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Properties of Minimizers of Nonlocal Interaction Energy

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Preface

This thesis is about two related objects of study in mathematical physics. One is the nonlocal aggregation equation, a nonlocal partial differential equation whose weak-solution gives an evolution of the positions of particles that are moving due to long-range interactions with the other particles. The other is interaction energies, which are energies on particle configurations that depend on the pairwise distances between particles. The nonlocal aggregation equation has come under investigation in recent years for its use modeling pattern formation with long-range interactions and as an example application of the theory of gradient flows of interaction energies in the space of probability measures.

The scientific motivations for this research comes from several phenomena due to long range interactions between particles or agents. In biology the nonlocal aggregation equation and the interaction energy are used to model or analyze models of animal swarm phenomena [45] such as bird flocks and fish schools [27, 35] as well as locust swarms [11, 52]. Beyond flocking, the interaction energy is used to model inelastic collisions of granular media in models first suggested from experimental results by the authors of [44], and is studied in later papers such as [8, 1, 2] and the nonlocal aggregation equation is also used to model robotic agent interactions such as the flocking of Dubins vehicles (that is, vehicles who's paths are constrained in their curvature).

In [39] the authors showed that changes in the parameters of interaction laws give a wide variety of different steady-states of the nonlocal aggregation equation in two dimensions. They found these by perturbing rings made of many particles and running simulations to see where the particles went. In addition, they gave necessary conditions for when discrete-particle and continuous ring steady-states are linearly stable. Though they are equivalent to conditions for displacement stability of N-particle rings discussed later in this thesis, they are simplified here by a change of coordinates.

In [29] the authors classify when finite particle steady-states of the nonlocal aggregation equation are locally stable in the space of probability measures in one dimension. They found two conditions that classify when these steady states are stable, and these conditions are in fact recovered from the conditions found in this thesis looking at finite particle steady-states in higher dimensions.

The conditions characterizing stability of radial steady-states under radial perturbations for any dimension were done in [6]. The same authors went on in [5] to classify the dimensionality of the support of local minimizers of the interaction energy. This is important since this thesis shows for a certain class of interaction laws that local minimizers of the energy are the stable steady-states of the nonlocal aggregation equation. They demonstrated that the size of the dimension of these local minimizers depends on the regularity of the interaction law at the origin. In particular the more singular the interaction kernel is at the origin (in the sense of non-differentiability or even blow-up) the higher the dimension of the support of the local

minimizers. For the interactions covered in this thesis their paper shows that local minimizers must have support that the dimension less than 1. It should be noted that all stable steady-states that have been analyzed or observed in this regime of interaction laws are finite particle steady-states, the steady-states for which this thesis characterizes stability.

The asymptotic stability of finite particles steady-states in dimensions greater than 1 was analyzed in the 2013 paper [38]. There they showed the necessity of the conditions given in this thesis for the stability of finite particle study states. They also studied the stability of 'spot' study-states, where mass concentrates on spots in the shape of ellipses, for interaction laws that have one less derivative at the origin than those interaction laws in this thesis.

In [52] the authors studied the nonlocal aggregation equation in one dimension but with both external forces as well as boundary. In this scenario they recovered the phenomenon of milling swarms that they saw in locust species that they were studying.

Results in this field sometimes look at the nonlocal aggregation equation as a system of ODE that give the movement of the positions of particles or agents, and sometimes they look at this system as a moving density of particles. This thesis looks at a general case that describes both, which is important because even initial data that is smooth at the initial time can collapse to measures with lower dimensional support in finite time. Thus this paper will study the nonlocal aggregation equation in the weak-solution sense where solutions are paths in the space of probability measures. In [46] the author motivates the expression of PDE as formal gradient flows of energy functionals according to a formal manifold structure on the space of probability measure in a discussion on the Porous Medium Equation. These formal notions were later made rigorous in [3], and for the interaction energy these results were later generalized to a wider class of interaction kernels in [20]. These results show that, given some regularity the interaction laws, then the space of probability measures can be endowed with a formal Riemannian manifold structure such that the nonlocal aggregation equation is the gradient flow of the interaction energy according to the associated Riemannian metric. This thesis also takes advantage of the opportunity provided by these results to get explicit rates of convergence for the evolutions.

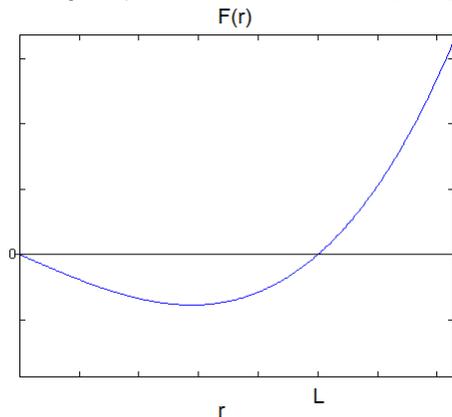
Introduction to the Nonlocal Aggregation Equation and the Interaction Energy

1.1. A Motivating Example of Nonlocal Aggregation

Suppose one has a system of two particles, whose identical masses sum to 1, that are obeying the following interaction force: if they are far away then they will move closer together, and if they are too close then they will move apart. More precisely, if their locations are $x_1(t)$ and $x_2(t)$ then

$$\dot{x}_i = \frac{1}{2} \sum_{j=1}^2 F(|x_j - x_i|) \frac{x_j - x_i}{|x_j - x_i|}$$

where $F : [0, \infty) \rightarrow \mathbb{R}$ has the following shape



This is a special case of dynamics described by nonlocal aggregation that exhibits long-range attraction and short-range repulsion. One can likewise write the dynamics for a system of N particles, each with position $x_i(t)$ at time t and mass m_i , where $i \in \{1, \dots, N\}$, and the sum of all masses being 1, as

$$\dot{x}_i = m_i \sum_{j=1}^N F(|x_j - x_i|) \frac{x_j - x_i}{|x_j - x_i|}.$$

One of the goals of this thesis is to study this system for arbitrary numbers of particles, and even “clouds” of particles where the system is described via the particle density function. The common

general structure under which these situations can be analyzed is to consider these particle distributions or densities of particles as probability measures. For example, the configuration of N particles above at time t can be written as the measure as the probability measure μ_t

$$\mu_t = \sum_{i=1}^N m_i \delta_{x_i(t)}$$

and a density of particles at time t , ρ_t , with total mass (i.e. L^1 norm) 1, can be written as a probability measure ν_t

$$\nu_t = \rho_t dx.$$

Thus it is natural to first understand the structure on probability measures motivated by the desire to study nonlocal aggregation.

1.2. Introduction to the Metric Topology on Probability Measures

As just mentioned, probability measures are useful to represent the configurations of particles of interest. In particular, the space that will be used here is \mathcal{P}_2 , where \mathcal{P}_2 is defined as

$$(1.2.1) \quad \mathcal{P}_2 := \left\{ \text{the space of second - moment bounded Borel probability measures on } \mathbb{R}^d \right\}.$$

This space is general enough that it allows, say, measures that are absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d , to evolve over time into measures that are singular with respect to the Lebesgue measure such as finite sums of delta masses.

\mathcal{P}_2 is a metric space with respect to the 2–Wasserstein metric. This metric measures the minimum cost, according to a distance-based cost, to transport mass between two measures. To define the metric explicitly it helps to first define what is meant by a “transportation plan” between two measures μ and $\nu \in \mathcal{P}_2$. A transportation plan between two such measures is itself a product measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that its first and second marginals are μ and ν , respectively, meaning the set of transportation plans between the two, $\Pi(\mu, \nu)$, is

$$(1.2.2) \quad \Pi(\mu, \nu) := \left\{ \pi \text{ is a probability measure on } \mathbb{R}^d \times \mathbb{R}^d \mid \begin{array}{l} \text{for borel sets } A, \cdot \\ \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A) \end{array} \right\}$$

With transportation plans defined, the 2–Wasserstein distance between measures is

$$(1.2.3) \quad d_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \|x - y\|_{L^2(d\pi(x, y))}.$$

If $\pi \in \Pi(\mu, \nu)$ is a transport plan that attains the infimum in (1.2.3), then it is called an “optimal transport plan” (with respect to the 2–Wasserstein metric) and belongs to the set of optimal transport plans denoted $\Gamma_{opt,2}(\mu, \nu)$, i.e.

$$(1.2.4) \quad \Gamma_{opt,2}(\mu, \nu) = \left\{ \pi \in \Pi \mid d_2(\mu, \nu) = \|x - y\|_{L^2(d\pi(x, y))} \right\}.$$

The nonlocal aggregation equation is a nonlocal continuity equation whose solutions can exhibit concentration of mass in finite time even for smooth initial data, so global-in-time weak solutions must be paths $\mu_t : [0, \infty) \rightarrow \mathcal{P}_2$. In order to define which curves should be expected to have derivatives in some weak sense one needs to define which curves are absolutely continuous. Given the 2-Wasserstein topology on \mathcal{P}_2 , one can define when paths $\mu_t : [0, \infty) \rightarrow \mathcal{P}_2$ are “absolutely continuous”, which is when there exists a function $m \in L^1_{loc}([0, \infty))$ such that

$$(1.2.5) \quad d_2(\mu_s, \mu_t) \leq \int_s^t m(r) dr.$$

It will be discussed later when weak solutions of the nonlocal aggregation equation are absolutely continuous paths in \mathcal{P}_2 .

1.3. Geometry on \mathcal{P}_2

A formal Riemannian structure can be placed on \mathcal{P}_2 . The tangent plane at $\mu \in \mathcal{P}_2$, denoted \mathcal{T}_μ , is the closure in the $L^2(d\mu)$ norm of gradient vector fields of smooth and compactly-supported functions on \mathbb{R}^d , i.e.

$$(1.3.1) \quad \mathcal{T}_\mu = \overline{\{\nabla\phi \mid \phi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(d\mu)}.$$

Then the metric defined at $\mu \in \mathcal{P}_2$ for $v, w \in \mathcal{T}_\mu$ is

$$(1.3.2) \quad g_\mu(v, w) = \int v(x) \cdot w(x) d\mu(x).$$

The geodesic between $\mu, \nu \in \mathcal{P}_2$ can be characterized in the following way: Take $\pi \in \Gamma_{opt,2}(\mu, \nu)$ as defined in (1.2.4). Then define the displacement interpolant π_s for $0 \leq s \leq 1$ as

$$(1.3.3) \quad \pi_s = ((1-s)x + sy)_\# \pi.$$

π_s is the constant speed geodesic between μ and ν .

