Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption

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Submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy in
Mathematical Sciences

Carnegie Mellon University
2012

Abstract

We consider a problem of optimal investment with intermediate consumption in the framework of an incomplete semimartingale model of a financial market. We show that a necessary and sufficient condition for the validity of key assertions of the theory is that the value functions of the primal and dual problems are finite.

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Acknowledgements

First and foremost, I would like to thank my advisor Dmitry Kramkov. He has systematically challenged me to consider the most general case in which one still can hope to achieve a result. Dmitry has helped me to overcome difficulties with his advice and provided critical insights and suggestions at the right moments. Studying and discussing with him, in particular his papers, gave me an idea of how central mathematical results are developed and helped to perform on my (local) maximum.

I would also like to thank Steven Shreve for teaching so much through his books and our conversations. Many other professors in the department have given me invaluable instruction and advice, including Giovanni Leoni, Kasper Larsen, Noel Walkington, Bill Hrusa, Ago Pisztora, and Scott Robertson.

Special thanks to Gerard Brunick, Sergio Pulido, Alexander Rand, and Pietro Siorpaes for useful, informal conversations on various topics in mathematics.

Finally, I am grateful to the entire mathematics department faculty and staff for creating such a stimulating atmosphere with its remarkable seminars, in particular, the Probability and Computational Finance Seminar, which allows one to see amazing speakers and absorb mathematical culture.

Contents

1 Introduction

A fundamental problem of mathematical finance is that of an investor who wants to invest and consume in a way that maximizes his expected utility. The first results for continuous time models were obtained by Merton [20, 21] in a Markovian setting via dynamic programming arguments. An alternative martingale approach was developed among others by Cox and Huang [3, 4], Karatzas, Lehoczky and Shreve [14], and Karatzas and Shreve [12] for complete markets and by Karatzas, Lehoczky, Shreve and Xu [15], He and Pearson [8, 9], Kramkov and Schachermayer [17, 18], Karatzas and Žitković [13], and Žitković [25] in an incomplete case. The main focus here was to establish conditions under which "key" results, such as the existence of primal and dual optimizers, hold.

When the consumption occurs only at maturity and the utility function is deterministic a necessary and sufficient condition has been obtained in Kramkov and Schachermayer [18]. It is stated as the finiteness of the dual value function. In the case of intermediate consumption and stochastic field utility, the latest sufficient conditions are due to Karatzas and Žitković [13] and Žitković [25]. They are formulated in the form of several regularity assumptions such as a uniform asymptotic elasticity.

This paper obtains necessary and sufficient conditions in the general framework of an incomplete financial model with a stochastic field utility and intermediate consumption occurring according to some stochastic clock. As in [18] we assume that the dual value function is finite (from above). Maybe surprisingly, the only other condition we need is the finiteness of the primal value function (from below). Note that the latter condition holds trivially in the setting of [18].

The remainder of the thesis is organized as follows. In Section 2 we describe the model and state the main results. Their proofs are given in Section 4 and are based on the abstract versions of the main theorems presented in Section 3.

2 Main Results

A model of a security market consists of $(d+1)$ assets: one bond and d stocks. We assume that the bond is chosen as a numéraire and denote by $S = (S^i)_{1\leq i\leq d}$ the discounted price processes of the stocks. We suppose that S is a semimartingale on a complete stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ with an infinite time horizon, $\mathscr{F}_0$ is the completion of the trivial σ-algebra.

Define a portfolio Π as a triple (x, H, c), where the constant x is an initial value, $H = (H^i)_{1\leq i\leq d}$ is a predictable S-integrable process of stocks' quantities, and $c = (c_t)_{t\geq 0}$ is a nonnegative and optional process that specifies the consumption rate in the units of the bond.

Hereafter we fix a *stochastic clock* $\kappa = (\kappa_t)_{t\geq 0}$, which is a non-decreasing, cádlág, adapted process such that

$$\kappa_0 = 0, \quad \mathbb{P}\left[\kappa_\infty > 0\right] > 0, \text{ and } \kappa_\infty \leq A \tag{2.1}$$

for some finite constant A. Stochastic clock represents the notion of time according to which consumption occurs.

The discounted value process $V = (V_t)_{t\geq 0}$ of a portfolio Π is defined as

$$V_t \triangleq x + \int_0^t H_u dS_u - \int_0^t c_u d\kappa_u, \quad t \geq 0. \tag{2.2}$$

A portfolio Π with $c \equiv 0$ is called *self-financing*. The collection of nonnegative value processes of self-financing portfolios with initial value 1 is denoted by $\mathscr{X}$, i.e.,

$$\mathscr{X} \triangleq \left\{ X \geq 0 : \; X_t = 1 + \int_0^t H_u dS_u, \quad t \geq 0 \right\}.$$

A pair (H, c), such that for a given $x > 0$ the corresponding value process V is nonnegative, is called an *x-admissible strategy*. If for a consumption process c we can find a predictable S-integrable process H such that (H, c) is an x-admissible strategy, we say that c is an *x-admissible consumption process*.

The set of the x-admissible consumption processes corresponding to a stochastic clock κ is denoted by $\mathscr{A}(x)$, that is,

$$\mathscr{A}(x) \triangleq \{c : \; c \text{ is } x\text{-admissible}\}, \quad x > 0. \tag{2.3}$$

We write $\mathscr{A} \triangleq \mathscr{A}(1)$ for brevity.

The set of *equivalent martingale deflators* is defined as

$$\begin{aligned} \mathscr{Z} \triangleq \{Z > 0 : \; & Z \text{ is a cádlág martingale, s.t. } Z_0 = 1 \text{ and} \\ & XZ = (X_t Z_t)_{t\geq 0} \text{ is a local martingale for every } X \in \mathscr{X}\}. \end{aligned} \tag{2.4}$$

We assume that

$$\mathscr{Z} \neq \emptyset. \tag{2.5}$$

This condition is closely related to the absence of arbitrage opportunities in the sense of [11].

We now introduce an economic agent whose consumption preferences are modeled with a *utility stochastic field* $U = U(t, \omega, x) : [0, \infty) \times \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ satisfying the conditions below.

Assumption 2.1. For every $(t, \omega) \in [0, \infty) \times \Omega$ the function $x \to U(t, \omega, x)$ is strictly concave, increasing, continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(t, \omega, x) = +\infty \quad \text{and} \quad \lim_{x \to \infty} U'(t, \omega, x) \triangleq 0, \tag{2.6}$$

where U' denotes the partial derivative with respect to the third argument. At $x = 0$ we have, by continuity, $U(t, \omega, 0) = \lim_{x \downarrow 0} U(t, \omega, x)$, this value may be $-\infty$. For every $x \geq 0$ the stochastic process $U(\cdot, \cdot, x)$ is optional.

For a given initial capital $x > 0$ the goal of the agent is to maximize his expected utility. The value function of this problem is denoted by

$$u(x) \triangleq \sup_{c \in \mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty U(t, \omega, c_t) d\kappa_t\right], \quad x > 0. \tag{2.7}$$

We use the convention

$$\mathbb{E}\left[\int_0^\infty U(t, \omega, c_t) d\kappa_t\right] \triangleq -\infty \quad \text{if} \quad \mathbb{E}\left[\int_0^\infty U^-(t, \omega, c_t) d\kappa_t\right] = +\infty.$$

Here and below, W^- and W^+ denote the negative and the positive parts of a stochastic field W, respectively.

Our goal is to find conditions on the financial market and the utility field U under which the key conclusions of the utility maximization theory hold, namely, u satisfies the Inada conditions and the solution $\hat{c}(x) \in \mathscr{A}(x)$ to (2.7) exists.

