CARNEGIE INSTITUTE OF TECHNOLOGY

COLLEGE OF ENGINEERING AND SCIENCE

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

TITLE ON THE PLASTIC COLLAPSE OF STRUCTURES AND LINEAR PROGRAMMING

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ACKNOWLEDGMENTS

The author would like to express his sincere gratitude to Professor H. J. Greenberg for his advice and suggestions during the investigation leading to this paper.

INTRODUCTION

This paper is concerned with the solution of various problems in the plastic collapse of plane structures.

In Chapter I the basic problems and theorems of limit analysis are reviewed and formulated in a convenient notation.

A pair of superposition principles are developed for limit analysis of structures in Chapter II. These principles lead to upper and lower bounds to the safety factor for a superimposed load system in terms of bounds to the safety factors for the individual loads. In addition several special problems are posed and solved in the second chapter. These include a minimax problem in which a safety factor which is valid for all load systems in a given range is found. Finally an iterative method is given for obtaining bounds to the safety factor for the proportional loading of frames when axial forces as well as bending moments are to be considered. Examples are included at the end of the chapter.

Chapter III reviews three basic methods of solution for linear programming problems. The problem of the plastic collapse of structures is reduced to forms suitable for the application of these three methods. A collapse problem is solved by the several linear programming methods in Chapter IV for demonstration and comparison.

A method for obtaining an initial feasible solution for Lemke's dual method of solving the linear programming problem is given in Appendix C. This method is analogous to a procedure developed by Dantzig for the simplex method.

Chapter I

FUNDAMENTAL PROBLEMS AND THEOREMS OF LIMIT ANALYSIS

The problems considered in this paper arise in the study of the plastic collapse of statically indeterminate plane structures, which are subjected to concentrated loads acting in the plane of the structure. These structures may be conveniently divided into three types: pin-jointed trusses, continuous beams, and frames.

Throughout, it will be assumed that all of the structural members are composed of an elastic-perfectly plastic material such as mild structural steel. For a member in pure bending this implies that the bending moment at any cross-section must lie between certain maximum and minimum values, the fully plastic moments. At a beam cross-section where the bending moment equals the fully plastic moment a yield hinge develops, and the beam segments adjacent to that cross-section are free to rotate about that point under constant moment.

Similarly for a bar in pure tension or compression the axial force is bounded by the fully plastic forces. A bar in which the axial force equals the fully plastic force (a yield bar) can undergo continuing change in length under constant force.

Since both bending moments and axial forces may be present in beams and frames, the yield condition for these structures in general involves both of these quantities. A more complete discussion of this situation is given in Section 4, Chapter II.

Most of the succeeding analysis, however, will deal specifically with frames in which the axial forces are assumed to be negligible compared with the bending moments. The yield criterion for pure bending may, therefore, be assumed. The remarks regarding such frames also apply to continuous beams, while a parallel discussion can be given for trusses with axial forces replacing bending moments and yield bars replacing yield hinges.

If in the course of a loading program a yield hinge develops at some point in a statically indeterminate frame, then the degree of redundancy of the frame is reduced by one. The appearance of a sufficient number of yield hinges, therefore, transforms the frame or some part of it into a mechanism, i.e., the structure is no longer rigid. When this phenomenon occurs, the frame is said to collapse.

For a given redundant frame with the fully plastic moments, i.e., bounds on the bending moments, specified at each cross-section and with a finite number of given loads applied at specified points; the basic problem is to determine the largest number by which all of the given loads may be multiplied before the structure will collapse. This type of loading program in which load ratios are maintained as the loads increase is designated proportional loading, and the maximum value of the multiplier is termed the safety factor against collapse.

Any value of the multiplier, for which there exists a bending moment distribution which nowhere exceeds the fully plastic moments and which together with the loads corresponding to this multiplier satisfies equilibrium everywhere, is called a statically admissible multiplier.

On the other hand, through the equation of virtual work, there exists a value of the multiplier corresponding to each mode of collapse of the structure in the form of a kinematically possible mechanism. Such a multiplier is called a kinematically sufficient multiplier.

The two fundamental theorems in the limit analysis of structures are:

THEOREM 1: The safety factor against collapse is the largest statically admissible multiplier.

THEOREM 2: The safety factor against collapse is the smallest kinematically sufficient multiplier.

There theorems were first stated and proved for frames by Greenberg and Prager [1][1].

The basic problem of determining the safety factor against collapse may, therefore, be formulated and solved in two ways.

It will be assumed for convenience of notation that the fully plastic moments are constant along each individual member, and that the loads are all transverse, i.e., perpendicular to the beam to which they are applied.

Under these assumptions yield hinges can develop only at discrete cross-sections in the frame, i.e., where the bending moment has a turning point. Since only concentrated loads are applied to the frame, these critical cross-sections occur under loads and at the ends of members. The critical cross-sections may, therefore, be enumerated.

The bending moment at the ith critical cross-section is M_i, and $M_{pi} > 0$ and $-M'_{pi} < 0$ are the fully plastic moments in the two directions of bending. Finally the applied load at the ith cross-section is λb_i (some of which may be zero).

The equations of equilibrium can be written as a system of linear equations in the quantities M_i and λb_i

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[1]Numbers in square brackets refer to the bibliography at the end of the discussion.

$$\sum_{j=1}^{n} a_{ij} M_j = \lambda \sum_{k=1}^{n} h_{ik} b_k \quad (i=1,2,\cdots,m) \quad , \quad n \geq m \qquad (1.1)$$

where there are n critical cross-sections. If equations (1.1) are linearly independent and if there n - m redundancies in the frame, then (1.1) is a complete set of equilibrium equations in the sense that all linear equilibrium relations between the bending moments can be expressed as a linear combination of this set. The a_{ij}, h_{ik} are constants which depend on the geometrical configuration of the structure and loads.

The yield criteria take the form

$$-M'_{pj} \leq M_j \leq M_{pj} \qquad (j=1,2,\ldots,n) \qquad (1.2)$$

Given a_{ij}, h_{ik}, b_k, M_{pi} and M'_{pi}; the problem reduces in the one case to finding the maximum value of λ for which a solution M_j to (1.1) and (1.2) exists. This value of λ is, by Theorem 1, the safety factor against collapse.

To formulate the problem in terms of the minimum principle expressed by Theorem 2, it is necessary to determine the value of the kinematically sufficient multiplier corresponding to every mechanism. Let v_j be the linear velocity of the load λb_j in a mechanism. Then if θ_j represents the relative rotational velocity of the beam segments adjacent to the jth cross-section and if the θ_j are kinematically compatible with the velocities v_j, the virtual work equation for this mechanism is

$$\frac{1}{2} \sum_{j=1}^{n} \left\{ M_{pj}\left[|\theta_j| + \theta_j\right] + M'_{pj}\left[|\theta_j| - \theta_j\right] \right\} = \lambda \sum_{j=1}^{n} b_j v_j \qquad (1.3)$$

and it is required that

$$\sum_{j=1}^{n} b_j v_j > 0 \tag{1.4}$$

The value of λ in (1.3) is then a kinematically sufficient multiplier and hence an upper bound for the actual safety factor. By Theorem 2 then the safety factor against collapse is the minimum of

$$\frac{\frac{1}{2}\sum_{j=1}^{n}\{M_{pj}[|\theta_j|+\theta_j]+M'_{pj}[|\theta_j|-\theta_j]\}}{\sum_{j=1}^{n} b_j v_j} \tag{1.5}$$

over all v_j, θ_j which represent mechanisms subject to (1.4).

It is important to note that in any assumed mechanism the absolute magnitudes of θ_j and v_j are undetermined. Multiplying both by the same arbitrary constant does not alter the mechanism and also yields the same multiplier. The contraint (1.4) is, therefore, a matter of sign convention since v_j, θ_j can always be multiplied by -1.

In cases where the mechanism has more than one degree of freedom, even the relative velocities of the different loads need not be uniquely determined. In such cases the value of the multiplier may depend on the ratios of the parameters representing the various degrees of freedom. This point will be discussed in more detail later.

In the above formulations the usual assumption has been made that the deformations prior to collapse are so small, i.e., of the order of elastic deformations, that the equilibrium equations are not significantly affected.

Throughout what follows it will be assumed for convenience that the cross-sections of the structural members are symmetric about the axis of bending. This implies that $M'_{pj} = M_{pj}$ for all j and the functional (1.5) reduces to

$$\frac{\sum_{j=1}^{n} M_{pj}|\theta_j|}{\sum_{j=1}^{n} b_j v_j} \tag{1.6}$$

Chapter II

SPECIAL COLLAPSE PROBLEMS AND SOLUTIONS

1. The Neal - Symonds Mechanism Technique.

A simple and rapid means of determining the safety factor against the collapse of plane frames which are not too complex is due to Neal and Symonds [2, 3, 4]. We will discuss their method, termed the "mechanism technique", in detail in this section, since it will be found useful in the solution of various collapse problems. In addition we will attempt to make this technique more rigorous by supplying proofs for certain heuristic arguments of the authors and we will extend the analysis to more complicated types of frames.

The mechanism technique is based on the minimum principle. Clearly, if the multipliers associated with all possible mechanisms could be found, then the smallest of these would be the safety factor. Since the virtual work equation yields the multiplier for any mechanism, it remains only to devise a technique for determining all possible mechanisms. This is supplied by the Neal and Symonds analysis.

A. Rectangular Frames. - A frame is called rectangular if it is constructed of rectangular bays or portals. The results and remarks regarding these frames apply also to any frame consisting of quadrilateral bays or portals.

For rectangular frames Neal and Symonds proposed the following three types of "elementary" or "basic" mechanisms:

(a) Beam: implying yield hinges at the end points of a beam and at some intermediate cross-section under a load (See Figure 1(a)).

(b) Frame: implying motion of a panel or story (See Figure 1(b)).

(c) Joint: implying rotation at a joint where two[1] or more beams unite (See Figures 1(c) and 1(d)).

If a frame mechanism is assumed for each story and each cantilevered section; a beam mechanism for each load not at a joint; and a joint mechanism for each joint; then the authors state that every possible mechanism is some combination of this set of basic mechanisms. Designate this set of basic mechanisms by B.

Because there are only a finite number of cross-sections where yielding can occur for a frame under concentrated loads, it is usually tacitly assumed that there are only a finite number of possible mechanisms. However, a mechanism requires specification of the relative velocities, v_i, of the cross-sections where loads are applied. It has already been mentioned that when the mechanism has more than one degree of freedom, the relative velocities may be arbitrary and there may exist an infinite set of allowable (v_i, θ_i) which are not simple multiples of each other.

Two questions, therefore, arise regarding the Neal-Symonds procedure for rectangular frames: (1) whether all mechanisms can be obtained by combining the mechanisms of the set B, and (2) whether an infinite number of mechanisms need be examined to determine the lowest multiplier. In the following we shall attempt to clarify these points.

A mechanism for a given set of loads b_j is defined by a set of ~~velocities, v_i,~~ and relative rotational velocities, θ_i, which are

[1]The authors originally proposed a joint mechanism only for points where three or more members are joined. The generalization here leads to a more systematic treatment.

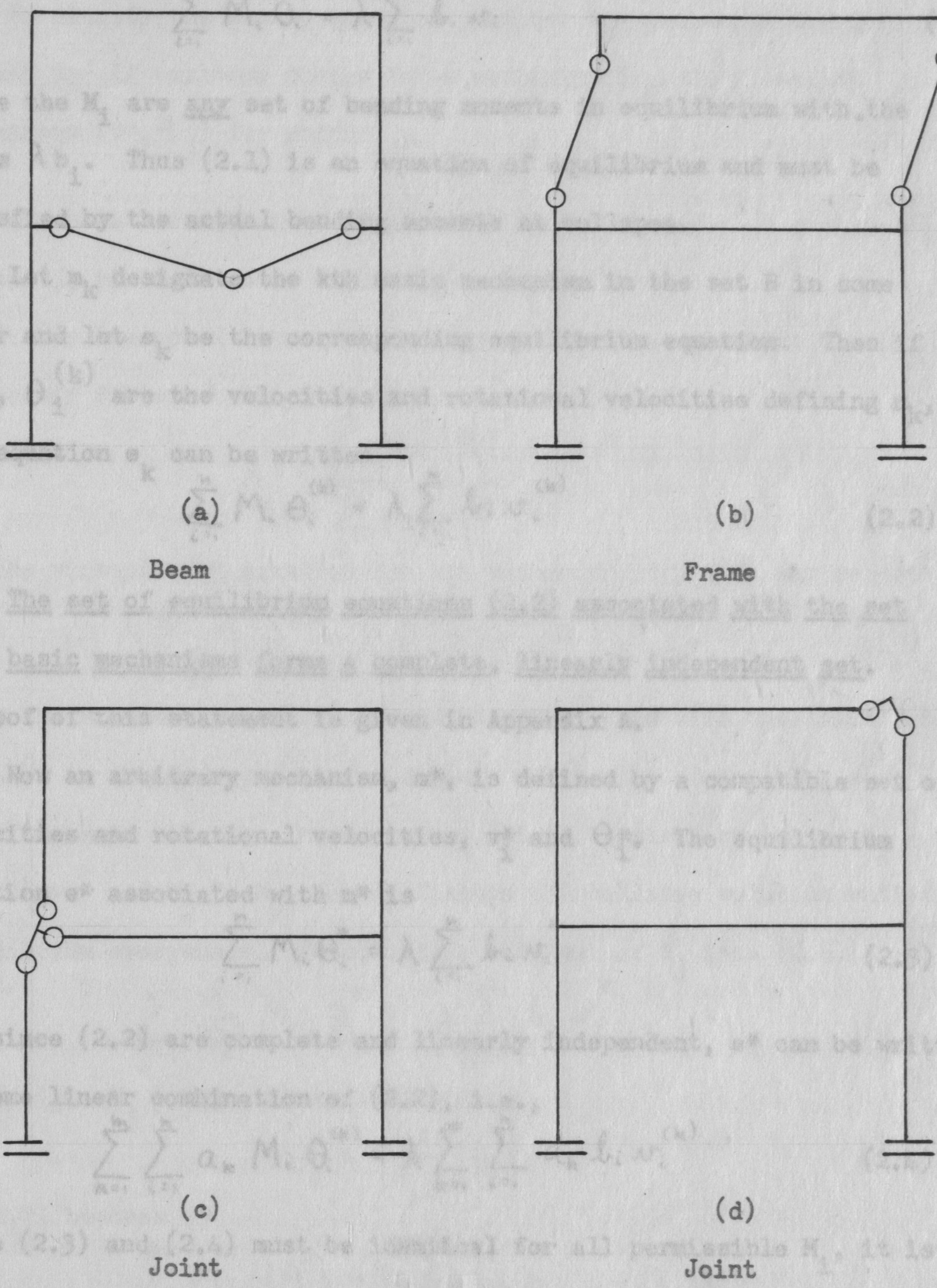

FIGURE 1

Types of Basic Mechanisms

compatible and satisfy (1.4). The virtual work principle requires that

$$\sum_{i=1}^{n} M_i \theta_i = \lambda \sum_{i=1}^{n} b_i v_i \tag{2.1}$$

where the M_i are <u>any</u> set of bending moments in equilibrium with the loads λb_i. Thus (2.1) is an equation of equilibrium and must be satisfied by the actual bending moments at collapse.

Let m_k designate the kth basic mechanism in the set B in some order and let e_k be the corresponding equilibrium equation. Then if $v_i^{(k)}$, $\theta_i^{(k)}$ are the velocities and rotational velocities defining m_k, the equation e_k can be written

$$\sum_{i=1}^{n} M_i \theta_i^{(k)} = \lambda \sum_{i=1}^{n} b_i v_i^{(k)} \tag{2.2}$$

<u>The set of equilibrium equations (2.2) associated with the set B of basic mechanisms forms a complete, linearly independent set.</u> A proof of this statement is given in Appendix A.

Now an arbitrary mechanism, m*, is defined by a compatible set of velocities and rotational velocities, v_i^* and θ_i^*. The equilibrium equation e* associated with m* is

$$\sum_{i=1}^{n} M_i \theta_i^* = \lambda \sum_{i=1}^{n} b_i v_i^* \tag{2.3}$$

Now since (2.2) are complete and linearly independent, e* can be written as some linear combination of (2.2), i.e.,

$$\sum_{k=1}^{m} \sum_{i=1}^{n} a_k M_i \theta_i^{(k)} = \lambda \sum_{k=1}^{m} \sum_{i=1}^{n} a_k b_i v_i^{(k)} \tag{2.4}$$

Since (2.3) and (2.4) must be identical for all permissible M_i, it is necessary that

$$\theta_i^* = \sum_{k=1}^{m} a_k \theta_i^{(k)} \quad \text{and} \quad v_i^* = \sum_{k=1}^{m} a_k v_i^{(k)}$$

This implies that m* is a linear combination of the m_k and resolves the first question.

To clarify the second question consider the following theorem:

THEOREM 3: If collapse occurs for a mechanism (v_j, θ_j) then all mechanisms (v_j^*, θ_j^*) for which

$$\sum_{j=1}^{n} b_j v_j^* > 0$$

with the same arrangement of hinges and such that sign θ_j = sign θ_j^*, for all j, yield the same value of the multiplier.

PROOF: By Theorem 2 since collapse occurs for the mechanism v_j, θ_j the safety factor λ is given by

$$\lambda = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j|}{\sum_{j=1}^{n} b_j v_j} \tag{2.5}$$

Now the virtual work equation for the mechanism (v_j^*, θ_j^*) may be written

$$\sum_{j=1}^{n} M_j \theta_j^* = \lambda \sum_{j=1}^{n} b_j v_j^* \tag{2.6}$$

where the M_j are any set of moments in equilibrium with the loads λb_j. Since λ is the safety factor and hence statically admissible, such a set of moments exists. Indeed such a set is obtained by taking $M_j = M_{pj}$ for $\theta_j > 0$ and $M_j = -M_{pj}$ for $\theta_j < 0$ since the collapse solution satisfies equilibrium everywhere. Substituting this set of M_j into (2.6)

$$\sum_{j=1}^{n} \frac{\theta_j}{|\theta_j|} M_{pj} \theta_j^* = \lambda \sum_{j=1}^{n} b_j v_j \tag{2.7}$$

Since sign θ_j = sign θ_j^* by assumption,

$$\theta_j^* = \frac{|\theta_j|}{\theta_j} |\theta_j^*|$$

so (2.7) becomes

$$\sum_{j=1}^{n} M_{pj} |\theta_j^*| = \lambda \sum_{j=1}^{n} b_j v_j^* \tag{2.8}$$

Now the multiplier, λ_*, associated with the mechanism (v_j^*, θ_j^*) is by the principle of virtual work

$$\lambda_* = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j^*|}{\sum_{j=1}^{n} b_j v_j^*}$$

Comparing this with (2.8) it follows that

$$\lambda_* = \lambda$$

If, therefore, one mechanism is considered for each possible arrangement of yield hinges, the safety factor will be the smallest of the multipliers computed for these mechanisms. There are only a finite number of such mechanisms. They may be found by considering all combinations of the m basic mechanisms of the set B taken any number at a time. The total number of these combinations is $\sum_{k=1}^{m}\binom{m}{k}$ where $\binom{m}{k}$ denotes the number of combinations of m things taken k at a time.