Furthermore, the exponential map defined at each $\mu \in \mathcal{P}_2$, denoted by $\exp_\mu(v; t) : \mathcal{T}_\mu \rightarrow \mathcal{P}_2$, is

$$(1.3.4) \quad \exp_\mu(v; t) = (x + tv(x))_\# \mu.$$

An energy functional $E : \mathcal{P}_2 \rightarrow \mathbb{R}$ is said to be geodesically convex (or semi-convex) if there exists a $\lambda > 0$ (respectively $\lambda \leq 0$) such that for any $\mu, \nu \in \mathcal{P}_2$ with $\pi \in \Gamma_{opt,2}(\mu, \nu)$ defining the geodesic path π_s as in (1.3.3) it holds that

$$(1.3.5) \quad E[\pi_s] \leq (1-s)E[\mu] + sE[\nu] - \frac{1}{2}\lambda s(1-s)d_2^2(\mu, \nu).$$

1.4. Definitions of Some Geometric Objects on Finite Dimensional Manifolds

The rest of this chapter will build to explain how a path μ_t that solves the nonlocal aggregation equation is also a path that evolves along the gradient flow of the associated interaction energy. In order to state that rigorously a geometry needs to be placed on \mathcal{P}_2 . To motivate the definitions of the gradient and Hessian for the interaction energy on \mathcal{P}_2 , consider the example of how the operators “grad” and “Hess”, the operators that send a function to it’s gradient and Hessian at a point, respectively, act on smooth functions on a finite dimensional Riemannian manifold \mathbb{M} .

To do this, fix a differentiable function $f : \mathbb{M} \rightarrow \mathbb{R}$ and $x \in \mathbb{M}$, and consider the smooth paths Φ_t in \mathbb{M} parametrized by t such that $\Phi_t = \exp_x(tv)$ for some vector $v \in \mathcal{T}_x\mathbb{M}$ and the exponential map \exp , meaning that where $\dot{\Phi}_0 = v$ and the covariant derivative of $\dot{\Phi}_t$ satisfies $\left. \frac{D\dot{\Phi}_t}{dt} \right|_{t=0} = 0$. Then $f \circ \Phi_t$ is a smooth function in t , and its derivative with respect to t at $t = 0$ is a linear functional on v . Thus, by the Riesz-Representation theorem, there exists a unique vector in \mathbb{R}^d called “the gradient of f at x ”, denoted by $\text{grad}f(x)$, such that for the Riemannian metric on \mathbb{M} , here called $g_{\mathbb{M}}$, is

$$(1.4.1) \quad \left. \frac{d(f \circ \Phi_t)}{dt} \right|_{t=0} = g_{\mathbb{M}}(\text{grad}f(x), v)$$

with $\text{grad}f$, an element of the tangent plane, defined one can then define the bilinear form “the Hessian of f at x ” by noting that

$$\begin{aligned} \left. \frac{d^2(f \circ \Phi_t)}{dt^2} \right|_{t=0} &= \left. \frac{d}{dt} \left(\left. \frac{d(f \circ \Phi_t)}{dt} \right) \right|_{t=0} \\ &= \left. \frac{d}{dt} \left(g_{\mathbb{M}} \left(\text{grad}f(x), \dot{\Phi}_t(x) \right) \right) \right|_{t=0} \\ &= g_{\mathbb{M}} \left(\left. \frac{D\text{grad}f(\Phi_t(x))}{dt} \right|_{t=0}, v \right). \end{aligned}$$

where $\left. \frac{D(\text{grad}f(\Phi_t(x)))}{dt} \right|_{t=0}$ is the covariant derivative of the gradient vector field of f and is itself a vector valued linear function, so by again applying the Riesz Representation theorem there is a unique operator $\text{Hess}f(x)$ (where operator in this sense just means a matrix) such that

$$\left. \frac{D(\text{grad}f(\Phi_t(x)))}{dt} \right|_{t=0} = \text{Hess}f(x) v$$

which implies that $\text{Hess}f(x)$ can be defined by using the bilinear form

$$(1.4.2) \quad \left. \frac{d^2(f \circ \Phi_t(x))}{dt^2} \right|_{t=0} = g_{\mathbb{M}}(\text{Hess}f(x)v, v)$$

Where the right hand side is also denoted by $\text{Hess}f(x)[v, v]$.

1.5. Introduction to the Interaction Energy

Each configuration $\mu \in \mathcal{P}_2$ has an associated interaction energy \mathcal{W} defined as

$$(1.5.1) \quad \mathcal{W}[\mu] := \frac{1}{2} \int \int W(x-y) d\mu(x) d\mu(y).$$

Note that unless ω grows at infinity at most quadratically, this integral may not be defined, so attention here will be restricted to those ω . The proceeding will follow the finite dimensional heuristics to define the gradient and the Hessian for the Interaction Energy \mathcal{W} . These computations follow the formal definitions stated in [55, Chapter 8.2]

Let $\Phi_t(x) : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be such that $\Phi_0(x) = x$ and $\ddot{\Phi}_t = 0$, so that $\Phi_t = x + tv(x)$ for some tangent vector $v \in \mathcal{T}_\mu$. Let γ_t be the path in \mathcal{P}_2 starting at μ and being pushed forward by Φ_t , i.e. $\gamma_t = \Phi_{t\#}\mu$. Then $\mathcal{W}(\gamma_t)$ is

$$\mathcal{W}[\gamma_t] = \frac{1}{2} \int \int W(\Phi_t(x) - \Phi_t(y)) d\mu(x) d\mu(y).$$

Now following the finite dimensional example above, to find the gradient of \mathcal{W} at μ first compute the first derivative of this functional along the path at $t = 0$:

$$\begin{aligned} \left. \frac{d\mathcal{W}}{dt} [\gamma_t] \right|_{t=0} &= \left. \frac{1}{2} \int \int \nabla W(\Phi_t(x) - \Phi_t(y)) \cdot ((v(x) - v(y))) d\mu(x) d\mu(y) \right|_{t=0} \\ &= \int (v(x))^T \left\{ \int \nabla W(x-y) d\mu(y) \right\} d\mu(x). \end{aligned}$$

has the explicit expression

$$(1.5.2) \quad \left. \frac{d\mathcal{W}}{dt} [\gamma_t] \right|_{t=0} = \int \left\{ \int \nabla W(x-y) d\mu(y) \right\} \cdot v(x) d\mu(x)$$

which by analogy with the chain rule in (1.4.1) gives that

$$(1.5.3) \quad \begin{aligned} \text{grad}\mathcal{W}|_\mu(x) &= \int \nabla W(x-y) d\mu(y) \\ &= \nabla \int W(x-y) d\mu(y). \end{aligned}$$

To define the Hessian one can copy the finite dimensional example again by computing the second derivative $\left. \frac{d^2\mathcal{W}}{dt^2} [\gamma_t] \right|_{t=0}$ at $t = 0$ which is

$$\begin{aligned} \left. \frac{d^2\mathcal{W}}{dt^2} [\gamma_t] \right|_{t=0} &= \int \int (v(x) - v(y))^T \text{Hess}W(\Phi_t(x) - \Phi_t(y)) \\ &\quad (v(x) - v(y)) d\mu(x) d\mu(y)_{t=0} \\ &\quad + \int \left\{ \int \nabla W(\Phi_t(x) - \Phi_t(y)) d\mu(y) \right\} \ddot{\Phi}_t(x) d\mu(x) \Big|_{t=0} \end{aligned}$$

where $\dot{\Phi}_t$ is constant along the path, so $\ddot{\Phi}_t = 0$, so plugging in $t = 0$ gives

$$\left. \frac{d^2 \mathcal{W}}{dt^2} [\gamma_t] \right|_{t=0} = \int \int (v(x) - v(y))^T \text{Hess} W(x - y) (v(x) - v(y)) d\mu(x) d\mu(y)$$

This is an explicit computation that shows how to define the Hessian of the energy at μ by analogy with (1.4.2). In particular, the Hessian is notated

$$(1.5.4) \quad \text{Hess} \mathcal{W}_\mu[v, v] = \int \int (v(x) - v(y))^T \text{Hess} W(x - y) (v(x) - v(y)) d\mu(x) d\mu(y).$$

1.6. The Nonlocal Aggregation Equation

1.6.1. Introduction to the Nonlocal Aggregation Equation. Now a rigorous formulation of the nonlocal aggregation equation can be stated. One says that the absolutely continuous path μ_t in \mathcal{P}_2 is the solution of the nonlocal aggregation equation

$$(1.6.1) \quad \begin{cases} \partial_t \mu_t + \nabla \cdot (\mu_t v_{\mu_t}) &= 0 \\ v_{\mu_t}(x) &= -\nabla W * \mu_t(x) \\ \mu_t &= \underline{\mu}, \text{ at } t = 0. \end{cases}$$

for the interaction kernel ω with initial data $\underline{\mu}$ when μ_t satisfies for all test functions $\varphi_t \in C_c^\infty([0, \infty) \times \mathbb{R}^d)$ the equation

$$(1.6.2) \quad \int_0^\infty \int_{\mathbb{R}^d} \frac{\partial \varphi_t}{\partial t}(x) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi_0(x) d\underline{\mu}(x) = \int_0^\infty \int_{\mathbb{R}^d} \nabla \varphi_t(x) \cdot v_{\mu_t}(x) d\mu_t(x).$$

In [20] it was shown that this result holds for ω such that

- (1) $W \in C^1(\mathbb{R}^d \setminus \{0\})$ with Lipschitz singularity at the origin.
- (2) $W(x) = W(-x)$
- (3) $W(x) \leq C|1 + |x|^2|$ for some $C > 0$.
- (4) $W(x) + \frac{1}{2}\lambda|x|^2$ is convex for some $\lambda \in \mathbb{R}$ (i.e. ω is semi-convex).

When studying the nonlocal aggregation equation in this thesis, however, all the interaction kernels ω of interest are in $C^{2,1}(\mathbb{R}^d)$, the space of twice differentiable functions with itself and each derivative being Lipschitz, and are radial, so they satisfy the requirements 1 and 2 above. Since the steady-states and neighborhoods of them that will be considered later are all compactly supported, any radial $W \in C^{2,1}$ can be replaced by a $\tilde{W} \in C^{2,1}$ that induces the same dynamics on the steady-states and their neighbors, and satisfy requirements 3 and 4 as well, so that for the purpose of this thesis it is sufficient to consider any $W \in C^{2,1}$ that is radial.