Remark 2.2. For simplicity of notations we assume throughout the paper that the argument x in $U(t, \omega, x)$ represents the consumption in the discounted units, that is, in the number of bonds. This does not restrict any generality. Indeed, suppose that the investor's stochastic field utility is given as $\tilde{U} = \tilde{U}(t, \omega, \tilde{x})$, where the consumption $\tilde{x}$ is measured in the number of units of a different asset, whose discounted value is given by a strictly positive semimartingale $A = (A_t)_{t \geq 0}$ .Then we arrive to our framework by setting

$$U(t, \omega, x) \triangleq \tilde{U}(t, \omega, x / A_t(\omega)).$$

To study (2.7) we employ standard duality arguments as in [17] and [25] and define the *conjugate stochastic field* V to U as

$$V(t, \omega, y) \triangleq \sup_{x > 0} (U(t, \omega, x) - xy), \quad (t, \omega, y) \in [0, \infty) \times \Omega \times [0, \infty). \tag{2.8}$$

It is well-known that $-V$ satisfies Assumption 2.1. We also denote

$$\begin{aligned}\mathscr{Y}(y) \triangleq \mathrm{cl}\{Y : Y &\text{ is càdlàg adapted and} \\ &0 \le Y \le yZ \;\; (d\kappa \times \mathbb{P}) \text{ a.e. for some } Z \in \mathscr{Z}\},\end{aligned} \tag{2.9}$$

where the closure is taken in the topology of convergence in measure $(d\kappa \times \mathbb{P})$ on the space of real-valued optional processes. We write $\mathscr{Y} \triangleq \mathscr{Y}(1)$ for brevity.

After these preparations, we define the value function of the dual optimization problem as

$$v(y) \triangleq \inf_{Y \in \mathscr{Y}(y)} \mathbb{E}\left[\int_0^\infty V(t, \omega, Y_t) d\kappa_t\right], \quad y > 0, \tag{2.10}$$

where we use the convention:

$$\mathbb{E}\left[\int_0^\infty V(t, \omega, Y_t) d\kappa_t\right] \triangleq +\infty \quad \text{if} \quad \mathbb{E}\left[\int_0^\infty V^+(t, \omega, Y_t) d\kappa_t\right] = +\infty.$$

Theorems 2.3 and 2.4 constitute our main results.

Theorem 2.3. *Assume that conditions* (2.1) *and (2.5) and Assumption 2.1 hold true and suppose*

$$v(y) < \infty \;\; \textit{for all } y > 0 \quad \textit{and} \quad u(x) > -\infty \;\; \textit{for all } x > 0. \tag{2.11}$$

Then we have:

1. $u(x) < \infty$ *for all* $x > 0$, $v(y) > -\infty$ *for all* $y > 0$. *The functions* u *and* v *are conjugate, i.e.,*

$$\begin{aligned} v(y) &= \sup_{x>0} (u(x) - xy), \quad y > 0, \\ u(x) &= \inf_{y>0} (v(y) + xy), \quad x > 0. \end{aligned} \tag{2.12}$$

   The functions u *and* $-v$ *are continuously differentiable on* $(0, \infty)$, *strictly increasing, strictly concave and satisfy the Inada conditions:*

$$\begin{aligned} u'(0) &\triangleq \lim_{x \downarrow 0} u'(x) = +\infty, & -v'(0) &\triangleq \lim_{y \downarrow 0} -v'(y) = +\infty, \\ u'(\infty) &\triangleq \lim_{x \to \infty} u'(x) = 0, & -v'(\infty) &\triangleq \lim_{y \to \infty} -v'(y) = 0. \end{aligned}$$

2. *For every* $x > 0$ *and* $y > 0$ *the optimal solutions* $\hat{c}(x)$ *to (2.7) and* $\hat{Y}(y)$ *to (2.10) exist and are unique. Moreover, if* $y = u'(x)$ *we have the dual relations*

$$\hat{Y}_t(y) = U'(t, \omega, \hat{c}_t(x)), \quad t \ge 0,$$

   and

$$\mathbb{E}\left[\int_0^\infty \hat{c}_t(x) \hat{Y}_t(y) d\kappa_t\right] = xy.$$

The finiteness conditions (2.11) are clearly necessary for the conclusions of either item 1 or 2. Notice that the condition $u(x) > -\infty$ for all $x > 0$ holds trivially if the utility stochastic field U is uniformly bounded from below by a real-valued function.

A natural question is whether one can use the set $\mathscr{Z}$ instead of $\mathscr{Y}$ as the dual domain and still obtain the same value function v. Theorem 2.4 below states that the answer is positive, however, the minimizer might lie outside of the set $\mathscr{Z}$ in general, see e.g. Example 5.1 in Kramkov and Schachermayer [17]. Furthermore, due to a certain *symmetry* between primal and dual problems (that is explored in more detail in Section 3) a similar conclusion is valid for the value function u. Let $\mathscr{B}$ be a subset of $\mathscr{A}$ such that

(i) for every $Y \in \mathscr{Y}$, we have

$$\sup_{c\in\mathscr{B}} \mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right] = \sup_{c\in\mathscr{A}} \mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right],$$

(ii) the set $\mathscr{B}$ is closed under the countable convex combinations, that is, for any sequence $(c^n)_{n\geq 1}$ of optional processes in $\mathscr{B}$ and any sequence of positive numbers $(a^n)_{n\geq 1}$ such that $\sum_{n=1}^\infty a^n = 1$, the process $\sum_{n=1}^\infty a^n c^n$ belongs to $\mathscr{B}$.

Observe that $\mathscr{Z}$ is closed under the countable convex combinations.

Theorem 2.4. *Under the conditions of Theorem 2.3, we have*

$$\begin{aligned} v(y) &= \inf_{Z\in\mathscr{Z}} \mathbb{E}\left[\textstyle\int_0^\infty V\left(t,\omega,yZ_t\right)d\kappa_t\right], \quad y > 0, \\ u(x) &= \sup_{c\in\mathscr{B}} \mathbb{E}\left[\textstyle\int_0^\infty U(t,\omega,xc_t)d\kappa_t\right], \quad\quad x > 0. \end{aligned}$$

The proofs of Theorems 2.3 and 2.4 will be given in Section 4 and will rely on Theorems 3.2 and 3.3, which are the "abstract" versions of Theorems 2.3 and 2.4, respectively. We conclude this section with examples of the investment problems (see e.g. Karatzas [10] as well as Karatzas and Shreve [12]) that are included in our formulation. Hereafter, 1_E denotes the indicator function of a set E.

Example 2.5. Maximization of the expected utility from consumption:

$$u(x) = \sup_{c\in\mathscr{A}(x)} \mathbb{E}\left[\int_0^T U(t,\omega,c_t)dt\right].$$

Here the clock κ is given by

$$\kappa(t) \triangleq \min\left(t,\ T\right), \quad t \geq 0.$$

Example 2.6. Maximization of the expected utility from consumption and terminal wealth:

$$u(x) = \sup_{c \in \mathscr{A}(x)} \mathbb{E}\left[\int_0^T U_1(t, \omega, c_t)dt + U_2(\omega, c_T)\right]. \tag{2.13}$$

Here the clock κ is given by

$$\kappa(t) \triangleq t 1_{[0,T)}(t) + (T+1)1_{[T,\infty)}(t), \quad t \geq 0.$$

Example 2.7. Maximization of the expected utility from terminal wealth:

$$u(x) = \sup_{X \in \mathscr{X}} \mathbb{E}\left[U(\omega, x X_T)\right], \tag{2.14}$$

The corresponding clock process is

$$\kappa(t) \triangleq 1_{[T,\infty)}(t), \quad t \geq 0.$$

Note that the formulation (2.14) extends the framework of Kramkov and Schachermayer (see [17, 18]) to stochastic utility.

