

**Optimal Execution in a General One-Sided Limit-Order Book and
Endogenous Dynamic Completeness of Financial Models**

by

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CHAPTER 1

Introduction

This thesis consists of two parts. The first one is a result obtained under the supervision of Steven Shreve and with the collaboration of Gennady Shaikhet. Our work yielded a detailed description of the optimal strategies for a large investor, when she needed to buy a large amount of shares of a stock over a finite time horizon. The dynamics of the limit order book of the underlying stock is a generalization of known results to continuous time and to arbitrary distributions of the said limit order book. See the introduction section in chapter 2 for a more detailed discussion.

The second part is a result obtained under the supervision of Dmitry Kramkov. Our work yielded a sufficient condition on the structure of the economic factors, dividends of traded assets and total endowment in a single-agent economy, such that in an Arrow - Debreu - Radner equilibrium the market is complete. The main result is formulated as an integral representation theorem. Our work generalizes and complements fairly recent results in this direction (at the time of this thesis) by requiring less smoothness of the driving diffusion process at the expense of seemingly stronger conditions on the terminal dividends of the assets. See the introduction section in chapter 3 for a more detailed discussion.

CHAPTER 2

Optimal Execution in a General One-Sided Limit-Order Book

1. Introduction

We consider optimal execution over a fixed time interval of a large asset purchase in the face of a one-sided limit-order book. We assume that the *ask price* (sometimes called the *best ask price*) for the underlying asset is a continuous martingale that undergoes two adjustments during the period of purchase. The first adjustment is that orders consume a part of the limit-order book, and this increases the ask price for subsequent orders. The second adjustment is that resilience in the limit-order book causes the effect of these prior orders to decay over time. In this paper, there is no permanent effect from the purchase we model. However, the temporary effect requires infinite time to completely disappear.

We assume that there is a fixed shadow limit-order book shape toward which resilience returns the limit-order book. At any time, the actual limit-order book relative to the martingale component of the ask price has this shape, but with some left-hand part missing due to prior purchases. An investor is given a period of time and a target amount of asset to be purchased within that period. His goal is to distribute his purchasing over the period in order to minimize the expected cost of purchasing the target. We permit purchases to occur in lumps or to be spread continuously over time. We show that the optimal execution strategy consists of three lump purchases, one or more of which may be of size zero, i.e., does not occur. One of these lump purchases is made at the initial time, one at an intermediate time, and one at the final time. Between these lump purchases, the optimal strategy purchases at a constant rate matched to the limit-order book recovery rate so that the ask price minus its martingale component remains constant. We provide a simple condition under which the intermediate lump purchase is of size zero (see Theorem 4.2 and Remark 4.4 below).

Bouchaud, et. al. [9] provide a survey of the empirical behavior of limit order books. Dynamic models for optimal execution designed to capture some of this behavior have been developed by several authors, including Bertsimas and Lo [8], Almgren and Chriss [6, 7], Grinold and Kahn [15] (Chapter 16), Almgren [5], Obizhaeva and Wang [10], and Alfonsi, Fruth and Schied [1, 4]. Trading in [8] is on a discrete-time grid, and the price impact of a trade is linear in the size of the trade and is permanent. In [8], the expected-cost-minimizing liquidation strategy for an order is to divide the order into equal pieces, one for each trading date. Trading in [6, 7] is also on a discrete-time grid, and there are linear permanent and temporary price impacts. In [6, 7] the variance of the cost of execution is taken into account. This leads to the construction of an efficient frontier of trading strategies.

In [15] and [5], trading takes place continuously and finding the optimal trading strategy reduces to a problem in the calculus of variations.

Other authors focus on the possibility of price manipulation, an idea that traces back to Huberman and Stanzl [16]. Price manipulation is a way of starting with zero shares and using a strategy of buying and selling so as to end with zero shares while generating income. Gatheral et. al. [13] permit continuous trading and use an integral of a kernel with respect to the trading strategy to capture the resilience of the book. In such a model, Gatheral [12] shows that exponential decay of market impact and absence of price manipulation opportunities are compatible only with linear market impact. In [14] this result is reconciled with the nonlinear market impact in models such as [2, 3, 4, 10] and this paper. Alfonsi, Schied and Slynko [3] discover in a discrete-time version of the model of [13], even under conditions that prevent price manipulation, it may still be optimal to execute intermediate sells while trying to execute an overall buy order, and they provide conditions to rule out this phenomenon.

For the type of model we consider in this paper, based on a shadow limit-order book, Alfonsi and Schied [2] show that price manipulation is not possible under very general conditions. Furthermore, it is never advantageous to execute intermediate sells while trying to execute an overall buy order. In [2], trading takes place at finitely many stopping times, and execution is optimized over these stopping times. In the present paper, where trading is continuous, we do not permit intermediate sells. This simplification of the model is justified by Remark 3.1 below, which argues that intermediate sells cannot reduce the total cost.

The present paper is inspired by Obizhaeva and Wang [10], who explicitly model the one-sided limit-order book as a means to capture the price impact of order execution. Empirical evidence for the model of [10] and its generalizations by Alfonsi, Fruth and Schied [1, 4] and Alfonsi and Schied [2] are reported in [1, 2, 4, 10]. In [10] and [1], the limit-order book has a block shape, and in this case the price impact of a purchase is linear, the same as in [8] and [7]. However, the change of mind set is important because it focuses attention on the shape of the limit order book as the determinant of price impact, rather than making assumptions about the price impact directly. This change of mind set was exploited by [2, 4], who permit more general limit-order book shapes, subject to the condition discussed in Remark 4.4 below. In [2, 4] trading is on a discrete-time grid and it is shown that for an optimal purchasing strategy all purchases except the first and last are of the same size. Furthermore, the size of the intermediate purchases is chosen so that the price impact of each purchase is exactly offset by the order book resiliency before the next purchase. Similar results are obtained in [2], although here trades are executed at stopping times.

In contrast to [2, 4, 10], we permit the order book shape to be completely general. However, in our model all price impact is transient; [4, 10] also include the possibility of a permanent linear price impact. In contrast to [2, 4], we do not assume that the limit order book has a positive density. It can be discrete or continuous and can have gaps. In contrast to [2, 4, 10], we permit the resilience in the order book to be a function of the adjustments to the martingale component of the ask price. Weiss [18] argues in a discrete-time model that this conforms better to empirical observations.

Finally, we set up our model so as to allow for both discrete-time and continuous-time trading, whereas [4, 10] begin with discrete-time trading and then study the limit of their optimal strategies as trading frequency approaches infinity. The simplicity afforded by a fully continuous model is evident in the analysis below. In particular, we provide constructive proofs of Theorems 4.2 and 4.5 that describe the form of the optimal purchasing strategies.

Section 2 of this paper presents our model. It contains the definition of the cost of purchasing in our more general framework, and that is preceded by a justification of the definition. Section 3 shows that randomness can be removed from the optimal purchasing problem and reformulates the cost function into a convenient form. In Section 4, we solve the problem, first in the case that is analogous to the one solved by [4], and then in full generality. Sections 4.1 and 4.3 contain examples.

2. The model

Let T be a positive constant. We assume that the ask price of some asset, in the absence of the large investor modeled by this paper, is a continuous nonnegative martingale A_t , $0 \leq t \leq T$, relative to some filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions. We assume that

$$(2.1) \quad \mathbb{E} \left[\max_{0 \leq t \leq T} A_t \right] < \infty.$$

We show below that for the optimal execution problem of this paper, one can assume without loss of generality that this martingale is identically zero. We make this assumption beginning in Section 3 in order to simplify the presentation.

For some extended positive real number M , let μ be an infinite measure on $[0, M)$ that is finite on each compact subset of $[0, M)$. Denote the associated left-continuous cumulative distribution function by

$$F(x) \triangleq \mu([0, x)), \quad x \geq 0.$$

This is the *shadow limit-order book*, in the sense described below. We assume $F(x) > 0$ for every $x > 0$. If B is a measurable subset of $[0, M)$, then in the absence of the large investor modeled in this paper, at time $t \geq 0$ the number of limit orders with prices in $B + A_t \triangleq \{b + A_t; b \in B\}$ is $\mu(B)$.

There is a strictly positive constant $\bar{X}$ such that our large investor must purchase $\bar{X}$ shares over the time interval $[0, T]$. His *purchasing strategy* is a non-decreasing right-continuous adapted process X with $X_T = \bar{X}$. We interpret X_t to be the cumulative amount of purchasing done by time t . We adopt the convention $X_{0-} = 0$, so that $X_0 = \Delta X_0$ is the number of shares purchased at time zero. Here and elsewhere, we use the notation ΔX_t to denote the jump $X_t - X_{t-}$ in X at time t .

The effect of the purchasing strategy X on the limit-order book is determined by a *resilience function* h , a strictly increasing, locally Lipschitz function defined on $[0, \infty)$ and satisfying

$$(2.2) \quad h(0) = 0, \quad h(\infty) \triangleq \lim_{x \rightarrow \infty} h(x) > \frac{\bar{X}}{T}.$$

The function h together with X determine the *volume effect process*¹ E satisfying

$$(2.3) \quad E_t = X_t - \int_0^t h(E_s) ds, \quad 0 \leq t \leq T.$$

It is shown in Appendix 1 that there is a unique nonnegative right-continuous finite-variation adapted process E satisfying (2.3). As with X , we adopt the convention $E_{0-} = 0$. We note that $\Delta X_t = \Delta E_t$ for $0 \leq t \leq T$.

Let B be a measurable subset of $[0, M)$. The interpretation of E is that in the presence of the large investor using strategy X , at time $t \geq 0$ the number of limit orders with prices in $B + A(t)$ is $\mu_t(B)$, where μ_t is the σ -finite infinite measure on $[0, M)$ with left-continuous cumulative distribution function $(F(x) - E_t)^+$, $x \geq 0$. In other words, E_t units of mass have been removed from the shadow limit-order book μ . In any interval in which no purchases are made, (2.3) implies $\frac{d}{dt} E_t = -h(E_t)$. Hence, in the absence of purchases, the volume effect process decays toward zero and the limit-order book tends toward the shadow limit-order book μ , displaced by the ask price A .

To calculate the cost to the investor of using the strategy X , we introduce the following notation. We first define the left-continuous inverse of F ,

$$\psi(y) \triangleq \sup\{x \geq 0 | F(x) < y\}, \quad y > 0.$$

We set $\psi(0) \triangleq \psi(0+) = 0$, where the second equality follows from the assumption that $F(x) > 0$ for every $x > 0$. The *ask price* in the presence of the large investor is defined to be $A_t + D_t$, where

$$(2.4) \quad D_t \triangleq \psi(E_t), \quad 0 \leq t \leq T.$$

This is the price after any lump purchases by the investor at time t (see Fig. 1). We give some justification for calling $A_t + D_t$ the ask price after the following three examples.

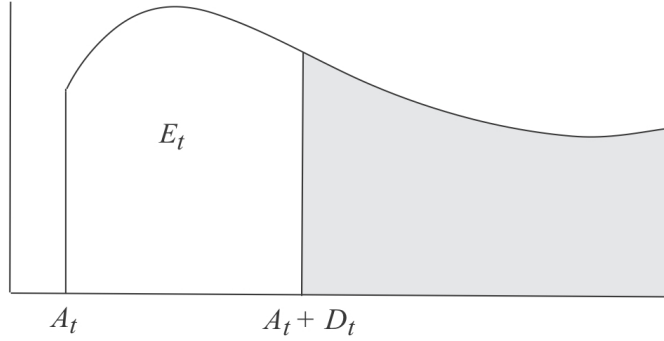


FIGURE 1. Limit order book at time t . The shaded region corresponds to the remaining shares. The white area E_t corresponds to the amount of shares missing from the order book at time t . The current ask price is $A_t + D_t$.

¹The case that resilience is based on price rather than volume is also considered in [2, 4].

EXAMPLE 2.1 (Block order book). Let q be a fixed positive number. If q is the quantity of shares available at each price, then for each $x \geq 0$, the quantity available at prices in $[0, x]$ is $F(x) = qx$. This is the *block order book* considered by [10]. In this case, $\psi(y) = y/q$ and $F(\psi(y)) = y$ for all $y \geq 0$. $\square$

EXAMPLE 2.2 (Modified block order book). Let $0 < a < b < \infty$ be given, and suppose

$$(2.5) \quad F(x) = \begin{cases} x, & 0 \leq x \leq a, \\ a, & a \leq x \leq b, \\ x - (b - a), & b \leq x < \infty. \end{cases}$$

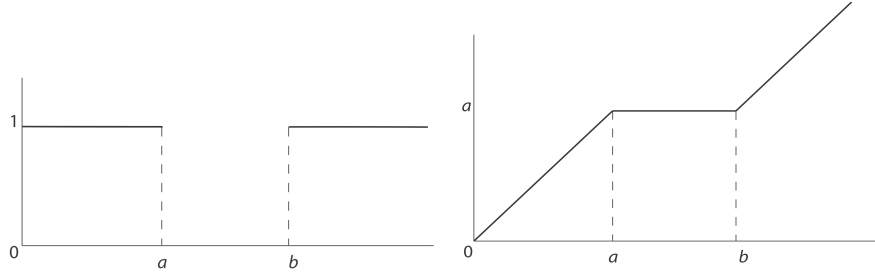


FIGURE 2. Density and cumulative distribution of the modified block order book

This is a block order book, except that the orders with prices between a and b are not present (see Fig 2). In this case,

$$(2.6) \quad \psi(y) = \begin{cases} y, & 0 \leq y \leq a, \\ y + b - a, & a < y < \infty. \end{cases}$$

We have $F(\psi(y)) = y$ for all $y \geq 0$. $\square$

EXAMPLE 2.3 (Discrete order book). Suppose that

$$(2.7) \quad F(x) = \sum_{i=0}^{\infty} \mathbb{I}_{(i, \infty)}(x), \quad x \geq 0,$$

which corresponds to an order of size one at each of the nonnegative integers (see Fig. 3). Then

$$(2.8) \quad \psi(y) = \sum_{i=1}^{\infty} \mathbb{I}_{(i, \infty)}(y), \quad y \geq 0.$$

For every nonnegative integer j , we have $F(j) = j$, $F(j+) = j + 1$, $\psi(j + 1) = j$, $\psi(j+) = j$, $F(\psi(j+)) = j$ and $\psi(F(j+)) = j$. $\square$

We return to the definition of the ask price as $A_t + D_t$ to provide some justification, leading up to Definition 2.4, for the total cost of a purchasing strategy. Suppose, as in Example 2.2, F is constant on an interval $[a, b]$, but strictly increasing to the left of a and to the right of b . Let $y = F(x)$ for $a \leq x \leq b$. Then $\psi(y) = a$ and $\psi(y+) = b$. Suppose at time t , we have $E_t = y$. Then $D_t = a$, but the measure μ_t assigns mass zero to $[a, b)$. The ask price is $A_t + D_t$, but there are no shares for sale at this price, nor in an interval to the right of this price. Nonetheless, it is

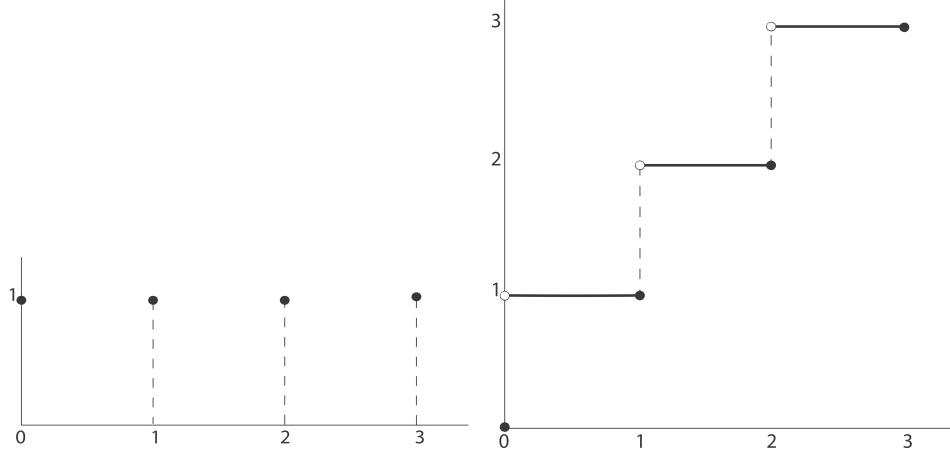


FIGURE 3. Measure and cumulative distribution function of the discrete order book

reasonable to call $A_t + D_t$ the ask price for an infinitesimal purchase because if the agent will wait an infinitesimal amount of time before making this purchase, shares will appear at the price $A_t + D_t$ due to resilience. We make this argument more precise.

Suppose the agent wishes to purchase a small number $\varepsilon > 0$ shares at time t at the ask price $A_t + D_t$. This purchase can be approximated by first purchasing zero shares in the time interval $[t, t + \delta]$, where δ is chosen so that $\int_t^{t+\delta} h(E_s) ds = \varepsilon$ and

$$E_s = X_t - \int_0^s h(E_u) du, \quad t \leq s < t + \delta.$$

In other words, E_s for $t \leq s < t + \delta$ is given by (2.3) with X held constant (no purchases) over this interval. With δ chosen this way, $E_{(t+\delta)-} = E_t - \varepsilon$. Resilience in the order book has created ε shares. Suppose the investor purchases these shares at time $t + \delta$, which means that $\Delta X_{t+\delta} = \Delta E_{t+\delta} = \varepsilon$ and $E_{t+\delta} = E_t$. Immediately before the purchase, the ask price is $A_{t+\delta} + \psi(E_t - \varepsilon)$; immediately after the purchase, the ask price is $A_{t+\delta} + \psi(E_t) = A_{t+\delta} + a$. The cost of purchasing these shares is

$$(2.9) \quad \varepsilon A_{t+\delta} + \int_{[\psi(E_t - \varepsilon), a]} \xi d(F(\xi) - E_t + \varepsilon)^+,$$

Because $\int_{[\psi(E_t - \varepsilon), a]} d(F(\xi) - E_t + \varepsilon)^+ = \varepsilon$, the integral in (2.9) is bounded below by $\varepsilon \psi(E_t - \varepsilon)$ and bounded above by εa . But $a = \psi(E_t) = D_t$ and ψ is left continuous, so the cost per share obtained by dividing (2.9) by ε converges to $A_t + a = A_t + D_t$ as ε (and hence δ) converge down to zero.

On the other hand, an impatient agent who does not wait before purchasing shares could choose a different method of approximating an infinitesimal purchase at time t that leads to a limiting cost per share $A_t + b$. In particular, it is not the case that our definition of ask price is consistent with all limits of discrete-time purchasing strategies. Our definition is designed to capture the limit of discrete-time purchasing strategies that seek to minimize cost.

To simplify calculations of the type just presented, we define the functions

$$(2.10) \quad \varphi(x) = \int_{[0,x)} \xi dF(\xi), \quad x \geq 0,$$

$$(2.11) \quad \Phi(y) = \varphi(\psi(y)) + [y - F(\psi(y))] \psi(y), \quad y \geq 0.$$

We note that $\Phi(0) = 0$, and we extend Φ to be zero on the negative half-line. In the absence of the large investor, the cost one would pay to purchase all the shares available at prices in the interval $[A(t), A(t) + x]$ at time t would be $A(t) + \varphi(x)$. The function $\Phi(y)$ captures the cost, in excess of A_t , of purchasing y shares in the absence of the large investor. The first term on the right-hand side of (2.11) is the cost less A_t of purchasing all the shares with prices in the interval $[A_t, A_t + \psi(y))$. If F has a jump at $\psi(y)$, this might be fewer than y shares. The difference, $y - F(\psi(y))$ shares, can be purchased at price $A_t + \psi(y)$, and this explains the second term on the right-hand side of (2.11). We present these functions in the three examples considered earlier.

EXAMPLE 2.1 (Block order book, continued). We have simply $\varphi(x) = q \int_0^x \xi d\xi = \frac{q}{2}x^2$ for all $x \geq 0$, and $\Phi(y) = \frac{q}{2}\psi^2(y) = \frac{1}{2q}y^2$ for all $y \geq 0$. Note that Φ is convex and $\Phi'(y) = \psi(y)$ for all $y \geq 0$, including at $y = 0$ because we define Φ to be identically zero on the negative half-line. $\square$

EXAMPLE 2.2 (Modified block order book, continued). With F and ψ given by (2.5) and (2.6), we have

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2, & 0 \leq x \leq a, \\ \frac{1}{2}a^2, & a \leq x \leq b, \\ \frac{1}{2}(x^2 + a^2 - b^2), & b \leq x < \infty, \end{cases}$$

and

$$\Phi(y) = \begin{cases} \frac{1}{2}y^2, & 0 \leq y \leq a, \\ \frac{1}{2}((y + b - a)^2 + a^2 - b^2), & a \leq y < \infty. \end{cases}$$

Note that Φ is convex with subdifferential

$$(2.12) \quad \partial\Phi(y) = \begin{cases} \{y\}, & 0 \leq y < a, \\ [a, b], & y = a, \\ \{y + b - a\}, & a < y < \infty. \end{cases}$$

In particular, $\partial\Phi(y) = [\psi(y), \psi(y+)]$ for all $y \geq 0$ (see Fig. 4). $\square$

EXAMPLE 2.3 (Discrete order book, continued). With F given by (2.7), we have $\varphi(x) = \sum_{i=0}^{\infty} i \mathbb{I}_{(i,\infty)}(x)$. In particular, $\varphi(0) = 0$ and for integers $k \geq 1$ and $k-1 < x \leq k$,

$$\varphi(x) = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}.$$

For $0 \leq y \leq 1$, $\psi(y) = 0$ and hence $\varphi(\psi(y)) = 0$, $[y - F(\psi(y))]\psi(y) = 0$, and $\Phi(y) = 0$. For integers $k \geq 1$ and $k < y \leq k+1$, (2.8) gives $\psi(y) = k$, and hence

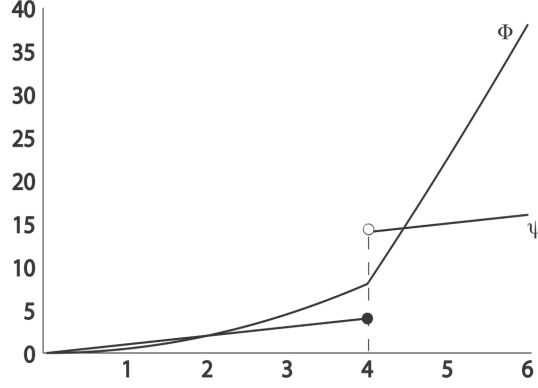


FIGURE 4. Functions Φ and ψ for the modified block order book with parameters $a = 4$ and $b = 14$

$\varphi(\psi(y)) = \frac{k(k-1)}{2}$. Finally, for y in this range, $[y - F(\psi(y))]\psi(y) = k(y - k)$. We conclude that

$$(2.13) \quad \Phi(y) = \sum_{k=1}^{\infty} k \left(y - \frac{1}{2}k - \frac{1}{2} \right) \mathbb{I}_{(k, k+1]}(y).$$

For each positive integer k , $\Phi(k-) = \Phi(k+) = \frac{1}{2}k(k-1)$, so Φ is continuous. Furthermore, $\partial\Phi(k) = [k-1, k] = [\psi(k), \psi(k+)]$. For nonnegative integers k and $k < y < k+1$, $\Phi'(y)$ is defined and is equal to $\psi(y) = k$. Furthermore $\Phi'(0) = \psi(0) = 0$. Once again we have $\partial\Phi(y) = [\psi(y), \psi(y+)]$ for all $y \geq 0$, and because ψ is nondecreasing, Φ is convex (see Fig. 5). $\square$

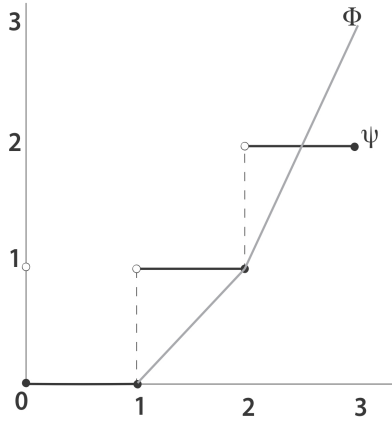


FIGURE 5. Functions Φ and ψ for the discrete order book

We decompose the purchasing strategy X into its continuous and pure jump parts $X_t = X_t^c + \sum_{0 \leq s \leq t} \Delta X_s$. The investor pays price $A_t + D_t$ for infinitesimal

purchases at time t , and hence the total cost of these purchases is $\int_0^T (A_t + D_t) dX_t^c$. On the other hand, if $\Delta X_t > 0$, the investor makes a lump purchase of size $\Delta X_t = \Delta E_t$ at time t . Because mass E_{t-} is missing in the shadow order book immediately prior to time t , the cost of this purchase is the difference between purchasing E_t and purchasing E_{t-} from the shadow order book, i.e., the difference in what the costs of these purchases would be in the absence of the large investor. Therefore, the cost of the purchase ΔX_t at time t is $A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t-})$. These considerations lead to the following definition.