Note that [20, Theorem 2.13] shows that these weak measure solutions have an exponential contraction property, namely give ω with λ as in requirement 4 above, it holds that for two

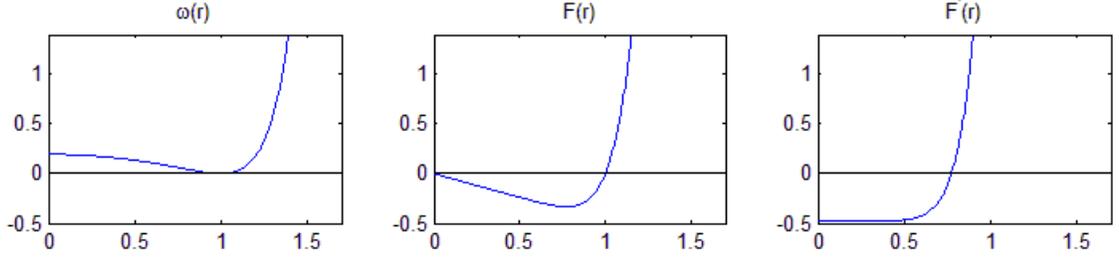


FIGURE 1.6.1. The figure shows an example ω with its derivatives F and F' , which model long-range attraction and short-range repulsion for the interactions between particles.

initial data μ_0 and ν_0 in \mathcal{P}_2 with the same center of mass, and μ_t and ν_t are their respective weak-measure global-in-time solutions of the nonlocal aggregation equation, then

$$(1.6.3) \quad d_2(\mu_t, \nu_t) \leq e^{-\lambda t} d_2(\mu_0, \nu_0).$$

Since this thesis will focus on $W \in C^{2,1}$ except for in chapter 2, the associated gradient flow map $\Phi_t(x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies the following proposition:

PROPOSITION 1. *There exists a $C^{1,1}$ in space and C^2 in space mapping $\Phi_t(x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ that is the flow map of the gradient flow the interaction energy, namely Φ_t is the flow map of $\dot{x} = -v_{\mu_t}(x(t))$.*

PROOF. First note that $v_{\mu_t}(x)$ is Lipschitz in space. One sees this since ω is $C^{2,1}$ in space, so be denoting with L_W the Lipschitz constant of ∇W one sees that

$$\begin{aligned} |v_{\mu_t}(x_1) - v_{\mu_t}(x_2)| &= \left| \int (-\nabla W(x_1 - y) + \nabla W(x_2 - y)) d\mu_t(y) \right| \\ &\leq \int |\nabla W(x_1 - y) - \nabla W(x_2 - y)| d\mu_t(y) \\ &\leq \int L_W |x_1 - x_2| d\mu_t(y) \\ &= L_W |x_1 - x_2|. \end{aligned}$$

Furthermore $v_{\mu_t}(x)$ is continuous in time, since

$$|v_\mu(x) - v_\nu(x)| = \left| \int \nabla W(x - y) d\mu(y) - \int \nabla W(x - \tilde{y}) d\nu(\tilde{y}) \right|.$$

Let $\pi \in \Gamma(\mu, \nu)$, then

$$\begin{aligned} |v_\mu(x) - v_\nu(x)| &= \left| \int (\nabla W(x - y) - \nabla W(x - \tilde{y})) d\pi(y, \tilde{y}) \right| \\ &\leq \int |\nabla W(x - y) - \nabla W(x - \tilde{y})| d\pi(y, \tilde{y}) \end{aligned}$$

and by using the fact that ∇W is Lipschitz continuous with Lipschitz constant L_W ,

$$|v_\mu(x) - v_\nu(x)| \leq L_W \int |y - \tilde{y}| d\pi(y, \tilde{y})$$

so by applying Cauchy-Schwartz inequality

$$|v_\mu(x) - v_\nu(x)| \leq L_W \sqrt{\int |y - \tilde{y}|^2 d\pi(y, \tilde{y})} = L_W d_2(\mu, \nu).$$

By applying this inequality where $\mu = \mu_t$ and $\nu = \mu_{t+h}$ one sees that

$$|v_{\mu_t}(x) - v_{\mu_{t+h}}(x)| \leq L_W d_2(\mu_t, \mu_{t+h}).$$

and since μ_t is an absolutely continuous function, recall from the definition of absolutely continuous (1.2.5) that there is then an L^1 function m such that

$$|v_{\mu_t}(x) - v_{\mu_{t+h}}(x)| \leq \int_t^{t+h} m(r) dr.$$

This shows that $v_{\mu_t}(x)$ is continuous in time, since as h goes to zero, so will the right side of the inequality above, and thus by the squeeze theorem so will the left side.

Thus since $v_{\mu_t}(x)$ is continuous in time and Lipschitz in space, by standard ODE theory there exists a C^1 in space and time flow map Φ_t associated with the ode $\dot{x} = v_{\mu_t}(x)$. Note that $\Phi_{t\#}\mu_0 = \mu_t$. Finally, using the change of variables $y = \Phi_t(z)$

$$v_{\mu_t}(x) = - \int \nabla W(x - y) d\mu_t(y) = - \int \nabla W(x - \Phi_t(z)) d\mu_0(z).$$

Thus v_{μ_t} is C^1 in time and thus Φ_t is C^2 in time. \square

Lastly, being radial, each ω has an associated function $F : [0, \infty) \rightarrow \mathbb{R}$ defined such that

$$\begin{aligned} W(z) &= \omega(|z|) \\ F(r) &= \omega'(r). \end{aligned}$$

This makes the velocity field defined at each point in the support of μ_t be

$$(1.6.4) \quad v_{\mu_t}(x) = - \int F(|x - y|) \frac{x - y}{|x - y|} d\mu_t(y).$$

1.6.2. The Nonlocal Aggregation Equation as a Gradient Flow of the Interaction Energy. Looking at the definition of the nonlocal aggregation equation, (1.6.1), and the definition of the gradient of the interaction energy, (1.5.3), it turns out that

$$(1.6.5) \quad v_{\mu_t} = -\text{grad}_{\mu_t} \mathcal{W}$$

justifying calling the solution to the nonlocal aggregation equation “the gradient flow of the interaction energy”.

Note that by (1.5.2) and (1.6.5)

$$\begin{aligned} \frac{d\mathcal{W}[\mu_t]}{dt} &= \int \text{grad}_{\mu_t} \mathcal{W} \cdot v_{\mu_t}(x) d\mu_t(x) \\ &= - \int |v_{\mu_t}(x)|^2 d\mu_t(x) \end{aligned}$$

implies that

$$\frac{d\mathcal{W}[\mu_t]}{dt} = 0 \iff v_t(x) = 0$$

for μ_t -a.e. x , so critical points of the evolution in time of the energy correspond to steady-states of the nonlocal aggregation equation.

1.7. Notation and Conventions

Throughout this thesis the following notation and conventions will be used.

- (1) \mathcal{W} is the interaction energy functional defined on $\mathcal{P}_2(\mathbb{R}^d)$, the space of Borel probability measures on \mathbb{R}^d with bounded second moment.
- (2) All measures in this thesis have their center of mass at zero. (Note that two paths in the space of probability measures used in this thesis are either rotations of a measure $\bar{\mu}$, or gradient flow evolutions with initial data μ_0 . Both of these paths preserve the center of mass.)
- (3) ω is the interaction kernel associated with the interaction energy \mathcal{W} . $W \in C^{2,1}(\mathbb{R}^d)$ the space of twice differential functions which are Lipschitz and have Lipschitz derivatives. Except for in Chapter 6 on global minimizers of the energy, where more general interaction kernels are considered.
- (4) L_W denotes the Lipschitz constant of ∇W , meaning for all $x, y \in \mathbb{R}^d$ that

$$|\nabla W(x) - \nabla W(y)| \leq L_W |x - y|.$$

- (5) c_W denotes the Lipschitz constant of $\text{Hess}W$, meaning that for all $x, y, z \in \mathbb{R}^d$ that

$$|(\text{Hess}W(x) - \text{Hess}W(y))z| \leq c_W |x - y| |z|.$$

- (6) d_2 refers to the 2-Wasserstein distance on the space of probability measures. If $\Pi(\mu, \nu)$ represents the transportation plans between μ and ν in \mathcal{P}_2 , then

$$d_2(\mu, \nu) := \min_{\pi \in \Pi(\mu, \nu)} \|x - \tilde{x}\|_{L^2(d\pi(x, \tilde{x}))}.$$

The existence of such a minimizer is a standard result in the theory of optimal transportation, for example as in [55, Theorem 1.3].

- (7) Likewise d_∞ refers to the ∞ -Wasserstein distance on the space of probability measures. If $\Pi(\mu, \nu)$ represents the transportation plans between μ and ν in \mathcal{P}_2 , then

$$d_\infty(\mu, \nu) := \min_{\pi \in \Pi(\mu, \nu)} \|x - \tilde{x}\|_{L^\infty(d\pi(x, \tilde{x}))}.$$

The existence of minimizers for μ, ν supported on a compact domain (as will be used in this thesis) was shown in [25, Proposition 2.1].

- (8) The set $\Gamma_{opt,2}(\mu, \nu)$ and $\Gamma_{opt,\infty}(\mu, \nu)$ represent the sets of optimal transport plans with respect to the 2-Wasserstein and ∞ -Wasserstein costs, respectively.
- (9) μ_t represents the gradient flow evolution in \mathcal{P}_2 of the interaction energy \mathcal{W} , starting from a given μ_0 its initial data.
- (10) Φ_t denotes the flow map of the gradient flow evolution, i.e. of the differential equation $\dot{x} = -\nabla W * \mu_t(x)$. They are $C^{1,1}$ in space and C^2 in time.
- (11) v_μ is the velocity vector field such that $v_\mu := -\nabla W * \mu$.

