mathbb{R}^*, \mathcal{F}) \models \varphi(\sigma(\beta)) \end{aligned}$$

But then $L[M_\infty, \sigma]$ can compute $L(\mathbb{R}^*, \mathcal{F}) \models \varphi(\sigma(\beta))$ using the homogeneity of the collapse. Hence $B \in L[M_\infty, \sigma]$ □

Also for an arbitrary $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact we can define the corresponding $\mathcal{D}_\mu$. Let $M_{\infty,\mu}$ be its direct limit (see for example Theorem 3.13. and subsequent discussion in [**25**]). Furthermore by [**25**] we have the following result.

Theorem 62. *Suppose $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then*

$$\mathrm{HOD}^{L(\mathbb{R},\mu)} = L[M_{\infty,\mu}, \Sigma_\mu]$$

where Σ_μ is defined in $L(\mathbb{R}, \mu)$ using the corresponding definition given in Lemma 57.

Let us fix a μ as in the discussion above from now on. Note that the construction recovering HOD can be relativized to any particular real y as follows. The existence of $\mathcal{M}^\sharp_{\omega^2}$ implies the existence of $\mathcal{M}^\sharp_{\omega^2}(y)$ and so one has $\mathrm{HOD}_y^{L(\mathbb{R},\mathcal{C})} = L[M_{\infty,\mu}(y), \Sigma_\mu(y)]$, where $M_{\infty,\mu}(y)$ is the direct limit of

$$D_\mu(y) = \{H(P, A) \mid P \text{ is a strongly } A\text{-iterable } y\text{-mouse and } A \in \mathrm{OD}_y^{L(\mathbb{R},\mathcal{C})}\}.$$

And $\Sigma_\mu(y)$ is the strategy whose domain consists of finite full stacks of trees on $M^-_{\infty,\mu}(y)$ that are in

$$M_{\infty,\mu}(y)|(\lambda_{\omega^2}^{M_{\infty,\mu}(y)})$$

and $\Sigma_\mu(y)$ picks branches b such that respect every A^* for $A \in \mathrm{OD}_y^{L(\mathbb{R},\mu)}$. We define A^* in $L(\mathbb{R}^*, \mathcal{F})$, the model given by a generic filter over $M_{\infty,\mu}$ for the collapse up to the sup of its Woodins. The crux of the main theorem of this section is the following observation.

Note that by Lemma 37 we have that $\mathcal{P}_\mathcal{C}(\mathbb{R}) = \mathcal{P}_\mu(\mathbb{R})$. This implies that the notion of suitability is the same in $L(\mathbb{R}, \mathcal{C})$ and $L(\mathbb{R}, \mu)$. The notion of ordinal definability might however be different. Define $\mathcal{T}_n^\mu$ to be the theory of n many indiscernibles of $L(\mathbb{R}, \mu)$.

We have that for any n there is k such that $\mathcal{T}_n^\mathcal{C} \leq_W \mathcal{T}_k^\mu$, and vice versa, for any n there is a k such that $\mathcal{T}_n^\mu \leq_W \mathcal{T}_k^\mathcal{C}$. From now on let us fix a real number x that codes all of these reductions in a natural way[2]. If P is a suitable initial segment

[2] Fix $z \mapsto \langle (z)_i \rangle_{i\in\omega}$ a recursive bijection between $\mathbb{R}$ and $\mathbb{R}^\omega$ and fix x such that given $n \in \omega$ there exists i and j naturals such that $(x)_i$ codes a continuous reduction witnessing $\mathcal{T}_n^\mu \leq_W \mathcal{T}_k^\mathcal{C}$ (for some k) and the similarly $(x)_j$ codes a reduction $\mathcal{T}_n^\mathcal{C} \leq_W \mathcal{T}_k^\mu$ (fore some other k).

of a Σ_x-iterate of $\mathcal{M}^\sharp_{\omega^2}(x)$ then by Lemmas 45 and 51 we have that P is strongly $\mathcal{T}^{\mathcal{C}}_n$-iterable. Furthermore as $x \in P$ by Lemma 5.9 of [**17**] we have that P captures $\mathcal{T}^\mu_n$ for every natural n. Moreover we have that if x codes a reduction $\mathcal{T}^\mu_n \leq_W \mathcal{T}^{\mathcal{C}}_k$ then for any $i < o(P)$, $\tau^P_{\mathcal{T}^\mu_n, i} \in H(P, \mathcal{T}^{\mathcal{C}}_k)$ and moreover every $\mathcal{T}^{\mathcal{C}}_k$-iteration of P is also a $\mathcal{T}^\mu_n$ iteration. The following lemma will show that as in the case of $L(\mathbb{R}, \mathcal{C})$ the pairs of the form $(P, \mathcal{T}^{\mathcal{C}}_n)$ are dense in $\mathcal{D}_\mu$ in the sense of Lemma 53. In other words.

Lemma 63. *Suppose $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Let A be $\mathrm{OD}^{L(\mathbb{R},\mu)}_x$ and P is an x-mouse that is A-iterable. Then there is a natural number n and a suitable initial segment of a correct iterate of $\mathcal{M}^\sharp_{\omega^2}(x)$, say Q, that is $A \oplus \mathcal{T}^{\mathcal{C}}_n$-iterable, $\tau^Q_A \in H(Q, \mathcal{T}^{\mathcal{C}}_n)$ and is an A-iterate of P.*

PROOF. Here just note that $\{\mathcal{T}^{\mathcal{C}}_n \mid n \in \omega\}$ is Wadge cofinal in the Wadge hierarchy of $L(\mathbb{R}, \mu)$. Also for every n we have that $\mathcal{T}^{\mathcal{C}}_n$ is $\mathrm{OD}^{L(\mathbb{R},\mu)}_x$. We can then apply Lemma 54 and comparison to get the desired Q. □

Theorem 64. *Suppose that $L(\mathbb{R}, \mu) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. Then for a Turing cone of $y \in \mathbb{R}$ we have that $\mathrm{HOD}^{L(\mathbb{R},\mu)}_y = \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}_y$.*

PROOF. Using the previous Lemma, the proof of Theorem 55 follows in the same way giving that $M_{\infty,\mu}(x) = M^+_\infty(x) | \lambda^{M^+_\infty(x)}_{\omega^2}$. But then Theorem 55 gives that $M_\infty(x) = M_{\infty,\mu}(x)$. Hence the covering limits agree. Now, we turn to see that the strategies agree as well.