DEFINITION 2.4. *The total cost incurred by the investor using purchasing strategy X over the interval $[0, T]$ is*

$$\begin{aligned} C(X) &\triangleq \int_0^T (A_t + D_t) dX_t^c + \sum_{0 \leq t \leq T} [A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t-})] \\ &= \int_0^T D_t dX_t^c + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \int_{[0, T]} A_t dX_t. \end{aligned}$$

Our goal is to determine the purchasing strategy X that minimizes $\mathbb{E}C(X)$.

3. Problem simplifications

To compute the expectation of $C(X)$ defined by (2.14), we invoke the integration by parts formula

$$\int_{[0, T]} A_t dX_t = A_T X_T - A_0 X_{0-} - \int_0^T X_t dA_t$$

for the bounded variation process X and the continuous martingale A . Our investor's strategies must satisfy $0 = X_{0-} \leq X_t \leq X_T = \bar{X}$, $0 \leq t \leq T$, and hence $\mathbb{E} \int_0^T X_t dA_t = 0$ (see Appendix 2) and $\mathbb{E} \int_0^T A_t dX_t = \bar{X} \mathbb{E} A_T = \bar{X} A_0$. It follows that

$$(3.1) \quad \mathbb{E}C(X) = \mathbb{E} \int_0^T D_t dX_t^c + \mathbb{E} \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \bar{X} A_0.$$

Since the third term on the right-hand side of (3.1) does not depend on X , minimization of $\mathbb{E}C(X)$ is equivalent to minimization of the first two terms. But the first two terms do not depend on A , and hence we may assume without loss of generality that A is identically zero. Under this assumption, the cost of using strategy X is

$$(3.2) \quad C(X) = \int_0^T D_t dX_t^c + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})].$$

But with $A \equiv 0$, there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time. Once we find a nonrandom purchasing strategy minimizing (3.4) below, then even if A is a continuous non-zero nonnegative martingale, we have found a purchasing strategy that minimizes the expected value of (2.14) over all (possibly random) purchasing strategies.

REMARK 3.1. We do not allow our agent to make intermediate sells in order to achieve the ultimate goal of purchasing $\bar{X}$ shares because doing so would not decrease the cost, at least when the total amount of buying and selling is bounded. Indeed, in addition to the purchasing strategy X , suppose the agent has a selling strategy Y , which we take to be a non-decreasing right-continuous adapted process with $Y_{0-} = 0$. We assume that both X and Y are bounded. For each t , X_t represents the number of shares bought by time t and Y_t is the number of shares sold. These processes must be chosen so that $X_T - Y_T = \bar{X}$. We have not modeled the limit buy order book, but if we did so in a way analogous to the model of the limit sell order book, then the bid price at each time t would be less than or equal to A_t . Therefore, the net cost of executing the strategy (X, Y) would satisfy

$$C(X, Y) \geq \int_0^T D_t dX_t^c + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \int_{[0, T]} A_t dX_t - \int_{[0, T]} A_t dY_t.$$

The integration by parts formula implies

$$\begin{aligned} \int_{[0, T]} A_t dX_t - \int_{[0, T]} A_t dY_t &= A_T(X_T - Y_T) - A_0(X_{0-} - Y_{0-}) \\ &\quad - \int_0^T (X_t - Y_t) dA_t \\ &= A_T \bar{X} - \int_0^T (X_t - Y_t) dA_t. \end{aligned}$$

Because we can apply Lemma 2.1 to both X and Y , the expectation of $\int_0^T (X_t - Y_t) dA_t$ is zero and

$$(3.3) \quad \mathbb{E}C(X, Y) \geq \mathbb{E} \int_0^T D_t dX_t^c + \mathbb{E} \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \bar{X} A_0.$$

The right-hand side of (3.3) is the formula (3.1) obtained for the cost of using the purchasing strategy X alone, but the X in inequality (3.3) makes a total purchase of $X_T = \bar{X} + Y_T \geq \bar{X}$. If we replace X by $\min\{X, \bar{X}\}$, we obtain a feasible purchasing strategy whose total cost is less than or equal to the right-hand side of (3.3). $\square$

THEOREM 3.2. *Under the assumption (made without loss of generality) that A is identically zero, the cost (3.2) associated with a nonrandom nondecreasing right continuous function X_t , $0 \leq t \leq T$, satisfying $X_{0-} = 0$ and $X_T = \bar{X}$ is equal to*

$$(3.4) \quad C(X) = \Phi(E_T) + \int_0^T D_t h(E_t) dt.$$

PROOF: The proof proceeds in two steps. In Step 1 we show that, as we have seen in the examples, Φ is a convex function with subdifferential

$$(3.5) \quad \partial\Phi(y) = [\psi(y), \psi(y+)], \quad y \geq 0.$$

In Step 2 we justify the integration formula

$$(3.6) \quad \Phi(E_T) = \int_0^T D^- \Phi(E_t) dE_t^c + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})],$$

where $D^-\Phi(E_t)$ denotes the left-hand derivative $\psi(E_t) = D_t$ of Φ at E_t , and E^c is the continuous part of E : $E_t^c = E_t - \sum_{0 \leq s \leq t} \Delta E_s$. From (2.3) and (3.6) we have immediately that

$$\Phi(E_T) = \int_{[0,T]} D_t dX_t^c - \int_0^T D_t h(E_t) dt + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})],$$

and (3.4) follows from (3.2).

Step 1. Using the integration by parts formula $xF(x) = \int_{[0,x]} \xi dF(\xi) + \int_0^x F(\xi) d\xi$, we write

$$\begin{aligned} \Phi(y) &= \int_{[0,\psi(y))} \xi dF(\xi) + [y - F(\psi(y))]\psi(y) \\ &= \int_0^{\psi(y)} (y - F(\xi)) d\xi \\ &= \int_0^{\psi(y)} \int_{F(\xi)}^y d\eta d\xi \\ &= \int_0^y \int_0^{\psi(\eta)} d\xi d\eta, \end{aligned}$$

where the last step follows from the fact that the symmetric difference of the sets $\{(\eta, \xi) | \xi \in [0, \psi(y)], \eta \in [F(\xi), y]\}$ and $\{(\eta, \xi) | \eta \in [0, y], \xi \in [0, \psi(\eta)]\}$ is at most a countable union of line segments and thus has two-dimensional Lebesgue measure 0. Therefore,

$$(3.7) \quad \Phi(y) = \int_0^y \psi(\eta) d\eta,$$

and by Problem 3.6.20, p. 213 of [17], with ψ and Φ extended to be 0 for the negative reals, we conclude that Φ is convex and that $\partial\Phi(y) = [\psi(y), \psi(y+)]$, as desired.

Step 2. We mollify ψ , taking ρ to be a nonnegative C^∞ function with support on $[-1, 0]$ and integral 1, defining $\rho_n(\eta) = n\rho(n\eta)$, and defining

$$\psi_n(y) = \int_{\mathbb{R}} \psi(y + \eta) \rho_n(\eta) d\eta = \int_{\mathbb{R}} \psi(\zeta) \rho_n(\zeta - y) d\zeta.$$

Then each ψ_n is a C^∞ function satisfying $0 \leq \psi_n(y) \leq \psi(y)$ for all $y \geq 0$. Furthermore, $\psi(y) = \lim_{n \rightarrow \infty} \psi_n(y)$ for every $y \in \mathbb{R}$. We set $\Phi_n(y) = \int_0^y \psi_n(\eta) d\eta$, so that each Φ_n is also a C^∞ function and $\lim_{n \rightarrow \infty} \Phi'_n(y) = D^-\Phi(y)$.

Because $\Phi_n(E_{0-}) = \Phi(0) = 0$, we have

$$(3.8) \quad \Phi_n(E_T) = \int_0^T \Phi'_n(E_t) dE_t^c + \sum_{0 \leq t \leq T} [\Phi_n(E_t) - \Phi_n(E_{t-})];$$

see, e.g., [11], p. 78. The function E_t , $0 \leq t \leq T$, is bounded. Letting $n \rightarrow \infty$ in (3.8) and using the bounded convergence theorem, we obtain

$$(3.9) \quad \Phi(E_T) = \int_0^T D^-\Phi(E_t) dE_t^c + \lim_{n \rightarrow \infty} \sum_{0 \leq t \leq T} [\Phi_n(E_t) - \Phi_n(E_{t-})].$$

To conclude the proof of (3.6), we divide the sum in (3.9) into two parts. Given $\delta > 0$, we define $S_\delta = \{t \in [0, T] : 0 < \Delta E_t \leq \delta\}$ and $S'_\delta = \{t \in [0, T] :$

$\Delta E_t > \delta\}$. The sum in (3.9) is over $t \in S_\delta \cup S'_\delta$, and because E has finite variation, $\sum_{t \in S_\delta \cup S'_\delta} \Delta E_t < \infty$. Let $\varepsilon > 0$ be given. We choose $\delta > 0$ so small that $\sum_{t \in S_\delta} \Delta E_t \leq \varepsilon$. Because ψ and hence each ψ_n is bounded on $[0, E_T]$, the function Φ and each Φ_n is Lipschitz continuous on $[0, E_T]$ with the same Lipschitz constant $L = \psi(E_T)$. It follows that

$$\begin{aligned} \sum_{t \in S_\delta} [\Phi(E_t) - \Phi(E_{t-})] &\leq L \sum_{t \in S_\delta} \Delta E_t \leq L\varepsilon, \\ \sum_{t \in S_\delta} [\Phi_n(E_t) - \Phi_n(E_{t-})] &\leq L \sum_{t \in S_\delta} \Delta E_t \leq L\varepsilon, \quad n = 1, 2, \dots \end{aligned}$$

Hence the difference between $\sum_{t \in S_\delta} [\Phi(E_t) - \Phi(E_{t-})]$ and any limit point as $n \rightarrow \infty$ of $\sum_{t \in S_\delta} [\Phi_n(E_t) - \Phi_n(E_{t-})]$ is at most $2L\varepsilon$. On the other hand, the set S'_δ contains only finitely many elements, and thus

$$\lim_{n \rightarrow \infty} \sum_{t \in S'_\delta} [\Phi_n(E_t) - \Phi_n(E_{t-})] = \sum_{t \in S'_\delta} [\Phi(E_t) - \Phi(E_{t-})].$$

Since $\varepsilon > 0$ is arbitrary, (3.9) reduces to (3.6). $\square$

4. Solution of the Optimization Problem

In view of Theorem 3.2, we want to minimize $\Phi(E_T) + \int_0^T D_t h(E_t) dt$ over the set of deterministic purchasing strategies. The main result of this paper is that there exists an optimal strategy X under which the trader buys a lump quantity $X_0 = E_0$ of shares at time 0, then buys at a constant rate $dX_t = h(E_0) dt$ up to time t_0 (so as to keep $E_t = E_0$ for $t \in [0, t_0)$), then buys another lump quantity of shares at time t_0 , subsequently trades again at a constant rate $dX_t = h(E_{t_0}) dt$ until time T (so as to keep $E_t = E_{t_0}$ for $t \in [t_0, T)$), and finally buys the remaining shares at time T . We shall call this strategy a *Type B strategy*. We further show that if the nonnegative function

$$(4.1) \quad g(y) \triangleq y\psi(h^{-1}(y))$$

is convex, then the purchase at time t_0 consists of 0 shares (so X has only jumps at times 0 and T). We call such a strategy a *Type A strategy*. Clearly the latter is a special case of the former.

Although g is naturally defined on $[0, h(\infty))$ by (4.1), we will want it to be defined on a compact set. Therefore we set

$$(4.2) \quad \bar{Y} = \max \left\{ h(\bar{X}), \frac{\bar{X}}{T} \right\}$$

and note that because of assumption (2.2), h^{-1} is defined on $[0, \bar{Y}]$. We specify the domain of the function g to be $[0, \bar{Y}]$. For future reference, we make three observations about the function g . First,

$$(4.3) \quad \lim_{y \downarrow 0} g(y) = g(0) = 0.$$

Secondly, using the definition (2.4) of D_t , we can rewrite the cost function formula (3.4) as

$$(4.4) \quad C(X) = \Phi(E_T) + \int_0^T g(h(E_t)) dt.$$

Lemma 1.1(iv) in the appendix shows that $0 \leq E_t \leq \bar{X}$, so the domain $[0, \bar{Y}]$ of g is large enough in order for (4.4) to make sense. Because h^{-1} is strictly increasing and continuous and ψ is nondecreasing and left continuous, the function g is nondecreasing and left continuous, hence lower semicontinuous. In particular,

$$(4.5) \quad g(\bar{Y}) = \lim_{y \uparrow \bar{Y}} g(y).$$

4.1. Convexity and Type A Strategies.

REMARK 4.1. A Type A strategy X^A can be characterized in terms of the terminal value E_T^A of the process E^A related to X^A by (2.3), and the cost of using a Type A strategy can be written as a function of E_T^A . It is this function of E_T^A we will minimize. To see that this is possible, let X^A be a Type A strategy and let E^A be related to X^A via (2.3), so that $E_t^A = X_0^A$ for $0 \leq t < T$. Then

$$(4.6) \quad X_{T-}^A = E_{T-}^A + \int_0^T h(E_t^A) dt = X_0^A + h(X_0^A)T,$$

$$(4.7) \quad \Delta X_T^A = \bar{X} - X_{T-}^A = \bar{X} - X_0^A - h(X_0^A)T,$$

$$(4.8) \quad E_T^A = E_{T-}^A + \Delta X_T^A = \bar{X} - h(X_0^A)T.$$

A Type A strategy is fully determined by its initial condition X_0^A , and from (4.8), we now see that choosing X_0^A is equivalent to choosing E_T^A . According to (4.4) and (4.8), the cost of this strategy

$$(4.9) \quad C(X^A) = \Phi(E_T^A) + Tg(h(X_0^A)) = \Phi(E_T^A) + Tg\left(\frac{\bar{X} - E_T^A}{T}\right)$$

can be written as a function of E_T^A .

We conclude this remark by determining the range of values that E_T^A can take for a Type A strategy. We must choose X_0^A so that $X_0^A \geq 0$ and X_{T-}^A given by (4.6) does not exceed $\bar{X}$. The function $k(x) \triangleq x + h(x)T$ is strictly increasing and continuous on $[0, \infty)$, and $k(\bar{X}) > \bar{X}$. Therefore, there exists a unique $\bar{e} \in (0, \bar{X})$ such that $k(\bar{e}) = \bar{X}$. i.e.,

$$(4.10) \quad \bar{e} + h(\bar{e})T = \bar{X}.$$

The constraint on the initial condition of Type A strategies that guarantees that the strategy is feasible is $0 \leq X_0^A \leq \bar{e}$. From (4.8) and (4.10) we see that the corresponding feasibility condition on E_T^A for Type A strategies is

$$(4.11) \quad \bar{e} \leq E_T^A \leq \bar{X}.$$

□

THEOREM 4.2. *If g given by (4.1) is convex on $[0, \bar{Y}]$, then there exists a Type A purchasing strategy that minimizes $C(X)$ over all purchasing strategies X . If g is strictly convex, this is the unique optimal strategy.*

PROOF: Assume that g is convex and let X be a purchasing strategy. Jensen's inequality applied to (4.4) yields the lower bound

$$C(X) = \Phi(E_T) + T \int_0^T g(h(E_t)) \frac{dt}{T} \geq \Phi(E_T) + Tg\left(\int_0^T h(E_t) \frac{dt}{T}\right).$$

From (2.3) we further have $\int_0^T h(E_t) dt = \bar{X} - E_T$, and thus the lower bound can be rewritten as

$$(4.12) \quad C(X) \geq \Phi(E_T) + Tg\left(\frac{\bar{X} - E_T}{T}\right).$$

Recall that $0 \leq E_T \leq \bar{X}$, so the argument of g in (4.12) is in $[0, \bar{Y}]$.

This leads us to consider minimization of the function

$$G(e) \triangleq \Phi(e) + Tg\left(\frac{\bar{X} - e}{T}\right)$$

over $e \in [0, \bar{X}]$. By assumption, the function g is convex on $[0, \bar{Y}]$ and hence continuous on $(0, \bar{Y})$. Equations (4.3) and (4.5) show that g is also continuous at the endpoints of its domain. Because Φ has the integral representation (3.7), it also is convex and continuous on $[0, \bar{X}]$. Therefore, G is a convex continuous function on $[0, \bar{X}]$, and hence the minimum is attained.

We show next that the minimum of G over $[0, \bar{X}]$ is attained in $[\bar{e}, \bar{X}]$. For this, we first observe that because g is convex,

$$D^+g(y) \geq \frac{g(y) - g(0)}{y} = \psi(h^{-1}(y)), \quad 0 < y \leq \bar{Y}.$$

This inequality together with (3.5) and (4.10) implies

$$(4.13) \quad D^-G(\bar{e}) = \psi(\bar{e}) - D^+g(y) \Big|_{y=\frac{\bar{X}-\bar{e}}{T}} \leq \psi(\bar{e}) - \psi\left(h^{-1}\left(\frac{\bar{X}-\bar{e}}{T}\right)\right) = 0.$$

Therefore, the minimum of the convex function G over $[0, \bar{X}]$ is obtained in $[\bar{e}, \bar{X}]$.

Let $e^* \in [\bar{e}, \bar{X}]$ attain the minimum of G over $[0, \bar{X}]$. The Type A strategy X^A with initial condition $X_0^A = h^{-1}(\frac{\bar{X}-e^*}{T})$ satisfies $E_T^A = e^*$ (see (4.8)), and hence the strategy is feasible (see (4.11)). The cost associated with this strategy is less than or equal to the right-hand side of (4.12) (see (4.9)). This strategy is therefore optimal.

If g is strictly convex at the point $\frac{\bar{X}-e^*}{T}$, where e^* minimizes G , then G is strictly convex at e^* , and this point is thus the unique minimizer of G . Therefore, every optimal strategy must satisfy $E_T = e^*$. By strict convexity of g , a strategy that does not keep $h(E)$ equal to $\frac{\bar{X}-e^*}{T}$ almost everywhere in $(0, T)$, would result in strict inequality in (4.12). Since h is strictly increasing and a process E does not have negative jumps, we conclude that the only optimal strategy is the Type A strategy constructed above. $\square$

If g is not strictly convex at the point $\frac{\bar{X}-e^*}{T}$ found in the proof of Theorem 4.2, then G might still be strictly convex at e^* , in which case there would be only one optimal strategy of Type A, but there could be optimal strategies that are not of Type A. We demonstrate this phenomenon with an example.

EXAMPLE 4.3 (Non-uniqueness of optimal purchasing strategy). Suppose

$$F(x) = \begin{cases} x, & 0 \leq x \leq 2, \\ \frac{4}{4-x}, & 2 \leq x \leq 3, \\ 4 + \frac{1}{8}(x-3), & x \geq 3. \end{cases}$$

This function is continuous and strictly increasing, and hence

$$\psi(y) = \begin{cases} y, & 0 \leq y \leq 2, \\ 4 - \frac{4}{y}, & 2 \leq y \leq 4, \\ 8y - 29, & y \geq 4, \end{cases}$$

is also continuous and strictly increasing. This implies that

$$\Phi(y) = \int_0^y \psi(\eta) d\eta = \begin{cases} \frac{1}{2}y^2, & 0 \leq y \leq 2, \\ 4y - 6 - 4 \log \frac{y}{2}, & 2 \leq y \leq 4, \\ 4y^2 - 29y + 62 - 4 \log 2, & y \geq 4. \end{cases}$$

We take $h(x) = x$, so that

$$g(y) = y\psi(y) = \begin{cases} y^2, & 0 \leq y \leq 2, \\ 4y - 4, & 2 \leq y \leq 4, \\ 8y^2 - 29y, & y \geq 4, \end{cases}$$

and

$$g'(y) = \begin{cases} 2y, & 0 \leq y \leq 2, \\ 4, & 2 \leq y < 4, \\ 16y - 29, & y > 4, \end{cases}$$

Note that g' is nondecreasing, so g is convex, but g is affine on the interval $[2, 4]$. Finally, we take $\bar{X} = 10\frac{1}{8}$ and $T=2$.