Global Minimizers of the Interaction Energy

2.1. Introduction to Global Minimizers

We consider the minimization of the nonlocal-interaction energy

$$\mathcal{W}[\mu] := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) d\mu(x) d\mu(y)$$

over the space of probability measures $\mathcal{P}(\mathbb{R}^N)$. Nonlocal-interaction energies arise naturally in descriptions of systems of interacting particles, as well as continuum descriptions of systems with long-range interactions. They play an important role in statistical mechanics [48, 50] and descriptions of crystallization [4, 47]. For semi-convex interaction potentials ω some systems governed by the energy E can be interpreted as a gradient flow of the energy with respect to Wasserstein metric and satisfy the nonlocal-interaction equation

$$\frac{\partial \mu}{\partial t} = 2 \operatorname{div}(\mu(\nabla W * \mu)).$$

Applications of the equation include models of collective behavior of many-agent systems [10, ?], granular media [9, 24, 53], self-assembly of nanoparticles [36, 37], and molecular dynamics simulations of matter [34].

Although the choice of the interaction potential ω depends on the phenomenon modeled by either (2.1) or (2.1), the interaction between two agents/particles is often determined only by the distance between them. This yields that the interaction potential ω is radially symmetric, i.e., $W(x) = \omega(|x|)$ for some $\omega : [0, +\infty) \rightarrow \mathbb{R} \cup \{+\infty\}$. Many potentials considered in the applications are repulsive at short distances ($\omega'(r) < 0$ for r small) and attractive at long distances ($\omega'(r) > 0$ for r large). While purely attractive potentials lead to finite-time or infinite time blow up [13] the attractive-repulsive potentials often generate finite-sized, confined aggregations [33, 40, 42]. On the other hand in statistical mechanics and in studies of crystallization it is the (attractive-repulsive) potentials that do not lead to confined states as the number of particles increases which are of interest [48, 51]. This highlights the importance of obtaining criteria for existence of global minimizers of the energy, for it is precisely those potentials which have a global minimizer that exhibit aggregation of particles into dense clumps.

The study of the nonlocal-interaction equation (2.1) in terms of well-posedness, finite or infinite time blow-up, and long-time behavior has attracted the interest of many research groups

in the recent years [7, 13, 14, 15, 16, 23, 30, 33, 38, 40, 41, 6, 20]. The energy (2.1) plays an important role in these studies as it governs the dynamics and as its (local) minima describe the long-time asymptotics of solutions. It has been observed that even for quite simple attractive–repulsive potentials the energy minimizers are sensitive to the precise form of the potential and can exhibit a wide variety of patterns [38, 40, 57]. In [5] Balagué, Carrillo, Laurent, and Raoul obtain conditions for the dimensionality of the support of local minimizers of (2.1) in terms of the repulsive strength of the potential ω at the origin. Properties of minimizers for a special class of potentials which blow up approximately like the Newtonian potential at the origin have also been studied [15, 22, 32, 33]. Particularly relevant to our study are the results obtained by Choksi, Fetecau and one of the authors [26] on the existence of minimizers of interaction energies in a certain form. There the authors consider potentials of the power-law form, $\omega(x) := |x|^a/a - |x|^r/r$, for $-N < r < a$, and prove the existence of minimizers in the class of probability measures when the power of repulsion r is positive. When the interaction potential has a singularity at the origin, i.e., for $r < 0$, on the other hand, they establish the existence of minimizers of the interaction energy in a restrictive class of uniformly bounded, radially symmetric L^1 -densities satisfying a given mass constraint. Carrillo, Chipot and Huang [21] also consider the minimization of nonlocal-interaction energies defined via power-law potentials and prove the existence of a global minimizer by using a discrete to continuum approach. The minimizers and their relevance to statistical mechanics were also considered in periodic setting (and on bounded sets) by Sütö [50].

Here (Theorems 4 and 5) we obtain criteria for the existence of minimizers in a very broad class of potentials. We employ the direct method of the calculus of variations. In Lemma 2 we establish the weak lower-semicontinuity of the energy with respect to weak convergence of measures. When the potential ω grows unbounded at infinity (case treated in Theorem 4) this provides enough confinement for a minimizing sequence to ensure the existence of minimizers. If ω asymptotes to a finite value (case treated in Theorem 5) then there is a delicate interplay between repulsion at some lengths (in most applications short lengths) and attraction at other length scales (typically long) which establishes whether the repulsion wins and a minimizing sequence spreads out indefinitely and “vanishes” or the minimizing sequence is compact and has a limit. We establish a simple, sharp condition, **(HE)** on the energy that characterizes whether a global minimizer exists. To establish compactness of a minimizing sequence we use Lions’ concentration compactness lemma.

The condition **(HE)** is closely related to the notion of stability (or H -stability) used in statistical mechanics [48]. Namely stability is a necessary condition for a many body system of interacting particles to exhibit a macroscopic thermodynamical behavior. As we show in Proposition 6 the condition **(HE)** is almost exactly the complement of H -stability. That is

if the energy (2.1) admits a global minimizer then the system of interacting particles is not expected to have a thermodynamic limit.

While the conditions **(H1)** and **(H2)** are easy-to-check conditions on the potential ω itself, the condition **(HE)** is a condition on the energy and it is not always easy to verify. Due to the above connection with statistical mechanics the conditions on H -stability (or lack thereof) can be used to verify if **(HE)** is satisfied for a particular potential. We list such conditions in Section 2.4. However only few general conditions are available. It is an important open problem to establish a more complete characterization of potentials ω which satisfy **(HE)**.

We finally remark that as this manuscript was being completed we learned that Cañizo, Carrillo, and Patacchini [18] have been working on the same problem and have obtained very similar conditions for the existence of minimizers, which they also show to be compactly supported. The proofs however are quite different.

2.2. Hypotheses and Preliminaries

The interaction potentials we consider are radially symmetric, that is, $W(x) = \omega(|x|)$ for some function $\omega : [0, +\infty) \rightarrow \mathbb{R} \cup \{+\infty\}$, and they satisfy the following basic properties:

- (H1)** ω is lower-semicontinuous.
- (H2)** The function $W(x)$ is locally integrable on \mathbb{R}^N .

Beyond the basic assumptions above, the behavior of the tail of ω will play an important role. We consider potentials which have a limit at infinity. If the limit is finite we can add a constant to the potential, which does not affect the existence of minimizers, and assume that the limit is zero. If the limit is infinite the proof of existence of minimizers is simpler, while if the limit is finite an additional condition is needed. Thus we split the condition on behavior at infinity into two conditions:

- (H3a)** $\omega(r) \rightarrow \infty$ as $r \rightarrow \infty$.
- (H3b)** $\omega(r) \rightarrow 0$ as $r \rightarrow \infty$.

By the assumptions **(H1)** and **(H3a)** or **(H3b)** the interaction potential ω is bounded from below. Hence

$$(2.2.1) \quad C_W := \inf_{r \in (0, \infty)} \omega(r) > -\infty.$$

If **(H3a)** holds, by adding $-C_W$ to ω from now on we assume that $\omega(r) \geq 0$ for all $r \in (0, \infty)$

As noted in the introduction the assumptions **(H1)**, **(H2)** with **(H3a)** or **(H3b)** allow us to handle a quite general class of interaction potentials ω .

In order to establish the existence of a global minimizer of E , for interaction potentials ω satisfying **(H1)**, **(H2)** and **(H3b)**, the following assumption on the interaction energy E is needed:

(HE) There exists a measure $\bar{\mu} \in \mathcal{P}(\mathbb{R}^N)$ such that $\mathcal{W}[\bar{\mu}] \leq 0$.

We establish that the conditions **(H1)**, **(H2)** and **(H3a)** or **(H3b)** imply the lower-semicontinuity of the energy with respect to weak convergence of measures. We recall that a sequence of probability measures μ_n converges weakly to measure μ , and we write $\mu_n \rightharpoonup \mu$, if for every bounded continuous function $\phi \in C_b(\mathbb{R}^N, \mathbb{R})$

$$\int \phi d\mu_n \rightarrow \int \phi d\mu \quad \text{as } n \rightarrow \infty.$$

LEMMA 2. *[Lower-semicontinuity of the energy] Assume $\omega : [0, \infty) \rightarrow (-\infty, \infty]$ is a lower-semicontinuous function bounded from below. Then the energy $\mathcal{W} : \mathcal{P}(\mathbb{R}^n) \rightarrow (-\infty, \infty]$ defined in (1.5.1) is weakly lower-semicontinuous with respect to weak convergence of measures.*

PROOF. Let μ_n be a sequence of probability measures such that $\mu_n \rightharpoonup \mu$ as $n \rightarrow \infty$. Then $\mu_n \times \mu_n \rightharpoonup \mu \times \mu$ in the set of probability measures on $\mathbb{R}^N \times \mathbb{R}^N$. If ω is continuous and bounded

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) d\mu_n(x) d\mu_n(y) \longrightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y) d\mu(x) d\mu(y) \quad \text{as } n \rightarrow \infty.$$

So, in fact, the energy is continuous with respect to weak convergence. On the other hand, if ω is lower-semicontinuous and ω is bounded from below then the weak lower-semicontinuity of the energy follows from the Portmanteau Theorem [54, Theorem 1.3.4]. \square

We remark that the assumption on boundedness from below is needed since if, for example, $\omega(r) = -r$ then for $\mu_n = (1 - \frac{1}{n})\delta_0 + \frac{1}{n}\delta_n$ the energy is $\mathcal{W}(\mu_n) = -1$ for all $n \in \mathbb{N}$, while $\mu_n \rightharpoonup \delta_0$ which has energy $\mathcal{W}(\delta_0) = 0$.

Finally, we state Lions' concentration compactness lemma for probability measures [43], [49, Section 4.3]. This lemma is the main tool in verifying that an energy-minimizing sequence is precompact in the sense of weak convergence of measures.

