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# Numerical Approximation of Valuation Equations Incorporating Stochastic Volatility Models

by

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## ABSTRACT

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This dissertation studies the problem of controlling far field boundary errors arising in partial differential equation approaches for pricing financial contracts written on stochastic volatility models.

Feynman-Kac type results are obtained by relating finite domain Dirichlet problems to options bearing barrier features. We then adopt a probabilistic framework to show convergence for strictly sublinear contracts even when the underlying process is a local martingale, and for linear contracts when it is a proper martingale. By restricting the stochastic volatility models to a smaller class, upper bounds for the far field boundary errors are derived for linear contracts. Convergence does not hold for linear contracts dependent on strict local martingales. While rigorous results for this case are unavailable, we conjecture inverse second order convergence in the far boundary distance when appropriate Neumann boundary conditions are imposed.

Effective use of a finite difference alternating direction implicit algorithm is dis-

cussed. This scheme is implemented to test convergence theories and conjectures on well known models, such as the Bessel model and the Heston model.

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# CHAPTER 1

# Introduction

## 1.1 Background and Overview of Thesis

Stochastic volatility models are widely used in fixed income pricing because of their abilities to capture the skews and smiles of the implied volatility in real financial markets. This phenomenon has been observed and studied by a number of practitioners and researchers, for example, [12], [2], [36].

Option valuation for stochastic volatility models often requires the use of numerical techniques. There are two basic numerical techniques: Monte Carlo simulation which considers the option price as the average of all simulated realizations, and solving partial differential equations (PDE) with finite difference methods. There is extensive literature discussing numerical algorithms for these two methods, to name a few, [11], [54], [52] for Monte Carlo simulation, and [39], [16], [34], [32] for finite difference schemes. The PDE approach outperforms Monte Carlo simulation for stochastic volatility models, because the dimension in this setting is only two. Our thesis is devoted to stochastic volatility models, using the PDE approach.

In the PDE approach for option valuation, the general idea is that one first obtains a Black-Scholes type partial differential equation (BS-PDE). As a second step, domain truncation incorporating artificial far field boundary conditions converts the

unbounded initial value problem (the BS-PDE) to an initial boundary value problem. Lastly, some finite difference scheme is applied to numerically solve the initial boundary value problem.

In the first step, the relationship between the option value function and BS-PDE valuation equation is known as "Feynman-Kac regularity". Janson and Tysk (2006) in [35] show that the stochastic representation of up-to-polynomial-growth payoffs yields a classical solution to the corresponding BS-PDE with appropriate terminal conditions under linear growth rates conditions in the diffusion models of the underlying asset. Ekström and Tysk showed similar results for certain term structure models in 2008 ([23]), and for one dimensional local volatility models in 2009 ([24]). Under certain strong restrictions, they further proved regularity results for stochastic volatility models in 2010 ([25]). In particular, they assume smoothness and boundedness of the payoff function. In [6], Bayraktar, Kardaras and Xing (2011) further refined these results by relaxing the assumptions, and providing conditions for the uniqueness of BS-PDE solutions.

Much existing literature studies finite difference methods to reduce the truncation errors arising from the last step of the process. The foundational work dates back to the classic explicit, implicit methods and Crank-Nicolson method ([17]). In order to address cross derivatives in the two-dimensional case, Douglas and Rachford ([21]) proposed their first order convergence "Do" scheme. Later, Craig and Sneyd ([16]) improved the precision to second order. A modified Craig-Sneyd scheme ([34]) and the Hundsdorfer-Verwer scheme ([32], [56]) allow more degrees of freedom.

For the second step of the process, to the best of our knowledge, very little rigorous work has been done on estimating the far field boundary errors. Kangro and Nicolaides [37] were among the first to consider the far field boundary errors. In [37],

they use pure PDE techniques, such as Schauder estimates and a priori estimates, to derive pointwise bounds for the far field boundary errors in a diffusion model with strong restrictions on uniform growth rates of both drifts and volatility matrix. The Black-Scholes model falls into their category, but most stochastic volatility models do not. Song (2011) in [52] shows the convergence of far field boundary errors for strictly sublinear payoffs written on one-dimensional local volatility models, and he proposes approximating linear growth payoffs with bounded payoffs in the case where the underlying process is a local martingale. Ekström, Lötstedt, Sydow and Tysk (2011) in [22] also notice the local martingale issue in one-dimensional local volatility models, for which they proposed to use a Neumann boundary condition for the PDE approach. They restrict the payoff functions to be non-decreasing so that the maximum principle can be applied to prove the convergence. Nevertheless, there is no existing systematic literature discussing the numerical pricing issues for stochastic volatility models in terms of controlling far field boundary errors.

In our thesis, we provide detailed theoretical results on controlling far field boundary errors generated from numerical pricing of at most linear growth contracts written on a large class of stochastic volatility models. Our results focus on the following questions:

1. What kind of boundary condition is suitable for numerical pricing under a particular stochastic volatility model?

2. Given a boundary condition type, what boundary conditions are appropriate?

3. If the solution of the initial boundary value problem converges to the option price, what is an upper bound for the far field boundary errors in the underlying direction and volatility direction?

4. Empirically, for good efficiency, how far away should one put the far field boundary, given the error estimates?

We adopt a different framework (and a different stochastic model class) from that of [37], [52] and [22]: specifically we use a probabilistic framework for stochastic volatility models rather than a pure PDE framework applied to Black-Scholes and one dimensional local volatility models. In the PDE approach, given the regularity result that the solution of an initial value problem is the option price, it is proved that a sequence of initial boundary value problems is convergent to the BS-PDE, which is the initial value problem. This method requires the comparison of two solutions to the partial differential equations. By contrast, our probabilistic framework compares two forms of stochastic representation with one being the original value function and the other one being the option price with barrier features. Since the real numerical scheme is applied to the initial boundary value problem, our framework requires the "Feynman-Kac regularity" which states that the solution to the initial boundary value problem is just the price of an option with barrier features. After we obtain the regularity, we use probability inequalities and stochastic differential equation techniques to estimate the difference in the solutions.

In our framework, we consider the convergence theorem based on an even more general model than [25] and [6]. In particular, we allow faster than linear growth rate in the volatility of the stochastic volatility process. The regularity results in [25] and [6] are developed for functions on an unbounded domain. In this work, we use probabilistic arguments to establish regularity on a bounded domain. We also consider the case of volatility of volatility growing super linearly. We show the required regularity results in Chapter 3. The tricky part is to rigorously show that the stochastic representation is continuous in three dimensions, due to the involvement

of stopping times and fast growth rates in the parameter functions of the model. Although, for a smaller model class, [6] shows that uniqueness for the BS-PDE may not hold, we show that the corresponding Dirichlet PDE always has a unique solution.

Based on our regularity foundation, we develop general convergence theorems without emphasizing rates. As an answer to Question 1 and Question 2, we show that if the payoff function is of strictly less than linear growth, then Dirichlet boundary conditions are always suitable for numerical pricing. In fact, any boundary condition of Dirichlet type works well. In addition, we bound the convergence rate, which partially answers Question 3. [52] shows the same convergence rate for one-dimensional local volatility models. When the underlying process is a strict local martingale, we give examples to show that such convergence fails on linear contracts. When the underlying process is a proper martingale, to give a better answer to Question 3, we restrict attention to the "Heston-type" models, which allows estimation of the tail distribution for both the underlying process and the stochastic volatility process. This may seem restrictive, but the Heston-type models are still rich enough to both preserve theoretical variety and reflect market behaviors. In particular, the stochastic volatility process can still have faster than linear growth. Since the payoff function does not depend on volatility, the far field boundary distance in the volatility direction can be handled in such a way that the overall far field boundary errors are dominated in the underlying direction. Under this setup, we shows some specific convergence rate theorems for some popular models that have practical value. Our proofs of convergence and error estimates are based on probability inequalities and stochastic calculus techniques, in contrast to the PDE techniques in [37]. We believe such an argument is more intuitive and straightforward.

In the case of numerical pricing of linear contracts based on strict local martin-

gales, we conjecture that Neumann type initial boundary value problems can approximate the value function to second order in the inverse distance of the far field boundary. However, those boundary conditions may not have arbitrary data and we provide heuristic explanations from a stochastic calculus and hedging point of view on how to choose an appropriate Neumann boundary condition. We believe similar rigorous results in this case can be obtained by an extension of the current techniques.

To verify our theoretical results and conjectures, we implement a finite difference alternating direction implicit algorithm for the Bessel and Heston models. These numerical experiments give empirical guidance how far away one should set up the far field boundary (Question 4).

In summary, our regularity results and convergence theorems are new, and they are a continuation and extension of similar existing theories in the Black-Scholes model and local volatility models. The new results provide theoretical support for practical use of stochastic volatility models.

### 1.2 Organization of Thesis

In Chapter 2, we provide the mathematical background for option pricing in a general sense. Specifically, we define two different types of numerical errors: the far field boundary errors and truncation errors. Two "toy" models are employed as examples to illustrate the importance of the different types of error. We also introduce some basic knowledge and examples of advanced stochastic models: the local volatility models and stochastic volatility models.

Chapter 3 is devoted to building the regularity results. Formal definitions of valuation functions and their relation to PDEs are given. We break the proof of the

regularity results into four sections to show continuity, interior regularity, boundary conditions and existence and uniqueness of Dirichlet problems respectively.

Convergence theories for Dirichlet problems are extensively studied in Chapter 4. We start this chapter with general convergence results without emphasizing the rate of convergence. The importance of being a proper martingale for the underlying process is revealed. By imposing more assumptions on the models, e.g., the "Heston-type" model, we obtain a rate of convergence. Supporting finite difference techniques and experiments are discussed to close this chapter.

Numerical pricing for strict local martingales is discussed in Chapter 5. This topic is open, and many of those results are to be developed. We make some conjectures on the use of Neumann problems for pricing linear contracts written on strict local martingales. Intuition from the stochastic point of view is supplemented. Numerical experiments demonstrates the validity of these conjectures in practice.

# CHAPTER 2

# Mathematical Background

This chapter sets up the mathematical background of the motivations discussed in the previous chapter. We first define the far field boundary error and truncation error. Then, we use two option pricing examples with the underlying process being a true martingale and strictly local martingale, respectively, to illustrate the importance of considerating the far field boundary error from a numerical perspective. At the end of the chapter, we introduce the Heston type of stochastic volatility model which shall serve as the focus of our study.

## 2.1 Far Field Boundary Error and Truncation Error

Far field boundary approximation errors and truncation errors are distinct types of error that occur when solving partial differential equations (PDE) by finite differences. We start with a discussion of these errors.

### 2.1.1 Far field boundary error

For simplicity, we assume the interest rate $r$ is flat at 0. Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be an $n$-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$ being the risk-neutral measure, and such that the asset process $X$ is an $n$-dimensional nonnegative $\{\mathcal{F}_t\}$-adapted continuous Markov process. Let

$g \in C((0,\infty)^n)$ be a payoff function.

According to risk-neutral pricing theory, there exists a measurable function $u(t,x)$ such that the option paying $g(X_T)$ at the maturity $T$ has the time $t$ price

$$u(t, X_t) = \mathbb{E}\left[g\left(X_T\right)|\mathcal{F}_t\right]. \tag{2.1}$$

If the expectation exists, the pricing problem reduces to solving for $u(t,x)$. In most cases, analytical calculation of the expectation in (2.1) is inefficient or impossible. When the dimension is not too large, one possibility is to derive a PDE for $u(t,x)$ and solve the PDE numerically. Heuristically, $u(t,x)$ is a solution of the following initial value problem

$$\begin{cases} v_t + \mathcal{L}v = 0, & (t,x) \in [0,T) \times D_\infty, \\ v(T,x) = g(x), & x \in D_\infty. \end{cases} \tag{2.2}$$

where $\mathcal{L}$ is the parabolic operator associated with the stochastic process $X$, $D_\infty = (0,\infty)^n$.

Virtually no numerical methods work on unbounded domains. In practice, we usually constrain the initial value problem to a finite domain, which results in an initial boundary value problem (IBVP), such as a Dirichlet problem, a Neumann problem or a combination of these two.

$$\begin{cases} v_t + \mathcal{L}v = 0, & (t,x) \in [0,T) \times D_M, \\ v(T,x) = g(x), & x \in D_M, \\ v(t,x) = h(x), \text{or}, v_x(t,x) = h(x), & t \in (0,T), x \in \partial D_M. \end{cases} \tag{2.3}$$

Here $D_M = (0,M)^n$, $M$ is a positive constant and $h(x) \in C((0,\infty)^n)$. We use $v^M(t,x)$, the solution to (2.3), as an approximation of $u(t,x)$, and find a numerical solution of $v^M(t,x)$. We are now ready to define the far field boundary error.

**Definition 2.1.** Let $u(t,x)$ be defined as in (2.1). Assume the operator $\mathcal{L}$ exists, and the Dirichlet problem (2.3) has a unique solution $v(t,x)$. Then

$$\mathcal{E}^M(t,x) \triangleq \left|v^M(t,x) - u(t,x)\right| \tag{2.4}$$

is defined to be the pointwise far field boundary error of the Dirichlet problem (2.3) with respect to pricing function $u(t,x)$ of (2.1).

**Definition 2.2.** A sequence of solutions $\{v^n(t,x); n = 1,2,3...\}$ to a family of initial boundary value problems is convergent to the pricing function $u(t,x)$ of (2.1) if and only if

$$\lim_{n\to\infty} \mathcal{E}^n(t,x) = 0. \tag{2.5}$$

The minimum requirement to use a sequence $v^n(t,x)$ to approximate $u(t,x)$ is that the equation (2.5) holds. The PDE method to verify this criterion is to compare solutions to (2.3) with those of (2.2). An immediate difficulty is that the solutions to (2.2) may not be unique while (2.3) usually has a unique solution. Our discussion of this and other issues will be probabilistic. In other words, we work with a stochastic representation of the solution to (2.3). We start with a heuristic guess of the representation, and we will provide rigorous arguments for some specific models in the next chapter.

We introduce the stopping time

$$\tau_{D_M} \triangleq \inf\{t \geq 0; X_t \in D_M^c\} \tag{2.6}$$

as the first exit time from $D_M$. Our guess for the stochastic representation of the Dirichlet problem is

$$v^M(t,X_t) = \mathbb{E}\left[g(X_T)\,1_{\left\{\tau_{D_M}>T\right\}} + h\left(X_{\tau_{D_M}}\right)1_{\{\tau_{D_M}\leq T\}}|\mathcal{F}_t\right], 0 \leq t < T. \tag{2.7}$$

Taking the difference of (2.1) and (2.7), we can express the far field boundary error of a Dirichlet problem with respect to the pricing equation as

$$\mathcal{E}^M(t, X_t) = \left| \mathbb{E}\left[ g\left(X_T\right) 1_{\left\{\tau_{D_M} \leq T\right\}} - h\left(X_{\tau_{D_M}}\right) 1_{\{\tau_{D_M} \leq T\}} | \mathcal{F}_t \right] \right|, 0 \leq t < T. \tag{2.8}$$

### 2.1.2 Truncation error

In the numerical solution of PDEs, truncation errors arise from using finite differences to approximate derivatives of continuous functions. For example, given $F(x) \in C^2(\mathbb{R})$, $x_0 < x_1 < x_2 < ... < x_m$, and $\Delta x_i = x_i - x_{i-1}$,

$$\begin{aligned} F''(x_i) &= \delta_{i,-1} F(x_{i-1}) + \delta_{i,0} F(x_i) + \delta_{i,1} F(x_{i+1}) + \epsilon_i, \\ \delta_{i,-1} &= \frac{2}{\Delta x_i (\Delta x_i + \Delta x_{i+1})}, \\ \delta_{i,0} &= \frac{-2}{\Delta x_i \Delta x_{i+1}}, \\ \delta_{i,1} &= \frac{2}{\Delta x_{i+1} (\Delta x_i + \Delta x_{i+1})}. \end{aligned}$$

Then the $\epsilon_i$ in the above equation is known as the truncation error for the approximation of the second derivative of $F(x)$ at $x_i$. Similarly, the explicit finite difference method to approximate the PDE in the initial boundary value problem (2.3) is

$$v_t + \mathcal{L}v\Big|_{(t,x_i)} = \frac{v(t + \Delta t, x_i) - v(t, x_i)}{\Delta t} + Lv(t, x_i) + \epsilon_i, \tag{2.9}$$

where $L$ is the discretization of operator $\mathcal{L}$, and $\epsilon_i$ is the truncation error.

## 2.2 Examples for Far Field Boundary Error

As it is illustrated in the section 2.1, the option pricing problem can be first converted to an initial boundary value problem, and then the initial boundary value problem can be solved numerically. This work-flow is

$$\text{Pricing Equation} \xrightarrow{Far Field Boundary Error} \text{IBVP} \xrightarrow{Truncation Error} \text{Discretized PDE}$$

In order to obtain a reliable numerical solution of a pricing problem, it is necessary to consider these two different types of errors. In this section, let us take a close look at the Far Field Boundary Error in two examples.

### 2.2.1 Geometric Brownian motion

Let us consider a classic pricing problem in the Black-Scholes world. We continue to assume the interest rate is 0. Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a one dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There is a single stock in the financial market governed by a geometric Brownian motion

$$dS_t = \sigma S_t dW_t, \sigma \in \mathbb{R}^+. \tag{2.10}$$

On a finite time horizon $T$, there is a European-style option $g(S_T)$ written on $S_T$, where the payoff function $g$ is nonnegative with at most of linear growth, i.e., $g(x) \leq c(1+x)$ for some constant $c$. By risk-neutral pricing, at time $0 \leq t \leq T$, this option is worth

$$u(t, S_t) \triangleq \mathbb{E}\left[g\left(S_T\right)|\mathcal{F}_t\right]. \tag{2.11}$$

The function $u(t,x)$ satisfies the Black-Scholes PDE

$$\begin{cases} u_t + \dfrac{1}{2}\sigma^2 x^2 u_{xx} = 0, & (t,x) \in [0,T) \times D_\infty, \\ u(T,x) = g(x), & x \in D_\infty. \end{cases} \tag{2.12}$$

Let us restrict the state space of the Black-Scholes equation to the finite interval $(0, M)$; we have the following Dirichlet problem

$$\begin{cases} v_t + \dfrac{1}{2}\sigma^2 x^2 v_{xx} = 0, & (t,x) \in [0,T) \times D_M, \\ v(T,x) = g(x), & x \in D_M, \\ v(t,M) = 0, & t \in [0,T). \end{cases} \tag{2.13}$$

**Proposition 2.3.** *The Dirichlet problem (2.13) has a unique solution $v^M(t,x)$. Furthermore,*

$$v^M(t,S_t) = \mathbb{E}\left[g\left(S_T\right)1_{\{\tau>T\}}|\mathcal{F}_t\right], \tag{2.14}$$

*where $\tau = \inf\{t \geq 0; S_t > M\}$.*

*Proof.* The existence, uniqueness and regularity of the solution to (2.13) can be proved by doing a log-transform of the second variable and then appealing to known results about the heat equation. It is then a straightforward application of Itô's formula to establish (2.14). □

**Theorem 2.4.** *For $M \geq 1 + S_0$, the far field boundary error of $v^M(t,x)$ relative to $u(t,x)$ satisfies*

$$\mathcal{E}^M\left(0,x\right) \leq C\sqrt{\frac{e^{-(\ln M)^2}}{\ln M}}. \tag{2.15}$$

*Here, $C$ is a constant independent of $M$.*

*Proof.* Equation (2.8), and at most linear growth property of the payoff function $g$ imply

$$\begin{aligned}
\mathcal{E}^M(0,x) &= \mathbb{E}\left[g\left(S_T\right)1_{\{\tau\leq T\}}\right] \\
&\leq c\mathbb{E}\left[\left(1+S_T\right)1_{\{\tau\leq T\}}\right] \\
&= c\mathbb{P}\left[\tau \leq T\right] + c\mathbb{E}\left[S_T 1_{\{\tau\leq T\}}\right]. \qquad (2.16)
\end{aligned}$$

Notice that

$$\begin{aligned}
\mathbb{E}\left[S_T\right] = S_0 = S_{0\wedge\tau} = \mathbb{E}\left[S_{T\wedge\tau}\right] &= \mathbb{E}\left[S_T 1_{\{\tau>T\}}\right] + \mathbb{E}\left[S_\tau 1_{\{\tau\leq T\}}\right] \\
&= \mathbb{E}\left[S_T 1_{\{\tau>T\}}\right] + M\mathbb{P}\left[\tau \leq T\right].
\end{aligned}$$

This gives

$$\mathbb{E}\left[S_T 1_{\{\tau \le T\}}\right] = M\mathbb{P}\left[\tau \le T\right]. \tag{2.17}$$

From (2.16) and (2.17), and with a slight abuse of notation, it suffices to show

$$\mathbb{E}\left[S_T 1_{\{\tau \le T\}}\right] \le C\sqrt{\frac{e^{-(\ln M)^2}}{\ln M}}. \tag{2.18}$$

Let us introduce the notation for the running maximum $S_T^* = \max_{0\le t\le T} S_t$, and a new probability measure $\tilde{\mathbb{P}}$ by

$$\begin{aligned}
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= Z_T, \\
dZ_t &= \frac{1}{2}\sigma Z_t dW_t, \\
Z_T &= e^{\frac{1}{2}\sigma W_T - \frac{1}{8}\sigma^2 T}.
\end{aligned}$$

An application of Girsanov theorem (Theorem 3.5.1 Karatzas and Shreve [38] ) gives $\tilde{\mathbb{P}}$ - Brownian motion $\tilde{W}_t = W_t - \frac{1}{2}\sigma t$. Therefore, the solution to the geometric Brownian motion (2.10) is

$$\begin{aligned}
S_T &= S_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} \\
&= S_0 e^{\sigma \tilde{W}_T}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\mathbb{E}\left[S_T 1_{\{\tau \le T\}}\right] &= \mathbb{E}\left[S_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} 1_{\left\{\left(S_0 e^{\sigma \tilde{W}_T}\right)^* \ge M\right\}}\right] \\
&= S_0 \mathbb{E}\left[e^{\sigma W_T - \frac{1}{2}\sigma^2 T} 1_{\left\{\tilde{W}_T^* \ge \frac{\ln \frac{M}{S_0}}{\sigma}\right\}}\right] \\
&= S_0 \mathbb{E}\left[Z_T e^{\frac{1}{2}\sigma W_T - \frac{3}{8}\sigma^2 T} 1_{\left\{\tilde{W}_T^* \ge \frac{\ln \frac{M}{S_0}}{\sigma}\right\}}\right] \\
&= S_0 e^{-\frac{1}{8}\sigma^2 T} \tilde{\mathbb{E}}\left[e^{\frac{1}{2}\sigma \tilde{W}_T} 1_{\left\{\tilde{W}_T^* \ge \frac{\ln \frac{M}{S_0}}{\sigma}\right\}}\right]
\end{aligned}$$

$$\leq S_0 e^{-\frac{1}{8}\sigma^2 T}\tilde{\mathbb{E}}\left[e^{\sigma\tilde{W}_T}\right]^{\frac{1}{2}}\tilde{\mathbb{E}}\left[1_{\left\{\tilde{W}_T^*\geq\frac{\ln\frac{M}{S_0}}{\sigma}\right\}}\right]^{\frac{1}{2}}, (Cauchy-Schwartz)$$

$$= S_0 e^{\frac{1}{8}\sigma^2 T}\tilde{\mathbb{P}}^{\frac{1}{2}}\left[\tilde{W}_T^*\geq\frac{\ln\frac{M}{S_0}}{\sigma}\right]$$

$$= S_0 e^{\frac{1}{8}\sigma^2 T}\sqrt{\int_{\frac{\ln\frac{M}{S_0}}{\sigma}}^{\infty}\sqrt{\frac{2}{\pi T}}e^{-\frac{z^2}{2T}}\,dz} \tag{2.19}$$

$$= S_0 e^{\frac{1}{8}\sigma^2 T}\sqrt{\sqrt{\frac{2T}{\pi}}\frac{\sigma}{\ln\frac{M}{S_0}}e^{-\frac{\left(\ln\frac{M}{S_0}\right)^2}{2\sigma^2 T}}} \tag{2.20}$$

$$\leq C(S_0,\sigma,T)\sqrt{\frac{e^{-(\ln M)^2}}{\ln M}},$$

Equation (2.19) follows from Problem 2.8.2 Karatzas and Shreve [38], and equation (2.20) results from Problem 2.9.22 of the same source. □

**Remark 2.5.** Note that Theorem 2.4 is similar to a result of Kangro and Nicolaides (2000) [37], where they adopted a PDE approach. The conclusion indicates that the far field boundary error for the Black-Scholes Model with at most linear growth in the contract is very small. The convergence rate, as shown by Theorem 2.4, is faster than any polynomial order, though slower than exponential order. In this case, the algebraic truncation error from discretization of the PDE becomes the main focus for controlling the accuracy of numerical solution.

### 2.2.2 Three dimensional Bessel process

Next we discuss a different PDE where the far field boundary error is more significant. This example arises from a three dimensional Bessel Process with a linear contract option.

Assume $B = \left\{B_t = \left(B_t^{(1)}, B_t^{(2)}, B_t^{(3)}\right), \mathcal{F}_t; 0 \leq t < \infty\right\}$ is a three dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $R_t$ the distance of

$B$ from the origin, i.e.

$$R_t = \sqrt{\left(B_t^{(1)}\right)^2 + \left(B_t^{(2)}\right)^2 + \left(B_t^{(3)}\right)^2}.$$

By Proposition 3.3.21 Karatzas and Shreve [38], $R_t$ satisfies the following SDE

$$dR_t = \frac{1}{R_t}dt + dW_t, \tag{2.21}$$

where $W$ is another one dimensional Brownian motion on the same probability space. One well-known property of the process $R_t$ is that the origin is non-attainable, and this suggests that we can define $X_t = \frac{1}{R_t}$. By Itô's formula

$$\begin{aligned} dX_t = d\frac{1}{R_t} &= -\frac{1}{R_t^2}dR_t + \frac{1}{R_t^3}dt \\ &= -\frac{1}{R_t^2}dW_t \\ &= -X_t^2 dW_t. \end{aligned} \tag{2.22}$$

The process $X$, also known as a special case of CEV models [15], is well studied and there is an analytical form for its probability density [14]

$$\mathbb{P}\left[X_T \in dz | X_t = x\right] = \frac{x}{z^3}\frac{dz}{\sqrt{2\pi(T-t)}}\left\{e^{-\frac{(1/z-1/x)^2}{2(T-t)}} - e^{-\frac{(1/z+1/x)^2}{2(T-t)}}\right\}. \tag{2.23}$$

Now, suppose there is a contract paying $X_T$ at maturity T, with the interest rate flat at 0. The price of this contract at any time $0 \leq t \leq T$ is

$$u(t, X_t) = \mathbb{E}\left[X_T | \mathcal{F}_t\right]. \tag{2.24}$$

Equations (2.23) and (2.24) imply (see also [47])

$$u(t,x) = 2x\Phi\left(\frac{1}{x\sqrt{T-t}}\right) - x, \tag{2.25}$$

where $\Phi$ is the cumulative standard normal density function.

**Proposition 2.6.** *The function $u(t,x)$ defined in (2.24) is a solution to the initial value problem*

$$\begin{cases} v_t + \dfrac{1}{2}x^4 v_{xx} = 0, & (t,x) \in [0,T) \times [0,\infty), \\ v(T,x) = x, & x \in [0,\infty). \end{cases} \tag{2.26}$$

*Furthermore, uniqueness within the class of linearly growing solutions does not hold for (2.26).*

*Proof.* From (2.25), we know that $u(t,x) \in C^{1,2}((0,T) \times (0,\infty))$. A direct computation of $u_t$ and $u_{xx}$ establishes the equation $u_t + \frac{1}{2}x^4 u_{xx} = 0$. By letting $t \to T$, $\Phi(\frac{1}{x\sqrt{T-t}}) \to 1$ and we arrive at (2.26). Clearly, $v_1(t,x) = x$ is another solution of (2.26). In fact, $u(t,x) + k(u(t,x) - v_1(t,x))$ is a solution to the initial value problem (2.26) for any $k \in \mathbb{R}$. □

**Remark 2.7.** Notice that $X$ is a nonnegative strict local martingale (see Exercise 3.3.36 Karatzas and Shreve [38]). An application of Fatou's lemma gives that $X$ is a strict super-martingale, and hence $u(t,X_t) = \mathbb{E}[X_T|\mathcal{F}_t] < X_t$. This implies that $u(t,x) = 2x\Phi\left(\frac{1}{x\sqrt{T-t}}\right) - x < x$ is a stochastic solution (see Definition 3.36) of (2.26), and is smaller than the "martingale solution".