Example 2.8. Maximization of the expected utility from consumption over the infinite time horizon, that is

$$u(x) = \sup_{c \in \mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\nu t} U(t, \omega, c_t)dt\right], \quad x > 0, \ \nu > 0, \tag{2.15}$$

where the clock is defined as

$$\kappa(t) \triangleq \int_0^t e^{-\nu s} ds = \frac{1}{\nu}\left(1 - e^{-\nu t}\right), \quad t \geq 0.$$

Example 2.9. Maximization of expected utility from consumption occurring at discrete times $(t_1, \dots, t_N)$:

$$u(x) = \sup_{c \in \mathscr{A}(x)} \mathbb{E}\left[\sum_{j=1}^N U(t_j, \omega, c_{t_j})\right], \quad x > 0. \tag{2.16}$$

Here the clock process is

$$\kappa(t) \triangleq \sum_{j=1}^N 1_{[t_j, +\infty)}(t), \quad t \geq 0.$$

3 Abstract versions of the main theorems

Let μ be a finite and positive measure on a measurable space $(\Omega, \mathscr{F})$. Denote by $\mathbf{L}^0 = \mathbf{L}^0(\Omega, \mathscr{F}, \mu)$ the vector space of (equivalence classes of) real-valued measurable functions on $(\Omega, \mathscr{F}, \mu)$ topologized by convergence in measure μ. Let $\mathbf{L}^0_+$ denote its positive orthant, i.e.,

$$\mathbf{L}^0_+ = \left\{\xi \in \mathbf{L}^0(\Omega, \mathscr{F}, \mu) : \ \xi \geq 0\right\}.$$

For any ξ and η in $\mathbf{L}^0$ we write

$$\langle \xi, \eta \rangle \triangleq \int_\Omega \xi \eta d\mu,$$

whenever the latter integral is well-defined. Let $\mathscr{C}, \mathscr{D}$ be subsets of $\mathbf{L}^0_+$ that satisfy the conditions below.

1. We have
$$\begin{array}{lcl} \xi \in \mathscr{C} & \Leftrightarrow & \langle \xi, \eta \rangle \leq 1 \text{ for all } \eta \in \mathscr{D}, \\ \eta \in \mathscr{D} & \Leftrightarrow & \langle \xi, \eta \rangle \leq 1 \text{ for all } \xi \in \mathscr{C}. \end{array} \tag{3.1}$$

2. $\mathscr{C}$ and $\mathscr{D}$ contain at least one strictly positive element:
$$\text{there are } \ \xi \in \mathscr{C}, \ \eta \in \mathscr{D} \ \text{ such that } \ \min(\xi, \ \eta) > 0 \ \ \mu \ a.e. \tag{3.2}$$

Observe that our construction of the abstract sets $\mathscr{C}$ and $\mathscr{D}$ is similar to the one in [17], however we do not require a constant to be an element of $\mathscr{C}$. This leads to a *symmetry* between the sets $\mathscr{C}$ and $\mathscr{D}$ that plays an important role in the proofs. Also notice that $\mathscr{C}$ and $\mathscr{D}$ are convex and bounded in $\mathbf{L}^0(\mu)$. For $x > 0$ and $y > 0$ we define the sets:

$$\begin{array}{lclcl} \mathscr{C}(x) & \triangleq & x\mathscr{C} & \triangleq & \{x\xi : \ \xi \in \mathscr{C}\}, \\ \mathscr{D}(y) & \triangleq & y\mathscr{D} & \triangleq & \{y\eta : \ \eta \in \mathscr{D}\}. \end{array} \tag{3.3}$$

Consider a *stochastic utility function* U: $\Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}$, which satisfies the following conditions.

Assumption 3.1. For every $\omega \in \Omega$ the function $x \to U(\omega, x)$ is strictly concave, increasing, continuously differentiable on $(0, \infty)$, and satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(\omega, x) = +\infty \quad \text{and} \quad \lim_{x \to \infty} U'(\omega, x) = 0, \tag{3.4}$$

where $U'(\cdot, \cdot)$ denotes the partial derivative with respect to the second argument. At $x = 0$ we have, by continuity, $U(\omega, 0) = \lim\limits_{x \downarrow 0} U(\omega, x)$, this value may be $-\infty$. For every $x \geq 0$ the function $U(\cdot, x)$ is measurable.

Define the *conjugate function* V to U as

$$V(\omega, y) \triangleq \sup_{x>0} (U(\omega, x) - xy), \quad (\omega, y) \in \Omega \times [0, \infty).$$

Observe that $-V$ satisfies Assumption 3.1. For a function W on $\Omega \times [0, \infty)$ and a function $\xi \in \mathbf{L}^0_+$ we will write $W(\xi) \triangleq W(\omega, \xi(\omega))$. Recall that W^+ and W^- denote the positive and the negative parts of W, respectively.

Now we can state the optimization problems:

$$u(x) = \sup_{\xi \in \mathscr{C}(x)} \int_\Omega U(\xi) d\mu, \quad x > 0, \tag{3.5}$$

$$v(y) = \inf_{\eta \in \mathscr{D}(y)} \int_\Omega V(\eta) d\mu, \quad y > 0, \tag{3.6}$$

where we used the convention:

$$\begin{array}{lll} \int_\Omega U(\xi) d\mu \triangleq -\infty & \text{if} & \int_\Omega U^-(\xi) d\mu = +\infty, \\ \int_\Omega V(\eta) d\mu \triangleq +\infty & \text{if} & \int_\Omega V^+(\eta) d\mu = +\infty. \end{array}$$

The following theorem is an abstract version of Theorem 2.3.

Theorem 3.2. *Assume that $\mathscr{C}$ and $\mathscr{D}$ satisfy conditions (3.1) and (3.2). Let Assumption 3.1 hold and suppose*

$$v(y) < \infty \quad \textit{for all } y > 0 \quad \textit{and} \quad u(x) > -\infty \quad \textit{for all } x > 0. \tag{3.7}$$

Then we have:

1. *$u(x) < \infty$ for all $x > 0$, $v(y) > -\infty$ for all $y > 0$. The functions u and v satisfy the biconjugacy relations, i.e.,*

$$\begin{array}{rcll} v(y) & = & \sup_{x>0} (u(x) - xy), & y > 0, \\ u(x) & = & \inf_{y>0} (v(y) + xy), & x > 0. \end{array} \tag{3.8}$$

The functions u and $-v$ are continuously differentiable on $(0, \infty)$, strictly increasing, strictly concave, and satisfy the Inada conditions:

$$\begin{array}{ll} u'(0) \triangleq \lim_{x \downarrow 0} u'(x) = +\infty, & -v'(0) \triangleq \lim_{y \downarrow 0} -v'(y) = +\infty, \\ u'(\infty) \triangleq \lim_{x \to \infty} u'(x) = 0, & -v'(\infty) \triangleq \lim_{y \to \infty} -v'(y) = 0. \end{array}$$

2. For every $x>0$ the optimal solution $\hat{\xi}(x)$ to (3.5) exists and is unique. For every $y>0$ the optimal solution $\hat{\eta}(y)$ to (3.6) exists and is unique. If $y=u'(x)$, we have the dual relations

$$\hat{\eta}(y)=U'\left(\hat{\xi}(x)\right) \quad \mu \text{ a.e.}$$

and

$$\langle\hat{\xi}(x),\hat{\eta}(y)\rangle = xy.$$

In order to state an abstract version of Theorem 2.4 we need the following definitions. Let $\tilde{\mathscr{D}}$ be a subset of $\mathscr{D}$ such that