In the notation of the proof of Theorem 4.2, we have $e^* = 4\frac{1}{8}$ and hence $\frac{\bar{X}-e^*}{T} = 3$. Indeed, $G'(4\frac{1}{8}) = \psi(4\frac{1}{8}) - g'(3) = 0$, and because ψ is strictly increasing, G is strictly convex, and hence $4\frac{1}{8}$ is the unique minimizer of G .

The Type A strategy with $E_T^A = 4\frac{1}{8}$ begins with an initial purchase of $X_0^A = 3$ and then consumes at rate 3 over the interval $[0, 2]$, so that $E_t^A = 3$ for $0 \leq t < T$. At the final time $T = 2$, there is an additional lump purchase of $1\frac{1}{8}$, so that $E_T^A = 4\frac{1}{8}$. The total cost of this strategy is

$$\Phi(E_T^A) + \int_0^T g(E_t^A) dt = \Phi\left(4\frac{1}{8}\right) + \int_0^2 (4E_t^A - 4) dt = \Phi\left(4\frac{1}{8}\right) + 16.$$

In particular, $\int_0^2 E_t^A dt = 6$.

In fact, any policy that satisfies $2 \leq E_t \leq 4$, $0 \leq t < 2$, and $\int_0^2 E_t dt = 6$ will result in the same cost. Indeed, for such a policy we will have

$$E_T = X_T - \int_0^T E_t dt = 10\frac{1}{8} - 6 = 4\frac{1}{8} = E_T^A$$

and

$$\int_0^T g(E_t) dt = \int_0^T (4E_t - 4) dt = 16 = \int_0^T g(E_t^A) dt,$$

so $\Phi(E_T) + \int_0^T g(E_t) dt = \Phi(E_T^A) + \int_0^T g(E_t^A) dt$. There are infinitely many policies like this. One such is to make an initial lump purchase of size 2, then purchase at rate 2 up to time $\frac{1}{2}$ so that $E_t = 2$, $0 \leq t < \frac{1}{2}$, make a lump purchase of size 1 at time $\frac{1}{2}$, then purchase at rate 3 up to time $\frac{3}{2}$ so that $E_t = 3$, $\frac{1}{2} \leq t < \frac{3}{2}$, make a lump purchase of size 1 at time $\frac{3}{2}$, then purchase at rate 4 up to time 2 so that $E_t = 4$, $\frac{3}{2} \leq t < 2$, and conclude with a lump purchase of size $\frac{1}{8}$ at time 2 so that $E_2 = 4\frac{1}{8}$. $\square$

REMARK 4.4. Alfonsi, Fruth and Schied [4] consider the case that the measure μ has a strictly positive density f . In this case, the function $F(x) = \int_0^x f(\xi) d\xi$ is strictly increasing and continuous with derivative $F'(x) = f(x)$, and its inverse ψ is likewise strictly increasing and continuous with derivative $\psi'(y) = 1/f(\psi(y))$. Furthermore, in [4] the resilience function is $h(x) = \rho x$, where ρ is a positive constant. In this case,

$$g'(y) = \psi(y/\rho) + \frac{y/\rho}{f(\psi(y/\rho))},$$

and Theorem 4.2 guarantees the existence of a Type A strategy under the assumption that g' is nondecreasing. This is equivalent to the condition that

$$\psi(y) + \frac{y}{f(\psi(y))}$$

is nondecreasing.

Alfonsi, Fruth and Schied [4] obtain a discrete-time version of a Type A strategy under the assumption that

$$h_1(y) \triangleq \psi(y) - e^{-\rho\tau}\psi(e^{-\rho\tau}y)$$

is strictly increasing, where τ is the time between trading dates. In order to study the limit of their model as $\tau \downarrow 0$, they observe that

$$\lim_{\tau \downarrow 0} h_1(y)/(1 - e^{-\rho\tau}) = \psi(y) + \frac{y}{f(\psi(y))},$$

which is thus nondecreasing. Thus g given by (4.1) is convex in their model.

To find a simpler formulation of the hypothesis of Theorem 4.2 under the assumption that μ has a strictly positive density f and $h(x) = \rho x$ for a positive constant ρ , we compute

$$\frac{d}{dy} \left(\psi(y) + \frac{y}{f(\psi(y))} \right) = \frac{2}{f(\psi(y))} - \frac{yf'(\psi(y))}{f^3(\psi(y))}.$$

This is nonnegative if and only if $2f^2(\psi(y)) \geq yf'(\psi(y))$. Replacing y by $F(x)$, we obtain the condition

$$2f^2(x) \geq F(x)f'(x), \quad x \geq 0.$$

This is clearly satisfied under the assumption of [10] that f is a positive constant. $\square$

Example 2.1 (Block order book, continued) In the case of the block order book with $h(x) = \rho x$, where ρ is a strictly positive constant,

$$g(y) = \frac{yh^{-1}(y)}{q} = \frac{y^2}{\rho q},$$

which is strictly convex. Theorem 4.2 implies that there is an optimal strategy of Type A, and this is the unique optimal strategy. From the formula $\Phi(e) = \frac{1}{2q}e^2$, we have

$$G(e) = \frac{e^2}{2q} + \frac{(\bar{X} - e)^2}{\rho q T}.$$

The minimizer is $e^* = \frac{2\bar{X}}{2+\rho T}$, which lies between $\bar{e} = \frac{\bar{X}}{1+\rho T}$ and $\bar{X}$, as expected. According to Remark 4.1, the optimal strategy of Type A is to make an initial purchase of size

$$X_0^A = h^{-1} \left(\frac{\bar{X} - e^*}{T} \right) = \frac{\bar{X}}{2 + \rho T},$$

then purchase continuously at rate $dX_t^A = h(X_0^A) dt = \frac{\rho \bar{X}}{2 + \rho T} dt$ over the time interval $[0, T]$, and conclude with a lump purchase

$$e^* - X_0^A = \frac{\bar{X}}{2 + \rho T}$$

at the final time T . In particular, the initial and final lump purchases are of the same size, and there is no intermediate lump purchase.

4.2. Type B Strategies.

THEOREM 4.5. *In the absence of the assumption that g given by (4.1) is convex, there exists a Type B purchasing strategy that minimizes $C(X)$ over all purchasing strategies X .*

The proof of Theorem 4.5 depends on the following lemma, whose proof is given in Appendix 3.

LEMMA 4.6. *The convex hull of g , defined by*

$$(4.14) \quad \hat{g}(y) \triangleq \sup \{ \ell(y) : \ell \text{ is an affine function and } \ell(\eta) \leq g(\eta) \forall \eta \in [0, \bar{Y}] \},$$

is the largest convex function defined on $[0, \bar{Y}]$ that is dominated by g there. It is continuous and nondecreasing on $[0, \bar{Y}]$, $\hat{g}(0) = g(0) = 0$, and $\hat{g}(\bar{Y}) = g(\bar{Y})$. If $y^ \in (0, \bar{Y})$ satisfies $\hat{g}(y^*) < g(y^*)$, then there exists a unique affine function ℓ lying below g on $[0, \bar{Y}]$ and agreeing with $\hat{g}$ at y^* . In addition, there exist numbers α and β satisfying*

$$(4.15) \quad 0 \leq \alpha < y^* < \beta \leq \bar{Y},$$

$$(4.16) \quad \ell(\alpha) = \hat{g}(\alpha) = g(\alpha), \quad \ell(\beta) = \hat{g}(\beta) = g(\beta),$$

$$(4.17) \quad \ell(y) = \hat{g}(y) < g(y), \quad \alpha < y < \beta.$$

PROOF OF THEOREM 4.5: Using $\hat{g}$ in place of g in (4.4), we define the *modified cost function*

$$\hat{C}(X) \triangleq \Phi(E_T) + \int_0^T \hat{g}(h(E_t)) dt.$$

For any purchasing strategy X , we obviously have $\hat{C}(X) \leq C(X)$. Analogously to (4.12), for any purchasing strategy X the lower bound

$$\hat{C}(X) \geq \Phi(E_T) + T\hat{g}\left(\frac{\bar{X} - E_T}{T}\right)$$

holds. This leads us to consider minimization of the function

$$(4.18) \quad \hat{G}(e) \triangleq \Phi(e) + T\hat{g}\left(\frac{\bar{X} - e}{T}\right)$$

over $e \in [0, \bar{X}]$. As in the proof of Theorem 4.2, this function attains its minimum at some $e^* \in [0, \bar{X}]$.

For the remainder of the proof, we use the notation

$$(4.19) \quad y^* = \frac{\bar{X} - e^*}{T}, \quad x^* = h^{-1}(y^*),$$

where it is assumed without loss of generality that e^* is the largest minimizer of $\hat{G}$ in $[0, \bar{X}]$. There are two cases. In both cases, we construct a strategy that satisfies $E_T^B = e^*$ and

$$(4.20) \quad C(X^B) = \hat{G}(e^*).$$

In the first case, the strategy is a Type A strategy, and it is Type B in the second case. In both cases, we exhibit the strategy explicitly.

Case I. $\hat{g}(y^*) = g(y^*)$.

It is tempting to claim that we are now in the situation of Theorem 4.2 with the convex function $\hat{g}$ replacing g . However, the proof needed here that $e^* \geq \bar{e}$, where $\bar{e}$ is determined by (4.10), cannot follow the proof of Theorem 4.2. In the proof of Theorem 4.2, this inequality was a consequence of (4.13), which ultimately depended on the definition (4.1) of $g(\bar{e})$. But we only have $\hat{g}(\bar{e}) \leq \bar{e}\psi(h^{-1}(\bar{e}))$; we do not have an equation analogous to (4.1) for $\hat{g}$. We thus provide a different proof, which depends on e^* being the largest minimizer of $\hat{G}$ in $[0, \bar{X}]$.

If $x^* = 0$, then $y^* = 0$, $e^* = \bar{X}$, and $\hat{G}(e^*) = G(e^*)$. The Type A strategy that waits until the final time T and then purchases $\bar{X}$ is optimal. In particular, this strategy satisfies the initial condition $X_0^A = x^*$.

If $x^* > 0$, we must consider two subcases. It could be that $0 < x^* \leq F(0+)$. In this subcase, $\hat{g}(y^*) = g(y^*) = y^*\psi(x^*) = 0$ because $\psi \equiv 0$ on $[0, F(0+)]$. But $\hat{g}(0) = 0$ and $\hat{g}$ is nondecreasing, so $\hat{g} \equiv 0$ on $[0, y^*]$. Furthermore, x^* is positive, so $e^* < \bar{X}$. For $e \in (e^*, \bar{X})$, the number $\frac{\bar{X}-e}{T}$ is in $(0, y^*)$, and by (3.5), $D^+\hat{G}(e) = D^+\Phi(e) = \psi(e+)$. On the other hand, e^* is the largest minimizer of $\hat{G}$ in $[0, \bar{X}]$, which implies $D^+\hat{G}(e) > 0$. This shows that $\psi(e+) > 0$ for every $e \in (e^*, \bar{X})$, which implies that $\psi(e) > 0$ for every $e \in (e^*, \bar{X})$ and further implies that $e \geq F(0+)$ for every $e \in (e^*, \bar{X})$. We conclude that $e^* \geq F(0+)$. Applying h to this inequality and using the subcase assumption $x^* \leq F(0+)$, we obtain

$$(4.21) \quad h(e^*) \geq h(F(0+)) \geq h(x^*) = \frac{\bar{X} - e^*}{T}.$$

In other words, $e^* + h(e^*)T \geq \bar{X}$, and by the defining equation (4.10) of $\bar{e}$, we conclude that $e^* \geq \bar{e}$. The corresponding optimal strategy, which is Type A, satisfies $X_0^A = x^*$ and $E_T^A = e^*$. The proof of optimality of this strategy follows the proof of Theorem 4.2 with $\hat{g}$ replacing g .

Finally, we consider the subcase $x^* > F(0+)$. Because $y^* = h(x^*)$ is positive, the left-hand derivative of $\hat{g}$ at y^* is defined, and it satisfies

$$(4.22) \quad D^-\hat{g}(y^*) \geq \frac{\hat{g}(y^*) - \hat{g}(0)}{y^*} = \frac{g(y^*)}{y^*} = \psi(x^*).$$

In fact, the inequality in (4.22) is strict. If it were not, the affine function

$$\ell(y) = \psi(x^*)(y - y^*) + \hat{g}(y^*) = y\psi(x^*)$$

would describe a tangent line to the graph of $\hat{g}$ at $(y^*, \hat{g}(y^*))$ lying below $\hat{g}(y)$, and hence below $g(y)$, for all $y \in [0, \bar{Y}]$. But the resulting inequality $y\psi(x^*) \leq g(y) =$

$y\psi(h^{-1}(y))$ yields $\psi(x^*) \leq \psi(h^{-1}(y))$ for all $y \in (0, \bar{Y}]$, and letting $y \downarrow 0$, we would conclude $\psi(x^*) = 0$. This violates the subcase assumption $x^* > F(0+)$. We conclude that $D^-\hat{g}(y^*) > \psi(x^*)$. The strict inequality, the fact that e^* minimizes $\hat{G}$, and (3.5) further imply

$$0 \leq D^+\hat{G}(e^*) = D^+\Phi(e^*) - D^-\hat{g}(y^*) < \psi(e^*) - \psi(x^*).$$

But $\psi(x^*) < \psi(e^*)$ implies $x^* \leq e^*$. Consequently, $h(e^*) \geq h(x^*) = \frac{\bar{X}-e^*}{T}$. This is the essential part of inequality (4.21), and we conclude as above, constructing an optimal Type A strategy with $X_0^A = x^*$ and $E_T^A = e^*$.

Case II. $\hat{g}(y^*) < g(y^*)$.

Recall from Lemma 4.6 that this case can occur only if $0 < y^* < \bar{Y}$. In particular, $x^* > 0$. We let ℓ to be the affine function and α and β be numbers as described in Lemma 4.6, and we construct a Type B strategy. To do this, we define $t_0 \in (0, T)$ by

$$(4.23) \quad t_0 = \frac{(\beta - y^*)T}{\beta - \alpha},$$

so that $\alpha t_0 + \beta(T - t_0) = y^*T$. Consider the Type B strategy that makes an initial purchase $X_0^B = h^{-1}(\alpha)$, then purchases at rate $dX_t^B = \alpha dt$ for $0 \leq t < t_0$ (so $E_t^B = h^{-1}(\alpha)$ for $0 \leq t < t_0$), follows this with a purchase $\Delta X_{t_0}^B = h^{-1}(\beta) - h^{-1}(\alpha)$ at time t_0 , thereafter purchases at rate $dX_t^B = \beta dt$ for $t_0 \leq t < T$ (so $E_t^B = h^{-1}(\beta)$ for $t_0 \leq t < T$), and makes a final purchase $\bar{X} - X_{T-}^B$ at time T . According to (2.3),

$$X_t^B = \begin{cases} h^{-1}(\alpha) + \alpha t, & 0 \leq t < t_0, \\ h^{-1}(\beta) + \alpha t_0 + \beta(t - t_0), & t_0 \leq t < T, \\ \bar{X}, & t = T. \end{cases}$$

In particular,

$$(4.24) \quad \Delta X_T^B = \bar{X} - h^{-1}(\beta) - \alpha t_0 - \beta(T - t_0) = \bar{X} - h^{-1}(\beta) - y^*T = e^* - h^{-1}(\beta).$$

We show at the end of this proof that

$$(4.25) \quad h^{-1}(\beta) \leq e^*.$$

This will ensure that ΔX_T^B is nonnegative, and since X^B is obviously nondecreasing on $[0, T)$, this will establish that X^B is a feasible purchasing strategy.

Accepting (4.25) for the moment, we note that (4.24) implies

$$(4.26) \quad E_T^B = E_{T-}^B + \Delta E_T^B = h^{-1}(\beta) + \Delta X_T^B = e^*.$$

Using (4.4), (4.26), (4.16), the linearity of ℓ , and (4.17) in that order, we compute

$$\begin{aligned}
C(X^B) &= \Phi(E_T^B) + \int_0^T g(h(E_t^B)) dt \\
&= \Phi(e^*) + g(\alpha)t_0 + g(\beta)(T - t_0) \\
&= \Phi(e^*) + \ell(\alpha)t_0 + \ell(\beta)(T - t_0) \\
&= \Phi(e^*) + T\ell\left(\frac{\alpha t_0 + \beta(T - t_0)}{T}\right) \\
&= \Phi(e^*) + T\ell(y^*) \\
&= \Phi(e^*) + T\widehat{g}(y^*) \\
&= \widehat{G}(e^*),
\end{aligned}$$

This is (4.20).

Finally, we turn to the proof of (4.25). Because e^* is the largest minimizer of the convex function $\widehat{G}$ in $[0, \overline{X}]$ and $e^* < \overline{X}$ (because $x^* > 0$), the right-hand derivative of $\widehat{G}$ at e^* must be nonnegative. Indeed, for all $e \in (e^*, \overline{X})$, this right-hand derivative must in fact be strictly positive. For e greater than but sufficiently close to e^* , $\frac{\overline{X}-e}{T}$ is in (α, y^*) , where $\widehat{g}$ is linear with slope $\frac{g(\beta)-g(\alpha)}{\beta-\alpha}$. For such e ,

$$\begin{aligned}
0 &< D^+\widehat{G}(e) \\
&= D^+\Phi(e+) - D^-\widehat{g}(y)\Big|_{y=\frac{\overline{X}-e}{T}} \\
&= \psi(e+) - \frac{g(\beta) - g(\alpha)}{\beta - \alpha} \\
&= \psi(e+) - \frac{\beta\psi(h^{-1}(\beta)) - \alpha\psi(h^{-1}(\alpha))}{\beta - \alpha} \\
&\leq \psi(e+) - \frac{\beta\psi(h^{-1}(\beta)) - \alpha\psi(h^{-1}(\beta))}{\beta - \alpha} \\
&= \psi(e+) - \psi(h^{-1}(\beta)).
\end{aligned}$$

This inequality $\psi(h^{-1}(\beta)) < \psi(e+)$ for all e greater than but sufficiently close to e^* implies (4.25). $\square$

REMARK 4.7 (Uniqueness). In Case I of the proof of Theorem 4.5, when $\widehat{g}(y^*) = g(y^*)$, strict convexity of $\widehat{g}$ at y^* implies uniqueness of the optimal purchasing strategy. The proof is similar to the uniqueness proof in Theorem 4.2.

However, in Case II $\widehat{g}$ is not strictly convex at y^* . In this case, if ψ is strictly increasing at e^* and if the affine function ℓ of Lemma 4.6 agrees with g only at α and β , then the optimal purchasing strategy is unique. Indeed, if ψ is strictly increasing at e^* , then Φ and hence $\widehat{G}$ are strictly convex at e^* , which implies that e^* is the unique minimizer of $\widehat{G}$. In order to be optimal, a purchasing strategy must satisfy the two inequalities

$$(4.27) \quad \int_0^T g(h(E_t)) dt \geq \int_0^T \widehat{g}(h(E_t)) dt \geq T\widehat{g}\left(\int_0^T h(E_t) \frac{dt}{T}\right)$$

with equality, as we explain below, and must also satisfy $E_T = e^*$. When the inequalities (4.27) hold, we can use (2.3) to obtain a lower bound on the cost of an

arbitrary purchasing strategy X by relations

$$\begin{aligned}
C(X) &= \Phi(E_T) + \int_0^T g(h(E_t))dt \\
&\geq \Phi(E_T) + T\hat{g}\left(\int_0^T h(E_t)\frac{dt}{T}\right) \\
&= \Phi(E_T) + T\hat{g}\left(\frac{\bar{X} - E_T}{T}\right) \\
&= \hat{G}(E_T).
\end{aligned}$$

The minimal cost is $\hat{G}(e^*) = \Phi(e^*) + T\hat{g}(\frac{\bar{X}-e^*}{T}) = \Phi(e^*) + T\hat{g}(y^*)$, and hence optimality of a strategy requires that equality hold in both parts of (4.27). The second inequality in (4.27) is Jensen's inequality, and equality holds if and only if $h(E_t)$, $0 \leq t < T$, stays in the region in which $\hat{g}$ is affine. But the average value of $h(E_t)$, $\frac{1}{T} \int_0^T h(E_t)dt$, is equal to y^* , and hence we cannot have $h(E_t) < y^*$ for all $t \in [0, T)$, nor can we have $h(E_t) > y^*$ for all $t \in [0, T)$. Hence the region in which $h(E_t)$ stays must be the region in which $\hat{g}$ agrees with ℓ . To get an equality in the first inequality in (4.27), $h(E_t)$, $0 \leq t < T$, must stay in the region where $\hat{g}$ agrees with g . If ℓ agrees with g only at the two points α and β , then $h(E_t)$, $0 \leq t < T$, must stay in the two-point set $\{\alpha, \beta\}$. Because $\Delta E_t = \Delta X_t \geq 0$ for all t , there must be some initial time interval $[0, t_0)$ on which $h(E_t) = \alpha$ and there must be some final time interval $[t_0, T)$ on which $h(E_t) = \beta$. In order to achieve this and to also have $\frac{1}{T} \int_0^T h(E_t)dt = y^*$, t_0 must be given by (4.23). $\square$

4.3. Examples of Type B optimal strategies.

EXAMPLE 2.2 (Modified block order book, continued). We continue Example 2.2 under the simplifying assumptions $T = 1$ and $h(x) = x$ for all $x \geq 0$, so $h^{-1}(y) = y$ for all $y \geq 0$ and $\bar{Y} = \bar{X}$. Recalling (2.6) and (4.1), we see that

$$g(y) = \begin{cases} y^2, & 0 \leq y \leq a, \\ y^2 + (b-a)y, & a < y < \infty. \end{cases}$$

The convex hull of g over $[0, \infty)$, given by (4.14), is

$$\hat{g}(y) = \begin{cases} y^2, & 0 \leq y \leq a, \\ (2\beta + b - a)(y - a) + a^2, & a \leq y \leq \beta, \\ y^2 + (b - a)y, & \beta \leq y < \infty, \end{cases}$$

where

$$(4.28) \quad \beta = a + \sqrt{a(b-a)}$$

(see Fig. 6). We take $\bar{X} = \bar{Y} > \beta$ so that this is also the convex hull of g over $[0, \bar{Y}]$.