Claim: $\Sigma_{\mu,x} = \Sigma_x$ when restricted to the relevant trees [3].

PROOF OF THE CLAIM. We will prove inductively that if $\vec{\mathcal{T}}$ is a stack of n trees, and is according to both Σ_x and $\Sigma_{\mu,x}$ then these strategies pick the next branch the same way. Note that by the definitions of Σ_x and $\Sigma_{\mu,x}$ we have that $\Sigma_x(\vec{\mathcal{T}}) = b$ if and only if $\vec{\mathcal{T}}^\frown b$ is an $\mathcal{T}^{\mathcal{C}*}_n$-iteration on M^-_x for all $n \in \omega$ (here again the key fact is that the $\xi^{M^-_x}_{\mathcal{T}^{\mathcal{C}}_n}$ and the $\xi^{M^-_x}_{\mathcal{T}^\mu_n}$ are cofinal in $\delta^{M^-_x}_o$). As we noted above this means that $\vec{\mathcal{T}}^\frown b$ is a $\mathcal{T}^{\mu*}_n$-iteration for all $n \in \omega$, in other words $\Sigma_{\mu,x}(\vec{\mathcal{T}}) = b$, which finishes the proof of the claim. □

But then this implies $\mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}_x = \mathrm{HOD}^{L(\mathbb{R},\mu)}_x$ by Theorem 64. Note also that if $y \geq_T x$ then the analogous results relative to y is still valid. This completes the proof.

□

PROOF OF THEOREM 1. First lets suppose that Σ is $(2^{\mathfrak{c}})^+$-universally Baire, so all the previous results of this section hold. By Theorem 64 we can fix a real x such that $\mathrm{HOD}^{L(\mathbb{R},\mu)}_x = \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}_x$. Then, by Theorem 24 we have that

$$L(\mathbb{R}, \mu) = \mathrm{HOD}^{L(\mathbb{R},\mu)}_x(\mathbb{R}) \text{ and } \mathrm{HOD}^{L(\mathbb{R},\mathcal{C})}_x(\mathbb{R}) = L(\mathbb{R}, \mathcal{C}),$$

which clearly implies $L(\mathbb{R}, \mathcal{C}) = L(\mathbb{R}, \mu)$.

Now, if Σ is just an $\omega_1 + 1$-iteration strategy, but not necessarily universally Baire. Pick γ such that V_γ reflects enough set theory, and let $N \prec V_\gamma$ be countable and H its transitive collapse. Then we are in the same situation as when proving

[3] We refer as Σ_x the strategy given by Lemma 57 and $\Sigma_{\mu,x}$ the one given in $L(\mathbb{R}, \mu)$.

Proposition 25. Hence the result follows word by word from the proof of Proposition 25. $\square$

CHAPTER 4

The $\mathrm{AD}+\mathrm{DC}_{\mathbb{R}}$ Case

We give in this Chapter a proof of Theorem 4. We will first assume AD^+ and for contradiction suppose that the theorem does not hold and then we reflect this statement to a Suslin co-Suslin set. Then we can use [**22**] and [**10**] to construct models with Woodin cardinals and run a version of the last chapter's arguments. We start by noting some preliminary facts. Lastly we show how to reduce the hypotheses to $\mathrm{AD}+\mathrm{DC}_{\mathbb{R}}$

Lemma 65. *Suppose $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$ and let μ be a filter such that $L(\mathbb{R},\mu)$ satisfies $\mathrm{AD}+\ \omega_1$ is $\mathbb{R}$-supercompact. Then $\mathcal{P}_\mu(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.*

Proof. Otherwise we have that $V = L(\mathcal{P}(\mathbb{R}))$ believes there is a supercompact measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Also $V = L(\mathbb{R},\mu)$, so by Theorem 9.103. of [**29**] (Theorem 15 here) $L(\mathbb{R},\mu) \models \mathrm{AD}_{\mathbb{R}}$ but this is impossible since we have $L(\mathbb{R},\mu) \models \Theta = \theta_0$. $\square$

From now on we will also assume that $V \models \Theta = \theta_0$, as otherwise there exists a non-tame mouse and hence $\mathcal{M}^{\sharp}_{\omega^2}$ exists and it is iterable so the results of last section would hold. Since $\Theta = \theta_0$ we have that, in particular, DC holds in V. We now prove the first approximation to our main result

Theorem 66. *Suppose $V = L(\mathcal{P}(\mathbb{R})) + \mathrm{AD}^+$. Then there is at most one model of the form $L(\mathbb{R},\mu)$ satisfying $\mathrm{AD}+\ \omega_1$ is $\mathbb{R}$-supercompact. Moreover if such model exists then the unique such model is $L(\mathbb{R},\mathcal{C})$ where $\mathcal{C}$ is the club filter on $\mathcal{P}_{\omega_1}(\mathbb{R})$.*

Proof. Suppose that there is $\mu \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ such that $L(\mathbb{R},\mu) \models \mathrm{AD}+$ ω_1 is $\mathbb{R}$-supercompact. Let μ be chosen such that $\mathcal{P}_\mu(\mathbb{R})$ is the minimal (in that given any ν such that $L(\mathbb{R},\nu) \models \mathrm{AD}+\ \omega_1$ is $\mathbb{R}$-supercompact then we have that $\mathcal{P}_\mu(\mathbb{R}) \subseteq \mathcal{P}_\nu(\mathbb{R})$). Note that by Lemma 65 we have that there is a set of reals A such that $L(\mathbb{R},\mu)$ is definable from parameters in $L(\mathbb{R},A)$ and moreover by AD^+ we have $(\mathbb{R},\mu)^\sharp \in L(\mathbb{R},A)$. Now by Theorem 11 and minimality of μ we may assume that A is Suslin and co-Suslin.