LEMMA 3. *[Concentration-compactness lemma for measures] Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on \mathbb{R}^N . Then there exists a subsequence $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ satisfying one of the three following possibilities:*

- (i) *(tightness up to translation)* There exists $y_k \in \mathbb{R}^N$ such that for all $\varepsilon > 0$ there exists $R > 0$ with the property that

$$\int_{B_R(y_k)} d\mu_{n_k}(x) \geq 1 - \varepsilon \quad \text{for all } k.$$

- (ii) (*vanishing*) $\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} d\mu_{n_k}(x) = 0$, for all $R > 0$;
- (iii) (*dichotomy*) There exists $\alpha \in (0, 1)$ such that for all $\varepsilon > 0$, there exist a number $R > 0$ and a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^N$ with the following property: Given any $R' > R$ there are nonnegative measures μ_k^1 and μ_k^2 such that

$$0 \leq \mu_k^1 + \mu_k^2 \leq \mu_{n_k}$$

$$\text{supp}(\mu_k^1) \subset B_R(x_k), \quad \text{supp}(\mu_k^2) \subset \mathbb{R}^N \setminus B_{R'}(x_k),$$

$$\limsup_{k \rightarrow \infty} \left(\left| \alpha - \int_{\mathbb{R}^N} d\mu_k^1(x) \right| + \left| (1 - \alpha) - \int_{\mathbb{R}^N} d\mu_k^2(x) \right| \right) \leq \varepsilon.$$

2.3. Existence of Minimizers

In this section we prove the existence of a global minimizer of \mathcal{W} . We use the direct method of the calculus of variations and utilize Lemma 3 to eliminate the “vanishing” and “dichotomy” of an energy-minimizing sequence. The techniques in our proofs, though, depends on the behavior of the interaction potential at infinity. Thus we prove two existence theorems: one for potentials satisfying **(H3a)** and another one for those satisfying **(H3b)**.

THEOREM 4. *Suppose ω satisfies the assumptions **(H1)**, **(H2)** and **(H3a)**. Then the energy (1.5.1) admits a global minimizer in $\mathcal{P}(\mathbb{R}^N)$.*

PROOF. Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a minimizing sequence, that is, $\lim_{n \rightarrow \infty} \mathcal{W}[\mu_n] = \inf_{\mu \in \mathcal{P}(\mathbb{R}^N)} \mathcal{W}[\mu]$.

Suppose $\{\mu_k\}_{k \in \mathbb{N}}$ has a subsequence which “vanishes”. Since that subsequence is also a minimizing sequence we can assume that $\{\mu_k\}_{k \in \mathbb{N}}$ vanishes. Then for any $\varepsilon > 0$ and for any $R > 0$ there exists $K \in \mathbb{N}$ such that for all $k > K$ and for all $x \in \mathcal{P}(\mathbb{R}^N)$

$$\mu_k(\mathbb{R}^N \setminus B_R(x)) \geq 1 - \varepsilon.$$

This implies that for $k > K$,

$$\iint_{|x-y| \geq R} d\mu_k(x) d\mu_k(y) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N \setminus B_R(x)} d\mu_k(y) \right) d\mu_k(x) \geq 1 - \varepsilon.$$

Given $M \in \mathbb{R}$, by condition **(H3a)** there exists $R > 0$ such that for all $r \geq R$, $\omega(r) \geq M$. Consider $\varepsilon \in (0, \frac{1}{2})$ and K corresponding to ε and R . Since $W \geq 0$ by Remark 2.2,

$$\begin{aligned} \mathcal{W}[\mu_k] &= \iint_{|x-y| \leq R} \omega(|x-y|) d\mu_k(x) d\mu_k(y) + \iint_{|x-y| \geq R} \omega(|x-y|) d\mu_k(x) d\mu_k(y) \\ &\geq \iint_{|x-y| \geq R} \omega(|x-y|) d\mu_k(x) d\mu_k(y) \\ &\geq (1 - \varepsilon)M. \end{aligned}$$

Letting $M \rightarrow \infty$ implies $\mathcal{W}[\mu_k] \rightarrow \infty$. This contradicts the fact that μ_k is a subsequence of a minimizing sequence of \mathcal{W} . Thus, “vanishing” does not occur.

Next we show that “dichotomy” is also not an option for a minimizing sequence. Suppose, that “dichotomy” occurs. As before we can assume that the subsequence along which dichotomy occurs is the whole sequence. Let R , sequence x_k and measures

$$\mu_k^1 + \mu_k^2 \leq \mu_k.$$

be as defined in Lemma 3(ii). For any $R' > R$, using Remark 2.2, we obtain

$$\begin{aligned} \liminf_{k \rightarrow \infty} \mathcal{W}[\mu_{n_k}] &\geq \liminf_{k \rightarrow \infty} \int_{B_R(x_{n_k})} \int_{B_{R'}^c(x_{n_k})} \omega(|x-y|) d\mu_k^2(x) d\mu_k^1(y) \\ &\geq \inf_{r \geq R'-R} \omega(r) \alpha (1-\alpha) \end{aligned}$$

where $B_{R'}^c(x_{n_k})$ simply denotes $\mathbb{R}^N \setminus B_{R'}(x_{n_k})$.

By **(H3a)**, letting $R' \rightarrow \infty$ yields that

$$\liminf_{k \rightarrow \infty} \mathcal{W}[\mu_{n_k}] \geq \infty,$$

which contradicts the fact that μ_k is an energy minimizing sequence.

Therefore “tightness up to translation” is the only possibility. Hence there exists $y_k \in \mathbb{R}^N$ such that for all $\varepsilon > 0$ there exists $R > 0$ with the property that

$$\int_{B(y_k, R)} d\mu_{n_k}(x) \geq 1 - \varepsilon \quad \text{for all } k$$

Let

$$\tilde{\mu}_{n_k} := \mu_{n_k}(\cdot - y_k).$$

Then the sequence of probability measures $\{\tilde{\mu}_{n_k}\}_{k \in \mathbb{N}}$ is tight. Since the interaction energy is translation invariant we have that

$$\mathcal{W}[\tilde{\mu}_{n_k}] = \mathcal{W}[\mu_{n_k}].$$

Hence, $\{\tilde{\mu}_{n_k}\}_{k \in \mathbb{N}}$ is also an energy-minimizing sequence. By the Prokhorov’s theorem (cf. [17, Theorem 4.1]) there exists a further subsequence of $\{\tilde{\mu}_{n_k}\}_{k \in \mathbb{N}}$ which we still index by k , and a measure $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$ such that

$$\tilde{\mu}_{n_k} \rightharpoonup \mu_0$$

in $\mathcal{P}(\mathbb{R}^N)$ as $k \rightarrow \infty$.

Since the energy is lower-semicontinuous with respect to weak convergence of measures, by Lemma 2, the measure μ_0 is a minimizer of E . \square

The second existence theorem involves interaction potentials which vanish at infinity.

THEOREM 5. *Suppose ω satisfies the assumptions **(H1)**, **(H2)** and **(H3b)**. Then the energy \mathcal{W} , given by (1.5.1), has a global minimizer in $\mathcal{P}(\mathbb{R}^N)$ if and only if it satisfies the condition **(HE)**.*

PROOF. Let us assume that \mathcal{W} satisfies condition **(HE)**. As before, our proof relies on the direct method of the calculus variations for which we need to establish precompactness of a minimizing sequence.

Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a minimizing sequence and let

$$I := \inf_{\mu \in \mathcal{P}(\mathbb{R}^N)} \mathcal{W}[\mu].$$

Condition **(HE)** implies that $I \leq 0$. If $I = 0$ then by assumption **(HE)** there exists $\bar{\mu}$ with $\mathcal{W}[\bar{\mu}] = 0$, which is the desired minimizer. Thus, we focus on case that $I < 0$. Hence there exists $\bar{\mu}$ for which $\mathcal{W}[\bar{\mu}] < 0$. Also note that by Remark 2.2, $I > -\infty$.

Suppose the subsequence $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ of the minimizing sequence $\{\mu_n\}_{n \in \mathbb{N}}$ “vanishes”. Since that subsequence is also a minimizing sequence we can assume that $\{\mu_k\}_{k \in \mathbb{N}}$ vanishes. That is, for any $R > 0$

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\mu_k(y) = 0.$$

Let

$$\bar{\omega}(R) = \inf_{r \geq R} \omega(r).$$

Since $\omega(r) \rightarrow 0$ as $r \rightarrow \infty$, $\bar{\omega}(r) \rightarrow 0$ as $r \rightarrow \infty$ and $\bar{\omega}(r) \leq 0$ for all $r \geq 0$. Then we have that

$$\begin{aligned} \mathcal{W}[\mu_k] &= \iint_{|x-y| \geq R} \omega(|x-y|) d\mu_k(x) d\mu_k(y) + \iint_{|x-y| \leq R} \omega(|x-y|) d\mu_k(x) d\mu_k(y) \\ &\geq \bar{\omega}(R) + C_W \iint_{|x-y| \leq R} d\mu_k(x) d\mu_k(y) \\ &= \bar{\omega}(R) + C_W \int_{\mathbb{R}^N} \left(\int_{B_R(x)} d\mu_k(y) \right) d\mu_k(x) \end{aligned}$$

Vanishing of the measures, (2.3), implies that $\liminf_{k \rightarrow \infty} \mathcal{W}[\mu_k] \geq \bar{\omega}(R)$ for all $R > 0$. Taking the limit as $R \rightarrow \infty$ gives

$$\liminf_{k \rightarrow \infty} \mathcal{W}[\mu_k] \geq 0.$$

This contradicts the fact that the infimum of the energy, namely I , is negative. Therefore “vanishing” in Lemma 3 does not occur.

Suppose the dichotomy occurs. Let α be as in Lemma 3 and C_W be the constant defined in (2.2.1). Let $\varepsilon > 0$ be such that

$$(2.3.1) \quad \varepsilon < \frac{|I|}{64|C_W|} \min \left\{ \frac{1}{\alpha} - 1, \frac{1}{1-\alpha} - 1 \right\}$$

and let R' be such that

$$(2.3.2) \quad |\bar{\omega}(R' - R)| = \left| \inf_{r \geq R' - R} \omega(r) \right| < \frac{|I|}{32} \min \left\{ \frac{1}{\alpha} - 1, \frac{1}{1-\alpha} - 1 \right\}.$$

As in the proof of Theorem 4, we can assume that dichotomy occurs along the whole sequence. Let μ_k^1 and μ_k^2 be measures described in Lemma 3. Let $\nu_k = \mu_k - (\mu_k^1 + \mu_k^2)$. Note that ν_k is a nonnegative measure with $|\nu_k| < \varepsilon$, where $|\nu_k| = \nu_k(\mathbb{R}^N)$.