For $a < y^* < \beta$, we have $\hat{g}(y^*) < g(y^*)$. For constants α and β from the statement of Lemma 4.6 (see (3.1)–(3.2) in Appendix 3), we have α of (3.1) is a , and β of (3.2) is given by (4.28). In order to illustrate a case in which a Type B purchasing strategy is optimal, we assume

$$(4.29) \quad a + 2\beta < \bar{X} < 3\beta.$$

The function $\hat{G}$ of (4.18) is minimized over $[0, \bar{X}]$ at e^* if and only if

$$0 \in \partial \hat{G}(e^*) = \partial \Phi(e^*) - \partial \hat{g}(\bar{X} - e^*),$$

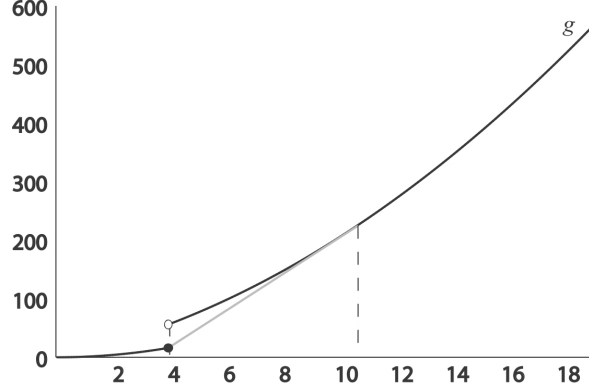


FIGURE 6. Function g for the modified block order book with parameters $a = 4$ and $b = 14$. The convex hull $\hat{g}$ is constructed by replacing a part $\{g(y), y \in (a, \beta)\}$ by a straight line connecting $g(a)$ and $g(\beta)$. Here $\beta = 10.3246$.

which is equivalent to $\partial\Phi(e^*) \cap \partial\hat{g}(\bar{X} - e^*) \neq \emptyset$. We show below that the largest value of e^* satisfying this condition is $e^* = 2\beta$. According to (4.29), $e^* = 2\beta$ is in $(\bar{X} - \beta, \bar{X} - a)$. Because $\beta > a$, e^* is also in (a, ∞) . We compute (recall (2.12))

$$\begin{aligned} \partial\Phi(e) &= \begin{cases} \{e\}, & 0 \leq e < a, \\ [a, b], & e = a, \\ \{e + b - a\}, & a < e < \infty, \end{cases} \\ \partial\hat{g}(\bar{X} - e) &= \begin{cases} \{2(\bar{X} - e) + b - a\}, & 0 \leq e \leq \bar{X} - \beta, \\ \{2\beta + b - a\}, & \bar{X} - \beta \leq e < \bar{X} - a, \\ [2a, 2\beta + b - a], & e = \bar{X} - a, \\ \{2(\bar{X} - e)\}, & \bar{X} - a < e \leq \bar{X}, \end{cases} \end{aligned}$$

and then evaluate

$$\partial\Phi(e^*) = \{e^* + b - a\} = \{2\beta + b - a\} = \partial\hat{g}(\bar{X} - e^*).$$

Therefore, $\hat{G}$ attains its minimum at e^* .

To see that there is no $e \in (2\beta, \bar{X}]$ where $\hat{G}$ attains its minimum, we observe that for $e \in (2\beta, \bar{X} - a)$, $\partial\Phi(e) \cap \partial\hat{g}(\bar{X} - e) = \{e + b - a\} \cap \{2\beta + b - a\} = \emptyset$. For $e \in [\bar{X} - a, \bar{X}]$, all points in $\partial\hat{g}(\bar{X} - e)$ lie in the interval $[0, 2a]$, whereas the only point in $\partial\Phi(e)$, which is $e + b - a$, lies in the interval $[\bar{X} + b - 2a, \bar{X} + b - a]$. Because of (4.29), we have $2a < \bar{X} + b - 2a$, and hence $\partial\Phi(e) \cap \partial\hat{g}(\bar{X} - e) = \emptyset$ for $e \in [\bar{X} - a, \bar{X}]$.

As in the proof of Theorem 4.5, we set $y^* = \bar{X} - e^* = \bar{X} - 2\beta$, $x^* = h^{-1}(y^*) = \bar{X} - 2\beta$. Condition (4.29) is equivalent to $a < y^* < \beta$, which in turn is equivalent to $\hat{g}(y^*) < g(y^*)$. The first inequality in (4.29) shows that $x^* > 0$, and we are thus in Case II of the proof of Theorem 4.5. In this case, we define

$$t_0 = \frac{\beta - y^*}{\beta - a} = \frac{3\beta - \bar{X}}{\beta - a}.$$

The optimal purchasing strategy is

$$X_t^B = \begin{cases} a(t+1), & 0 \leq t < t_0, \\ at_0 + \beta(t+1-t_0), & t_0 \leq t < 1, \\ \bar{X}, & t = 1. \end{cases}$$

In particular, $\Delta X_0 = a$, $\Delta X_{t_0} = \beta - a$, $\Delta X_1 = \beta$ (see (4.24) for the last equality). The corresponding E^B process is

$$E_t^B = \begin{cases} a, & 0 \leq t < t_0, \\ \beta, & t_0 \leq t < 1, \\ 2\beta, & t = 1. \end{cases}$$

The initial lump purchase moves the ask price to the left endpoint a of the gap in the order book. Purchasing is done to keep the ask price at a until time t_0 , when another lump purchase moves the ask price to β , beyond the right endpoint b of the gap in the order book. Purchasing is done to keep the ask price at β until the final time, when another lump purchase is executed. $\square$

EXAMPLE 2.3 (Discrete order book, continued). We continue Example 2.3 under the simplifying assumptions that $T = 1$ and $h(x) = x$ for all $x \geq 0$, so that $h^{-1}(y) = y$ for all $y \geq 0$ and $\bar{Y} = \bar{X}$. From (2.8) and (4.1) we see that $g(0) = 0$, and $g(y) = ky$ for integers $k \geq 0$ and $k < y \leq k+1$. In particular, $g(k) = (k-1)k$ for nonnegative integers k . The convex hull of g interpolates linearly between the points $(k, (k-1)k)$ and $(k+1, k(k+1))$, i.e., $\hat{g}(y) = k(2y - (k+1))$ for $k \leq y \leq k+1$, where k ranges over the nonnegative integers (see Fig. 7).

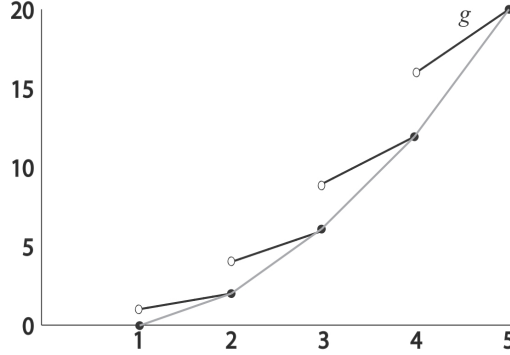


FIGURE 7. Function g for the discrete order book. The convex hull $\hat{g}$ interpolates linearly between the points $(k, (k-1)k)$ and $(k+1, k(k+1))$.

Therefore,

$$\partial \hat{g}(y) = \begin{cases} \{0\}, & y = 0, \\ [2(k-1), 2k], & y = k \text{ and } k \text{ is a positive integer,} \\ \{2k\}, & k < y < k+1 \text{ and } k \text{ is a nonnegative integer.} \end{cases}$$

Recall from the discussion following (2.13) that

$$\partial\Phi(y) = \begin{cases} \{0\}, & y = 0, \\ [k-1, k], & y = k \text{ and } k \text{ is a positive integer,} \\ \{k\}, & k < y < k+1 \text{ and } k \text{ is a nonnegative integer.} \end{cases}$$

We seek the largest number $e^* \in [0, \bar{X}]$ for which $\partial\Phi(e^*) \cap \partial\hat{g}(\bar{X} - e^*) \neq \emptyset$. This is the largest minimizer of $\hat{G}(e) = \Phi(e) + \hat{g}(\bar{X} - e)$ in $[0, \bar{X}]$. We define k^* to be the largest integer less than or equal to $\frac{\bar{X}}{3}$, so that

$$3k^* \leq \bar{X} < 3k^* + 3.$$

We divide the analysis into three cases:

Case A:: $3k^* \leq \bar{X} \leq 3k^* + 1$,

Case B:: $3k^* + 1 < \bar{X} < 3k^* + 2$,

Case C:: $3k^* + 2 \leq \bar{X} < 3k^* + 3$.

We show below that in Cases A and B, the optimal strategy makes an initial lump purchase of size k^* , which executes the orders at prices $0, 1, \dots, k^* - 1$. In Case A the optimal strategy then purchases at rate k^* over the interval $(0, 1)$, and at time 1 makes a final lump purchase of size $\bar{X} - 2k^*$, which is in the interval $[k^*, k^* + 1]$. This is a Type A strategy. In Case B there is an intermediate lump purchase of size one at time $3k^* + 2 - \bar{X}$. Before this intermediate purchase, the rate of purchase is k^* and after this purchase, the rate of purchase is $k^* + 1$. In Case B at time 1 there is a final lump purchase of size k^* . In Case B we have a Type B strategy. In Case C, the optimal strategy makes a lump purchase of size $k^* + 1$ at time 0, which executes the orders at prices $0, 1, \dots, k^* - 1, k^*$. The optimal strategy then purchases continuously at rate $k^* + 1$ over the interval $(0, 1)$, and at time 1 makes a final lump purchase of size $\bar{X} - 2k^* - 2$, which is in the interval $[k^*, k^* + 1)$. This is a Type A strategy.

Case A: $3k^* \leq \bar{X} \leq 3k^* + 1$.

We define $e^* = \bar{X} - k^*$, so that $2k^* \leq e^* \leq 2k^* + 1$ and $k^* = \bar{X} - e^*$. Then $2k^* \in \partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*) = [2(k^* - 1), 2k^*]$, so the intersection of $\partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^*, \bar{X}]$ and $\partial\hat{g}(\bar{X} - e) \subset [0, 2(k^* - 1)]$, so the intersection of these two sets is empty.

In this case, y^* and x^* defined by (4.19) are both equal to k^* and hence $\hat{g}(y^*) = g(y^*)$. If $k^* = 0$, we are in the first subcase of Case I of the proof of Theorem 4.5. The optimal purchasing strategy is to do nothing until time 1, and then make a lump purchase of size $\bar{X}$. If $k^* = 1$, which is equal to $F(0+)$, we are in the second sub-case of Case I of the proof of Theorem 4.5. We should make an initial purchase of size $x^* = 1$, purchase continuously over the time interval $(0, 1)$ at rate 1 so that that $E_t \equiv 1$ and $D_t \equiv 0$, and make a final purchase of size $\bar{X} - 2$. If $k^* \geq 2$, we are in the third subcase of Case I of the proof of Theorem 4.5. We should make an initial purchase of size k^* , purchase continuously over the time interval $(0, 1)$ at rate k^* so that $E_t \equiv k^*$ and $D_t \equiv k^* - 1$, and make a final purchase of size $\bar{X} - 2k^*$.

Case B: $3k^* + 1 < \bar{X} < 3k^* + 2$.

We define $e^* = 2k^* + 1$, so that $k^* < \bar{X} - e^* < k^* + 1$. Then $\partial\Phi(e^*) = [2k^*, 2k^* + 1]$ and $2k^* \in \partial\hat{g}(\bar{X} - e^*)$, so the intersection of $\partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^* + 1, \bar{X}]$ and $\partial\hat{g}(\bar{X} - e) \subset [0, 2k^*]$, so the intersection of these two sets is empty.

In this case, y^* and x^* defined by (4.19) are both equal to $\bar{X} - e^*$. Hence $k^* < y^* < k^* + 1$, $\hat{g}(y^*) < g(y^*)$, and we are in Case II of the proof of Theorem 4.5 with $\alpha = k^*$ and $\beta = k^* + 1$ (see (4.14)–(4.17) and (3.1)–(3.2)). The optimal purchasing strategy is Type B. In particular, with $t_0 = \beta - y^* = k^* + 1 - x^* = 3k^* + 2 - \bar{X}$, the optimal purchasing strategy makes an initial lump purchase $\alpha = k^*$, which executes the orders at prices $0, 1, \dots, k^* - 1$, then purchases continuously over the interval $(0, t_0)$ at rate k^* so that $E_t \equiv k^*$ and $D_t \equiv k^* - 1$, at time t_0 makes a lump purchase of size $\beta - \alpha = 1$, which consumes the order at price k^* , then purchases continuously over the interval $(t_0, 1)$ at rate $k^* + 1$ so that $E_t \equiv k^* + 1$ and $D_t \equiv k^*$, and finally executes a lump purchase of size $e^* - \beta = k^*$ (see (4.24)) at time 1. The total quantity purchased is

$$k^* + k^* t_0 + 1 + (k^* + 1)(1 - t_0) + k^* = \bar{X},$$

as required.

Case C: $3k^* + 2 \leq \bar{X} < 3k^* + 3$.

We define $e^* = \bar{X} - k^* - 1$, so that $2k^* + 1 \leq e^* < 2k^* + 2$ and $\bar{X} - e^* = k^* + 1$. Then $2k^* + 1 \in \partial\Phi(e^*)$ and $\hat{g}(\bar{X} - e^*) = [2k^*, 2k^* + 2]$, and the intersection of $\partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^* + 1, \bar{X}]$ and $\partial\hat{g}(\bar{X} - e) \subset [0, 2k^*]$, so the intersection of these two sets is empty. In this case, y^* and x^* are both equal to $k^* + 1$. The optimal purchasing strategy falls into either second (if $k^* = 0$) or third (if $k^* \geq 1$) subcases of Case I of the proof of Theorem 4.5. $\square$

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1. The process E

In this appendix we prove that there exists a unique adapted process E satisfying (2.3) pathwise, and we provide a list of its properties.

LEMMA 1.1. *Let h be a nondecreasing, real-valued, locally Lipschitz function defined on $[0, \infty)$ such that $h(0) = 0$. Let X be a purchasing strategy. Then there exists a unique bounded adapted process E depending pathwise on X such that (2.3) is satisfied. Furthermore,*

- (i) E is right continuous with left limits;
- (ii) $\Delta E_t = \Delta X_t$ for all t ;

- (iii) E has finite variation on $[0, T]$;
- (iv) E takes values in $[0, \bar{X}]$.

PROOF: Because we do not know a priori that E is nonnegative, we extend h to all of $\mathbb{R}$ by defining $h(x) = 0$ for $x < 0$. This extended h is nondecreasing and locally Lipschitz.

In Section 2 we introduced the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. The purchasing strategy X is right-continuous and adapted to this filtration, and hence is an optional process, i.e., $(t, \omega) \mapsto X_t(\omega)$ is measurable with respect to the optional σ -algebra, the σ -algebra generated by the right-continuous adapted processes. For any bounded optional process Y , $h(Y)$ and $\int_0^t h(Y_s) ds$ are also bounded optional processes. Optional processes are adapted, and hence $\int_0^t h(Y_s) ds$ is $\mathcal{F}_t$ -measurable for each $t \in [0, T]$.

We first prove uniqueness. If E and $\hat{E}$ are bounded processes satisfying (2.3), then there is a local Lipschitz constant K , chosen taking the bounds on E and $\hat{E}$ into account, such that

$$|E_t - \hat{E}_t| = \left| \int_0^t (h(E_s) - h(\hat{E}_s)) ds \right| \leq K \int_0^t |E_s - \hat{E}_s| ds.$$

Gronwall's inequality implies $E = \hat{E}$.

For the existence part of the proof, we assume for the moment that h is globally Lipschitz with Lipschitz constant K , and we construct E as a limit of a recursion. Let $E_t^0 \equiv X_0$. For $n = 1, 2, \dots$, define recursively

$$E_t^n = X_t - \int_0^t h(E_s^{n-1}) ds, \quad 0 \leq t \leq T.$$

Since X is bounded and optional, each E^n is bounded and optional. For $n = 1, 2, \dots$, let $Z_t^n = \sup_{0 \leq s \leq t} |E_s^n - E_s^{n-1}|$. A proof by induction shows that

$$Z_t^n \leq \frac{K^{n-1} t^{n-1}}{(n-1)!} \max \{ \bar{X}, Th(X_0) + X_0 \}.$$

Because this sequence of nonrandom bounds is summable, E^n converges uniformly in $t \in [0, T]$ and ω to a bounded optional process E that satisfies (2.3). In particular, E_t is $\mathcal{F}_t$ -measurable for each t , and since X is nondecreasing and right-continuous with left limits and the integral in (2.3) is continuous, (i), (ii) and (iii) hold.

It remains to prove (iv). For $\varepsilon > 0$, let $X_t^\varepsilon = X_t + \varepsilon t$ and define $t_0^\varepsilon = \inf \{ t \in [0, T] : E_t^\varepsilon < 0 \}$. Assume this set is not empty. Then the right-continuity of E^ε combined with the fact that E^ε has no negative jumps implies that $E_{t_0^\varepsilon}^\varepsilon = 0$. Let $t_n^\varepsilon \downarrow t_0^\varepsilon$ be such that $E_{t_n^\varepsilon}^\varepsilon < 0$ for all n . Then

$$\int_{t_0^\varepsilon}^{t_n^\varepsilon} h(E_s^\varepsilon) ds = X_{t_n^\varepsilon}^\varepsilon - X_{t_0^\varepsilon}^\varepsilon - (E_{t_n^\varepsilon}^\varepsilon - E_{t_0^\varepsilon}^\varepsilon) > X_{t_n^\varepsilon}^\varepsilon - X_{t_0^\varepsilon}^\varepsilon \geq \varepsilon(t_n^\varepsilon - t_0^\varepsilon).$$

But since

$$\int_{t_0^\varepsilon}^{t_n^\varepsilon} h(E_s^\varepsilon) ds \leq K \left(\max_{t_0^\varepsilon \leq s \leq t_n^\varepsilon} E_s^\varepsilon \right) (t_n^\varepsilon - t_0^\varepsilon),$$

there must exist $s_n^\varepsilon \in (t_0^\varepsilon, t_n^\varepsilon)$ such that $E_{s_n^\varepsilon}^\varepsilon \geq \frac{\varepsilon}{K}$. This contradicts the right continuity of E^ε at t_0^ε . Consequently, the set $\{t \in [0, T] : E_t^\varepsilon < 0\}$ must be empty. We conclude that $E_t^\varepsilon \geq 0$ for all $t \in [0, T]$.

Now notice that for $0 \leq t \leq T$,

$$E_t^\varepsilon - E_t = \varepsilon t - \int_0^t (h(E_s^\varepsilon) - h(E_s)) ds,$$

and hence

$$|E_t^\varepsilon - E_t| \leq \varepsilon t + K \int_0^t |E_s^\varepsilon - E_s| ds.$$

Gronwall's inequality implies that $E^\varepsilon \rightarrow E$ as $\varepsilon \downarrow 0$. Since $E_t^\varepsilon \geq 0$, we must have $E_t \geq 0$ for all t . Equation (2.3) now implies that $E_t \leq X_t$, and therefore $E_t \leq \overline{X}$. The proof of (iv) is complete.

When h is locally but not globally Lipschitz, we let $\tilde{h}$ be equal to h on $[0, \overline{X}]$, $\tilde{h}(x) = 0$ for $x < 0$, and $\tilde{h}(x) = h(\overline{X})$ for $x > \overline{X}$. We apply the previous arguments to $\tilde{h}$, and we observe that the resulting $\tilde{E}$ satisfies the equation corresponding to h . $\square$

REMARK 1.2. The pathwise construction of E in the proof of Lemma 1.1 shows that if X is deterministic, then so is E .

$$2. \mathbb{E} \int_0^T X_t dA_t = 0$$

LEMMA 2.1. *Under the assumptions that $0 \leq X_t \leq \bar{X}$, $0 \leq t \leq T$, and that the continuous nonnegative martingale A satisfies (2.1), we have $\mathbb{E} \int_0^T X_t dA_t = 0$.*

PROOF: The Burkholder-Davis-Gundy inequality implies that the continuous local martingale $M_t = \int_0^t X_s dA_s$ satisfies

$$\begin{aligned} \mathbb{E} \left[\max_{0 \leq t \leq T} |M_t| \right] &\leq C \mathbb{E} [\langle M \rangle_T^{1/2}] \\ &= C \mathbb{E} \left[\left(\int_0^T X_t^2 d\langle A \rangle_t \right)^{1/2} \right] \\ &\leq C \bar{X} \mathbb{E} [\langle A \rangle_T^{1/2}] \\ &= C' \bar{X} \mathbb{E} \left[\max_{0 \leq t \leq T} A_t \right], \end{aligned}$$

where C and C' are positive constants. By virtue of being a local martingale, M has the property that $\mathbb{E} M_{\tau_n} = 0$ for a sequence of stopping times $\tau_n \uparrow T$. The dominated convergence theorem implies $\mathbb{E} M_T = 0$. $\square$

3. Convex hull of g

PROOF OF LEMMA 4.6: Recall the definition

$$\hat{g}(y) \triangleq \sup \{ \ell(y) : \ell \text{ is an affine function and } \ell(\eta) \leq g(\eta) \forall \eta \in [0, \bar{Y}] \} \quad (4.14)$$

of the convex hull of g , defined for $y \in [0, \bar{Y}]$. The function $\hat{g}$ is the largest convex function defined on $[0, \bar{Y}]$ that is dominated by g there.

For each $0 \leq y < \bar{Y}$, the supremum in (4.14) is obtained by the support line of $\hat{g}$ at y . For $y = 0$ the zero function is such a support line, and hence $0 \leq \hat{g}(0) \leq g(0) = 0$ (recall (4.3)). At $y = \bar{Y}$ the only support line might be vertical, in which case the supremum in (4.14) is not attained. Because $\hat{g}(0) = 0$, $\hat{g}$ is nonnegative, and $\hat{g}$ is convex, we know that $\hat{g}$ is also nondecreasing. Being convex, $\hat{g}$ is continuous on $(0, \bar{Y})$, upper semi-continuous on $[0, \bar{Y}]$, and we have continuity at 0 because of (4.3). We also have continuity of $\hat{g}$ at $\bar{Y}$, as we now show. Given $\varepsilon > 0$, the definition of $\hat{g}$ implies that there exists an affine function $\ell \leq g$ such that $\ell(\bar{Y}) \geq \hat{g}(\bar{Y}) - \varepsilon$. But $\hat{g} \geq \ell$, and thus $\liminf_{y \uparrow \bar{Y}} \hat{g}(y) \geq \lim_{y \uparrow \bar{Y}} \ell(y) = \ell(\bar{Y}) \geq \hat{g}(\bar{Y}) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we must in fact have $\liminf_{y \uparrow \bar{Y}} \hat{g}(y) \geq \hat{g}(\bar{Y})$. Coupled with the upper semicontinuity of $\hat{g}$ at $\bar{Y}$, this gives us continuity.