Let us work from now on in $L(\mathbb{R},A)$. By minimality of μ we get that $(\mathbb{R},\mu)^\sharp$ is Suslin and co-Suslin in $L(\mathbb{R},A)$. The presence of $(\mathbb{R},\mu)^\sharp$ implies trivially the existence of $\mathcal{N} = (M_{\infty,\mu})^\sharp$. Here we identify $\mathcal{N}$ with the least active mouse extending $M_{\infty,\mu}$. Let Γ be $\Sigma_1^{L(\mathbb{R},\mu)}$ and $\vec{B}$ a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Let us fix ζ to be the largest Suslin cardinal in $L(\mathbb{R},A)$.

Claim 1: $\mathbf{Env}(\Gamma) \subset Lp(\mathbb{R})$

Proof of Claim 1: Let $B \in Env(\Gamma)$. First, note that B is in $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$. So, for any $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ we have that $B \cap \sigma \in M_\sigma$ where $M_\sigma = \mathrm{HOD}^{L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))}_{\{B,\sigma\}\cup\sigma}$.

By the definition of **Env** we have that $B \cap \sigma \in \mathrm{OD}^{L(\mathbb{R},\mu)}_{\{\sigma,A\}\cup\sigma}$, and so by mouse capturing in $L(\mathbb{R},\mu)$ we have that $B \cap \sigma \in Lp(\sigma)$. Define $\mathcal{M}_\sigma \lhd Lp(\sigma)$ to be the least initial segment of $Lp(\sigma)$ having $B \cap \sigma$ as an element. Note that $\mathcal{M}_\sigma \in M_\sigma$ and $M_\sigma \models$ "$\mathcal{M}_\sigma$ is countably iterable" because the unique iteration strategy for $\mathcal{M}_\sigma$ is definable.

Also the club filter $\mathcal{C}$ is an ultrafilter on $\mathcal{P}_\zeta(\mathbb{R})$. So, we can define $M = \prod_{\sigma\in\mathcal{P}_{\omega_1}} M_\sigma/\mathcal{C}$, where the functions of this ultraproduct are $f : \mathcal{P}_{\omega_1}(\mathbb{R}) \to \prod_{\sigma\in\mathcal{P}_{\omega_1}} M_\sigma$ and $f \in L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$. Note that by [**24**], $\mathcal{C}$ is normal and countably complete. Then we have that Σ_1-Łos holds, since $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$ satisfies Σ_1-replacement. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_\mathcal{C}$; we claim that M believes "$\mathcal{M}$ is countably iterable". To see this let $\bar{\mathcal{M}}$ be a countable transitive hull of $\mathcal{M}$, then we have that $\bar{\mathcal{M}} \in \sigma$ for club-many σ. Also $[\sigma \mapsto \bar{\mathcal{M}}]_\mathcal{C} = \bar{\mathcal{M}}$ (by countable completeness of $\mathcal{C}$). Now by Σ_1 -Łos we have that for club-many σ, $\bar{\mathcal{M}}$ is a countable hull of $\mathcal{M}_\sigma$ and so $M_\sigma \models$ "$\bar{\mathcal{M}}$ is ω_1-iterable". Let Σ_σ be the unique iteration strategy of $\bar{\mathcal{M}}$, then the function $\sigma \mapsto \Sigma_\sigma$ is in $L_\zeta(\mathcal{P}_\zeta(\mathbb{R}))$ and is such that $M_\sigma \models (HC, \Sigma_\sigma)$" $\models \Sigma_\sigma$is an ω_1 strategy for $\bar{\mathcal{M}}$". By Łos, again, we get that $M \models$ "$\bar{\mathcal{M}}$ is ω_1-iterable".

Also, $B = [\sigma \mapsto B \cap \sigma]_\mathcal{C}$ hence $B \in \mathcal{M}$. Note that in $L(\mathbb{R}, A)$, $\mathcal{M}$ is actually countably iterable, so we have $\mathcal{M} \lhd Lp(\mathbb{R})$ and so $B \in Lp(\mathbb{R})$. □

Arguing as in the proof of the Claim in Lemma 37 we then get that $\mathbf{Env}(\Gamma) = \mathcal{P}_\mu(\mathbb{R})$. Let $\vec{B}$ be a self-justifying system sealing $\mathbf{Env}(\Gamma)$. Recall that $\mathcal{N}$ captures every B in $\vec{B}$, say via τ_B. Define then

$$\mathcal{M} = Hull^{\mathcal{N}}(\{\tau_B^{\mathcal{N}} \mid B \in \vec{B}\}).$$

Here we think of $\mathcal{M}$ as the transitive collapse of this Hull. Then we have that $\mathcal{M}$ is $\omega_1 + 1$ iterable and so $\mathcal{M}^\sharp_{\omega^2}$ exists and is $\omega_1 + 1$-iterable.

Claim 2: $L(\mathbb{R}, \mathcal{C})$ is a model of AD + ω_1 is $\mathbb{R}$-supercompact and the only such model.

Proof of Claim 2. Here we use the results of Section 2. The key point is that the iteration strategy for $\mathcal{M}^\sharp_{\omega^2}$ might not extend to big generic collapses. For this though we use instead a countable elementary substructure of $L_\alpha(\mathbb{R}, A)$, where α is such that $L_\alpha(\mathbb{R}, A)$ reflects enough set theory. Let $N \prec L_\alpha(\mathbb{R}, A)$ be countable and elementary such that $\mathcal{M}^\sharp_{\omega^2} \in N$ (here we use that DC holds in V). Let $\bar{H}$ be the transitive collapse of N. Then as in the proof of Proposition 7 the results of Section 2 give that $\bar{H}$ models "$L(\mathbb{R}, \mathcal{C})$ satisfies AD + ω_1 is $\mathbb{R}$-supercompact", but then N does and so does V.