(i) $\tilde{\mathscr{D}}$ is closed under the countable convex combinations,

(ii) for every $\xi\in\mathscr{C}$ we have

$$\sup_{\eta\in\mathscr{D}}\langle\xi,\eta\rangle=\sup_{\eta\in\tilde{\mathscr{D}}}\langle\xi,\eta\rangle. \tag{3.9}$$

Likewise, define $\tilde{\mathscr{C}}$ to be a subset of $\mathscr{C}$ such that

(iii) $\tilde{\mathscr{C}}$ is closed under the countable convex combinations,

(iv) for every $\eta\in\mathscr{D}$ we have

$$\sup_{\xi\in\mathscr{C}}\langle\xi,\eta\rangle=\sup_{\xi\in\tilde{\mathscr{C}}}\langle\xi,\eta\rangle.$$

Theorem 3.3. *Under the conditions of Theorem 3.2, we have*

$$\begin{aligned} v(y) &= \inf_{\eta\in\tilde{\mathscr{D}}}\int_\Omega V\left(y\eta\right)d\mu, \qquad y>0.\\ u(x) &= \sup_{\xi\in\tilde{\mathscr{C}}}\int_\Omega U\left(x\xi\right)d\mu, \qquad x>0. \end{aligned}$$

The proofs of Theorem 3.2 and 3.3 are given via several lemmas.

Lemma 3.4. *Under the conditions of Theorem 3.2, we have*

$$v(y)\geq\sup_{x>0}\left(u(x)-xy\right), \qquad y>0. \tag{3.10}$$

As a result, both u and v are real-valued functions, such that

$$\limsup_{x\to\infty}\frac{u(x)}{x}\leq 0 \quad \text{and} \quad \liminf_{y\to\infty}\frac{v(y)}{y}\geq 0.$$

Proof. Fix $x > 0$ and $y > 0$. We have

$$\sup_{\xi \in \mathscr{C}(x)} \inf_{\eta \in \mathscr{D}(y)} \int_\Omega (U(\xi) - \xi\eta)\, d\mu \leq \inf_{\eta \in \mathscr{D}(y)} \sup_{\xi \in \mathscr{C}(x)} \int_\Omega (U(\xi) - \xi\eta)\, d\mu. \tag{3.11}$$

Using (3.1) we can bound the left-hand side from below by $u(x) - xy$:

$$\sup_{\xi \in \mathscr{C}(x)} \inf_{\eta \in \mathscr{D}(y)} \textstyle\int_\Omega (U(\xi) - \xi\eta)\, d\mu \geq \sup_{\xi \in \mathscr{C}(x)} \left(\int_\Omega U(\xi) d\mu - xy \right) = u(x) - xy.$$

Since $V(\eta) \geq U(\xi) - \xi\eta$ for every $\xi \geq 0$ and $\eta \geq 0$, we can bound the right-hand side of (3.11) from above by $v(y)$:

$$\inf_{\eta \in \mathscr{D}(y)} \sup_{\xi \in \mathscr{C}(x)} \int_\Omega (U(\xi) - \xi\eta)\, d\mu \leq \inf_{\eta \in \mathscr{D}(y)} \int_\Omega V(\eta) d\mu = v(y),$$

and the result follows. □

The techniques in Kramkov and Schachermayer [18] inspired the proof of the following lemma.

Lemma 3.5. *Under the conditions of Theorem 3.2, for every $y > 0$ the family $\left(V^-(h)\right)_{h \in \mathscr{D}(y)}$ is uniformly integrable.*

Proof. Fix $y > 0$. Assume by contradiction that $\left(V^-(h)\right)_{h \in \mathscr{D}(y)}$ is not a uniformly integrable family. Then we can find a sequence $(\eta^n)_{n \geq 2} \subset \mathscr{D}(y)$, a sequence $(A^n)_{n \geq 2}$ of disjoint subsets of $(\Omega, \mathscr{F})$ and a constant $\alpha > 0$ such that

$$\int_\Omega V^-(\eta^n)\, 1_{A^n} d\mu \geq \alpha, \quad n \geq 2.$$

Since $v(y) < \infty$, there exists $\eta^1 \in \mathscr{D}(y)$ such that

$$M \triangleq \int_\Omega V^+\left(\eta^1\right) d\mu < \infty.$$

Define a sequence $(\zeta^n)_{n \geq 1}$ as $\zeta^n \triangleq \sum_{k=1}^n \eta^k$, $n \geq 1$. Then by (3.1) for every $\xi \in \mathscr{C}$ we have

$$\langle \zeta^n, \xi \rangle = \sum_{k=1}^n \langle \eta^k, \xi \rangle \leq ny.$$

Thus $\zeta^n \in \mathscr{D}(ny)$, $n \geq 1$. Now, since V^- is nonnegative and nondecreasing we get

$$\begin{aligned}
\textstyle\int_\Omega V^-(\zeta^n)\, d\mu &\geq \textstyle\int_\Omega \sum_{k=2}^n V^- \left(\sum_{j=1}^n \eta^j \right) 1_{A^k} d\mu \\
&\geq \textstyle\int_\Omega \sum_{k=2}^n V^-\left(\eta^k\right) 1_{A^k} d\mu \\
&\geq \alpha(n-1), \quad n \geq 2.
\end{aligned}$$

On the other hand, since V^+ is nonincreasing we obtain

$$\int_\Omega V^+ (\zeta^n)\, d\mu \le \int_\Omega V^+ \left(\eta^1\right) d\mu = M < \infty.$$

Therefore we deduce that

$$\int_\Omega V (\zeta^n)\, d\mu \le M - \alpha(n-1), \quad n \ge 2.$$

Consequently,

$$\liminf_{z\to\infty} \frac{v(z)}{z} \le \liminf_{n\to\infty} \frac{\int_\Omega V(\zeta^n)\, d\mu}{ny} \le \liminf_{n\to\infty} \frac{M - \alpha(n-1)}{ny} = -\frac{\alpha}{y} < 0,$$

which contradicts to the conclusion of Lemma 3.4. □

We need a version of Komlós' lemma for the set $\mathscr{D}$. Some other formulations of Komlós' lemma are proven in [16, 5, 1, 23].

Lemma 3.6. *Assume that the sets $\mathscr{C}$ and $\mathscr{D}$ satisfy (3.1) and (3.2). Let $(\eta^n)_{n\ge1} \subset \mathscr{D}$. Then there exists a sequence of convex combinations $\zeta^n \in \operatorname{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \ge 1$, and an element $\hat\eta \in \mathscr{D}$, such that $(\zeta^n)_{n\ge1}$ converges μ a.e. to $\hat\eta$.*

Proof. Using Lemma A1.1 p.515 in [5] we can construct a sequence $\zeta^n \in \operatorname{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \ge 1$, such that $(\zeta^n)_{n\ge1}$ converges μ a.e. to an element $\hat\eta$. By convexity of the set $\mathscr{D}$ we obtain that $(\zeta^n)_{n\ge1}$ is a subset of $\mathscr{D}$. By Fatou's lemma for every $\xi \in \mathscr{C}$ we have

$$\langle \xi, \hat\eta\rangle \le \liminf_{n\to\infty} \langle \xi, \zeta^n\rangle \le 1.$$

Hence, $\hat\eta \in \mathscr{D}$. □

Lemma 3.7. *Under conditions of Theorem 3.2 for each $y > 0$ there exists a unique $\hat\eta(y) \in \mathscr{D}(y)$, such that*

$$v(y) = \int_\Omega V(\hat\eta(y))\, d\mu. \tag{3.12}$$

As a consequence, v is strictly convex.