If we were going to solve the initial value problem (2.26) numerically, it is reasonable to consider the following Dirichlet problem

$$\begin{cases} v_t + \dfrac{1}{2}x^4 v_{xx} = 0, & x \in [0,M), t \in [0,T), \\ v(T,x) = x, & x \in [0,M), \\ v(t,M) = M, & t \in [0,T), \end{cases} \tag{2.27}$$

where $M$ is a positive constant serving as the far field boundary location. This choice of the far field boundary condition maintains the continuity of $v(t,x)$ at the corner

$t = T, x = M$. A direct verification shows that $v^M(t,x) = x$, no matter how large $M$ is, and by the maximum principle ( Friedman 1964 [29] ) this is the unique solution.

**Proposition 2.8.** *A Dirichlet problem for a model with the underlying process being a strict local martingale can fail to be convergent for linear contracts.*

*Proof.* Use Bessel process as a counter-example. In light of Definition 2.1, equation (2.25), and the solution to Dirichlet problem (2.27), the far field boundary error of (2.27) with respect to the Bessel pricing equation (2.25)

$$\begin{aligned}\mathcal{E}^M(t,x) &= \left|v^M(t,x) - u(t,x)\right| \\ &= 2x\left(1 - \Phi\left(\frac{1}{x\sqrt{T-t}}\right)\right).\end{aligned}$$

Therefore,

$$\lim_{M\to\infty} \mathcal{E}^n(t,x) \neq 0,$$

for any $0 \leq t < T$. □

**Remark 2.9.** Although the example is set up with the far field boundary condition at $x = M$ to be $M$, numerical experiments show that an arbitrary Dirichlet far field boundary condition, if not $u(t,M)$, has the IBVP solutions not converging to $u(t,x)$ of the Bessel process case. The failure of convergence for the Bessel model suggests that the far field boundary condition is much more significant than the truncation error of any specific numerical scheme. Therefore, it is particularly important to discuss when the Dirichlet problem solutions converge, and how fast they converge in terms of the distance of the far field boundary from the origin.

## 2.3 One Dimensional Local Volatility Models

In section 2.1, we used two examples to illustrate the necessity of understanding the far field boundary error. The two models have the form

$$dX_t = \alpha(X_t)dW_t, \tag{2.28}$$

where $\alpha$ is a function of one variable. This is often called a "one-dimensional time homogeneous local volatility model". In this section, we describe the regularity and properties of the one-dimensional model (2.28) as a guideline for the stochastic volatility models in future chapters.

Let us assume $\alpha$ is nonnegative, $\alpha(x) \neq 0$ for $x \neq 0$, and $\alpha^{-2} \in L^1_{loc}(0,\infty)$. Under such assumptions, (2.28) has a unique weak solution absorbed at zero. A full discussion of existence and uniqueness of (2.28) is in Engelbert and Schmidt [26], and can be found also in Section 5.5 of [38]. Delbaen and Shirakawa [20] found a condition for $X$ to be a true martingale.

**Proposition 2.10** (Delbaen and Shirakawa, 2002)**.** *The local martingale $X$ is a true martingale if and only if*

$$\int_c^\infty \frac{x}{\alpha^2(x)}\,dx = \infty, \text{ for some } c > 0. \tag{2.29}$$

If the model (2.28) is not time homogeneous, i.e.,

$$dX_t = \alpha(X_t, t)dW_t, \tag{2.30}$$

By imposing some regularity conditions on $\alpha(\cdot)$, Ekstrom and Tysk ([24]) have a necessary condition for it to be a strict local martingale in the following proposition.

**Proposition 2.11.** *If the $\alpha$ satisfies*

$$\alpha^2(x,t) \geq \epsilon x^\eta$$

*for all* $(x,t) \in [\frac{1}{\epsilon},\infty) \times [0,T]$, *where* $\epsilon > 0$ *and* $\eta > 2$ *are constants, then the underlying process* $X$ *is a strict local martingale. Moreover, for any time bounded away from expiry, the stock option (the option paying the stock itself) price is* $o(x^\delta)$ *for any positive* $\delta$, *and if* $\eta > 3$ *then the stock option price is bounded.*

**Remark 2.12.** Since the geometric Brownian motion satisfies (2.29), while the Bessel process does not, the Bessel process is inherently different from the geometric Brownian motion in terms of its martingale properties.

Again, if there is an option written on $X$ that pays $g(X_T)$ at maturity $T$, where $g$ is of at most linear growth, the option price is a solution to the following initial value problem [24]:

$$\begin{cases} v_t + \dfrac{1}{2}\alpha^2(x)v_{xx} = 0, & (t,x) \in [0,T) \times (0,\infty), \\ v(t,0) = g(0), & t \in [0,T), \\ v(T,x) = g(x), & x \in (0,\infty). \end{cases} \tag{2.31}$$

**Definition 2.13.** A function $u : [0,\infty) \times [0,T] \to \mathbb{R}$ is called a classical solution to the initial value problem (2.31) if

(1) $u \in C^{1,2}([0,T) \times (0,\infty))$,

(2) $u \in C([0,T] \times [0,\infty))$,

(3) $u$ solves (2.31).

Bayraktar and Xing [7] have provided the following regularity results which are useful for understanding the different behaviors in the examples from Section 2.2,

**Theorem 2.14** (Bayraktar and Xing, 2010)**.**

*(a) The initial value problem (2.31) has a unique classical solution ( if any ) in the class of functions with at most linear growth in* $x$ *if (2.29) is satisfied.*

*(b) If we further assume that* $\alpha : [0,\infty) \to [0,\infty)$ *is locally Hölder continuous with*

*exponent 1/2 and g is of linear growth, then the initial value problem (2.31) has a unique classical solution if and only if (2.29) is satisfied.*

*Proof.* See [7]. □

**Remark 2.15.** As a consequence of this result it follows that the initial value problem (2.26) for the Bessel process has more than one solution, a fact we observed earlier by exhibiting two solutions for the payoff function $g(x) = x$. It is quite tricky to find a family of IBVPs to approximate a particular solution of the initial value problem when there is no uniqueness.

## 2.4 Introduction to Stochastic Volatility Models

In real financial markets, especially in the areas of interest rates, foreign exchange and commodities, single factor stochastic process models are rarely used, simply because they are not able to capture market behavior. Vanilla options based on single factor stochastic process models are exceptional. These options are so simple that most models can fully characterize their features. However, single factor models may sometime fail to recover the volatility, even though they are calibrated to the vanilla options.

To price and hedge exotic options, many of which are explicit (exotic) volatility options, volatility models are invariably necessary. It is desirable to model the volatility in a way that reflects market behaviors. In fact, the prices of many derivatives and exotics are explicitly related to future volatility levels; however the forward smile and skew of the volatility is often underestimated by local volatility models.

In contrast to local volatility models, stochastic volatility models use an independent process to model the volatility. Consequently, they are less computationally tractable than local volatility models in terms of having closed or semi closed solu-

tions. Nevertheless, they are able to fit the market volatility better than the local volatility models. Some stochastic volatility models will be introduced in this section.

**Hull-White model**

The Hull-White model [31] is one of the earliest stochastic volatility models used in financial modeling. Rather than assuming the underlying follows a geometric Brownian motion, they model the volatility by geometric Brownian motion. The Hull-White model has the following dynamics:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t, \\ d\nu_t &= \kappa \nu_t dt + \beta \nu_t dB_t, \\ dW_t dB_t &= \rho dt, \end{aligned}$$

where $W$ and $B$ are correlated Brownian motions, and $\sigma = \sqrt{\nu}$ is the stochastic volatility. It is not difficult to show that the stochastic volatility is log-normally distributed with the following properties:

$$\begin{aligned} \mathbb{E}[\sigma_t] &= \sigma_0 e^{\frac{1}{2}\kappa t - \frac{1}{8}\beta^2 t}, \\ \mathbb{V}[\sigma_t] &= \sigma_0^2 e^{\kappa t}(1 - e^{-\frac{1}{4}\beta^2 t}), \end{aligned}$$

where $\mathbb{V}[\sigma_t]$ denotes to the variance of $\sigma_t$.

It can be seen that this model suffers some deficiencies. If $\kappa < \frac{1}{4}\beta^2$ the volatility expectation converges as $t \to \infty$ to level 0, and the volatility expectation diverges for $\kappa > \frac{1}{4}\beta^2$. Also, the variance of volatility is either unbounded or diminishes over time. These phenomena rarely happen in real markets.

**Stein and Stein's model**

Stein and Stein [53] adopt a mean reverting process to model the square of stochastic volatility. By assuming $W$ and $B$ are Brownian motions on some probability

space, the model takes the following form:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t, \\ d\sigma_t &= \kappa(\theta - \sigma_t)dt + \beta dB_t, \\ dW_t dB_t &= \rho dt. \end{aligned}$$

By Itô's lemma, we obtain that

$$d\left(e^{\kappa t}\sigma_t\right) = e^{\kappa t}\kappa\theta dt + \beta e^{\kappa t} dW_t.$$

This further indicates the stochastic volatility follows a Gaussian distribution with mean and variance

$$\begin{aligned} \mathbb{E}[\sigma_t] &= \theta(1 - e^{-\kappa t}) + \sigma_0 e^{-\kappa t}, \\ \mathbb{V}[\sigma_t] &= \frac{\beta^2}{2\kappa}(1 - e^{-2\kappa t}). \end{aligned}$$

One observation is that when $t \to \infty$, both the expectation and the variance approach constant values. Roughly, the model suggests

$$\mathbb{E}[\sigma_t] \approx \theta, \text{ and } \mathbb{V}[\sigma_t] \approx \frac{\beta^2}{2\kappa}.$$

Also, since the stochastic volatility is Gaussian and can go negative the correlation between the underlying asset price and the stochastic volatility can suddenly change sign.

The simple form of the volatility and the geometric Brownian motion style of the underlying asset price suggests the existence of analytical solution for Stein and Stein's model. In fact, they have given a probability distribution for the stock price process $S$ [53]. However, practitioners may still seek numerical pricing as the closed form solution is rather complicated.

**Heston's model**

Heston's model has gained much attention in the fixed income and currencies market. Although this model itself is no longer used by practitioners in the original form, many of the practical currency models are derived from it. Heston [30] assumes the spot price satisfies

$$\begin{cases} dS_t = \sqrt{\nu_t} S_t dW_t, \\ d\nu_t = \kappa(\theta - \nu_t)dt + \beta\sqrt{\nu_t} dB_t, \\ dW_t dB_t = \rho dt. \end{cases} \tag{2.32}$$

The stochastic volatility model in (2.32) is known as a Cox-Ingersoll-Ross model. A nice property is that if the coefficients $\kappa, \theta, \beta$ are appropriately chosen, i.e., $2\kappa\theta \geq \beta^2$, the process $\nu$ will never go to zero.

Empirically, when the Heston model is calibrated to the market, $\kappa$ is quite small and the correlation $\rho$ is comparably high in order to generate sufficient skew. Therefore, $2\kappa\theta \geq \beta^2$ will not always hold, and stochastic volatility will stay near zero or very high for some time.

Although Heston gave a semi-closed form solution for European options in his paper, in practice, people still prefer numerical solutions. That's because the semi-closed form solution is given as an inverse Fourier transform and involves complicated integrations. Therefore, it is especially meaningful to obtain an efficient numerical pricing engine for the Heston model. In our thesis, the Heston model will serve as a test problem for many of the results to be developed in future chapters.

The three models discussed above are only a few among the widely used stochastic volatility models in industry and academia. Starting in the next Chapter, we will

devote our efforts to study the far field boundary error for a general class of stochastic volatility models.

# CHAPTER 3

# PDEs for Valuation Equations of Stochastic Volatility Models

This chapter is devoted to building the connection between the stochastic representation of an option price with an up-and-out feature, whose underlying asset price is modeled by a general stochastic volatility model, and the solution of an initial boundary value problem of parabolic type. For standard models, such as the Black-Scholes model, this connection is traditionally known as the Feynman-Kac theorem. We will prove similar connections for a larger class of models with a barrier feature. As discussed in the previous chapter, when we move the barrier towards infinity the stochastic representation is expected to converge to the European style option price (without any exotic feature). Bayraktar, Kardaras and Xing (2011) [6] showed the connections for initial value problems, and we will show similar results for IBVPs with a larger class of stochastic volatilities. Once the Feynman-Kac style results are proved, we will study the convergence in the following chapter.

## 3.1 Assumptions on the Stochastic Volatility Models

In this chapter, we shall exclusively consider two real-valued one-dimensional continuous processes, the underlying process $X = \{X_t; 0 \leq t < \infty\}$ and the stochastic volatility process $Y = \{Y_t; 0 \leq t < \infty\}$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

adapted to a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by two Brownian motions $W = \{W_t; 0 \leq t < \infty\}$ and $B = \{B_t; 0 \leq t < \infty\}$. We further assume that $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual condition, which refers to it being right continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathcal{F}$.

Our model has the following dynamics:

**Assumption 3.1.** *The underlying process and the stochastic volatility process satisfy:*

$$dX_t = b(Y_t)X_t dW_t, \tag{3.1}$$

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \tag{3.2}$$

$$dW_t dB_t = \rho dt.$$

**Remark 3.2.** In fact, the correlation between two Brownian motions does not have to be constant. The results in this chapter apply for the correlation being a deterministic function of time. For simplicity, we continue to consider financial markets with zero interest rates, but extension to the deterministic interest rate case is straightforward.

**Assumption 3.3.** *The parameter functions defined in (3.1) and (3.2) have the following properties:*

*(a) The* $b : [0,\infty) \to [0,\infty)$ *is strictly positive on* $(0,\infty)$*, and* $b(0) = 0$*. Also, there exists* $m > 0$ *such that*

$$|(b^2)'(\cdot)| \leq C(1 + (\cdot)^m).$$

*(b)* $\mu : [0,\infty) \to \mathbb{R}$*, and* $\sigma : [0,\infty) \to [0,\infty)$*.* $\sigma$ *is strictly positive on* $(0,\infty)$*.* $\sigma(0) = 0$*, and* $\mu(0) \geq 0$*. Further these two functions satisfy either (i) or (ii) below*

*(i)* $|\mu(\cdot)| + \sigma(\cdot) \leq C(1 + \cdot)$*, for some constant* $C$*.*

*(ii)* $|\mu(\cdot)| \leq C(1 + \cdot)$*, for some constant* $C$*, and* $\sigma(\cdot) = (\beta)(\cdot)^p$*, where* $p > 1$*.*

*(c) $\mu, \sigma^2, b^2$, and $b\sigma$ are continuously differentiable on $[0, \infty)$ with locally $\alpha-$Hölder continuous derivatives for some $\alpha \in (0, 1]$.*

**Remark 3.4.** We see that the three classic models introduced in Chapter 2 Section 2.4 satisfy Assumption 3.1 and Assumption 3.3. The Assumption 3.3.(b).(ii) clearly includes the classic model

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^p dB_t, \tag{3.3}$$

where the constant $p$ can be any real number greater or equal than $\frac{1}{2}$.

We restrict consideration to European-style payoff options written on the stock process $X$ having the following properties:

**Assumption 3.5.** *The payoff function $g : [0, \infty) \to [0, \infty)$ is nonnegative and continuous. $g$ is also of at most linear growth, i.e., there exists a positive constant $C$ such that $g(\cdot) \leq C(1 + \cdot)$, or equivalently $\limsup_{x\to\infty} \frac{g(x)}{x} \in [0, \infty)$ .*

**Lemma 3.6.** *For the at most linear growth nonnegative function $g$ in the previous assumption, there exists a smallest concave nonnegative majorant $\bar{g}$ that is nondecreasing and satisfies:*

$$\limsup_{x\to\infty} \frac{g(x)}{x} = \limsup_{x\to\infty} \frac{\bar{g}(x)}{x} = \liminf_{x\to\infty} \bar{g}'(x),$$

*Additionally, $\bar{g}(\cdot) \leq C(1 + \cdot)$.*

*Proof.* For existence, refer to [18]. Lemma 5.3 of [6] gives the equalities. A linear function is surely a majorant of $g$. Since $\bar{g}$ is the smallest majorant of $g$, we have $\bar{g}(\cdot) \leq C(1 + \cdot)$. □

It is seen that the underlying process (3.1) resembles geometric Brownian motion, in that the constant volatility is replaced with a function $b(\cdot)$ of the stochastic

volatility process (3.2). Therefore, it is relevant to discuss some properties of the stochastic volatility process, such as non-explosion, existence and uniqueness so that the underlying process can be uniquely determined up to some time $t$.

**Proposition 3.7.** *Under the Assumption 3.1 and Assumption 3.3, the process (3.2) does not blow up to infinity in finite time.*

*Proof.* If the process (3.2) satisfies the condition (b).(i) of Assumption 3.3, then the volatility process will not explode at any finite time by Remark 5.5.19 of Karatzas and Shreve [38].

Let us assume the stochastic volatility process $Y$ satisfies (b).(ii) of Assumption 3.3, i.e., $Y$ takes the form

$$dY_t = \mu(Y_t)dt + \beta Y_t^p dB_t,$$

for some $p > 1$. We prove this case by Feller's test for explosions. Define auxiliary functions

$$P(y) \triangleq \int_1^y \exp\{-2\int_1^x \frac{\mu(z)}{\beta^2 z^{2p}}\, dz\}\, dx. \tag{3.4}$$

For a sufficiently large constant $c$, we have $\mu(z)/\beta^2 \leq c(1+z)$, and therefore

$$\begin{aligned} P(\infty) &\geq \int_1^\infty \exp\{-2\int_1^x \frac{c(1+z)}{z^{2p}}\, dz\}\, dx \\ &= \infty. \end{aligned}$$

By Feller's theorem, the process $Y$ cannot explode to infinity at any finite time. □

**Corollary 3.8.** *The volatility process (3.3) does not explode to infinity at any finite time.*

*Proof.* This is the consequence of the proposition 3.7 and the remark 3.4. □

**Remark 3.9.** The reason we assume the growth rate for the drift of the volatility process is at most linear is that a higher growth rate usually results in explosion in finite time for the volatility process.

**Proposition 3.10.** *Under the Assumption 3.1 and the Assumption 3.3, the stochastic volatility model admits a unique nonnegative strong solution.*

*Proof.* Since $X$ in (3.1) resembles a geometric Brownian motion for the stochastic volatility process $Y$ governed by (3.2), it suffices to prove the existence and uniqueness of the process $Y$.

Assumption 3.3.(c) implies $\mu$ is locally Lipschitz continuous and $\sigma$ is locally $\frac{1}{2}$-Hölder continuous on $[0, \infty)$. For any $n > 0$, define $Y^n$ by

$$dY_t^n = \mu^n(Y_t^n)dt + \sigma^n(Y_t^n)dB_t, Y_0^n = y, y \geq 0,$$

where $\mu^n(y) = \mu(y)$, for $y \in [0, n]$, $\mu^n(y) = \mu(n)$, for $y > n$ and $\sigma^n$ is defined similarly. Let $\delta_n = \inf\{t \geq 0 | Y_t^n \geq n\}$. $Y^n$ stopped at $\delta_n$ is a bounded process. A theorem of Yamada and Watanabe (1971)(Proposition 5.2.13 in [38]) says pathwise uniqueness holds for $Y_n$, and a theorem of Stroock and Varadhan (Theorem 5.4.22 in [38]) guarantees the existence of a weak solution for $Y^n$. Thus, $Y^n$ admit a unique strong solution up to $\delta_n$ (Proposition 5.3.20 in [38]) and $Y^n$ agrees with $Y^{n+1}$ and on $[0, \delta_n]$. By constructing $Y_t \triangleq Y_t^n$ on $[0, \delta_n]$ for each $n$, we obtain a process $Y$ that is the unique strong solution for the volatility process up to $\lim_{n\to\infty} \delta_n$, and $\mathbb{P}\left[\lim_{n\to\infty} \delta_n = \infty\right] = 1$ (Proposition 3.7), we conclude that $Y$ has a unique strong solution.

The process $X$ is surely nonnegative. For the process $Y$, nonnegativity is guaranteed by Assumption 3.3.(b). □

### 3.2 The Valuation Equations and the Partial Differential Equations

Now we have shown that the model has a unique non-exploding nonnegative strong solution, we are ready to define the valuation equation. Throughout the rest of the thesis, we denote by $X_s^{t,x,y}$ the time $s$ random variable of the process $X$ starting from time $t$ with $X_t = x$ and $Y_t = y$. $X_s^{x,y}$ is similar to $X_s^{t,x,y}$ except that the starting time is 0. We define $\mathbb{E}^{t,x,y}$ as the conditional expectation on $\mathcal{F}_t$, so that the process $X$ is $x$, and $Y$ is $y$ at time $t$. Let $D_M = (0, M) \times (0, M)$, $\bar{D}_M = [0, M] \times [0, M]$, $D_\infty = (0, \infty) \times (0, \infty)$ and $\bar{D}_\infty = [0, \infty) \times [0, \infty)$.

Let

$$H_t^y \triangleq e^{\int_0^t b(Y_t^y)\, dW_t - \frac{1}{2}\int_0^t b^2(Y_t^y)\, dt}. \tag{3.5}$$

Then the solution to (3.1) can be expressed as

$$X_t^{x,y} = xH_t^y. \tag{3.6}$$

Let us further introduce some stopping times for the underlying process (3.6). Define

$$\tau_M^{t,x,y} = \inf\{s | t \le s < \infty, (X_s^{t,x,y}, Y_s^y) \notin \bar{D}_M\}, \tag{3.7}$$

$$\tau_0^{t,y} = \inf\{s | t \le s < \infty, Y_s^{t,y} = 0\}, \tau_0^y \triangleq \tau_0^{0,y}. \tag{3.8}$$

Let the function $u : [0, T] \times \bar{D}_\infty \to [0, \infty)$ be the price for the European style option for the payoff function $g$ under Assumption 3.5, and the functions $v^M : [0, T] \times \bar{D}_M \to [0, \infty)$ be the prices for the barrier option for the same payoff function $g$ in a truncated domain, i.e.,

$$u(t, x, y) = \mathbb{E}^{t,x,y}[g(X_T^{t,x,y})], \tag{3.9}$$

$$v^M(t, x, y) = \mathbb{E}^{t,x,y}[g(X_{T\wedge\tau_M^{t,x,y}}^{t,x,y})]. \tag{3.10}$$

For ease of notation, we will omit the superscript $\{t, x, y\}$ whenever there is no confusion.

Notice that

$$\begin{aligned} u(0,x,y) &= \mathbb{E}^{x,y}[g(X_T)] \\ &\leq \mathbb{E}^{x,y}[\bar{g}(X_T)] \\ &\leq \bar{g}(\mathbb{E}^{x,y}[X_T]) \\ &\leq \bar{g}(x) \\ &< \infty. \end{aligned}$$

It follows that the process $u(t, X_t, Y_t)$ is a martingale on $[0, T]$, thanks to the tower property and integrability. Similarly, this argument can be ported to the fact that $v^M(t, X_t, Y_t)$ stopped at the boundary is a martingale.

To derive an appropriate PDE for the functions $u$ and $v^M$, we start with a heuristic argument by assuming these two functions are sufficiently smooth. An application of Itôs lemma to $u$ (same for $v^M$) leads to

$$du(t, X_t, Y_t) = (\mathcal{L}u(t, X_t, Y_t) + u_t)dt + u_x(t, X_t, Y_t)b(Y_t)X_t dW_t + u_y(t, X_t, Y_t)\sigma(Y_t)dB_t,$$

where

$$\mathcal{L} \triangleq \frac{1}{2}b^2(y)x^2\partial^2_{xx} + \frac{1}{2}\sigma^2(y)\partial^2_{yy} + \mu(y)\partial_y + \rho b(y)\sigma(y)x\partial^2_{xy}. \tag{3.11}$$

Now, by setting the drift term of (3.11) to zero and combining the terminal and boundary conditions, we see that $u, v^M$ satisfy the following initial boundary prob-

lems respectively:

$$
(3.12)\qquad \begin{cases} u_t + \mathcal{L}u = 0, & (t,x,y) \in (0,T) \times D_\infty, \\ u(T,x,y) = g(x), & (x,y) \in \bar{D}_\infty, \\ u(t,0,y) = g(0), & (t,y) \in [0,T) \times (0,\infty), \\ u(t,x,0) = g(x), \text{ if } \mu(0) = 0, & (t,x) \in [0,T) \times [0,M), \end{cases}
$$

and

$$
(3.13)\qquad \begin{cases} v^M_t + \mathcal{L}v^M = 0, & (t,x,y) \in (0,T) \times D_M, \\ v^M(T,x,y) = g(x), & (x,y) \in \bar{D}_M, \\ v^M(t,0,y) = g(0), & (t,y) \in [0,T) \times (0,M), \\ v^M(t,x,0) = g(x), \text{ if } \mu(0) = 0, & (t,x) \in [0,T) \times [0,M), \\ v^M(t,M,y) = g(M), & (t,y) \in [0,T) \times (0,M], \\ v^M(t,x,M) = g(x), & (t,x) \in [0,T) \times (0,M]. \end{cases}
$$

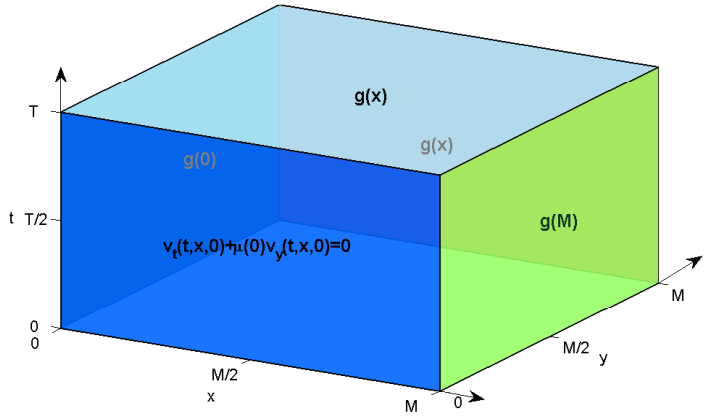


Figure 3.1: Boundary conditions for stochastic volatility model

Figure 3.1 shows how the boundary condition is set up on each face.

We will prove that $v^M$ defined in (3.10) solves (3.13) later.

**Remark 3.11.** We have been vague on the near field boundary condition for the $y$-direction in the PDEs. In fact, the boundary conditions of the PDEs at $y = 0$ is determined by the paths of the process $Y$. If $\mathbb{P}[\tau_0^y = \infty] = 1$, then the left boundary condition is never needed. From a probability point of view, almost surely no path of the process $Y$ will hit zero. When $\mathbb{P}[\tau_0^y = \infty] < 1$, the behavior of the process varies according to the value of $\mu(0)$. In case of $\mu(0) = 0$, the volatility just vanishes at zero after the hitting time, and the underlying process remains flat thereafter. In other words, zero is an absorbing point for $Y$. Therefore, the boundary condition is needed and as it is described in the equations (3.12) and (3.13). When $\mathbb{P}[\tau_0^y = \infty] < 1$ and $\mu(0) > 0$, the process $Y$ is instantaneously reflecting at the point 0. In this case, additional near field boundary requirements must be specified into the PDEs. When Assumption 3.3.(b).(i) prevails, Bayraktar Kardaras and Xing (2011) [6] shows that the value function $u$ of the initial value problem is in the closure of a set of functions $v^n$, each of which satisfies the equation (3.12) and has vanishing second order derivative terms at $y = 0$.

**Remark 3.12.** In Chapter 2, we discussed that one possible way to approximate an option price is to solve the initial boundary value problem derived from the option price valuation equation. The reason for constructing the initial boundary value problem for the stochastic volatility model is that we wish to study its far field boundary error for our approximation. If the function $v^M$ is a solution to an initial boundary value problem, then by Definition 2.1, the far field boundary error of $v^M$ with respect to $u$ is

$$\mathcal{E}^M(t, x, y) \triangleq |v^M(t, x, y) - u(t, x, y)|.$$

We expect this error to decrease by some order of $\frac{1}{M}$ for fixed $t, x, y$. A detailed

discussion of this issue is in Chapter 4.