The same argument combined with the results of Section 3 will show that since $\mathcal{M}^\sharp_{\omega^2}$ exists $L(\mathbb{R}, \mathcal{C})$ is the unique model of AD + ω_1 is $\mathbb{R}$-supercompact. This concludes the proof. □

□

Let us mention that the key fact about AD^+ we used in the proof of Theorem 66 is that given μ such that $L(\mathbb{R}, \mu) \models$ AD $+\omega_1$ is $\mathbb{R}$-supercompact, then one can reflect the existence of such a μ to the Suslin co-Suslin part of a model of the form $L(\mathbb{R}, A)$, where A is a set of reals. This is particularly useful as then one can take ultraproducts using the club filter. In the absence of AD^+ this can be a little bit more tricky, but we show how to overcome this difficulty and get the proof of the result under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$.

PROOF OF THEOREM 4. First let us assume AD^+ holds, and then we will use this proof to get a proof under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$. Suppose that there are μ and ν such that $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ are models of $\mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact. We may assume with no loss that $V = L(\mathbb{R}, \mu, \nu)$ and $\Theta = \theta_0$, as otherwise there is a non-tame mouse [1].

Note that the proof of Lemma 37 holds in this case too, so $\mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\nu(\mathbb{R})$.

Claim: $\mathcal{P}(\mathbb{R})$ is strictly larger that $\mathcal{P}_\mu(\mathbb{R})$.

PROOF OF THE CLAIM. Otherwise we have that $\mathcal{P}(\mathbb{R}) = \mathcal{P}_\mu(\mathbb{R}) = \mathcal{P}_\nu(\mathbb{R})$. We can fix then an $\mathrm{OD}^{L(\mathcal{P}(\mathbb{R}))}$ tree T that projects to a universal Σ^2_1. Following [**25**] we let $\mathbb{D} = \{\langle d_i \,|\, i \in \omega\rangle \,|\, \forall i \in \omega \; d_i \text{ is a } \Sigma^2_1 \text{ degree and } d_i < d_{i+1}\}$. We recall in the following lines the definition of the auxiliary measures $\bar{\mu}$ and $\bar{\nu}$ on $\mathbb{D}$.

For $A \subseteq \mathbb{D}$, let $S \subset \mathrm{ON}$ be an ∞-Borel code for A, then

$$A \in \bar{\mu} \text{ iff } \forall^*_\mu \sigma L[T,S](\sigma) \models \text{“}\mathrm{AD}^+ + \sigma = \mathbb{R} \text{ and } \exists(\emptyset, U) \in \bar{\mathbb{P}}(\emptyset, U) \Vdash \dot{G} \in \mathcal{A}_S\text{”}$$

Where $\bar{\mathbb{P}}$ is the usual Prikry forcing using Σ^2_1-degrees in $L[S,T](\sigma)$ and the Martin measure (see section 6.3 of [**3**]), also $\dot{G}$ is the name of the corresponding Prikry sequence and $\mathcal{A}_S$ is the interpretation of the set of reals coded by S.

By results of [**25**] we have:

- For any $S \subset \mathrm{ON}$ we have that $\forall^*_\mu \sigma L[T,S](\sigma) \models$ “$\mathrm{AD}^+ + \sigma = \mathbb{R}$”.
- Whether $A \in \bar{\mu}$ does not depend on the code S.
- Let $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ and for $d \in \mathbb{D}$ let

  $$\sigma_d = \{y \,|\, \text{there are } i \text{ and } x \text{ such that } y \leq d(i)\}.$$

  Then we have that if $\bar{A} = \{d \in \mathbb{D} \,|\, \sigma_d \in A\}$

  $$A \in \mu \text{ if and only if } \bar{A} \in \bar{\mu}.$$

Let us recall the construction of the Prikry Forcing done in Section 2 of [**25**]; we however, will alternate using μ and ν when choosing measure one sets. More precisely, given $X \subseteq \mathbb{D}^{n+1}$ we say $X \in \mathcal{U}_n$ if

$$\forall^*_{\bar{\mu}} \vec{z}(0) \forall^*_{\bar{\nu}} \vec{z}(1) \cdots \forall^*_{\bar{\mu}} \vec{z}(n-1) \forall^*_{\bar{\nu}} \vec{z}(n) \left(\langle \vec{z}(i) \,|\, i < n+1\rangle \in X\right)$$

We also define $\mathbb{P}$ as follows. Conditions will be pairs $(p, \vec{U})$, with $\vec{U}(n) \in \mathcal{U}_n$ for all $n \in \omega$ and such that $p = \langle \vec{d}_i \,|\, i < n\rangle$ is a sequence of elements in $\mathbb{D}$, such that $\vec{d}_i$ is in $L[x,T]$ for any (all) $x \in \vec{d}_{i+1}(0)$ and it is countable there. We say $(q, \vec{W}) \leq_{\mathbb{P}} (p, \vec{U})$ if $q = p^\frown r$ and $r^\frown s \in \vec{U}(n+k)$ for all k, and $s \in \vec{W}(k)$. As in Section 6 of [**3**] we will have that $\mathbb{P}$ has the Prikry property, which is to say that given a forcing statement Φ and a condition $(p, \vec{U}) \in \mathbb{P}$, there is $\vec{W}$ such that $(p, \vec{W})$ decides Φ. We summarize the facts of this forcing that we will use (see [**25**]).

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[1] Here $L(\mathbb{R}, \mu, \nu)$ is constructed by induction as follows. $L_0(\mathbb{R}, \mu, \nu) = \mathbb{R}$, for $\alpha \in \mathrm{ON}$ we let $L_{\alpha+1}(\mathbb{R}, \mu, \nu)$ be the collection definable sets over $(L_\alpha(\mathbb{R}, \mu, \nu), \in, \nu \cap L_\alpha(\mathbb{R}, \mu, \nu), \mu \cap L_\alpha(\mathbb{R}, \mu, \nu))$ and taking unions at limit stages.