Actually, all of these mechanisms need not be considered. If n - m + 1 hinges appear, then the bending moments throughout the frame are uniquely determined. Therefore, for a mode of collapse involving more than n - m + 1 hinges the moment distribution and multiplier can be determined from some mechanism where only n - m + 1 hinges occur. It is, therefore, not necessary to consider any mechanisms involving more than n - m + 1 hinges.

The number of combinations which must be tested can be reduced still further. Neal and Symonds have stated and shown by example that only those combinations of mechanisms for which a hinge is eliminated at some cross-section need be considered. This is a generally valid principle and may be stated and proved as follows.

THEOREM 4[1]: If two mechanisms, in which the rotational velocities at all common hinges are in the same sense, are combined in positive amounts;

[1] A similar theorem proved by R.M. Haythornthwaite in the discussion of [4] applies only to combinations of quite restricted types of mechanisms.

then the multiplier associated with the combined mechanism cannot be less than the smaller of the multipliers associated with the original mechanisms.

PROOF: Let the multipliers associated with the original mechanisms be

$$\lambda' = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j'|}{\sum_{j=1}^{n} b_j v_j'} \tag{2.9}$$

and

$$\lambda'' = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j''|}{\sum_{j=1}^{n} b_j v_j''} \tag{2.10}$$

where

$$\sum_{j=1}^{n} b_j v_j' > 0 \quad , \quad \sum_{j=1}^{n} b_j v_j'' > 0$$

Since the rotational velocities are all in the same sense

$$\text{Sign } \theta_j' = \text{Sign } \theta_j'' \tag{2.11}$$

Now a positive combination of the two mechanisms is defined by relative velocities $\theta_j' + g\,\theta_j''$ and $v_j' + g v_j''$ where $g > 0$. The multiplier associated with the combined mechanism is

$$\lambda_c = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j' + g\,\theta_j''|}{\sum_{j=1}^{n} b_j (v_j' + g v_j'')}$$

From (2.11) it follows that

$$|\theta_j' + g\theta_j''| = |\theta_j'| + g\,|\theta_j''|$$

and thus

$$\lambda_c = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j'| + g \sum_{j=1}^{n} M_{pj} |\theta_j''|}{\sum_{j=1}^{n} b_j v_j' + g \sum_{j=1}^{n} b_j v_j''}$$

Now, assuming without loss of generality that $\lambda' \leq \lambda''$, then from (2.9) and (2.10) the last equation yields

$$\lambda' \leq \lambda_c \leq \lambda''$$

B. <u>General Frames</u>. - The mechanism technique may also be applied to more general frames which are composed of straight members. The concept of beam and joint mechanisms carries over immediately and such mechanisms are identified as before. The idea of a frame mechanism requires generalization and the identification of such mechanisms is no longer obvious since the structure may no longer consist of simple bays and portals.

In order to carry out the mechanism technique it is necessary to find $n - r$ mechanisms whose corresponding equilibrium equations are linearly independent. The particular choice of basic mechanisms for rectangular frames was merely a matter of convenience. If the beam and joint mechanisms are retained as basic mechanisms in the general case, then exactly $2v - b$ more independent mechanisms are needed to form a complete set[1].

The additional set of $2v - b$ mechanisms can be chosen so that hinges occur only at the ends of beams. A necessary and sufficient condition that such a set of mechanisms be independent is that none may be obtained from any of the others by rotation of joints.

With these criteria at hand it is usually not difficult to choose $2v - b$ mechanisms, which may be arbitrarily termed frame mechanisms.

2. Plastic Superposition.

In the elastic analysis of frames the solution of problems can often be reduced to the solution of several simpler problems by use of the principle of superposition. This principle cannot be extended to limit analysis since for elastic-plastic behavior there is no longer a one-to-one correspondence between stresses and strains. However, we will now develop a pair of superposition principles for the limit analysis

[1] See Appendix A. The number of bars in the frame is b and the number of verticies, i.e., joints, is v.

of proportionally loaded frames. While neither principle determines the safety factor for superimposed loads, the two together yield upper and lower bounds for that quantity.

A. <u>Lower Bounds</u>. - Consider w different load systems $b_j^{(1)}$, $b_j^{(2)}$, . . ., $b_j^{(w)}$ all applied to the same frame. Let $\lambda_1', \lambda_2', \ldots, \lambda_w'$ be statically admissible multipliers for each of these load systems respectively. Then there exist bending moments $M_j^{(1)}$, $M_j^{(2)}$, . . ., $M_j^{(w)}$ such that

$$\sum_{j=1}^{n} a_{ij} M_j^{(k)} = \lambda_k' \sum_{j=1}^{n} h_{ij} b_j^{(k)} \qquad \begin{matrix}(i = 1, 2, \cdots, m) \\ (k = 1, 2, \cdots, w)\end{matrix}$$

and

$$-M_{pj} \le M_j^{(k)} \le M_{pj} \qquad \begin{matrix}(j = 1, 2, \ldots, n) \\ (k = 1, 2, \ldots, w)\end{matrix} \tag{2.12}$$

Multiplying the equilibrium equations by c/λ_k' and summing over k

$$\sum_{j=1}^{n} a_{ij} \sum_{k=1}^{w} c \frac{M_j^{(k)}}{\lambda_k'} = c \sum_{j=1}^{n} h_{ij} \sum_{k=1}^{w} b_j^{(k)}$$

Now let

$$\bar{M}_j = c \sum_{k=1}^{w} \frac{M_j^{(k)}}{\lambda_k'}$$

$$\bar{b}_j = \sum_{k=1}^{w} b_j^{(k)} \tag{2.13}$$

so that the last equation becomes

$$\sum_{j=1}^{n} a_{ij} \bar{M}_j = c \sum_{j=1}^{n} h_{ij} \bar{b}_j$$

Since $\bar{b}_j$ is the load system obtained from superimposing the load systems $b_j^{(k)}$, c is a statically admissible multiplier for the superimposed loads provided it is chosen so that

$$-M_{pj} \le \bar{M}_j \le M_{pj} \qquad (j = 1, 2, \ldots, n)$$

If λ_s is the safety factor for the loads $\bar{b}_j$, then

$$c \le \lambda_s \tag{2.14}$$

To get the largest lower bound to λ_s, c is chosen as large as possible so that

$$-M_{pj} \le c \sum_{k=1}^{w} \frac{M_j^{(k)}}{\lambda_k'} \le M_{pj} \qquad (j = 1, 2, \ldots, n) \tag{2.15}$$

A lower bound $\bar{c}$ to λ_s which will in general be smaller than c and therefore not as good a bound can be more easily obtained. Notice that (2.12) implies

$$-\bar{c} M_{pj} \sum_{k=1}^{w} \frac{1}{\lambda_k'} \le \bar{c} \sum_{k=1}^{w} \frac{M_j^{(k)}}{\lambda_k'} \le \bar{c} M_{pj} \sum_{k=1}^{w} \frac{1}{\lambda_k'}$$

where $\bar{c} > 0$. The continued inequality (2.15) is satisfied then if

$$\bar{c} \sum_{k=1}^{w} \frac{1}{\lambda_k'} \le 1$$

Now $\bar{c}$ is a lower bound to λ_s as was c before, and the largest value of $\bar{c}$ is obtained if equality is taken in the last inequality. From (2.14) therefore

$$\frac{1}{\lambda_s} \le \sum_{k=1}^{w} \frac{1}{\lambda_k'} \tag{2.16}$$

As previously stated, this bound is in general not as good as that obtained by maximizing c in (2.15).

The largest lower bound in (2.16) is obtained if the λ_k' are the safety factors for the load systems $b_j^{(k)}$ for all k. In general, however, this lower bound $\bar{c}$ will not equal λ_s since the bending moments in equilibrium with loads $\bar{c}b_j$ may not equal the yield moment at sufficient cross-sections to produce collapse.

B. <u>Upper</u> <u>Bounds</u>. - Consider now a mechanism (v_j, θ_j) such that

$$\sum_{j=1}^{n} b_j^{(k)} v_j > 0 \qquad (k = 1, 2, \ldots, w)$$

for all k. Let λ_i^* be the kinematically sufficient multiplier associated with this mechanism for the load system $b_j^{(i)}$. Then $\lambda_i^* > 0$ and

$$\frac{1}{\lambda_i^*} = \frac{\sum_{j=1}^{n} b_j^{(i)} v_j}{\sum_{j=1}^{n} M_{pj} |\theta_j|}$$

Summing this over i and using (2.13)

$$\sum_{i=1}^{w} \frac{1}{\lambda^*} = \frac{\sum_{j=1} \bar{b}_j v_j}{\sum_{j=1}^{n} M_{pj} |\theta_j|}$$

The right hand member is the reciprocal of the multiplier, λ^*, for this mechanism when associated with the superimposed loads. By Theorem 2

$$\frac{1}{\lambda_a} \geqslant \frac{1}{\lambda^*}$$

so

$$\frac{1}{\lambda_a} \geqslant \sum_{i=1}^{w} \frac{1}{\lambda_i^*}$$

This may be combined with (2.16) to form the continued inequality

$$\sum_{k=1}^{w} \frac{1}{\lambda_k^*} \leq \frac{1}{\lambda_a} \leq \sum_{k=1}^{w} \frac{1}{\lambda_k'} \tag{2.17}$$

An example of the use of these superposition principles is given in Section 5, Part A of this chapter.

THEOREM 5: If for a given frame the mode of collapse is the same for two different load systems, $b_j^{(1)}$ and $b_j^{(2)}$, and if the safety factors against collapse are λ_1 and λ_2 respectively; then the collapse mode for the combined load system $\bar{b}_j = b_j^{(1)} + b_j^{(2)}$ is the same, and the safety factor against collapse, $\bar{\lambda}$, is

$$\bar{\lambda} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$

PROOF: By the upper bound principle

$$\bar{\lambda} \leq \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \tag{2.18}$$

Now since λ_1 and λ_2 are safety factors, they are statically admissible multipliers, so bending moments $M_j^{(1)}$ and $M_j^{(2)}$ exist such that

$$\sum_{j=1}^{n} a_{ij} M_j^{(1)} = \lambda_1 \sum_{k=1}^{n} h_{ik} b_k^{(1)}$$

$$\sum_{j=1}^{n} a_{ij} M_j^{(2)} = \lambda_2 \sum_{k=1}^{n} h_{ik} b_k^{(2)} \tag{2.19}$$

and

$$-M_{pj} \leq M_j^{(1)} \leq M_{pj}$$

$$-M_{pj} \leq M_j^{(2)} \leq M_{pj}$$

Consider the bending moment distribution

$$\bar{M}_j = \frac{\lambda_2 M_j^{(1)} + \lambda_1 M_j^{(2)}}{\lambda_1 + \lambda_2}$$

Obviously

$$-M_{pj} \leq \bar{M}_j \leq M_{pj}$$

Moreover

$$\sum_{j=1}^{n} a_{ij} \bar{M}_j = \frac{\lambda_2}{\lambda_1+\lambda_2} \sum_{j=1}^{n} a_{ij} M_j^{(1)} + \frac{\lambda_1}{\lambda_1+\lambda_2} \sum_{j=1}^{n} a_{ij} M_j^{(2)}$$

By (2.19)

$$\sum_{j=1}^{n} a_{ij} \bar{M}_j = \frac{\lambda_1 \lambda_2}{\lambda_1+\lambda_2} \sum_{k=1}^{n} h_{ik} \left[b_k^{(1)} + b_k^{(2)} \right]$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1+\lambda_2} \sum_{k=1}^{n} h_{ik} \bar{b}_k$$

Thus $\lambda_1\lambda_2/\lambda_1+\lambda_2$ is a statically admissible multiplier for the load system $\bar{b}_k$ and

$$\frac{\lambda_1 \lambda_2}{\lambda_1+\lambda_2} \leq \bar{\lambda}$$

Combining this with (2.18), the result follows.

An example of the use of this theorem is given in Section 5, Part C.

The theorem is easily extended to the case of w load systems where the mode of collapse is the same for each system.

3. A Minimax Problem.

Consider now the following generalization of the proportional loading problem defined by (1.1) and (1.2): To find the minimum over b_j of the maxima over M_j of λ subject to

$$\sum_{j=1}^{n} a_{ij} M_j = \lambda \sum_{j=1}^{n} h_{ij} b_j \qquad (i=1,2,\ldots,m) \tag{2.20}$$

$$-M_{pj} \leq M_j \leq M_{pj} \qquad (j=1,2,\ldots,m) \tag{2.21}$$

$$\bar{b}_j - \Delta_j \leq b_j \leq \bar{b}_j + \Delta_j \qquad (j=1,2,\ldots,n) \tag{2.22}$$

$$\bar{b}_j \geq \Delta_j \geq 0 \qquad (j = 1, 2, \cdots, n) \tag{2.23}$$

where a_{ij}, h_{ij}, M_{pj}, $\bar{b}_j$, Δ_j are given.

For every set of b_j satisfying (2.22) a value of the safety factor is obtained by maximizing λ over the statically admissible M_j, i.e., M_j satisfying (2.20) and (2.21). The present problem seeks the smallest of all these safety factors over all possible sets of loads b_j satisfying (2.22). Let λ_c be the solution to this minimax problem. Then λ_c is a true safety factor for <u>all</u> load systems in the given range, i.e., for all $\lambda < \lambda_c$ collapse will not occur for any set of loads satisfying (2.22). The final restriction (2.23) prevents the loads from changing direction and is a necessary simplification for the treatment which follows.

For a fixed set of loads b_j which satisfies (2.22), the problem reduces to the simple proportional loading problem posed in (1.1) and (1.2). It may therefore be formulated as a minimum problem given by (1.4) and (1.6), i.e., to minimize

$$\lambda = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j|}{\sum_{j=1}^{n} b_j v_j} \tag{2.24}$$

over all possible mechanisms (v_j, θ_j), where the v_j and θ_j are kinematically compatible velocities and rotational velocities respectively, and

$$\sum_{j=1}^{n} b_j v_j > 0 \tag{2.25}$$

The original minimax problem is therefore equivalent to minimizing (2.24) first over all mechanisms (v_j, θ_j) and then over all loads b_j subject to (2.22), (2.23), and (2.25).

Although this minimum problem is actually an iterated one, it is equivalent to the minimization of the functional (2.24) over the two sets of quantities (v_j, θ_j) and b_j without regard to order. If this latter minimum exists then the iterated one does also, and the two are equal.

The problem as stated is non-linear. All attempts to reduce it to a linear programming problem both in the present and in previous[1] investigations have failed. A solution is obtained here by use of the Neal-Symonds technique.

THEOREM 6: Only the end points of the intervals of the loads, b_j, need by considered in seeking λ_c.

PROOF: Suppose in contradiction to the theorem that the functional (2.24) takes on its minimum for $b_j = \overline{b}_j + \epsilon_j$ where $|\epsilon_j| < \Delta_j$ for some j, say j = p. Then

$$\lambda_c = \frac{\sum_{j=1}^{n} M_{p_j} |\theta_j|}{\sum_{j=1}^{n} \overline{b}_j v_j + \sum_{j=1}^{n} \epsilon_j v_j}$$

Three cases arise: (1) $v_p > 0$, in which case

$$\Delta_p v_p > \epsilon_p v_p$$

and the value of λ_c can be decreased by replacing ϵ_p by Δ_p contrary to assumption. Similarly if, (2), $v_p < 0$, then

$$-\Delta_p v_p > \epsilon_p v_p$$

and a contradiction is reached. Finally if, (3), $v_p = 0$, then the functional takes on the same value for all values of b_p and it is sufficient to consider the end points.

The problem therefore reduces to minimizing

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[1] See pp. 53-58 of [5].

$$\lambda_c = \frac{\sum_{j=1}^{n} M_{pj} |\theta_j|}{\sum_{j=1}^{n} \bar{b}_j v_j + \sum_{j=1}^{n} \Delta_j |v_j|} \tag{2.26}$$

over all mechanisms (v_j, θ_j) subject to (2.25). Notice, however, that (2.25) may be replaced by the equivalent restriction

$$\sum_{j=1}^{n} \bar{b}_j v_j > 0$$

These mechanisms are, therefore, identical with those arising in the Neal-Symonds solution for the proportionally loaded frame with loads $\bar{b}_j$. They are finite in number and thus the minimum exists.

In order to obtain a more convenient form of the functional in (2.26), let $v_j^{(k)}$, $\theta_j^{(k)}$ designate the velocities and rotational velocities of the kth mechanism arising from the Neal-Symonds analysis, and let the $v_j^{(k)}$ be so normalized that

$$\sum_{j=1}^{n} \bar{b}_j v_j^{(k)} = 1 \tag{2.27}$$

for all k. Then

$$\frac{1}{\lambda_c} = \text{Maximum} \frac{1}{\sum M_{pj} |\theta_j^{(k)}|} \left[1 + \sum_{j=1}^{n} \Delta_j |v_j^{(k)}| \right]$$

where the maximum is taken over all k.

The value of the multiplier associated with the kth mechanism for the loads $\bar{b}_j$ is

$$\bar{\lambda}^{(k)} = \sum_{j=1}^{n} M_{pj} |\theta_j^{(k)}|$$

so

$$\frac{1}{\lambda_c} = \text{Maximum} \frac{1}{\bar{\lambda}^{(k)}} \left[1 + \sum_{j=1}^{n} \Delta_j |v_j^{(k)}| \right] \tag{2.28}$$

over all k.

The procedure for obtaining the solution to the minimax problem then is:

(1) Proceed as in the solution of the proportional loading collapse

problem for loads $\bar{b}_j$ using the Neal-Symonds technique. Tabulate all mechanisms and the corresponding multipliers, $\bar{\lambda}^{(k)}$.

(2) Normalize the velocities, $v_j^{(k)}$, for each mechanism according to (2.27).

(3) Form the right hand member of (2.28) for each mechanism using the normalized velocities and select the largest of these numbers. This is the reciprocal of λ_c.

An example of this type of minimax problem and solution follows in Section 5, Part B of this chapter.

4. The Effect of Axial Forces in Frames.

Another modification of the Neal-Symonds technique leads to bounds on the collapse solution of a proportionally loaded frame when axial forces are significant. As mentioned in Chapter I, the addition of axial force effects requires a yield condition involving both moments and forces at each cross section.

A mechanism may now involve yield bars as well as yield hinges. It is, therefore, necessary to examine cross-sections where the bending moment and/or axial force has a turning point. Since all loads are assumed to be transverse, these are identical with the critical cross-sections neglecting axial forces, i.e., at the ends of each member and under each load.

Let M_i be the bending moment at the ith critical cross-section, and let N_j be the axial force, considered positive for a tensile force, in the jth member.[1] Furthermore, let M_{pi} be the fully plastic moment

---

[1] Since the loads are transverse, the axial force is constant along

at the ith cross-section and let N_{pj} be the yield force in the jth member.