We next argue that $\hat{g}(\bar{Y}) = g(\bar{Y})$. Suppose, on the contrary, we had $\hat{g}(\bar{Y}) < g(\bar{Y})$. The function g is continuous at $\bar{Y}$ (see (4.5)) and $\hat{g}$ is upper semicontinuous. Therefore, there is a one-sided neighborhood $[\gamma, \bar{Y}]$ of $\bar{Y}$ (with $\gamma < \bar{Y}$) on which $g - \hat{g}$ is bounded away from zero by a positive number ε . The function

$$\hat{g}(y) + \frac{\varepsilon(y - \gamma)}{\bar{Y} - \gamma}, \quad 0 \leq y \leq \bar{Y},$$

is convex, lies strictly above $\hat{g}$ at $\bar{Y}$, and lies below g everywhere. This contradicts the fact that $\hat{g}$ is the largest convex function dominated by g . We must therefore have $\hat{g}(\bar{Y}) = g(\bar{Y})$.

Finally, we describe the situation when for some $y^* \in [0, \bar{Y}]$, we have $\hat{g}(y^*) < g(y^*)$. We have shown that this can happen only if $0 < y^* < \bar{Y}$. Let ℓ be a support

line of $\widehat{g}$ at y^* , which is an affine function that attains the maximum in (4.14) at the point y^* . In particular, $\ell \leq \widehat{g} \leq g$ and $\ell(y^*) = \widehat{g}(y^*)$. Define

$$(3.1) \quad \alpha = \sup\{\eta \in [0, y^*] : g(\eta) - \ell(\eta) = 0\},$$

$$(3.2) \quad \beta = \inf\{\eta \in [y^*, \overline{Y}] : g(\eta) - \ell(\eta) = 0\}.$$

Because g is continuous, the minimum of $g - \ell$ over $[0, \overline{Y}]$ is attained. This minimum cannot be a positive number ε , for then $\ell + \varepsilon$ would be an affine function lying below g . Therefore, either the supremum in (3.1) or the infimum in (3.2) is taken over a nonempty set. In the former case, we must have $g(\alpha) = \ell(\alpha)$, whereas in the latter case $g(\beta) = \ell(\beta)$.

Let us consider first the case that $g(\alpha) = \ell(\alpha)$. Define $\gamma = \frac{1}{2}(\alpha + y^*)$. Like α , γ is strictly less than y^* . The function $g - \ell$ attains its minimum over $[\gamma, \overline{Y}]$. If this minimum were a positive number ε , then the affine function

$$\ell(y) + \frac{\varepsilon(y - \gamma)}{\overline{Y} - \gamma}, \quad 0 \leq y \leq \overline{Y},$$

would lie below g but have a larger value at y^* than ℓ , violating the choice of ℓ . It follows that $g - \ell$ attains the minimum value zero on $[\gamma, \overline{Y}]$, and since this function is strictly positive on $[\gamma, y^*]$, the minimum is attained to the right of y^* . This implies that $g(\beta) = \ell(\beta)$. Similarly, if we begin with the assumption that $g(\beta) = \ell(\beta)$, we can argue that $g(\alpha) = \ell(\alpha)$.

In conclusion, α and β defined by (3.1) and (3.2) satisfy (4.15) and (4.16). Finally, (4.16) shows that ℓ restricted to $[\alpha, \beta]$ is the largest affine function lying below g on this interval, and hence (4.17) holds.

Because of (4.16), every affine function lying below g on $[0, \overline{Y}]$ must lie below ℓ on $[\alpha, \beta]$. If such an affine function agrees with $\widehat{g}$ and hence with ℓ at y^* , it must in fact agree with ℓ everywhere. Hence, ℓ is the only function lying below g on $[0, \overline{Y}]$ and agreeing with $\widehat{g}$ at y^* . $\square$

CHAPTER 3

Integral Representation of Martingales and Endogenous Completeness of Financial Models

1. Introduction

Let $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ be a complete filtered probability space, $\mathbb{Q}$ be an equivalent probability measure, and $S = (S_t^j)$ be a J -dimensional martingale under $\mathbb{Q}$. It is often important to know whether any local martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to S , that is,

$$(1) \quad M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0, 1],$$

for some predictable S -integrable process $H = (H_t^j)$. For instance, in mathematical finance, which is the topic of a particular interest to us, the existence of such a martingale representation corresponds to the *completeness* of the market model driven by stock prices S , see Harrison and Pliska [5].

A general answer is given in Jacod [7], Section XI.1(a). Jacod's theorem states that the integral representation property holds if and only if $\mathbb{Q}$ is the *only* equivalent martingale measure for S . In mathematical finance this result is sometimes referred to as the 2nd fundamental theorem of asset pricing.

In many applications, including those in finance, the process S is defined in terms of its predictable characteristics under $\mathbb{P}$. The construction of a martingale measure $\mathbb{Q}$ for S is then accomplished through the use of the Girsanov theorem and its generalizations, see Jacod and Shiryaev [8]. The verification of the existence of integral representations for all $\mathbb{Q}$ -martingales under S is often straightforward. For example, if S is a diffusion process under $\mathbb{P}$ with the drift vector-process $b = (b_t)$ and the volatility matrix-process $\sigma = (\sigma_t)$, then such a representation exists if and only if the volatility matrix-process σ has full rank $d\mathbb{P} \times dt$ almost surely.

In this paper we assume that the inputs are the random variables $\xi > 0$ and $\psi = (\psi^j)_{j=1,\dots,J}$, while $\mathbb{Q}$ and S are defined as

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} &\triangleq \frac{\xi}{\mathbb{E}[\xi]}, \\ S_t &\triangleq \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t], \quad t \in [0, 1]. \end{aligned}$$

We are looking for (easily verifiable) conditions on ξ and ψ guaranteeing the integral representation of all $\mathbb{Q}$ -martingales with respect to S .

Our study is motivated by the problem of *endogenous completeness* in financial economics, see Anderson and Raimondo [1]. Here ξ is an equilibrium state price density, usually defined implicitly by a fixed point argument, and $\psi = (\psi^j)$ is the random vector of the cumulative discounted dividends for traded stocks. The term

“endogenous” is used because the stock prices S are now *computed* as an output of equilibrium.

We focus on the case when ξ and ψ are defined in terms of a weak solution X to a d -dimensional stochastic differential equation. With respect to x the coefficients of this equation satisfy classical conditions guaranteeing weak existence and uniqueness: the drift vector $b(t, \cdot)$ is measurable and bounded and the volatility matrix $\sigma(t, \cdot)$ is uniformly continuous and bounded and has a bounded inverse. With respect to t our assumptions are more stringent: $b(\cdot, x)$ and $\sigma(\cdot, x)$ are required to be analytic functions. We give an example showing that the t -analyticity assumption on the volatility matrix σ cannot be removed.

Our results complement and generalize those in Anderson and Raimondo [1], Hugonnier, Malamud, and Trubowitz [6], and Riedel and Herzberg [18]. In the pioneering paper [1], X is a Brownian motion. The proofs in this paper rely on non-standard analysis. In [6] and [18], among other conditions, the diffusion coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$ are assumed to be analytic functions with respect to (t, x) . In one important aspect, however, the assumptions in [1], [6], and [18] are less restrictive. If $\psi = F(X_1)$, where $F = F^j(x)$ is a J -dimensional vector-function on $\mathbb{R}^d$, then these papers require the Jacobian matrix of F to have rank d only on some open set. In our framework, this property needs to hold almost everywhere on $\mathbb{R}^d$. We provide an example showing that in the absence of the x -analyticity assumption on b and σ this stronger condition cannot be relaxed.

2. Main results

Let $\mathbf{X}$ be a Banach space and D be a subset of either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. We remind the reader that a map $f : D \rightarrow \mathbf{X}$ is called *analytic* if for any $x \in D$ there exist a number $\epsilon > 0$ and a sequence $A = (A_n)_{n \geq 0}$ in $\mathbf{X}$ (both ϵ and A depend on x) such that

$$f(y) = \sum_{n=0}^{\infty} A_n (y - x)^n, \quad y \in D, |y - x| < \epsilon,$$

where the series converges in the norm $\|\cdot\|_{\mathbf{X}}$ of $\mathbf{X}$.

In the statements of our results, $D = [0, 1]$ and $\mathbf{X}$ will be one of the following spaces:

$\mathbf{L}_{\infty} = \mathbf{L}_{\infty}(\mathbb{R}^d, dx)$: the Lebesgue space of bounded measurable real-valued functions f on an Euclidean space $\mathbb{R}^d$ with the norm $\|f\|_{\mathbf{L}_{\infty}} \triangleq \text{ess sup}_{x \in \mathbb{R}^d} |f(x)|$.

As usual, the term *a measurable function* is used for an equivalence class of Borel measurable functions indistinguishable with respect to the Lebesgue measure.

$\mathbf{C} = \mathbf{C}(\mathbb{R}^d)$: the Banach space of bounded and continuous real-valued functions f on $\mathbb{R}^d$ with the norm $\|f\|_{\mathbf{C}} \triangleq \sup_{x \in \mathbb{R}^d} |f(x)|$.

We shall use standard notations of linear algebra. If x and y are vectors in an Euclidean space $\mathbb{R}^n$, then xy denotes the scalar product and $|x| \triangleq \sqrt{xx}$. If $a \in \mathbb{R}^{m \times n}$ is a matrix with m rows and n columns, then ax denotes its product on the (column-)vector x , a^* stands for the transpose, and $|a| \triangleq \sqrt{\text{trace}(aa^*)}$.

Let $\mathbb{R}^d$ be an Euclidean space and the functions $b = b(t, x) : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma = \sigma(t, x) : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be such that for all $i, j = 1, \dots, d$:

(A1) $t \mapsto b^i(t, \cdot)$ is an analytic map of $[0, 1]$ to $\mathbf{L}_{\infty}$.

(A2) $t \mapsto \sigma^{ij}(t, \cdot)$ is an analytic map of $[0, 1]$ to $\mathbf{C}$. For $t \in [0, 1]$ and $x \in \mathbb{R}^d$ the matrix $\sigma(t, x)$ has the inverse $\sigma^{-1}(t, x)$ and there exists a constant $N > 0$, same for all t and x , such that

$$(2) \quad |\sigma^{-1}(t, x)| \leq N.$$

Moreover, there exists a strictly increasing function $\omega = (\omega(\epsilon))_{\epsilon > 0}$ such that $\omega(\epsilon) \rightarrow 0$ as $\epsilon \downarrow 0$ and, for all $t \in [0, 1]$ and all $x, y \in \mathbb{R}^d$,

$$|\sigma(t, x) - \sigma(t, y)| \leq \omega(|x - y|).$$

Note that (2) is equivalent to the uniform ellipticity of the matrix-function $a \triangleq \sigma \sigma^*$: for all $y \in \mathbb{R}^d$ and $(t, x) \in [0, 1] \times \mathbb{R}^d$,

$$ya(t, x)y = |\sigma(t, x)y|^2 \geq \frac{1}{N^2}|y|^2.$$

Let $X_0 \in \mathbb{R}^d$. The classical results of Stroock and Varadhan [19], Theorem 7.2.1 and Krylov [16, 14] imply that under (A1) and (A2) there exist a complete filtered probability space $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, 1]}, \mathbb{P})$, a Brownian motion W , and a stochastic process X , both with values in $\mathbb{R}^d$, such that

$$(3) \quad X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, 1],$$

and, moreover, all finite dimensional distributions of X are defined uniquely. In view of (2), we can (and will) assume that the filtration $\mathbf{F}$ is generated by X :

$$(4) \quad \mathbf{F} = \mathbf{F}^X.$$

In this case, $\mathbb{P}$ is defined uniquely in the sense that if $\mathbb{Q} \sim \mathbb{P}$ is an equivalent probability measure on $(\Omega, \mathcal{F}_1) = (\Omega, \mathcal{F}_1^X)$ such that

$$W_t = \int_0^t \sigma^{-1}(s, X_s)(dX_s - b(s, X_s)ds), \quad t \in [0, 1],$$

is a Brownian motion under $\mathbb{Q}$, then $\mathbb{Q} = \mathbb{P}$.

REMARK 2.1. With respect to x , (A1) and (A2) are, essentially, the minimal classical assumptions guaranteeing the existence and the uniqueness of the weak solution to (3). This weak solution is also well-defined when b and σ are only measurable functions with respect to t . As we shall see in Example 2.6, the requirement on σ to be t -analytic is, however, essential for the validity of our main results, Theorems 2.3 and 2.5.

REMARK 2.2. It is well-known that any local martingale M adapted to the filtration $\mathbf{F}^W$, generated by the Brownian motion W , is a stochastic integral with respect to W , that is, there exists an $\mathbf{F}^W$ -predictable process H with values in $\mathbb{R}^d$ such that

$$(5) \quad M_t = M_0 + \int_0^t H_u dW_u \triangleq M_0 + \sum_{i=1}^d \int_0^t H_u^i dW_u^i, \quad t \in [0, 1].$$

The example in Barlow [2] shows that under (A1) and (A2) the filtration $\mathbf{F}^W$ may be strictly smaller than $\mathbf{F} = \mathbf{F}^X$. Nevertheless, for every local martingale M adapted to $\mathbf{F}$ the integral representation (5) still holds with some $\mathbf{F}$ -predictable H . This follows from the mentioned fact that any $\mathbb{Q} \sim \mathbb{P}$ such that W is a $\mathbb{Q}$ -local

martingale (equivalently, a $\mathbb{Q}$ -Brownian motion) coincides with $\mathbb{P}$ and the integral representation theorems in Jacod [7], Section XI.1(a).

Recall that a locally integrable function f on $(\mathbb{R}^d, dx)$ is *weakly differentiable* if for any index $i = 1, \dots, d$ there is a locally integrable function g^i such that the identity

$$\int_{\mathbb{R}^d} g^i(x) h(x) dx = - \int_{\mathbb{R}^d} f(x) \frac{\partial h}{\partial x^i}(x) dx$$

holds for any function $h \in \mathbf{C}^\infty$ with compact support, where $\mathbf{C}^\infty$ is the space of infinitely many times differentiable functions. In this case, we set $\frac{\partial f}{\partial x^i} \triangleq g^i$. The weak derivatives of higher orders are defined recursively.

Let $J \geq d$ be an integer and the functions $F^j, G : \mathbb{R}^d \rightarrow \mathbb{R}$ and $f^j, \alpha^j, \beta : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$, $j = 1, \dots, J$, be such that for some $N \geq 0$

- (A3) The functions F^j are weakly differentiable, $e^{-N|\cdot|} \frac{\partial F^j}{\partial x^i} \triangleq (e^{-N|x|} \frac{\partial F^j}{\partial x^i}(x))_{x \in \mathbb{R}^d} \in \mathbf{L}_\infty$, $i = 1, \dots, d$, and the Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i} \right)_{i=1, \dots, d, j=1, \dots, J}$ has rank d almost surely under the Lebesgue measure on $\mathbb{R}^d$.
- (A4) The function G is strictly positive and weakly differentiable and $e^{-N|\cdot|} \frac{\partial G}{\partial x^i} \in \mathbf{L}_\infty$, $i = 1, \dots, d$.
- (A5) The maps $t \mapsto e^{-N|\cdot|} f^j(t, \cdot) \triangleq (e^{-N|x|} f^j(t, x))_{x \in \mathbb{R}^d}$, $t \mapsto \alpha^j(t, \cdot)$, and $t \mapsto \beta(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$ are analytic.

We now define the random variables

$$(6) \quad \psi^j \triangleq F^j(X_1) e^{\int_0^1 \alpha^j(t, X_t) dt} + \int_0^1 e^{\int_0^t \alpha^j(s, X_s) ds} f^j(t, X_t) dt, \quad j = 1, \dots, J,$$

$$(7) \quad \xi \triangleq G(X_1) e^{\int_0^1 \beta(t, X_t) dt},$$

and state the main results of the paper.

THEOREM 2.3. *Suppose that (4) and (A1)–(A5) hold. Then the equivalent probability measure $\mathbb{Q}$ with the density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\xi}{\mathbb{E}[\xi]},$$

and the $\mathbb{Q}$ -martingale

$$S_t \triangleq \mathbb{E}^\mathbb{Q}[\psi | \mathcal{F}_t], \quad t \in [0, 1],$$

with values in $\mathbb{R}^J$ are well-defined and any local martingale M under $\mathbb{Q}$ is a stochastic integral with respect to S , that is, (1) holds.

REMARK 2.4. The t -analyticity condition on f^j in (A5) cannot be relaxed even if X is a one-dimensional Brownian motion, see Example 2.7 below. By contrast, the x -regularity assumptions on the functions F^j , G , and f^j in (A3), (A4), and (A5) admit weaker formulations with the $\mathbf{L}_\infty$ space being replaced by appropriate $\mathbf{L}_p$ spaces (with the power $p > 1$ different for each of these functions). This generalization leads, however, to more delicate and longer proofs and will be dealt with elsewhere.

The proof of Theorem 2.3 will be given in Section 5 and will rely on the study of parabolic equations in Section 4. In Section 3.2 we shall apply Theorem 2.3 to the problem of endogenous completeness in an economy with terminal consumption.

The following result, which, in fact, is an easy corollary of Theorem 2.3, will be used in Section 3.3 to study the endogenous completeness in an economy with intermediate consumption. For $i = 1, \dots, d$ let the functions $\gamma^i = \gamma^i(t, x)$ on $[0, 1] \times \mathbb{R}^d$ be such that

(A6) the maps $t \mapsto \gamma^i(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$ are analytic.

THEOREM 2.5. *Suppose that (4), (A1)–(A3), and (A5)–(A6) hold. Then the equivalent probability measure $\mathbb{Q}$ with the density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(\int_0^1 \gamma(s, X_s) dW_s - \frac{1}{2} \int_0^1 |\gamma(s, X_s)|^2 ds \right)$$

and the $\mathbb{Q}$ -martingale

$$S_t \triangleq \mathbb{E}^\mathbb{Q}[\psi | \mathcal{F}_t], \quad t \in [0, 1],$$

with values in $\mathbb{R}^J$ are well-defined and any local martingale under $\mathbb{Q}$ is a stochastic integral with respect to S .

PROOF. By Girsanov's theorem,

$$W_t^\mathbb{Q} = W_t - \int_0^t \gamma(s, X_s) ds$$

is a Brownian motion under $\mathbb{Q}$. After this substitution the equation (3) becomes

$$dX_t = (b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t))dt + \sigma(t, X_t)dW_t^\mathbb{Q}, \quad X_0 = x.$$

The result now follows from Theorem 2.3, where we can assume $\xi = 1$, if we observe that, similarly with b , each component of $\tilde{b} \triangleq b + \sigma\gamma$ defines an analytic map of $[0, 1]$ to $\mathbf{L}_\infty$. $\square$

We conclude with a few counter-examples illustrating the sharpness of the conditions of the theorems. Our first two examples show that the time analyticity assumptions on the volatility coefficient $\sigma = \sigma(t, x)$ and on the functions $f^j = f^j(t, x)$ in Theorems 2.3 and 2.5 cannot be relaxed. In both cases, we take $b(t, x) = \alpha(t, x) = \beta(t, x) = \gamma(t, x) = 0$ and $G(x) = 1$; in particular, $\mathbb{Q} = \mathbb{P}$.

EXAMPLE 2.6. We show that the assertions of Theorems 2.3 and 2.5 can fail to hold when all their conditions are satisfied except the t -analyticity of the volatility matrix σ . In our construction, $d = J = 2$ and both σ and its inverse σ^{-1} are $\mathbf{C}^\infty$ -matrices on $[0, 1] \times \mathbb{R}^2$ which are bounded with all their derivatives and have analytic restrictions to $[0, \frac{1}{2}] \times \mathbb{R}^2$ and $(\frac{1}{2}, 1] \times \mathbb{R}^2$.

Let $g = g(t)$ be a $\mathbf{C}^\infty$ -function on $[0, 1]$ which equals 0 on $[0, \frac{1}{2}]$ and is analytic and strictly positive on $(\frac{1}{2}, 1]$. Let $h = h(t, y)$ be a non-constant analytic function on $[0, 1] \times \mathbb{R}$ such that $0 \leq h \leq 1$ and

$$\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial^2 h}{\partial y^2} = 0.$$

For instance, we can take

$$h(t, y) = \frac{1}{2} (1 + e^{\frac{t-1}{2}} \sin y).$$

Define a 2-dimensional diffusion (X, Y) on $[0, 1]$ by

$$\begin{aligned} X_t &= \int_0^t \sqrt{1 + g(s)h(s, Y_s)} dB_s, \\ Y_t &= W_t, \end{aligned}$$

where B and W are independent Brownian motions. Clearly, the volatility matrix

$$\sigma(t, x, y) = \begin{pmatrix} \sqrt{1 + g(t)h(t, y)} & 0 \\ 0 & 1 \end{pmatrix}$$

has the announced properties and coincides with the identity matrix for $t \in [0, \frac{1}{2}]$.

Define the functions $F = F(x, y)$ and $H = H(x, y)$ on $\mathbb{R}^2$ as

$$F(x, y) = x,$$

$$H(x, y) = x^2 - 1 - h(1, y) \int_0^1 g(t) dt.$$

As $h(1, \cdot)$ is non-constant and analytic, the set of zeros for $\frac{\partial h}{\partial y}(1, \cdot)$ is at most countable. Since the determinant of the Jacobian matrix for (F, H) is given by

$$\frac{\partial F}{\partial x} \frac{\partial H}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial H}{\partial x} = -\frac{\partial h}{\partial y}(1, y) \int_0^1 g(t) dt,$$

it follows that this Jacobian matrix has full rank almost surely.

Observe now that

$$S_t \triangleq \mathbb{E}[F(X_1, Y_1) | \mathcal{F}_t] = X_t,$$

$$R_t \triangleq \mathbb{E}[H(X_1, Y_1) | \mathcal{F}_t] = X_t^2 - t - h(t, Y_t) \int_0^t g(s) ds,$$

which can be verified by Ito's formula. As $g(t) = 0$ for $t \in [0, \frac{1}{2}]$, it follows that $S_t = B_t$ and $R_t = B_t^2 - t$ on $[0, \frac{1}{2}]$. Hence, the Brownian motion $Y = W$ cannot be written as a stochastic integral with respect to (S, R) .

EXAMPLE 2.7. This counter-example shows the necessity of the t -analyticity assumption on $f^j = f^j(t, x)$ in (A5). Let $g = g(t)$ be a $\mathbf{C}^\infty$ -function on $[0, 1]$ which equals 0 on $[0, \frac{1}{2}]$, is analytic on $(\frac{1}{2}, 1]$, and is such that $g(1) \neq 0$. For the functions

$$\begin{aligned} f(t, x) &= -(g'(t)x + \frac{1}{2}g^2(t))e^{g(t)x}, \\ F(x) &= e^{g(1)x}, \end{aligned}$$

the conditions (A3) and (A5) hold except the time analyticity of the map $t \rightarrow e^{-N|\cdot|}f(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$. This map belongs instead to $\mathbf{C}^\infty$ and has analytic restrictions to $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$.