Proof. Fix $y > 0$. Let $(\eta^n)_{n=1}^\infty \subset \mathscr{D}(y)$ be a minimizing sequence, i.e.,

$$v(y) = \lim_{n\to\infty} \int_\Omega V(\eta^n)\, d\mu.$$

It follows from Lemma 3.6 that there exists a sequence of convex combinations $\zeta^n \in \operatorname{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \geq 1$, and an element $\hat{\eta}(y) \in \mathscr{D}(y)$, such that $(\zeta^n)_{n=1}^\infty$ converges μ a.e. to $\hat{\eta}(y)$.

Using convexity of V, Lemma 3.5, and Fatou's lemma we get

$$v(y) = \liminf_{n\to\infty} \int_\Omega V(\eta^n)\, d\mu \geq \liminf_{n\to\infty} \int_\Omega V(\zeta^n)\, d\mu \geq \int_\Omega V(\hat{\eta}(y))\, d\mu.$$

Therefore (3.12) holds. Uniqueness of the minimizer to (3.6) follows from the strict convexity of V.

To show the strict convexity of v fix $y_1 < y_2$. Since $\frac{\hat{\eta}(y_1)+\hat{\eta}(y_2)}{2} \in \mathscr{D}\left(\frac{y_1+y_2}{2}\right)$ and V is strictly convex we obtain

$$v\left(\frac{y_1+y_2}{2}\right) \leq \int_\Omega V\left(\frac{\hat{\eta}(y_1)+\hat{\eta}(y_2)}{2}\right) d\mu < \frac{v(y_1)+v(y_2)}{2}.$$

□

By the symmetry between the optimization problems (3.5) and (3.6), the following result is a corollary to Lemma 3.7.

Lemma 3.8. *Under the assumptions of Theorem 3.2, for every $x > 0$ there exists a unique maximizer to the primal problem (3.5). As a consequence, u is strictly concave.*

Lemma 3.9. *Under the assumptions of Theorem 3.2, we have*

$$v(y) = \sup_{x>0} (u(x) - xy), \quad y > 0. \tag{3.13}$$

Proof. The two-step proof is based on the change of numéraire ideas.

Step 1. Let us show (3.13) assuming that

$$\text{the constant function } 1 \in \mathscr{C} \quad \text{and} \quad \int_\Omega U(1) d\mu > -\infty.$$

In this case $\int_\Omega U(x) d\mu$ is finite for any constant $x \geq 1$. Let $\mathscr{S}_n$ be the set of all nonnegative, measurable functions $\xi : \Omega \to [0, n]$, i.e.,

$$\mathscr{S}_n \triangleq \left\{\xi \in \mathbf{L}^0 : \xi(\omega) \in [0, n] \text{ for all } \omega \in \Omega\right\}, \quad n > 0. \tag{3.14}$$

The sets $\mathscr{S}_n$ are $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ compact. Fix $y > 0$. Since $\mathscr{D}(y)$ is convex and U is concave, the minimax theorem (see [24], Theorem 45.8) gives the following equality

$$\sup_{\xi\in\mathscr{S}_n} \inf_{\eta\in\mathscr{D}(y)} \int_\Omega (U(\xi) - \xi\eta)\, d\mu = \inf_{\eta\in\mathscr{D}(y)} \sup_{\xi\in\mathscr{S}_n} \int_\Omega (U(\xi) - \xi\eta)\, d\mu. \tag{3.15}$$

Denote

$$\mathscr{C}'(x) \triangleq \left\{ \xi \in \mathscr{C}(x) : \sup_{\eta \in \mathscr{D}(y)} \langle \xi, \eta \rangle = xy \right\}.$$

It follows from (3.3) that $\bigcup_{x>0} \mathscr{C}'(x) \bigcup \{\xi \equiv 0\} = \bigcup_{x>0} \mathscr{C}(x)$. As a result, we get

$$\begin{aligned} \sup_{x>0} (u(x) - xy) &= \sup_{x>0} \sup_{\xi \in \mathscr{C}'(x)} \left(\int_\Omega U(\xi) d\mu - xy \right) \\ &\geq \lim_{n\to\infty} \sup_{\xi \in \mathscr{S}_n} \inf_{\eta \in \mathscr{D}(y)} \int_\Omega (U(\xi) - \xi\eta)\, d\mu. \end{aligned} \tag{3.16}$$

In view of (3.15), (3.16), and Lemma 3.4 it suffices to show that

$$v(y) = \lim_{n\to\infty} \inf_{\eta \in \mathscr{D}(y)} \sup_{\xi \in \mathscr{S}_n} \int_\Omega (U(\xi) - \xi\eta)\, d\mu. \tag{3.17}$$

For each $n \geq 1$ define V^n as follows:

$$V^n(z) \triangleq \sup_{0<x\leq n} (U(x) - xz), \quad z > 0.$$

Then via pointwise maximization we get

$$\inf_{\eta \in \mathscr{D}(y)} \sup_{\xi \in \mathscr{S}_n} \int_\Omega (U(\xi) - \xi\eta)\, d\mu = \inf_{\eta \in \mathscr{D}(y)} \int_\Omega V^n(\eta) d\mu \triangleq v^n(y).$$

Notice that $v^n \leq v$ and $(v^n(y))_{n\geq 1}$ is an increasing sequence. Let $(\eta^n)_{n\geq 1} \subset \mathscr{D}(y)$ be such that

$$\lim_{n\to\infty} v^n(y) = \lim_{n\to\infty} \int_\Omega V^n(\eta^n) d\mu. \tag{3.18}$$

It follows from Lemma 3.6, that there exists a sequence $\zeta^n \in \operatorname{conv}(\eta^n, \eta^{n+1}, \dots)$, $n \geq 1$, such that $(\zeta^n)_{n\geq 1}$ converges μ a.e. to a function $\hat{\zeta} \in \mathscr{D}(y)$.

We claim that $(V^n)^- (\zeta^n)$, $n \geq 2$, is a uniformly integrable sequence. Indeed, for $n \geq 2$ we have

$$V^n(\zeta) \geq V^2(\zeta) \geq V(\zeta) 1_{\{\zeta \geq U'(2)\}} + (U(2) - 2U'(2))\, 1_{\{\zeta < U'(2)\}}.$$

The concavity of U yields that $U'(2) \leq U(2) - U(1)$. Therefore,

$$V^n(\zeta) \geq \min (V(\zeta),\ 2U(1) - U(2)), \quad n \geq 2.$$

The uniform integrability of $(V^n)^- (\zeta^n)$, $n \geq 2$, follows now from Lemma 3.5 and the integrability of $U(1)$ and $U(2)$.

Therefore from the convexity of V^n and Fatou's lemma we get

$$\lim_{n\to\infty}\int_\Omega V^n(\eta^n)d\mu \geq \liminf_{n\to\infty}\int_\Omega V^n(\zeta^n)d\mu \geq \int_\Omega V(\hat{\zeta})d\mu \geq v(y),$$

which in view of (3.18) implies (3.17).