It should be noted that M_{pi} and N_{pi} are not independent but are related by

$$M_{pi} = k_i N_{pi}$$

where k_i is a constant which depends on the cross-section.[2]

The state of stress at a generic cross-section can be completely specified by a point, the stress point, in a two-dimensional Euclidean space whose rectangular coordinates are N_i/N_{pi} and M_i/M_{pi}. Onat and Prager [6] have shown that for a beam of rectangular cross-section the stresses at a yielding cross-section must satisfy one of the two equations

$$\left(\frac{N_i}{N_{pi}}\right)^2 \pm \frac{M_i}{M_{pi}} = 1$$

In the stress point plane these are represented by two intersecting parabolas (dashed curves in Figure 2). All statically admissible stress states must be represented by stress points interior to these curves, designated the yield curves for rectangular cross-sections.

For beams with symmetric cross-sections, the yield curves are closed, convex curves symmetric about both axes. Some empirical curves have been given by Baker [7].

As a linear approximation to all of these convex yield curves,

~~each member.~~ Specification of the axial force in each beam, therefore, is sufficient for the determination of N_i at every cross-section

[2]For rectangular cross-sections $k_i = h/4\alpha$ where h is the length and α the length-depth ratio of the beam in which the cross-section is located. For idealized I-beams, $k_i = w/2$ where w is the heighth of the web.

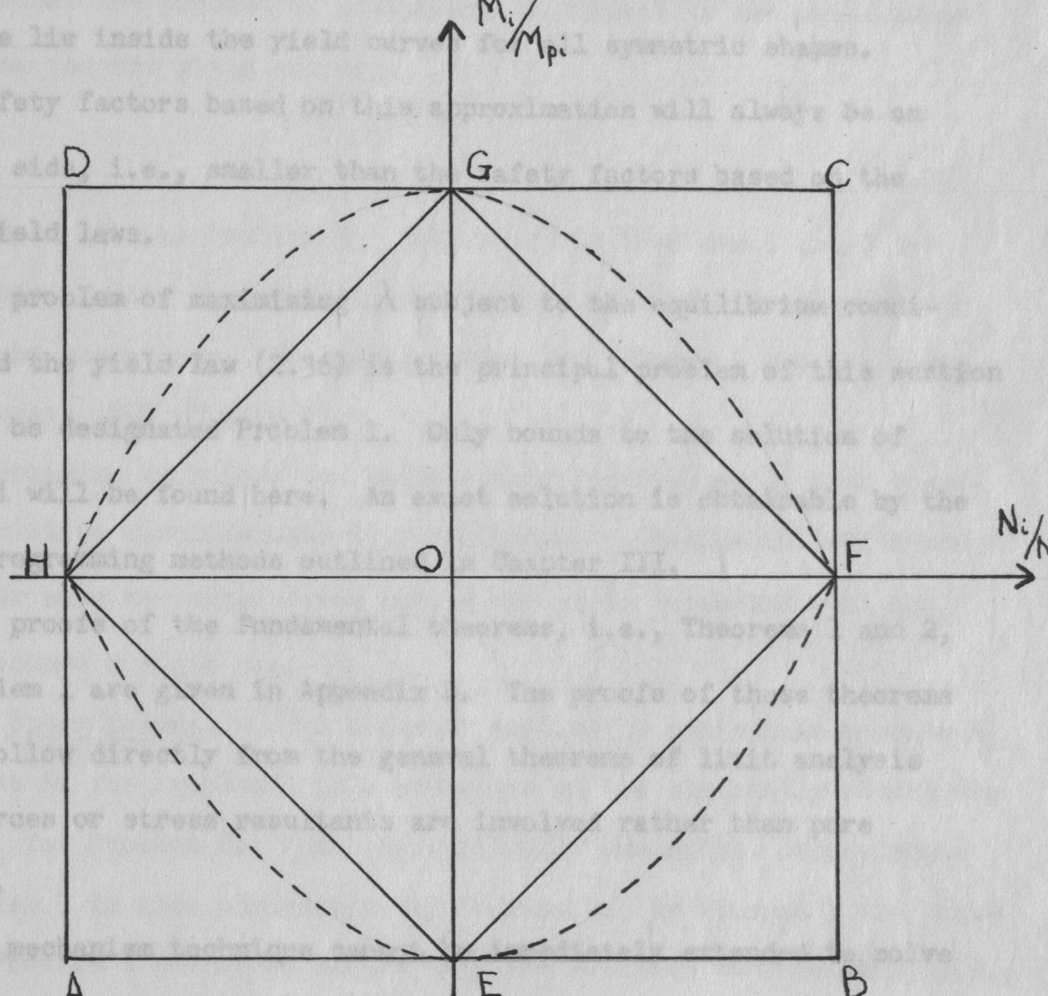

FIGURE 2

Yield Curves

the yield condition[1]

$$\frac{|M_i|}{M_{pi}} + \frac{|N_i|}{N_{pi}} \leq 1 \tag{2.36}$$

may be taken. The yield curves for this criterion are the sides of the square EFGH (Figure 2). A stress point satisfying (2.36) must therefore lie inside the yield curves for all symmetric shapes. Hence safety factors based on this approximation will always be on the safe side, i.e., smaller than the safety factors based on the actual yield laws.

The problem of maximizing λ subject to the equilibrium conditions and the yield law (2.36) is the principal problem of this section and will be designated Problem 1. Only bounds to the solution of Problem 1 will be found here. An exact solution is obtainable by the linear programming methods outlined in Chapter III.

The proofs of the fundamental theorems, i.e., Theorems 1 and 2, for Problem 1 are given in Appendix B. The proofs of these theorems do not follow directly from the general theorems of limit analysis since forces or stress resultants are involved rather than pure stresses.

The mechanism technique cannot be immediately extended to solve Problem 1, however, since it is necessary to allow for relative displacements of the cross-sections adjacent to a yield hinge as well as for rotations of beam segments about the hinge.[2] The axial force

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[1] A mechanical interpretation of this yield law has been given by Onat and Prager in [8].

[2] See discussion of flow vectors in Appendix B.

across a yield hinge, therefore, does work in a mechanism and must be included in the virtual work equation.

We now introduce a problem for which a mechanism technique is available and which will in turn lead to bounds on Problem 1.

Consider the problem of maximizing λ subject to the equilibrium conditions and the yield criteria

$$|M_i| \leq M_{pi}$$

and

$$|N_i| \leq N_{pi} \tag{2.37}$$

We designate this as Problem 2. The proofs of Theorems 1 and 2 for this problem follow the proofs given in Appendix B for Problem 1 with only slight variations. The yield curves for Problem 2 are the sides of the square ABCD (Figure 2).

A mechanism technique can be developed for Problem 2 since there are no relative displacements at yield hinges. Changes in length can only occur when the axial force equals the yield force and thus the member becomes a yield bar.

A. <u>Upper</u> <u>Bounds</u>. - The class of statically admissible moments M_i and forces N_i for Problem 1 is a sub-class of the statically admissible M_i and N_i for Problem 2. Thus any statically admissible stress state for Problem 1 is also admissible for Problem 2. By Theorem 1 the solution to Problem 1 cannot be larger than the solution to Problem 2, i.e.,

$$\lambda_1 \leq \lambda_2 \tag{2.38}$$

where λ_1 and λ_2 are the safety factors for Problems 1 and 2 respectively.

As previously mentioned, the mechanism technique can be immediately extended to solve Problem 2 and therefore to find an upper bound for λ_1. It should be noted that the mechanisms for Problem 2 are not necessarily kinematically possible mechanisms for Problem 1.

The number of basic mechanisms necessary to form a complete set is equal to the number of independent equilibrium equations. In addition to the set B of beam, joint and frame mechanisms defined by Neal and Symonds[1], b more independent mechanisms are required where there are b members in the frame.

An <u>axial failure mechanism</u> is defined as a mechanism containing one and only one yield bar together with sufficient yield hinges so that the frame or some part of it is no longer rigid. There exist therefore b independent axial failure mechanisms and these are independent of the mechanisms of the set B. The set of mechanisms $\overline{B}$ composed of the set B together with b independent axial failure mechanisms is a complete set for Problem 2.

All possible mechanisms are obtained as linear combinations of the set $\overline{B}$, and the combinatorial techniques of Neal and Symonds (Section 1) may be used to examine all combinations of $\overline{B}$ necessary to determine λ_2.

B. <u>Lower Bounds</u>. - The solution of Problem 2 can also be used to find a lower bound for Problem 1. Let $M_j^{(2)}$, $N_j^{(2)}$ be a set of moments and forces statically compatible with λ_2. Define numbers μ_i at each critical cross-section as

$$\mu_i = \text{Maximum} \left\{ 1 - \frac{|N_i^{(2)}|}{N_{pi}} \; , \; \frac{1}{2} \right\} \tag{2.39}$$

, i.e., the larger of the two numbers in brackets. Notice that $\frac{1}{2} \leq \mu_i \leq 1$ for $i = 1, 2, \ldots, n$.

---

[1] There are n - r such mechanisms, where n is the number of critical cross-sections and r is the number of redundancies in the frame.

Consider now a new problem, Problem 3: to maximize λ subject to the equilibrium conditions and the yield criteria

$$\left.\begin{aligned} |M_i| &\le M_{pi}^* \\ |N_i| &\le N_{pi}^* \end{aligned}\right\} \quad (i = 1, 2, \ldots, n) \tag{2.40}$$

where

$$M_{pi}^* = \mu_i M_{pi} \tag{2.41}$$

$$N_{pi}^* = \mu_i N_{pi} \tag{2.42}$$

Problem 3 is, therefore, Problem 2 for a weakened frame.

Let λ_3 be the solution to Problem 3 and let $M_j^{(3)}$, $N_j^{(3)}$ be moments and forces in equilibrium with loads $\lambda_3 b_j$ and satisfying the yield criteria (2.40)

For a cross-section where $\mu_i = \frac{1}{2}$

$$\begin{aligned} |M_i^{(3)}| &\le \tfrac{1}{2} M_{pi} \\ |N_i^{(3)}| &\le \tfrac{1}{2} N_{pi} \end{aligned} \tag{2.43}$$

so

$$\frac{|M_i^{(3)}|}{M_{pi}} + \frac{|N_i^{(3)}|}{N_{pi}} \le 1 \tag{2.44}$$

and (2.36) is satisfied.

At cross-sections where $\mu_i > \frac{1}{2}$ from (2.39)

$$|N_i^{(2)}| = (1 - \mu_i) N_{pi} \tag{2.45}$$

Now from (2.41) and (2.42)

$$\frac{|M_i^{(3)}|}{M_{pi}} + \frac{|N_i^{(3)}|}{N_{pi}} = \mu_i \left[\frac{|M_i^{(3)}|}{M_{pi}^*} + \frac{|N_i^{(3)}|}{N_{pi}^*} \right]$$

and since $M_i^{(3)}$ satisfy (2.40)

$$\frac{|M_i^{(3)}|}{M_{pi}} + \frac{|N_i^{(3)}|}{N_{pi}} \leq \mu_i \left[1 + \frac{|N_i^{(3)}|}{N_{pi}^*} \right] \tag{2.46}$$

Two cases arise: (1) at <u>all</u> cross-sections

$$|N_i^{(3)}| \leq |N_i^{(2)}|$$

or (2) at <u>some</u> cross-section

$$|N_i^{(3)}| > |N_i^{(2)}|$$

If (1) holds then from (2.45) and (2.42)

$$|N_i^{(2)}| \leq (1-\mu_i) N_{pi} = (1-\mu_i) \frac{N_{pi}^*}{\mu_i}$$

thus

$$\frac{|N_i^{(3)}|}{N_{pi}^*} \leq \frac{1-\mu_i}{\mu_i}$$

Therefore in (2.46)

$$\frac{|M_i^{(3)}|}{M_{pi}} + \frac{|N_i^{(3)}|}{N_{pi}} \leq \mu_i \left[1 + \frac{1-\mu_i}{\mu_i} \right] = 1$$

Thus $M_i^{(3)}$, $N_i^{(3)}$ satisfy the yield criteria (2.36) at every cross-section and since they are in equilibrium with loads $\lambda_3 b_j$, λ_3 is a statically admissible multiplier for Problem 1 and

$$\lambda_3 \leq \lambda_1 \tag{2.47}$$

If, however, (2) holds at some cross-section a similar analysis results in

$$\frac{|M_i^{(3)}|}{M_{pi}} + \frac{|N_i^{(3)}|}{N_{pi}} > 1$$

This does not indicate that λ_3 is an upper bound since the multiplier may not be kinematically sufficient. The procedure can be iterated, however, by further weakening the strength at all cross-sections.

Usually one or two iterations are sufficient to achieve a lower bound. If, however, the convergence is slow, a lower bound can be immediately obtained by letting $\mu_i = \frac{1}{2}$ for i = 1, 2, . . ., n. Then

(2.43) and (2.44) hold everywhere. It also follows that

$$\lambda_3 = \tfrac{1}{2}\lambda_2$$

and thus

$$\tfrac{1}{2}\lambda_2 \le \lambda_1 \le \lambda_2$$

Better bounds are obtained in general by using the μ_i defined in (2.39).

An example of the use of these techniques is given in Section 5, Part D of this chapter.

Notice that a different lower bound than λ_3 could be found by defining quantities

$$\bar{\mu}_i = \text{Maximum}\left\{1 - \frac{|M_i^{(2)}|}{M_{pi}},\ \frac{1}{2}\right\}$$

at each cross-section and proceeding in an entirely analagous way with the rolls played by the moments and forces interchanged. For frames where the normalized axial forces, N_i/N_{pi}, are smaller than the normalized moments, M_i/M_{pi}, however the best lower bound is obtained from Problem 3 with the yield criteria (2.40).

C. <u>Distributed Forces</u>. - The methods developed in this section can also be extended to transverse distributed forces.

Following Neal and Symonds [2], an upper bound to Problem 2 can be found by assuming that a hinge appears at the midpoint of the beam in each beam mechanism involving a distributed load. After choosing a collapse mechanism in the usual way, the bound may be improved by letting each hinge appearing under a distributed load be at some distance x_i from the center of the member in which it appears. After computing the multiplier as a function of all of the x_i for beams subjected to distributed loads from the virtual work equation, the multiplier is minimized with respect to each x_i seperately.

Since this is an upper bound for λ_2 it is also greater than λ_1.

To obtain a lower bound, assume on members where distributed loads act a set of concentrated forces whose resultant is the same as that of the distributed load [2] . This can be shown to yield a lower bound to λ_2. Problem 3 is then formulated for this frame which now is subjected to concentrated forces. The solution is a lower bound to λ_1 for distributed loads.

5. Examples.

A. Superposition. - Consider the two-bay frame in Figure 3(a) where the fully plastic moment in each member is h. The 12 critical cross-sections are labeled and the sign convention is chosen so that positive moments cause compression in the fibers adjacent to the dotted lines.

To find an upper bound to the safety factor, λ , choose the mechanism in Figure 3(b). The kinematically sufficient multipliers associated with the loads of magnitude 1, 2, and 3 respectively are

$$\lambda_*^{(1)} = 6 \quad , \quad \lambda_*^{(2)} = 9/2 \quad , \quad \lambda_*^{(3)} = 6$$

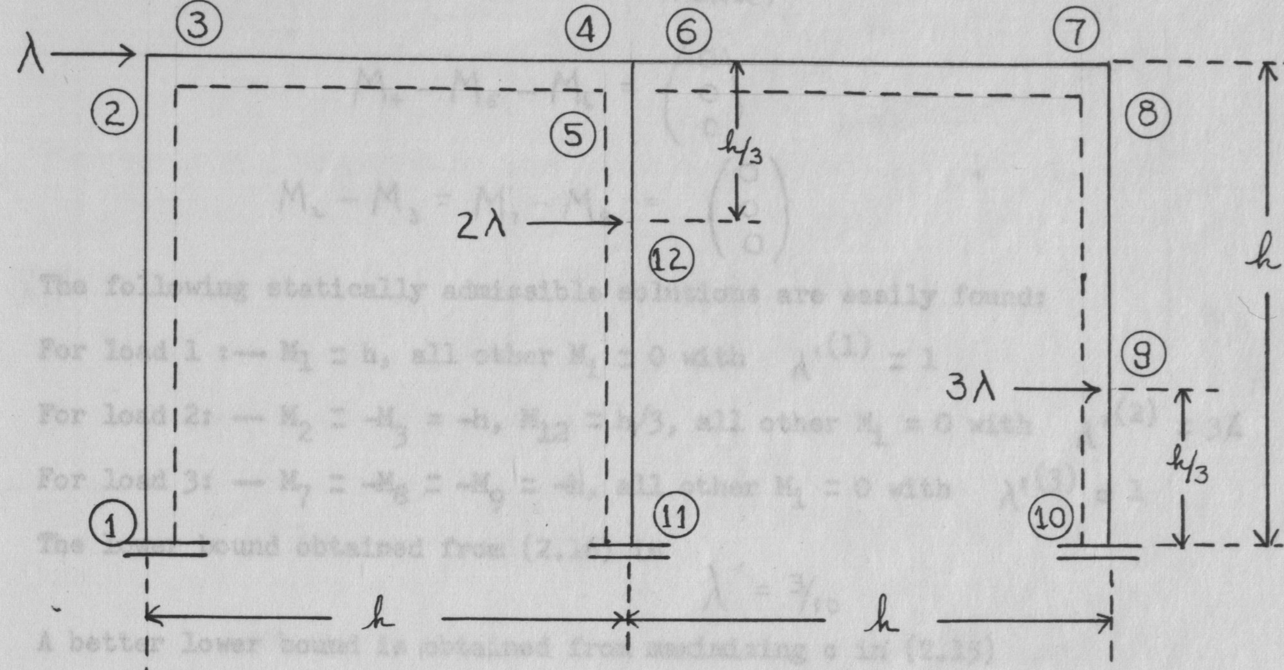
By superposition, an upper bound for the combined loads is

$$\lambda_* = 9/5$$

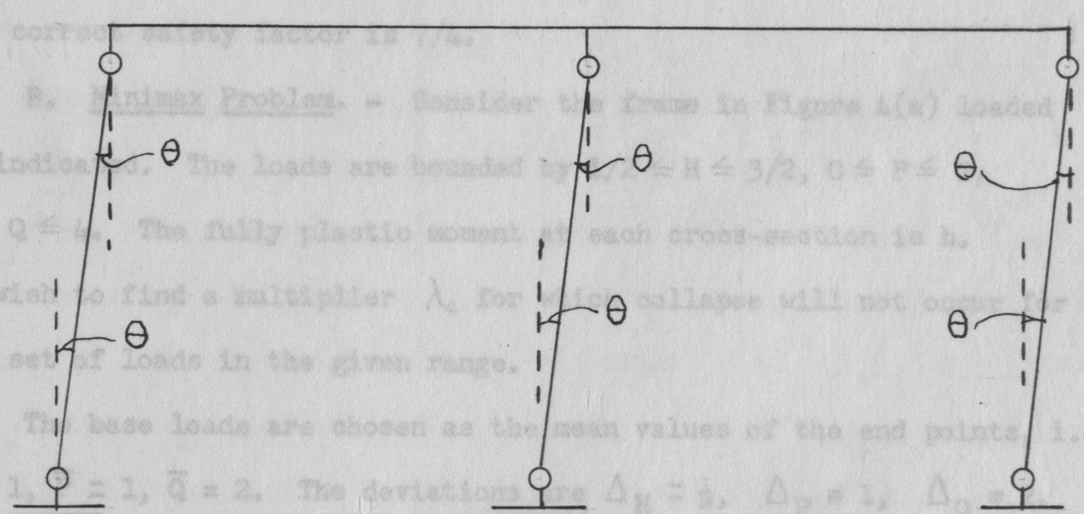
The equilibrium equations for the loads 1, 2, 3 are

$$M_1 - M_2 + M_5 + M_8 - M_{10} - M_{11} = \begin{pmatrix} \lambda h \\ \frac{4}{3}\lambda h \\ \lambda h \end{pmatrix}$$

$$-2M_5 + 3M_{12} - M_{11} = \begin{pmatrix} 0 \\ \frac{4}{3}\lambda h \\ 0 \end{pmatrix}$$



(a)



(b)

FIGURE 3

Plastic Superposition of Forces

$$-M_8 + 3M_9 - 2M_{10} = \begin{pmatrix} 0 \\ 0 \\ 2\lambda h \end{pmatrix}$$

$$M_4 - M_5 - M_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$M_2 - M_3 = M_7 - M_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The following statically admissible solutions are easily found:

For load 1 :-- $M_1 = h$, all other $M_i = 0$ with $\lambda'^{(1)} = 1$

For load 2: -- $M_2 = -M_3 = -h$, $M_{12} = h/3$, all other $M_i = 0$ with $\lambda'^{(2)} = 3/4$

For load 3: -- $M_7 = -M_8 = -M_9 = -h$, all other $M_i = 0$ with $\lambda'^{(3)} = 1$

The lower bound obtained from (2.16) is

$$\lambda' = 3/10$$

A better lower bound is obtained from maximizing c in (2.15)

$$c = 1$$

Thus

$$1 \leq \lambda \leq 9/5$$

The correct safety factor is 7/4.