**Remark 3.13.** Notice that the far field boundary conditions are uniformly set to the payoff function $g$. Actually, they do not have to be the same function as the payoff function, if they have the same growth rate of $g$. By changing the far field boundary conditions to another function, a revision to the stochastic representation (3.10) is needed. For convenience of notation, we might as well just use the function $g$ as the far field boundary conditions.

## 3.3 Regularity of the Value Function - the Dirichlet Problem

In Section 3.2 of this chapter, we used Itô's lemma for the valuation equations $u$ and $v^M$ without specifying appropriate smoothness properties. In this section, we will establish those properties, and argue that the value function (3.10) is a solution to the initial boundary value problem (3.13). Bayraktar, Kardaras and Xing (2011) [6], and Ekström and Tysk [25] have proved the regularity for the initial value problems derived from a family of stochastic volatility models. Our results shown in this section are different from those in [6] or [25] in the sense that we work in a finite domain for an initial boundary value problem instead of an initial value problem in an infinite domain. What is more important is that we allow the growth rate for the volatility of the stochastic volatility process to be faster than linear.

As we pointed out in Remark 3.11, when $\mathbb{P}[\tau_0^y = \infty] < 1$ and $\mu(0) > 0$, the near field boundary condition at $y = 0$ needs extra treatment. Since our focus is on the far field boundary error, we avoid the discussion of the regularity in this scenario by the following assumption.

**Assumption 3.14.** *If* $\mu(0) > 0$, $\mathbb{P}[\tau_0^y = \infty] < 1$.

It is essential to define what kind of solution to the initial boundary value problem

is a good solution. For example, in the Black-Scholes setting, the Black-Scholes PDE has more than one solution, but there is only one solution that has at most linear growth. Let us define the concept of classical solution for (3.13).

**Definition 3.15.** Under the Assumption 3.1, 3.3, 3.5 and 3.14, the function $v^M : [0,T] \times \bar{D}_M \to [0,\infty)$ is a classical solution to (3.13) if

(a) $v^M \in C([0,T] \times \bar{D}_M) \cap C^{1,2,2}((0,T) \times D_M)$,

(b) $v^M(t,x,y) \leq \bar{g}(x)$,

(c) $v^M$ solves (3.13).

**Theorem 3.16.** *The value function $v^M$ defined in equation (3.10) is a classical solution, and also the unique classical solution, to the Dirichlet problem (3.13).*

In order to prove this statement, we break the argument into some subsections. We first show the continuity, then the interior regularity as well as the boundary conditions and lastly the uniqueness result.

### 3.3.1 Continuity

Let us begin by showing that $v^M \in C([0,T] \times \bar{D}_M)$, i.e., for a family of triplets $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}} \to (t,x,y) \in [0,T] \times \bar{D}_M$, $v^M(t_n,x_n,y_n) \to v^M(t,x,y)$. In fact, instead of assuming the processes start from different times, and end up with the same time, we rather assume they all start from 0, and end up with different times $\{T_n\}_{n\in\mathbb{N}}$.

Define

$$\xi_n = \inf\{t \geq 0 | Y_t^{y_n} > M\},$$

$$\xi = \inf\{t \geq 0 | Y_t^{y} > M\},$$

$$\zeta_n = \inf\{t \geq 0 | X_t^{x_n,y} > M\},$$

$$\zeta_n^n = \inf\{t \geq 0 | X_t^{x_n, y_n} > M\},$$

$$\zeta^n = \inf\{t \geq 0 | X_t^{x, y_n} > M\},$$

$$\zeta = \inf\{t \geq 0 | X_t^{x, y} > M\}.$$

To ease the notation without causing confusion, we denote by $\tau_n$ the stopping time defined in (3.7) for process starting from $(0, x_n, y_n)$, since we have fixed the far field boundary to be $M$. Therefore,

$$\tau_n = \xi_n \wedge \zeta_n^n,$$

$$\tau = \xi \wedge \zeta. \tag{3.14}$$

**Lemma 3.17.** *Let $\{y_n\}_{n\in\mathbb{N}}$ and $y_n \to y$. Then in case of Assumption 3.3 (b).(i) $\lim_{n\to\infty} \mathbb{E}\left[|Y_t^{y_n} - Y_t^y|^2\right] = 0$ and $\lim_{n\to\infty} \mathbb{E}\left[|\xi_n \wedge t - \xi \wedge t|^{\bar{m}}\right] = 0$ for any $t \in [0, \infty)$ and $\bar{m} \in \mathbb{N}$.*

*Proof.* We work with a fixed $t \in [0, \infty)$ in the proof. In fact, Theorem 2.4 in [3] gives a stronger result that,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq u\leq t} |Y_u^{y_n} - Y_u^y|^2\right] = 0, \tag{3.15}$$

which in turn implies $Y_t^{y_n} \xrightarrow{L^2} Y_t^y$ under measure $\mathbb{P}$ and further $Y_t^{y_n} \to Y_t^y$ in probability.

In the case of $y_n \uparrow y$, we have $\xi_n \geq \xi$ and $\xi_n$ is decreasing. Pathwise uniqueness implies $Y_u^{y_1} \leq Y_u^{y_2} \leq \cdots \leq Y_u^y$ for every $u$. This implies

$$\sup_{0\leq u\leq t} |Y_u^y - Y_u^{y_1}|^2 \geq \sup_{0\leq u\leq t} |Y_u^y - Y_u^{y_2}|^2 \geq \cdots \geq \sup_{0\leq u\leq t} |Y_u^y - Y_u^{y_n}|^2 \geq \cdots \geq 0.$$

The dominated convergence theorem gives

$$0 = \lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq u\leq t} |Y_u^{y_n} - Y_u^y|^2\right] = \mathbb{E}\left[\lim_{n\to\infty} \sup_{0\leq u\leq t} |Y_u^{y_n} - Y_u^y|^2\right].$$

Hence, almost surely,

$$\lim_{n\to\infty}\sup_{0\le u\le t}\left(Y_u^y - Y_u^{y_n}\right) = 0.$$

With the fixed $t > 0$, define

$$\Omega_1 \triangleq \left\{\omega\in\Omega : \lim_{n\to\infty}\sup_{0\le u\le t}\left(Y_u^y - Y_u^{y_n}\right) = 0\right\},$$

and thus $\mathbb{P}\left[\Omega_1\right] = 1$. Let

$$\xi^{M+\frac{1}{n}} \triangleq \inf\left\{t\ge 0 | Y_t^y > M + \frac{1}{n}\right\}.$$

For any large $n\in\mathbb{N}$, and $\omega\in\Omega_1$, there exists a large positive number $k(n,\omega)$ such that for all $k > k(n,\omega)$

$$\sup_{0\le u\le t}\left(Y_u^y(\omega) - Y_u^{y_k}(\omega)\right) < \frac{1}{n}.$$

Thus,

$$Y^y_{\xi^{M+\frac{1}{n}}(\omega)\wedge t}(\omega) - Y^{y_k}_{\xi^{M+\frac{1}{n}}(\omega)\wedge t}(\omega) < \frac{1}{n},\ \forall k > k(n,\omega),$$

If $\xi^{M+\frac{1}{n}}(\omega)\le t$,

$$Y^{y_k}_{\xi^{M+\frac{1}{n}}(\omega)\wedge t}(\omega) > M,$$

and thus

$$\xi^{M+\frac{1}{n}}(\omega)\wedge t \ge \xi_k(\omega) \ge \xi_k(\omega)\wedge t.$$

If $\xi^{M+\frac{1}{n}}(\omega) > t$,

$$\xi^{M+\frac{1}{n}}(\omega)\wedge t = t \ge \xi_k(\omega)\wedge t.$$

Therefore, for all $k > k(n,\omega)$,

$$\xi^{M+\frac{1}{n}}(\omega)\wedge t \ge \xi_k(\omega)\wedge t \ge \xi(\omega)\wedge t.$$

Define $\xi^+(\omega)\triangleq \lim_{n\to\infty}\xi^{M+\frac{1}{n}}(\omega)$. Let $k\to\infty$, and then $n\to\infty$

$$\xi^+(\omega)\wedge t = \lim_{n\to\infty}\xi^{M+\frac{1}{n}}(\omega)\wedge t \ge \lim_{k\to\infty}\xi_k(\omega)\wedge t \ge \xi(\omega)\wedge t. \tag{3.16}$$

In the case of $y_n \downarrow y$, we have $Y_\cdot^{y_n} \geq Y_\cdot^y$, $\xi_n \leq \xi$ and $\xi_n$ is increasing. Define $\Omega_1$ similarly. With the same fixed $t$, for any $n \in \mathbb{N}$ and $\omega \in \Omega_1$, by possible different choice of $k(n,\omega)$, similar argument results in

$$\sup_{0\leq u\leq t} \left(Y_u^{y_k}(\omega) - Y_u^y(\omega)\right) < \frac{1}{n},\ \forall k > k(n,\omega).$$

$$Y^{y_k}_{\xi_k(\omega)\wedge t}(\omega) - Y^{y}_{\xi_k(\omega)\wedge t}(\omega) < \frac{1}{n}$$

If $\xi_k(\omega) \leq t$,

$$Y^{y}_{\xi_k(\omega)\wedge t}(\omega) > M - \frac{1}{n},$$

and thus

$$\xi(\omega)\wedge t \geq \xi_k(\omega)\wedge t \geq \xi^{M-\frac{1}{n}}(\omega) \geq \xi^{M-\frac{1}{n}}(\omega)\wedge t.$$

If $\xi_k(\omega) > t$,

$$\xi(\omega)\wedge t \geq \xi_k(\omega)\wedge t = t \geq \xi^{M-\frac{1}{n}}(\omega)\wedge t.$$

Therefore, for all $k > k(n,\omega)$,

$$\xi(\omega)\wedge t \geq \xi_k(\omega)\wedge t \geq \xi^{M-\frac{1}{n}}(\omega)\wedge t.$$

Similarly define $\xi^-(\omega) \triangleq \lim_{n\to\infty} \xi^{M-\frac{1}{n}}(\omega)$. Let $k \to \infty$ and then $n \to \infty$

$$\xi(\omega)\wedge t \geq \lim_{k\to\infty} \xi_k(\omega)\wedge t \geq \lim_{n\to\infty} \xi^{M-\frac{1}{n}}(\omega)\wedge t = \xi^-(\omega)\wedge t. \tag{3.17}$$

For a fixed $M > 0$, we have $\mathbb{P}[\xi^+ = \xi = \xi^-] = 1$, since the volatility of the process $Y$ does not disappear in a small neighborhood of $M$. From equation (3.16) and equation (3.17), We have

$$\xi_k \wedge t \to \xi \wedge t \text{ a.s.}$$

Then by the dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{E}\left[|\xi_n \wedge t - \xi \wedge t|^{\bar{m}}\right] = 0$$

for any $t \in [0,\infty)$ and $\bar{m} \in \mathbb{N}$. □

**Lemma 3.18.** *For $T > 0$ and $y_n \uparrow y$, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[|Y^{y_n}_{t\wedge\xi} - Y^{y}_{t\wedge\xi}|^2\right] = 0.$$

*Proof.* This is obviously true if Assumption 3.3 (b).(i) holds. In case of Assumption 3.3 (b).(ii), we have

$$dY_t = \mu(Y_t)dt + \beta Y_t^p dB_t, p > 1.$$

With $y_n \uparrow y$, both $Y^{y_n}_{t\wedge\xi}$ and $Y^{y}_{t\wedge\xi}$ are bounded by $M$. A subtraction of these two processes gives

$$Y^{y}_{t\wedge\xi} - Y^{y_n}_{t\wedge\xi} = y - y_n + \int_0^{t\wedge\xi} \left(\mu\left(Y^y_s\right) - \mu\left(Y^{y_n}_s\right)\right) ds + \beta \int_0^{t\wedge\xi} \left((Y^y_s)^p - (Y^{y_n}_s)^p\right) dB_s.$$

By the Lipschitz property of $\mu(\cdot)$, we further have

$$Y^{y}_{t\wedge\xi} - Y^{y_n}_{t\wedge\xi} = y - y_n + c\int_0^{t} \left(Y^y_{s\wedge\xi} - Y^{y_n}_{s\wedge\xi}\right) ds + \beta \int_0^{t} \left((Y^y_{s\wedge\xi})^p - (Y^{y_n}_{s\wedge\xi})^p\right) dB_s,$$

where $c$ is the Lipschitz constant for $\mu(\cdot)$. The stochastic integral above is square integrable, since the integrand is bounded by $2M^p$. Thus,

$$\begin{aligned}\mathbb{E}\left[\left(Y^{y}_{t\wedge\xi} - Y^{y_n}_{t\wedge\xi}\right)^2\right] \le 2(y-y_n)^2 &+ 2c^2 \int_0^t \mathbb{E}\left[\left(Y^{y}_{s\wedge\xi} - Y^{y_n}_{s\wedge\xi}\right)^2\right] ds \\ &+ C\beta^2 \int_0^t \mathbb{E}\left[\left(Y^{y}_{s\wedge\xi} - Y^{y_n}_{s\wedge\xi}\right)^2\right] ds,\end{aligned}$$

where C depends only on $M, p$, and $c$. Gronwall's inequality yields

$$\lim_{n\to\infty} \mathbb{E}\left[|Y^{y_n}_{t\wedge\xi} - Y^{y}_{t\wedge\xi}|^2\right] = 0.$$

□

**Remark 3.19.** In case of $y_n \downarrow y$, similar argument to the previous lemma implies

$$\lim_{n\to\infty} \mathbb{E}\left[|Y^{y_n}_{t\wedge\xi_n} - Y^{y}_{t\wedge\xi_n}|^2\right] = 0.$$

**Lemma 3.20.** [1] *For a fixed $t$, let $\{y_n\}_{n\in\mathbb{N}}$ and $y_n \to y$, then*

$$\xi_n \wedge t \to \xi \wedge t \text{ -a.s.}$$

*In particular,* $\lim_{n\to\infty} \mathbb{E}\left[|\xi_n \wedge t - \xi \wedge t|^{\bar{m}}\right] = 0$, *for any* $t \in [0,\infty)$ *and* $\bar{m} \in \mathbb{N}$.

*Proof.* This claim holds when Assumption 3.3 (b).(i) prevails, because of Lemma 3.17. Now let us prove the case of Assumption 3.3 (b).(ii).

Define $\psi^M \in C^\infty([0,\infty))$ with $0 \le \psi^M \le 1$ and

$$\psi^M(y) = \begin{cases} 1, & y \le 2M, \\ 0, & y > 3M. \end{cases}$$

Let $\sigma^M(y) \triangleq \beta y^p \psi^M(y)$. Define a stochastic process $Y^M$

$$dY_s^M = \mu(Y_s^M)dt + \sigma^M(Y_s^M)dB_s, Y_0^M = y.$$

$Y^M$ satisfies Assumption 3.3 (b).(i). The observation is that $Y$ agrees with $Y^M$ before the hitting time of level $2M$, then by Lemma 3.17

$$\xi_n \wedge t \to \xi \wedge t \text{ -a.s.}$$

and the rest of the statement follows naturally. □

**Lemma 3.21.** *For $T > 0$, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[|Y^{y_n}_{t\wedge\xi_n} - Y^y_{t\wedge\xi}|^2\right] = 0.$$

*Proof.* Without loss of generality, assume $y_n \uparrow y$. By Lemma 3.18, we have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{t\wedge\xi_n} - Y^y_{t\wedge\xi}\right|^2\right] \\ \le &\lim_{n\to\infty} 2\mathbb{E}\left[\left|Y^{y_n}_{t\wedge\xi_n} - Y^{y_n}_{t\wedge\xi}\right|^2\right] + \lim_{n\to\infty} 2\mathbb{E}\left[\left|Y^{y_n}_{t\wedge\xi} - Y^y_{t\wedge\xi}\right|^2\right] \\ = &0. \end{aligned}$$

[1] This lemma is from private conversation with Professor Steven Shreve.

The limit of the first expectation vanishes because of Lemma 3.20 and boundedness of the process $Y_{\xi}^{y_n}, Y_{\xi_n}^{y_n}$.

In case of $y_n \downarrow y$, we use the triangle inequality in the above argument by replacing $Y_{t\wedge\xi}^{y_n}$ with $Y_{t\wedge\xi_n}^{y}$, and invoke Remark 3.19 instead. □

**Lemma 3.22.** $\lim_{n\to\infty} \mathbb{E}\left[\left|b(Y_{t\wedge\xi_n}^{y_n}) - b(Y_{t\wedge\xi}^{y})\right|^2\right] = 0.$

*Proof.* The processes $Y_{t\wedge\xi_n}^{y_n}$ and $Y_{t\wedge\xi}^{y}$ are all bounded by $M$. By the locally Hölder $\alpha$ continuity of the function $b$, Jensen's inequality, and Lemma 3.17

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|b(Y_{t\wedge\xi_n}^{y_n}) - b(Y_{t\wedge\xi}^{y})\right|^2\right] \\
\leq & c \lim_{n\to\infty} \mathbb{E}\left[\left|Y_{t\wedge\xi_n}^{y_n} - Y_{t\wedge\xi}^{y}\right|^{2\alpha}\right] \\
\leq & c \lim_{n\to\infty} \left(\mathbb{E}\left[\left|Y_{t\wedge\xi_n}^{y_n} - Y_{t\wedge\xi}^{y}\right|^2\right]\right)^{\alpha} \\
= & 0.
\end{aligned}$$

We used Lemma 3.21 in the last step. □

**Proposition 3.23.** *Fix a triplet $(T, x, y) \in [0,\infty) \times \bar{D}_M$. Then for any sequence of triplets $\{(T_n, x_n, y_n)\}_{n\in\mathbb{N}}$ that converges to $(T, x, y)$, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi_n}^{y_n} - Y_{T\wedge\xi}^{y}\right|^2\right] = 0, \tag{3.18}$$

$$\mathbb{P}\text{-}\lim_{n\to\infty} Y_{T_n\wedge\xi_n}^{y_n} = Y_{T\wedge\xi}^{y}. \tag{3.19}$$

*Proof.* We start by estimating the approximation in $L^2$. Without loss of generality (see Remark 3.24), let $y_n \uparrow y$.

(3.20)

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi_n}^{y_n} - Y_{T\wedge\xi}^{y}\right|^2\right] \\
= & \lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi_n}^{y_n} - Y_{T_n\wedge\xi}^{y} + Y_{T_n\wedge\xi}^{y} - Y_{T\wedge\xi}^{y}\right|^2\right] \\
\leq & 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi_n}^{y_n} - Y_{T_n\wedge\xi}^{y}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi}^{y} - Y_{T\wedge\xi}^{y}\right|^2\right] \\
\leq & 4\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi_n}^{y_n} - Y_{T_n\wedge\xi}^{y_n}\right|^2\right] + 4\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi}^{y_n} - Y_{T_n\wedge\xi}^{y}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y_{T_n\wedge\xi}^{y} - Y_{T\wedge\xi}^{y}\right|^2\right].
\end{aligned}$$

Let us now show that each of the three limits in the above equation is zero. Firstly,

$$
\begin{aligned}
(3.21)\qquad & \lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y_n}_{T_n\wedge\xi}\right|^2\right] \\
\leq & 2\lim_{n\to\infty} \mathbb{E}\left[\left|\int_{T_n\wedge\xi}^{T_n\wedge\xi_n} \mu(Y^{y_n}_s)\,ds\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|\int_{T_n\wedge\xi}^{T_n\wedge\xi_n} \sigma(Y^{y_n}_s)\,dW_s\right|^2\right] \\
\leq & C_1\lim_{n\to\infty} \mathbb{E}\left[|\xi_n\wedge T_n - \xi\wedge T_n|^2\right] + C_2\lim_{n\to\infty} \mathbb{E}\left[|\xi_n\wedge T_n - \xi\wedge T_n|\right] \\
= & 0,
\end{aligned}
$$

where we used the facts that functions $\mu, \sigma$ are continuous and bounded between the stopping time $\xi_n \wedge t$ and $\xi \wedge t$, $C_1, C_2$ are constants dependent on $M$, and Lemma 3.17. Secondly, by Lemma 3.18

(3.22)

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi} - Y^{y}_{T_n\wedge\xi}\right|^2\right] \\
\leq & 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi} - Y^{y_n}_{T\wedge\xi}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T\wedge\xi} - Y^{y}_{T\wedge\xi}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T\wedge\xi} - Y^{y}_{T_n\wedge\xi}\right|^2\right] \\
\leq & 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi} - Y^{y_n}_{T\wedge\xi}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T\wedge\xi} - Y^{y}_{T_n\wedge\xi}\right|^2\right],
\end{aligned}
$$

Lastly, Problem 5.3.15 in Karatzas and Shreve [38] gives

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T_n\wedge\xi} - Y^{y}_{T\wedge\xi}\right|^2\right] + \lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi} - Y^{y_n}_{T\wedge\xi}\right|^2\right] \\
\leq & \lim_{n\to\infty} C_3(1+y^2)\mathbb{E}\left[|T_n\wedge\xi - T\wedge\xi|\right] \\
= & 0.
\end{aligned}
$$

In conclusion, we have

$$
\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y}_{T\wedge\xi}\right|^2\right] = 0,
$$

and this gives (3.19). □

**Remark 3.24.** We often assume $y_n \uparrow y$, and $x_n \uparrow x$ in some proofs to make them concise. This is because doing the other way only requires some changes in the

arguments slightly such as the use of triangle inequality, the order of integration interval and the order of their stopping times in the proofs. For example, in case of $y_n \downarrow y$ in Proposition 3.23, we modify its proof as the following:

We change the use of triangle inequality in equation (3.20) to

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y}_{T\wedge\xi}\right|^2\right] \\
\leq& 4\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y}_{T_n\wedge\xi_n}\right|^2\right] + 4\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T_n\wedge\xi_n} - Y^{y}_{T_n\wedge\xi}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T_n\wedge\xi} - Y^{y}_{T\wedge\xi}\right|^2\right].
\end{aligned}$$

Accordingly, the first inequality relationship in equation (3.21) becomes

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T_n\wedge\xi_n} - Y^{y}_{T_n\wedge\xi}\right|^2\right] \\
\leq& 2\lim_{n\to\infty} \mathbb{E}\left[\left|\int_{T_n\wedge\xi_n}^{T_n\wedge\xi} \mu(Y^y_s)\,ds\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|\int_{T_n\wedge\xi_n}^{T_n\wedge\xi} \sigma(Y^y_s)\,dW_s\right|^2\right].
\end{aligned}$$

Equation (3.22) is now using Remark 3.19

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y}_{T_n\wedge\xi_n}\right|^2\right] \\
\leq& 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y_n}_{T\wedge\xi_n}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T\wedge\xi_n} - Y^{y}_{T\wedge\xi_n}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T\wedge\xi_n} - Y^{y}_{T_n\wedge\xi_n}\right|^2\right] \\
\leq& 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y_n}_{T_n\wedge\xi_n} - Y^{y_n}_{T\wedge\xi_n}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|Y^{y}_{T\wedge\xi_n} - Y^{y}_{T_n\wedge\xi_n}\right|^2\right],
\end{aligned}$$

and, for the same reason, [38] implies these two limits disappear.

We shall not elaborate the reasons for making such assumptions from now on when only similar changes are needed.

Now, we turn our attention to the underlying process $X$.

**Lemma 3.25.** *Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *satisfy* $x_n \to x$ *and* $y_n \to y$. *Then* $\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n} - X^{x,y_n}_{t\wedge\xi_n}\right|^2\right] = 0$, *and* $\lim_{n\to\infty} \mathbb{E}\left[|t\wedge\zeta^n_n\wedge\xi_n - t\wedge\zeta^n\wedge\xi_n|^{\bar{m}}\right] = 0$, *for any* $t \in [0,\infty)$ *and* $\bar{m} \in \mathbb{N}$.

*Proof.* From the dynamics (3.1) for $X$,

$$X^{x,y_n}_{t\wedge\xi_n} = x + \int_0^{t\wedge\xi_n} b(Y^{y_n}_s) X^{x,y_n}_s dW_s,$$
$$X^{x_n,y_n}_{t\wedge\xi_n} = x_n + \int_0^{t\wedge\xi_n} b(Y^{y_n}_s) X^{x_n,y_n}_s dW_s.$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n} - X^{x,y_n}_{t\wedge\xi_n}\right|^2\right] \\
=&\lim_{n\to\infty} \mathbb{E}\left[\left|x - x_n + \int_0^{t\wedge\xi_n} b(Y^{y_n}_s)(X^{x,y_n}_s - X^{x_n,y_n}_s)\, dW_s\right|^2\right] \\
\leq& 2\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t\wedge\xi_n} b(Y^{y_n}_s)^2 (X^{x,y_n}_s - X^{x_n,y_n}_s)^2\, ds\right] + 2\lim_{n\to\infty} (x - x_n)^2 \\
\leq& C(1+M^m)^2 \int_0^t \lim_{n\to\infty} \mathbb{E}\left[\left(X^{x,y_n}_{s\wedge\xi_n} - X^{x_n,y_n}_{s\wedge\xi_n}\right)^2\right] ds,
\end{aligned}$$

where $C, m$ are positive constants.

By Gronwall's inequality, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n} - X^{x,y_n}_{t\wedge\xi_n}\right|^2\right] = 0.$$

Doob's maximal martingale inequality further implies

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq s\leq t} |X^{x_n,y_n}_{s\wedge\xi_n} - X^{x,y_n}_{s\wedge\xi_n}|^2\right] \leq 4 \lim_{n\to\infty} \mathbb{E}\left[|X^{x_n,y_n}_{t\wedge\xi_n} - X^{x,y_n}_{t\wedge\xi_n}|^2\right] = 0, t > 0.$$

Following the same techniques as Lemma 3.17, we have

$$\lim_{n\to\infty} \mathbb{E}\left[|t\wedge\zeta^n_n\wedge\xi_n - t\wedge\zeta^n\wedge\xi_n|^{\bar{m}}\right] = 0.$$

□

**Corollary 3.26.** $\lim_{n\to\infty} \mathbb{E}\left[|t\wedge\zeta_n\wedge\xi - t\wedge\zeta\wedge\xi|^{\bar{m}}\right] = 0, \bar{m}\in\mathbb{N}.$

*Proof.* It follows immediately from the previous lemma by setting $y_n = y$ for any $n \in \mathbb{N}$. □

**Lemma 3.27.** $\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n\wedge\zeta^n_n} - X^{x,y_n}_{t\wedge\xi_n\wedge\zeta^n}\right|^2\right] = 0.$

*Proof.* Without loss of generality, we continue to assume that $x_n \uparrow x$. Since both $X^{x_n,y_n}$ and $X^{x,y_n}$ start from the same initial stochastic volatility level $y_n$, the pathwise uniqueness implies $\zeta^n_n$ is decreasing and $\zeta^n_n \geq \zeta^n$ for every $n$. Once again, an integration of (3.1) for $X$ gives

$$\begin{aligned} X^{x_n,y_n}_{t\wedge\xi_n\wedge\zeta^n_n} &= x_n + \int_0^{t\wedge\xi_n\wedge\zeta^n_n} b(Y^{y_n}_s) X^{x_n,y_n}_s dW_s, \\ X^{x,y_n}_{t\wedge\xi_n\wedge\zeta^n} &= x + \int_0^{t\wedge\xi_n\wedge\zeta^n} b(Y^{y_n}_s) X^{x,y_n}_s dW_s. \end{aligned}$$

And then,

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n\wedge\zeta^n_n} - X^{x,y_n}_{t\wedge\xi_n\wedge\zeta^n}\right|^2\right] \\ =&\lim_{n\to\infty} \mathbb{E}\left[\left|x_n - x + \int_0^{t\wedge\xi_n\wedge\zeta^n} b(Y^{y_n}_s)(X^{x_n,y_n}_s - X^{x,y_n}_s)dW_s + \int_{t\wedge\xi_n\wedge\zeta^n}^{t\wedge\xi_n\wedge\zeta^n_n} b(Y^{y_n}_s) X^{x_n,y_n}_s dW_s\right|^2\right] \\ \leq& 2\lim_{n\to\infty} (x_n - x)^2 + 2\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t\wedge\xi_n\wedge\zeta^n} b(Y^{y_n}_s)^2 (X^{x_n,y_n}_s - X^{x,y_n}_s)^2 ds\right] \\ &+ 2\lim_{n\to\infty} \mathbb{E}\left[\int_{t\wedge\xi_n\wedge\zeta^n}^{t\wedge\xi_n\wedge\zeta^n_n} (b(Y^{y_n}_s) X^{x_n,y_n}_s)^2 ds\right]. \end{aligned}$$

Notice that

$$b(Y^{y_n}_s) \leq C(1+M)^m, s \in [0, t\wedge\xi_n],$$

where $C, m$ are positive constants, and

$$X^{x_n,y_n}_s \leq M, s \in [\zeta^n, \zeta^n_n].$$

We have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t\wedge\xi_n\wedge\zeta^n} b(Y^{y_n}_s)^2 (X^{x_n,y_n}_s - X^{x,y_n}_s)^2 ds\right] \\ \leq& C^2(1+M^m)^2 \int_0^t \lim_{n\to\infty} \mathbb{E}\left[\left(X^{x_n,y_n}_{s\wedge\xi_n} - X^{x,y_n}_{s\wedge\xi_n}\right)^2\right] ds = 0, \end{aligned}$$

where we used the first part of the Lemma 3.25.