Step 2. Here we show how the general case can be reduces to the one in Step 1. Let $\hat{\xi} \triangleq \arg\min_{\xi\in\mathscr{C}(1/2)} \int_\Omega U(\xi)d\mu$ and ξ_0 be a strictly positive element of $\mathscr{C}(1/2)$. Both $\hat{\xi}$ and ξ_0 exist by Lemma 3.8 and assumption (3.2) respectively. Define

$$\zeta \triangleq \max(\hat{\xi}, \xi_0).$$

Then $\zeta \in \mathscr{C}$ and $\int_\Omega U(\zeta)d\mu$ is finite. Let

$$\begin{aligned} \tilde{U}(x) &\triangleq U(\zeta x), \\ \tilde{\mathscr{C}}(x) &\triangleq \{\xi:\ \xi\zeta \in \mathscr{C}(x)\}, \end{aligned}$$

then

$$u(x) = \sup_{\xi\in\tilde{\mathscr{C}}(x)}\int_\Omega \tilde{U}(\xi)d\mu, \quad x>0.$$

Similarly, define

$$\begin{aligned} \tilde{V}(y) &\triangleq V(y/\zeta), \\ \tilde{\mathscr{D}}(y) &\triangleq \{\eta:\ \eta/\zeta \in \mathscr{D}(y)\}, \end{aligned}$$

then we have

$$v(y) = \inf_{\eta\in\tilde{\mathscr{D}}(y)}\int_\Omega \tilde{V}(\eta)d\mu, \quad y>0.$$

Observe that $\tilde{U}$ satisfies assumption 3.1, $\tilde{V}$ is the conjugate function to $\tilde{U}$, whereas the sets $\tilde{\mathscr{C}}(1)$ and $\tilde{\mathscr{D}}(1)$ satisfy the bipolar relations (3.1) and (3.2). Moreover,

$$1 \in \tilde{\mathscr{C}}(1) \quad \text{and} \quad \int_\Omega \tilde{U}(1)d\mu > -\infty.$$

Now (3.13) follows from Step 1.

□

Proof of Theorem 3.2. Observe that by Lemmas 3.8 and 3.7 both functions u and $-v$ are strictly concave. Thus, conjugacy relations (3.8) follow from Lemma 3.9 and Theorem 12.2 in Rockafellar [22] (if we extend u by the value $-\infty$ on $(-\infty,0]$). In turn, the strict concavity of u and $-v$, (3.8), and Theorem 26.3 in [22] imply differentiability of u and v everywhere in their domains.

Fix $x > 0$ and take $y = u'(x)$. Let $\hat{\eta} \in \mathscr{D}(y)$ be the optimizer to the dual problem (3.6) and $\hat{\xi} \in \mathscr{C}(x)$ be the optimizer to the primal problem (3.5). Both $\hat{\eta}$ and $\hat{\xi}$ exist by Lemmas 3.7 and 3.8 respectively. Using the definition of V, (3.1), (3.3), and Theorem 23.5 in [22] we get

$$0 \leq \int_{\Omega} \left(V\left(\hat{\eta}\right) - U\left(\hat{\xi}\right) + \hat{\xi}\hat{\eta} \right) d\mu \leq v(y) - u(x) + xy = 0.$$

Therefore, for μ a.e. $\omega \in \Omega$ we have

$$V\left(\hat{\eta}\right) = U\left(\hat{\xi}\right) - \hat{\xi}\hat{\eta}.$$

This implies the remaining assertions of the theorem:

$$\begin{gathered} U'\left(\hat{\xi}\right) = \hat{\eta} \quad \mu \text{ a.e.}, \\ \langle \hat{\xi}, \hat{\eta} \rangle = \textstyle\int_{\Omega} U\left(\hat{\xi}\right) d\mu - \int_{\Omega} V\left(\hat{\eta}\right) d\mu = u(x) - v(y) = xy. \end{gathered}$$

□

In order to prove Theorem 3.3 we proceed in a way that is similar to the proof of Proposition 1 in Kramkov and Schachermayer [18]. Define the *polar* of a set $A \subseteq \mathbf{L}^0_+$ as

$$A^o \triangleq \left\{ \xi \in \mathbf{L}^0_+ : \ \langle \xi, \eta \rangle \leq 1 \ \text{ for all } \eta \in A \right\}.$$

A subset A of $\mathbf{L}^0_+$ is called *solid* if $0 \leq \eta \leq \zeta$ and $\zeta \in A$ implies that $\eta \in A$. Observe that the sets $\mathscr{C}$ and $\mathscr{D}$ satisfy the bipolar relations. We will use a version of the *bipolar theorem* that was proven by Brannath and Schachermayer in [2]: for a subset A of $\mathbf{L}^0_+$ the bipolar A^{oo} is the smallest subset of $\mathbf{L}^0_+$ containing A, which is convex, solid, and closed with respect to the topology of convergence in measure.

Lemma 3.10. *Under the conditions of Theorem 3.2, for every fixed $y > 0$ let $\hat{\eta}(y)$ be the minimizer to the dual problem (3.6). Then there exists a sequence $(\zeta^n)_{n \geq 1}$ in $\tilde{\mathscr{D}}$ that μ a.e. converges to $\hat{\eta}(y)/y$.*

Proof. Fix $y > 0$. By assumption $\tilde{\mathscr{D}}$ is a convex set that satisfies (3.9). Therefore, applying the bipolar theorem (see [2]) we deduce that $\mathscr{D}$ is the smallest convex, closed and solid subset of $\mathbf{L}^0_+ (\Omega, \mathscr{F}, \mu)$ containing $\tilde{\mathscr{D}}$. Thus for any $\eta \in \mathscr{D}$ there exists a sequence $(\zeta^n)_{n \geq 1}$ in $\tilde{\mathscr{D}}$ such that $\zeta = \lim\limits_{n \to \infty} \zeta^n$ exists μ a.e. and $\zeta \geq \eta$. In particular such a sequence exists for $\eta = \hat{\eta}(y)/y$. We deduce from optimality of $\hat{\eta}(y)$ that $\eta = \zeta = \lim\limits_{n \to \infty} \zeta^n$. □

Lemma 3.11. *Under the conditions of Theorem 3.2 for each $y > 0$ we have*

$$\inf_{\eta \in \tilde{\mathscr{D}}} \int_\Omega V(y\eta) d\mu < \infty.$$

Proof. To simplify notations we will assume that $y = 1$. Let $(a^n)_{n \geq 1}$ be a sequence of strictly positive numbers such that $\sum\limits_{n=1}^{\infty} a^n = 1$. By Lemma 3.7, for each $n \geq 1$ there exists $\hat{\eta}(a^n)$, the minimizer to the dual problem (3.6) when $y = a^n$. One can construct a sequence of strictly positive numbers $(\delta_n)_{n \geq 2}$ that decreases to 0, such that

$$\sum_{n=1}^{\infty} \int_\Omega V\left(\hat{\eta}(a^n)\right) 1_{A_n} d\mu < \infty, \quad \text{if } A_n \in \mathscr{F}, \text{and } \mu(A_n) \leq \delta_n, \ n \geq 2. \quad (3.19)$$

From Lemma 3.10 we deduce the existence of a sequence $(\eta^n)_{n \geq 1} \subset \tilde{\mathscr{D}}$ such that

$$\mu\left(V\left(a^n \eta^n\right) > V\left(\hat{\eta}(a^n)\right) + 1\right) \leq \delta_{n+1}, \ \ n \geq 1.$$

Define the sequences of measurable sets $(B_n)_{n \geq 1}$ and $(A_n)_{n \geq 1}$ as follows:

$$B_n \triangleq \left\{V\left(a^n \eta^n\right) \leq V\left(\hat{\eta}(a^n)\right) + 1\right\}, \quad n \geq 1,$$

$$A_1 \triangleq B_1, \ldots, A_n \triangleq B_n \backslash \left(\bigcup_{k=1}^{n-1} A_k\right), \ldots.$$

Then $(A_n)_{n \geq 1}$ is a measurable partition of Ω and $\mu\left(A_n\right) \leq \delta_n$ for $n \geq 2$.