B. <u>Minimax Problem</u>. - Consider the frame in Figure 4(a) loaded as indicated. The loads are bounded by $1/2 \leq H \leq 3/2$, $0 \leq P \leq 2$, $0 \leq Q \leq 4$. The fully plastic moment at each cross-section is h. We wish to find a multiplier λ_c for which collapse will not occur for any set of loads in the given range.

The base loads are chosen as the mean values of the end points, i.e., $\overline{H} = 1$, $\overline{P} = 1$, $\overline{Q} = 2$. The deviations are $\Delta_H = \frac{1}{2}$, $\Delta_P = 1$, $\Delta_Q = 2$. The three basic mechanisms are shown in Figures 4(b), (c), (d). The two combinations of these which follow from the Neal-Symonds analysis $\{$(b) added to (d), and (c) added to (d)$\}$ are not shown. The multipliers associated with the base loads and the normalized angles are

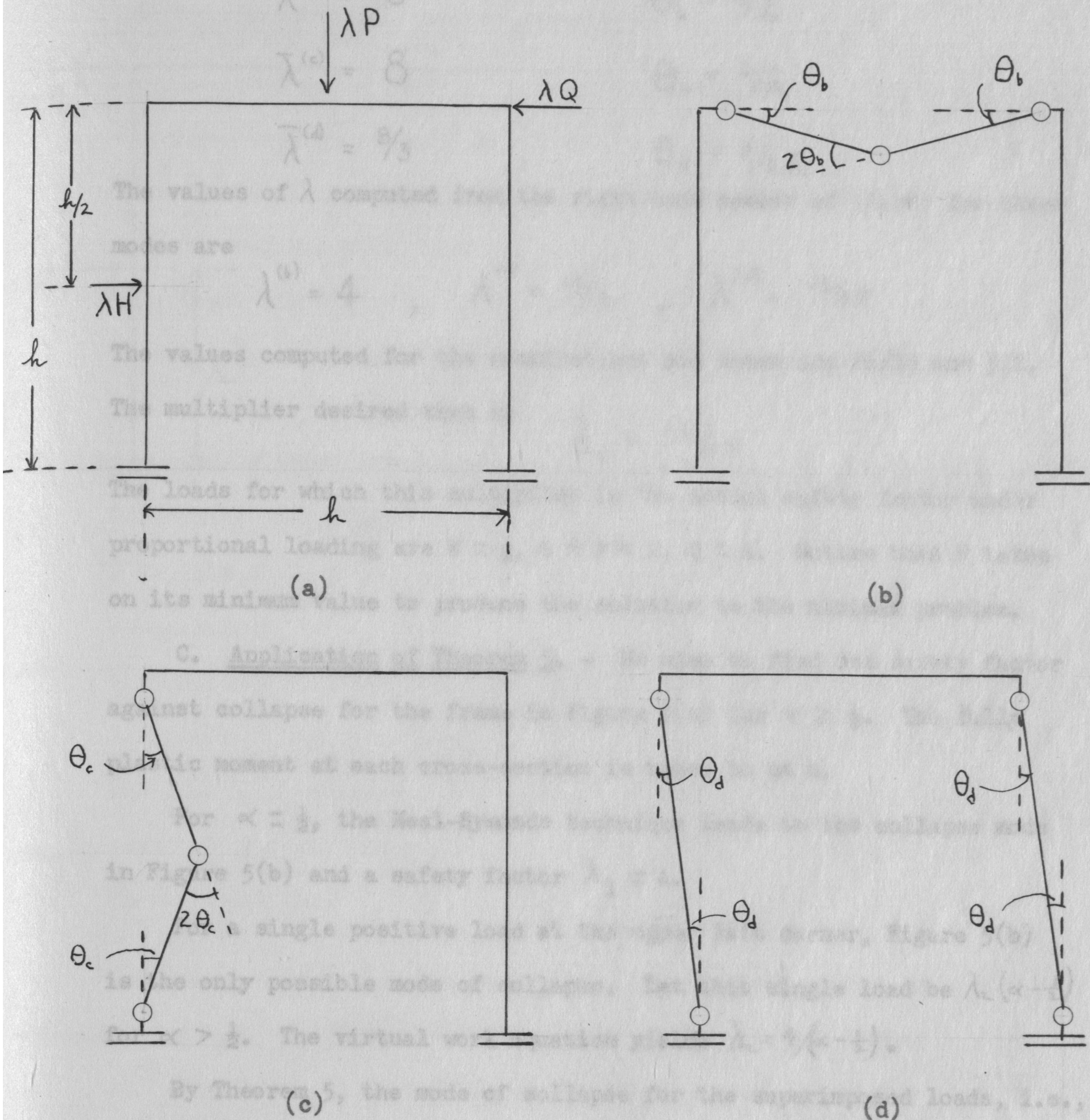


FIGURE 4

A Minimax Problem

$$\bar{\lambda}^{(b)} = 8 \qquad\qquad \theta_b = 2/h$$

$$\bar{\lambda}^{(c)} = 8 \qquad\qquad \theta_c = 2/h$$

$$\bar{\lambda}^{(d)} = 8/3 \qquad\qquad \theta_d = 2/3h$$

The values of λ computed from the right hand member of (2.28) for these modes are

$$\lambda^{(b)} = 4 \quad , \quad \lambda^{(c)} = 16/3 \quad , \quad \lambda^{(d)} = 16/15$$

The values computed for the combinations not shown are 24/19 and 3/2. The multiplier desired then is

$$\lambda_c = 16/15$$

The loads for which this multiplier is the actual safety factor under proportional loading are $H = \frac{1}{2}$, $0 \leq P \leq 2$, $Q = 4$. Notice that H takes on its minimum value to produce the solution to the minimax problem.

C. <u>Application of Theorem 5</u>. - We wish to find the safety factor against collapse for the frame in Figure 5(a) for $\alpha \geq \frac{1}{2}$. The fully plastic moment at each cross-section is taken to be h.

For $\alpha = \frac{1}{2}$, the Neal-Symonds technique leads to the collapse mode in Figure 5(b) and a safety factor $\lambda_1 = 4$.

For a single positive load at the upper left corner, Figure 5(b) is the only possible mode of collapse. Let this single load be $\lambda_2(\alpha - \frac{1}{2})$ for $\alpha > \frac{1}{2}$. The virtual work equation yields $\lambda_2 = 4/(\alpha - \frac{1}{2})$.

By Theorem 5, the mode of collapse for the superimposed loads, i.e., $\alpha\lambda$ at the upper left and λ at the midpoint of the left leg, is also that shown in Figure 5(b) and the safety factor is

$$\lambda = \frac{8}{2\alpha + 1} \qquad\qquad \alpha > \frac{1}{2}$$

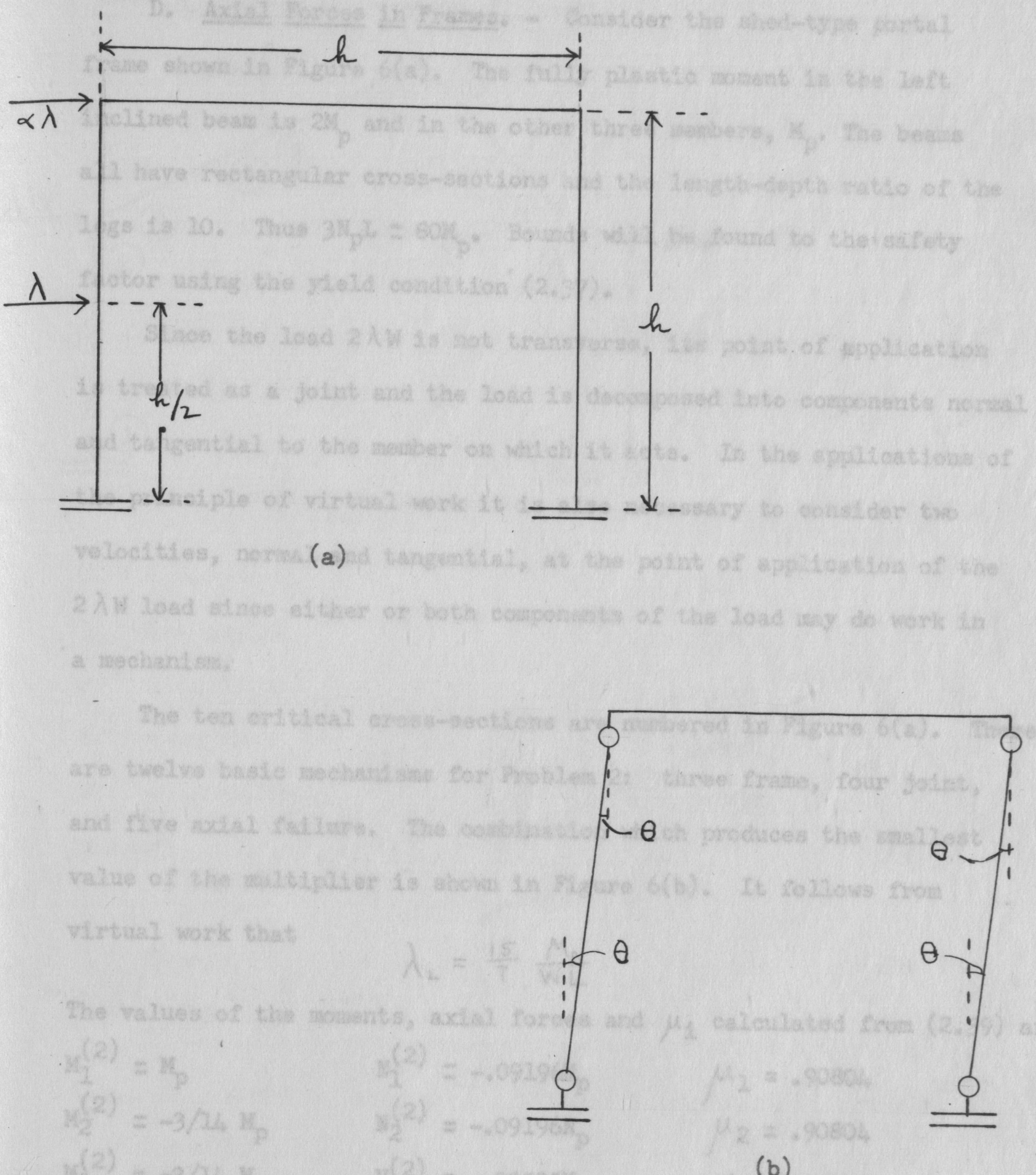


FIGURE 5

Application of Theorem 5

D. Axial Forces in Frames. - Consider the shed-type portal frame shown in Figure 6(a). The fully plastic moment in the left inclined beam is $2M_p$ and in the other three members, M_p. The beams all have rectangular cross-sections and the length-depth ratio of the legs is 10. Thus $3N_pL = 80M_p$. Bounds will be found to the safety factor using the yield condition (2.37).

Since the load $2\lambda W$ is not transverse, its point of application is treated as a joint and the load is decomposed into components normal and tangential to the member on which it acts. In the applications of the principle of virtual work it is also necessary to consider two velocities, normal and tangential, at the point of application of the $2\lambda W$ load since either or both components of the load may do work in a mechanism.

The ten critical cross-sections are numbered in Figure 6(a). There are twelve basic mechanisms for Problem 2: three frame, four joint, and five axial failure. The combination which produces the smallest value of the multiplier is shown in Figure 6(b). It follows from virtual work that

$$\lambda_2 = \frac{15}{7}\frac{M_p}{WL}$$

The values of the moments, axial forces and μ_i calculated from (2.39) are

$M_1^{(2)} = M_p$ $\quad N_1^{(2)} = -.09196N_p$ $\quad \mu_1 = .90804$

$M_2^{(2)} = -3/14\ M_p$ $\quad N_2^{(2)} = -.09196N_p$ $\quad \mu_2 = .90804$

$M_3^{(2)} = -3/14\ M_p$ $\quad N_3^{(2)} = -.08585N_p$ $\quad \mu_3 = .95708$

$M_4^{(2)} = -2\ M_p$ $\quad N_4^{(2)} = -.08585N_p$ $\quad \mu_4 = .95708$

$M_5^{(2)} = -2\ M_p$ $\quad N_5^{(2)} = -.01397N_p$ $\quad \mu_5 = .99301$

$M_6^{(2)} = \frac{1}{2}\ M_p$ $\quad N_6^{(2)} = -.01397N_p$ $\quad \mu_6 = .99301$

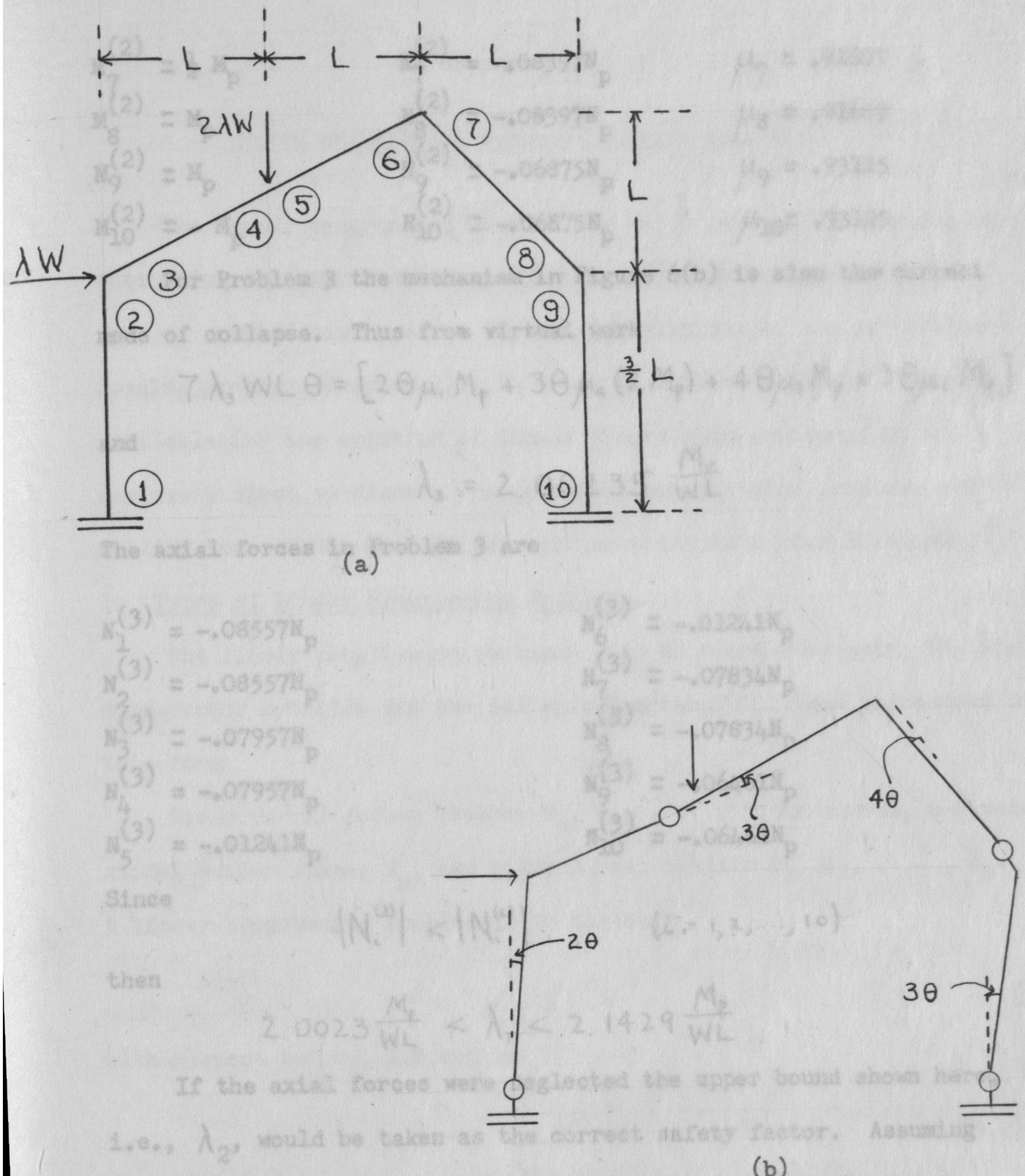


FIGURE 6

Axial Force Effects in Frames

$M_7^{(2)} = \frac{1}{2} M_p$ $N_7^{(2)} = -.08397N_p$ $\mu_7 = .91603$

$M_8^{(2)} = M_p$ $N_8^{(2)} = -.08397N_p$ $\mu_8 = .91603$

$M_9^{(2)} = M_p$ $N_9^{(2)} = -.06875N_p$ $\mu_9 = .93125$

$M_{10}^{(2)} = - M_p$ $N_{10}^{(2)} = -.06875N_p$ $\mu_{10} = .93125$

For Problem 3 the mechanism in Figure 6(b) is also the correct mode of collapse. Thus from virtual work

$$7\lambda_3 WL\theta = [2\theta\mu_1 M_p + 3\theta\mu_4 (2M_p) + 4\theta\mu_8 M_p + 3\theta\mu_{10} M_p]$$

and

$$\lambda_3 = 2.00235 \frac{M_p}{WL}$$

The axial forces in Problem 3 are

$N_1^{(3)} = -.08557N_p$ $N_6^{(3)} = -.01241N_p$

$N_2^{(3)} = -.08557N_p$ $N_7^{(3)} = -.07834N_p$

$N_3^{(3)} = -.07957N_p$ $N_8^{(3)} = -.07834N_p$

$N_4^{(3)} = -.07957N_p$ $N_9^{(3)} = -.06461N_p$

$N_5^{(3)} = -.01241N_p$ $N_{10}^{(3)} = -.06461N_p$

Since

$$|N_i^{(3)}| < |N_i^{(2)}| \qquad (i = 1, 2, \ldots, 10)$$

then

$$2.0023 \frac{M_p}{WL} < \lambda_1 < 2.1429 \frac{M_p}{WL}$$

If the axial forces were neglected the upper bound shown here, i.e., λ_2, would be taken as the correct safety factor. Assuming that the solution is the mean of the two bounds, then the error committed by neglecting the axial forces is approximately 3.5 %.