Also,

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\left[\int_{t\wedge\xi_n\wedge\zeta_n^n}^{t\wedge\xi_n\wedge\zeta_n^n}\left(b\left(Y_s^{y_n}\right)X_s^{x_n,y_n}\right)^2 ds\right]\\
\leq& C^2(1+M^m)^2M^2\lim_{n\to\infty}\mathbb{E}\left[|t\wedge\xi_n\wedge\zeta^n - t\wedge\xi_n\wedge\zeta_n^n|\right]\\
=&0,
\end{aligned}$$

according to the second part of the Lemma 3.25. Thus,

$$\lim_{n\to\infty}\mathbb{E}\left[\left|X_{t\wedge\xi_n\wedge\zeta_n^n}^{x_n,y_n} - X_{t\wedge\xi_n\wedge\zeta^n}^{x,y_n}\right|^2\right] = 0.$$

□

**Remark 3.28.** Notice that the previous lemma still holds when the process $Y$ starts from $y$ with the corresponding stopping times $\xi, \zeta, \zeta_n$.

**Lemma 3.29.** $\lim_{n\to\infty}\mathbb{E}\left[\left|X_{t\wedge\xi_n\wedge\zeta^n}^{x,y_n} - X_{t\wedge\xi\wedge\zeta}^{x,y}\right|^2\right] = 0.$

*Proof.* Without loss of generality, assume $y_n \uparrow y$. It is seen that

$$\begin{aligned}
&\mathbb{E}\left[\left|X_{t\wedge\xi_n\wedge\zeta^n}^{x,y_n} - X_{t\wedge\xi\wedge\zeta}^{x,y}\right|^2\right]\\
\leq& 2\mathbb{E}\left[\left|X_{t\wedge\xi_n\wedge\zeta^n}^{x,y_n} - X_{t\wedge\xi\wedge\zeta^n}^{x,y_n}\right|^2\right] + 2\mathbb{E}\left[\left|X_{t\wedge\xi\wedge\zeta^n}^{x,y_n} - X_{t\wedge\xi\wedge\zeta}^{x,y}\right|^2\right]\\
\leq& 2\mathbb{E}\left[\left|X_{t\wedge\xi_n\wedge\zeta^n}^{x,y_n} - X_{t\wedge\xi\wedge\zeta^n}^{x,y_n}\right|^2\right] + 4\mathbb{E}\left[\left|X_{t\wedge\xi\wedge\zeta^n}^{x,y_n} - X_{t\wedge\xi\wedge\zeta}^{x,y_n}\right|^2\right] + 4\mathbb{E}\left[\left|X_{t\wedge\xi\wedge\zeta}^{x,y_n} - X_{t\wedge\xi\wedge\zeta}^{x,y}\right|^2\right].
\end{aligned}$$

Because of Lemma 3.17 and Corollary 3.26, it remains to prove

$$\lim_{n\to\infty}\mathbb{E}\left[\left|X_{t\wedge\xi\wedge\zeta}^{x,y_n} - X_{t\wedge\xi\wedge\zeta}^{x,y}\right|^2\right] = 0.$$

$$
\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x,y_n}_{t\wedge\xi\wedge\zeta}-X^{x,y}_{t\wedge\xi\wedge\zeta}\right|^2\right]\\
=&\lim_{n\to\infty}\mathbb{E}\left[\left|\int_0^{t\wedge\xi\wedge\zeta} b(Y^{y_n}_s)X^{x,y_n}_s-b(Y^{y}_s)X^{x,y}_s dW_s\right|^2\right]\\
\leq&2\lim_{n\to\infty}\mathbb{E}\left[\left|\int_0^{t\wedge\xi\wedge\zeta} b(Y^{y_n}_s)(X^{x,y_n}_s-X^{x,y}_s) dW_s\right|^2\right]\\
&+2\lim_{n\to\infty}\mathbb{E}\left[\left|\int_0^{t\wedge\xi\wedge\zeta} (b(Y^{y_n}_s)-b(Y^{y}_s))X^{x,y}_s dW_s\right|^2\right]\\
\leq&c(1+M^m)^2\int_0^t\lim_{n\to\infty}\mathbb{E}\left[\left|\left(X^{x,y_n}_{s\wedge\xi\wedge\zeta}-X^{x,y}_{s\wedge\xi\wedge\zeta}\right)\right|^2\right]ds\\
&+2M^2\int_0^t\lim_{n\to\infty}\mathbb{E}\left[\left|\left(b\left(Y^{y_n}_{s\wedge\xi\wedge\zeta}\right)-b\left(Y^{y}_{s\wedge\xi\wedge\zeta}\right)\right)\right|^2\right]ds\\
=&c(1+M^m)^2\int_0^t\lim_{n\to\infty}\mathbb{E}\left[\left|\left(X^{x,y_n}_{s\wedge\xi\wedge\zeta}-X^{x,y}_{s\wedge\xi\wedge\zeta}\right)\right|^2\right]ds,
\end{aligned}
$$

where we used the Lemma 3.22. An application of Gronwall's inequality completes the proof. □

**Lemma 3.30.** $\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\tau_n}-X^{x,y}_{t\wedge\tau}\right|^2\right]=0.$

*Proof.* The Lemma 3.27 and the Lemma 3.29 indicate

$$
\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\tau_n}-X^{x,y}_{t\wedge\tau}\right|^2\right]\\
=&\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n\wedge\zeta^n_n}-X^{x,y}_{t\wedge\xi\wedge\zeta}\right|^2\right]\\
\leq&2\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x_n,y_n}_{t\wedge\xi_n\wedge\zeta^n_n}-X^{x,y_n}_{t\wedge\xi_n\wedge\zeta^n}\right|^2\right]+2\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x,y_n}_{t\wedge\xi_n\wedge\zeta^n}-X^{x,y}_{t\wedge\xi\wedge\zeta}\right|^2\right]\\
=&0.
\end{aligned}
$$

□

**Proposition 3.31.** *Fix a triplet $(T,x,y)\in[0,\infty)\times\bar{D}_M$. Then for any sequence of triplets $\{(T_n,x_n,y_n)\}_{n\in\mathbb{N}}$ convergent to $(T,x,y)$, we have*

$$\lim_{n\to\infty}\mathbb{E}\left[\left|X^{x_n,y_n}_{T_n\wedge\tau_n}-X^{x,y}_{T\wedge\tau}\right|^2\right]=0, \tag{3.23}$$

$$\mathbb{P}\text{-}\lim_{n\to\infty}X^{x_n,y_n}_{T_n\wedge\tau_n}=X^{x,y}_{T\wedge\tau}. \tag{3.24}$$

*Proof.* Without loss of generality, we assume the triplet $\{(T_n, x_n, y_n)\}_{n\in\mathbb{N}}$ is increasing to $(T, x, y)$. Using the triangle inequality, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{T_n\wedge\tau_n} - X^{x,y}_{T\wedge\tau}\right|^2\right] \\
\leq& 2\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{T_n\wedge\tau_n} - X^{x,y}_{T_n\wedge\tau}\right|^2\right] + 2\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x,y}_{T_n\wedge\tau} - X^{x,y}_{T\wedge\tau}\right|^2\right]
\end{aligned}$$

The first limit in the equation above is zero according to the Lemma 3.30. For the second limit,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x,y}_{T_n\wedge\tau_n} - X^{x,y}_{T\wedge\tau}\right|^2\right] \\
=&\lim_{n\to\infty} \mathbb{E}\left[\left|\int_{T_n\wedge\tau}^{T\wedge\tau} b(Y^{x,y}_s)X^{x,y}_s\, dW_s\right|^2\right] \\
\leq&\lim_{n\to\infty} \mathbb{E}\left[\left|\int_{T_n\wedge\tau}^{T\wedge\tau} c(1+M^m)^2M^2\, ds\right|^2\right] \\
=&c(1+M^m)^2M^2\lim_{n\to\infty} \mathbb{E}\left[T\wedge\tau - T_n\wedge\tau\right] \\
=&0.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|X^{x_n,y_n}_{T_n\wedge\tau_n} - X^{x,y}_{T\wedge\tau}\right|^2\right] = 0,$$

and as a corollary,

$$\mathbb{P}\text{-}\lim_{n\to\infty} X^{x_n,y_n}_{T_n\wedge\tau_n} = X^{x,y}_{T\wedge\tau}.$$

□

**Theorem 3.32.** *The value function (3.10) for the Dirichlet problem (3.13) is continuous in $(t, x, y)$. In other words, $v^M \in C([0, T] \times \bar{D}_M)$. Also, this value function satisfies $v^M(t, x, y) \leq \bar{g}(x)$.*

*Proof.* First, let us show that the growth rate of $v^M$ is bounded by that of $\bar{g}$.

$$\begin{aligned} v^M(t,x,y) =& \mathbb{E}^{t,x,y}\left[g\left(X_{T\wedge\tau}\right)\right] \\ \leq& \mathbb{E}^{t,x,y}\left[\bar{g}\left(X_{T\wedge\tau}\right)\right] \\ \leq& \bar{g}\left(\mathbb{E}^{t,x,y}\left[X_{T\wedge\tau}\right]\right) \\ \leq& \bar{g}(x). \end{aligned}$$

For $\{(T_n, x_n, y_n)\} \to (T^*, x, y) \in (0,T)\times D_M$, Proposition 3.31 indicates that $X^{x_n,y_n}_{T_n\wedge\tau_n}$ converges to $X^{x,y}_{T^*\wedge\tau}$ in probability. Recall that the payoff function $g$ is continuous, nonnegative and at most of linear growth. Thus, $g(X^{x_n,y_n}_{T_n\wedge\tau_n})$ converges to $g(X^{x,y}_{T^*\wedge\tau})$ in probability. Notice that $\{g(X^{x_n,y_n}_{T_n\wedge\tau_n})\}_{n\in\mathbb{N}}$ are bounded by $C(1+M)$, for some constant $C$. Therefore, $g(X^{x_n,y_n}_{T_n\wedge\tau_n})$ converges in $L^1$, which shows the continuity of $v^M$.

Next, we show that the continuity extends to boundaries. Now, let $(T^*, x, y) \in \partial([0,T]\times\bar{D}_M)$ In fact, since the family $\{X^{x_n,y_n}_{T_n\wedge\tau_n}\}_{n\in\mathbb{N}}$ is uniformly bounded, and $g(x)$ is a continuous function, the family $\{g(X^{x_n,y_n}_{T_n\wedge\tau_n})\}_{n\in\mathbb{N}}$ is uniformly integrable. Continuous extensions to the boundaries other than the far field boundary are obvious. For the far field boundary face, since $\tau_n \to 0$, if $x_n \to M$,

$$g(M) = \mathbb{E}\left[\lim_{n\to\infty} g\left(X^{x_n,y_n}_{T_n\wedge\tau_n}\right)\right],$$

thanks to the bounded convergence theorem.

Therefore, the continuity holds on the far field boundary $x = M$. A similar argument shows the continuity on the boundary $y = M$. In conclusion, $v^M \in C([0,T]\times\bar{D}_M)$. □

### 3.3.2 Interior regularity

In the previous subsection, we have shown that the value function $v^M$ defined in (3.10) is continuous inside the domain $[0,T]\times\bar{D}_M$. Also, recall that when we derive

the parabolic PDE for $v^M$, we heuristically assumed $v^M$ is sufficiently smooth. In this subsection, we prove that $v^M$ is indeed regular inside the interior of the domain $[0,T] \times \bar{D}_M$.

**Theorem 3.33.** *The value function $v^M(t,x,y)$ defined in equation (3.10) is in $C^{1,2,2}((0,T) \times D_M)$, and moreover it satisfies the parabolic PDE $v_t^M + \mathcal{L}v^M = 0$ for all points in $(0,T) \times D_M$, where $\mathcal{L}$ is defined in (3.11).*

*Proof.* We will perform a standard verification-type of argument to prove this result. Let us pick a triplet $(t,x,y)$ in the interior of $(0,T) \times D_M$, and an open cylindrical volume $\mathcal{D} \triangleq (t_1,t_2) \times (x_1,x_2) \times (y_1,y_2)$ that contains this triplet.

It is clear from the discussion of Theorem 3.32 that $v^M$ is uniformly bounded in the domain $(0,T) \times D_M$, and hence in the domain $\mathcal{D}$. Also, we observe that all the coefficients in the operator $\mathcal{L}$ are uniformly bounded away from 0 in the domain $\mathcal{D}$. Thanks to the continuity of $v^M$, it follows from parabolic theory [29] that the boundary value problem

$$\begin{cases} w_t(t,x,y) + \mathcal{L}w(t,x,y) = 0, & (t,x,y) \in \mathcal{D}, \\ w(t,x,y) = v^M(t,x,y), & (t,x,y) \in \partial\mathcal{D}, \end{cases} \tag{3.25}$$

admits a unique solution $w$.

Define $Z_t \triangleq w(t, X_t, Y_t)$ and a stopping time $\tau_{\mathcal{D}}$ as

$$\tau_{\mathcal{D}} = \inf\{s \geq t | (s, X_s, Y_s) \notin \mathcal{D}\}.$$

Obviously, the region $\mathcal{D} \subseteq [0,T] \times \bar{D}_M$, and hence $\tau_{\mathcal{D}} \leq \tau_M$. Thus,

$$\begin{aligned} v^M(t,x,y) =& \mathbb{E}^{t,x,y}\left[g\left(X^{x,y}_{T\wedge\tau_M}\right)\right] \\ =& \mathbb{E}^{t,x,y}\left[v^M\left(X^{x,y}_{T\wedge\tau_M}\right)\right] \\ =& \mathbb{E}^{t,x,y}\left[v^M\left(X^{x,y}_{T\wedge\tau_{\mathcal{D}}}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^{t,x,y}\left[Z^{t,x,y}_{T\wedge\tau_M}\right] \\
&= Z_t \\
&= w(t,x,y)
\end{aligned}
$$

Therefore, $v^M$ enjoys the same regularity as $w$ in the domain $\mathcal{D}$. Because of the arbitrary choices of the triplet $(t,x,y)$ and $\mathcal{D}$, $v^M \in C^{1,2,2}\left([0,T)\times D_M\right)$ and $v^M_t + \mathcal{L}v^M = 0$ for all points in $(0,T)\times D_M$. □

**Remark 3.34.** Actually, we do not have to avoid the boundary of $t = 0$ for the proof of the interior regularity. This is because if we reverse the time by changing the backward parabolic equation to a forward parabolic equation, the time domain for the forward parabolic equation doesn't have to be bounded.

#### 3.3.3 Boundary conditions

Subsection 3.3.2 and subsection 3.3.1 have shown that the value function $v^M$ satisfies the conditions (a) and (b), and the PDE in condition (c) of Definition 3.15. In order to complete the proof of the Theorem 3.16, we need to further verify the boundary conditions in condition (c), and condition (d) of Definition 3.15.

**Theorem 3.35.** *The value function $v^M(t,x,y)$ defined in (3.10) satisfies the boundary conditions required by Definition 3.15.*

*Proof.* Recall that we have shown in Theorem 3.32 that $v^M(t,x,y)$ is continuous in the region $[0,T]\times\bar{D}_M$. In other words, the value function extends its continuity to the boundary of the region. Let us enumerate the verifications of the boundary conditions for all faces except the one with $y = 0$. When the underlying process $X$ starts from 0, it is absorbed at 0 as it is implied from (3.6), and the option clearly pays $g(0)$. Same as either $X$ or $Y$ starts from the level $M$, the clock is stopped by the

stopping time at the starting time $t$ and the underlying process freezes at its starting point. Thus the option holder gets paid $g(x)$ with $x$ being the starting point of $X$. The terminal condition is trivially satisfied. We discussed earlier in Remark 3.11 that the boundary behaviors on the face of $y = 0$ are strongly tied to the properties of the stochastic volatility process $Y$. From Assumption 3.3, we know that the lower bound for the process $Y$ is 0. The stopping time $\tau_0^{t,y}$ fully describes the explosion of $Y$ to the bound 0.

If $\mathbb{P}[\tau_0^y = \infty] = 1$, the process never explodes to 0, and the Dirichlet problem (3.13) is well defined without a near field boundary condition. If $\mathbb{P}[\tau_0^y = \infty] < 1$ and $\mu(0) = 0$, the volatility vanishes once it hits 0. This is the situation of 0 being an absorbing point of $Y$, and the boundary condition given in (3.13) follows naturally. □

### 3.3.4 Existence and uniqueness of the Dirichlet problem

In this subsection, we prove Theorem 3.16, which is one of the main theorems of the thesis. Recall that the theorem states that the value function $v^M$ in (3.10) (the one with a barrier feature) is the unique classical solution to the Dirichlet problem (3.13). The fact that $v^M$ is a classical solution is detailed in subsections 3.3.1, 3.3.2 as well as 3.3.3, and we are going to summarize them to establish the existence result. For the uniqueness we need the notion of stochastic solutions and an auxiliary lemma.

**Definition 3.36.** A continuous function $\omega^M\colon [0,T] \times \bar{D}_M \to [0,\infty)$ is a stochastic solution of the initial boundary value problem (3.13) if for each $(x,y) \in D_M$:

1. $\omega^M(\cdot, X^{x,y}_{\cdot\wedge\tau}, Y^y_{\cdot\wedge\tau})$ is a local martingale, where $\tau$ is defined in (3.14),
2. $\omega^M(t,x,M) = g(x)$,
3. $\omega^M(t,M,y) = g(M)$,

4. $\omega^M(T,x,y) = g(x)$.

**Proposition 3.37.** *The value function $v^M$ defined in (3.10) is the unique stochastic solution of the initial boundary value problem (3.13).*

*Proof.* From the definition of $v^M$, we know that $v^M(\cdot, X^{x,y}_{\cdot\wedge\tau}, Y^y_{\cdot\wedge\tau})$, $(x,y) \in \bar{D}_M$ is a martingale by the tower property. Other conditions in Definition 3.36 are consequences of Theorem 3.35.

Let $\omega^M$ be another stochastic solution. Then for $(x,y) \in \bar{D}_M$, $\omega^M(t, X^{x,y}_{t\wedge\tau}, Y^y_{t\wedge\tau})$, $0 \le t \le T$, is a local martingale. Since $\omega^M$ is a continuous function on compact set $[0,T] \times \bar{D}_M$ by Definition 3.36, $\omega^M$ is a uniformly bounded function and $\omega^M(t, X^{x,y}_{t\wedge\tau}, Y^y_{t\wedge\tau})$, $0 \le t \le T$, is a uniformly integrable martingale. For any $(t,\bar{x},\bar{y}) \in [0,T] \times D_M$, by conditions 2, 3, 4 of Definition 3.36,

$$\omega^M(t,\bar{x},\bar{y}) = \mathbb{E}^{t,\bar{x},\bar{y}}\left[\omega^M(T, X^{x,y}_{T\wedge\tau}, Y^y_{T\wedge\tau})\right] = \mathbb{E}^{t,\bar{x},\bar{y}}\left[g\left(X^{x,y}_{T\wedge\tau}\right)\right] = v^M(t,\bar{x},\bar{y}).$$

Thanks to the continuity of $\omega^M$ and $v^M$, $\omega^M$ agrees with $v^M$ on $[0,T] \times \bar{D}_M$, and uniqueness holds. □

**Proposition 3.38.** *Any classical solution of the initial boundary value problem (3.13) is a stochastic solution.*

*Proof.* For $(x,y) \in D_M$ and a fixed $M$, and $n$ large enough so that $x \wedge y > \frac{1}{n}$, define the stopping time

$$\eta_n = \inf_{0\le t\le\tau}\left\{X^{x,y}_t \wedge Y^y_t \le \frac{1}{n}\right\}.$$

Let $v$ be a classical solution to the initial boundary value problem (3.13) defined on $[0,T] \times \bar{D}_M$. An application of Itô's lemma yields the martingale property of $v(\cdot\wedge\eta_n, X^{x,y}_{\cdot\wedge\tau\wedge\eta_n}, Y^y_{\cdot\wedge\tau\wedge\eta_n})$ on $[0,T]$. By the definition of $\tau^y_0$ in (3.8), we have $\mathbb{P}\left[\lim_{n\to\infty}\eta_n = \tau^y_0\right] = 1$. In case that $\mathbb{P}[\tau^y_0 = \infty] = 1$, $v(\cdot, X^{x,y}_{\cdot\wedge\tau}, Y^y_{\cdot\wedge\tau})$ is a local martingale on $[0,T]$. If $\mathbb{P}[\tau^y_0 = \infty] < 1$, by Assumption 3.14, $\mu(0) = 0$ and $Y$ process is

absorbing once it hits zero and $X$ freezes at the level on the hitting time thereafter. The boundary condition $v(t,x,0) = g(x)$ in the initial boundary value problem (3.13) implies $\lim_{n\to\infty} v(\cdot \wedge \eta_n, X^{x,y}_{\cdot\wedge\tau\wedge\eta_n}, Y^{y}_{\cdot\wedge\tau\wedge\eta_n}) = v(\cdot \wedge \tau_0^y, X^{x,y}_{\cdot\wedge\tau\wedge\tau_0^y}, Y^{y}_{\cdot\wedge\tau\wedge\tau_0^y}) = v(\cdot, X^{x,y}_{\cdot\wedge\tau}, Y^{y}_{\cdot\wedge\tau})$ on $[0,T]$. Other boundary and terminal conditions in the Definition 3.36 are inferred directed from (3.13). □

Now let us prove theorem 3.16.

***Proof of Theorem 3.16***. To see that $v^M$ is a classic solution to the Dirichlet problem (3.13), Theorem 3.32 and Theorem 3.35 imply that $v^M$ satisfies the continuity and boundary conditions, which is the first part of (a) in Definition 3.15, and $v^M(t,x,y) \leq \bar{g}(x)$, which is (b) in this definition; Conditions (a) and (c) are the results shown in Theorem 3.33.

The initial boundary value problem admits a unique classical solution. This is because any classical solution is a stochastic solution and uniqueness holds for stochastic solution, according to Proposition 3.38 and Proposition 3.37, respectively. □

## 3.4 The Initial Value Problem

We close this chapter by stating some results from Bayraktar, Kardaras and Xing for the initial value problem (3.12). For an unbounded domain, we shall exclusively confine the discussion to at most linear growth parametric functions for the stochastic volatility process, i.e., for (b) in Assumption 3.3, we only consider case (i).

**Proposition 3.39** (Proposition 3.3. from [6])**.**

*The following statements are equivalent:*

*(1) $H^y_{\cdot\wedge T}$ is a strict local martingale for some, and then all (y, T) in $(0,\infty)\times(0,\infty)$.*

*(2)* $\mathfrak{v}(\infty) < \infty$*, where*

$$\mathfrak{v}(y) \triangleq 2\int_c^y \frac{\mathfrak{p}(y)-\mathfrak{p}(z)}{\mathfrak{p}'(z)\sigma^2(z)}\,dz, y > 0,$$

$$\mathfrak{p}(y) \triangleq \int_c^y exp\{-2\int_c^x \frac{\bar{\mu}(z)}{\sigma^2(z)}\,dz\}\,dx, y > 0,$$

$$\bar{\mu}(y) \triangleq \mu(y) + \rho b(y)\sigma(y).$$

**Remark 3.40.** The first statement in the above proposition tells us that the underlying cannot be a strictly local martingale for some $(y, T)$, but remain a true martingale anywhere else. In fact, the loss of martingale property results from the assumptions on the stochastic volatility process. The second statement is a consequence of Feller's test for explosions.

**Theorem 3.41** (Existence: Theorem 2.8 from [6])**.** *The value function u defined in (3.9) is a classical solution to the initial value problem (3.12). Moreover, it is the smallest classical solution.*

**Theorem 3.42** (Uniqueness :Theorem 2.9 from [6])**.** *The following two statements hold:*

*(a) When g is of strictly sublinear growth, u is the unique classical solution with growth domination h.*

*(b) When g is of linear growth, u is the unique classical solution with growth domination h if and only if the underlying process X is a true martingale.*

*Uniqueness holds if and only if the following comparison result holds. Let v and w be classical super/sub-solutions with growth domination h. If* $v(0, x, y) \geq g(x) \geq w(0, x, y)$ *for* $(x, y) \in [0, \infty) \times [0, \infty)$*, then* $v \geq w$ *on* $[0, T] \times [0, \infty) \times [0, \infty)$.

**Remark 3.43.** It is important to see that the initial value problem may not have a unique solution while Dirichlet problem always has a unique solution. A heuristic

way to understand that Dirichlet problems always have unique classical solutions is that the stopped underlying process $X_{\cdot\wedge\tau}$ is a bounded martingale, and this situation can somehow fall into the category of martingale underlying in case of (b) in Theorem 3.42. We say this is a heuristic point of view, because when we stop the $X$, we essentially make all parametric functions in Assumption 3.1 discontinuous, and Assumption 3.3 can no longer be valid.

We will see in the next chapter that the martingale property is also critical to numerical pricing in terms of controlling the far field boundary errors.

# CHAPTER 4

# Convergence of the Dirichlet Problem

Under all the assumptions we made Chapter 3, we have built the relationship between the valuation equation (3.9) and the initial value problem (3.12), and that between the valuation equation (3.10) with the barrier feature and the Dirichlet problem (3.13). Defined in Chapter 2, the far field boundary error is the difference of Dirichlet problem (3.13) and the value function (3.9). This type of error is implicit in a numerical PDE perspective, since numerical schemes are usually applied in a finite domain problem, e.g. Dirichlet problems. The situation in stochastic volatility model differs from the scenario discussed in Kangro and Nicolaides (2000) [37], because, unlike Black-Scholes case, the uniqueness for (3.12) does not generally hold. Thus, it is not convenient to consider the convergence of the Dirichlet problem (3.13) to the initial value problem (3.12).

In this chapter, we alternatively study the convergence of the barrier option valuation equation (3.10) to the valuation equation (3.9) by following the context in Chapter 2, thanks to the uniqueness property of Dirichlet problem (3.13). We begin this chapter with some convergence results for a general stochastic volatility model under the assumptions we made in Chapter 3. Then, in order to calculate an upper bound for the convergence rates, we make the stochastic volatility model more spe-

cific than that in Chapter 2. Lastly in this chapter, we discuss a feasible numeric scheme on solving the Dirichlet problem by using finite difference method. Note that all results are obtained under the Assumption 3.1, Assumption 3.3, Assumption 3.5 and Assumption 3.14, and we will not state these explicitly unless necessary.