Let $B[\cdot, \cdot]$ denote the symmetric bilinear form

$$B[\mu, \nu] := 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(|x - y|) d\mu(x) d\nu(y).$$

By the definition of energy

$$(2.3.3) \quad \begin{aligned} \mathcal{W}(\mu_k) &= \mathcal{W}(\mu_k^1) + \mathcal{W}(\mu_k^2) + B(\mu_k^1, \mu_k^2) + B(\mu_k^1 + \mu_k^2, \nu_k) + \mathcal{W}(\nu_k) \\ &\geq \mathcal{W}(\mu_k^1) + \mathcal{W}(\mu_k^2) - |\bar{\omega}(R' - R)| - 2|C_W|\varepsilon \end{aligned}$$

where we used that the supports of μ_k^1 and μ_k^2 are at least $R' - R$ apart. We can also assume, without the loss of generality, that $E(\mu_k) < \frac{1}{2}I$ for all k . Let $\alpha_k = |\mu_k^1|$, $\beta_k = |\mu_k^2|$.

Let us first consider the case that $\frac{1}{\alpha_k} \mathcal{W}(\mu_k^1) \leq \frac{1}{\beta_k} \mathcal{W}(\mu_k^2)$. Note that the energy has the following scaling property:

$$\mathcal{W}[c\sigma] = c^2 \mathcal{W}[\sigma]$$

for any constant $c > 0$ and measure σ . Our goal is to show that for some $\lambda > 0$, for all large enough k , $E(\frac{1}{\alpha_k} \mu_k^1) < E(\mu_k) - \lambda|I|$ which contradicts the fact that μ_k is a minimizing sequence.

Let us consider first the subcase that $E(\mu_k^2) \geq 0$ along a subsequence. By relabeling we can assume that the subsequence is the whole sequence. From (2.3.1), (2.3.2), and (2.3.3) it follows that $\frac{1}{\alpha_k} E(\mu_k^1) < I/4$ for all k . Using the estimates again, we obtain

$$\begin{aligned} E(\mu_k) - E\left(\frac{1}{\alpha_k} \mu_k^1\right) &\geq \left(1 - \frac{1}{\alpha_k^2}\right) E(\mu_k^1) - |\bar{\omega}(R' - R)| - 2|C_W|\varepsilon \\ &\geq \left(\frac{1}{\alpha_k} - 1\right) \frac{|I|}{4} - |\bar{\omega}(R' - R)| - 2|C_W|\varepsilon \\ &\geq \left(\frac{1}{\alpha} - 1\right) \frac{|I|}{16}. \end{aligned}$$

Thus μ_k is not a minimizing sequence. Contradiction. Let us now consider the subcase $E(\mu_k^2) \leq 0$ for all k . Using (2.3.3) and $\frac{\beta_k}{\alpha_k} E(\mu_k^1) \leq E(\mu_k^2)$ we obtain

$$\frac{I}{2} \geq E(\mu_k) \geq \left(1 + \frac{\beta_k}{\alpha_k}\right) E(\mu_k^1) - |\bar{\omega}(R' - R)| - 2|C_W|\varepsilon.$$

From (2.3.1) and (2.3.2) follows that for all k

$$\frac{1}{\alpha_k} E(\mu_k^1) \leq \frac{I}{8}.$$

Combining with above inequalities gives

$$\begin{aligned} E(\mu_k) - E\left(\frac{1}{\alpha_k} \mu_k^1\right) &\geq \left(1 + \frac{\beta_k}{\alpha_k} - \frac{1}{\alpha_k^2}\right) E(\mu_k^1) - |\bar{\omega}(R' - R)| - 2|C_W|\varepsilon \\ &\geq \left(\frac{1}{\alpha_k} - \alpha_k - \beta_k\right) \frac{|I|}{8} - \left(\frac{1}{\alpha} - 1\right) \left(\frac{|I|}{32} + \frac{|I|}{32}\right) \\ &\geq \frac{|I|}{32} \left(\frac{1}{\alpha} - 1\right) \end{aligned}$$

for k large enough. This contradicts the assumption that μ_k is a minimizing sequence.

The case $\frac{1}{\alpha_k} E(\mu_k^1) > \frac{1}{\beta_k} E(\mu_k^2)$ is analogous. In conclusion the dichotomy does not occur. Therefore “tightness up to translation” is the only possibility. As in the proof of Theorem 4, we can translate measures μ_{n_k} to obtain a tight, energy-minimizing sequence $\tilde{\mu}_{n_k}$.

By Prokhorov’s theorem, there exists a further subsequence of $\{\tilde{\mu}_{n_k}\}_{k \in \mathbb{N}}$, still indexed by k , such that

$$\mu_{n_k} \rightharpoonup \mu_0 \quad \text{as } k \rightarrow \infty$$

for some measure $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$ in $\mathcal{P}(\mathbb{R}^N)$ as $k \rightarrow \infty$. Therefore, by lower-semicontinuity of the energy, μ_0 is a minimizer of E in the class $\mathcal{P}(\mathbb{R}^N)$.

We now show the necessity of condition **(HE)**. Assume that $\mathcal{W}[\mu] > 0$ for all $\mu \in \mathcal{P}(\mathbb{R}^N)$. To show that the energy E does not have a minimizer consider a sequence of measures which “vanishes” in the sense of Lemma 3(ii). Let

$$\rho(x) = \frac{1}{\omega_N} \chi_{B_1(0)}(x),$$

where ω_N denotes the volume of the unit ball in \mathbb{R}^N and $\chi_{B_R(0)}$ denotes the characteristic function of $B_R(0)$, the ball of radius R centered at the origin. Consider the sequence

$$\rho_n(x) = \frac{1}{n^N} \rho\left(\frac{x}{n}\right)$$

for $n \geq 1$. Note that ρ_n are in $\mathcal{P}(\mathbb{R}^N)$. We estimate

$$\begin{aligned}
0 < \mathcal{W}[\rho_n] &= \frac{1}{\omega_N^2 n^{2N}} \int_{B_n(0)} \int_{B_n(0)} \omega(|x-y|) dx dy \\
&\leq \frac{1}{\omega_N^2 n^{2N}} \int_{B_n(0)} \left(\int_{B_n(y)} |\omega(|x|)| dx \right) dy \\
&\leq \frac{1}{\omega_N n^N} \left(\int_{B_R(0)} |\omega(|x|)| dx + \int_{B_{2n}(0) \setminus B_R(0)} |\omega(|x|)| dx \right) \\
&\leq \frac{C(R)}{\omega_N n^N} + \frac{2^N}{\omega_N} \sup_{r \geq R} |\omega(r)|
\end{aligned}$$

Since $\sup_{r \geq R} |\omega(r)| \rightarrow 0$ as $R \rightarrow \infty$, for any $\varepsilon > 0$ we can choose R so that $\frac{2^N}{\omega_N} \sup_{r \geq R} |\omega(r)| < \frac{\varepsilon}{2}$. We can then choose n large enough for $\frac{C(R)}{\omega_N n^N} < \frac{\varepsilon}{2}$ to hold. Therefore $\lim_{n \rightarrow \infty} \mathcal{W}[\rho_n] = 0$, that is, $\inf_{\mu \in \mathcal{P}(\mathbb{R}^N)} \mathcal{W}[\mu] = 0$. However, since $\mathcal{W}[\cdot]$ is positive for any measure in $\mathcal{P}(\mathbb{R}^N)$ the energy does not have a minimizer. \square

2.4. Stability and Condition (HE)

The interaction energies of the form (1.5.1) have been an important object of study in statistical mechanics. For a system of interacting particles to have a macroscopic thermodynamic behavior it is needed that it does not accumulate mass on bounded regions as the number of particles goes to infinity. Ruelle called such potentials stable (a.k.a. H -stable). More precisely, a potential $\omega : [0, \infty) \rightarrow (-\infty, \infty]$ is defined to be *stable* if there exists $B \in \mathbb{R}$ such that for all n and for all sets of n distinct points $\{x_1, \dots, x_n\}$ in \mathbb{R}^N

$$(2.4.1) \quad \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \omega(x_i - x_j) \geq -\frac{1}{n} B.$$

We show that for a large class of potentials the stability is equivalent with nonnegativity of energies. Our result is a continuum analogue of a part of Lemma 3.2.3 [48].

PROPOSITION 6. *[Stability conditions] Let $\omega : [0, \infty) \rightarrow \mathbb{R}$ be an upper-semicontinuous function such that ω is bounded from above or there exists \bar{R} such that ω is nondecreasing on $[\bar{R}, \infty)$. Then the conditions*

- (S1) ω is a stable potential as defined by (2.4.1),
- (S2) for any probability measure $\mu \in \mathcal{P}(\mathbb{R}^N)$, $\mathcal{W}(\mu) \geq 0$

are equivalent.

Note that all potentials considered in the proposition are finite at 0. We expect that the condition can be extended to a class of potentials which converge to infinity at zero. Doing so is an open problem. We also note that condition (S2) is not exactly the complement of (HE), as the nonnegative potentials whose minimum is zero satisfy both conditions. Such potentials

indeed exist: for example consider any smooth nonnegative ω such that $\omega(0) = 0$. Then the associated energy is nonnegative and $\mathcal{W}(\delta_0) = 0$ so any singleton is an energy minimizer. Note that E satisfies both **(HE)** and stability. To further remark on connections with statistical mechanics we note that such potentials ω are not *super-stable*, but are *tempered* if ω decays at infinity (both notions are defined in [48]).

PROOF. To show that **(S2)** implies **(S1)** consider $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$. Then from $\mathcal{W}(\mu) \geq 0$ it follows that $\frac{1}{n^2} \sum_{1 \leq i < j \leq n} \omega(x_i - x_j) \geq -\frac{1}{2n} \omega(0)$ so **(S1)** holds with $B = \frac{1}{2} \omega(0)$.

We now turn to showing that **(S1)** implies **(S2)**. Let us recall the definition of Lévy–Prokhorov metric, which metrizes the weak convergence of probability measures: Given probability measures ν and σ

$$d_{LP}(\nu, \sigma) = \inf\{\varepsilon > 0 : (\forall A - \text{Borel}) \nu(A) \leq \sigma(A + \varepsilon) + \varepsilon \text{ and } \sigma(A) \leq \nu(A + \varepsilon) + \varepsilon\}$$

where $A + \varepsilon = \{x : d(x, A) < \varepsilon\}$.