Chapter III

LINEAR PROGRAMMING METHODS IN LIMIT ANALYSIS

The linear programming problem may be defined as the problem of optimizing a linear functional subject to linear constraints. This type of problem has appeared in both formulations of the proportional loading problem (Chapter I). In order to utilize the special methods available for the solution of linear programming problems, it is necessary first to discuss the general types of these problems and to outline some of the methods of solution which have been developed.

1. Types of Linear Programming Problems.

The linear programming problems will be formulated using the standard vector notation and the collapse problems will then be reduced to this form.

Given $(n+1)$ column vectors P_o, P_1, . . ., P_n in a real, m-dimensional vector space, V_m, and given n real scalars c_1, c_2, . . ., c_n. A linear programming problem is to minimize

$$z_o = \sum_{j=1}^{n} \rho_j c_j \tag{3.1}$$

with respect to ρ_j, subject to

$$P_o = \sum_{j=1}^{n} \rho_j P_j \tag{3.2}$$

$$\rho_j \geq 0 \qquad (j=1,2,\ldots,n) \tag{3.3}$$

This will be called the simplex problem or a problem of Type I.

The dual problem to the simplex problem stated in (3.1), (3.2), (3.3) is to maximize

$$P_0' w \tag{3.4}$$

where

$$P_j' w \leq c_j \qquad (j = 1, 2, \ldots, n) \tag{3.5}$$

The prime denotes transpose and w is a real m-tuple (or alternatively a vector in V_m). $P_j'w$ is the inner product of P_j and w considered as vectors. This problem is called a problem of Type II.

The dual theorem[1] relating the above problems states that if either the minimum of z_0 or the maximum of $P_0'w$ exists and is finite, then

$$\text{Minimum } z_0 = \text{Maximum } P_0' w \tag{3.6}$$

Type III problems are defined to be identical with Type I problems except that every variable is constrained by an upper bound, i.e.,

$$0 \leq \rho_j \leq b_j \qquad (j = 1, 2, \ldots, n) \tag{3.7}$$

This is alternately called a <u>bounded variables problem</u>. It includes the type of problems where some but not all of the variables are bounded since one may prescribe arbitrarily large bounds for the unbounded variables.

2. <u>Computational Techniques</u>.

Three finite iterative techniques for solving the above problems will be discussed here. They are included for completeness and for ease of reference in the limit analysis examples solved later by these methods.

A. <u>The Simplex Method</u>. - A problem of Type I is naturally adapted

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[1] See for example Chapter VIII of [10].

to the simplex technique. A complete discussion of this method may be found in [9, 10]. Two assumptions will be made: (1) that the solution z_o is finite[1] and (2) that P_o is linearly independent of any m - 1 vectors selected from among the P_j. The latter assumption avoids the degenerate cases. Such occurrences, however, can be handled by the ϵ-procedure of Charnes [10].

From these assumptions it follows that there exists a set of m of the vectors P_j which are linearly independent and therefore form a basis for the vector space, V_m. Let such a set of basis vectors be designated by a_i for i = 1, 2, . . ., m. The vector P_o may then be expressed as a linear combination of the a_i. This representation for P_o and the corresponding value of z_o will be a feasible solution to the simplex problem in (3.1), (3.2), (3.3) provided the coefficients of the a_i in (3.2) are all positive, i.e., $P_o'a_i > 0$. The second assumption assures that a basis satisfying this constraint exists. A method of determining one such basis and hence a feasible solution is given in Appendix C.

Consider now m vectors α_j[2] in V_m such that

$$a_i'\alpha_j = \delta_{ij} \qquad (i, j = 1, 2, \ldots, m)$$

The α_j are uniquely determined since the a_i are linearly independent.

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[1] It will be shown later that this is indeed the case for the collapse problems considered.

[2] The matrix $[\alpha_1, \alpha_2, \ldots, \alpha_m]$ is the transpose of the inverse to the matrix $[a_1, a_2, \ldots, a_m]$. Note also that in [11] the α_j are referred to as a^j.

Then the P_j and P_o may be written

$$P_j = \sum_{i=1}^{m} \frac{(P_j'a_i)a_i}{a_i'a_i} = \sum_{i=1}^{m}(P_j'\alpha_i)a_i \qquad (j=1,2,\ldots,n) \tag{3.8}$$

$$P_o = \sum_{i=1}^{m} \frac{(P_o'a_i)a_i}{a_i'a_i} = \sum_{i=1}^{m}(P_o'\alpha_i)a_i \tag{3.9}$$

where $P_o'\alpha_i > 0$ for $i = 1, 2, \ldots, m$. Hence

$$\rho_i = \begin{cases} P_o'\alpha_i & i=1,2,\ldots,m \\ 0 & i=m+1,\ldots,n \end{cases} \tag{3.10}$$

is a feasible solution to the problem. The corresponding value of the functional is

$$z_o = \sum_{i=1}^{m}(P_o'\alpha_i)c_i \tag{3.11}$$

In order to obtain another feasible solution which yields a smaller value of the functional, consider first the n scalar quantities

$$z_j = \sum_{i=1}^{m}(P_j'\alpha_i)c_i \tag{3.12}$$

Either (i) $z_j - c_j \leq 0$ for <u>all</u> j, or (ii) $z_j - c_j > 0$ for <u>some</u> j. If (i) holds then

$$P_j' \sum_{i=1}^{m} \alpha_i c_i \leq c_j \qquad (j=1,2,\ n)$$

Therefore the point $\omega = \sum_{i=1}^{m} \alpha_i c_i$ in V_m is a feasible solution to the dual problem, (3.5). The value of the functional, z_o, is

$$z_o = \sum_{i=1}^{m}(P_o'\alpha_i)c_i = P_o'\sum_{i=1}^{m}\alpha_i c_i = P_o'\omega$$

However from the dual theorem $P_o'w \leq z_o$ and therefore z_o takes on its minimum and $P_o'w$ its maximum whenever equality holds. Thus if (i) holds the solution is optimum and the value of z_o in (3.11) is the minimum value of the functional and cannot be further decreased.

Suppose, therefore, that (ii) holds for some j, say j = k. Then rewrite (3.9) as

$$P_0 = \sum_{i=1}^{m} (P_0' \alpha_i) a_i - \Theta P_k + \Theta P_k$$

where $\Theta > 0$ and P_k is not an a_i. Using (3.8) with $j = k$,

$$P_0 = \sum_{i=1}^{m} (P_0' \alpha_i - \Theta P_k' \alpha_i) a_i + \Theta P_k \tag{3.13}$$

If the coefficients of a_i, P_k are non-negative, then the set of ρ_j defined by

$$\rho_j = \begin{cases} P_0' \alpha_j - \Theta P_k' \alpha_j & j = 1, 2, \dots, m \\ \Theta & j = k \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

are a feasible set for the simplex problem. The value of the functional associated with this feasible set is

$$z_0 = \sum_{i=1}^{m} (P_0' \alpha_i) c_i - \Theta \left[\sum_{i=1}^{m} (P_k' \alpha_i) c_i - c_k \right]$$

$$z_0 = \sum_{i=1}^{m} (P_0' \alpha_i) c_i - \Theta (z_k - c_k) \tag{3.15}$$

Now if $P_k' \alpha_i \leq 0$ for all i, then the ρ_j defined in (3.14) are positive for arbitrarily large positive Θ. Since $z_k - c_k > 0$, the value of the functional z_0 given in (3.15) can be made arbitrarily small contrary to assumption (1). Therefore $P_k' \alpha_i > 0$ for some i.

Since the coefficients of a_i in (3.13) must be non-negative

$$\Theta \leq \frac{P_0' \alpha_i}{P_k' \alpha_i} \quad \text{for } P_k' \alpha_i > 0$$

Therefore in (3.15) the smallest permissible value of the functional is obtained if Θ is chosen as

$$\Theta = \text{Minimum } \frac{P_0' \alpha_i}{P_k' \alpha_i} \quad \text{for } P_k' \alpha_i > 0$$

This minimum is taken on for one and only one value of i, say $i = s$, because of assumption (2). For this value of θ the coefficient of a_s in (3.13) vanishes and

$$P_o = \sum_{\substack{i=1 \\ i \neq s}}^{m} (P_o'\alpha_i - \theta P_k'\alpha_i) a_i + \theta P_k$$

The a_i, $i \neq s$, and P_k are easily shown to form a basis for V_m[1].

Notice that if (ii) holds for more than one value of j, there is a choice as to which vector shall enter the basis. Any choice will result in a decrease in the functional. The choice may be governed by experience and the physical interpretation of the problem. This will be discussed later in the structures examples.

Finally notice that the vectors α_j were not needed explicitly in the above analysis. It is only necessary to express the P_o, P_j in terms of the a_i, i.e., to find $P_o'\alpha_i$ and $P_j'\alpha_i$. P_o, P_j may be expressed in terms of the new basis, a_i for $i \neq s$ and P_k, by the algorithm

$$P_j = \sum_{\substack{i=1 \\ i \neq s}}^{m} \left[P_j'\alpha_i - \frac{P_j'\alpha_s}{P_k'\alpha_s} P_k'\alpha_i \right] a_i + \frac{P_j'\alpha_s}{P_k'\alpha_s} P_k \qquad (j = 0, 1, \ldots, n) \tag{3.16}$$

The coefficients on the right involve only quantities already computed. The z_j may then be formed as before and the entire process iterated. Only a finite number of bases, in fact at most $\binom{n}{m}$[2], are possible and no basis will reappear since the functional decreases at each iteration.

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[1]See for example, Lecture IV of [10].

[2]$\binom{n}{m}$ represents the number of combinations of n things taken m at a time.

Consequently the process converges in a finite number of iterations.

To proceed from one iteration to the next it is convenient to assemble the information in tableau form[1] as follows:

$c_j \rightarrow$			c_1	. . . .	c_k	. . . .	c_n
$\downarrow$	a_i	P_o	P_1	. . . .	P_k	. . . .	P_n
c_1	a_1	$P'_o\alpha_1$	$P'_1\alpha_1$	. . . . . .	$P'_k\alpha_1$	. . . . . .	$P'_n\alpha_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_s	a_s	$P'_o\alpha_s$	$P'_1\alpha_s$	. . . . . .	$P'_k\alpha_s$	. . . . . .	$P'_n\alpha_s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_m	a_m	$P'_o\alpha_m$	$P'_1\alpha_m$	. . . . .	$P'_k\alpha_m$	. . . . .	$P'_n\alpha_m$
		z_o	$z_1 - c_1$	. . . . . .	$z_k - c_k$	. . . . . .	$z_n - c_n$

TABLE I

Simplex Tableau

The entry in the row labeled a_s $(s = 1, 2, . . ., m)$ and column P_k $(k = 1, 2, . . ., n)$ is $P'_k\alpha_s$, the component of P_k along the vector a_s. The entry z_k at the base of a column is computed by taking the scalar product of the entries in that column with the column of c_j at the left. The entries in the P_o column constitute values of the variables ρ_j corresponding to the vectors a_i. All other ρ_j vanish at this stage.

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[1] This arrangement was developed by Orden, Dantzig, and Hoffman.

A positive element is selected from the last row (if none exist an optimum value of the functional is given by z_o). The positive entries in that column are divided into the corresponding entries in the P_o column, and the minimum of these quotients is selected as θ. To proceed to a new tableau the algorithm (3.16) is used.

In each iteration there are in general $(m + 1)(n + 1)$ multiplications required to complete the new tableau once the replaced and replacing vectors have been chosen.

B. <u>The Dual Method</u>. - A Type II Problem is well suited to solution by the dual method [11]. This method will be briefly reviewed here.

A point w_o is an extreme point of the set w satisfying (3.5) if from among the vectors P_j for which equality is satisfied in $P_j' w_o \leq c_j$, there exist m vectors which are linearly independent and hence form a basis for V_m. It can be shown that the functional $P_o' w$ takes on its maximum at an extreme point of the set w satisfying (3.5).

It will be assumed that (1) the maximum value of the functional (3.4) is finite and also that (2) for every extreme point equality is satisfied in <u>exactly</u> m of (3.5). This latter assumption avoids the problem of "dual degeneracy". For a discussion of this case see Appendix I of [11].

Consider an extreme point w_o. Let the m P_j for which equality is satisfied in (3.5) be designated by a_i, i.e., $a_i w = c_i$ for $i = 1, \ldots, m$. The a_i are then a basis for V_m and the P_j may be expressed in terms of this basis as in (3.8) and (3.9).

Two cases arise: (i) $P_o' \alpha_i \geq 0$ for <u>all</u> i or (ii) $P_o' \alpha_i < 0$ for <u>some</u> i.

Now if (i) holds then the value of the functional associated with the point w_o is

$$P_o' w_o = \sum_{i=1}^{m} (P_o' \alpha_i) a_i w_o = \sum_{i=1}^{m} (P_o' \alpha_i) c_i$$

Now defining n scalars ρ_j as

$$\rho_j = \begin{cases} P_o' \alpha_j & j = 1, 2, \ldots, m \\ 0 & j = m+1, \ldots, n \end{cases}$$

It follows from (3.1) that

$$P_o' w_o = \sum_{j=1}^{n} \rho_j c_j = z_o$$

and moreover

$$P_o = \sum_{j=1}^{n} \rho_j P_j$$

$$\rho_j \geq 0$$

The ρ_j are, therefore, a feasible solution to the problem in (3.1), (3.2), (3.3) and by the dual theorem then $P_o'w_o$ is the maximum value of the functional for the Type II problem defined in (3.4), (3.5).

Suppose, therefore, that (ii) holds for some r, i.e., $P_o'\alpha_r < 0$. Let

$$\bar{w} = w_o - \phi \alpha_r \qquad , \phi \geq 0 \tag{3.17}$$

where ϕ is chosen so that

$$P_j' \bar{w} \leq c_j \qquad (j = 1, 2, \ldots, n) \tag{3.18}$$

The value of the functional (3.4) associated with the point $\bar{w}$ is

$$P_o' \bar{w} = P_o' w_o - \phi P_o' \alpha_r \geq P_o' w_o \tag{3.19}$$

Now if $P_j'\alpha_r \geq 0$ for all j, then

$$P_j' \bar{w} = P_j' w_o - \phi P_j' \alpha_r \leq P_j' w_o \leq c_j$$

for any $\phi > 0$, i.e., $\bar{w}$ satisfies (3.18) for arbitrarily large positive ϕ. In (3.19), therefore, the functional $P_o'\bar{w}$ may be made arbitrarily large contrary to assumption (i).

Thus for some j, $P_j'\alpha_r < 0$. From (3.17) and (3.18) therefore it is necessary that

$$\phi \leq \frac{P_j'w_o - c_j}{P_j'\alpha_r} \qquad \text{for } P_j'\alpha_r < 0$$

The largest permissible increase in the functional is obtained then if ϕ is chosen as

$$\phi = \text{Minimum } \frac{P_j'w_o - c_j}{P_j'\alpha_r} \qquad , P_j'\alpha_r < 0$$

If ϕ takes on its minimum for j = q, then from (3.17)

$$P_q'\bar{w} = P_q'w_o - \phi(P_q'\alpha_r) = c_q$$

Moreover for i = 1, 2, . ., r-1, r+1, . ., m

$$a_i'\bar{w} = a_i'w_o - \phi(a_i'\alpha_r) = a_i'w_o = c_i$$

and thus $\bar{w}$ is an extreme point. By assumption (2) it also follows that ϕ takes on its minimum for a unique value of j, i.e., j = q. The vectors $a_1, \ldots, a_{r-1}, P_q, a_{r+1}, \ldots, a_m$ form the basis for V_m associated with the extreme point $\bar{w}$.

The algorithm for computing the vectors P_o, P_j in terms of this new basis is given by (3.16) if r replaces s and q replaces k.

Recall that the points w_o, $\bar{w}$ were not needed explicitly in the analysis. It is sufficient to compute $P_j'w_o$ and $P_j'\bar{w}$ for j = 0, 1, . ., n.

The tableau arrangement is identical with that for the simplex method (Table I). The quantities in the last row $(z_j - c_j)$ are now identified as $P_j'w - c_j$.

The procedure, however, is to select a negative element in the P_o column. The negative entries in that row are divided into the corresponding entries in the last row, and the minimum of these quotients is selected as ϕ. The algorithm (3.16) is then used to find the entries of the new tableau.

If $P_0'\alpha_i < 0$ for more than one value of i then there is a certain freedom in choosing the vector to leave the basis. This will be discussed later in the structures example.

The number of multiplications per iteration is identical with that for the simplex technique applied to the dual problem.

C. <u>The Bounded Variables Technique</u>. - The problem stated in (3.1), (3.2), (3.7) can be transformed into a simplex problem by introducing non-negative variables x_j such that

$$\rho_j + x_j = b_j$$

Now if b is the vector in V_n whose jth component is b_j and if Q_j is a unit vector in V_n with a 1 as the jth component and all others zero, we define the following vectors in $V_{m\,n}$

$$\overline{P}_o = \begin{bmatrix} P_o \\ b \end{bmatrix}, \quad \overline{P}_j = \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \quad \overline{Q}_j = \begin{bmatrix} 0 \\ Q_j \end{bmatrix} \quad (j = 1, 2, . ., n)$$

The bounded variables problem may then be written:

Maximize

$$z_o = \sum_{j=1}^{n} \rho_j c_j$$

subject to

$$P_o = \sum_{j=1}^{n} \rho_j \overline{P}_j + \sum_{j=1}^{n} x_j \overline{Q}_j$$

$$\rho_j \geq 0$$
$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n)$$

This is a problem of Type I of size (m + n) x (2n).

Charnes and Lemke [12] have shown that this may be treated as an m x n problem, i.e., the inequalities, $\rho_j \leq b_j$, may be suppressed.

A brief outline of the computational procedure will be given here. The motivation and rigorous treatment can be found in [12].