In order to ease the notation, we often omit the superscripts which indicate the starting points of some processes. To keep the notation consistent, define

$$\xi_M \triangleq \inf\{t \geq 0 | Y_t^y > M\}, \tag{4.1}$$

$$\zeta_M \triangleq \inf\{t \geq 0 | X_t^{x,y} > M\}, \tag{4.2}$$

$$X_t^* \triangleq \max\{X_s | 0 \leq s \leq t\}, \tag{4.3}$$

$$Y_t^* \triangleq \max\{Y_s | 0 \leq s \leq t\}. \tag{4.4}$$

The stopping time $\tau_M^{x,y}$ defined (3.7) has the relationship

$$\tau_M = \xi_M \wedge \zeta_M.$$

By the continuity of the processes,

$$\{X_T^* \geq M\} = \{\zeta_M \leq T\},$$

and

$$\{Y_T^* \geq M\} = \{\xi_M \leq T\}.$$

## 4.1 The General Convergence Results

Let the underlying process and the stochastic volatility process satisfy all the assumptions in Section 3.1, Chapter 3. Recall our Remark 3.13 about the far field boundary conditions, that they do not have to have the same function $g$. But for simplicity, we assume the payoff function is still $g(\cdot)$, when the far field boundary is

hit. All results in this section are carried out for any far field boundary condition functions that have the same growth rates as $g$.

Combining the definition of far field boundary error and the convention for the stopping time from (3.7), for $0 \leq t < T$, equation (2.8) can be rewritten as

$$\begin{aligned}\mathcal{E}^M(t, X_t) &= \left|\mathbb{E}\left[g\left(X_T\right) 1_{\{\tau_M \leq T\}} - g\left(X_{\tau_M}\right) 1_{\{\tau_M \leq T\}} | \mathcal{F}_t\right]\right| \\ &= \left|\mathbb{E}\left[\left(g\left(X_T\right) - g\left(X_{\tau_M}\right)\right) 1_{\{\tau_M \leq T\}} | \mathcal{F}_t\right]\right|.\end{aligned}$$

Without loss of generality, we uniformly consider the error at time 0. A direct calculation gives

$$\begin{aligned}\mathcal{E}^M(x) \triangleq & \mathcal{E}^M(0, X_0) \\ = & \left|\mathbb{E}\left[\left(g\left(X_T\right) - g\left(X_{\tau_M}\right)\right) 1_{\{\tau_M \leq T\}}\right]\right| \\ \leq & \mathbb{E}\left[\left|g\left(X_T\right) - g\left(X_{\tau_M}\right)\right| 1_{\{\tau_M \leq T\}}\right] \qquad (4.5)\end{aligned}$$

We break our discussion of the convergence results for $\mathcal{E}^M(x)$ into two different types of contracts, strictly sublinear contract and linear contract.

### 4.1.1 Sublinear growth contract

Suppose the payoff function $g$ is strictly sublinear, then there exists a constant $C$ such that

$$g(x) \leq C(1 + x),$$

and

$$\limsup_{x \to \infty} \frac{g(x)}{x} = 0.$$

Before we state the convergence result, we start with some auxiliary technical results.

**Lemma 4.1.** *Given a function $g$ that satisfies the Assumption 3.5 and $g$ is of strictly sublinear growth, then*

$$\lim_{M\to\infty} \mathbb{E}\left[g\left(X_T\right) 1_{\{X_T>M\}}\right] = 0.$$

*Proof.* For any $\epsilon > 0$, because of the sublinear property of $g$, i.e.

$$\limsup_{x\to\infty} \frac{g(x)}{x} = 0,$$

there exist a constant $M$, such that for any $x \geq M$, $\frac{g(x)}{x} < \epsilon$.

For such a $M$,

$$\begin{aligned}
\mathbb{E}\left[g\left(X_T\right) 1_{\{X_T>M\}}\right] =& \mathbb{E}\left[\frac{g\left(X_T\right)}{X_T} X_T 1_{\{X_T>M\}}\right] \\
<& \epsilon \mathbb{E}\left[X_T 1_{\{X_T>M\}}\right] \\
<& \epsilon \mathbb{E}\left[X_T\right] \\
\leq& \epsilon X_0.
\end{aligned}$$

□

**Lemma 4.2.** *Let $Y$ start from some point $y < M$, then*

$$\mathbb{P}\left[Y_T^* \geq M\right] \leq \frac{1}{M} \mathbb{E}\left[Y_{\xi_M \wedge T}\right].$$

*Proof.* Because the process is continuous and nonnegative, $Y_{\xi_M} \geq M$. We have

$$\begin{aligned}
\mathbb{P}\left[Y_T^* \geq M\right] =& \mathbb{E}\left[1_{\{\xi_M \leq T\}}\right] \\
=& \mathbb{E}\left[1_{\{\xi_M \leq T\}} \frac{Y_{\xi_M}}{M}\right] \\
\leq& \frac{1}{M} \mathbb{E}\left[1_{\{\xi_M \leq T\}} Y_{\xi_M} + 1_{\{\xi_M > T\}} Y_T\right] \\
=& \frac{1}{M} \mathbb{E}\left[Y_{\xi_M \wedge T}\right]
\end{aligned}$$

□

**Remark 4.3.** In fact, Lemma 4.2 works for process $X$ as well.

**Lemma 4.4.** $\mathbb{E}\left[Y_{\xi_M \wedge T}\right]$ *is bounded by a constant only depends on* $T$, $x$, *and* $C$ *(in Assumption 3.3), not on* $M$.

*Proof.* Under Assumption 3.3.(b)(i), an application of Problem 5.3.15 (Karatzas and Shreve 1991 [38]) leads

$$\begin{aligned}\mathbb{E}\left[Y_{\xi_M \wedge T}\right] \leq & C_1 \mathbb{E}\left[\max_{0 \leq t \leq T} |Y_t|^2\right] \\ \leq & C_2(1 + Y_0^2)e^{C_2 T}.\end{aligned}$$

If Assumption 3.3.(b)(ii) prevails, $Y$ admits the following dynamics

$$dY_t = \mu(Y_t)dt + \beta Y_t^p dB_t, Y_0 = y.$$

Since $\mu(y) \leq C(1 + y)$, consider an auxiliary process

$$d\tilde{Y}_t = C(1 + \tilde{Y}_t)dt + \beta \tilde{Y}_t^p dB_t, \tilde{Y}_0 = y.$$

The comparison principle (Proposition 5.2.18 [38]) implies

$$\mathbb{P}\left[Y_t \leq \tilde{Y}_t, \forall 0 \leq t < \infty\right] = 1.$$

It suffices to derive a bound for $\mathbb{E}\left[\tilde{Y}_{\xi_M \wedge T}\right]$. Notice that

$$de^{-Ct}\tilde{Y}_t = Ce^{-Ct}dt + \beta e^{-Ct}\tilde{Y}_t^p dB_t.$$

Thus,

$$e^{-CT \wedge \xi_M}\tilde{Y}_{T \wedge \xi_M} = y + \int_0^{T \wedge \xi_M} Ce^{-Cs}\, ds + \int_0^{T \wedge \xi_M} \beta e^{-Cs}\tilde{Y}_s^p\, dB_s.$$

The stochastic integral above is a martingale because the integrand is bounded before the stopping time $\xi_M$. Therefore,

$$\begin{aligned}e^{-CT}\mathbb{E}\left[\tilde{Y}_{T \wedge \xi_M}\right] \leq & \mathbb{E}\left[e^{-CT \wedge \xi_M}\tilde{Y}_{T \wedge \xi_M}\right] \\ \leq & y + CT.\end{aligned}$$

This is equivalent to saying that

$$\mathbb{E}\left[\tilde{Y}_{T\wedge\xi_M}\right] \leq e^{CT}(y+CT).$$

In either case, the statement of the lemma holds. □

**Theorem 4.5.** *For a fixed $x$, and a strictly sublinear growth payoff function $g$, the solution to the Dirichlet problem (3.13) converges to the value function $u$ in (3.9), as the far field boundary $M$ approaches to infinity.*

*Proof.* Since $g$ can be bounded by $\bar{g}$ as in Lemma 3.6, we can assume $g$ is nonnegative and nondecreasing. Also, without loss of generality, we maintain the choice of far field boundary conditions, although the convergence results applies to any function in Assumption 3.5 with strictly sublinear growth used in the following argument.

By following the discussion from equation (4.5), we have

$$\begin{aligned}
\mathcal{E}^M(x) \leq & \mathbb{E}\left[\left|g\left(X_T\right)-g\left(X_{\tau_M}\right)\right| 1_{\{\tau_M\leq T\}}\right] \\
\leq & \mathbb{E}\left[\left(g\left(X_T\right)+g\left(X_{\tau_M}\right)\right) 1_{\{\tau_M\leq T\}}\right] \\
\leq & \mathbb{E}\left[\left(g\left(X_T\right)+g\left(X_{\tau_M}\right)\right) 1_{\left\{X_T^*\geq M\right\}}\right] \\
& + \mathbb{E}\left[\left(g\left(X_T\right)+g\left(X_{\tau_M}\right)\right) 1_{\left\{X_T^*<M\right\}} 1_{\left\{Y_T^*\geq M\right\}}\right]
\end{aligned}$$

Notice that $X_{\tau_M} \leq M$, and on the set $\{X_T^* < M\}$, $X_T$ is bounded by $M$, the second expectation can be bounded by

$$2g(M)\mathbb{P}[Y_T^* \geq M].$$

Thus,

$$\begin{aligned}
\mathcal{E}^M(x) \leq & \mathbb{E}\left[\left(g\left(X_T\right)+g\left(X_{\tau_M}\right)\right) 1_{\left\{X_T^*\geq M\right\}}\right] + 2g(M)\mathbb{P}[Y_T^* \geq M] \\
\leq & \mathbb{E}\left[g\left(X_T\right) 1_{\left\{X_T^*\geq M\right\}}\right] + g\left(M\right)\mathbb{P}[X_T^* \geq M] + 2g(M)\mathbb{P}[Y_T^* \geq M]
\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\left[g\left(X_T\right)1_{\left\{X_T^*\geq M\right\}}1_{\{X_T>M\}}\right]+\mathbb{E}\left[g\left(X_T\right)1_{\left\{X_T^*\geq M\right\}}1_{\{X_T\leq M\}}\right]\\
&+g\left(M\right)\mathbb{P}[X_T^*\geq M]+2g(M)\mathbb{P}[Y_T^*\geq M]\\
\leq&\mathbb{E}\left[g\left(X_T\right)1_{\left\{X_T^*\geq M\right\}}1_{\{X_T>M\}}\right]\\
&+2g\left(M\right)\mathbb{P}[X_T^*\geq M]+2g(M)\mathbb{P}[Y_T^*\geq M]\\
=&\mathbb{E}\left[g\left(X_T\right)1_{\{X_T>M\}}\right]+2g\left(M\right)\mathbb{P}[X_T^*\geq M]+2g(M)\mathbb{P}[Y_T^*\geq M]. \qquad (4.6)
\end{aligned}
$$

Because of Lemma 4.1, the first term has a zero limit, as $M$ approaches infinity. According to Lemma 4.2, Lemma 4.4 and supermartingale nature of $X$, we have

$$g\left(M\right)\mathbb{P}[X_T^*\geq M]\leq\frac{g\left(M\right)}{M}\mathbb{E}\left[X_{T\wedge\zeta_M}\right]=\frac{g\left(M\right)}{M}x,$$

and

$$2g(M)\mathbb{P}[Y_T^*\geq M]\leq\frac{2g\left(M\right)}{M}\mathbb{E}\left[Y_{T\wedge\xi_M}\right]\leq\frac{2g\left(M\right)}{M}\tilde{C},$$

where $\tilde{C}$ is a constant that derived from Lemma 4.4. Since $g$ is of strictly sublinear growth, the last too terms go to zero as well. Hence,

$$\lim_{M\to\infty}\mathcal{E}^M(x)=0,$$

and this means that the Dirichlet problem is convergent to the value function $u$. □

**Remark 4.6.** Theorem 4.5 tells us that we can apply appropriate numeric scheme to the Dirichlet problem (3.13) when the payoff function is of sublinear growth, even though the underlying or the stochastic volatility process could carry a strictly local martingale feature. This is because that when $M$ is large enough the solution to the Dirichlet problem is close to the value function $u$ in (3.9). Admittedly, this does not give any evidence about the speed of the convergence. In fact, the convergence rate can be very slow for local martingale underlying processes. If, however, we know the growth rate for the function $g$, some convergence rates are implied from the proof of the previous theorem, as indicated in Corollary 5.4.

### 4.1.2 Linear growth contract

In Subsection 4.1.1, we have shown the convergence result (Theorem 4.5) under the assumption that

$$\limsup_{x\to\infty} \frac{g(x)}{x} = 0.$$

However, if

$$\limsup_{x\to\infty} \frac{g(x)}{x} = \alpha \in (0, \infty),$$

the proof in Theorem 4.5 does not work any more. In fact, we have discussed a situation that a solution to Dirichlet problem (2.27) fails to converge to the value function (2.24) in Subsection 2.2.2 of Chapter 2. Thus, we should not expect Theorem 4.5 to work for linear contract. It is worth pointing out that this does not mean that Dirichlet boundary can never be used for pricing of options written on local martingale process. Rather, the strictly local martingale property would require a really nice Dirichlet boundary condition instead of an arbitrary function in Assumption 3.5. More often than not, such a precise boundary function is difficult to find. Even it can be found, the truncation errors generated from numerical scheme will not be stable.

In fact, the theorem below shows that the conclusion of Theorem 4.5 will hold if the underlying process is a true martingale.

**Theorem 4.7.** *For a fixed $x$, and an at most linear growth payoff function $g$, the solution to the Dirichlet problem (3.13) (possible extending the far field boundary for $Y$ to $M^{1+\epsilon}$, $\epsilon \geq 0$) converges to the value function $u$ in (3.9), as the far field boundary $M$ approaches to infinity, if the underlying process $X$ is a true martingale.*

*Proof.* We slightly modify the hitting level for the process $Y$ inside in the proof, by

defining

$$\xi_M^\epsilon \triangleq \inf\{t \geq 0 | Y_t^y > M^{1+\epsilon}\}, \epsilon \geq 0.$$

By the exact same argument as Theorem 4.5, we arrive the equation (4.6)

$$\begin{aligned}
&\mathcal{E}^M(x) \\
\leq &\mathbb{E}\left[g\left(X_T\right) 1_{\{X_T > M\}}\right] + 2g\left(M\right) \mathbb{P}[X_T^* \geq M] + 2g(M)\mathbb{P}[Y_T^* \geq M^{1+\epsilon}]. \\
\leq &\mathbb{E}\left[C\left(1 + X_T\right) 1_{\{X_T > M\}}\right] + 2C\left(1 + M\right) \mathbb{P}[X_T^* \geq M] + 2C(1 + M)\mathbb{P}[Y_T^* \geq M^{1+\epsilon}].
\end{aligned}$$

In case of Assumption 3.3.(b).(i), $\epsilon$ can be chosen as 0. $M\mathbb{P}[Y_T^* \geq M]$ has limit zero as M approaches infinity.

In case of Assumption 3.3.(b).(ii), choose $\epsilon > 0$. An application of Lemma 4.2 and Lemma 4.4, we have

$$2C(1 + M)\mathbb{P}[Y_T^* \geq M^{1+\epsilon}] \leq \frac{2C\left(1 + M\right)}{M^{1+\epsilon}} \mathbb{E}\left[Y_{T \wedge \xi_M'}\right] \leq \frac{2C\left(1 + M\right)}{M^{1+\epsilon}} \tilde{C},$$

which has a zero limit when $M$ goes to infinity.

Because $X$ is a true margingale by assumption, the optional sampling theorem leads to

$$\mathbb{E}\left[X_T\right] = X_0 = X_{0 \wedge \zeta_M} = \mathbb{E}\left[X_{T \wedge \zeta_M}\right] = \mathbb{E}\left[X_T 1_{\{T \leq \zeta_M\}}\right] + \mathbb{E}\left[M 1_{\{T > \zeta_M\}}\right],$$

and hence

$$\mathbb{E}\left[X_T 1_{\left\{X_T^* \geq M\right\}}\right] = M\mathbb{P}[X_T^* \geq M]. \tag{4.7}$$

With this equation and $\mathbb{E}\left[X_T 1_{\{X_T \geq M\}}\right] \leq \mathbb{E}\left[X_T 1_{\left\{X_T^* \geq M\right\}}\right]$, it suffices to show that

$$\mathbb{E}\left[X_T 1_{\left\{X_T^* \geq M\right\}}\right] \xrightarrow{M \to \infty} 0.$$

By either the monotone convergence theorem or the uniformly integrability of $\left\{ X_T 1_{\left\{X_T^* \geq M\right\}} \right\}_{M>0}$,

$$\lim_{M \to \infty} \mathbb{E}\left[X_T 1_{\left\{X_T^* \geq M\right\}}\right] = \mathbb{E}\left[X_T \lim_{M \to \infty} 1_{\left\{X_T^* \geq M\right\}}\right].$$

Since the process $Y$ does not explode to infinity (Proposition 3.7) and $X$ is in exponential form of $Y$, neither $X$ nor $X^*$ explodes to infinity at finite time. Therefore,

$$\lim_{M\to\infty} 1_{\{X_T^*\geq M\}} = 0$$

almost surely, and this completes the proof. □

**Remark 4.8.** This proof won't work for strictly local martingale underlying simply because the equation (4.7) does not hold any more. Instead, the relationship becomes

$$\mathbb{E}\left[X_T 1_{\{X_T^*\geq M\}}\right] < M\mathbb{P}[X_T^* \geq M].$$

Thus, even though the left hand side of this equation goes to zero when $M$ goes to infinity, the right hand side can remain non-zero.

#### 4.1.3 Faster than linear growth contract

Stochastic volatility models have gained vast application in the fixed income, currencies and commodities business. Many ongoing models on the trading floors are just some variations of stochastic variation models. Although a majority of traded options have at most linear growth payoffs, it is not uncommon to see faster than linear growth contracts. One category of such examples are the quantos in foreign exchange markets. A quanto is a derivative that involves two or more currencies with the underlying asset in one currency and the instrument is settled in another currency at time of maturity. These financial instruments are popular among speculators who want to expose themselves to more volatilities in the foreign exchange rates, and overseas investors and companies who do business in another currency and try to hedge their exposure to the fluctuation of the foreign exchange rates. We shall use the self-quanto to serve as an example of faster than linear growth contract. For full discussion of the empirical knowledge of quantos, see Clark(2011) [12].

Consider a EUR/USD self-quanto call option with the foreign currency euros being the overlying and domestic currency dollars being the underlying. Denote by $X$ from (3.1) the exchange rate for EUR/USD, and constant $K$ the strike. If it were a normal standard European call option, the value of the option should be

$$u(x) = \mathbb{E}^x \left[ (X_T^x - K)^+ \right].$$

However, for a quanto option, the notional amount $(X_T^x - K)^+$ is paid in foreign currency euros instead of dollars. By converting the euro notional to a dollar equivalence, the quanto pays

$$u(x) = \mathbb{E}^x \left[ X_T^x \, (X_T^x - K)^+ \right].$$

From a mathematical point of the view, the payoff is a quadratic function, and has faster than linear growth with respect to $X$.

In terms of mathematical valuation of this option, an immediate problem with the quanto is square integrability of $X$, which is not guaranteed. Regularity for faster than linear growth contracts under general stochastic volatility models are not available. And even though by ignoring potential regularity problem, we would have to consider the following tail estimation for sake of far field boundary error

$$\mathbb{E}^x \left[ (X_T^x)^2 \, 1_{\left\{ (X_T^x)^* \geq M \right\}} \right].$$

One does not expect this equation to have a zero limit as $M$ goes to infinity, because $X$ often has a fat tail. Empirically, the tail for foreign exchange rate involving emerging market currencies is usually heavy, because of the high volatility for that rate. In fact, if the tail is small, and quadratic payoff can be integrable, then the linear contract may have faster convergence rate in terms of far field boundary error, and thus the above expectation may be negligible.

## 4.2 Convergence Rates for Heston-type Stochastic Volatility Model

We obtained some convergence conclusions in the previous section without any results on the convergence rates of Dirichlet problem (3.13) to the value function (3.9). In fact, the study of convergence rates is more difficult than just the convergence results. Meanwhile, we have seen that the convergence results are sensitive to the growth rates of the payoff function, to the martingale property of the underlying process and the existence of higher moments for the underlying process and the volatility process. Therefore, we expect that the convergence rates can heavily depend on these quantities as well. What's more, those properties are indirectly related to the choices of the parametric functions of the model (3.1). For example, Proposition 3.39 shows such a relation.

We continue to assume that the payoff function $g$ is of at most linear growth. According to the proof of Theorem 4.7, we need to estimate the following three error terms in order of $\frac{1}{M}$:

$$M\mathbb{P}[X_T^* \geq M], \tag{4.8}$$

$$\mathbb{E}[X_T 1_{\{X_T \geq M\}}],$$

$$M\mathbb{P}[Y_T^* \geq M]. \tag{4.9}$$

We have discussed in Theorem 4.7 that the second term is bounded by the first term. Therefore, it suffices to derive an estimation for (4.8) and (4.9).

For the model under Assumption 3.3, the parametric functions are too general to perform calculation and estimation. As an example, $\mathfrak{v}(\cdot)$, and $\mathfrak{p}(\cdot)$ in Proposition 3.39 are usually hard to simplify. Throughout this section, we confine our consideration of convergence rates estimation in a more specific and representative model.

### 4.2.1 Further assumptions

We assume the model (3.1) take a specific form as stated below, and we call it the **Heston-type** model.

**Assumption 4.9.** *The underlying process and the stochastic volatility process satisfies the following dynamics:*

$$dX_t = \lambda\sqrt{Y_t}X_t dW_t, \tag{4.10}$$

$$dY_t = \kappa(\theta - Y_t)dt + \beta Y_t^p dB_t, \tag{4.11}$$

$$dW_t dB_t = \rho dt,$$

*where* $\lambda, \kappa, \beta$ *are positive constants,* $\rho \in (-1, 1)$*, and* $p \geq \frac{1}{2}$*.*

**Example 4.10** (Heston model)**.** See Subsection 2.4

The reason that we use the CIR type of volatility process (4.11) is that we believe the stochastic volatility is mean reverting in real financial markets. We make $\lambda, \kappa, \beta$ and $\rho$ positive constants solely for the purpose of ease the notation. All results in this section are able to be reproduced when those constants are nonnegative deterministic functions.

**Remark 4.11.** One can verify that the model described in Assumption 4.9 also satisfies Assumption 3.3. It is straightforward to check that $b(y) = \sqrt{y}$ and $\mu(y) = \kappa(\theta - y)$ satisfies $(a), (b)$ and $(c)$ of Assumption 3.3. For $\sigma(y) = \beta y^p, p \leq 1$, $\sigma(\cdot)$ satisfies $(c)$ of Assumption 3.3 if and only if $p \geq \frac{1}{2}$. Actually, the stochastic volatility process $Y$ in (4.11) does not admit a unique solution when $p < \frac{1}{2}$ (see [38]). One way to make the process have unique solution is to impose a boundary condition when the process hits level zero, e.g., the process is reflected at the origin. In [8], Bhattacharya and Waymire indicate that this property is equivalent to imposing a

boundary condition on the forward Kolmogorov equation for its transition probability density, which is

$$\lim_{y\downarrow 0}\left(\frac{1}{2}\beta^2\frac{\partial}{\partial y}\left(y^{2p}\mathbb{P}\left(t,y_0,y\right)\right)-\kappa\left(\theta-\mathbb{P}\left(t,y_0,y\right)\right)\right)=0.$$

However, it is unclear how this boundary condition will generate a boundary condition for the pricing PDE. We only consider the case when the process admits a unique strong solution.

As a special case of the model discussed in Chapter 3, the model is non-exploding and has unique strong solution. In addition, the following proposition also characterizes the non-explosion to zero property of the process $Y$.

**Proposition 4.12.** *Infinity is unattainable for the stochastic process $Y$ in (4.11), and zero is attainable if and only if $p=\frac{1}{2}$, and $2\kappa\theta<\beta^2$.*

*Proof.* Similar to Proposition 3.7, the result is fully covered by the Feller's test for explosion. □

**Remark 4.13.** Assumption 4.9 is not so restrictive as it appears to be. Although it is sometimes more attempting to consider the following model in terms of convergence rates,

$$\begin{aligned}
&dX_t=\lambda Y_t^{\gamma}X_t dW_t,\gamma>0,\\
&dY_t=\mu(Y_t)dt+\beta Y_t^{p}dB_t,p\geq\frac{1}{2},\\
&dW_t dB_t=\rho dt,
\end{aligned}$$

but some algebraic calculation can transfer this model to a similar one to Assumption 4.9. For narrative simplicity, we adopt the Assumption 4.9 follow this subsection.

### 4.2.2 The importance of being a true martingale

In Section 4.1, we have seen the differences in terms of convergence between martingale asset and strictly local martingale asset. It is significant to filter out cases in which the asset is a martingale, because it might not converge at all for local martingales. In [7], it is proved that $X$ is a strictly local martingale for some, and then all all $\{y, T\} \in (0, \infty) \times (0, \infty)$. Thus, it suffices to study the martingale property on a fixed horizon.

When $p = \frac{1}{2}$, i.e., the model (3.1) is the Heston model, the underlying process $X$ is a true martingale as it is shown in the following proposition.

**Proposition 4.14.** *The underlying process in Heston model (2.32) is a true martingale.*

*Proof.* Recall that $X_t^{x,y} = xH_t^y$, and the goal is to show that $H_t^y$ is a martingale, where

$$H_t^y \triangleq \exp\left(\int_0^t b(Y_t^y)\, dW_t - \frac{1}{2}\int_0^t b^2(Y_t^y)\, dt\right).$$

If we can show the weak form of Novikov condition holds for $H$, then the martingale property of $H$ follows immediately. Thus, we need to show there is an increasing sequence $\{t_n\}_{n=0}^{\infty}$ of real numbers with $t_n \uparrow \infty$, such that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} Y_t\, dt\right)\right] < \infty.$$

Let us first derive a bound on

$$\mathbb{E}\left[\exp\left(\nu Y_u\right)\right],$$

for a fixed small $\nu$, and fixed time $u$.

It is well-known that the CIR process $Y$ is a affine process (refer to [28]). Thus

there are deterministic functions $A(\cdot)$ and $C(\cdot)$ such that

$$\mathcal{M}_s \triangleq \mathbb{E}\left[\exp\left(\nu Y_u\right)|\mathcal{F}_s\right] = \exp\left(Y_s A\left(u-s\right) + C\left(u-s\right)\right).$$

An application of Itô's formula gives

$$d\mathcal{M}_s = \mathcal{M}_s\left(Y_s\left(-A' - \kappa A + \frac{1}{2}\beta^2 A^2\right) + \left(\kappa\theta A - C'\right)\right)ds + \mathcal{M}_s\beta A\sqrt{Y_s}dB_s.$$

Because $\mathcal{M}$ is a martingale, the following ODEs holds:

$$\begin{cases} A' = -\kappa A + \frac{1}{2}\beta^2 A^2, \\ C' = \kappa\theta A, \\ A(0) = \nu, \\ C(0) = 0. \end{cases}$$

By choosing $\nu = \frac{\beta^2}{2\kappa}$, and solving the ODEs, we arrive at

$$\begin{cases} A(s) = \frac{2\kappa}{\beta^2}, \\ C(s) = \frac{2\kappa}{\beta^2}s. \end{cases}$$

Thus,

$$\mathbb{E}\left[\exp\left(\nu Y_u\right)\right] = \exp\left(yA\left(u\right) + C\left(u\right)\right) = \exp\left(y\frac{2\kappa}{\beta^2} + \frac{2\kappa}{\beta^2}u\right) < \infty.$$

Now, let $\{t_n\}_{n=0}^{\infty}$ be an arithmetic progression with the common difference of successive members is $2\nu$. By Jensen's inequality

$$\begin{aligned} \mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} Y_t\, dt\right)\right] \leq & \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \frac{1}{2\nu}\exp\left(\nu Y_u\right)\, du\right] \\ = & \frac{1}{2\nu}\int_{t_{n-1}}^{t_n} \mathbb{E}\left[\exp\left(\nu Y_u\right)\right]\, du \\ = & \frac{1}{2\nu}\int_{t_{n-1}}^{t_n} \exp\left(y\frac{2\kappa}{\beta^2} + \frac{2\kappa}{\beta^2}u\right)\, du \\ < & \infty. \end{aligned}$$

This completes the verification of Novikov's condition, and the martingale property of $X$ follows consequently. $\square$

In order to derive similar results when $p > \frac{1}{2}$, we need the following lemma.