- For a given set a that admits a well order rudimentary in a, there is a cone of reals x such that $\mathrm{HOD}_{T,a}^{L[T,x]} \models \omega_2^{L[T,x]}$ is Woodin. For a real x we let $\delta(x) = \omega_2^{L[T,x]}$. And for a Σ_1^2-degree d, we let $\delta(d) = \delta(x)$ for any (all) $x \in d$.
- Given $\langle \vec{d}_i \mid i < n\rangle \in \mathbb{D}^n$, we let
$$Q_0(\vec{d}) = \mathrm{HOD}_{\vec{d}_0,T}^{L[\vec{d}_0,T]} \mid \sup\{\delta(d_0(n)) \mid n \in \omega\},$$
and
$$Q_{i+1}(\vec{d}) = \mathrm{HOD}_{Q_i,\vec{d}_{i+1},T}^{L[T,\vec{d}_{i+1}]} |(\sup\{\delta(d_{i+1}(n)) \mid n \in \omega\}).$$
- Given G generic for $\mathbb{P}$ define $g = \bigcup\{p \mid (p, \vec{U}) \in G \text{ for some } \vec{U}\}$. Let $Q_i(g)$ be $Q_i(g \restriction i)$. Then $L[\cup_{i\in\omega} Q_i(g), T]$ has ω^2 many Woodin cardinals.
- If $\sigma_i = \{x \mid \exists i, n(x \in \vec{d}_i(n))\}$ then the tail filter $\mathcal{F}$ generated by σ_i is such that $L(\mathbb{R}, \mathcal{F}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact.

Let us fix G a V-generic filter for $\mathbb{P}$ and let $\mathcal{F}$ be its associated tail filter. We claim that $L(\mathbb{R}, \mu) = L(\mathbb{R}, \mathcal{F}) = L(\mathbb{R}, \nu)$. For this, suppose that $A \in \mathcal{F} \cap V$, we will show $A \in \mu$. Otherwise we have $A \notin \mu$, let $(p, \vec{U}) \Vdash A \in \mathcal{F}$. Let $\vec{W}$ be defined as $\vec{W}(2n) = \vec{U}(2n) \cap \mathbb{D} \setminus \bar{A}$, and $\vec{W}(2n+1) = \vec{U}(2n+1)$ for $n \in \omega$ (here $\bar{A}$ is the translation to of A to $\mathbb{D}$ as defined before). But then it is clear that $(p, \vec{W}) \Vdash A \notin \mathcal{F}$, a contradiction. Hence $L(\mathbb{R}, \mu) = L(\mathbb{R}, \nu)$ and so $V = L(\mathbb{R}, \mu)$ which is impossible. $\square$

Hence $\mathcal{P}(\mathbb{R})$ is strictly larger than $\mathcal{P}_\mu(\mathbb{R})$, and we can choose $A \subseteq \mathbb{R}$ such that $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ are definable (from parameters) in $L(\mathbb{R}, A)$ and hence the result follows from Theorem 66.

Now, assume AD^+ does not hold, then we have that $\mathcal{P}_\mu(\mathbb{R})$ is strictly smaller than $\mathcal{P}(\mathbb{R})$ (because AD^+ holds in $L(\mathbb{R}, \mu)$). Let Γ be $\{A \subset \mathbb{R} \mid L(\mathbb{R}, A) \models \mathrm{AD}^+\}$ by Theorem 9.14. of [**29**] we have that $L(\mathbb{R}, \Gamma) \models \mathrm{AD}^+$. We have two cases. If Γ strictly contains $\mathcal{P}_\mu(\mathbb{R})$, then we have that $L(\mathbb{R}, \mu)$ is definable from parameters in $L(\mathbb{R}, \Gamma)$ and hence one can work in $L(\mathbb{R}, \Gamma)$ and the theorem follows from Theorem 66.

If $\Gamma = \mathcal{P}_\mu(\mathbb{R})$, then $\Gamma \neq \mathcal{P}(\mathbb{R})$ and, by Theorem 9.14 of [**29**] again, we get $L(\mathbb{R}, \Gamma) \models \mathrm{AD}_{\mathbb{R}}$, and so $L(\mathbb{R}, \mu) \models \mathrm{AD}_{\mathbb{R}}$, which is a contradiction.

$\square$

CHAPTER 5

The ZFC case without $\mathcal{M}^\sharp_{\omega^2}$

This chapter can be seen as a continuation of Chapter 3. We remind the reader that in this case M^+_∞ does not exists as the existence of $\mathcal{M}^\sharp_{\omega^2}$ is not assumed. However, if $L(\mathbb{R},\mu)$ is a model of AD $+\omega_1$ is $\mathbb{R}$-supercompact then $\mathrm{HOD}^{L(\mathbb{R},\mu)} = L[M_{\infty,\mu}, \Sigma_\mu]$ (see Theorem 61), where $M_{\infty,\mu}$ is the direct limit of the internal system $\mathcal{D}_\mu$, with partial order $\preceq$ and maps $\pi^\mu_{(P,A)(Q,B)}$. For technical reasons that will be clear later we will consider subsystems of this system. We say that a set $\mathfrak{B}$ of $\mathrm{OD}^{L(\mathbb{R},\mu)}$ sets of reals is *Wadge cofinal in* $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$ if and only if for every $A \in \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$ there is a $B \in \mathfrak{B}$ with $A \leq_W B$. $M_{\infty,\mu}$ can be computed using $\mathfrak{B}$ as follows. Note that by Lemma 54 given any (P, A) there is a $B \in \mathfrak{B}$ and Q, an A-iterate of P, such that Q is strongly $A \oplus B$-iterable, and moreover $H(Q, A \oplus B) = H(Q, B)$.

Definition 67. $\mathcal{D}^{\mathfrak{B}}_\mu$ is the set of $(P, B \oplus A) \in \mathcal{D}_\mu$ such that

(1) $B \in \mathfrak{B}$ and $A \in \mathrm{OD}^{L(\mathbb{R},\mu)}$
(2) $A \leq_W B$, and
(3) $H(Q, A \oplus B) = H(Q, B)$.

By the observations before Definition 67, given $\mathfrak{B}$ Wadge cofinal in $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$ we have that $\mathcal{D}^{\mathfrak{B}}_\mu$ is a directed system, and its direct limit is exactly $M_{\infty,\mu}$.