Select first a basis, a_i, for V_m from among the P_j as before in the simplex method, and compute $P'_j \alpha_i$ for $j = 0, 1, 2, \ldots, n$. Then the vector $\overline{P}_j$ corresponding to the a_i and all of the $\overline{Q}_j$ for $j = 1, 2, \ldots, n$ constitute a basis for V_{m+n}. Designate the basis for V_{m+n} by B_{m+n}. Compute the quantities

$$\left.\begin{aligned} \varphi_i &= P_0'\alpha_i \\ &b_i - \varphi_i \end{aligned}\right\} \qquad (i = 1, 2, \ldots, m)$$

$$z(P_j) = \sum_{i=1}^{m} (P_j'\alpha_i)\, c_i - c_j \qquad (j = 1, 2, \ldots, n)$$

The above information is then assembled in the following tableau:

$b_j \longrightarrow$				$b_1 \ldots\ldots b_m$	$b_{m+1} \ldots\ldots b_n$
$c_i \longrightarrow$				$c_1 \ldots\ldots c_m$	$c_{m+1} \ldots\ldots c_n$
$\downarrow$	B_m	P_o		$P_1 \ldots\ldots P_m$	$P_{m+1} \ldots\ldots P_n$
c_1	a_1	φ_1	$b_1 - \varphi_1$	$P'_1\alpha_1 \ldots\ldots P'_m\alpha_1$	$P'_{m+1}\alpha_1 \ldots\ldots P'_n\alpha_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \qquad \vdots$	$\vdots \qquad \vdots$
c_r	a_r	φ_r	$b_r - \varphi_r$	$P'_1\alpha_r \ldots\ldots P'_m\alpha_r$	$P'_{m+1}\alpha_r \ldots\ldots P'_n\alpha_r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \qquad \vdots$	$\vdots \qquad \vdots$
c_m	a_m	φ_m	$b_m - \varphi_m$	$P'_1\alpha_m \ldots\ldots P'_m\alpha_m$	$P'_{m+1}\alpha_m \ldots\ldots P'_n\alpha_m$
		z_o		$z(P_1) \ldots\ldots z(P_m)$	$z(P_{m+1}) \ldots\ldots z(P_n)$
$B_{m+n} \longrightarrow$				$+,- \ldots\ldots +,-$	$- \ldots\ldots -$

TABLE II

Bounded Variables Tableau

The (+, −) sign in the last row under the P_j column indicates that both $\overline{P}_j$ and $\overline{Q}_j$ are in B_{m+n}, while a single + (or −) means that $\overline{P}_j$ (or $\overline{Q}_j$) is in the basis, B_{m+n}.

An optimum solution has been reached if both

(a) $z(P_j) \geqslant 0$ in each column having the sign − in the final entry and

(b) $z(P_j) \leq 0$ in each column having the sign + in the final entry.

If either of the above are violated then the following procedure is used to increase z_o.

<u>Case I</u>: $z(P_k) > 0$ when a + sign appears under this quantity. Then $\overline{Q}_k$ enters B_{m+n}. Choose

$$\Theta = \text{Minimum}\begin{cases} (i)\ \text{Min}\ \dfrac{\varphi_i}{(-P_k'\alpha_i)}, & P_k'\alpha_i < 0 \\ (ii)\ \text{Min}\ \dfrac{b_i - \varphi_i}{P_k'\alpha_i}, & P_k'\alpha_i > 0 \\ (iii)\ b_k \end{cases}$$

If the minimum Θ occurs for $i = q$ in (i) then $\overline{P}_q$ leaves B_{m+n}. If this minimum appears for $i = q$ in (ii) then $\overline{Q}_q$ leaves the basis. Finally if $\Theta = b_k$ then $\overline{P}_k$ is removed.

<u>Case II</u>: $z(P_k) < 0$ when a − sign appears. Then $\overline{P}_k$ enters B_{m+n}. Let

$$\Theta = \text{Minimum}\begin{cases} (i)\ \text{Min}\ \dfrac{\varphi_i}{P_k'\alpha_i}, & P_k'\alpha_i > 0 \\ (ii)\ \text{Min}\ \dfrac{b_i - \varphi_i}{(-P_k'\alpha_i)}, & P_k'\alpha_i < 0 \\ (iii)\ b_k \end{cases}$$

If the minimum θ occurs in (i) for $i = q$ then $\overline{P}_q$ is replaced; if in (ii) for $i = q$, then $\overline{Q}_q$ is replaced. Finally if $\Theta = b_k$, then $\overline{Q}_k$ is removed.

To proceed to a new tableau three cases are distinguished:

Case A: $\overline{Q}_k$ replaces $\overline{P}_k$. There is no change in B_m, but (1) the + sign under P_k is changed to a −, (2) φ_i is replaced by $\varphi_i + b_k(P_k'\alpha_i)$, and (3) z_o is replaced by $z_o + b_k z(P_k)$.

Case B: $\overline{P}_k$ replaces $\overline{Q}_k$. There is no change in B_m, but (1) the − under P_k is changed to a +, (2) φ_i is replaced by $\varphi_i - b_k(P_k'\alpha_i)$, and (3) z_o is replaced by $z_o - b_k z(P_k)$.

Case C: Either $\overline{P}_k$ or $\overline{Q}_k$ replaces either $\overline{P}_q$ or $\overline{Q}_q$. Then P_k replaces P_q in B_m. The tableau changes are:

(1) P_k replaces P_q in the B_m column and c_k replaces c_q.

(2) Both a + and − appear under P_k. The P_q column has a + if $\overline{Q}_q$ has been replaced or a − if $\overline{P}_q$ has left B_{m+n}.

(3) φ_i is replaced by $\varphi_i - \theta(P_k'\alpha_i)$ for $i \neq q$. φ_q is replaced by θ and $b_q - \varphi_q$ is replaced by $b_k - \theta$.

(4) The P_j are expressed by the algorithm (3.16).

This completes the new tableau and the process is iterated.

The number of multiplications per iteration is $(m + 1)(n + 1)$ for Case C.

It should be noted that the "modified" simplex and dual methods [5] may be used in all of the techniques outlined in this section. This modified technique has the advantage of controlling round-off errors without adding to the number of computations.

3. Collapse Under Proportional Loading.

The problem of finding the safety factor against collapse for proportionally loaded frames has been formulated in (1.1) and (1.2). We turn now to the formulation of the same problem for pin-jointed trusses, which furnish simple examples for the purpose of the illustration of the use of linear programming methods. This is due to the fact that pin-jointed trusses may have a single degree of redundancy while frames cannot.

A. Equilibrium Equations and Yield Criteria. - Consider a plane truss with no external redundancies and composed of s bars and k joints. If this truss is subjected to a finite number of concentrated loads at the joints, the equilibrium equations may be written[1]

$$\sum_{j=1}^{s} a_{ij} S_j = \lambda p_i \qquad (i = 1, 2, \ldots, 2k-3) \tag{3.20}$$

where S_j is the axial force in the jth bar considered positive for tensile forces, p_i are the fixed loads, and λ is the multiplier common to each load. The a_{ij} depend on the geometrical configuration of the truss and are direction cosines of the angles between the bars and the coordinate axes. If $s > 2k - 3$ then the truss is redundant and equations (3.20) admit a non-trivial solution.

The yield conditions are

$$-L_j \leq S_j \leq U_j \qquad (j = 1, 2, \ldots, s) \tag{3.21}$$

where U_j $(-L_j)$ is the fully plastic force in tension (compression).

A value of λ for which there exist S_j satisfying (3.20) and (3.21) is a statically admissible multiplier. By Theorem 1, therefore, the largest value of λ for which a solution S_j exists, is the safety factor against collapse.

B. Reduction to a Linear Programming Form. - Without loss of generality, the first 2k - 3 bars are assumed to form a statically

[1] See for example, pp. 115-122 of [13].

determinate truss. Rewriting (3.20)

$$\sum_{j=1}^{2k-3} a_{ij} S_j = \lambda p_i - \sum_{q=2k-2}^{s} a_{iq} S_q \qquad (3.22)$$

Now by the above assumption there exist[1] elements a_{ti}^{-1} such that

$$\sum_{i=1}^{2k-3} a_{ti}^{-1} a_{ij} = \delta_{tj} \qquad (t, j = 1, 2, \ldots, 2k-3)$$

Multiplying (3.22) by a_{ti}^{-1} and summing over i

$$S_t = \lambda \sum_{i=1}^{2k-3} a_{ti}^{-1} p_i - \sum_{i=1}^{2k-3} \sum_{q=2k-2}^{s} a_{ti}^{-1} a_{iq} S_q$$

Substituting this into (3.21), the yield conditions become

$$-L_j \leq -\sum_{i=1}^{2k-3} \sum_{q=2k-2}^{s} a_{ji}^{-1} a_{iq} S_q + \lambda \sum_{i=1}^{2k-3} a_{ji}^{-1} p_i \leq U_j \qquad (j = 1, 2, \ldots, 2k-3)$$

$$-L_q \leq S_q \leq U_q \qquad (q = 2k-2, \ldots, s)$$

The unknowns are $S_{2k-2}, \ldots, S_s, \lambda$ which are $s - 2k + 4$ in number. Now if the number of redundancies is r then $s = 2k - 3 + r$, so that the number of unknowns is $r + 1$. We are led, therefore, to consider the following vectors in a space of $r + 1$ - dimensions, V_{r+1}.

$$Q_j = \begin{bmatrix} -\sum_{i=1}^{2k-3} a_{ji}^{-1} a_{i,2k-2} \\ -\sum_{i=1}^{2k-3} a_{ji}^{-1} a_{i,2k-1} \\ \cdot \\ \cdot \\ -\sum_{i=1}^{2k-3} a_{ji}^{-1} a_{is} \\ \sum_{i=1}^{2k-3} a_{ji}^{-1} p_i \end{bmatrix} \qquad (j = 1, \ldots, 2k-3)$$

---

[1] See Theorem III, p. 122 of [13].

$$Q_{2k-3+j} = \begin{bmatrix} \delta_{j1} \\ \vdots \\ \delta_{jr} \\ 0 \end{bmatrix} \quad (j = 1, \ldots, r) \qquad x = \begin{bmatrix} S_{2k-2} \\ \vdots \\ S_s \\ \lambda \end{bmatrix}$$

Then the yield criteria become

$$\begin{aligned} Q_j' x &\leq U_j \\ -Q_j' x &\leq L_j \end{aligned} \qquad (j = 1, 2, \ldots, s)$$

Finally let

$$P_j = \begin{cases} Q_j & (j = 1, 2, \ldots, s) \\ -Q_{j-s} & (j = s+1, s+2, \ldots, 2s) \end{cases}$$

$$c_j = \begin{cases} U_j & (j = 1, 2, \ldots, s) \\ L_{j-s} & (j = s+1, s+2, \ldots, 2s) \end{cases}$$

and

$$P_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The problem then reduces to maximizing

$$\lambda = P_0' x \tag{3.23}$$

subject to

$$P_j' x \leq c_j \quad (j = 1, 2, \ldots, 2s) \tag{3.24}$$

This is a problem of Type II with the following special properties

$$P_{j+s} = -P_j \quad (j = 1, 2, \ldots, s) \tag{3.25}$$

and

$$c_j \geq 0 \quad (j = 1, 2, \ldots, 2s) \tag{3.26}$$

It is now clear that a finite maximum value of λ exists. Recall that such is not the case if and only if $P_j'\alpha_r \geq 0$ for all j where $P_o'\alpha_r < 0$. Without loss of generality assume that $a_r = P_r$ for $r \leq s$. Then from (3.25) $P_{r+s}'\alpha_r \equiv -1$ and the maximum value of the functional is therefore finite.

Now only the first s of the P_j need be carried in the tableau. The other entries can be computed from

$$P_{j+s}'\alpha_i = -P_j'\alpha_i \qquad (j = 1, 2, \ldots, s) \tag{3.27}$$

$$\sum_{i=1}^{r+1}(P_{j+s}'\alpha_i)c_i - c_{j+s} = -\left\{\sum_{i=1}^{r+1}(P_j'\alpha_i)c_i - c_j\right\} - \left\{U_j + L_j\right\} \qquad (j = 1, 2, \ldots, s) \tag{3.28}$$

The entries on the right of (3.27) and (3.28) all appear in the first s columns.

This problem may, therefore, be treated as a Type II problem of size $(r+1) \times s$.

To formulate this as a Type I or simplex problem consider the dual problem to (3.23) and (3.24), i.e., to minimize

$$\lambda = \sum_{j=1}^{2s}\rho_j c_j$$

where

and

$$\sum_{j=1}^{2s}\rho_j P_j \geq P_o$$

$$\rho_j \geq 0 \qquad (j = 1, 2, \ldots, 2s)$$

This minimum problem was shown to be equivalent to the kinematic principle (Theorem 2) by Charnes and Greenberg [14].

A significant difference in the simplex and dual methods is that in the former a certain freedom of choice may be available in the vectors

entering the basis, while in the dual method a choice may exist in the vectors **leaving** the basis. A physical interpretation of the presence of certain vectors in the basis for the optimum solution will clarify the significance of this distinction.

If P_j for $j = 1, 2, \ldots, s$ is in the basis at the final solution then the jth bar yields in tension. Similarly if P_j is present for $j = s+1, \ldots, 2s$ then the $(j - s)$th bar is yielding in compression.

In the simplex technique when a choice is available then it is best to bring in those vectors corresponding to bars which experience or intuition indicates should yield in the collapse solution. In the dual technique, of course, one removes vectors when it appears that the corresponding bars should not be yielding at collapse.

The other major distinction between the two computational techniques lies in the method of obtaining initial solutions (Appendix C).

C. The Bounded Variables Problem. - To formulate the collapse problem as a bounded variables problem, return to equations (3.20) and (3.21) and let

$$x_j = \frac{S_j}{L_j} + 1 \qquad (j = 1, 2, \ldots, s)$$

$$x_{s+1} = \lambda$$

Then (3.20) may be written

$$\sum_{j=1}^{s+1} a_{ij}^* x_j = d_i \qquad (i = 1, 2, \ldots, 2k-3) \qquad (3.29)$$

where

$$a_{ij}^* = a_{ij} L_j \qquad (j = 1, 2, \ldots, s)$$

$$a_{i,s+1}^* = -p_i$$

and

$$d_i = \sum_{j=1}^{s} a_{ij} L_j \qquad (i = 1, 2, \ldots, 2k-3)$$

The yield conditions (3.21) are

$$0 \leq x_j \leq \frac{U_j + L_j}{L_j} \qquad (j = 1, 2, \ldots, s) \tag{3.30}$$

Since λ is non-negative

$$0 \leq x_{s+1} \leq M \tag{3.31}$$

where M is an arbitrarily large positive number. Since the problem is to maximize x_{s+1}, if $c_j = 0$ for $j = 1, \ldots, s$ and $c_{s+1} = 1$, the problem defined in (3.29), (3.30), (3.31) may be written to maximize

$$\lambda = \sum_{j=1}^{s+1} x_j c_j$$

subject to

$$P_o = \sum_{j=1}^{s+1} x_j P_j$$

$$0 \leq x_j \leq b_j \qquad (j = 1, \ldots, s+1)$$

where

$$P_o = \begin{bmatrix} \sum_{j=1}^{s} a_{1j} L_j \\ \cdot \\ \cdot \\ \sum_{j=1}^{s} a_{2k-3,j} L_j \end{bmatrix} \qquad P_j = \begin{bmatrix} a_{1j} L_j \\ \cdot \\ \cdot \\ \cdot \\ a_{2k-3,j} L_j \end{bmatrix} \qquad (j = 1, \ldots, s)$$

$$P_{s+1} = \begin{bmatrix} -p_1 \\ \cdot \\ \cdot \\ -p_{2k-3} \end{bmatrix} \qquad b_j = \begin{cases} \dfrac{U_j + L_j}{L_j} & j = 1, \ldots, s \\ M & j = s+1 \end{cases}$$

This is a problem of Type III and the tableau is $(2k - 3) \times (s + 1)$.

A physical interpretation of the presence of certain vectors in the basis is also possible in this case. If in the optimum solution $\overline{P}_j$ but not $\overline{Q}_j$ is in the basis (only a + sign in the P_j column), then the jth bar yields in tension. Correspondingly if $\overline{Q}_j$ but not $\overline{P}_j$ is in the basis (only a - sign in the P_j column), then the jth bar yields in compression. With these facts at hand the analyst may use his experience and intuition in selecting vectors to enter the basis when a choice is available.

D. Comparison of Methods. - As a measure of the number of arithmatical operations per iteration for the proportional loading problem, we may use the number of multiplications to be performed.

For Type I and II formulations the number of multiplications in each iteration is $(r+2)(s+1)$, while for a Type III formulation this number is $(s - r + 1)(s + 2)$. The preference for formulation on the basis of number of arithmatical operations, therefore, depends on the relationship between s, the number of bars, and r, the number of redundancies.

A seeming disadvantage of the first two type formulations is the apparent need for inverting the a_{ij} matrix in order to eliminate the equalities. In general, however, the a_{ij}^{-1} are never found explicitly, and because of the special nature of the equilibrium equations the solution for the redundant forces in terms of the non-redundant ones is usually not difficult.

E. Frames and Beams. - For frames and beams where axial force effects are assumed negligible a parallel discussion can be given. If the number of beams in the frame is b; the number of joints, v; and the number of loads not at joints, ℓ ; then the number of multiplications for a Type I or II formulation is $(3b - 3v + 2)x(2b + \ell + 1)$. For a Type III formulation this number is $(3v - b + \ell + 1)x(2b + \ell + 2)$.

If axial forces are to be considered, it is necessary to introduce a linearized yield criterion. A convenient choice which offers a good approximation is

$$\frac{|M_i|}{M_{pi}} + \frac{|N_i|}{N_{pi}} \leq 1$$

This has already been discussed in Section 4, Chapter II where bounds were found for the safety factor. Here a technique for determining the exact safety factor is briefly outlined.

The equilibrium equations are first solved for the non-redundant moments and forces in terms of a set of redundant moments and forces. Introducing these into the yield conditions, a Type II problem results. This may be solved by the techniques in Section 2, Parts A and B of this chapter.

Because of the nature of the yield conditions it is not possible to reduce this problem to a bounded variables problem (Type III).

Chapter IV

EXAMPLES OF LINEAR PROGRAMMING METHODS

To illustrate and compare the three methods of solution of linear programming problems described in Chapter III as applied to structural collapse problems, we consider a simple example and solve it by the three methods.