**Lemma 4.15.** *For $p > 0$ and $\eta \geq 1$,*

$$\mathbb{E}\left[X_T^\eta\right] = x^\eta \mathbb{E}^{(\eta)}\left[1_{\{\xi_\infty > T\}} \exp\left(\alpha \int_0^T Y_s \, ds\right)\right],$$

*where $\alpha = \frac{\lambda^2}{2}\eta(\eta - 1)$, and $\xi_\infty = \lim_{n\to\infty} \xi_n$ with $\xi_n$ defined in (4.1).*

*What's more, under a new probability measure $\mathbb{P}^{(\eta)}$, the stochastic volatility process $Y$ has the dynamic*

$$dY_t = \left(\kappa\left(\theta - Y_t\right) + \rho\beta\lambda\eta Y_t^{p+\frac{1}{2}}\right) dt + \beta Y_t^p dB_t^{(\eta)}, \tag{4.12}$$

*Proof.* We borrow the ideas from Andersen, Piterbarg (see [1]), and Sin (see [51]).

Let $\bar{B}$ be a Brownian motion such that

$$dW_t = \rho dB_t + \bar{\rho} d\bar{B}_t, \bar{\rho} = \sqrt{1 - \rho^2}.$$

Then,

$$X_T^\eta = x^\eta \exp\left(\rho\eta\lambda \int_0^T \sqrt{Y_t}\, dB_t + \bar{\rho}\eta\lambda \int_0^T \sqrt{Y_t}\, d\bar{B}_t - \frac{1}{2}\lambda^2\eta \int_0^T Y_t \, dt\right).$$

By the definition of $\xi_n$, the process $Y$ is bounded by constant $n$ on the set $\{T < \xi_n\}$. Denote by $\{\mathcal{F}_t^B\}$ the filtration generated by $B$, then tower property gives

$$\begin{aligned}
&\mathbb{E}\left[X_T^\eta 1_{\{T<\xi_n\}}\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[X_T^\eta 1_{\{T<\xi_n\}} | \mathcal{F}_T^B\right]\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[x^\eta \exp\left(\rho\eta\lambda \int_0^T \sqrt{Y_t}\, dB_t + \bar{\rho}\eta\lambda \int_0^T \sqrt{Y_t}\, d\bar{B}_t - \frac{1}{2}\lambda^2\eta \int_0^T Y_t \, dt\right) 1_{\{T<\xi_n\}} \middle| \mathcal{F}_T^B\right]\right] \\
=&\mathbb{E}\left\{x^\eta \exp\left(\rho\eta\lambda \int_0^T \sqrt{Y_t}\, dB_t - \frac{1}{2}\lambda^2\eta \int_0^T Y_t \, dt\right) 1_{\{T<\xi_n\}} \mathbb{E}\left[\exp\left(\bar{\rho}\eta\lambda \int_0^T \sqrt{Y_t}\, d\bar{B}_t\right) \middle| \mathcal{F}_T^B\right]\right\}
\end{aligned}$$

$$=\mathbb{E}\left[x^{\eta}\exp\left(\rho\eta\lambda\int_0^T\sqrt{Y_t}\,dB_t-\frac{1}{2}\lambda^2\eta\int_0^T Y_t\,dt\right)1_{\{T<\xi_n\}}\exp\left(\frac{1}{2}\bar{\rho}^2\eta^2\lambda^2\int_0^T Y_t\,dt\right)\right]$$
$$=x^{\eta}\mathbb{E}\left[\exp\left(\rho\eta\lambda\int_0^T\sqrt{Y_t}\,dB_t-\frac{1}{2}\rho^2\lambda^2\eta^2\int_0^T Y_t\,dt\right)1_{\{T<\xi_n\}}\exp\left(\alpha\int_0^T Y_t\,dt\right)\right],$$

where $\alpha=\frac{\lambda^2}{2}\eta(\eta-1)$.

Let

$$\mathcal{M}_T\triangleq\exp\left(\rho\eta\lambda\int_0^T\sqrt{Y_t}\,dB_t-\frac{1}{2}\rho^2\lambda^2\eta^2\int_0^T Y_t\,dt\right).$$

Notice that on the set $\{T<\xi_n\}$, $\mathcal{M}_T=\mathcal{M}_{T\wedge\xi_n}$, and $\mathbb{E}\left[\mathcal{M}_{T\wedge\xi_n}\right]=1$. Define a new probability measure $\mathbb{P}_n^{(\eta)}$ by

$$\mathcal{M}_{T\wedge\xi_n}=\frac{d\mathbb{P}_n^{(\eta)}}{d\mathbb{P}}.$$

By Girsonav Theorem, under the measure $\mathbb{P}_n^{(\eta)}$, $Y$ has the dynamic

$$dY_t=\left(\kappa\left(\theta-Y_t\right)+\rho\beta\lambda\eta Y_t^{p+\frac{1}{2}}\right)dt+\beta Y_t^p dB_t^{(\eta)}.$$

Thus,

$$\mathbb{E}\left[X_T^{\eta}1_{\{T<\xi_n\}}\right]=x^{\eta}\mathbb{E}_n^{(\eta)}\left[1_{\{T<\xi_n\}}\exp\left(\alpha\int_0^T Y_t\,dt\right)\right].$$

It follows from Lemma 4.2 in [51] that $\mathbb{P}_n^{(\eta)}$ is consistent with $\mathbb{P}^{(\eta)}$, and therefore,

$$\mathbb{E}\left[X_T^{\eta}1_{\{T<\xi_n\}}\right]=x^{\eta}\mathbb{E}^{(\eta)}\left[1_{\{T<\xi_n\}}\exp\left(\alpha\int_0^T Y_t\,dt\right)\right].$$

Now by the monotone convergence theorem and the fact that $Y$ does not explode to infinity under $\mathbb{P}$,

$$\begin{aligned}\mathbb{E}\left[X_T^{\eta}\right]=&\lim_{n\to\infty}\mathbb{E}\left[X_T^{\eta}1_{\{T<\xi_n\}}\right]\\=&\lim_{n\to\infty}x^{\eta}\mathbb{E}^{(\eta)}\left[1_{\{T<\xi_n\}}\exp\left(\alpha\int_0^T Y_t\,dt\right)\right]\\=&x^{\eta}\mathbb{E}^{(\eta)}\left[1_{\{T<\xi_\infty\}}\exp\left(\alpha\int_0^T Y_t\,dt\right)\right].\end{aligned}$$

□

**Proposition 4.16.** *$X$ is a martingale if one of the following condition holds:*

*(a) $p \in (0, \frac{1}{2}]$;*

*(b) $p \in (\frac{1}{2}, \frac{3}{2})$, and $\rho \leq 0$;*

*(c) $p = \frac{3}{2}$, and $\rho \leq \frac{\beta}{2\lambda}$;*

*(d) $p \in (\frac{3}{2}, \infty)$.*

*Proof.* Setting $\eta = 1$ in Lemma 4.15, we see that

$$\mathbb{E}\left[X_T\right] = x\mathbb{E}^{(1)}\left[1_{\{T<\xi_\infty\}}\right],$$

and under measure $\mathbb{P}^{(1)}$,

$$dY_t = \left(\kappa\left(\theta - Y_t\right) + \rho\beta\lambda Y_t^{p+\frac{1}{2}}\right)dt + \beta Y_t^p dB_t^{(1)}.$$

Hence, $X$ is a martingale if and only if $Y$ does not explode to infinity under the new measure $\mathbb{P}^{(1)}$.

Therefore, the explosion behavior of process $Y$ is once again covered by Feller's test for explosions, and we refer our readers to [1], [51], and [41] for details of the calculation. □

**Remark 4.17.** We discuss some intuition why in those cases $X$ is a martingale in the cases listed in previous proposition.

When $p < \frac{1}{2}$, the drift in $Y$ is dominated by $-\kappa Y_t$, as $\rho\beta\lambda Y_t^{p+\frac{1}{2}}$ has a slower growth rate. And, the process will not explode to infinity under $\mathbb{P}^{(1)}$.

When case (b) holds, the correlation is non-positive. We know if $Y$ did not have drift, it would not explode. And a strong negative drift in (4.12) gives the process less chance of exploding to infinity.

For case (c) and (d), the situation is less intuitive. We see that the squared volatility has index $2p$, which is larger than $p + \frac{1}{2}$. And, the positive drift $Y_t^{p+\frac{1}{2}}$ is less important than the comparatively large variance $Y_t^{2p}$.

Starting from the next subsection, we will derive a upper bound for the far field boundary error when the model falls into some of the cases in Proposition 4.16. Recall our discussion in Subsection 4.1.2 that linear growth options written on local martingales can fail to be approximated by Dirichlet problems, and consequently, it is not needed to consider their convergence rates.

We study the convergence rates for $p = \frac{1}{2}$ in Subsection 4.2.3, and then the case of (b) in Subsection 4.2.4. Case (c) is quite similar to the case of (b), and similar results are implied from the case (b). We do not discuss case (d), because the convergence rates in the $Y$ direction may be quite slow already.

**Remark 4.18.** We indicated before that far field boundary error estimation is strongly tied to the tail distribution of the underlying process $X$, and the stochastic volatility process $Y$. And since the distribution itself of the stochastic volatility model is difficult to track, to compute a tight order of far field boundary error is usually difficult. It worth pointing out that a lower bound for the far field boundary error of Dirichlet problem is zero, as one can use the value function $u$ in (3.9) for the far field boundary and there is no such error incurred with a correct boundary condition.

### 4.2.3 Convergence rates for $p = \frac{1}{2}$ - Heston model

In case (a) of Proposition 4.16, we only discuss the case of the Heston model, because of Assumption 4.9 that requires $p \geq \frac{1}{2}$. Let us start by estimating the moment stability of $X$, and $Y$.

**Proposition 4.19.** *For any fixed $T$, $\eta \geq 0$, and $p \in [\frac{1}{2}, 1]$*

$$\mathbb{E}\left[Y_T^{\eta}\right] < \infty.$$

*Consequently the far field boundary error (4.9) in $Y$ direction converges to zero faster than any polynomial.*

*Proof.* Define a new process $\tilde{Y}$ by

$$d\tilde{Y}_t = \kappa(\theta + \tilde{Y}_t)dt + \tilde{Y}_t^p dB_t.$$

The moment stability of $Y$ and $\tilde{Y}$ follows directly from Problem 5.3.15 in [38]. By the comparison principle (Proposition 5.2.18 [38]) $Y \leq \tilde{Y}$ almost surely. Now $\tilde{Y}$ has positive drift always, and by Doob's maximum inequality

$$\begin{aligned} M\mathbb{P}[Y_T^* \geq M] \leq & M\mathbb{P}[\tilde{Y}_T^* \geq M] \\ \leq & \frac{1}{M^{\eta-1}}\mathbb{E}\left[\tilde{Y}_T^{\eta}\right] \\ \leq & \frac{1}{M^{\eta-1}}C, \end{aligned}$$

where $C$ is a constant independent of $M$. Because $\eta$ and $T$ are arbitrary, this completes the proof. □

Proposition 4.19 implies that the bottleneck for the convergence rates is in the $X$ direction. The estimation of far field boundary error in the $X$ direction results in the following theorem.

**Theorem 4.20.** *Assume $2\kappa\theta \geq \beta^2$. The far field boundary error for Heston model decreases uniformly in $T$ at least in order of*

$$\frac{1}{M^{\eta^*}},$$

*and, for a fixed $T$, in order of*

$$\frac{1}{M^{\eta_T^*}}, \quad \textit{with } \eta_T^* = \eta_1^* \vee \eta_2^*,$$

*where*

$$\eta^* = \max\left\{\eta > 1 \middle| f_\eta^+ \leq \frac{\kappa}{\beta\lambda}\right\},$$

$$\eta_1^* = \max\left\{\eta > 1 \middle| f_\eta^- \geq \frac{\kappa}{\beta\lambda}, T < \frac{\log\frac{\rho\beta\eta\lambda-\kappa+\sqrt{d_\eta}}{\rho\beta\eta\lambda-\kappa-\sqrt{d_\eta}}}{\sqrt{d_\eta}}\right\},$$

$$\eta_2^* = \max\left\{\eta > 1 \middle| f_\eta^+ > \frac{\kappa}{\beta\lambda} > f_\eta^-, T < \frac{\pi 1_{\left\{\rho\eta<\frac{\kappa}{\beta\lambda}\right\}} + \arctan\frac{\sqrt{-d_\eta}}{\rho\beta\eta\lambda-\kappa}}{2\sqrt{-d_\eta}}\right\},$$

*and*

$$f_\eta^\pm \triangleq \rho\eta \pm \sqrt{\eta\left(\eta-1\right)}, \quad d_\eta \triangleq \left(\rho\beta\eta\lambda-\kappa\right)^2 - \lambda^2\beta^2\eta\left(\eta-1\right).$$

*Proof.* $2\kappa\theta \geq \beta^2$ implies the regularity results in Chapter 3 applies. Similar to Lemma 4.15,

$$\begin{aligned}\mathbb{E}\left[X_T^\eta\right] =& x^\eta \mathbb{E}\left[\exp\left(\eta\lambda\int_0^T \sqrt{Y_t}dW_t - \frac{1}{2}\eta\lambda^2\int_0^T \sqrt{Y_t}dt\right)\right] \\ =& x^\eta \mathbb{E}^{(\eta)}\left[1_{\{T<\xi_\infty\}}\exp\left(\alpha\int_0^T Y_t\,dt\right)\right],\end{aligned}$$

where $\alpha = \frac{1}{2}\lambda^2\eta(\eta-1)$. Under the measure $\mathbb{P}^{(\eta)}$, $Y$ has the following dynamic

$$dY_t = \left(\kappa\left(\theta - Y_t\right) + \rho\beta\lambda\eta Y_t\right)dt + \beta\sqrt{Y_t}dB_t^{(\eta)}.$$

Because $Y$ does not explode to infinity almost surely under $\mathbb{P}^{(\eta)}$, the moment of $X$ can be simplified to

$$\mathbb{E}\left[X_T^\eta\right] = x^\eta \mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^T Y_t\,dt\right)\right].$$

Following the same routine as Proposition 4.14, and using the affine property of $Y$,

$$\mathcal{M}_s \triangleq \mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^T Y_t\,dt\right)|\mathcal{F}_s\right] = \exp\left(\alpha\int_0^s Y_t\,dt + Y_s A\left(u-s\right) + C\left(u-s\right)\right).$$

Suppose that $\mathbb{E}\left[X_T^\eta\right]$ is bounded. Then $\mathcal{M}_s$ is a martingale, and Itô's lemma gives

$$d\mathcal{M}_s = \mathcal{M}_s \left( Y_s \left( \alpha - A' - \kappa A + \rho\beta\eta\lambda A + \frac{1}{2}\beta^2 A^2 \right) + (\kappa\theta A - C') \right) ds + \mathcal{M}_s \beta A \sqrt{Y_s} dB_s^{(\eta)}.$$

The martingale property of $M$ indicates that

$$\begin{cases} A' = \alpha + \rho\beta\eta\lambda A - \kappa A + \frac{1}{2}\beta^2 A^2, \\ C' = \kappa\theta A, \\ A(0) = 0, \\ C(0) = 0. \end{cases} \tag{4.13}$$

The moment of the process $X$ does not explode if and only if the deterministic functions $A$ and $C$ do not explode before the maturity time $T$. It is straightforward to see from the ODE that function $C$ has the same explosion time as $A$. It suffices to study the explosion behavior of $A$. The Riccati equation for $A$ here appears more complicated than the one in Proposition 4.14. Nevertheless, its explosion behavior is fully known (see [4] and [1]).

Define

$$\begin{aligned} b &\triangleq \frac{2\alpha}{\beta^2}, \\ a &\triangleq \frac{2(\rho\beta\eta\lambda - \kappa)}{\beta^2}, \\ d &\triangleq a^2 - 4b = \frac{4\left(\rho\beta\eta\lambda - \kappa\right)^2 - 8\alpha\beta^2}{\beta^4}. \end{aligned}$$

The explosion time $T^*$ of (4.13) falls into the following three categories:

(1) $d \geq 0$, $a < 0$:

$$T^* = \infty;$$

(2) $d \geq 0$, $a > 0$:

$$T^* = 2\frac{1}{\sqrt{d}\beta^2} \log\left( \frac{a + \sqrt{d}}{a - \sqrt{d}} \right);$$

(3) $d < 0$:

$$T^* = \frac{1}{\sqrt{-d}\beta^2}\left(\pi 1_{\{a<0\}} + \arctan\left(\frac{\sqrt{-d}}{a}\right)\right).$$

Define

$$f_\eta^+ \triangleq \rho\eta + \sqrt{\eta\,(\eta-1)},$$

$$f_\eta^- \triangleq \rho\eta - \sqrt{\eta\,(\eta-1)},$$

$$d_\eta \triangleq (\rho\beta\eta\lambda - \kappa)^2 - \lambda^2\beta^2\eta\,(\eta-1).$$

Hence the moment $\mathbb{E}\left[X_T^{\eta^*}\right]$ exists uniformly for all $T > 0$, where

$$\eta^* = \max\left\{\eta > 1 \middle| f_\eta^+ \leq \frac{\kappa}{\beta\lambda}\right\}.$$

Denote $\eta_1^*, \eta_2^*$ by

$$\eta_1^* = \max\left\{\eta > 1 \middle| f_\eta^- \geq \frac{\kappa}{\beta\lambda}, T < \frac{\log\frac{\rho\beta\eta\lambda-\kappa+\sqrt{d_\eta}}{\rho\beta\eta\lambda-\kappa-\sqrt{d_\eta}}}{\sqrt{d_\eta}}\right\},$$

$$\eta_2^* = \max\left\{\eta > 1 \middle| f_\eta^+ > \frac{\kappa}{\beta\lambda} > f_\eta^-, T < \frac{\pi 1_{\left\{\rho\eta<\frac{\kappa}{\beta\lambda}\right\}} + \arctan\frac{\sqrt{-d_\eta}}{\rho\beta\eta\lambda-\kappa}}{2\sqrt{-d_\eta}}\right\},$$

$$\eta_T^* = \eta_1^* \vee \eta_1^*.$$

Therefore, for fixed $T$, the moment $\mathbb{E}\left[X_T^{\eta_T^*}\right]$ exists. Because of the martingale property of $X$ and Doob's maximum inequality, the far field boundary error (4.8) in the underlying direction can be approximated as

$$M\mathbb{P}[X_T^* \geq M] \leq \frac{1}{M^\eta}\mathbb{E}[X_T^\eta].$$

By Proposition 4.19, we conclude that the far field boundary error for Heston model decreases uniformly in T in order of at least

$$\frac{1}{M^{\eta^*}},$$

and for fixed $T$, it decreases in order of at least

$$\frac{1}{M^{\eta_T^*}}.$$

□

#### 4.2.4 Convergence rates for $p \in \left(\frac{1}{2}, \frac{3}{2}\right)$

The model in Assumption 4.9 is a martingale if $\rho \leq 0$ under the condition $p \in \left(\frac{1}{2}, \frac{3}{2}\right)$, and we shall assume $\rho \leq 0$ in this subsection.

By using the same approach to estimate a convergence rate in this case, we need the following proposition.

**Proposition 4.21.** *For $p \in \left(\frac{1}{2}, \frac{3}{2}\right)$, $\eta \geq 0$, and $\forall\ T > 0$*

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} X_T^\eta\right] \leq \infty,$$

*if*

$$\rho < -\sqrt{\frac{\eta - 1}{\eta}} \tag{4.14}$$

*Proof.* This is a direct application of the Theorem 3.2.(iv) of Lions and Musiela [43].

In fact for those parameters defined in [43], let $\delta = \frac{1}{2}$, $\gamma = p$, $m = \eta$, and $b(\cdot) = \mu(\cdot)$, etc, we arrive at the conclusion that if

$$\rho < -\sqrt{\frac{\eta - 1}{\eta}} - \frac{\mu_\infty}{\beta\eta}, \tag{4.15}$$

then

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} X_T^\eta\right] \leq \infty,$$

for all $T > 0$, with $\mu_\infty$ given by

$$\mu_\infty = \limsup_{y \to \infty} \frac{\mu(y)}{y^{p + \frac{1}{2}}}.$$

Because $\mu(y) = \kappa(\theta - y)$, and $p \in \left(\frac{1}{2}, \frac{3}{2}\right)$, we have

$$\mu_\infty = 0.$$

Now, condition (4.15) reduces to (4.14), and this completes the proof of the statement. □

Notice that Proposition 4.19 no longer holds for $p \in \left(1, \frac{3}{2}\right)$, as the growth condition for the volatility of the stochastic volatility process is faster than linear.

Having the previous proposition, a theorem analogous to Theorem 4.20 can be stated below.

**Theorem 4.22.** *The far field boundary error for model described in Assumption 4.9 with $p \in \left(\frac{1}{2}, 1\right]$, $\rho < 0$ decreases uniformly for all $T$ at least in order of*

$$\frac{1}{M^{\eta^*}},$$

*where*

$$\eta^* = \frac{\rho^2}{1 - \rho^2}.$$

*Further more, by an extension of the far field boundary domain for the $Y$ direction, this convergence rate applies to all $p \in \left(\frac{1}{2}, \frac{3}{2}\right)$.*

*Proof.* If $p \in \left(\frac{1}{2}, 1\right]$, then Proposition 4.19 implies that the far field error in the $Y$ direction is very small, and it converges faster than any polynomial orders of $\frac{1}{M}$.

To estimate (4.8), we again use Doob's maximal inequalities on the martingale process $X$. If $\rho < 0$, according to Proposition 4.21,

$$\begin{aligned} M\mathbb{P}[X_T^* \geq M] \leq & \frac{1}{M^{\eta^*}} \mathbb{E}\left[X_T^{\eta^*+1}\right] \\ \leq & \frac{1}{M^{\eta^*}} \mathbb{E}\left[\max_{0 \leq s \leq T} X_s^{\eta^*+1}\right] \\ \leq & \frac{C}{M^{\eta^*}}, \end{aligned}$$

where $\eta^* = \frac{\rho^2}{1-\rho^2}$.

In the situation that $p \in \left(1, \frac{3}{2}\right]$, Proposition 4.19 does not apply any more. Since the payoff does not depend on $Y$, we can extend the far field boundary from $M$ to $M^{1+\eta^*}$. Arguing analogously to Theorem 4.7,

$$M\mathbb{P}[Y_T^* \geq M^{1+\eta^*}] \leq \frac{1}{M^{\eta^*}}\mathbb{E}\left[Y_{T\wedge\xi_{M^{1+\eta^*}}}\right] \leq \frac{C}{M^{\eta^*}},$$

and thus we obtain the convergence rates. □

Notice that in case of $\rho = 0$, and $p \in (\frac{1}{2}, \frac{3}{2})$, the underlying process $X$ is also a martingale. By Theorem 4.7, we would have the Dirichlet problem to approximiate the pricing equation well. Meanwhile, we have seen that the tail distribution of $X$ is intimately related to the integrability of its moments. Yet, the following proposition implies that the convergence rate may be very slow.

**Proposition 4.23.** *For all $p \in (\frac{1}{2}, 1]$, $T > 0$, and $\eta > 1$,*

$$\mathbb{E}\left[X_T^\eta\right] = \infty,$$

*if $\rho = 0$.*

*Proof.* We provide a proof reconstructed from [1].

According to Lemma 4.15 and the fact that $Y_t$ under the new measure $\mathbb{P}^{(\eta)}$ (see equation (4.12) ) does not explode to infinity at any finite time , it suffices to prove that under the same condition,

$$\mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^T Y_t\,dt\right)\right] = \infty,$$

for any $\alpha > 0$.

In case of $p = 1$, the stochastic volatility process resembles a geometric Brownian motion. Denote by $\tilde{Y}$ the solution of the following SDE

$$d\tilde{Y}_t = -\kappa\tilde{Y}_t dt + \beta\tilde{Y}_t dB_t^{(\eta)}.$$

The comparison principle (Theorem 5.2.18 in [38]) implies $Y^y \geq \tilde{Y}^y$ almost surely, so that

$$\mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^T Y_t\,dt\right)\right] \geq \mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^T \tilde{Y}_t\,dt\right)\right] = \infty$$

where the second equality follows from the integration instability of exponential functions under the geometric Brownian motion model ( see [49]).

Now, let's assume $p \in (\frac{1}{2}, 1)$. Denote

$$q \triangleq 2(1-p) \in (0,1), \qquad \bar{Y}_t \triangleq \frac{Y_t^q}{q}.$$

We have

$$d\bar{Y}_t = \left(\kappa\theta q^{\frac{q-1}{q}}\bar{Y}_t^{\frac{q-1}{q}} + \rho\beta\lambda\eta q^{\frac{q+p-\frac{1}{2}}{q}}\bar{Y}_t^{\frac{p+q-\frac{1}{2}}{q}} - \kappa q\bar{Y}_t + \beta^2\frac{q-1}{2}\right)dt + \beta\sqrt{q}\sqrt{\bar{Y}_t}dB_t^{(\eta)}.$$

Notice that for all $y \geq 0$, the drift of $d\bar{Y}_t$ can be lower bounded by

$$\left(-\kappa q\bar{Y}_t + \beta^2\frac{q-1}{2}\right)dt.$$

Denoting $\kappa' \triangleq \kappa q$, and $\beta' \triangleq \beta\sqrt{q}$, there exists $\theta'$ such that the drift is further lower bounded by

$$\kappa'\left(\theta' - \bar{Y}_t\right)dt.$$

Once again by the comparison principle $Y^y$ is lower bounded by $\hat{Y}^y$, where

$$d\hat{Y}_t = \kappa'(\theta' - \hat{Y}_t)dt + \beta'\sqrt{\hat{Y}_t}dB_t^{(\eta)}.$$

This is again the CIR process, and we replicate the part of proof in Theorem 4.20 to obtain the fact that there exists $T_\alpha^*$, such that when $\bar{T} > T_\alpha^*$,

$$\mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^{\bar{T}} \bar{Y}_t\,dt\right)\right] \geq \mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^{\bar{T}} \hat{Y}_t\,dt\right)\right] = \infty.$$

Thus

$$\mathbb{E}^{(\eta)}\left[\exp\left(\frac{\alpha}{q}\int_0^{\bar{T}} Y_t^q\,dt\right)\right] = \infty.$$

To prove the proposition, we choose an arbitrary $T > 0$, $c > q$, and a large enough $\alpha_0$ such that

$$\mathbb{E}^{(\eta)}\left[\exp\left(\frac{\alpha_0}{T^{\frac{q}{c}-1}}\int_0^T Y_t^q\,dt\right)\right] = \infty.$$

An application of Hölder's inequality gives,

$$\int_0^T Y_t^q\,dt \le T^{1-\frac{q}{c}}\left(\int_0^T Y_t^c\,dt\right)^{\frac{q}{c}},$$

and thus,

$$\mathbb{E}^{(\eta)}\left[\exp\left(\alpha_0\left(\int_0^T Y_t^c\,dt\right)^{\frac{q}{c}}\right)\right] = \infty.$$

Pick an arbitrary $\alpha > 0$, and denote $Z \triangleq \int_0^T Y_t^c\,dt$, and $z \triangleq \left(\frac{\alpha}{\alpha_0}\right)^{\frac{c}{q-c}}$, to get

$$\begin{aligned}\mathbb{E}^{(\eta)}\left[\exp\left(\alpha Z\right)\right] =& \mathbb{E}^{(\eta)}\left[1_{\{z<Z\}}\exp\left(\alpha Z\right) + 1_{\{z\ge Z\}}\exp\left(\alpha Z\right)\right]\\ \ge& e^{\alpha z} + \mathbb{E}^{(\eta)}\left[1_{\{z<Z\}}\exp\left(\alpha Z\right)\right].\end{aligned}$$

Observing that

$$\{z < Z\} = \{\alpha Z > \alpha_0 Z^{\frac{q}{c}}\},$$

and

$$\mathbb{E}^{(\eta)}\left[1_{\{z<Z\}}\exp\left(\alpha_0 Z^{\frac{q}{c}}\right)\right] = \infty,$$

we conclude

$$\mathbb{E}^{(\eta)}\left[\exp\left(\alpha Z\right)\right] \ge e^{\alpha z} + \mathbb{E}^{(\eta)}\left[1_{\{z<Z\}}\exp\left(\alpha_0 Z^{\frac{q}{c}}\right)\right] = \infty.$$

Therefore, when $p \in (\frac{1}{2}, 1]$, we have

$$\mathbb{E}^{(\eta)}\left[\exp\left(\alpha\int_0^T Y_t\,dt\right)\right] = \infty,$$

which in turn implies the statement of this proposition. □

**Remark 4.24.** Our discussion on the convergence rates of Dirichlet problem as an approximation of value function reveals that we can expect fast convergence in the Heston model with short maturity, and under Assumption 4.9 with $p \in (\frac{1}{2}, 1)$ and $\rho < 0$. The power index $p$ in the stochastic volatility process is preferably small for better controlling the far field boundary error.