Lemma 68. *Let* $L(\mathbb{R},\mu) \models \mathrm{AD} + \omega_1$ *is* $\mathbb{R}$*-supercompact and suppose that* $\mathfrak{B} \subset \mathrm{OD}^{L(\mathbb{R},\mu)}$ *is Wadge cofinal in* $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$*. Then the direct limit of* $\mathcal{D}^{\mathfrak{B}}_\mu$ *is* $M_{\infty,\mu}$*.*

Now, we have the main ingredients to start the path for Theorem 5. Let us prove our first approximation to the main result.

Lemma 69. *If* $L(\mathbb{R},\mu)$ *and* $L(\mathbb{R},\nu)$ *are both models of* $\mathrm{AD} + \omega_1$ *is* $\mathbb{R}$*-supercompact and* $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$*. Then* $M_{\infty,\mu} = M_{\infty,\nu}$*.*

Proof. Let us define Γ to be $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$. First, by Corollary 39 we have that $L(\mathbb{R},\Gamma) \models \Theta = \theta_0$. In other words $\mathrm{OD}^{L(\mathbb{R},\Gamma)}$ is Wadge cofinal in Γ. Note also that if A is an $\mathrm{OD}^{L(\mathbb{R},\Gamma)}$ set then it is also $\mathrm{OD}^{L(\mathbb{R},\mu)}$ and $\mathrm{OD}^{L(\mathbb{R},\nu)}$. We will conclude the proof of the lemma by constructing an isomorphism $\sigma : M_{\infty,\mu} \to M_{\infty,\nu}$.

Let $z \in M_{\infty,\mu}$. Then by Lemma 53 there exist a $B \in \Gamma$, $A \in \mathrm{OD}^{L(\mathbb{R},\mu)}$ and $\bar{z} \in H(P, B)$ such that $\sigma^\mu_{(P,A\oplus B),\infty}(\bar{z}) = z$. Note that $(P, B) \in \mathcal{D}_\nu$ (as the notion of suitability is the same in both models). Now, we can define $\sigma(z) = \sigma^\nu_{(P,B),\infty}(\bar{z})$. This map clearly respects the membership relation so we need to see that it is well defined and surjective.

Claim 1: σ is well defined:

Proof of Claim 1. Suppose that for $i = 0, 1$ there are $(P_i, A_i \oplus B_i) \in \mathcal{D}^\Gamma_\mu$ and $z_i \in H(P_i, B_i)$ such that $\sigma^\mu_{(P_i,A_i\oplus B_i),\infty}(z_i) = z$.

Because $\sigma^\mu_{(P_i,A_i\oplus B_i),\infty}(z_i) = z$ for $i = 0,1$, by directness of $\mathcal{D}^\mu$ there exists P_3 and $z_3 \in P_3$, such that $(P_3, A_1 \oplus B_1 \oplus A_2 \oplus B_2) \in \mathcal{D}^\mu$, and if $\sigma_i = \sigma^\mu_{(P_i,A_i\oplus B_i),(P_3,A_1\oplus B_1\oplus A_2\oplus B_2)}$, then $\sigma_i(z_i) = z_3$. Note that σ_i is in particular a B_i-iteration of P_i. Note also that since $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$ then the strategy giving raise to σ_i is also in $L(\mathbb{R},\nu)$ and $L(\mathbb{R},\nu) \models$ "σ_i is a B_i-iteration and $(P_3, B_1 \oplus B_2) \in \mathcal{D}_\nu$". In other words $\sigma^\nu_{(P_i,B_i),(P_3,B_1\oplus B_2)}(z_i) = z_3$, which implies σ assigns a unique value to z. □

Claim 2: σ is surjective:

Proof of Claim 2. Let $z \in M_{\infty,\nu}$ then by Lemma 53 (applied in $L(\mathbb{R},\nu)$) there exists a $B \in \Gamma$, $A \in \mathrm{OD}^{L(\mathbb{R},\nu)}$ and $\bar{z} \in H(P,B)$ such that $\sigma^\nu_{(P,A\oplus B),\infty}(\bar{z}) = z$. Now $(P,B) \in \mathcal{D}_\mu$ hence $\sigma^\nu_{(P,B),\infty}(\bar{z})$ is in the range of σ but of course the identity on P is a B-iteration, hence $\sigma^\nu_{(P,B),\infty}(\bar{z}) = \sigma^\nu_{(P,A\oplus B),\infty}(\bar{z})$. Hence z is in the range of σ as wanted. □

These claims finish the proof of the Lemma.

□

Theorem 70. *If $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ are both models of* AD + ω_1 *is* $\mathbb{R}$- *supercompact and $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$. Then $L(\mathbb{R},\mu) = L(\mathbb{R},\nu)$.*

Proof. By Lemma 69 we have that $M = M_{\infty,\mu} = M_{\infty,\nu}$ and moreover as Σ_μ and Σ_ν can be defined by M and $\Gamma = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$ via the same formula, then $\Sigma_\mu = \Sigma_\nu$. So, by Theorem 64 we have $\mathrm{HOD}^{L(\mathbb{R},\mu)} = \mathrm{HOD}^{L(\mathbb{R},\nu)}$. Finally by [**29**] we have that

$$L(\mathbb{R},\mu) = \mathrm{HOD}^{L(\mathbb{R},\mu)}(\mathbb{R}) = \mathrm{HOD}^{L(\mathbb{R},\mu)}(\mathbb{R}) = L(\mathbb{R},\nu)$$

□

We will conclude the proof of our main Theorem by showing that it is the case that models of "AD + ω_1 is $\mathbb{R}$-supercompact" share the same sets of reals. We begin with the following approximations.