Consider the once redundant truss in Figure 7 loaded with a single concentrated force as shown. The members are numbered as indicated, and the equilibrium equations (3.20) may be written

$$-S_2 - \frac{4}{5} S_5 = \lambda b$$

$$S_2 + \frac{4}{5} S_6 = 0$$

$$S_4 + \frac{4}{5} S_5 = 0$$

$$S_1 + \frac{3}{5} S_5 = 0$$

$$S_3 + \frac{3}{5} S_6 = 0$$

The fully plastic forces in tension and compression are taken to be the same and equal to N_p. The yield criteria (3.21) become then

$$|S_i| \leq N_p \qquad (i = 1, 2, \ldots, 6)$$

Now letting $x_1 = S_6/N_p$ and $x_2 = \lambda b/N_p$ and solving the equilibrium equations for $S_1, S_2, \ldots, S_5$ in terms of these, the yield criteria may be written

$$\left| -\frac{3}{5} x_1 + \frac{3}{4} x_2 \right| \leq 1$$

$$\left| -\frac{4}{5} x_1 \right| \leq 1$$

$$\left| -\frac{3}{5} x_1 \right| \leq 1$$

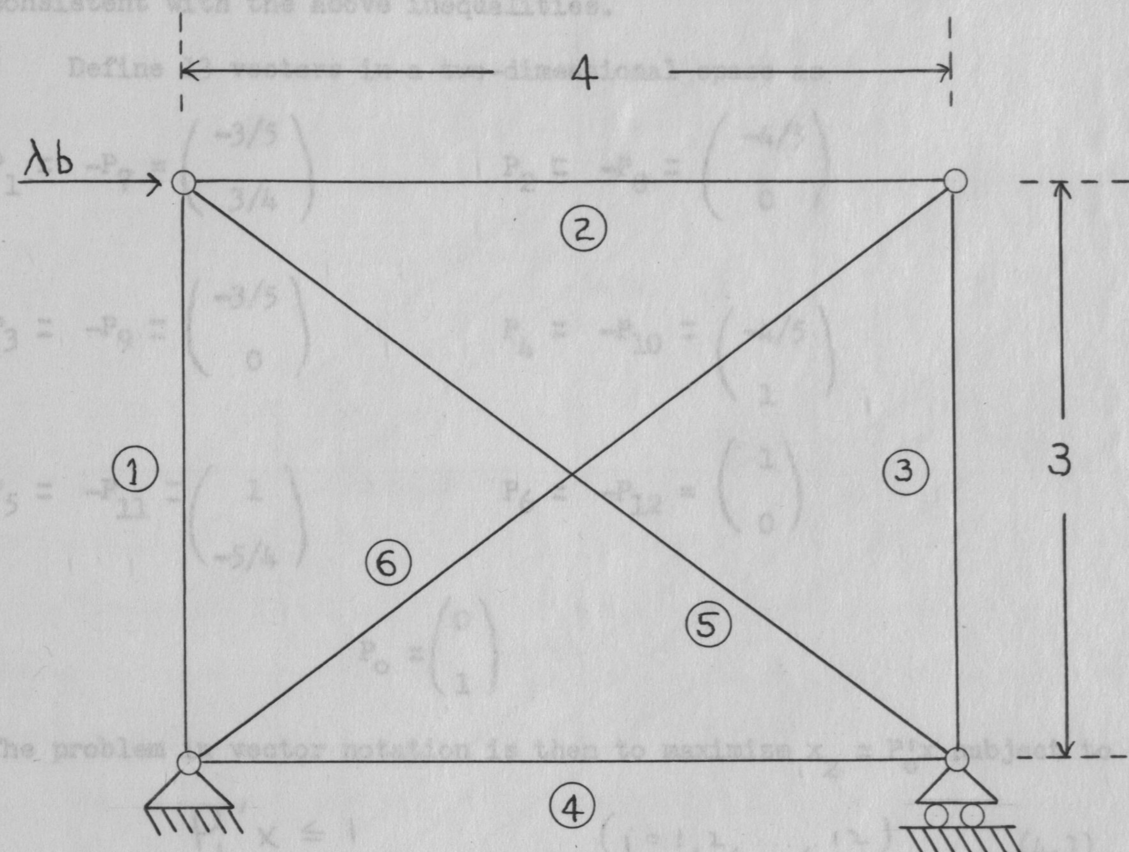


FIGURE 7

A Once Redundant Truss

$$\left|-\frac{4}{5}x_1 + x_2\right| \le 1$$

$$\left|x_1 - \frac{5}{4}x_2\right| \le 1$$

$$|x_1| \le 1$$

The safety factor against collapse is the largest value of x_2N_p/b consistent with the above inequalities.

Define 13 vectors in a two-dimensional space as

$$P_1 = -P_7 = \begin{pmatrix} -3/5 \\ 3/4 \end{pmatrix} \qquad P_2 = -P_8 = \begin{pmatrix} -4/5 \\ 0 \end{pmatrix}$$

$$P_3 = -P_9 = \begin{pmatrix} -3/5 \\ 0 \end{pmatrix} \qquad P_4 = -P_{10} = \begin{pmatrix} -4/5 \\ 1 \end{pmatrix}$$

$$P_5 = -P_{11} = \begin{pmatrix} 1 \\ -5/4 \end{pmatrix} \qquad P_6 = -P_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P_o = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The problem in vector notation is then to maximize $x_2 = P_o'x$ subject to

$$P_j'x \le 1 \qquad (j = 1, 2, \ldots, 12) \tag{4.1}$$

The equation $P_j'x = 1$ defines a line in two-space for each j. The vector P_j is normal to this line and points into the half-space for which the corresponding inequality is violated. All of the lines defined by equality in (4.1) are shown in Figure 8. The set of points Λ for which all of (4.1) are satisfied is the parallelogram ABCD, and the maximum value of x_2 is obtained at the point D (1, 8/5).

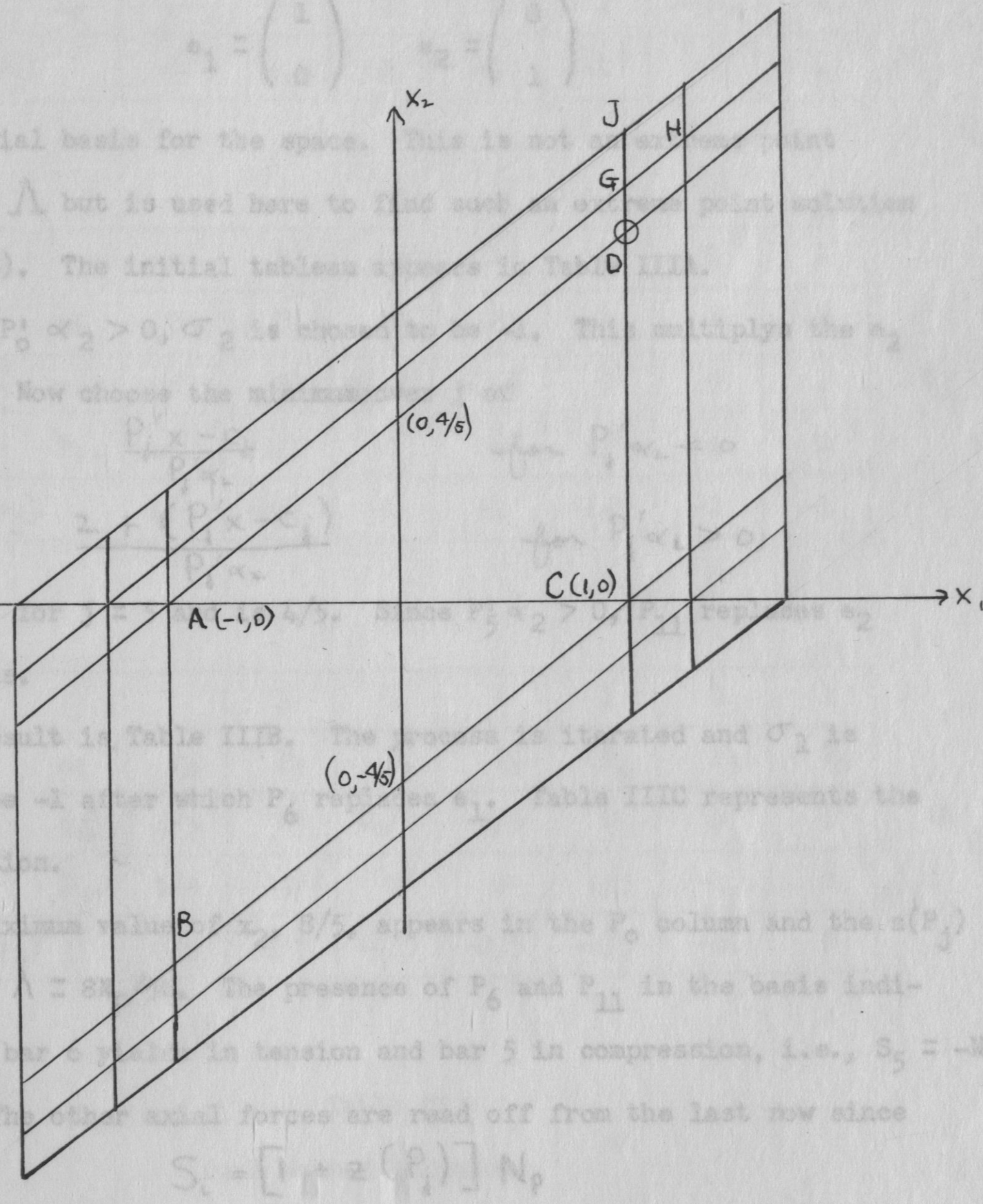


FIGURE 8

Graphic Solution of Truss Problem

The safety factor is then

$$\lambda = \frac{8}{5}\frac{N_p}{b}$$

An analytic solution is obtained by the dual method (Section 2B, Chapter III). Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be the initial basis for the space. This is not an extreme point solution of Λ but is used here to find such an extreme point solution (Appendix C). The initial tableau appears in Table IIIA.

Since $P_0' \alpha_2 > 0$, σ_2 is chosen to be -1. This multiplys the e_2 row by -1. Now choose the minimumoover j of

$$\frac{P_j' x - c_j}{P_j' \alpha_2} \qquad \text{for } P_j' \alpha_2 < 0$$

$$\frac{2 + (P_j' x - c_j)}{P_j' \alpha_2} \qquad \text{for } P_j' \alpha_2 > 0$$

This occurs for j = 5 and is 4/5. Since $P_5' \alpha_2 > 0$, P_{11} replaces e_2 in the basis.

The result is Table IIIB. The process is iterated and σ_1 is chosen to be -1 after which P_6 replaces e_1. Table IIIC represents the final solution.

The maximum value of x_2, 8/5, appears in the P_0 column and the $z(P_j)$ row. Thus $\lambda = 8N_p/5b$. The presence of P_6 and P_{11} in the basis indicates that bar 6 yields in tension and bar 5 in compression, i.e., $S_5 = -N_p$, $S_6 = N_p$. The other axial forces are read off from the last row since

$$S_i = [1 + z(P_j)]\, N_p$$

Thus $S_1 = 3N_p/5$, $S_2 = -4N_p/5$, $S_3 = -3N_p/5$, $S_4 = 4N_p/5$.

The dual to this problem is a simplex problem and the same initial tableau (Table IIIA) may be used if the vectors e_1, e_2 are given large

	C_j	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
	$C_j \rightarrow$			1	1	1	1	1	1	A
	0	e_1	0	-3/5	-4/5	-3/5	-4/5	1	1	
-->	0	e_2	1	3/4	0	0	1	-5/4	0	
	Z (P_j)		0	-1	-1	-1	-1	-1	-1	

	C_j	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
	$C_j \rightarrow$			1	1	1	1	1	1	B
-->	0	e_1	4/5	0	-4/5	-3/5	0	0	1	
	1	P_{11}	4/5	3/5	0	0	4/5	-1	0	
	Z (P_j)		4/5	-2/5	-1	-1	-1/5	-2	-1	

C_j	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6	
			1	1	1	1	1	1	C
1	P_6	4/5	0	-4/5	-3/5	0	0	1	
1	P_{11}	4/5	3/5	0	0	4/5	-1	0	
Z (P_j)		8/5	-2/5	-9/5	-8/5	-1/5	-2	0	

Table III

Dual Method Solution

positive weights, M. This changes the last row, $z(P_j)$ only. Now many vectors, i.e. those for which $z(P_j) > 0$, may enter the basis. P_6 is chosen on an intuitive basis and thus e_1 leaves the basis. The resulting tableau is shown in Table IVA. P_4 then replaces e_2 and Table IVB results. Finally P_5 replaces P_4 and the final tableau will be identical with Table IIIC.

Notice that the solution in Table IVA corresponds to the point (1, M) in Figure 8 and the solution in Table IVB to the point G(1, 9/5) in that figure.

It should also be noted that J, H and all other intersections of the lines lying above D are feasible solutions to the above simplex problem.

Finally, to formulate the problem as a bounded variables problem (Type III), let $w_j = 1 + (S_j/N_p)$ for j = 1, 2, . . ., 6 and let $w_7 = \lambda b/N_p$. The equilibrium equations become

$$w_2 + \frac{4}{5} w_5 + w_7 = 9/5$$

$$w_2 + \frac{4}{5} w_6 = 9/5$$

$$w_4 + \frac{4}{5} w_5 = 9/5$$

$$w_1 + \frac{3}{5} w_5 = 8/5$$

$$w_3 + \frac{3}{5} w_6 = 8/5$$

And the yield criteria are

$$0 \leq w_i \leq 2 \qquad (i = 1, 2, \ldots, 6)$$

To this we add

$$0 \leq w_7 \leq M$$

where M is an arbitrarily large positive number.

C_j	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6
			1	1	1	1	1	1
1	P_6	0	-3/5	-4/5	-3/5	-4/5	1	1
M	e_2	1	3/4	0	0	1	-5/4	0
$Z(P_j)$		M	3M/4 -8/5	-9/5	-8/5	M - 9/5	-5M/4	0

A

C_j	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6
			1	1	1	1	1	1
1	P_6	4/5	0	-4/5	-3/5	0	0	1
1	P_4	1	3/4	0	0	1	-5/4	0
$Z(P_j)$		9/5	-1/4	-9/5	-8/5	0	-9/4	0

B

Table IV

Simplex Method Solution

To find an initial basis, it is necessary to find a linearly independent set of column vectors defined by the matrix of the equilibrium equations. Using unit vectors $e_1, \ldots, e_5$ in V_5 with large negative weights, $-N$, these column vectors are expressed in Table VA. In one iteration e_1 is replaced by P_7, e_3 by P_4, e_4 by P_1, and e_5 by P_3. Table VB shows the result. Then P_2 replaces e_2.

This then leads to the initial bounded variables solution in Table VIA. The values in the φ column are values of w_i, i.e., $w_1 = 8/5$, $w_2 = 1/5$, $w_3 = 2/5$, $w_4 = 9/5$, $w_7 = 8/5$. Since a $-$ appears under P_5 and a $+$ under P_6 then $S_1 = 3N_p/5$, $S_2 = -4N_p/5$, $S_3 = -3N_p/5$, $S_4 = 4N_p/5$, $S_5 = -N_p$, $S_6 = N_p$ and $\lambda = 8N_p/5b$.

C_j →			0	0	0	0	0	0	1
↓	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
$-N$	e_1	9/5		1			4/5		1
$-N$	e_2	9/5		1				4/5	
$-N$	e_3	9/5				1	4/5		
$-N$	e_4	8/5	1				3/5		
$-N$	e_5	8/5			1			3/5	
$Z\ (P_j)$ →			$-N$	$-2N$	$-N$	$-N$	$-\frac{11N}{5}$	$-\frac{7N}{5}$	$-N$

A

C_j →			0	0	0	0	0	0	1
↓	a_i	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1	P_7	9/5		1			4/5		1
$-N$	e_2	9/5		1				4/5	
0	P_4	9/5				1	4/5		
0	P_1	8/5	1				3/5		
0	P_3	8/5			1			3/5	
$Z\ (P_j)$ →			0	$-N + 1$	0	0	4/5	$-4N/5$	0

B

Table VI

Initial Solution to Bounded Variable Problem

C_j	B_m	P_0	b −	P_1	P_2	P_3	P_4	P_5	P_6	P_7
			$b_j \rightarrow$	2	2	2	2	2	2	M
$C_j \rightarrow$				0	0	0	0	0	0	1
1	P_7	0	M					4/5	−4/5	1
0	P_2	9/5	1/5		1				4/5	
0	P_4	9/5	1/5				1	4/5		
0	P_1	8/5	2/5	1				3/5		
0	P_3	8/5	2/5			1			3/5	
$Z(P_j) \rightarrow$				0	0	0	0	4/5	−4/5	0
$B_m + n \rightarrow$				+, −	+, −	+, −	+, −	−	−	+, −

C_j	B_m	P_0	b −	P_1	P_2	P_3	P_4	P_5	P_6	P_7
			$b_j \rightarrow$	2	2	2	2	2	2	M
$C_j \rightarrow$				0	0	0	0	0	0	1
1	P_7	8/5	M − 8/5					4/5	−4/5	1
0	P_2	1/5	9/5		1				4/5	
0	P_4	9/5	1/5				1	4/5		
0	P_1	8/5	2/5	1				3/5		
0	P_3	2/5	8/5			1			3/5	
$Z(P_j) \rightarrow$				0	0	0	0	4/5	−4/5	0
$B_m + n \rightarrow$				+, −	+, −	+, −	+, −	−	+	+, −

Table VIII

Bounded Variables Solution

Appendix A

THE DETERMINATION OF COMPLETE SETS OF BASIC MECHANISMS FOR FRAMES

A set of basic mechanisms is called complete if the equilibrium equations associated with the set form a complete, linearly independent set of equilibrium equations. For quadrilateral or rectangular frames it will be shown that the set B defined in Section 1, Chapter II is a complete set of basic mechanisms. For more general frames a definition of basic mechanisms which leads to a complete set will be given.

For quadrilateral frames the equilibrium equations associated with the set B of basic mechanisms form a complete, linearly independent set.

It will be shown first that the equations are linearly independent. Every beam mechanism equation contains a bending moment at a cross-section not at a joint. Moreover this bending moment does not appear in any other beam equation nor in the equilibrium equation for any joint or frame mechanism. Each beam mechanism equation is, therefore, independent of all others of the set. The equation for a joint mechanism contains at least one bending moment which does not appear in any frame mechanism since side-sway can occur in only one direction at each joint. Since the moments appearing in the joint equations are mutually exclusive, these equations are independent of all others.

Finally, the moments appearing in each frame mechanism equation are mutually exclusive. It follows that all of the equilibrium equations are linearly independent.

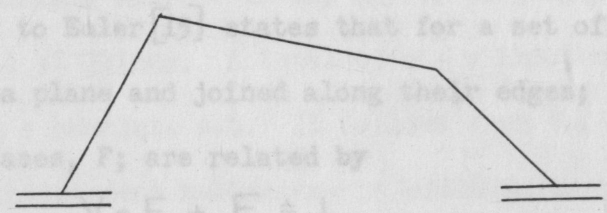
To show that these equations form a complete set it is necessary to show there are exactly n - r basic mechanisms where there are n critical cross-sections and r redundancies in the frame (Chapter I).

Let F, J, B be the number of frame, joint, and beam mechanisms respectively. It must be shown that $F + B + J = n - r$. We let b represent the number of members in the frame; v, the number of joints or vertices; s, the number of supports; and f, the number of closed quadrilaterals in the frame.

Now the number of frame mechanisms, F, can be shown to be

$$F = 2v - b \tag{a.1}$$

Starting with the simple frame



$F = 1$, $v = 2$, $b = 3$, $f = 0$ so the relationship is valid. Now all quadrilateral frames can be constructed from this simple frame by adding a sequence of either closed quadrilaterals and/or open quadrilaterals plus supports. During any addition in the sequence, bars can be appended to the existing frame only at existing joints, i.e., beams may be joined only at their end points.

If a closed face is added then the increase in the number of joints, Δv, is either 1 or 2. If $\Delta v = 1$, then the increase in frame mechanisms is $\Delta F = 0$; and if $\Delta v = 2$, $\Delta F = 1$. On the other hand, for the addition of an open face and support, Δv is 0 or 1. In the first case $\Delta F = -1$ and in the latter, $\Delta F = 0$. In every case then

$$\Delta F = \Delta v - 1$$

Now the number of bars added, Δb, is related to Δv by

$$\Delta b = \Delta v + 1$$

Thus

$$\Delta F = 2\Delta v - \Delta b$$

and hence the number of frame mechanisms is given by (a.1) for any rectangular frame.