To finish the proof, let $\eta \triangleq \sum\limits_{n=1}^{\infty} a^n \eta^n$. Then $\eta \in \tilde{\mathscr{D}}$, since $\tilde{\mathscr{D}}$ is closed under countable convex combinations. From the construction of $(A_n)_{n \geq 1}$, monotonicity of V, and (3.19) we obtain

$$\begin{aligned}
\int_\Omega V(\eta) d\mu &= \sum_{n=1}^{\infty} \int_\Omega V\left(\sum_{j=1}^{\infty} a^j \eta^j\right) 1_{A_n} d\mu \\
&\leq \sum_{n=1}^{\infty} \int_\Omega V\left(a^n \eta^n\right) 1_{A_n} d\mu \\
&\leq \sum_{n=1}^{\infty} \int_\Omega V\left(\hat{\eta}(a^n)\right) 1_{A_n} d\mu + \mu(\Omega) \\
&< \infty.
\end{aligned}$$

This concludes the proof of the lemma. □

Proof of Theorem 3.3. By symmetry between the primal and dual problems, it suffices to prove that

$$v(y) = \inf_{\eta \in \tilde{\mathscr{D}}} \int_\Omega V(y\eta) d\mu, \quad y > 0.$$

Fix $y > 0$ and $\varepsilon > 0$. We will show that there exists $\eta \in \tilde{\mathscr{D}}$ such that

$$\int_\Omega V\left((y+\varepsilon)\eta\right) d\mu \leq v(y) + \varepsilon.$$

Let $\hat{\eta} \in \mathscr{D}(y)$ be the minimizer to the dual problem (3.6), ζ be an element of $\tilde{\mathscr{D}}$, such that

$$\int_\Omega V(\varepsilon\zeta)\, d\mu < \infty,$$

whose existence follows from Lemma 3.11. Let $\delta > 0$ be such that

$$\int_\Omega \left(|V(\hat{\eta})| + |V(\varepsilon\zeta)|\right) 1_A d\mu \leq \frac{\varepsilon}{2}, \quad \text{if } A \in \mathscr{F} \text{ with } \mu(A) \leq \delta.$$

By Lemma 3.10 there exists $\theta \in \tilde{\mathscr{D}}$ such that the set

$$B \triangleq \left\{ V(y\theta) > V(\hat{\eta}) + \frac{\varepsilon}{2\mu(\Omega)} \right\}$$

has measure $\mu(B) \leq \delta$. Define

$$\eta \triangleq \frac{y\theta + \varepsilon\zeta}{y + \varepsilon}.$$

Since $\tilde{\mathscr{D}}$ is convex it follows that $\eta \in \tilde{\mathscr{D}}$. By construction of the set B and monotonicity of V we obtain

$$\begin{aligned}
\textstyle\int_\Omega V\left((y+\varepsilon)\eta\right) d\mu &= \textstyle\int_\Omega V(y\theta + \varepsilon\zeta)\, d\mu \\
&\leq \textstyle\int_\Omega V(y\theta)\, 1_{B^c} d\mu + \int_\Omega V(\varepsilon\zeta)\, 1_B d\mu \\
&\leq \textstyle\frac{\varepsilon}{2} + \int_\Omega V(\hat{\eta})\, d\mu + \int_\Omega \left(V(\varepsilon\zeta) - V(\hat{\eta})\right) 1_B d\mu \\
&\leq v(y) + \varepsilon.
\end{aligned}$$

□

4 Proofs of the main theorems

Let us recall the concept of Fatou convergence of stochastic processes, see [7].

Definition 4.1. Let τ be a dense subset of $[0,\infty)$. A sequence of processes $(Y^n)_{n\geq 1}$ is *Fatou convergent on* τ to a process Y, if $(Y^n)_{n\geq 1}$ is uniformly bounded from below and

$$Y_t = \limsup_{s\downarrow t,\ s\in\tau} \limsup_{n\to\infty} Y^n_s = \liminf_{s\downarrow t,\ s\in\tau} \liminf_{n\to\infty} Y^n_s$$

almost surely for every $t \geq 0$. If $\tau = [0,\infty)$, then the sequence $(Y^n)_{n\geq 1}$ is called *Fatou convergent*.

We also recall that a probability measure $\mathbb{Q}$ is called an *equivalent local martingale measure* for $\mathscr{X}$, if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and every $X \in \mathscr{X}$ is a local martingale under $\mathbb{Q}$. We denote the set of equivalent local martingale measures by $\mathscr{M}^e$.

The following lemma can be thought as an extension of Theorem 5.12 in [6] to our settings. The proof of Lemma 4.2 is based on an application of Fatou convergence and the optional decomposition theorem, see [19, 7]. However, since assumption (2.5) is weaker than the condition $\mathscr{M}^e \neq \emptyset$ in [19, 7], we need to do extra work.

Lemma 4.2. *Let c be a nonnegative optional process and κ be a stochastic clock. Under the assumptions (2.1) and (2.5), the following conditions are equivalent:*

(i) $c \in \mathscr{A}$,

(ii) $\sup\limits_{Z\in\mathscr{Z}} \mathbb{E}\left[\int_0^\infty c_t Z_t d\kappa_t\right] \leq 1$.

Proof. Let $c \in \mathscr{A}$. Then there exists a predictable S-integrable process H, s.t.

$$1 + \int_0^t H_u dS_u \geq \int_0^t c_u d\kappa_u \geq 0, \quad t \geq 0.$$

Take an arbitrary $Z \in \mathscr{Z}$. Using supermartingale property of $Z_t(1+\int_0^t H_u dS_u)$, $t \geq 0$, we obtain for every $T \geq 0$

$$1 \geq \mathbb{E}\left[Z_T\left(1 + \int_0^T H_u dS_u\right)\right] \geq \mathbb{E}\left[Z_T \int_0^T c_u d\kappa_u\right].$$

Using localization and integration by parts we deduce

$$\mathbb{E}\left[Z_T \int_0^T c_u d\kappa_u\right] = \mathbb{E}\left[\int_0^T c_u Z_u d\kappa_u\right].$$

Taking $T \to \infty$ and using the monotone convergence theorem, we get (ii).

Conversely, assume that $\sup\limits_{Z\in\mathscr{Z}} \mathbb{E}\left[\int_0^\infty c_t Z_t d\kappa_t\right] \leq 1$. Using localization and integration by parts we deduce from (ii):

$$\mathbb{E}\left[Z_n \int_0^n c_u d\kappa_u\right] = \mathbb{E}\left[\int_0^n c_u Z_u d\kappa_u\right], \quad n \geq 0.$$