Lemma 71. *Suppose that $L(\mathbb{R},\mu)$ models* "AD + ω_1 *is $\mathbb{R}$-supercompact" and $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ is club in $L(\mathbb{R},\mu)$. Then $A \in \mu$*

Proof. Otherwise let A be a counterexample. As the statement "A is club and $A \notin \mu$" is Σ_1, by Theorem 23 we have that there is such a counterexample in $L_{\boldsymbol{\delta}^2_1}(\mathbb{R},\mu)$. But again, Theorem 23 implies that μ restricted to $L_{\boldsymbol{\delta}^2_1}(\mathbb{R},\mu)$ is the club filter (of $L(\mathbb{R},\mu)$) restricted to $L_{\boldsymbol{\delta}^2_1}(\mathbb{R},\mu)$, contradiction. □

We have the following refinement of Lemma 34 that we will use. The proof is essentially the same except we use Lemma 71 instead of assuming that μ or ν is the club filter.

Lemma 72. *Suppose that $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ are models of* "AD + ω_1 *is $\mathbb{R}$-supercompact". Then either $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} \subseteq \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$ or $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)} \subseteq \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$.*

Proof. Suppose neither $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} \subseteq \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$ nor $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)} \subseteq \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$. Let $\Gamma = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} \cap \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$. By Theorem 3.7.1 of [**26**] $L(\mathbb{R},\Gamma) \models \mathrm{AD}_\mathbb{R}$. Hence by a theorem of Solovay, if $\mathcal{C}$ is the club filter defined in $L(\mathbb{R},\Gamma)$, then $L(\mathbb{R},\mathcal{C}) \models \mathrm{AD} + \omega_1$ is $\mathbb{R}$-supercompact (see [**12**]). Moreover, by Lemma 71, $\mathcal{C} \subseteq \mu$. This implies $L(\mathbb{R},\mathcal{C}) = L(\mathbb{R},\mu)$, a contradiction. □

Theorem 73. *Suppose that $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ are models of "$\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact". Then $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)} = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$.*

Proof. Suppose for a contradiction that there are μ and ν such that $L(\mathbb{R},\mu)$ and $L(\mathbb{R},\nu)$ are models of "$\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact", but $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$ is strictly contained in $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\nu)}$.

Claim 1: $L(\mathbb{R},\mu)$ is definable (from parameters) in $L(\mathbb{R},\nu)$.

Proof of Claim 1. Let us call $\Gamma = \mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$. Recall that, by Corollary 39, $L(\mathbb{R},\Gamma)$ is a model of $\Theta=\theta_0$ and hence $\mathrm{OD}^{L(\mathbb{R},\Gamma)}$ is Wadge cofinal in $\mathcal{P}(\mathbb{R})^{L(\mathbb{R},\mu)}$. Now, in $L(\mathbb{R},\nu)$ we can define $\mathcal{D}$ to be the set of (P,A) such that $A\in\mathrm{OD}^{L(\mathbb{R},\Gamma)}$ and P is a Γ-suitable and strongly A-iterable mouse. Note that $\mathcal{D}\subseteq\mathcal{D}_\mu$, and is directed. Let M be its direct limit, then essentially by the argument in Lemma 69 we have that $M=M_{\infty,\mu}$. Also $\Gamma\in L(\mathbb{R},\nu)$ hence $\mathrm{HOD}^{L(\mathbb{R},\mu)}$ is definable in $L(\mathbb{R},\nu)$, and so since $L(\mathbb{R},\mu)=\mathrm{HOD}^{L(\mathbb{R},\mu)}(\mathbb{R})$ it is definable in $L(\mathbb{R},\nu)$ as well. □

Note that as $L(\mathbb{R},\mu)\subset L(\mathbb{R},\nu)$, there is $A\subset\mathbb{R}$ in $L(\mathbb{R},\nu)$ such that $L(\mathbb{R},A)$ can define $L(\mathbb{R},\nu)$ (any $A\subset\mathbb{R}$ of Wadge rank $\Theta(\mu)$ would suffice). Moreover, as $L(\mathbb{R},A)\neq L(\mathbb{R},\nu)$, its easy to see that $A^\sharp$ exists and is in $L(\mathbb{R},\nu)$. So $(\mathbb{R},\mu)^\sharp\in L(\mathbb{R},\nu)$ (here by $(\mathbb{R},\mu)^\sharp$ we mean the theory of indiscernibles for $L(\mathbb{R},\mu)$). Also, recall that by Theorem 23 we have that $L_{\boldsymbol{\delta}^2_1(\nu)}(\mathbb{R},\nu)\prec_1 L(\mathbb{R},\nu)$ (where $\boldsymbol{\delta}^2_1(\nu)$ is the least stable of $L(\mathbb{R},\nu)$). Hence there is a sharp $(\mathbb{R},\bar{\mu})^\sharp$, for a model of "$\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact", in $L_{\boldsymbol{\delta}^2_1(\nu)}(\mathbb{R},\nu)$. By the the proof of Lemma 37 we have that $\mathcal{M}^\sharp_{\omega^2}$ exists and is ω_1+1-iterable in $L(\mathbb{R},\nu)$. Hence the proof of Theorem 1 will imply that if $\mathcal{C}$ is the club filter of $L(\mathbb{R},\nu)$ then $L(\mathbb{R},\mathcal{C})$ is a model of "$\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact" but as in the proof of Lemma 72 this is a contradiction.

□

CHAPTER 6

Complementary Results

6.1. μ is not necessarily the club filter

In this section we prove that ZFC plus the existence of a model of the form $L(\mathbb{R},\mu)$ where μ does not contain only elements containing a club is relatively consistent. Here there is a delicate line as by [**8**] the existence of $\mathcal{M}^{\sharp}_{\omega^2}$ implies that $L(\mathbb{R},\mathcal{C}) \models \mathrm{ZF}+\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact and hence the unique such model.