Now, from the definitions, $J = v$ and $B = \ell$ where ℓ is the number of loads not at joints. The total number of basic mechanisms is therefore

$$F + B + J = 3v - b + \ell \tag{a.2}$$

A formula due to Euler [15] states that for a set of closed polygons lying in a plane and joined along their edges; the vertices, V; edges, E; and faces, F; are related by

$$V - E + F = 1$$

Quadrilateral frames are just such a collection of polygons with some of the edges removed and supports placed at the free ends of beams. Now removing an edge also removes a face, so counting supports as vertices the Euler formula remains valid. In the notation we have used for frames

$$s + v - b + f = 1 \tag{a.3}$$

Thus

$$3v = 3(1 + b - s - f)$$

and (a.2) becomes

$$F + B + J = (2b + \ell) - (3f + 3s - 3)$$

The number of critical cross-sections is

$$n = 2b + \ell \tag{a.4}$$

and the number of redundancies is

$$r = 3f + 3(s-1) = 3f + 3s - 3 \tag{a.5}$$

Therefore,

$$F + B + J = n - r$$

This completes the proof of the statement.

A complete set of basic mechanisms can be obtained for more general frames consisting of any arrangement of straight beams rigidly joined as follows.

For general frames, joint and beam mechanisms are defined exactly as they are for quadrilateral frames (Section 1, Chapter II). There are v ℓ such mechanisms where v is the number of joints and ℓ the number of loads not at joints. A total of n - r independent mechanisms are needed to form a complete set. It follows from (a.3), (a.4), (a.5) then that 2v - b independent mechanisms in addition to the joint and beam mechanisms are required.

For general frames the set of frame mechanisms is defined as 2v - b independent mechanisms which are also independent of all joint and beam mechanisms. These can be found by considering mechanisms for which hinges appear only at ends of beams. A necessary and sufficient condition that they be independent of each other is that none may be obtained from the others by a rotation of joints.

For quadrilateral frames, of course, the set of frame mechanisms defined by Neal and Symonds satisfies this definition as well.

Appendix B

PROOF OF THE FUNDAMENTAL THEOREMS

For a beam cross-section which yields under the action of a bending moment and an axial force, the <u>flow vector</u> is defined as a two-dimensional vector whose first component is proportional to the relative axial velocity of the adjacent cross-sections and whose second component is in the same proportion to the relative rotational velocity of the beam segments adjacent to that cross-section.

An essential requirement for the theorems of limit analysis to apply is that the flow vector be orthogonal to the yield curve [16].

For a yield curve defined by (2.36) then the relative velocity, δ_i, and the relative rotational velocity, θ_i, at a yielding cross-section must satisfy

$$\frac{|\delta_i|}{M_{pi}} = \frac{|\theta_i|}{N_{pi}} \qquad \text{(b.1)}$$

Notice that if a cross section yields and either $N_i = 0$ or $M_i = 0$, the flow vector is arbitrary to within an angle $\pi/2$, i.e., at a point on the yield curve where the tangent is discontinuous the flow vector is not uniquely determined. However for $N_i = 0$

$$|\delta_i| \leq \frac{M_{pi}}{N_{pi}} |\theta_i|$$

and for $M_i = 0$

$$|\delta_i| \geqslant \frac{M_{pi}}{N_{pi}} |\theta_i|$$

THEOREM: A statically admissible multiplier for Problem 1 is less than or equal to the safety factor for that problem.

PROOF: Let λ be the safety factor against collapse and let M_i, N_i

be the bending moments and axial forces in equilibrium with the loads λb_j. If the collapse mechanism is defined by velocities, v_j, relative rotational velocities, θ_j, and relative axial velocities, δ_j, then from the equation of virtual work

$$\sum_{i=1}^{n} (M_i \theta_i + N_i \delta_i) = \lambda \sum_{i=1}^{n} b_i v_i$$

and

$$\sum_{i=1}^{n} b_i v_i > 0 \qquad \text{(b.2)}$$

Let λ' be a statically admissible multiplier and let M_i', N_i' be moments and forces in equilibrium with loads $\lambda' b_j$. Then again from the principle of virtual work

$$\sum_{i=1}^{n} (M_i' \theta_i + N_i' \delta_i) = \lambda' \sum_{i=1}^{n} b_i v_i$$

Subtracting the two virtual work equations there results

$$(\lambda - \lambda') \sum_{i=1}^{n} b_i v_i = \sum_{i=1}^{n} (M_i - M_i')\theta_i + (N_i - N_i')\delta_i \qquad \text{(b.3)}$$

Now since M_i', N_i' are statically admissible

$$\frac{|M_i'|}{M_{pi}} + \frac{|N_i'|}{N_{pi}} \leq \frac{|M_i|}{M_{pi}} + \frac{|N_i|}{N_{pi}} = 1 \qquad \text{(b.4)}$$

for each i,

$$M_i \theta_i + N_i \delta_i = |M_i||\theta_i| + |N_i||\delta_i|$$

Three cases arise: (i) $M_i \neq 0$, $N_i \neq 0$; (ii) $M_i \neq 0$, $N_i = 0$; and (iii) $M_i = 0$, $N_i \neq 0$.

For case (i)

$$|\delta_i| = \frac{M_{pi}}{N_{pi}} |\theta_i| \qquad \text{(b.5)}$$

and

$$M_i\,\theta_i + N_i\,\delta_i = \left\{|M_i| + \frac{M_{pi}}{N_{pi}}\,|N_i|\right\}|\theta_i|$$

Now from equation (b.5)

$$M_i'\theta_i + N_i'\delta_i \leq \left\{\frac{\theta_i}{|\theta_i|}\,M_i' + \frac{\delta_i}{|\delta_i|}\,N_i'\,\frac{M_{pi}}{N_{pi}}\right\}|\theta_i|$$

$$\leq \left\{|M_i'| + \frac{M_{pi}}{N_{pi}}\,|N_i'|\right\}|\theta_i|$$

Each term in the sum on the right of (b.3) is then greater than or equal to

$$\left[\left(|M_i| + \frac{M_{pi}}{N_{pi}}|N_i|\right) - \left(|M_i'| + \frac{M_{pi}}{N_{pi}}|N_i'|\right)\right]|\theta_i|$$

But from (b.4), the term in equare brackets is non-negative and, therefore, the corresponding term in the sum in (b.3) is also non-negative.

For case (ii)

$$|\delta_i| \leq \frac{M_{pi}}{N_{pi}}\,|\theta_i| \qquad \text{(b.6)}$$

and

$$|M_i| = M_{pi}$$

Thus

$$M_i\theta_i + N_i\delta_i = M_{pi}\,|\theta_i|$$

and from (b.6)

$$M_i'\theta_i + N_i'\delta_i \leq \left\{\frac{\theta_i}{|\theta_i|}\,M_i' + \frac{\delta_i}{|\delta_i|}\,\frac{M_{pi}}{N_{pi}}\,N_i'\right\}|\theta_i|$$

$$\leq \left\{|M_i'| + \frac{M_{pi}}{N_{pi}}\,|N_i'|\right\}|\theta_i|$$

Thus each term in the sum on the right of (b.3) is greater than or equal to

$$\left[M_{pi} - \left\{|M_i'| + \frac{M_{pi}}{N_{pi}}|N_i'|\right\}\right]|\theta_i|$$

From (b.4) the term in square brackets is non-negative and thus so is the corresponding term in (b.3).

Finally for case (iii)

$$|\delta_i| \geq \frac{M_{pi}}{N_{pi}}|\theta_i| \tag{b.7}$$

and

$$|N_i| = N_{pi}$$

Thus

$$M_i\theta_i + N_i\delta_i = N_{pi}|\delta_i|$$

and from (b.7)

$$M_i'\theta_i + N_i'\delta_i \leq \left\{\frac{\theta_i}{|\theta_i|}\frac{N_{pi}}{M_{pi}}M_i' + \frac{\delta_i}{|\delta_i|}N_i'\right\}|\delta_i|$$

$$\leq \left\{\frac{N_{pi}}{M_{pi}}|M_i'| + |N_i'|\right\}|\delta_i|$$

Thus each term in the sum on the right of (b.3) is greater than or equal to

$$\left[N_{pi} - \left\{\frac{N_{pi}}{M_{pi}}|M_i'| + |N_i'|\right\}\right]|\delta_i|$$

From (b.4) the term in square brackets is non-negative and thus so is the corresponding term in (b.3).

Therefore, every term in the sum on the right of (b.3) is non-negative. Combining this result with (b.2), it follows that

$$\lambda - \lambda' \geq 0$$

This completes the proof of the theorem.

THEOREM: A kinematically sufficient multiplier for Problem 1 is greater than or equal to the safety factor against collapse for that problem.

PROOF: Consider a mechanism defined by velocities, v_j^*; rotational velocities, θ_j^*; and axial velocities, δ_j^*; such that

$$|\delta_j^*| = \frac{M_{pj}}{N_{pj}} |\theta_j^*| \qquad \text{(b.7)}$$

The kinematically sufficient multiplier associated with this mechanism is

$$\lambda^* = \frac{\sum_{j=1}^{n} M_j^* |\theta_j^*| + \sum_{j=1}^{n} N_j^* |\delta_j^*|}{\sum_{j=1}^{n} b_j v_j^*}$$

where M_j^*, N_j^* are a system of moments and forces compatible with the hinge distribution.

Now if λ is the safety factor then there exist moments and forces, M_j, N_j, in equilibrium with loads λb_j such that

$$\frac{|M_j|}{M_{pj}} + \frac{|N_j|}{N_{pj}} \leq \frac{|M_j^*|}{M_{pj}} + \frac{|N_j^*|}{N_{pj}} \qquad \text{(b.8)}$$

at the yield hinges in the given mechanism. Moreover,

$$\sum_{j=1}^{n} (M_j \theta_j^* + N_j \delta_j^*) = \lambda \sum_{j=1}^{n} b_j v_j^*$$

from the principle of virtual work.

We will assume now that only isolated hinges appear in the given mechanism, i.e., there are no yield bars present. An argument similar to the one below can be given for the excluded case.

The last equation given a bove can be written

$$\lambda = \frac{\sum_{j=1}^{n} (M_j \theta_j^* + N_j \delta_j^*)}{\sum_{j=1}^{n} b_j v_j^*}$$

and using (b.7)

$$\lambda \leq \frac{\sum_{j=1}^{n} \left\{ |M_j| |\theta_j^*| + |N_j| \frac{M_{pj}}{N_{pj}} |\theta_j^*| \right\}}{\sum_{j=1}^{n} b_j v_j^*}$$

$$\lambda \leq \frac{\sum_{j=1}^{n} M_{pj} \left\{ \frac{|M_j|}{M_{pj}} + \frac{|N_j|}{N_{pj}} \right\} |\theta_j^*|}{\sum_{j=1}^{n} b_j v_j^*}$$

Then from (b.8)

$$\lambda \leq \frac{\sum_{j=1}^{n} M_{pj} \left\{ \frac{|M_j^*|}{M_{pj}} + \frac{|N_j^*|}{N_{pj}} \right\} |\theta_j^*|}{\sum_{j=1}^{n} b_j v_j^*}$$

and using (b.7) again

$$\lambda \leq \frac{\sum_{j=1}^{n} |M_j^*||\theta_j^*| + |N_j^*||\delta_j^*|}{\sum_{j=1}^{n} b_j v_j^*} = {\lambda'}^{*}$$

This completes the proof.

Appendix C

INITIAL SOLUTIONS TO LINEAR PROGRAMMING PROBLEMS

We present here methods for obtaining initial feasible solutions to Type I and Type II linear programming problems. For Type III problems it is sufficient to find a basis for the vector space of the equalities (3.2). The method developed for Type I produces such a basis.

1. Type I Problems.

Consider the following modification[1] of the Type I problem formulated in (3.1), (3.2), (3.3): To minimize

$$z_0 = \sum_{j=1}^{n} \rho_j c_j + \sum_{i=1}^{m} \gamma_i M \tag{c.1}$$

subject to

$$P_0 = \sum_{j=1}^{n} \rho_j P_j + \sum_{i=1}^{m} \gamma_i e_i \tag{c.2}$$

$$\begin{aligned} \rho_j &\geq 0 & (j = 1, 2, \ldots, n) \\ \gamma_i &\geq 0 & (i = 1, 2, \ldots, m) \end{aligned} \tag{c.3}$$

where e_i is a unit vector with plus or minus one as the ith component depending on whether the ith component of P_0 is positive or negative, and where M is an arbitrarily large positive number.

The solution to this problem is identical with the one phrased in (3.1), (3.2), (3.3) since the minimum will occur for $\gamma_i = 0$ for all i.

A basis for this problem, however, is readily available. Indeed, the basis a_i may be taken to be

---

[1]This modification was first suggested by Dantzig [9], see footnote to page 340.

$$a_i = e_i \qquad (i = 1, 2, \ldots, m)$$

It follows from the definitions of e_i and α_i that

$$\alpha_i = e_i \qquad (i = 1, 2, \ldots, m)$$

The entries of the tableau (Table I) are, therefore, easily computed.

The vectors e_i need not be carried in the tableau since if an e_i leaves the basis it cannot return because it carries a large positive weight, M. It will require exactly m iterations in order to obtain a basis comprised entirely of vectors chosen from among the P_j. However, in many cases these iterations are trivial and require a minimum of calculations.

As mentioned previously a basis a_i for V_m in a Type III problem can be obtained in the same way. The basis for V_{m+n} is then $\overline{Q}_j$ for $j = 1, 2, \ldots, n$ and $\overline{P}_i = \begin{pmatrix} a_i \\ Q_i \end{pmatrix}$.

2. Type II Problems.

We now develop a method for finding an initial extreme point solution to Type II problems.

If any point w satisfying (3.5) can be found then a simple change of variables will translate this point to the origin. Starting from the origin, the following method then produces an extreme point solution by use of the dual method applied to a modified problem.

The Type II problem arising from the simple proportional loading collapse problem is given by (3.23) and (3.24). From (3.26), $c_j \geqslant 0$ and this implies that $x \equiv 0$ is a solution to (3.24).

Consider now the modified[1] problem to maximize $P_o'x$ subject to

[1]The author is indebted to Dr. C. E. Lemke for suggesting this modification.

$$P_j' x \leq c_j \qquad (j = 1, 2, \ldots, n) \qquad \text{(c.4)}$$

$$\sigma_i e_i' x \leq 0 \qquad (i = 1, 2, \ldots, m) \qquad \text{(c.5)}$$

$$c_j \geqslant 0 \qquad \text{(c.6)}$$

where e_i is a unit vector in V_m with $+1$ as the ith component and σ_i is either equal to $+1$ or -1 and is chosen according to the criteria described below.

Notice that the origin is an extreme point of the modified problem regardless of the choice of sign for σ_i. The basis vectors associated with this point are e_i for $i = 1, 2, \ldots, m$.

The set of points x satisfying (c.4) is designated by Λ. The set satisfying both (c.4) and (c.5) is Λ^*. Note that (c.5) is just a restriction to some orthant of V_m once the signs of σ_i have been chosen. In order that the solution to the modified problem coincide with the original problem, it is necessary and sufficient that the σ_i be chosen in (c.5) so that the point x in Λ for which $P_o'x$ takes on its maximum also is contained in Λ^*, i.e., the correct orthant of V_m must be chosen.

The advantage of the modified problem is, of course, that $x \equiv 0$ is an extreme point solution. Starting from this solution and using the dual method a value of σ_i for some i is chosen at each iteration, and the corresponding e_i leaves the basis in favor of some P_j.

The procedure for accomplishing this is as follows: Let the basis at some stage be $e_1, \ldots, e_s, P_{s+1}, \ldots, P_m$ where the signs of $\sigma_{s+1}, \ldots, \sigma_m$ have already been properly chosen. Let the dual vectors to this basis be α_j, i.e.,

$$
\begin{aligned}
e_i'\alpha_j &= \delta_{ij} \qquad \begin{pmatrix} i = 1, 2, \dots, s \\ j = 1, 2, \dots, m \end{pmatrix} \\
P_i'\alpha_j &= \delta_{ij} \qquad \begin{pmatrix} i = s+1, \dots, m \\ j = 1, 2, \dots, m \end{pmatrix}
\end{aligned}
\qquad (c.7)
$$

Then

$$P_0 = \sum_{i=1}^{s} (P_0'\alpha_i) e_i + \sum_{i=s+1}^{m} (P_0'\alpha_i) P_i$$

and if x_0 is the extreme point of Λ^* corresponding to this basis then by definition

$$
\begin{aligned}
e_i' x_0 &= 0 \qquad (i = 1, 2, \dots, s) \\
P_i' x_0 &= c_i \qquad (i = s+1, \dots, m)
\end{aligned}
\qquad (c.8)
$$

Let

$$\bar{x} = x_0 - \theta \alpha_q \qquad 1 \le q \le s$$

then

$$
\begin{aligned}
P_0'\bar{x} = {} & \sum_{\substack{i=1 \\ i \neq q}}^{s} (P_0'\alpha_i)(e_i'x_0) - \theta \sum_{\substack{i=1 \\ i \neq q}}^{s} (P_0'\alpha_i)(e_i'\alpha_q) + (P_0'\alpha_q)(e_q'\bar{x}) \\
& + \sum_{i=s+1}^{m} (P_0'\alpha_i)(P_i'x_0) - \theta \sum_{i=s+1}^{m} (P_0'\alpha_i)(P_i'\alpha_q)
\end{aligned}
$$

Now from (c.7)

$$
\begin{aligned}
e_i'\alpha_q &= 0 \qquad (i = 1, \dots, q-1, q+1, \dots, s) \\
P_i'\alpha_q &= 0 \qquad (i = s+1, \dots, m)
\end{aligned}
$$

and using (c.8)

$$P_0'\bar{x} = (P_0'\alpha_q)(e_q'\bar{x}) + \sum_{i=s+1}^{m} (P_0'\alpha_i)(P_i'x_0).$$

The last sum is $P_0' x_0$ so

$$P_0'\bar{x} = P_0'x_0 + (P_0'\alpha_q)(e_q'\bar{x})$$

Three cases arise: (i) $P_0'\alpha_q < 0$; (ii) $P_0'\alpha_q > 0$; (iii) $P_0'\alpha_q = 0$.

For case (i) if $e_q'\bar{x} > 0$ then $P_0'\bar{x} < P_0'x_0$. It follows that any $\bar{x}$ yielding a larger value of the functional than x_0 cannot lie in the half-space $e_q'\bar{x} > 0$, but must satisfy

$$+ e_q' \bar{x} \leq 0$$

Thus e_q is removed from the basis and we pick $\sigma_q = +1$.

For case (ii) if $e_q'\bar{x} < 0$ then again $P_o'\bar{x} < P_o'x_o$. Thus similarly it is necessary that

$$- e_q' \bar{x} \leq 0$$

Again e_q is removed from the basis but σ_q is chosen to be -1.

Finally for case (iii), $P_o'\bar{x} = P_o'x_o$ and the choice of σ_q is deferred for the present.

If at some stage of the computations, case (iii) holds for all e_i remaining in the basis then the choice of σ_i for those e_i is arbitrary.

The usual algorithm (3.16) is used to proceed to a new tableau.

In this way in m iterations all of the e_i are eliminated from the basis and the appropriate values of σ_i equal to $+1$ or -1 are chosen.

Notice that the "modified" simplex or dual techniques for control of round-off error [5] are readily adaptable to the methods of this appendix.

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