One can see that $\left\{(Z_t)_{t\in[0,n]} : \ Z \in \mathscr{Z}\right\}$ coincides with the set of cádlág densities of equivalent local martingale measures for $\mathscr{X}$ on $(\Omega, \mathscr{F}_n)$. Let us denote the set of such measures by $\mathscr{M}_n^e$. Then, by Proposition 4.2 in [19], there exists a cádlág process V^n on $[0, n]$ given by

$$V_t^n = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{M}_n^e} \mathbb{E}^{\mathbb{Q}}\left[\int_0^n c_u d\kappa_u | \mathscr{F}_t\right], \quad t \in [0, n],$$

which is a supermartingale under every $\mathbb{Q} \in \mathscr{M}_n^e$. Notice that $V_t^n \geq \int_0^t c_u d\kappa_u$, $t \in [0, n]$, and $V_0^n \leq 1$. Now, applying Theorem 4.1 in [7], we can write V^n as

$$V_t^n = V_0^n + \int_0^t H_u^n dS_u - A_t^n, \quad t \in [0, n],$$

where H^n is predictable S-integrable and A^n is optional and increasing, s.t. $A_0^n = 0$. Let us extend H^n to $[0, \infty)$ by setting $H_t^n \triangleq 0$ for $t > n$. Using Lemma 5.2 in [7], we can construct a sequence of stochastic processes $Y^n \in \operatorname{conv}\left(1 + \int_0^\cdot H_u^n dS_u, 1 + \int_0^\cdot H_u^{n+1} dS_u, \dots\right)$, $n \geq 1$, and a process Y, such that $(ZY^n)_{n\geq1}$ is Fatou convergent on the set of positive rational numbers to a supermartingale ZY for every $Z \in \mathscr{Z}$. Then, we have $Y_t \geq \int_0^t c_u d\kappa_u$, $t \geq 0$, and $Y_0 \leq 1$. Now, on $[0, n]$ using Theorem 4.1 in [7], we get

$$Y_t = Y_0 + \int_0^t G_u^n dS_u - B_t^n, \quad t \in [0, n],$$

where G^n is predictable S-integrable and B^n is optional and increasing with $B_0^n = 0$. Let us set $G_t^n \triangleq 0$ for $t > n$. Denoting

$$n(t) \triangleq \min\{n \in \mathbb{N} : \ n > t\}, \quad t \geq 0,$$

we deduce that the process

$$\tilde{G}_t \triangleq \sum_{k=1}^{n(t)} \left(G_t^k - G_t^{k-1}\right), \quad t \geq 0,$$

is such that $1 + \int_0^t \tilde{G}_u dS_u \geq \int_0^t c_u d\kappa_u$, $t \geq 0$. Thus, $c \in \mathscr{A}$. $\square$

Lemma 4.3. *Let κ be a stochastic clock. Under the assumptions (2.1) and (2.5), for every $c \in \mathscr{A}$ we have*

$$\sup_{Z\in\mathscr{Z}} \mathbb{E}\left[\int_0^\infty c_t Z_t d\kappa_t\right] = \sup_{Y\in\mathscr{Y}} \mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right] \le 1.$$

Proof. By definition (2.9) for an arbitrary $Y \in \mathscr{Y}$ we can find a sequence $(Y^n)_{n\ge1}$ in the solid hull of $\mathscr{Z}$ (i.e., such that $Y^n \le Z^n$ $(d\kappa\times\mathbb{P})$ a.e. for some $Z^n \in \mathscr{Z}$), such that $(Y^n)_{n\ge1}$ converges $(d\kappa\times\mathbb{P})$ a.e. to Y. Using Fatou's lemma and Lemma 4.2 we get

$$\mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right] \le \liminf_{n\to\infty} \mathbb{E}\left[\int_0^\infty c_t Y_t^n d\kappa_t\right] \le \sup_{Z\in\mathscr{Z}} \mathbb{E}\left[\int_0^\infty c_t Z_t d\kappa_t\right] \le 1.$$

□

Denote by $\mathbf{L}^0 = \mathbf{L}^0(d\kappa\times\mathbb{P})$ the linear space of (equivalence classes of) real-valued optional processes on the stochastic basis $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge0}, \mathbb{P}\right)$ which we equip with the topology of convergence in measure $(d\kappa\times\mathbb{P})$. Let $\mathbf{L}^0_+$ be the positive orthant of $\mathbf{L}^0$. Recall that a *polar* of a set $A \subseteq \mathbf{L}^0_+$ is defined as:

$$A^o \triangleq \left\{Y \in \mathbf{L}^0_+ :\ \mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right] \le 1 \text{ for all } c \in A\right\}.$$

In view of Theorems 3.2 and 3.3 in order to complete the proofs of Theorems 2.3 and 2.4 it suffices to establish the following proposition. Note that the sets $\mathscr{C}$, $\mathscr{D}$ and measure μ correspond to the sets $\mathscr{A}$, $\mathscr{Y}$ and measure $(d\kappa\times\mathbb{P})$, the sets $\tilde{\mathscr{C}}$ and $\tilde{\mathscr{D}}$ accord with the sets $\mathscr{B}$ and $\mathscr{Z}$, respectively.

Proposition 4.4. *Assume that an $\mathbb{R}^d$-valued semimartingale S satisfies (2.5). Under the condition (2.1), the sets $\mathscr{A}$ and $\mathscr{Y}$, defined in (2.3) and (2.9), respectively, have the following properties:*

(i) $\mathscr{A}$ and $\mathscr{Y}$ are subsets of $\mathbf{L}^0_+$ that are convex, solid and closed in the topology of convergence in measure $(d\kappa\times\mathbb{P})$.

(ii) The sets $\mathscr{A}$ and $\mathscr{Y}$ satisfy the bipolar relations:

$$\begin{array}{lcl} c \in \mathscr{A} & \Leftrightarrow & \mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right] \le 1 \quad \text{for all} \quad Y \in \mathscr{Y}, \\ Y \in \mathscr{Y} & \Leftrightarrow & \mathbb{E}\left[\int_0^\infty c_t Y_t d\kappa_t\right] \le 1 \quad \text{for all} \quad c \in \mathscr{A}. \end{array}$$

(iii) There exists $c \in \mathscr{A}$ such that $c > 0$ and there exists $Y \in \mathscr{Y}$ such that $Y > 0$.

Proof. *(i)* It is enough to show closedness of $\mathscr{A}$. Let $(c^n)_{n\geq 1}$ be a sequence in $\mathscr{A}$ that $(d\kappa \times \mathbb{P})$ a.e. converges to c. For an arbitrary $Z \in \mathscr{Z}$ using Fatou's lemma and Lemma 4.2 we get:

$$\mathbb{E}\left[\int_0^\infty c_t Z_t d\kappa_t\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_0^\infty c_t^n Z_t d\kappa_t\right] \leq 1.$$

Therefore by Lemma 4.2, $c \in \mathscr{A}$, and thus $\mathscr{A}$ is closed.

(ii) It follows from Lemma 4.2 that

$$\mathscr{A} = \mathscr{Z}^o,$$

whereas from Lemma 4.3 we deduce

$$\mathscr{Y} \subseteq \mathscr{A}^o = \mathscr{Z}^{oo}. \tag{4.1}$$

Since $\mathscr{Y}$ is closed, convex, and solid and $\mathscr{Z} \subset \mathscr{Y}$, it follows from the bipolar theorem of Brannath and Schachermayer that $\mathscr{Z}^{oo} \subseteq \mathscr{Y}$. Combining this with (4.1) we conclude that

$$\mathscr{Y} = \mathscr{A}^o. \tag{4.2}$$

On the other hand it follows from part *(i)* that $\mathscr{A}$ is also convex, closed and solid. Thus $\mathscr{A} = \mathscr{A}^{oo}$ by the bipolar theorem. Therefore, from (4.2) we get

$$\mathscr{A} = \mathscr{Y}^o.$$

(iii) Since $\mathscr{X}$ contains a constant function $\mathbf{1} = (1)_{t\geq 0}$, the existence of $c \in \mathscr{A}$, such that $c > 0$, follows from the definition of the set $\mathscr{A}$. The existence of $Y \in \mathscr{Y}$, such that $Y > 0$, follows from assumption (2.5). This completes the proof of Proposition 4.4. $\square$

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