Take X to be a one-dimensional Brownian motion:

$$X_t = W_t, \quad t \in [0, 1],$$

and observe that, by Ito's formula,

$$S_t \triangleq \mathbb{E}[\psi | \mathcal{F}_t] = e^{g(t)W_t} - \int_0^t (g'(s)W_s + \frac{1}{2}g^2(s))e^{g(s)W_s} ds,$$

where

$$\psi = F(X_1) + \int_0^1 f(t, X_t) dt.$$

For $t \in [0, \frac{1}{2}]$ we have $g(t) = 0$ and, therefore, $S_t = 1$. Hence, any local martingale M which is non-constant on $[0, \frac{1}{2}]$ cannot be a stochastic integral with respect to S .

When the diffusion coefficients $\sigma^{ij} = \sigma^{ij}(t, x)$ and $b^i = b^i(t, x)$ and the functions $f^j = f^j(t, x)$ in (A5) are also analytic with respect to the state variable x , the results in [6] and [18] show that in (A3) it is sufficient to require the Jacobian matrix of $F = F(x)$ to have rank d only on an open set. The following example shows that in the case of $\mathbf{C}^\infty$ functions this simplification is not possible anymore.

EXAMPLE 2.8. Let $d = J = 2$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a $\mathbf{C}^\infty$ function such that $g(x) = 0$ for $x \leq 0$, while $g'(x) > 0$ and $g''(x)$ is bounded for $x > 0$.

Define the diffusion processes X and Y on $[0, 1]$ by

$$\begin{aligned} X_t &= B_t, \\ Y_t &= \int_0^t g''(X_s) ds + W_t, \end{aligned}$$

where B and W are independent Brownian motions. Clearly, the diffusion coefficients of (X, Y) satisfy (A1) and (A2).

Define the functions $F = F(x, y)$ and $H = H(x, y)$ on $\mathbb{R}^2$ as

$$\begin{aligned} F(x, y) &= y, \\ H(x, y) &= y - 2g(x), \end{aligned}$$

and the function $f = f(t, x, y)$ on $[0, 1] \times \mathbb{R}^2$ as

$$f(t, x, y) = -g''(x).$$

Observe that the determinant of the Jacobian matrix for (F, H) is given by

$$\frac{\partial F}{\partial x} \frac{\partial H}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial H}{\partial x} = 2g'(x),$$

and, hence, this Jacobian matrix has full rank on the set $(0, \infty) \times \mathbb{R}$.

A simple application of Ito's formula yields

$$\begin{aligned} S_t &\triangleq \mathbb{E}[F(X_1, Y_1) + \int_0^1 f(s, X_s, Y_s) ds | \mathcal{F}_t] = W_t, \\ R_t &\triangleq \mathbb{E}[H(X_1, Y_1) | \mathcal{F}_t] = W_t - 2 \int_0^t g'(X_s) dB_s. \end{aligned}$$

Hence, any martingale in the form

$$M_t = \int_0^t h(X_s) dB_s,$$

where the function $h = h(x)$ is different from zero for $x \leq 0$, cannot be written as a stochastic integral with respect to (S, R) .

3. Endogenous completeness

In this section, Theorems 2.3 and 2.5 will be applied to the problem of *endogenous completeness* in financial economics.

As before, the uncertainty and the information flow are modeled by the filtered probability space $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, 1]}, \mathbb{P})$ with the filtration $\mathbf{F}$ generated by the solution X to (3).

3.1. Financial market with exogenous prices. Recall first the “standard” model of mathematical finance, where the prices of traded securities are given as model inputs or, in more economic terms, *exogenously*.

Consider a financial market with $J + 1$ traded assets: a bank account and J stocks. The bank account pays the continuous interest rate $r = (r_t)$ and the stocks pay the continuous dividends $\theta = (\theta_t^j)$ and have the prices $P = (P_t^j)$, where $t \in [0, 1]$ and $j = 1, \dots, J$. We assume that P is a continuous semimartingale with values in $\mathbb{R}^J$ and

$$\int_0^1 (|r_t| + |\theta_t|) dt < \infty.$$

We shall use the abbreviation (r, θ, P) for such a model.

The wealth of a (self-financing) strategy evolves as

$$(8) \quad V_t = v + \int_0^t \zeta_u (dP_u + \theta_u du) + \int_0^t (V_u - \zeta_u P_u) r_u du, \quad t \in [0, 1],$$

where $v \in \mathbb{R}$ is the initial wealth and $\zeta = (\zeta_t^j)$ is the predictable process with values in $\mathbb{R}^J$ of the number of stocks such that the integrals in (8) are well-defined. This balance equation can be written more compactly in terms of discounted values:

$$V_t e^{-\int_0^t r_u du} = v + \int_0^t \zeta_u dS_u, \quad t \in [0, 1],$$

where, for $j = 1, \dots, J$,

$$S_t^j \triangleq P_t^j e^{-\int_0^t r_u du} + \int_0^t \theta_s^j e^{-\int_0^s r_u du} ds, \quad t \in [0, 1],$$

denotes the discounted wealth of the “buy and hold” strategy for j th stock, that is, the strategy where we start with one unit of such a stock and reinvest the continuous dividends $\theta = (\theta_t)$ in the bank account.

It is common to assume that the family $\mathcal{Q}$ of the equivalent martingale measures for S is not empty:

$$\mathcal{Q} = \mathcal{Q}(r, \theta, P) \triangleq \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q}\text{-martingale}\} \neq \emptyset.$$

This is equivalent to the absence of arbitrage if one is allowed to sell short both the bank account and the stock until the maturity; see [4].

The following property is the primary focus of our study.

DEFINITION 3.1. The model (r, θ, P) is called *complete* if for any random variable μ such that $|\mu| \leq 1$ there is a self-financing strategy such that $|V_t e^{-\int_0^t r_u du}| \leq 1$, $t \in [0, 1]$, and $V_1 e^{-\int_0^1 r_u du} = \mu$.

Recall, see Harrison and Pliska [5] and Jacod [7], Section XI.1(a), that for a (r, θ, P) -model with $\mathcal{Q} \neq \emptyset$ the completeness is equivalent to any of the following conditions:

- (1) there exists only one $\mathbb{Q} \in \mathcal{Q}$;
- (2) if $\mathbb{Q} \in \mathcal{Q}$ then any $\mathbb{Q}$ -local martingale is a discounted wealth process or, equivalently, is a stochastic integral with respect to S .

3.2. Economy with terminal consumption. Consider an economy with a single (representative) agent. We assume that the agent consumes only at maturity 1 and denote by $U = (U(x))_{x>0}$ his utility function for terminal wealth.

- (B1) The utility function $U = U(x)$ is twice weakly differentiable on $(0, \infty)$ and $U' > 0$. Moreover, it has a bounded relative risk aversion, that is, for some constant $N > 0$,

$$\frac{1}{N} \leq A(x) \triangleq -\frac{xU''(x)}{U'(x)} \leq N, \quad x \in (0, \infty).$$

Note that (B1) implies that U is strictly increasing, strictly concave, and continuously differentiable, that it satisfies the Inada conditions:

$$\lim_{x \downarrow 0} U'(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(x) = 0,$$

and that its asymptotic elasticity is strictly less than 1:

$$\limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

Given an (r, θ, P) -market, a basic problem of financial economics is to determine an optimal investment strategy $\widehat{V}(v)$ of the agent starting with the initial capital $v > 0$. More formally, if

$$\mathcal{V}(v) \triangleq \{V \geq 0 : (8) \text{ holds for some } \zeta\}$$

denotes the family of positive wealth processes starting from $v > 0$, then $\widehat{V}(v)$ is defined as an element of $\mathcal{V}(v)$ such that

$$(9) \quad \infty > \mathbb{E}[U(\widehat{V}_1(v))] \geq \mathbb{E}[U(V_1)] \quad \text{for all } V \in \mathcal{V}(v),$$

where we used the convention:

$$\mathbb{E}[U(V_1)] \triangleq -\infty \quad \text{if} \quad \mathbb{E}[\min(U(V_1), 0)] = -\infty.$$

We are interested in an inverse problem: given a terminal wealth Λ for the agent and final dividends $\Theta = (\Theta^j)$ for the stocks find a price process $P = (P_t^j)$ such that $P_1 = \Theta$ and, in the (r, θ, P) -market, $\widehat{V}_1(v) = \Lambda$ for some initial wealth $v > 0$. We particularly want to know whether the family $\mathcal{Q}(r, \theta, P)$ is a singleton and, hence, the (r, θ, P) -model is complete. Since the price process P is now an outcome, rather than an input, the latter property is referred to as an *endogenous completeness*.

We make the following assumptions:

- (B2) The interest rate $r_t = \beta(t, X_t)$, $t \in [0, 1]$, where the function $\beta = \beta(t, x)$ satisfies (A5).
 (B3) The continuous dividends $\theta = (\theta_t^j)$ and the terminal dividends $\Theta = (\Theta^j)$ are such that, for $t \in [0, 1]$ and $j = 1, \dots, J$,

$$\begin{aligned} \theta_t^j &= f^j(t, X_t) e^{\int_0^t \alpha^j(s, X_s) ds}, \\ \Theta^j &= F^j(X_1) e^{\int_0^1 \alpha^j(s, X_s) ds}, \end{aligned}$$

where the functions $F^j = F^j(x)$ satisfy (A3) and the functions f^j and α^j satisfy (A5).

- (B4) The terminal wealth $\Lambda = e^{H(X_1)}$, where the function $H = H(x)$ on $\mathbb{R}^d$ is weakly differentiable and $\frac{\partial H}{\partial x^i} \in \mathbf{L}^\infty$, $i = 1, \dots, d$.

Note that a function $H = H(x)$ on $\mathbb{R}^d$ satisfies (B4) if and only if it is Lipschitz continuous, that is, there is $N \geq 0$ such that

$$|H(x) - H(y)| \leq N|x - y|, \quad x, y \in \mathbb{R}^d.$$

For $j = 1, \dots, J$ denote

$$(10) \quad \psi^j \triangleq \Theta^j e^{-\int_0^1 r_u du} + \int_0^1 \theta_u^j e^{-\int_0^u r_s ds} du,$$

the cumulative values of the discounted cash flows generated by the stocks.

THEOREM 3.2. *Let (4), (A1)–(A2), and (B1)–(B4) hold. Then there exists a continuous process $P = (P_t^j)$ with the terminal value $P_1 = \Theta$ such that, in the (r, θ, P) -market, for some initial capital $v_0 > 0$ the optimal terminal wealth $\hat{V}_1(v_0)$ in (9) equals Λ and such that the set of martingale measures $\mathcal{Q} = \mathcal{Q}(r, \theta, P)$ is a singleton; in particular, the (r, θ, P) -market is complete.*

Further, $P = (P_t^j)$, $\mathbb{Q} \in \mathcal{Q}$, and v_0 are unique and given by

$$(11) \quad P_t = S_t e^{\int_0^t r_u du} - \int_0^t e^{\int_s^t r_u du} \theta_s ds, \quad t \in [0, 1],$$

$$(12) \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'(\Lambda) e^{\int_0^1 r_u du}}{\mathbb{E}[U'(\Lambda) e^{\int_0^1 r_u du}]},$$

$$(13) \quad v_0 = \mathbb{E}^{\mathbb{Q}}[\Lambda e^{-\int_0^1 r_u du}],$$

where, for $\psi = (\psi^j)$ from (10),

$$(14) \quad S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t], \quad t \in [0, 1].$$

PROOF. It is well-known, see [9, Theorem 3.7.6] and [11, Theorem 2.0], that for the utility function $U = U(x)$ as in (B1) and a complete market with unique $\mathbb{Q} \in \mathcal{Q}$ the optimal terminal wealth equals Λ if and only if (12) holds. Clearly, the martingale property of the discounted wealth process of an optimal strategy yields (13). Hence, it remains only to verify the completeness of the (r, θ, P) -market with $P = (P_t^j)$ given by (11).

Define the function

$$G(x) \triangleq U'(e^{H(x)}), \quad x \in \mathbb{R}^d,$$

and observe that

$$\frac{\partial \ln G}{\partial x^i} = \frac{U''}{U'}(e^H) e^H \frac{\partial H}{\partial x^i} = -A(e^H) \frac{\partial H}{\partial x^i} \in \mathbf{L}_\infty,$$

by the boundedness of A and $\frac{\partial H}{\partial x^i}$. This implies the existence of $N \geq 0$ such that

$$e^{-N|\cdot|} \left(G + \sum_{i=1}^d \left| \frac{\partial G}{\partial x^i} \right| \right) \in \mathbf{L}_\infty,$$

which, in particular, yields (A4).

Since $e^{-N|\cdot|}(G + e^H + |F|) \in \mathbf{L}_\infty$ for some $N \geq 0$, we deduce the existence of $N \geq 0$ such that

$$U'(\Lambda)(1 + \Lambda + |\psi|) \leq e^{N(1 + \sup_{t \in [0, 1]} |X_t|)}.$$

As the diffusion coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$ are bounded, the random variable $\sup_{t \in [0, 1]} |X_t|$ has all exponential moments. It follows that

$$\mathbb{E}[U'(\Lambda)(1 + \Lambda + |\psi|)] < \infty,$$

and, in particular, P , $\mathbb{Q}$, v_0 , and S are well-defined by (11)–(14).

By construction, $\mathbb{Q} \in \mathcal{Q}(r, \theta, P)$. With (A4) verified above, the assumptions of Theorem 2.3 for $\mathbb{Q}$ and S hold trivially. The results cited after Definition 3.1 then imply that the (r, θ, P) -market is complete and that $\mathbb{Q}$ is the only martingale measure. $\square$

We conclude this section with an important corollary of Theorem 3.2. Theorem 3.3 below yields dynamic completeness of all Pareto equilibria in an economy where M investors trade in the *exogenous* bank account paying the interest rate r and in the *endogenous* stocks paying the continuous dividends θ and the terminal dividends Θ . The economic agents have utility functions $U_m = U_m(x)$, $m = 1, \dots, M$, and they collectively possess the terminal wealth Λ . A result of this kind plays a crucial role in the proof of the existence of a continuous-time Arrow-Debreu-Radner equilibrium, see [1], [6], and [18].

THEOREM 3.3. *Let (4), (A1)–(A2), and (B2)–(B4) hold. Suppose each utility function U_m , $m = 1, \dots, M$, satisfies (B1). Fix $w \in (0, \infty)^N$ and define the function*

$$(15) \quad U(x) \triangleq \sup_{x^1 + \dots + x^M = x} \sum_{m=1}^M w^m U_m(x^m), \quad x \in (0, \infty).$$

Let the price process P be defined by (11), (12), and (14). Then the (r, θ, P) -market is complete.

PROOF. The result is an immediate consequence of Theorem 3.2 as soon as we verify that U satisfies (B1). This follows from the well-known identity for the relative risk-aversions:

$$\sum_{m=1}^M \frac{\hat{x}_m(x)}{A_m(\hat{x}_m(x))} = \frac{x}{A(x)}, \quad x \in (0, \infty).$$

Here $\hat{x}_1(x) > 0, \dots, \hat{x}_M(x) > 0$ are the arguments of maximum in (15) and A_m is the relative risk-aversion of U_m . The arguments leading to this equality will be recalled in the proof of Lemma 3.6. $\square$

3.3. Economy with intermediate consumption. Consider now an economy where a single (representative) agent consumes *continuously* on $[0, 1]$. We denote by $u(t, x) : [0, 1] \times (0, \infty) \rightarrow \mathbb{R}$ the agent's utility function for intermediate consumption and assume that

- (B5) $u = u(t, x)$ is analytic in t and 3-times weakly differentiable in x . Moreover, $u_x > 0$ and $u_{xx} < 0$ and $t \mapsto a(t, \cdot)$, $t \mapsto \frac{1}{a(t, \cdot)}$, $t \mapsto p(t, \cdot)$, and

$t \mapsto q(t, \cdot)$ are analytic maps of $[0, 1]$ to $\mathbf{L}_\infty$, where

$$\begin{aligned} a(t, x) &\triangleq -\frac{xu_{xx}(t, x)}{u_x(t, x)}, \\ p(t, x) &\triangleq -\frac{xu_{xxx}(t, x)}{u_{xx}(t, x)}, \\ q(t, x) &\triangleq -\frac{\partial \ln u_x(t, x)}{\partial t} = -\frac{u_{xt}}{u_x}(t, x), \end{aligned}$$

are, respectively, the relative risk aversion, the relative prudence, and an “impatience” rate of the utility function u .

Note that (B5) implies that $u(t, \cdot)$ is twice continuously differentiable, strictly increasing, and strictly concave and that there is a constant $N > 0$ such that

$$(16) \quad a(t, x) + \frac{1}{a(t, x)} + |p(t, x)| + |q(t, x)| \leq N, \quad (t, x) \in [0, 1] \times \mathbb{R}.$$

Recall the formulation of the investment problem with continuous consumption in a *given* (r, θ, P) -market. Let $\eta = (\eta_t)$ be a non-negative adapted process such that $\int_0^1 \eta_t dt < \infty$. The wealth process of a strategy with the consumption process η is defined as

$$(17) \quad V_t = v + \int_0^t \zeta_u(dP_u + \theta_u du) + \int_0^t (V_u - \zeta_u P_u)r_u du - \int_0^t \eta_u du,$$

or, in discounted terms,

$$V_t e^{-\int_0^t r_s ds} = v + \int_0^t \zeta_u dS_u - \int_0^t \eta_u e^{-\int_0^u r_s ds} du, \quad t \in [0, 1].$$

Here, as before, v and $\zeta = (\zeta_t^j)$ stand, respectively, for the initial wealth and the process of the number of stocks. We consider the optimization problem

$$(18) \quad \mathbb{E}\left[\int_0^1 u(t, \eta_t) dt\right] \rightarrow \max, \quad \eta \in \mathcal{W}(v),$$

where $\mathcal{W}(v)$ denotes the family of consumption processes obtained from the initial wealth v , that is,

$$\mathcal{W}(v) \triangleq \{\eta \geq 0 : (17) \text{ holds for some } V \geq 0 \text{ and } \zeta\},$$

and we have used the convention:

$$\mathbb{E}\left[\int_0^1 u(t, \eta_t) dt\right] \triangleq -\infty \quad \text{if} \quad \mathbb{E}\left[\int_0^1 \min(u(t, \eta_t), 0) dt\right] = -\infty.$$

As in the previous section, we study an inverse problem to (18): given a consumption process $\lambda = (\lambda_t)$ for the agent and final dividends $\Theta = (\Theta^j)$ for the stocks, find an interest rate process $r = (r_t)$ and a price process $P = (P_t^j)$ such that $P_1 = \Theta$ and, in the (r, θ, P) -model, the upper bound in (18) is attained at $\lambda = (\lambda_t)$ for some initial wealth $v > 0$. We are particularly interested in the completeness of the resulting (r, θ, P) -market.

(B6) The consumption process $\lambda_t = e^{g(t, X_t)}$, $t \in [0, 1]$, where the function $g = g(t, x)$ on $[0, 1] \times \mathbb{R}^d$ is analytic in t and twice weakly differentiable in x . Moreover, $t \mapsto \frac{\partial g}{\partial t}(t, \cdot)$, $t \mapsto \frac{\partial g}{\partial x^i}(t, \cdot)$, and $t \mapsto \frac{\partial^2 g}{\partial x^i \partial x^j}(t, \cdot)$ are analytic maps of $[0, 1]$ to $\mathbf{L}_\infty$ for all $i, j = 1, \dots, J$.

THEOREM 3.4. *Suppose that (4), (A1)–(A2), (B3), and (B5)–(B6) hold. Then there exist a bounded process $r = (r_t)$ and a continuous process $P = (P_t^j)$ with the terminal value $P_1 = \Theta$ such that, in the (r, θ, P) -market, the set of martingale measures $\mathcal{Q}$ is a singleton and, for some initial wealth $v_0 > 0$, the consumption process $\lambda = (\lambda_t)$ solves (18).*

The interest rate process $r = (r_t)$ and the density process $Z = (Z_t)$ of $\mathbb{Q} \in \mathcal{Q}$ are uniquely determined from the decomposition

$$(19) \quad u_x(t, \lambda_t) = u_x(0, \lambda_0) Z_t e^{-\int_0^t r_s ds}, \quad t \in [0, 1].$$

The price process $P = (P_t)$ is unique and given, in terms of $r = (r_t)$ and $\mathbb{Q}$, by (11), (14), and (10). Finally, the initial wealth v_0 is unique and given by

$$(20) \quad v_0 = \mathbb{E}^{\mathbb{Q}} \left[\int_0^1 e^{-\int_0^t r_u du} \lambda_t dt \right] < \infty.$$

PROOF. The well-known results on optimal consumption in complete markets, see [9, Theorem 3.7.3], imply that for a utility function $u = u(t, x)$ as in (B1) and a complete (r, θ, P) -market with unique $\mathbb{Q} \in \mathcal{Q}$, a non-negative process $\lambda = (\lambda_t)$ solves (18) if and only if (19) holds. Moreover, the initial wealth of an optimal strategy yielding the consumption process $\lambda = (\lambda_t)$ is given by (20).

The function

$$w(t, x) \triangleq u_x(t, e^{g(t, x)}), \quad (t, x) \in [0, 1] \times \mathbf{R}^d,$$

is analytic in t and twice weakly differentiable in x . Further, there is $N > 0$ such that the second derivatives $\frac{\partial^2 w}{\partial x^i \partial x^j}$ are bounded by $e^{N|x|}$. Although the second derivatives are not continuous, a version of Ito's formula from Krylov [15], Theorem 2.10.1 can still be applied to

$$Y_t \triangleq u_x(t, \lambda_t) = u_x(t, e^{g(t, X_t)}) = w(t, X_t), \quad t \in [0, 1],$$

yielding

$$(21) \quad dY_t = Y_t(-\beta(t, X_t)dt + \gamma(t, X_t)dW_t).$$

The functions $\beta = \beta(t, x)$ and $\gamma^i = \gamma^i(t, x)$, $i = 1, \dots, d$, on $[0, 1] \times \mathbb{R}^d$ are given by

$$\begin{aligned} \beta &= q(t, e^g) + a(t, e^g) \left(\frac{\partial g}{\partial t} + \sum_{k=1}^d \frac{\partial g}{\partial x^k} b^k + \frac{1}{2} \sum_{k, l, m=1}^d \sigma^{km} \sigma^{lm} c^{kl} \right), \\ \gamma^i &= -a(t, e^g) \sum_{k=1}^d \frac{\partial g}{\partial x^k} \sigma^{ki}, \end{aligned}$$

where we omitted the common argument (t, x) and

$$c^{kl} = (1 - p(t, e^g)) \frac{\partial g}{\partial x^k} \frac{\partial g}{\partial x^l} + \frac{\partial^2 g}{\partial x^k \partial x^l}.$$

The assumptions of the theorem imply that $\beta = \beta(t, x)$ and $\gamma^i = \gamma^i(t, x)$, $i = 1, \dots, d$, satisfy the conditions (A5) and (A6), respectively.