## 4.3 A Finite Difference Scheme for Stochastic Volatility Model

Once the far field boundary is set up such that the far field boundary error is convergent to zero as boundary moves towards infinity, we are able to solve the Dirichlet problem (3.13) numerically by using a finite difference scheme.

In the well-known Black-Scholes PDE case, a finite difference scheme is easy to set up and the calculation cost is small. Some basic finite difference schemes, such as the explicit method, implicit method, and Crank-Nicolson method ([17]), are all applicable to solving Black-Scholes PDE numerically. Stochastic volatility models, as a two dimensional extension to Black-Scholes PDE, are more difficult to solve efficiently with a numerical method. One complication is that adding a dimension increase the complexities of those basic schemes dramatically, and usually requires coarser mesh sizes. On the other hand, stochastic volatility models often do not preserve the fast convergence of far field boundary error, and hence require larger state space domains, which results in more mesh points. Another complication is that there are cross derivatives in two dimensional models, which are harder to tackle by traditional schemes.

In this section, we apply an ADI (Alternating Directions Implicit) scheme and a non-uniform mesh grid to numerically solve (3.13). Notice that classical ADI scheme does not consider cross derivatives in the PDE. The Do scheme introduced

by Douglas and Rachford ([21]) was an extension to address the cross derivatives. However the Do scheme is only a first order scheme. Instead, the Craig-Sneyd scheme ([16]) guarantees second order accuracy addresses cross derivative issues. There are also more advanced schemes that allow more degrees of freedoms in the parameters, such as the modified Craig-Sneyd scheme([34]), and the Hundsdorfer-Verwer scheme ([32], [56]), but the Craig-Sneyd scheme is good enough for our purposes.

To illustrate the work-flow of the numerical mechanism, we start by talking about the non-uniform mesh followed by approximation of derivative, and then cover the Craig-Sneyd iteration. Throughout this section, we use a European call option written on Heston model (2.32) as an example, and follow the idea of In't Hout and Foulon ([33]).

#### 4.3.1 Non-uniform grid

A non-uniform grid is usually preferred when there are significantly many points in each direction of the state space. Non-uniform grids locate more mesh points in intervals that are numerically sensitive, and fewer mesh points elsewhere. Often, a generating function is used to map the non-uniform grid to a uniform grid.

Recall that $\bar{D}_M = [0, M] \times [0, M]$. Let $\Pi : [0, M] \to [0, M]$, a generating function, be a smooth function such that $\Pi(0) = 0$, $\Pi(M) = M$. Suppose $\mathbf{x} = \{x_0 = 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = M\}$ is the uniform mesh satisfies $x_i - x_{i-1} = \frac{M}{n}$, then $\mathbf{x}^* = \Pi(\mathbf{x})$ is the non-uniform mesh associated with the generating function $\Pi$.

For a European call option price is very sensitive at $x = K$, since there is a discontinuity point at $x = K$ for the first derivative of the payoff function. And in the stochastic volatility direction, this price is sensitive when the volatility is approaching to zero, i.e. $Y_0 = 0$.

We employ the non-uniform mesh introduced by Tavella and Randall ([55]), and

Kluge ([39]).

For the underlying direction, define

$$\Delta x = \frac{1}{n}\left(\sinh^{-1}\left(\frac{M-K}{c}\right) - \sinh^{-1}\left(\frac{-K}{c}\right)\right), c > 0,$$

and

$$x_i = \sinh^{-1}\left(\frac{-K}{c}\right) + i \cdot \Delta x.$$

Then the non-uniform mesh $\mathbf{x}^*$ is given by

$$x_i^* = K + c\sinh\left(x_i\right).$$

The constant $c$ is the parameter to adjust the approximate ratio $\frac{\Delta x^*}{\Delta x}$. Namely, $\frac{\Delta x^*}{\Delta x} \approx c$, around the point $x = K$.

Similarly for the stochastic volatility direction, define

$$\Delta y = \frac{1}{n}\sinh^{-1}\left(\frac{M}{d}\right), d > 0,$$

and

$$y_j = j \cdot \Delta y.$$

Then the non-uniform mesh $\mathbf{y}^*$ is given by

$$y_i^* = d\sinh\left(y_i\right).$$

The constant $d$ controls the density of number of points around $y = 0$.

Figure 4.1 is an example of non-uniform grid we discussed.

For a different payoff function and model, it is usually optimal to choose a particular non-uniform grid adapted to the features of this payoff function and this model. The Tavella and Randall mesh grid is just a suitable non-uniform mesh for European call written on the Heston model. In the following context, we use notation $x_i$ and $y_i$ to denote the mesh once the mesh grid is chosen.

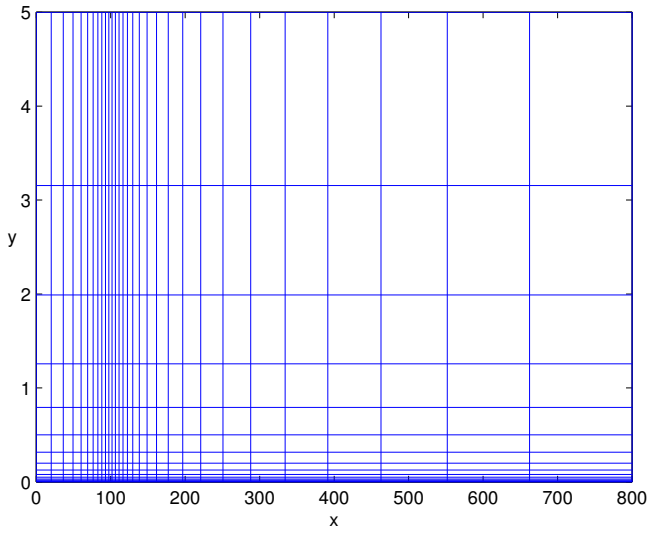


Figure 4.1: Non-uniform mesh grid

### 4.3.2 Approximating the derivatives

The pricing PDE for the European call option written on Heston model takes the following form:

$$\frac{\partial u}{\partial t} + \frac{1}{2}x^2 y \frac{\partial^2 u}{\partial x^2} + \rho\beta xy \frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2}\beta^2 y \frac{\partial^2 u}{\partial y^2} + \kappa\left(\theta - y\right)\frac{\partial u}{\partial y} = 0.$$

Theorem 4.20 implies that a Dirichlet type of far field boundary can be used for the Heston model in both $X$ and $Y$ directions, i.e.,

$$u(t, x = M, y) = g(M), t \in [0, T), y \in [0, M)$$

and

$$u(t, x, y = M) = g(x), t \in [0, T), x \in [0, M),$$

where $g$ is the payoff function

$$g(x) = (x - K)^+.$$

Even though the near field boundary condition on $y = 0$ might not be needed as it is discussed in Chapter 3, something have to be specified numerically. We use the

condition from [25]

$$u_t(t, x, y = 0) + \kappa\theta u_y(t, x, y = 0) = 0, t \in [0, T), x \in [0, M).$$

Also on the near field boundary $x = 0$, we use,

$$u(t, x = 0, y) = g(0).$$

To approximate the PDE and boundary conditions using finite difference methods, we approximate first and second order derivatives inside the domain $D_M$, and the first derivative on the boundary $y = 0$.

For any derivative inside $D_M$, we use the following central scheme,

$$u'(x_i) \approx a_i^- u(x_{i-1}) + a_i u(x_i) + a_i^+ u(x_{i+1}),$$

and

$$u''(x_i) \approx b_i^- u(x_{i-1}) + b_i u(x_i) + b_i^+ u(x_{i+1}),$$

where,

$$a_i^- = \frac{-\Delta x_{i+1}}{\Delta x_i \left(\Delta x_i + \Delta x_{i+1}\right)}, a_i = \frac{\Delta x_{i+1} - \Delta x_i}{\Delta x_i \Delta x_{i+1}}, a_i^+ = \frac{\Delta x_i}{\Delta x_{i+1} \left(\Delta x_i + \Delta x_{i+1}\right)},$$

$$b_i^- = \frac{2}{\Delta x_i \left(\Delta x_i + \Delta x_{i+1}\right)}, a_i = \frac{-2}{\Delta x_i \Delta x_{i+1}}, a_i^+ = \frac{2}{\Delta x_{i+1} \left(\Delta x_i + \Delta x_{i+1}\right)}.$$

An upwind scheme shown below is used to approximate the derivative in the left hand side boundary $y = 0$ to maintain the second order accuracy.

$$u'(x_i) \approx c_i u(x_i) + c_i^+ u(x_{i+1}) + b_i^{++} u(x_{i+2}),$$

with

$$c_i = \frac{-2\Delta x_{i+1} - \Delta x_{i+2}}{\Delta x_{i+1} \left(\Delta x_{i+1} + \Delta x_{i+1}\right)}, c_i^+ = \frac{\Delta x_{i+1} + \Delta x_{i+2}}{\Delta x_{i+1} \Delta x_{i+2}}, c_i^{++} = \frac{-\Delta x_{i+1}}{\Delta x_{i+2} \left(\Delta x_{i+1} + \Delta x_{i+2}\right)}.$$

Applying this discretization scheme to the Heston PDE and its boundary conditions, the problem reduces to a system of first order ordinary differential equations

$$U'(t) + AU(t) + b(t) = 0, U(T) = U_T. \tag{4.16}$$

Here, U is an $n^2$ dimensional vector representing the values of the mesh grid points at some fixed time $t$. A is a time homogeneous matrix of size $n^2 \times n^2$, resulting from the discretization. $U_T$ is the terminal condition for the mesh grid points.

In such a discretization, another type of error - trucation error is incurred aside from the far field boundary error. A strightforward taylor expansion calculation on the approximation of $u', u''$ gives the truncation error in order of

$$O\left(\Delta x^2\right) + O\left(\Delta y^2\right).$$

#### 4.3.3 The iteration scheme

As we discussed earlier that there are a number of different time discretization schemes for the equation (4.16).

One simple and straightforward scheme is the explicit method; it calculates the approximation of $U^{(n-1)}$ from the upper layer $U^{(n)}$ by

$$\frac{U^{(n)} - U^{(n-1)}}{\Delta t} + AU^{(n)} + b\left(t^{(n)}\right) = 0.$$

This scheme is not unconditionally stable, and one needs to pick up the time step $\Delta t$ very carefully. For the explicit method, the time step is normally very small, and significantly increases the computational effort.

The implicit method is plausible for its unconditional stability. It requires solving $U^{(n-1)}$ implicitly from

$$\frac{U^{(n)} - U^{(n-1)}}{\Delta t} + AU^{(n-1)} + b\left(t^{(n-1)}\right) = 0.$$

Aside from its drawback of first order accuracy in time steps, this method requires inverting a huge matrix ($n^2 \times n^2$). The complexity on each time step is at least $O\left(n^{4.6}\right)$, with the known best matrix inversion algorithm ([13]).

The famous Crank-Nicolson method is able to remedy the truncation error order and make it $O\left(\Delta t^2\right)$, by using central difference scheme on time discretization, i.e.,

$$\frac{U^{(n)} - U^{(n-1)}}{\Delta t} + A\frac{U^{(n-1)} + U^{(n)}}{2} + \frac{b\left(t^{(n-1)}\right) + b\left(t^{(n)}\right)}{2} = 0.$$

However, it can not escape the necessity of inverting a large matrix. And, it is not practical when $n$ is large.

Instead of using those standard scheme, we split the matrix $A$, and apply ADI scheme on equation (4.16).

The matrix $A$ can be decomposed into the sum of three matrices,

$$A = A_0 + A_1 + A_2,$$

such that $A_0$ contains all entries in $A$ resulting from discretization of cross derivatives in the Heston PDE, $A_1$ includes those from discretization of spatial derivatives in the $X$ direction and $A_2$ takes the rest entries, which correspond to the $Y$ directions.

Analogously, we can decompose $b(t)$ as $b(t) = b_0(t) + b_1(t) + b_2(t)$, and define operators

$$F_i(t, \omega) \triangleq A_i\omega + b_i(t), i = 0, 1, 2, t \in [0, T], \omega \in \mathbb{R}^{n^2}.$$

And hence

$$F(t, \omega) \triangleq F_0(t, \omega) + F_1(t, \omega) + F_2(t, \omega) = A\omega + b(t).$$

By denote $U^{(n)}$ the known layer, and $U^{(n-1)}$ the layer to be calculated, the Craig-Sneyd scheme to solve (4.16) consists the following five steps.

*Step one:*

Explicitly derive $V_0$ from

$$\frac{U^{(n)} - V_0}{\Delta t} + F\left(t^{(n)}, U^{(n)}\right) = 0.$$

The auxiliary layer $V_0$ might not be a stable approximation of the real value of $u(t^{(n-1)}, \cdot)$, but there are more correction steps to stabilize it.

*Step two:*

Implicitly solve $V_1, V_2$ from

$$\begin{cases} \dfrac{V_1 - V_0}{\Delta t} = \dfrac{1}{2}\left(F_1\left(t^{(n-1)}, V_1\right) - F_1\left(t^{(n)}, U^{(n)}\right)\right), \\ \dfrac{V_2 - V_1}{\Delta t} = \dfrac{1}{2}\left(F_2\left(t^{(n-1)}, V_2\right) - F_2\left(t^{(n)}, U^{(n)}\right)\right). \end{cases}$$

This step is a correction step on $X, Y$ directions separately. In fact, if there were no cross derivative in the Heston PDE, $V_2$ is already a finished ADI approximation of $u(t^{n-1}, \cdot)$. It worth pointing out that the $V_2$ is already stable.

*Step three:*

This is a explicit step for correction on the cross derivatives, and it obtains $\tilde{V}_0$ from

$$\frac{\tilde{V}_0 - V_0}{\Delta t} = \frac{1}{2}\left(F_0\left(t^{(n-1)}, V_2\right) - F_0\left(t^{(n)}, U^{(n)}\right)\right).$$

Notice that this step could breach the stability again, and further refinements are required.

*Step four:*

Similar to step two, we need to solve $\tilde{V}_1, \tilde{V}_2$ from

$$\begin{cases} \dfrac{\tilde{V}_1 - \tilde{V}_0}{\Delta t} = \dfrac{1}{2}\left(F_1\left(t^{(n-1)}, \tilde{V}_1\right) - F_1\left(t^{(n)}, U^{(n)}\right)\right), \\ \dfrac{\tilde{V}_2 - \tilde{V}_1}{\Delta t} = \dfrac{1}{2}\left(F_2\left(t^{(n-1)}, \tilde{V}_2\right) - F_2\left(t^{(n)}, U^{(n)}\right)\right). \end{cases}$$

*Step five:*

The $\tilde{V}_2$ from step four is the value for the new mesh layer, i.e.

$$U^{(n-1)} = \tilde{V}_2.$$

### 4.3.4 Complexity and stability

Although the matrix $A$ is size of $n^2 \times n^2$, it is sparse matrix. In the Craig-Sneyd scheme, it seems like the explicit steps (step one and three) requires a multiplication of $A$ and $U$. Since there are no more than 9 non-zero entries, it only takes $O(n^2)$ operations for the explicit steps. During the implicit steps (step two and step four), the scheme needs to invert matrices $A_1$ and $A_2$. However, one of the matrices is tridiagonal, and the other one is pentadiagonal. In the case of the pentadiagonal matrix, a permutation of the order of the elements in the vector $U$ can rearrange the matrix to a tridiagonal once again. Hence, in fact, for those steps, only solving tridiagonal systems is necessary, which takes $O(n^2)$ operations. Therefore, the whole algorithm is of complexity of $O(n^2)$, and it takes linear time in the size of mesh grid points to accomplish each 5-steps iteration between two time steps.

Theoretical stability proofs can be found in [34], [16], [44], [45]. Those references indicate the Craig-Sneyd scheme is unconditionally stable for any $\Delta t$, and the accuracy is in order of $\Delta t^2$. Combining the analysis of spatial discretization, the total trucation error of the scheme is of order $O(\Delta t^2 + \Delta x^2 + \Delta y^2)$.

## 4.4 Numerical Experiments for Dirichlet Boundary Type

In this section, we continue to take Heston model as an example to conduct numerical experiments. We compute call option prices written on the model to assess convergence and its rates of the Dirichlet problem (3.13) to the value function (3.9). In other words, we emphasize the far field boundary error versus to the truncation error.

Specifically, by assuming the all interest rates and market price risk of volatility are zero and $M > K$, the Dirichlet problem for call option strike at $K$ written on

Heston model is

$$\begin{cases} v_t^M + \frac{1}{2}x^2 y v_{xx}^M + \rho\beta x y v_{xy}^M + \frac{1}{2}\beta^2 y v_{yy}^M + \kappa(\theta - y)v_y^M = 0, & (t,x,y) \in (0,T)\times D_M, \\ v^M(T,x,y) = (x-K)^+, & (x,y) \in \bar{D}_M, \\ v^M(t,0,y) = 0, & (t,y) \in [0,T)\times(0,M), \\ v_t^M(t,x,0) = -\kappa\theta v_y^M(t,x,0), & (t,x) \in [0,T)\times(0,M], \\ v^M(t,M,y) = M-K, & (t,y) \in [0,T)\times(0,M], \\ v^M(t,x,M) = (x-K)^+, & (t,x) \in [0,T)\times(0,M]. \end{cases}$$

It is proved that

$$v^M(0,x,y) = \mathbb{E}^{x,y}\left[(X_T - K)^+ 1_{\{\tau > T\}} + (M-K)1_{\{\tau \le T\}}\right].$$

We perform the numerical experiments with fixed the horizon $T = 0.5$, the strike for the European call $K = 5$. The uniform mesh grid is adopted for both $X$ and $Y$ directions with mesh size $\Delta x = \Delta y = 0.01$. The time step size is also chosen as $\Delta t = 0.01$.

We employ the Craig-Snyde ADI from previous section to carry out the numerical computation. The truncation error is in order of $O(\Delta t^2 + \Delta x^2 + \Delta y^2)$, which, in our case, has precision up to $10^{-4}$.

Figure 4.2 shows the numerical solution surface $v^M(0,x,y)$ with the parameters values

$$M = 10, \kappa = 2, \theta = 0.2, \beta = 0.8, \lambda = 1, \text{ and } \rho = 0.1. \tag{4.17}$$

The graph shows that the parabolic operator smoothes the payoff function inside the domain $D_M$. It is seen that the stochastic volatility nature gives obvious dependency of the option price on the initial stochastic volatility level $y$.

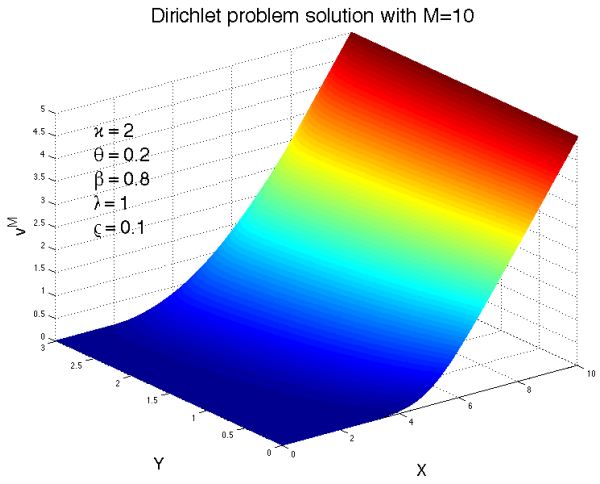


Figure 4.2: Call option price surface written on Heston model

Observe that the growth rate order of the volatility in the stochastic volatility process is less than linear. The far field boundary error in the $Y$-direction is more tamed than the $X$-direction, as it is implied from Proposition 4.19. We conduct a far field boundary error test separately for $X$-direction and $Y$-direction. We denote the far field boundaries by $M^x, M^y$ respectively for $X$ and $Y$ directions.

### 4.4.1 Far field boundary error in $Y$-direction

Let us fix the far field boundary $M^x = 7.5$, and we continue to use the same parameters as in (4.17). The Feller's condition holds for this set of parameters, and thus the boundary $y = 0$ never hits by the process $Y$. In the experiments, the PDE holds on the boundary, and this generate no near field boundary error. We compute the call option prices on the points $(t, x, y) = (0, 5, 0.5), (0, 5, 1), (0, 5, 1.5), (0, 5, 2)$ for far field boundary $M^y = 2.5, 3, 4.5, 6$, and arrive at the results shown in Table 4.4.1.

Table 4.4.1 implies that the numerical prices stop changing after the far field boundary $M^y$ in the $Y$-direction is some distance away from the point that the value is computed. Although that the far field boundary in the $X$-direction is not good

| $M^y$ | $v^{M^y}(0,5,0.5)$ | $v^{M^y}(0,5,1)$ | $v^{M^y}(0,5,1.5)$ | $v^{M^y}(0,5,2)$ |
|---|---|---|---|---|
| 2.5 | 0.81974 | 1.03213 | 1.15733 | 1.21567 |
| 3 | 0.81974 | 1.03215 | 1.15805 | 1.24134 |
| 4.5 | 0.81974 | 1.03215 | 1.15807 | 1.24214 |
| 6 | 0.81974 | 1.03215 | 1.15807 | 1.24214 |

enough, the data in the table show that it does not improve the precision by setting the far field boundary further far away.

Empirically, the far field boundary in the $Y$-direction should be set two to three times the initial value for the stochastic volatility $y$. Proposition 4.19 suggests the far field boundary error decreases faster than any polynomial, which is a similar convergence speed as the situation in Black-Scholes model.

### 4.4.2 Far field boundary error in $X$-direction

We evaluate the convergence in the $X$-direction around the mesh point $(t,x,y) = (0,5,1)$, by fixing the initial volatility $y = 1$. In view of last subsection, it is safe enough to put the far field boundary $M^y = 3$ through out the tests for assessments of the far field boundary error in the $X$-direction.

Upon working out the computation of European call option prices on the points $(t,x,y) = (0,5,1),(0,4,1),(0,6,1)$ with $M^x = 7.5, 10, 12.5, 15, 17.5, 20$, we generate the data in Table 4.1.

Table 4.1: Call Option Prices with fixed $M^y$ and variate $M^x$

| $M^x$ | $v^{M^x}(0,4,1)$ | $v^{M^x}(0,5,1)$ | $v^{M^x}(0,6,1)$ |
|---|---|---|---|
| 7.5 | 0.570238 | 1.03215 | 1.59159 |
| 10 | 0.615636 | 1.1397 | 1.79847 |
| 12.5 | 0.620006 | 1.15165 | 1.82465 |
| 15 | 0.620563 | 1.15331 | 1.82858 |
| 17.5 | 0.620653 | 1.1536 | 1.8293 |
| 20 | 0.620671 | 1.15366 | 1.82945 |

The numbers in Talbe 4.1 suggests that the far field boundary error is decreasing much faster than linear with $M^x$ increasing linearly. According to Theorem 4.20, the

far field boundary error is in order of at least $O\left(\frac{1}{M^{\eta^*}}\right)$, where

$$\eta^* = \max\left\{\eta > 1 \middle| \rho\eta + \sqrt{\eta\left(\eta - 1\right)} \leq \frac{\kappa}{\beta\lambda}\right\}.$$

In our example, $\eta^* \approx 3$.

Figure 4.3 plots the logarithm of far field boundary error against the inverted far field boundary distance. It implies that the error is decreasing in order of $\frac{1}{M^{60}}$, which is faster than implies by the theorem.

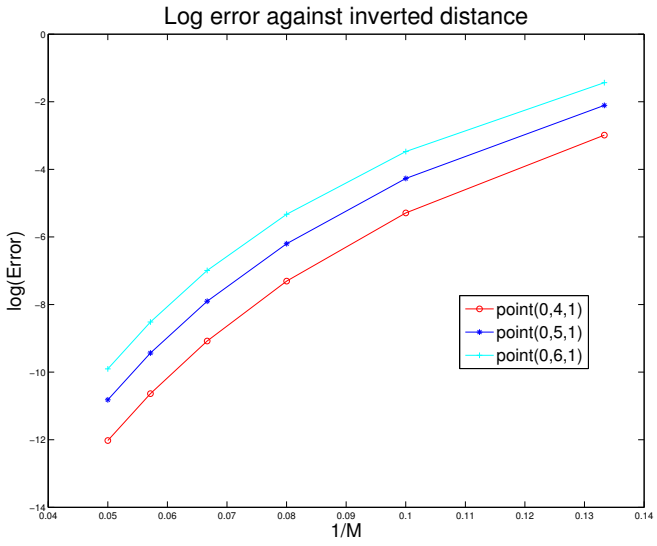


Figure 4.3: Far field boundary error plot in Heston model

**Remark 4.25.** Notice that the numerical results agree with the theorems we developed. Our theorem gives an upper bound for the error. It is not surprising to see the error rates difference between theoretic upper bound and the experiments, since the theorem holds for any $T > 0$ and the choice of $T$ in the experiment is small.

# CHAPTER 5

## Convergence of the Neumann Problem - Local Martingale

This chapter conducts a discussion on solving for the value function (3.9) by using numerical methods, when the underlying process $X$ in (2.28) or in (3.1) is a strict local martingale.

There is extensive literature on when a local martingale process serves as underlying process for some contingent claims. Assets governed by local martingales often make the arbitrage opportunities possible in the markets. Such assets are known as bubble-assets. Delbaen and Schachermayer (in [19]) explored the arbitrage opportunities in the Bessel process. Some optimal strategies relative to market order were discussed by Fernholz and Karatzas ([27]). Ruf ([47]) discusses about hedge strategies under arbitrage. American options with local martingale underlyings are considered in ([5], [46]). Yet, very limited literatures addresses the numerical pricing techniques for local martingale assets, and it is still an open question to find a general reliable method.

According to Theorem 4.5 and Theorem 4.7 in the previous chapter, the solution to Dirichlet problem, e.g. (3.13), is not an appropriate approximation for (3.9) in local martingale case. At the same time, the Bessel process discussed in Section 2.2.2 provides an example of divergence. Besides the convergence issue of classical finite

difference methods, classical Monte Carlo simulation does not work well either for local martingale asset (see [52]). This is not surprising, because classic Monte Carlo simulation is approximating stochastic integrals with random walks, and symmetric random walks with equal probability are proper martingales while stochastic integrals can be strictly local martingales.

For the one-dimensional case, Song suggests (in [52]) a method to approximate the linear payoff function with sublinear growth function, or even bounded function. However, the convergence rates are demonstrated to be slow in [52] for both Monte Carlo method and finite difference method. Ekström, Tysk, etc. use a Neumann type far field boundary in their finite difference method. Empirically, the convergence rate for Neumann type boundary is quadratic in terms of the inverse distance of the far field boundary.

We recall some existing theories for pricing contingent claims written on local martingales, and present our conjectures and intuitions for this problem.

## 5.1 One Dimensional Local Volatility Model

In this section, we consider that the underlying process follows the one-dimensional local volatility model (2.30)

$$dX_t = \alpha(t, X_t)dW_t,$$

where $\alpha$ is continuous and locally Hölder-$\frac{1}{2}$ in the $x$-variable and non-zero for all $x > 0, t > 0$. We further assume that $\alpha(t, 0) = 0$.

Recall that Proposition 2.11 gives a necessary condition for the underlying process to be a strict local martingale. Also, Proposition 2.29 gives an equivalent condition for the time homogeneous process to be a strict local martingale.