Theorem 74. *Suppose that $L(\mathbb{R},\mu)$ is a model of* $\mathrm{ZF}+\mathrm{AD}+\omega_1$ *is $\mathbb{R}$ supercompact as witnessed by μ. Then there is a universe M of* ZFC*, with $\mu \in M$ but $L(\mathbb{R},\mu) \neq L(\mathbb{R},\mathcal{C})$.*

Proof. Let $N = L(\mathbb{R},\mu)$ be as in the statement, we will work in N. Fix $A \in \mu$ such that $N \models$ "A does not contain a club" and consider the following club shooting poset $\mathbb{P}_A$. Conditions are:

- $p = \langle \sigma_i \mid i < \beta\rangle$ for some $\beta \in \omega_1$,
- for all $i \in \beta$, $\sigma_i \in \mathcal{P}_{\omega_1}(\mathbb{R}) \setminus A$,
- for $i < j$, $i,j \in \beta$ we have $\sigma_i \subseteq \sigma_j$, and
- for $i \in \beta$ limit $\sigma_i = \bigcup_{j<i} \sigma_j$

The order is reverse inclusion. Let G be N generic for $\mathbb{P}_A$, then by assumption $\mathcal{P}_{\omega_1}(\mathbb{R}) \setminus A$ is stationary and so $M = N[G]$ has the same reals numbers as N and M has a well-order of the reals hence $M \models$ ZFC. Also $\mu \in M$ and $A \in \mu$ but does not contain a club (in M) as $\mathcal{P}_{\omega_1}(\mathbb{R}) \setminus A$ is in the club filter, yet $L(\mathbb{R},\mu) \models \mathrm{ZF}+\mathrm{AD}+\omega_1$ is $\mathbb{R}$ supercompact as witnessed by μ. $\square$

6.2. Two Different Models of ZF+ω_1 is $\mathbb{R}$-supercompact

By results of the previous section there is at most a model of $\mathrm{ZF}+\mathrm{AD}+\omega_1$ is $\mathbb{R}$-supercompact of the form $L(\mathbb{R},\mu)$, where $\mu \subset \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$. A natural question is whether models of ZF+ ω_1 is $\mathbb{R}$-supercompact are also unique. We show this is not the case if there is a measurable κ of Mitchell order 2. Let us recall that the existence of a model of ZF $+\omega_1$ is $\mathbb{R}$-supercompact is exactly that of a measurable (see [**25**]). Let us start by reviewing a construction found in [**25**].

Let κ be a measurable cardinal, $\mathcal{U}$ a normal κ-complete filter over κ and G be V-generic for $\mathrm{col}(\omega,<\kappa)$. Let $j_{\mathcal{U}} = V :\to Ult(V,\mathcal{U}) = M$ be the ultra-power embedding. Note that if H is such that $G \times H$ is M-generic for $\mathrm{col}(\omega,< j_{\mathcal{U}}(\kappa))$ one can extend $j_{\mathcal{U}}$ to $j^+_{\mathcal{U}} : V[G] \to M[G][H]$. We define then $\mathcal{F}$ on $\mathcal{P}_{\omega_1}(\mathbb{R}^{V[G]})$ as follows. $A \in \mathcal{F}$ if and only if

$$\mathbb{R}^{V[G]} \in j^+_{\mathcal{U}}(A), \text{ for any } H, \text{ such that } G \times H \subset \mathrm{col}(\omega,< j_{\mathcal{U}}(\kappa)) \text{ is } M\text{-generic}$$

The fact of [**25**] we will use is that $L(\mathbb{R}^{V[G]}, \mathcal{F})$ is a model of ZF + ω_1 is $\mathbb{R}$-supercompact.

Proof of Theorem 6. Suppose there is such a measurable and without loss of generality that there is no Woodin cardinal. Then by maximality of the the core model K (as defined in [**2**]) we have that K has a measurable of Mitchell order 2. For now we will work in K.

Now, let un fix U_0 and U_1 the measures on κ such that

$$B := \{\alpha \in \kappa \,|\, \alpha \text{ is measurable}\} \in U_1 \setminus U_0.$$

Let $j_0 : K \to M_0$ and $j_1 : K \to M_1$ be the ultrapower maps. Also let G be a K-generic filter for $\mathrm{col}(\omega, < \kappa)$ and, let us define $\mathbb{R}^* = \mathbb{R}^{K[G]}$. Then given any H_0 such that $G \times H_0$ is M_0-generic for $\mathrm{col}(\omega, < j_0(\kappa))$, there is a unique extension $j_0^+ : K \to M_0[G][H_0]$ of j_0. Let us define $\mathbb{P}_0$ to be the factor poset such that $\mathrm{col}(\omega, < j_0(\kappa)) = \mathrm{col}(\omega, < \kappa) \times \mathbb{P}_0$ and define $\mathcal{F}_0$ (in $K[G]$) as follows:

$$A \in \mathcal{F}_0 \text{ if and only if, } ||\!\!-^{M_0}_{\mathbb{P}_0} (\mathbb{R}^* \in j_0^+(A))$$

In other words, $A \in \mathcal{F}_0$ if for any $H_0 \subset \mathbb{P}_0$ and M_0-generic, then $\mathbb{R}^* \in j_0^+(A)$. We also define $\mathcal{F}_1$ in the analogous way. Also by the discussion above, we know that $L(\mathbb{R}^*, \mathcal{F}_i)$ is a model of ZF + ω_1 is $\mathbb{R}^*$-supercomapct (for $i = 0, 1$). Now since $K^K = K$ by generic absoluteness we also have that $K^{K[G]} = K$, besides $\omega_1^{K[G]} = \kappa$, also by Theorem 1.1. of [**2**], we have that $K|\kappa$ is definable in $L(\mathbb{R}^*)$ and hence $B \in L(\mathbb{R}^*, \mathcal{F}_0) \cap L(\mathbb{R}^*, \mathcal{F}_1)$. Let us define B^* as

$$B^* = \{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}^*) \mid \sup \sigma \in B\}.$$

Here the $\sup \sigma$ denotes the supremum of the order types of the reals in σ coding a well order (under some reasonable coding). Then from the definition of B and the fact that $\sup \mathbb{R}^* = \kappa$ we have that $B^* \in \mathcal{F}_1 \setminus \mathcal{F}_0$ so $L(\mathbb{R}^*, \mathcal{F}_0)$ and $L(\mathbb{R}, \mathcal{F}_1)$ are different models of the theory ZF + ω_1 is $\mathbb{R}^*$-supercompact. $\square$

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