From (21) we deduce that a local martingale Z such that $Z_0 = 1$ and a predictable process $r = (r_t)$ are uniquely determined by (19) and are given by

$$Z_t = \exp \left(\int_0^t \gamma(s, X_s) dW_s - \frac{1}{2} \int_0^t |\gamma(s, X_s)|^2 ds \right),$$

$$r_t = \beta(t, X_t).$$

Since $\gamma = \gamma(t, x)$ is bounded on $[0, 1] \times \mathbb{R}^d$, we obtain that Z is, in fact, a martingale and, hence, is a density of some $\mathbb{Q} \sim \mathbb{P}$. Given $r = (r_t)$ and $\mathbb{Q}$ we define $P = (P_t^j)$ and $S = (S_t^j)$ by (11) and (14), respectively. By construction, $\mathbb{Q} \in \mathcal{Q}(r, \theta, P)$. Observe now that the conditions of Theorem 2.5 hold trivially for these $\mathbb{Q}$ and S . Hence the (r, θ, P) -market is complete and $\mathbb{Q}$ is its only martingale measure.

Finally, from (B6) we deduce the existence of $N \geq 0$ such that

$$\lambda_t = e^{g(t, X_t)} \leq e^{N(1+|X_t|)},$$

which, in view of the boundedness of the functions β and γ^i and of the diffusion coefficients b^i and σ^{ij} , easily yields the finiteness of v_0 in (20). $\square$

We conclude with a criteria for dynamic completeness of Pareto equilibria in the case of intermediate consumption. Consider an economy populated by M investors who trade in the bank account and the stocks; both are defined *endogenously*. The stocks pay the continuous dividends θ and the terminal dividends Θ . The economic agents jointly consume with the rate $\lambda = (\lambda_t)$ and have the utility functions $u^m = u^m(t, x)$, $m = 1, \dots, M$.

We are interested in the validity of the assertions of Theorem 3.4 when the function $u = u(t, x)$ is given by

$$(22) \quad u(t, x) \triangleq \sup_{x^1 + \dots + x^M = x} \sum_{m=1}^M w^m u^m(t, x^m), \quad (t, x) \in [0, 1] \times (0, \infty),$$

for some $w \in (0, \infty)^M$. The delicacy of the situation is that the t -analyticity of u does not follow automatically from the t -analyticity of u^m , $m = 1, \dots, M$. We consider two special cases:

- (B7) For every $m = 1, \dots, M$ the function $u^m = u^m(t, x)$ satisfies (B5) and is jointly analytic in (t, x) .
- (B8) For every $m = 1, \dots, M$ the function $u^m = u^m(t, x)$ is given by

$$u^m(t, x) = e^{-\nu(t)} U_m(x), \quad (t, x) \in [0, 1] \times (0, \infty),$$

where $\nu = \nu(t)$ is an analytic function on $[0, 1]$ and the function $U_m = U_m(x)$ satisfies (B1) and has a bounded relative risk-prudence:

$$-N \leq -\frac{x U_m'''(x)}{U_m''(x)} \leq N, \quad x \in (0, \infty),$$

for some $N > 0$.

THEOREM 3.5. *Assume (4), (A1)–(A2), (B3), and (B6). Suppose also that the utility functions $u^m = u^m(t, x)$ satisfy either (B7) or (B8). Fix $w \in (0, \infty)^M$ and define $u = u(t, x)$ by (22). Then the assertions of Theorem 3.4 hold.*

The result is an immediate corollary of Theorem 3.4 and the following

LEMMA 3.6. *Assume the utility functions $u^m = u^m(t, x)$ satisfy either (B7) or (B8) and let $w \in (0, \infty)^M$. Then for $u = u(t, x)$ defined by (22) condition (B5) holds true.*

PROOF. We shall focus on the case when (B7) holds. The proof under (B8) is analogous. Denote by a^m , p^m , and q^m the coefficients for u^m from (B5).

Condition (B5) for u^m implies that

$$\lim_{x \downarrow 0} u_x^m(t, x) = \infty, \quad \lim_{x \rightarrow \infty} u_x^m(t, x) = 0.$$

It follows that the upper bound in (22) is attained at unique $\hat{x}(t, x) = (\hat{x}^m(t, x))_{m=1, \dots, M}$ determined by

$$(23) \quad \sum_{m=1}^M \hat{x}^m(t, x) = x,$$

$$(24) \quad w^m u_x^m(t, \hat{x}^m(t, x)) = w^M u_x^M(t, \hat{x}^M(t, x)), \quad m = 1, \dots, M-1.$$

On $[0, 1] \times (0, \infty) \times (0, \infty)^M$ define the functions

$$h^m(t, x, y) = w^m u_x^m(t, y^m) - w^M u_x^M(t, y^M), \quad m = 1, \dots, M-1,$$

$$h^M(t, x, y) = \sum_{m=1}^M y^m - x.$$

Clearly,

$$h^m(t, x, \hat{x}(t, x)) = 0, \quad m = 1, \dots, M,$$

and

$$\frac{\partial h^m}{\partial y^l}(t, x, \hat{x}(t, x)) = w^m u_{xx}^m(t, \hat{x}^m(t, x)) 1_{\{l=m\}}, \quad m, l = 1, \dots, M-1,$$

$$\frac{\partial h^m}{\partial y^M}(t, x, \hat{x}(t, x)) = -w^M u_{xx}^M(t, \hat{x}^M(t, x)) \quad m = 1, \dots, M-1,$$

$$\frac{\partial h^M}{\partial y^m}(t, x, \hat{x}(t, x)) = 1, \quad m = 1, \dots, M.$$

As $u_{xx}^m < 0$ the Jacobian matrix of $h^1(t, x, \cdot), \dots, h^M(t, x, \cdot)$ at $\hat{x}(t, x)$ has a full rank. Since the functions h^m are analytic in (t, x, y) the implicit function theorem yields that the functions $\hat{x}^m$ are analytic in (t, x) , see Krantz and Parks [12], Theorem 2.3.5. Moreover, standard computations in the implicit function theorem show that

$$(25) \quad \frac{\partial \hat{x}^l}{\partial x}(t, x) = \frac{\hat{x}^l}{a^l(t, \hat{x}^l)} / \left(\sum_{m=1}^M \frac{\hat{x}^m}{a^m(t, \hat{x}^m)} \right).$$

Since

$$u(t, x) = \sum_{m=1}^M w^m u^m(t, \hat{x}^m(t, x)),$$

the function u is analytic in (t, x) . Hence, to complete the proof it only remains to verify (16) for this function.

Accounting for (23) and (24) we obtain

$$\begin{aligned} u_t(t, x) &= \sum_{m=1}^M w^m u_t^m(t, \hat{x}^m), \\ u_x(t, x) &= w^m u_x^m(t, \hat{x}^m), \quad m = 1, \dots, M. \end{aligned}$$

By differentiating these equalities a necessary number of times with respect to x and accounting for (25) we arrive to the identities:

$$\begin{aligned} \frac{1}{a(t, x)} &= \sum_{m=1}^M \frac{1}{a^m(t, \hat{x}^m)} \frac{\hat{x}^m}{x}, \\ p(t, x) &= \sum_{m=1}^M p^m(t, \hat{x}^m) \left(\frac{a(t, x)}{a^m(t, \hat{x}^m)} \right)^2 \frac{\hat{x}^m}{x}, \\ q(t, x) &= \sum_{m=1}^M q^m(t, \hat{x}^m) \frac{a(t, x)}{a^m(t, \hat{x}^m)} \frac{\hat{x}^m}{x}, \end{aligned}$$

which readily imply (16). $\square$

4. A time analytic solution of a parabolic equation

The proof of Theorem 2.3 will rely on the study of a parabolic equation in Theorem 4.4 below.

For reader's convenience, recall the definition of the classical Sobolev spaces $\mathbf{W}_p^m$ on $\mathbb{R}^d$ where $m \in \{0, 1, \dots\}$ and $p \geq 1$. When $m = 0$ we get the classical Lebesgue spaces $\mathbf{L}_p = \mathbf{L}_p(\mathbb{R}^d, dx)$ with the norm

$$\|f\|_{\mathbf{L}_p} \triangleq \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

When $m \in \{1, \dots\}$ the Sobolev space $\mathbf{W}_p^m$ consists of all m -times weakly differentiable functions f such that

$$\|f\|_{\mathbf{W}_p^m} \triangleq \|f\|_{\mathbf{L}_p} + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha f\|_{\mathbf{L}_p} < \infty$$

and is a Banach space with such a norm. The summation is taken with respect to multi-indexes $\alpha = (\alpha_1, \dots, \alpha_d)$ of non-negative integers, $|\alpha| \triangleq \sum_{i=1}^d \alpha_i$ and

$$D^\alpha \triangleq \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Recall also that a function $h = h(t) : [0, 1] \rightarrow \mathbf{X}$ with values in a Banach space $\mathbf{X}$ is called Hölder continuous if there is $0 < \gamma < 1$ such that

$$\sup_{t \in [0, 1]} \|h(t)\|_{\mathbf{X}} + \sup_{0 \leq s < t \leq 1} \frac{\|h(t) - h(s)\|_{\mathbf{X}}}{|t - s|^\gamma} < \infty.$$

For $t \in [0, 1]$ and $x \in \mathbb{R}^d$ consider an elliptic operator

$$A(t) \triangleq \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(t, x) \frac{\partial}{\partial x^i} + c(t, x),$$

where a^{ij} , b^i , and c are measurable functions on $[0, 1] \times \mathbb{R}^d$ such that

- (C1) $t \mapsto a^{ij}(t, \cdot)$ is an analytic map of $[0, 1]$ to $\mathbf{C}$, $t \mapsto b^i(t, \cdot)$ and $t \mapsto c(t, \cdot)$ are analytic maps of $[0, 1]$ to $\mathbf{L}_\infty$. The matrix a is symmetric: $a^{ij} = a^{ji}$, uniformly elliptic: there exists $N > 0$ such that

$$ya(t, x)y \geq \frac{1}{N^2}|y|^2, \quad (t, x) \in [0, 1] \times \mathbb{R}^d, \quad y \in \mathbb{R}^d,$$

and is uniformly continuous with respect to x : there exists a decreasing function $\omega = (\omega(\epsilon))_{\epsilon > 0}$ such that $\omega(\epsilon) \rightarrow 0$ as $\epsilon \downarrow 0$ and for all $t \in [0, 1]$ and $y, z \in \mathbb{R}^d$

$$|a^{ij}(t, y) - a^{ij}(t, z)| \leq \omega(|y - z|).$$

Let $g = g(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ and $f = f(t, x) : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable functions such that for some $p > 1$

- (C2) the function g belongs to $\mathbf{W}_p^1$ and $t \mapsto f(t, \cdot)$ is a Hölder continuous map from $[0, 1]$ to $\mathbf{L}_p$ whose restriction to $(0, 1]$ is analytic.

THEOREM 4.1. *Let $p > 1$ and suppose the conditions (C1) and (C2) hold. Then there exists a unique measurable function $u = u(t, x)$ on $[0, 1] \times \mathbb{R}^d$ such that*

- (1) $t \mapsto u(t, \cdot)$ is a Hölder continuous map of $[0, 1]$ to $\mathbf{L}_p$,
- (2) $t \mapsto u(t, \cdot)$ is a continuous map of $[0, 1]$ to $\mathbf{W}_p^1$,
- (3) $t \mapsto u(t, \cdot)$ is an analytic map of $(0, 1]$ to $\mathbf{W}_p^2$,

and such that $u = u(t, x)$ solves the parabolic equation:

$$(26) \quad \frac{\partial u}{\partial t} = A(t)u + f, \quad t \in (0, 1],$$

$$(27) \quad u(0, \cdot) = g.$$

The proof is essentially a compilation of references to known results. We first introduce some notations and state a few lemmas.

Let $\mathbf{X}$ and $\mathbf{D}$ be Banach spaces. By $\mathcal{L}(\mathbf{X}, \mathbf{D})$ we denote the Banach space of bounded linear operators $T : \mathbf{X} \rightarrow \mathbf{D}$ endowed with the operator norm. A shorter notation $\mathcal{L}(\mathbf{X})$ is used for $\mathcal{L}(\mathbf{X}, \mathbf{X})$. We shall write $\mathbf{D} \subset \mathbf{X}$ if $\mathbf{D}$ is *continuously embedded* into $\mathbf{X}$, that is, the elements of $\mathbf{D}$ form a subset of $\mathbf{X}$ and there is a constant $N > 0$ such that $\|x\|_{\mathbf{X}} \leq N\|x\|_{\mathbf{D}}$, $x \in \mathbf{D}$. We shall write $\mathbf{D} = \mathbf{X}$ if $\mathbf{D} \subset \mathbf{X}$ and $\mathbf{X} \subset \mathbf{D}$.

Let $\mathbf{D} \subset \mathbf{X}$. A Banach space $\mathbf{E}$ is called an *interpolation space* between $\mathbf{D}$ and $\mathbf{X}$ if $\mathbf{D} \subset \mathbf{E} \subset \mathbf{X}$ and any linear operator $T \in \mathcal{L}(\mathbf{X})$ whose restriction to $\mathbf{D}$ belongs to $\mathcal{L}(\mathbf{D})$ also has its restriction to $\mathbf{E}$ in $\mathcal{L}(\mathbf{E})$; see Bergh and Löfström [3], Section 2.4.

The following lemma will be used in the proof of item 2 of the theorem.

LEMMA 4.2. *Let $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{X}$ be Banach spaces such that $\mathbf{D} \subset \mathbf{X}$, $\mathbf{E}$ is an interpolation space between $\mathbf{D}$ and $\mathbf{X}$, and $\mathbf{D}$ is dense in $\mathbf{E}$. Let $(T_n)_{n \geq 1}$ be a sequence of linear operators in $\mathcal{L}(\mathbf{X})$ such that $\lim_{n \rightarrow \infty} \|T_n x\|_{\mathbf{X}} = 0$ for any $x \in \mathbf{X}$ and $\lim_{n \rightarrow \infty} \|T_n x\|_{\mathbf{D}} = 0$ for any $x \in \mathbf{D}$. Then $\lim_{n \rightarrow \infty} \|T_n x\|_{\mathbf{E}} = 0$ for any $x \in \mathbf{E}$.*

PROOF. The uniform boundedness theorem implies that the sequence $(T_n)_{n \geq 1}$ is bounded both in $\mathcal{L}(\mathbf{X})$ and $\mathcal{L}(\mathbf{D})$. Due to the Banach property, $\mathbf{E}$ is a uniform interpolation space between $\mathbf{D}$ and $\mathbf{X}$, that is, there is a constant $M > 0$ such that

$$\|T\|_{\mathcal{L}(\mathbf{E})} \leq M \max(\|T\|_{\mathcal{L}(\mathbf{X})}, \|T\|_{\mathcal{L}(\mathbf{D})}) \text{ for any } T \in \mathcal{L}(\mathbf{X}) \cap \mathcal{L}(\mathbf{D});$$

see Theorem 2.4.2 in [3]. Hence, $(T_n)_{n \geq 1}$ is also bounded in $\mathcal{L}(\mathbf{E})$. The density of $\mathbf{D}$ in $\mathbf{E}$ then yields the result. $\square$

Let A be an (unbounded) closed linear operator on $\mathbf{X}$. We denote by $\mathbf{D}(A)$ the domain of A and assume that it is endowed with the graph norm of A :

$$\|x\|_{\mathbf{D}(A)} \triangleq \|Ax\|_{\mathbf{X}} + \|x\|_{\mathbf{X}}.$$

Then $\mathbf{D}(A)$ is a Banach space. Recall that the resolvent set $\rho(A)$ of A is defined as the set of complex numbers λ for which the operator $\lambda I - A : \mathbf{D}(A) \rightarrow \mathbf{X}$, where I is the identity operator, is invertible; the inverse operator is called the resolvent and is denoted by $R(\lambda, A)$. The bounded inverse theorem implies that $R(\lambda, A) \in \mathcal{L}(\mathbf{X}, \mathbf{D}(A))$ and, in particular, $R(\lambda, A) \in \mathcal{L}(\mathbf{X})$.

The operator A is called *sectorial* if there are constants $M > 0$, $r \in \mathbb{R}$, and $\theta \in (0, \frac{\pi}{2})$ such that the sector

$$(28) \quad S_{r,\theta} \triangleq \{\lambda \in \mathbb{C} : \lambda \neq r \text{ and } |\arg(\lambda - r)| \leq \pi - \theta\}$$

of the complex plane $\mathbb{C}$ is a subset of $\rho(A)$ and

$$(29) \quad \|R(\lambda, A)\|_{\mathcal{L}(\mathbf{X})} \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in S_{r,\theta}.$$

The set of such sectorial operators will be denoted by $\mathcal{S}(M, r, \theta)$. Sectorial operators are important, because when their domains are dense in $\mathbf{X}$ they coincide with generators of analytic semi-groups, see Pazy [17], Section 2.5.

The following lemma will enable us to use the results from Kato and Tanabe [10] to verify item 3 of the theorem.

LEMMA 4.3. *Let $\mathbf{X}$ and $\mathbf{D}$ be Banach spaces such that $\mathbf{D} \subset \mathbf{X}$ and let $A = (A(t))_{t \in [0,1]}$ be closed linear operators on $\mathbf{X}$ such that $\mathbf{D}(A(t)) = \mathbf{D}$ for all $t \in [0, 1]$. Suppose $A : [0, 1] \rightarrow \mathcal{L}(\mathbf{D}, \mathbf{X})$ is an analytic map, and there are $M > 0$, $r < 0$, and $\theta \in (0, \frac{\pi}{2})$ such that $A(t) \in \mathcal{S}(M, r, \theta)$ for all $t \in [0, 1]$.*

Then there exist a convex open set U in $\mathbb{C}$ containing $[0, 1]$ and an analytic extension of A to U such that $A(z) \in \mathcal{S}(2M, r, \theta)$ for all $z \in U$ and the function $A^{-1} : [0, 1] \rightarrow \mathcal{L}(\mathbf{X}, \mathbf{D})$ is analytic.

PROOF. If $A \in \mathcal{S}(M, r, \theta)$, then for $\lambda \in S_{r,\theta}$

$$\|R(\lambda, A)\|_{\mathcal{L}(\mathbf{X}, \mathbf{D}(A))} = \|R(\lambda, A)\|_{\mathcal{L}(\mathbf{X})} + \|AR(\lambda, A)\|_{\mathcal{L}(\mathbf{X})} \leq M + 1,$$

where we used (29) and the identity $AR(\lambda, A) = \lambda R(\lambda, A) - I$. As $A : [0, 1] \rightarrow \mathcal{L}(\mathbf{D}, \mathbf{X})$ is a continuous function, the Banach spaces $\mathbf{D}$ and $\mathbf{D}(A(t))$, $t \in [0, 1]$, are uniformly equivalent, that is, there is $L > 0$ such that $\|x\|_{\mathbf{D}(A(t))} \leq L\|x\|_{\mathbf{D}}$ and $\|x\|_{\mathbf{D}} \leq L\|x\|_{\mathbf{D}(A(t))}$ for every $t \in [0, 1]$ and every $x \in \mathbf{D}$. It follows that one can find $N > 0$ such that

$$(30) \quad \|R(\lambda, A(t))\|_{\mathcal{L}(\mathbf{X}, \mathbf{D})} \leq N, \quad \lambda \in S_{r,\theta}, t \in [0, 1].$$

Since $r < 0$, the operator $A(t)$ is invertible for every $t \in [0, 1]$. As $A : [0, 1] \rightarrow \mathcal{L}(\mathbf{D}, \mathbf{X})$ is analytic, the inverse function $B = A^{-1} : [0, 1] \rightarrow \mathcal{L}(\mathbf{X}, \mathbf{D})$ is well-defined and analytic. Clearly, there is an open convex set U in $\mathbb{C}$ containing $[0, 1]$ on which both A and B can be analytically extended. Then $B = A^{-1}$ on U , as AB is an analytic function on U with values in $\mathcal{L}(\mathbf{X})$ which on $[0, 1]$ equals the identity

operator. Of course, we can choose U so that for any $z \in U$ there is $t \in [0, 1]$ such that

$$(31) \quad \|A(z) - A(t)\|_{\mathcal{L}(\mathbf{D}, \mathbf{X})} \leq \frac{1}{2N},$$

where the constant $N > 0$ is taken from (30).

Fix $\lambda \in S_{r, \theta}$ and take $t \in [0, 1]$ and $z \in U$ satisfying (31). By (30) and (31)

$$\|(A(z) - A(t))R(t, A(t))\|_{\mathcal{L}(\mathbf{X})} \leq \frac{1}{2}.$$

Hence the operator $I - (A(z) - A(t))R(t, A(t))$ in $\mathcal{L}(\mathbf{X})$ is invertible and its inverse has norm less than 2. Since

$$\lambda I - A(z) = (I - (A(z) - A(t))R(t, A(t)))(\lambda I - A(t)),$$

we obtain that the resolvent $R(\lambda, A(z))$ is well-defined and

$$\|R(\lambda, A(z))\|_{\mathcal{L}(\mathbf{X})} \leq \frac{2M}{1 + |\lambda|}.$$

This completes the proof. $\square$

PROOF OF THEOREM 4.1. It is well-known that under (C1) for every $t \in [0, 1]$ the operator $A(t)$ is closed in $\mathbf{L}_p$ and has $\mathbf{W}_p^2$ as its domain:

$$(32) \quad \mathbf{D}(A(t)) = \mathbf{W}_p^2.$$

Moreover, the operators $(A(t))_{t \in [0, 1]}$ are *sectorial* with the same constants $M > 0$, $r \in \mathbb{R}$, and $\theta \in (0, \frac{\pi}{2})$:

$$(33) \quad A(t) \in \mathcal{S}(M, r, \theta), \quad t \in [0, 1].$$

These results can be found, for example, in Krylov [13], see Section 13.4 and Exercise 13.5.1.