For a given measure μ , we first show that it can be approximated in the Lévy–Prokhorov metric by an empirical measure of a finite set with arbitrarily many points. That is, we show that for any $\varepsilon > 0$ and any n_0 there exists $n \geq n_0$ and a set of distinct points $X = \{x_1, \dots, x_n\}$ such that the corresponding empirical measure $\mu_X = \frac{1}{n} \sum_{j=1}^n \delta_{x_j}$ satisfies $d_{LP}(\mu_X, \mu) < \varepsilon$.

Let $\varepsilon > 0$. We can assume that $\varepsilon < \frac{1}{2}$. There exists $R > 0$ such that for $Q_R = [-R, R]^N$, $\mu_X(\mathbb{R}^N \setminus Q_R) < \frac{\varepsilon}{2}$. For integer l such that $\sqrt{N} \frac{2R}{l} < \varepsilon$ divide Q_R into l^N disjoint cubes Q_i , $i = 1, \dots, l^N$ with sides of length $2R/l$. While cubes have the same interiors, they are not required to be identical, namely some may contain different parts of their boundaries, as needed to make them disjoint. Note that the diameter of each cube, $\sqrt{N} \frac{2R}{l}$, is less than ε . Let $n > n_0$ be such that $\frac{l^N}{n} < \frac{\varepsilon}{2}$. Let $p = \frac{1}{n}$. For $i = 1, \dots, l^N$ let $p_i = \mu(Q_i)$, $n_i = \lfloor p_i n \rfloor$, and $q_i = n_i p$. Note that $0 \leq p_i - q_i \leq p$ and thus $s_q = \sum_i q_i \geq \sum_i p_i - l^N p > 1 - \frac{\varepsilon}{2}$. In each cube Q_i place n_i distinct points and let \tilde{X} be the set of all such points. Note that $\tilde{n} = \sum_i n_i = s_q n > (1 - \varepsilon)n$. Let \hat{X} be an arbitrary set of $n - \tilde{n}$ distinct points in $Q_{2R} \setminus Q_R$. Let $X = \tilde{X} \cup \hat{X}$. Note that X is a set of n distinct points. Then for any Borel set A

$$\mu(A) \leq \sum_{i: \mu(A \cap Q_i) > 0} \mu(Q_i) + \frac{\varepsilon}{2} \leq \sum_{i: \mu(A \cap Q_i) > 0} (\mu_X(Q_i) + p) + \frac{\varepsilon}{2} \leq \mu_X(A + \varepsilon) + \varepsilon.$$

Similarly

$$\mu_X(A) \leq \mu(A + \varepsilon) + \varepsilon.$$

Therefore $d_{LP}(\mu, \mu_X) \leq \varepsilon$.

Consequently there exists a sequence of sets X_m with $n(m)$ points satisfying $n(m) \rightarrow \infty$ as $m \rightarrow \infty$ for which the empirical measure $\mu_m = \mu_{X_m}$ converges weakly $\mu_m \rightarrow \mu$ as $m \rightarrow \infty$. By

assumption **(S1)**

$$\iint_{x \neq y} W(x-y) d\mu_m(x) d\mu_{X_m}(y) \geq -\frac{1}{n(m)} B.$$

Let us first consider the case that ω is an upper-semicontinuous function bounded from above. It follows from Lemma 2 that the energy E is an upper-semicontinuous functional. Therefore

$$\mathcal{W}(\mu) \geq \limsup_{m \rightarrow \infty} \mathcal{W}(\mu_m) \geq \limsup_{m \rightarrow \infty} -\frac{1}{n(m)} (B - \omega(0)) = 0$$

as desired.

If ω is an upper-semicontinuous function such that there exists \bar{R} such that ω is nondecreasing on $[\bar{R}, \infty)$ we first note that we can assume that $\omega(r) \rightarrow \infty$ as $r \rightarrow \infty$, since otherwise ω is bounded from above which is covered by the case above. If μ is a compactly supported probability measure then there exists L such that for all m , $\text{supp } \mu_m \subseteq [-L, L]^N$. Since ω is upper-semicontinuous it is bounded from above on compact sets and thus upper-semicontinuity of the energy holds. That is $\mathcal{W}(\mu) \geq \limsup_{m \rightarrow \infty} \mathcal{W}(\mu_m) \geq 0$ as before.

If μ is not compactly supported it suffices to show that there exists a compactly supported measure $\tilde{\mu}$ such that $\mathcal{W}(\mu) \geq \mathcal{W}(\tilde{\mu})$, since by above we know that $\mathcal{W}(\tilde{\mu}) \geq 0$. Note that since $\mathcal{W}(\frac{1}{2}(\delta_x + \delta_0)) \geq 0$, $\omega(|x|) \geq -\omega(0)$. Therefore ω is bounded from below by $-\omega(0)$ and $\omega(0) \geq 0$.

Since $\omega(r) \rightarrow \infty$ as $r \rightarrow \infty$ there exists $R_1 \geq \bar{R}$ such that $\omega(R_1) \geq \max\{1, \max_{r \leq R_1} \omega(r)\}$ and $m_1 = \mu(\bar{B}_{R_1}(0)) > \frac{7}{8}$. Let R_2 be such that $\omega(R_2) > 2\omega(R_1)$, and define the constants $m_2 = \mu(\bar{B}_{R_2}(0) \setminus \bar{B}_{R_1}(0))$ and $m_3 = \mu(\mathbb{R}^N \setminus \bar{B}_{R_2}(0))$. Note that $m_1 + m_2 + m_3 = 1$. Consider the mapping

$$P(x) = \begin{cases} x & \text{if } |x| \leq R_2 \\ 0 & \text{if } |x| > R_2. \end{cases}$$

Let $\tilde{\mu} = P_{\#} \mu$. Estimating the interaction of particles between the regions provides:

$$\begin{aligned} \mathcal{W}(\tilde{\mu}) &\leq \mathcal{W}(\mu) + 2\omega(0)m_3^2 + 2(\omega(R_2) + \omega(0))m_2m_3 - 2(\omega(R_2) - \omega(R_1))m_1m_3 \\ &\leq \mathcal{W}(\mu) + \omega(R_2)m_3(m_3 + 4m_2 - m_1) < \mathcal{W}(\mu). \end{aligned}$$

□

As we showed in Theorem 5 the property **(HE)** is necessary and sufficient for the existence of a global minimizer when E is defined via an interaction potential satisfying **(H1)**, **(H2)** and **(H3b)**. The property **(HE)** is posed as a condition directly on the energy E , and can be difficult to verify for a given ω . It is then natural to ask what conditions the interaction potential ω needs to satisfy so that the energy E has the property **(HE)**. In other words, how can one characterize interaction potentials ω for which E admits a global minimizer? We do not address that question in detail, but just comment on the partial results established in the

context of H -stability of statistical mechanics and how they apply to the minimization of the nonlocal-interaction energy.

Perhaps the first condition which appeared in the statistical mechanics literature states that absolutely integrable potentials which integrate to a negative number over the ambient space are not stable (cf. [28, Theorem 2] or [48, Proposition 3.2.4]). In our language these results translate to the following proposition.

PROPOSITION 7. *Consider an interaction potential $W(x) = \omega(|x|)$ where ω satisfies the hypotheses **(H1)**, **(H2)** and **(H3b)**. If ω is absolutely integrable on \mathbb{R}^N and*

$$\int_{\mathbb{R}^N} \omega(|x|) dx < 0,$$

*then the energy \mathcal{W} defined by (1.5.1) satisfies the condition **(HE)**.*

PROOF. Since $\int_{\mathbb{R}^N} \omega(|x|) dx < 0$, given $\varepsilon > 0$ there exists a constant $R > 0$ such that

$$\int_{B_R(0)} \omega(|x|) dx < \varepsilon.$$

Consider the function $\rho(x) := \frac{1}{\omega_N R^N} \chi_{B_R(0)}(x)$, i.e., the scaled characteristic function of the ball of radius R . Since $\rho \in L^1(\mathbb{R}^N)$ with $\|\rho\|_{L^1(\mathbb{R}^N)} = 1$ it defines a probability measure. Estimating at the energy of ρ we obtain

$$\begin{aligned} \mathcal{W}[\rho] &= \frac{1}{\omega_N^2 n^{2N}} \int_{B_R(0)} \int_{B_R(0)} \omega(|x-y|) dx dy \\ &= \frac{1}{\omega_N^2 n^{2N}} \int_{B_R(0)} \left(\int_{B_R(y)} |\omega(|x|)| dx \right) dy < \varepsilon \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ shows that the energy E satisfies **(HE)**. □

An alternative condition for instability of interaction potential is given in [19, Section II]. This condition, which we state and prove in the following proposition, extends the result of Proposition 7 to interaction potentials which are not absolutely integrable.

PROPOSITION 8. *Suppose the interaction potential ω satisfies the hypotheses **(H1)**, **(H2)** and **(H3b)**. If there exists $p \geq 0$ for which*

$$(2.4.2) \quad \int_{\mathbb{R}^N} \omega(|x|) e^{-p^2|x|^2} dx < 0$$

*then the energy E defined by (1.5.1) satisfies the condition **(HE)**.*

PROOF. Let $p \geq 0$ be given such that the inequality (2.4.2) holds. Since the case $p = 0$ has been considered in Proposition 7 we can assume $p > 0$. Consider the function

$$\rho(x) = \frac{p^N}{\pi^{N/2}} e^{-2p^2|x|^2}.$$

Clearly $\rho \in L^1(\mathbb{R}^N)$ and $\|\rho\|_{L^1(\mathbb{R}^N)} = 1$; hence, it defines a probability measure on \mathbb{R}^N . Using the linear transformation on \mathbb{R}^{2N} given by

$$u = x - y, \quad v = x + y$$

for x and y in \mathbb{R}^N and denoting by C the Jacobian of this transformation we get that

$$\begin{aligned} \mathcal{W}[\rho] &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(|x - y|) e^{-2p^2|x|^2} e^{-2p^2|y|^2} dx dy \\ &= C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(|u|) e^{-2p^2|u+v|^2/2} e^{-2p^2|u-v|^2/2} du dv \\ &= C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(|u|) e^{-2p^2(|u|^2+|v|^2)} du dv \\ &= C \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \omega(|u|) e^{-2p^2|u|^2} du \right) e^{-2p^2|v|^2} dv < 0 \end{aligned}$$

Hence, the energy E satisfies **(HE)**. □

REMARK 9. Another useful criterion can be obtained by using the Fourier transform, as also noted in [48]. Namely if $W \in L^2(\mathbb{R}^N)$, for measure μ that has a density $\rho \in L^2(\mathbb{R}^N)$, by Plancharel's theorem

$$\mathcal{W}(\mu) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x - y) d\mu(x) d\mu(y) = \int_{\mathbb{R}^N} \hat{W}(\xi) |\hat{\rho}(\xi)|^2 d\xi.$$

So if real part of \hat{W} is positive, the energy does not have a minimizer.