Let $g$ be a payoff function satisfying Assumption 3.5, which indicates that $g$ can

be a linear growth function. By assuming the interest rate is flat at zero, the time $t$ option price of the contract that pays $g(X_T)$ at maturity $T$ is the value function (2.1),

$$u(t,x) = \mathbb{E}\left[g\left(X_T^{t,x}\right)\right].$$

The earlier Bessel process example we discussed indicates that a Dirichlet type far field boundary is not appropriate for approximating $u$, because of the possible linear growth condition on $g$.

There are generally two different ways around this difficulty. One method is to use a sequence of bounded or less than linear growth functions as payoff functions, whose option values converge to $u$, to replace $g$. The other method is to choose a different kind of far field boundary condition.

### 5.1.1 The method of payoff function modification

This method is introduced by Song ([52]). Denote by $x = M$ the far field boundary and $\tau = \inf\{t \geq 0; X_t > M\}$, and define a revised payoff function $g^M$ by

$$g^M(x) = g(x)\,1_{\left\{x \leq \frac{M}{2}\right\}} + \frac{2(M-x)\,g(x)}{M} 1_{\left\{\frac{M}{2} < x \leq M\right\}}. \tag{5.1}$$

Assume $g \in C_\gamma(\mathbb{R}+)$ for some $\gamma \in [0,1]$, and notice that $g^M$ is a bounded continuous function, where $\gamma$ denotes the growth rates.

Let

$$v^M(t,x) = \mathbb{E}\left[g^M\left(X_T^{t,x}\right) 1_{\{\tau > T\}}\right].$$

Our regularity results in Chapter 3 still apply, and $v^M$ solves the following PDE

$$\begin{cases} v_t^M + \frac{1}{2}\alpha^2(t,x)\, v_{xx}^M = 0, & (t,x) \in [0,T) \times (0,M), \\ v^M(t,0) = g^M(0), & t \in [0,T), \\ v^M(t,M) = 0, & t \in [0,T), \\ v^M(T,x) = g^M(x), & x \in (0,M). \end{cases}$$

The choice of far field boundary condition makes sure that it agrees with the terminal condition on the corner, which avoids the error resulting from the discontinuity of boundary and terminal functions. This Dirichlet type parabolic PDE can be solved by using classical finite difference methods, and the only thing we need to worry about is the error incurred by modifying the terminal payoff.

Song ([52]) has shown the following theorem regarding the convergence of $v^M$ to $u$ and its rates.

**Theorem 5.1** (Song 2011)**.** *If $\alpha$ is time homogeneous, $v^M(t,x) \in C^{1,2}((0,T) \times (0,M)) \cap C([0,T] \times [0,M])$ and*

$$\lim_{M \to \infty} v^M(t,x) = v(t,x), (t,x) \in (0,T) \times (0,\infty),$$

*moreover,*

$$|v(t,x) - v^M(t,x)| \leq K \left(\frac{1}{M}\right)^{1-\gamma},$$

*where $K$ is a positive constant independent of $M$.*

In the case of the one-dimensional model, this theorem shows both convergence and its rate for underlying process governed by local martingale assets. Song's results matches our previous remark on stochastic volatility model. However, this theorem still does not provide any meaningful information regarding linear contract in terms of convergence rates. Hence, we believe modification of terminal payoff function is

not effective or robust enough to evaluate linear growth contracts written on local martingale processes.

### 5.1.2 The method of imposing Neumann boundary conditions

Let us assume the one-dimensional local volatility model (2.30), and $\alpha$ satisfies the regularity conditions we discussed in the beginning of this chapter.

We know that $u(t,x)$ is the smallest non-negative solution of the initial value problem (see [24]).

$$\begin{cases} u_t + \frac{1}{2}\alpha^2(t,x)u_{xx} = 0, & x \in (0,\infty), t \in [0,T), \\ u(t,0) = g(0), & t \le T, \\ u(T,x) = g(x), & x \in (0,\infty). \end{cases} \tag{5.2}$$

Define $v^M(t,x)$ the solution to the Neumann problem

$$\begin{cases} v^M_t + \frac{1}{2}\alpha^2(t,x)v^M_{xx} = 0, & x \in (0,M], t \in [0,T), \\ v^M(t,0) = g(0), & t \le T, \\ v^M(T,x) = g(x), & x \in (0,M], \\ v^M_x(t,M) = 0, & t \in [0,T). \end{cases} \tag{5.3}$$

By setting $v^M(t,x) = v^M(t,M)$ for $x \ge M$, the definition of $v^M$ extends to $[0,\infty) \times [0,T]$. For a fixed $M$, the Neumann problem (5.3) has a unique solution by the maximum principle.

From the assumptions imposed on $\alpha$, it can be seen that the underlying process $X$ can be either a strict local martingale or a proper martingale. Regardless of the martingale property, the value function $u(t,x)$ can be approximated by a sequence of solutions to the Neumann problems, as it is proved by the following theorem.

**Theorem 5.2** (Ekoström, Lötstedt, Sydow and Tysk [22]). *Let the payoff function $g(\cdot)$ satisfy the Assumption 3.5, and assume that $g(\cdot)$ can be written as a linear combination of non-decreasing functions. Then the sequence of solutions $\left\{v^M(t,x)\right\}_{\{M\in\mathbb{N}\}}$ to the Neumann problems (5.3) is convergent to the value function $u(t,x)$,*

$$\lim_{M\to\infty} v^M(t,x) = u(t,x)$$

*Proof.* It suffices to prove this result for a payoff function $g$ satisfying Assumption 3.5 and being non-decreasing.

Let $\{M_i\}_{\{i\in\mathbb{N}\}}$ be a increasing real sequence, and assume $v^{M_i}(t,x)$ solves

$$\begin{cases} v_t^{M_i} + \dfrac{1}{2}\alpha^2(t,x)v_{xx}^{M_i} = 0, & x\in(0,M_i], t\in[0,T), \\ v^{M_i}(t,0) = g(0), & t\le T, \\ v^{M_i}(T,x) = g(x), & x\in(0,M_i], \\ v_x^{M_i}(t,M_i) = 0, & t\in[0,T). \end{cases}$$

Because $g(\cdot)$ is non-decreasing, $v^{M_i}(t,x)$ is non-decreasing in $x$ for fixed $t$, and $i$. Thus, for any $i,j\in\mathbb{N}$, s.t. $i<j$, $v_x^{M_j}(t,M_i)\ge 0$. We claim that $v^{M_i}(t,x) < v^{M_j}(t,x), (t,x)\in[0,T]\times[0,\infty)$. In fact, $v^{M_j}(t,x)$ satisfies

$$\begin{cases} v_t^{M_j} + \dfrac{1}{2}\alpha^2(t,x)v_{xx}^{M_j} = 0, & x\in(0,M_i], t\in[0,T), \\ v^{M_j}(t,0) = g(0), & t\le T, \\ v^{M_j}(T,x) = g(x), & x\in(0,M_i], \\ v_x^{M_j}(t,M_i) \ge 0, & t\in[0,T). \end{cases}$$

By the maximum principle, $v^{M_i}(t,x)\le v^{M_j}(t,x)$. Monotonicity implies that the limit

$$v(t,x) \triangleq \lim_{i\to\infty} v^{M_i}(t,x)$$

exists.

Notice that it has been shown ([27], [48]) that it is never optimal to short the underlying in the replication portfolio of the option paying $g(X_T)$. Therefore, the delta of the option $u_x(t,x) \geq 0$, for all $(t,x) \in [0,T] \times [0,\infty)$. And hence, $v(t,x) \leq u(t,x)$.

On the other hand, $v(t,x)$ solves the PDE in (5.2) by an application of interior Schauder estimates ([40], [10]). Because of $v^{M_i}(t,x) \leq v(t,x) \leq u(t,x)$ for all $i$, and the continuity of $v^{M_i}$, $u$ at the boundary, $v$ has to be continuous at the boundary. Consequently, $v(T,x) = g(x)$ and $v(t,0) = g(0)$. Therefore, $v(t,x)$ is also a solution to the initial value problem (5.2), and $u(t,x) \leq v(t,x)$, since $u(t,x)$ is the smallest nonnegative solution to (5.2).

In conclusion, $v(t,x) = u(t,x)$, which proves the theorem. □

Theorem 5.2 only gives convergence without showing the speed of the convergence, and this theorem essentially requires the payoff function to be nondecreasing. Empirically, the Neumann problem (5.3) approaches the value function (2.1) in order of $\frac{1}{M^2}$ without the nonnegativity restriction on $g$, when the underlying process is a strictly local martingale. In other words, the far field boundary error for Neumann problem is roughly $O\left(\frac{1}{M^2}\right)$. We make the following conjecture, although there are technical difficulties for the proof.

**Conjecture 5.3.** *Let the payoff function $g(\cdot)$ satisfy the Assumption 3.5, and $X$ is a strictly local martingale. For fixed $(t,x)$, the sequence of solutions $\left\{v^M(t,x)\right\}_{\{M\in\mathbb{N}\}}$ to the Neumann problems (5.3) is convergent to the value function $u(t,x)$ in the sense that*

$$|v^M(t,x) - u(t,x)| \leq \frac{K}{M^2},$$

*where $K$ is a positive constant independent of $M$.*

We will show a numerical example in Section 5.3.

## 5.2 Stochastic Volatility Models

Stochastic volatility models are generally more complicated than one dimensional models, and hence it is more challenging to develop theoretical evidence for far field error when the underlying process is a strict local martingale. It is an open problem to develop a robust approximation method. We focus on some intuition and conjectures for the method of imposing Neumann type boundary condition.

In this section, we continue to assume the underlying process and the stochastic volatility process satisfy Assumption 3.1

$$\begin{aligned} dX_t &= b(Y_t)X_t dW_t, \\ dY_t &= \mu(Y_t)dt + \sigma(Y_t)dB_t, \\ dW_t dB_t &= \rho dt, \end{aligned}$$

where the parameter functions satisfy Assumption 3.3.

The goal is to solve the pricing equation (3.9)

$$u(t,x,y) = \mathbb{E}[g(X_T^{t,x,y})],$$

with $g$ being of at most linear growth. Once again, we wish to obtain a numerical approximation for $u$ in the case that the underlying process $X$ is a strictly local martingale.

In fact, we can arrive at a similar convergence rate result to Theorem 5.1 as a corollary of Theorem 4.5, without the aid of modifying the payoff function.

**Corollary 5.4.** *Let $v^M$ be as defined in (3.10) and let $g$ satisfy Assumption 3.5. Then the far field boundary error of $v^M$ with respect to $u$ is in order of $O\left(M^{\gamma-1}\right)$,*

*i.e.*

$$|v^M(t,x,y) - u(t,x,y)| \leq K\frac{1}{M^{1-\gamma}},$$

*where $K$ is a constant independent of $M$, and $\gamma \in [0,1]$ is the growth rate for function $g$.*

**Remark 5.5.** This corollary implies that we can not expect a fast convergence rate by modifying the payoff function as in the one-dimensional model case. Also, it fails to show convergence when the payoff function is of strictly linear growth. However, this does not mean that the modification of payoff function method is not practical at all. Actually, this method can make the Monte Carlo simulation pricing work for strict local martingale models, as it is pointed out by Song ([52]). Admittedly, the convergence rate of this method for Monte Carlo pricing is also far from satisfactory.

When the payoff function is of linear growth and the underlying process $X$ is a strict local martingale, here is a heuristic argument suggest the conjecture that Neumann problems can give good approximations to the value function $u$.

Notice that $u(t, X_t, Y_t)$ is a martingale with

$$\mathbb{E}\left[u(t,X_t,Y_t)|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[g(X_T)|\mathcal{F}_t\right]|\mathcal{F}_s\right] = u(s,X_s,Y_s), 0 \leq s \leq t$$

by the tower property.

It has been shown that $u$ is smooth enough to apply Itô's lemma, which gives

$$\begin{aligned} du(t,X_t,Y_t) = &\left(u_t\left(t,x,y\right) + \mathcal{L}u\left(t,x,y\right)\right)\Big|_{\{x=X_t,y=Y_t\}} dt \\ &+ u_x(t,X_t,Y_t)b(Y_t)X_t dW_t + u_y(t,X_t,Y_t)\sigma(Y_t)dB_t, \end{aligned}$$

where $\mathcal{L}$ is given by (3.11), i.e.,

$$\mathcal{L} = \frac{1}{2}b^2(y)x^2\partial^2_{xx} + \frac{1}{2}\sigma^2(y)\partial^2_{yy} + \mu(y)\partial_y + \rho b(y)\sigma(y)x\partial^2_{xy}.$$

Define

$$I_t \triangleq \int_0^t u_x(s, X_s, Y_s) b(Y_s) X_s \, dW_s,$$

$$J_t \triangleq \int_0^t u_y(s, X_s, Y_s) \sigma(Y_s) dB_s.$$

The martingale property requires that

$$u_t(t, x, y) + \mathcal{L}u(t, x, y) = 0,$$

and $I_t, J_t$ are true martingales so that

$$\mathbb{E}[I_t] = \mathbb{E}[J_t] = 0. \tag{5.4}$$

Recall that Theorem 3.42 implies that uniqueness does not hold for the initial value problem (3.12), and the value function $u(t, x, y)$ is one of the many solutions, when the underlying process $X$ is strictly local martingale and $g$ is of linear growth. This is true because not every solution $\bar{u}(t, x, y)$ satisfies equation (5.4). In fact, the option price function should be one of the solutions such that (5.4) holds.

As an example, let $g(x) = x$, i.e., the option pays the underlying process itself. Assume that $X$ is a strict local martingale (See Proposition 3.39). By direct verification, $\bar{u}(t, x, y) = x$ is a solution to the initial value problem (3.12), and $u(t, x, y) < \bar{u}(t, x, y) = x, \forall\ 0 \leq t < T$.

For a proof by contradiction, assume $\bar{u}(t, x, y)$ is the value of the expectation equation (3.9). Because $\bar{u}_t + \mathcal{L}\bar{u} = 0$,

$$d\bar{u}(t, X_t, Y_t) = \bar{u}_x(t, X_t, Y_t) b(Y_t) X_t dW_t + \bar{u}_y(t, X_t, Y_t) \sigma(Y_t) dB_t.$$

Plugging in $\bar{u}(t, x, y) = x$, it follows that

$$dX_t = b(Y_t) X_t dW_t,$$

which is a strict local martingale, and hence

$$\mathbb{E}[I_t] \neq 0.$$

This contradicts the assumption that $\bar{u}(t,x,y) = \mathbb{E}^{0,x,y}[X_T]$.

In finance, $u_x$ is called the Delta of an option, which represents the number of shares of the underlying asset to hold in order to replicate the option. The empirical idea behind the example is that the volatility of the local martingale asset is in high order of the asset price. Thus, this asset has large tendency for its price goes down from high level, and it remains in low level for some time. Based on its bubble essence, the replicating portfolio suggests a low Delta value if the maturity time is not soon to protect from a huge price slump. In the previous example, having a constant Delta $\bar{u}_x = 1$ is clearly not a good choice, and that's why $\bar{u}$ is not even a close approximation of the option price.

From a PDE point of view, imposing a restriction on $u_x$ is similar to adding a Neumann type condition on the initial value problem. Thus, a natural approach is to use the Neumann problem, instead of the Dirichlet problem, to approximate the initial value problem. The key point here is that to choose $u_x$ such that $I_t$ is a martingale. While there might be many choices, the easiest one is to set $u_x = 0$.

Let $\omega^M(t,x,y)$ be the solution to the following Neumann problem

$$\begin{cases} \omega^M_t + \mathcal{L}\omega^M = 0, & (t,x,y) \in (0,T) \times D_M, \\ \omega^M(T,x,y) = g(x), & (x,y) \in \bar{D}_M, \\ \omega^M(t,0,y) = g(0), & (t,y) \in [0,T) \times (0,M), \\ \omega^M_t(t,x,0) = -\omega^M_y(t,x,0), & (t,x) \in [0,T) \times (0,M], \\ \omega^M_x(t,M,y) = 0, & (t,y) \in [0,T) \times (0,M], \\ \omega^M(t,x,M) = g(x), & (t,x) \in [0,T) \times (0,M]. \end{cases} \tag{5.5}$$

We make a similar conjecture to Conjecture 5.3 as follows.

**Conjecture 5.6.** *Let the payoff function $g(\cdot)$ satisfy the Assumption 3.5, and $X$ is a strictly local martingale. The sequence of solutions $\left\{\omega^M(t,x,y)\right\}_{\{M\in\mathbb{N}\}}$ to the Neumann problems (5.5) is convergent to the value function $u(t,x,y)$ in the sense that*

$$|\omega^M(t,x,y) - u(t,x,y)| \leq K\frac{1}{M^2},$$

*where $K$ is a positive constant independent of $M$.*

**Remark 5.7.** In Neumann problem (5.5), we still use the Dirichlet boundary problem in the $Y$ direction. However, same treatment for the far field boundary condition can be done to the $Y$ direction, if the volatility of the process $Y$ grows faster than linear.

**Remark 5.8.** There are technical difficulties proving the statement in Conjecture 5.6. In fact, even though we can mimic the argument in Theorem 5.2, there are at least two new issues here: one is that it is unknown ([27]) under what conditions the replication portfolio never requires a negative delta $u_x$; the other one is that we do not know the properties of $u_y$, and hence the $Y$-direction is hard to tackle as well. However, if we try the method used in Chapter 4, there is no straightforward intuition how to formulate the Neumann boundary condition into the payoff expectation in probability language.

The theories of robust numerical techniques solving a linear option price written on a local martingale assets remains an open topic for future research. In the following section, we discuss Neumann problem experiment as an approximation of initial value problem driven by local martingale process.

## 5.3 Numerical Experiments for Neumann Boundary Type

In this section, we illustrate Theorem 5.2 and Conjecture 5.3 by computing the option price paying the stock itself, which follows the Bessel process

$$dX_t = X_t^2 dW_t.$$

According to equations (2.24) and (2.25), the option has the pricing formula

$$u(t, X_t) = \mathbb{E}\left[X_T | \mathcal{F}_t\right],$$

and it has a closed form solution as in equation (2.25):

$$u(t,x) = 2x\Phi\left(\frac{1}{x\sqrt{T-t}}\right) - x,$$

where $\Phi$ is the standard normal cumulative density function. $u$ is a martingale and hence admits the following dynamics

$$du(t, X_t) = u_x X_t^2 dW_t.$$

Further more, $u$ solves the initial value problem

$$\begin{cases} u_t + \dfrac{1}{2}x^4 u_{xx} = 0, & (t,x) \in [0,T) \times [0,\infty), \\ u(T,x) = x, & x \in [0,\infty). \end{cases}$$

Experimentally, we use Crank-Nicolson method solving the following Neumann problem

$$\begin{cases} v_t^M + \dfrac{1}{2}x^4 v_{xx}^M = 0, & (t,x) \in [0,T) \times [0,\infty), \\ v^M(T,x) = x, & x \in [0,\infty), \\ v^M(t,0) = 0, & t \in [0,T), \\ v_x^M(t,M) = 0, & t \in [0,T). \end{cases} \tag{5.6}$$

**Remark 5.9.** It is not necessary to use the Neumann condition $v_x^M(t, M) = 0$. For example, we can alternatively use $v_x^M(t, M) = \frac{1}{M^2}$. A heuristic reason is that the choice of $u_x = \frac{1}{x^2}$ also makes $\int_0^t u_x X_s^2 dW_s$ a martingale.

In this example, numerical solutions of Neumann problems (5.6) are computed in a uniform grid with

$$\Delta t = \Delta x = 0.001.$$

The Neumann far field boundary condition is used, and we apply an upwind style derivative approximation on $v_x(t, M)$, i.e.,

$$u_x(t, M) \approx \frac{1}{2\Delta x} u(t, M - 2\Delta x) - \frac{2}{\Delta x} u(t, M - \Delta x) + \frac{3}{2\Delta x} u(t, M).$$

Approximation of derivatives in other cases than the far field boundary is the same as we discussed in Subsection 4.3.2.

For simplicity, the maturity $T$ is set as 1. Since the Crank-Nicolson method is unconditionally stable and has truncation error in order of $O(\Delta t^2 + \Delta x^2)$, which is small enough on the grid we choose, we focus our attention on the far field boundary error for the Neumann problem.

Given the analytical solution for $u$ in (2.25), direct calculation implies that the delta $u_x$ satisfies the equation

$$u_x(t, x) = 2\Phi\left(\frac{1}{x\sqrt{T-t}}\right) - 1 - \frac{2}{x\sqrt{T-t}}\phi\left(\frac{1}{x\sqrt{T-t}}\right), \tag{5.7}$$

with $\Phi$ being the cumulative density function and $\phi$ the probability density function of the standard normal distribution.

We conduct a numerical computation for the far field boundary $M = 1, 2, 4, 8$.

Figure 5.1 shows the analytical delta as a function of $t$ for some fixed state space level $M$. It is seen that, as the far field boundary moves towards infinity, the delta

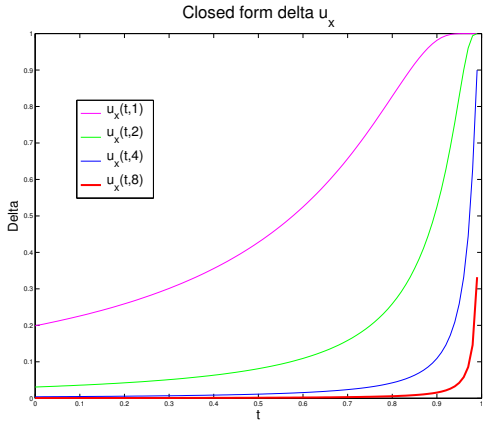


Figure 5.1: Closed form deltas as function of $t$ for $x = 1, 2, 4, 8$ in Bessel model

$u_x(t, x \to \infty)$ is almost flattened off at 0, if $t$ is away from $T$. However, when the time approaches maturity $T$, the delta sharply increases from almost 0 to nearly 1. Therefore, imposing a Neumann condition either $v_x^M(t, M) = 0$ or $v_x^M(t, M) = \frac{1}{M^2}$ looks a good approximation of the true delta on the boundary.

**Remark 5.10.** From the hedging point of view, the optimal strategy is that of always investing a small amount of capital in the underlying, and the rest in the money market when the current time is away from maturity and the underlying is in a high level. The situation changes rapidly to using a delta of value one in the replicating portfolio when the time nears maturity.

Figure 5.2 displays the numerical results solving the Neumann problem (5.6) for $v^M(0, x)$. Calculations are carried out for far field boundaries set as $M = 1, 2, 4, 8$. It is no surprising that the numerical solutions are increasing in $M$, as it appears in the figure and also proved in Theorem 5.2. The top line in Figure 5.2 represents the analytical solution $u(0, x)$, and the numerical solution $v^8(0, x)$ by setting the far field boundary $M = 8$ is already a good approximation.

If alternatively we choose $v^M(t, M) = \frac{1}{M^2}$ as the Neumann boundary condition,

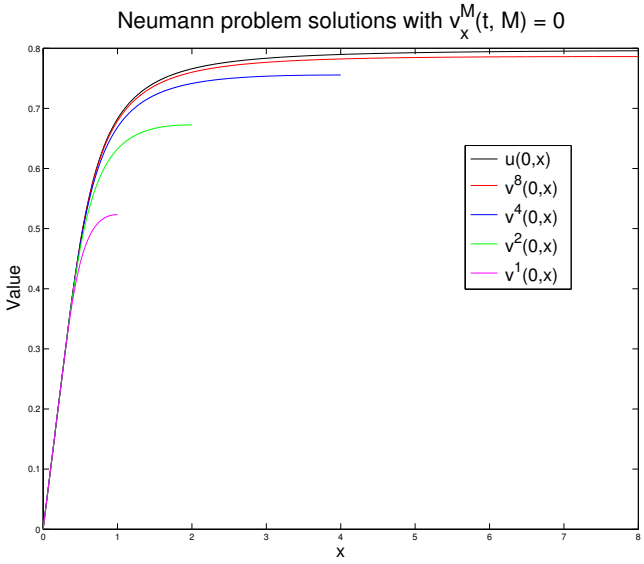


Figure 5.2: Numerical solutions with Neumann boundary $v_x^M(t, M) = 0, M = 1, 2, 4, 8$

similar findings can be seen from Figure 5.3. In fact, this boundary condition gives better convergence than the previous one, especially when the boundary distance is small, simply because it is a better guess for the closed form for the delta on the boundary.

To better show the rate of convergence, let us fix the initial point for the underlying process, e.g. $x = 0.25, 0.5, 1$, and look at the far field boundary error. Table 5.1 shows the speed far field boundary error decreases while $M$ increases. Roughly speaking, the far field boundary error decreases to $\frac{1}{4}$, when the far field boundary $M$ doubles. This confirms the Conjecture 5.3, which states that the far field boundary error of the Neumann problem (5.3) with respect to the value function $u(t, x)$ is bounded by $O\left(\frac{1}{M^2}\right)$.

Table 5.1: Far Field Boundary Errors with Boundary $v_x^M(t, M) = 0$

| M | $u(0, 0.25) - v^M(0, 0.25)$ | $u(0, 0.5) - v^M(0, 0.5)$ | $u(0, 1) - v^M(0, 1)$ |
|---|---|---|---|
| 1 | 1.0e-03 * 0.1343 | 0.0339 | 0.1595 |
| 2 | 1.0e-03 * 0.0234 | 0.0095 | 0.0502 |
| 4 | 1.0e-03 * 0.0059 | 0.0027 | 0.0140 |
| 8 | 1.0e-03 * 0.0016 | 0.0007 | 0.0036 |

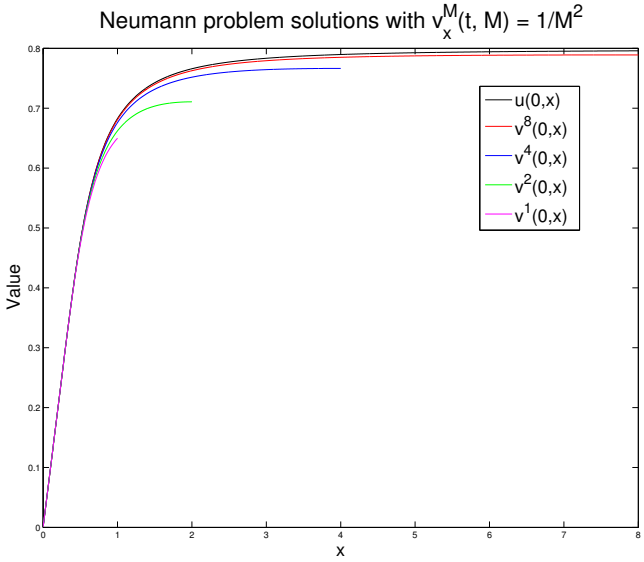


Figure 5.3: Numerical solutions with Neumann boundary $v_x^M(t,M) = \frac{1}{M^2}, M = 1, 2, 4, 8$

Similarly, Table 5.2 displays the far field boundary errors if the Neumann boundary is set as $v_x^M(t,M) = \frac{1}{M^2}$. In general, this boundary condition gives smaller far field boundary error than the boundary condition used in Table 5.1. It starts with much less error comparatively, but, when $M$ is big, the error still decreases quadratically. The reason that in the case $x = 0.5$ and $x = 1$ do not look like quadratic is because the boundary is too close the point that the value is calculated.

Table 5.2: Far Field Boundary Errors with Boundary $v_x^M(t,M) = \frac{1}{M^2}$

| M | $u(0,0.25) - v^M(0,0.25)$ | $u(0,0.5) - v^M(0,0.5)$ | $u(0,1) - v^M(0,1)$ |
|---|---|---|---|
| 1 | 1.0e-04 * 0.1769 | 0.0052 | 0.0328 |
| 2 | 1.0e-04 * 0.0370 | 0.0023 | 0.0205 |
| 4 | 1.0e-04 * 0.0147 | 0.0010 | 0.0067 |
| 8 | 1.0e-04 * 0.0059 | 0.0003 | 0.0018 |

In fact, In't Hout and Foulon has a similar numerical example for Heston model in [33]. Experimentally, they also confirm the Neumann problem converges quadratically to the value function.

In conclusion, Neumann problem serves as a good approximation to the option

value function (3.9) while Dirichlet problem loses convergence, if the underlying process is a strict local martingale. And, we conjecture that such an approximation is in quadratic order of the inverse distance of the far field boundary.

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