It will be convenient for us to assume that the sector $S_{r, \theta}$ defined in (28) contains 0 or, equivalently, that $r < 0$. This does not restrict any generality as for $s \in \mathbb{R}$ the substitution $u(t, x) \rightarrow e^{st}u(t, x)$ in (26) corresponds to the shift $A(t) \rightarrow A(t) + s$ in the operators $A(t)$. Among other benefits, this assumption implies the existence of inverses and fractional powers for the operators $-A(t)$; see Section 2.6 in [17] on fractional powers of sectorial operators.

From (C1) we clearly deduce the existence of $M > 0$ such that for any $v \in \mathbf{W}_p^2$

$$(34) \quad \|(A(t) - A(s))v\|_{\mathbf{L}_p} \leq M|t - s|\|v\|_{\mathbf{W}_p^2}, \quad s, t \in [0, 1].$$

Conditions (32), (33), and (34) for the operators $A = A(t)$ and condition (C2) for f and g imply the existence and uniqueness of the classical solution $u = u(t, x)$ to the initial value problem (26)–(27) in $\mathbf{L}_p$; see Theorem 7.1 in Section 5 of [17]. Recall that $u = u(t, x)$ is the *classical solution* to (26) and (27) if $u(t, \cdot) \in \mathbf{W}_p^2$ for $t \in (0, 1]$, the map $t \mapsto u(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_p$ is continuous, the restriction of this map to $(0, 1]$ is continuously differentiable, and the equations (26) and (27) hold.

To verify item 1 we use Theorem 3.10 in Yagi [20] dealing with maximal regularity properties of solutions to evolution equations. This theorem implies the existence of constants $\delta > 0$ and $M > 0$ such that

$$(35) \quad \left\| \frac{\partial u}{\partial t}(t, \cdot) \right\|_{\mathbf{L}_p} \leq Mt^{\delta-1}, \quad t \in (0, 1],$$

provided that the operators $A = A(t)$ satisfy (32)–(34), the function f is Hölder continuous as in (C2), and for some $0 < \gamma < 1$

$$(36) \quad g \in \mathbf{D}((-A(0))^\gamma),$$

where $\mathbf{D}((-A(0))^\gamma)$ is the domain of the fractional power γ of the operator $-A(0)$ acting in $\mathbf{L}_p$. The inequality (35) clearly implies the Hölder continuity of $u(t, \cdot) : [0, 1] \rightarrow \mathbf{L}_p$ and, hence, to complete the proof of item 1 we only need to verify (36).

Since $g \in \mathbf{W}_p^1$, we obtain (36) if

$$\mathbf{W}_p^1 \subset \mathbf{D}((-A(0))^\gamma), \quad \gamma \in (0, \frac{1}{2}).$$

This embedding is an immediate corollary of the classical characterization of Sobolev spaces $\mathbf{W}_p^m$ as the domains of $(1 - \Delta)^{m/2}$ in $\mathbf{L}_p$:

$$\mathbf{W}_p^m = \mathbf{D}((1 - \Delta)^{m/2}), \quad m \in \{0, 1, \dots\},$$

where $\Delta \triangleq \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and the fact that for $0 < \alpha < \beta < 1$ and sectorial operators A and B such that $\mathbf{D}(B) \subset \mathbf{D}(A)$ and such that the fractional powers $(-A)^\alpha$ and $(-B)^\beta$ are well-defined we have $\mathbf{D}((-B)^\beta) \subset \mathbf{D}((-A)^\alpha)$. These results can be found, respectively, in [13, Theorem 13.3.12] and [20, Theorem 2.25]. This finishes the proof of item 1.

Another consequence of the maximal regularity properties of u given in [20, Theorem 3.10] is that the map $u(t, \cdot) : [0, 1] \rightarrow \mathbf{W}_p^2$ is continuous if $g \in \mathbf{W}_p^2 = \mathbf{D}(A(0))$. We shall apply this result shortly to prove item 2.

For $t \in [0, 1]$ define a linear operator $T(t)$ on $\mathbf{L}_p$ such that for $h \in \mathbf{L}_p$ the function $v = v(t, x)$ given by $v(t, \cdot) = T(t)h$ is the unique classical solution in $\mathbf{L}_p$ of the homogeneous problem:

$$(37) \quad \frac{\partial v}{\partial t} = A(t)v, \quad v(0, \cdot) = h.$$

Actually, $T(t) = U(t, 0)$, where $U = (U(t, s))_{0 \leq s \leq t \leq 1}$ is the evolution system for $A = A(t)$; see Pazy [17], Chapter 5. However, we shall not use this relation. Of course, the properties established above for $u = u(t, x)$ will also hold for the solution $v = v(t, x)$ to (37). It follows that for any $h \in \mathbf{L}_p$ the map $t \mapsto T(t)h$ is well-defined and continuous in $\mathbf{L}_p$ and if $h \in \mathbf{W}_p^2$ then the same map is also continuous in $\mathbf{W}_p^2$. Recall now that $\mathbf{W}_p^1$ is an interpolation space between $\mathbf{L}_p$ and $\mathbf{W}_p^2$, more precisely, a midpoint in complex interpolation, see, for example, Bergh and Löfström [3], Theorem 6.4.5. Since $\mathbf{W}_p^2$ is dense in $\mathbf{W}_p^1$, Lemma 4.2 yields the continuity of the map $t \mapsto T(t)h$ in $\mathbf{W}_p^1$.

Observe now that $u = u(t, x)$ can be decomposed as

$$u(t, \cdot) = T(t)g + w(t, \cdot),$$

where $w(t, \cdot)$ is the unique classical solution in $\mathbf{L}_p$ of the inhomogeneous problem:

$$\frac{\partial w}{\partial t} = A(t)w + f, \quad w(0, \cdot) = 0.$$

Since w coincides with u in the special case $g = 0$, the map $t \mapsto w(t, \cdot)$ is continuous in $\mathbf{W}_p^2$ and, hence, also continuous in $\mathbf{W}_p^1$. This completes the proof of item 2.

Finally, let us prove item 3. To simplify notations suppose that the map $f = f(t, \cdot) : [0, 1] \rightarrow \mathbf{L}_p$ is actually analytic; otherwise, we repeat the same arguments on $[\epsilon, 1]$ for $0 < \epsilon < 1$. The condition (C1) implies the analyticity of the function

$A = A(t) : [0, 1] \rightarrow \mathcal{L}(\mathbf{W}_p^2, \mathbf{L}_p)$. Let U be an open convex set in $\mathbb{C}$ containing $[0, 1]$ on which there is an analytic extension of A satisfying the assertions of Lemma 4.3. We choose U so that $f = f(t, \cdot) : [0, 1] \rightarrow \mathbf{L}_p$ can also be analytically extended on U . Theorem 2 in Kato and Tanabe [10] now implies the analyticity of the map $t \mapsto u(t, \cdot)$ in $\mathbf{L}_p$. However, as

$$u(t, \cdot) = (A(t))^{-1} \left(\frac{\partial u}{\partial t} - f(t, \cdot) \right),$$

and since, by Lemma 4.3, the $\mathcal{L}(\mathbf{L}_p, \mathbf{W}_p^2)$ -valued function $(A(t))^{-1}$ on $[0, 1]$ is analytic, the map $t \mapsto u(t, \cdot)$ is also analytic in $\mathbf{W}_p^2$.

The proof is completed. $\square$

In the proof of our main Theorem 2.3 we actually need Theorem 4.4 below, which is a corollary of Theorem 4.1. Instead of (C2) we assume that the measurable functions $g = g(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ and $f = f(t, x) : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ have the following properties:

- (C3) There is a constant $N \geq 0$ such that $e^{-N|\cdot|} \frac{\partial g}{\partial x^i}(\cdot) \in \mathbf{L}_\infty$ and for any $p \geq 1$ we have $t \mapsto e^{-N|\cdot|} f(t, \cdot)$ is a Hölder continuous map from $[0, 1]$ to $\mathbf{L}_p$ whose restriction to $(0, 1]$ is analytic.

Fix a function $\phi = \phi(x)$ such that

$$(38) \quad \phi \in \mathbf{C}^\infty(\mathbb{R}^d) \text{ and } \phi(x) = |x| \text{ when } |x| \geq 1.$$

THEOREM 4.4. *Suppose the conditions (C1) and (C3) hold. Let $\phi = \phi(x)$ be as in (38). Then there exists a unique continuous function $u = u(t, x)$ on $[0, 1] \times \mathbb{R}^d$ and a constant $N \geq 0$ such that for any $p \geq 1$*

- (1) $t \mapsto e^{-N\phi} u(t, \cdot)$ is a Hölder continuous map of $[0, 1]$ to $\mathbf{L}_p$,
- (2) $t \mapsto e^{-N\phi} u(t, \cdot)$ is a continuous map of $[0, 1]$ to $\mathbf{W}_p^1$,
- (3) $t \mapsto e^{-N\phi} u(t, \cdot)$ is an analytic map of $(0, 1]$ to $\mathbf{W}_p^2$,

and such that $u = u(t, x)$ solves the Cauchy problem (26) and (27).

PROOF OF THEOREM 4.4. From (C3) we deduce the existence of $M > 0$ such that

$$\left| \frac{\partial g}{\partial x^i} \right|(x) \leq M e^{M|x|}, \quad x \in \mathbb{R}^d,$$

and, therefore, such that

$$|g(x) - g(0)| \leq M|x|e^{M|x|}, \quad x \in \mathbb{R}^d.$$

Hence, for any $N > M$ and any function $\phi = \phi(x)$ as in (38)

$$e^{-N\phi} g \in \mathbf{W}_p^1, \quad p \geq 1.$$

Hereafter, we choose the constant $N \geq 0$ so that in addition to (C3) it also has the property above.

Define the functions $\tilde{b}^i = \tilde{b}^i(t, x)$ and $\tilde{c} = \tilde{c}(t, x)$ so that for any $t \in [0, 1]$ and any $v \in \mathbf{C}^\infty$

$$\tilde{A}(t)(e^{-N\phi} v) = e^{-N\phi} A(t)v,$$

where

$$\tilde{A}(t) \triangleq \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d \tilde{b}^i(t, x) \frac{\partial}{\partial x^i} + \tilde{c}(t, x).$$

It is easy to see that $\tilde{b}^i$ and $\tilde{c}$ satisfy the same conditions as b^i and c in (C1). From Theorem 4.1 we deduce the existence of a measurable function $\tilde{u} = \tilde{u}(t, x)$ which for any $p > 1$ complies with the items 1–3 of this theorem and solves the Cauchy problem:

$$\frac{\partial \tilde{u}}{\partial t} = \tilde{A}(t)\tilde{u} + e^{-N\phi}f, \quad \tilde{u}(0, \cdot) = e^{-N\phi}g.$$

For $p > d$, by the classical Sobolev's embedding, the continuity of the map $t \mapsto \tilde{u}(t, \cdot)$ in $\mathbf{W}_p^1$ implies its continuity in $\mathbf{C}$. In particular, we obtain that the function $\tilde{u} = \tilde{u}(t, x)$ is continuous on $[0, 1] \times \mathbb{R}^d$.

To conclude the proof it only remains to observe that $u = u(t, x)$ complies with the assertions of the theorem for $p > 1$ if and only if $\tilde{u} \triangleq e^{-N\phi}u$ has the properties just established. The case $p = 1$ follows trivially from the case $p > 1$ by taking N slightly larger. $\square$

5. Proof of Theorem 2.3

Throughout this section we assume the conditions and the notations of Theorem 2.3. We fix a function ϕ satisfying (38). We also denote by $L(t)$ the infinitesimal generator of X at $t \in [0, 1]$:

$$L(t) = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(t, x) \frac{\partial}{\partial x^i},$$

where $a \triangleq \sigma\sigma^*$ is the covariation matrix of X . The proof is divided into several lemmas.

LEMMA 5.1. *There exist unique continuous functions $u = u(t, x)$ and $v^j = v^j(t, x)$, $j = 1, \dots, J$, on $[0, 1] \times \mathbb{R}^d$ and a constant $N \geq 0$ such that*

- (1) *For any $p \geq 1$ the maps $t \mapsto e^{-N\phi}u(t, \cdot)$ and $t \mapsto e^{-N\phi}v^j(t, \cdot)$ are*
 - (a) *Hölder continuous maps of $[0, 1]$ to $\mathbf{L}_p$;*
 - (b) *continuous maps of $[0, 1]$ to $\mathbf{W}_p^1$.*
 - (c) *analytic maps of $[0, 1]$ to $\mathbf{W}_p^2$.*
- (2) *The function $u = u(t, x)$ solves the Cauchy problem:*

$$(39) \quad \frac{\partial u}{\partial t} + (L(t) + \beta)u = 0, \quad t \in [0, 1],$$

$$(40) \quad u(1, \cdot) = G,$$

- (3) *The function $v^j = v^j(t, x)$ solves the Cauchy problem:*

$$(41) \quad \frac{\partial v^j}{\partial t} + (L(t) + \alpha^j + \beta)v^j + u f^j = 0, \quad t \in [0, 1],$$

$$(42) \quad v^j(1, \cdot) = F^j G.$$

PROOF. Observe first that (A2) on $\sigma = \sigma(t, x)$ implies (C1) on the covariation matrix $a = a(t, x)$. The assertions for $u = u(t, x)$ and, then, for $v^j = v^j(t, x)$, $j = 1, \dots, J$, follow now directly from Theorem 4.4, where we need to make the time change $t \rightarrow 1 - t$. $\square$

Hereafter, we denote by $u = u(t, x)$ and $v^j = v^j(t, x)$, $j = 1, \dots, J$, the functions defined in Lemma 5.1.

LEMMA 5.2. *The matrix-function $w = w(t, x)$, with d rows and J columns, given by*

$$(43) \quad w^{ij}(t, x) \triangleq \left(u \frac{\partial v^j}{\partial x^i} - v^j \frac{\partial u}{\partial x^i} \right) (t, x), \quad i = 1, \dots, d, \quad j = 1, \dots, J,$$

has rank d almost surely with respect to the Lebesgue measure on $[0, 1] \times \mathbb{R}^d$.

PROOF. Denote

$$g(t, x) \triangleq \det(w w^*)(t, x), \quad (t, x) \in [0, 1] \times \mathbb{R}^d,$$

the determinant of the product of w on its transpose, and observe that the result holds if and only if the set

$$A \triangleq \{(t, x) \in [0, 1] \times \mathbb{R}^d : g(t, x) = 0\}$$

has the Lebesgue measure zero on $[0, 1] \times \mathbb{R}^d$ or, equivalently, the set

$$B \triangleq \{x \in \mathbb{R}^d : \int_0^1 1_A(t, x) dt > 0\}$$

has the Lebesgue measure zero on $\mathbb{R}^d$.

From Lemma 5.1 we deduce that the existence of a constant $N \geq 0$ such that for any $p \geq 1$ the map $t \mapsto e^{-N\phi} g(t, \cdot)$ from $[0, 1]$ to $\mathbf{W}_p^1$ is analytic and the same map of $[0, 1]$ to $\mathbf{L}_p$ is continuous. Taking $p \geq d$, we deduce from the classical Sobolev embedding of $\mathbf{W}_p^1$ into $\mathbf{C}$ that this map is also analytic from $[0, 1]$ to $\mathbf{C}$. It follows that if $x \in B$ then $g(t, x) = 0$ for all $t \in [0, 1]$ and, in particular,

$$\lim_{t \uparrow 1} g(t, x) = 0, \quad x \in B.$$

Since

$$\|g(t, \cdot) - g(1, \cdot)\|_{\mathbf{L}^p} = \|g(t, \cdot) - \det(w w^*)(1, \cdot)\|_{\mathbf{L}^p} \rightarrow 0, \quad t \uparrow 1,$$

the Lebesgue measure of B is zero if the matrix-function $w(1, \cdot)$ has rank d almost surely. This follows from the expression for $w(1, \cdot)$:

$$w^{ij}(1, \cdot) = G \frac{\partial(F^j G)}{\partial x^i} - F^j G \frac{\partial G}{\partial x^i} = G^2 \frac{\partial F^j}{\partial x^i},$$

and the assumptions (A3) and (A4) on $F = (F^j)$ and G . $\square$

Recall the notations ψ^j , $j = 1, \dots, J$, and ξ for the random variables defined in (6) and (7).

LEMMA 5.3. *The processes Y and R^j , $j = 1, \dots, J$, on $[0, 1]$ defined by*

$$\begin{aligned} Y_t &\triangleq e^{\int_0^t \beta(s, X_s) ds} u(t, X_t), \\ R_t^j &\triangleq e^{\int_0^t (\alpha^j + \beta)(s, X_s) ds} v^j(t, X_t) + Y_t \int_0^t e^{\int_0^s \alpha^j(r, X_r) dr} f^j(s, X_s) ds, \end{aligned}$$

are continuous uniformly integrable martingales with the terminal values $Y_1 = \xi$ and $R_1^j = \xi\psi^j$. Moreover, for $t \in [0, 1]$,

$$(44) \quad Y_t = Y_0 + \sum_{i,k=1}^d \int_0^t e^{\int_0^s \beta(r, X_r) dr} \left(\frac{\partial u}{\partial x^i} \sigma^{ik} \right) (s, X_s) dW_s^k,$$

$$(45) \quad \begin{aligned} R_t^j &= R_0^j + \sum_{i,k=1}^d \int_0^t e^{\int_0^s (\alpha^j + \beta)(r, X_r) dr} \left(\frac{\partial v^j}{\partial x^i} \sigma^{ik} \right) (s, X_s) dW_s^k \\ &\quad + \int_0^t \left(\int_0^s e^{\int_0^r \alpha^j(q, X_q) dq} f^j(r, X_r) dr \right) dY_s. \end{aligned}$$

PROOF. From the continuity of u and v^j on $[0, 1] \times \mathbb{R}^d$ we obtain that Y and R^j are continuous processes on $[0, 1]$. The expressions (40) and (42) for $u(1, \cdot)$ and $v^j(1, \cdot)$ imply that $Y_1 = \xi$ and $R_1^j = \xi\psi^j$.

Let $N \geq 0$ be the constant in Lemma 5.1. Choosing $p = d + 1$ in Lemma 5.1 we deduce that the maps $t \mapsto e^{-N\phi}u(t, \cdot)$ and $t \mapsto e^{-N\phi}v^j(t, \cdot)$ of $[0, 1]$ to $\mathbf{W}_{d+1}^2$ are continuously differentiable. This enables us to use a variant of the Ito formula due to Krylov, see [15, Section 2.10, Theorem 1]. Direct computations, where we account for (39) and (41), then yield the integral representations (44) and (45).

In particular, we have shown that Y and R^j are continuous local martingales. It only remains to verify their uniform integrability. By Sobolev's embeddings, since $t \mapsto e^{-N\phi}u(t, \cdot)$ and $t \mapsto e^{-N\phi}v^j(t, \cdot)$ are continuous maps of $[0, 1]$ to $\mathbf{W}_{d+1}^1$, they are also continuous maps of $[0, 1]$ to $\mathbf{C}$. This implies the existence of $c > 0$ such that

$$\sup_{t \in [0, 1]} (|Y_t| + |R_t^j|) \leq e^{c(1 + \sup_{t \in [0, 1]} |X_t|)}.$$

The result now follows from the well-known fact that, for bounded b^i and σ^{ij} , the random variable $\sup_{t \in [0, 1]} |X_t|$ has all exponential moments. $\square$

PROOF OF THEOREM 2.3. Let Y and R be the processes defined in Lemma 5.3. This lemma implies, in particular, that

$$\mathbb{E}[|\xi| + \sum_{j=1}^J |\xi\psi^j|] < \infty,$$

and, hence, the probability measure $\mathbb{Q}$ and the $\mathbb{Q}$ -martingale $S = (S^j)$ are well-defined. Since $\xi > 0$, the measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and Y is a strictly positive martingale. Observe that

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi | \mathcal{F}_t] = \frac{\mathbb{E}[\xi\psi | \mathcal{F}_t]}{\mathbb{E}[\xi | \mathcal{F}_t]} = \frac{R_t}{Y_t}, \quad t \in [0, 1].$$

From (44) and (45) we deduce, after some computations, that

$$(46) \quad dS_t^j = d \frac{R_t^j}{Y_t} = e^{\int_0^t \alpha^j(s, X_s) ds} \frac{1}{u^2(t, X_t)} \sum_{i,k=1}^d (w^{ij} \sigma^{ik})(t, X_t) dW_t^{\mathbb{Q}, k},$$

where the matrix-function $w = w(t, x)$ is defined in (43) and

$$W_t^{\mathbb{Q}, k} \triangleq W_t^k - \sum_{l=1}^d \int_0^t \left(\frac{1}{u} \frac{\partial u}{\partial x^l} \sigma^{lk} \right) (t, X_t) dt, \quad k = 1, \dots, d, \quad t \in [0, 1].$$

By Girsanov's theorem, $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. Note that the division on $u(t, X_t)$ is safe as the process $u(t, X_t) = Y_t e^{-\int_0^t \beta(s, X_s) ds}$, $t \in [0, 1]$, is strictly positive.

As we have already observed in Remark 2.2, any $\mathbb{P}$ -local martingale is a stochastic integral with respect to W . This readily implies that any $\mathbb{Q}$ -local martingale M is a stochastic integral with respect to $W^{\mathbb{Q}}$. Indeed, since $L \triangleq YM$ is a local martingale under $\mathbb{P}$, there is a predictable process ζ with values in $\mathbb{R}^d$ such that

$$L_t = L_0 + \int_0^t \zeta_u dW_u \triangleq L_0 + \sum_{i=1}^d \int_0^t \zeta_u^i dW_u^i$$

and then

$$dM_t = d\frac{L_t}{Y_t} = \frac{1}{Y_t} \sum_{i=1}^d \left(\zeta_t^i - L_t \sum_{k=1}^d \left(\frac{1}{u} \frac{\partial u}{\partial x^k} \sigma^{ki} \right) (t, X_t) \right) dW_t^{\mathbb{Q}, i}.$$

In view of (46), to conclude the proof we only have to show that the matrix-process $((w^* \sigma)(t, X_t))_{t \in [0, 1]}$ has rank d on $\Omega \times [0, 1]$ almost surely under the product measure $dt \times d\mathbb{P}$. Observe first that by (2) and Lemma 5.2 the matrix-function $w^* \sigma = (w^* \sigma)(t, x)$ has rank d almost surely under the Lebesgue measure on $[0, 1] \times \mathbb{R}^d$. The result now follows from the well-known fact that under (A1) and (A2) the distribution of X_t has a density under the Lebesgue measure on $\mathbb{R}^d$, see [19, Theorem 9.1.9]. $\square$

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