This criterion can be refined. By Bochner's theorem the Fourier transforms of finite non-negative measures are precisely the positive definite functions. Thus we know which family of functions, $\hat{\rho}$ belongs to. Hence we can formulate the following criterion:

If $W \in L^2(\mathbb{R}^N)$ and there exists a positive definite complex valued function ψ such that $\int \hat{W}(\xi) |\psi^2(\xi)| d\xi \leq 0$ then the energy \mathcal{W} satisfies the condition **(HE)**.

CHAPTER 3

Formulation of the Problem of Stability of Steady-States

3.1. Introduction to the Stability Problem

The stability problem this thesis is concerned with is the following:

Suppose $\bar{\mu} \in \mathcal{P}_2$ is a steady state of the nonlocal aggregation equation. What are some conditions on ω that are sufficient to guarantee exponential convergence in the 2-Wasserstein distance of gradient flow evolutions that start near to $\bar{\mu}$ in some sense?

To answer this question this thesis explores how to linearize the dynamics in such a way to find explicit conditions for nonlinear stability, as well as obtain explicitly the rates of convergence.

Recall from chapter 1 that a typical result in the evolutions of gradient flows of energies in the space probability measures is that if an energy is λ -geodesically convex as defined in (1.3.5), and if $\lambda > 0$, then one gets the exponential contraction (1.6.3) of the evolutions, i.e.

$$d_2(\mu_t, \nu_t) \leq e^{-\lambda t} d_2(\mu, \nu)$$

where μ_t and ν_t are gradient flow evolutions of the energy with initial data μ and ν .

However, if one were to try to apply this to a particle interaction system that exhibits the long-range attraction and short range repulsion dynamics described above then one would run into two immediate problems:

- (1) Since ω is semi-convex (due to the short-range repulsion), \mathcal{W} is in general only semi-convex: meaning it only satisfies (1.3.5) for some $\lambda < 0$. Thus motivating the search for λ -convexity for positive λ only in a neighborhood of $\bar{\mu}$, and in particular the neighborhoods we considered are in the ∞ -Wasserstein distance, d_∞ , defined as:

$$d_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \|x - \tilde{x}\|_{L^\infty(d\pi(x, \tilde{x}))}.$$

- (2) Further, \mathcal{W} is never even locally λ -convex for any $\lambda > 0$ either, because there are rotations and translations of a measure in any p -Wasserstein neighborhood of a measure (whether p is 2, ∞ , or any of the other possible p -Wasserstein metrics that one might think were reasonable), and these rigid motions always leave the energy invariant since the energy of a configuration only depends on the distances between the particles.

Thus the search for λ -convexity for positive λ can only be along a restricted subset of perturbations of the steady-state.

Given these issues, it is still possible to obtain some meaningful convergence, which is the subject of this thesis. One can show that λ -convexity in only a (∞ -Wasserstein) neighborhood of a steady state, and only on a restricted subset of paths, is sufficient to give the exponential convergence results desired.

Furthermore, this thesis will later discuss how these local convexity conditions on the interaction energy that need to (in principle) be evaluated at all measures in a neighborhood of the steady-state can be pulled-back to conditions at the steady-state. They are pulled back in such a way that conditions on the interaction kernel that are evaluated only at the steady-state are shown to be sufficient -given enough smoothness on the interaction kernel- to guarantee an exponential rate of convergence in the 2-Wasserstein metric to a rotation of said steady-state.

3.2. Proving the Finite Dimensional Version of the Stability Result

The following chapters of the thesis will cover the convergence of the gradient flow to a locally asymptotically stable steady-state. To motivate the argument, here is the finite dimensional version of the argument.

Consider a potential on \mathbb{R}^d , say f , that is $C^{2,1}$ such that the Lipschitz constant of ∇f is L_f and the Lipschitz constant of $\text{Hess} f$ is c_f , and with f is associated the gradient flow trajectory Φ_t such that

$$\begin{cases} \dot{\Phi}_t(x) = -\text{grad} f(\Phi_t(x)) \\ \Phi_0(x) = x \end{cases}$$

with a steady-state of the evolution at \bar{x} . To further motivate the connection to the current problem, suppose that there is a Lipschitz vector field C that is nowhere zero and such that at each point x in the domain,

$$C[f] = \nabla f \cdot C|_x = 0.$$

This vector field is here to motivate how to deal with the translation and rotation invariance of the interaction energy. C likewise induces C^1 curves φ_t that lie in level sets of the potential f , namely φ_t solves

$$\begin{cases} \dot{\varphi}_t(x) = C(\varphi_t(x)) \\ \varphi_0(x) = x. \end{cases}$$

Since C is Lipschitz, a_ε is a well-defined continuous nondecreasing and bounded function in ε such that

$$a_\varepsilon := \sup_{x \in \{|\bar{x}-x| < \varepsilon\}} \frac{|C(\bar{x}) - C(x)|}{|\bar{x} - x|}.$$

This gives a potential that is λ -convex in directions perpendicular to the vector field C , whose flow gives the levels sets of the potential f . For example, if the domain is \mathbb{R}^2 then

$$f(x) = x_2^2$$

with

$$C(x) = e_1$$

would be an example of such a potential. In this case the potential is 2-convex in the e_2 direction and the potential stays constant along the e_1 direction.

LEMMA 10. *If \bar{x} is a steady-state such that for some $\lambda > 0$,*

$$\bar{v}^T \text{Hess} f(\bar{x}) \bar{v} \geq \lambda \bar{v}^T \bar{v}$$

for all $\bar{v} \perp C(\bar{x})$, then there is a $\varepsilon > 0$ such that

$$\lambda' := \lambda - \lambda \frac{a_\varepsilon^2}{|C(\bar{x})|^2} \varepsilon^2 - c_f \varepsilon$$

is positive and for x such that $|x - \bar{x}| < \varepsilon$ then

$$\nabla f(x)^T \text{Hess} f(x) \nabla f(x) \geq \lambda' \nabla f(x)^T \nabla f(x).$$

PROOF. Using that $\text{Hess} f$ is Lipschitz, note that

$$\begin{aligned} \nabla f(x)^T \text{Hess} f(x) \nabla f(x) &\geq \nabla f(x)^T \text{Hess} f(\bar{x}) \nabla f(x) - c_f |x - \bar{x}| \nabla f(x)^T \nabla f(x) \\ &\geq \nabla f(x)^T \text{Hess} f(\bar{x}) \nabla f(x) - c_f \varepsilon \nabla f(x)^T \nabla f(x). \end{aligned}$$

Now, just because $\nabla f(x) \perp C(x)$ it does not follow $\nabla f(x) \perp C(\bar{x})$. However $\nabla f(x)$ can be decomposed. Define $w \in \text{span}\{C(\bar{x})\}$ and $\bar{v} \perp C(\bar{x})$ such that

$$\begin{aligned} w &= \text{proj}_{C(\bar{x})} \nabla f(x) = \frac{(\nabla f(x) \cdot C(\bar{x}))}{|C(\bar{x})|^2} C(\bar{x}) \\ \bar{v} &= \nabla f(x) - w. \end{aligned}$$

Note that since $\nabla f(x) \perp C(x)$ one gets the estimate

$$\begin{aligned} w^T w &= \frac{1}{|C(\bar{x})|^2} (\nabla f(x) \cdot C(\bar{x}))^2 \\ &= (\nabla f(x) \cdot (C(\bar{x}) - C(x)))^2 \\ &\leq \nabla f(x)^T \nabla f(x) |C(\bar{x}) - C(x)|^2 \\ &\leq \left(\frac{a_\varepsilon^2}{|C(\bar{x})|^2} |\bar{x} - x|^2 \right) \nabla f(x)^T \nabla f(x) \end{aligned}$$

which also implies that

$$\begin{aligned}\bar{v}^T \bar{v} &= \nabla f(x)^T \nabla f(x) - w^T w \\ &\geq \left(1 - \frac{a_\varepsilon^2}{|C(\bar{x})|^2} |\bar{x} - x|^2\right) \nabla f(x)^T \nabla f(x) \\ &= \left(1 - \frac{a_\varepsilon^2}{|C(\bar{x})|^2} \varepsilon^2\right)\end{aligned}$$

and it is easily seen that

$$\text{Hess}f(\bar{x}) w = 0$$

so the above estimate on $\nabla f(x)^T \text{Hess}f(x) \nabla f(x)$ becomes

$$\begin{aligned}\nabla f(x)^T \text{Hess}f(x) \nabla f(x) &\geq (\bar{v} + w)^T \text{Hess}f(\bar{x}) (\bar{v} + w) - c_f \varepsilon \nabla f(x)^T \nabla f(x) \\ &= \bar{v}^T \text{Hess}f(\bar{x}) \bar{v} - c_f \varepsilon \nabla f(x)^T \nabla f(x) \\ &\geq \lambda \bar{v}^T \bar{v} - c_f \varepsilon \nabla f(x)^T \nabla f(x) \\ &= \left(\lambda - \lambda \frac{a_\varepsilon^2}{|C(\bar{x})|^2} \varepsilon^2 - c_f \varepsilon\right) \nabla f(x)^T \nabla f(x) \\ &= \lambda' \nabla f(x)^T \nabla f(x)\end{aligned}$$

as was to be shown. □

LEMMA 11. *Suppose that \bar{x} is a steady-state that satisfies the hypothesis of the previous lemma, i.e. there is a $\lambda > 0$ such that*

$$\bar{v}^T \text{Hess}f(\bar{x}) \bar{v} \geq \lambda \bar{v}^T \bar{v}$$

for $\bar{v} \perp C(\bar{x})$, which implies that there is a $\varepsilon > 0$ and $\lambda' > 0$ such that for all z such that $|z - \bar{x}| < \varepsilon$ it is true that

$$\nabla f(z)^T \text{Hess}f(z) \nabla f(z) \geq \lambda' \nabla f(z)^T \nabla f(z),$$

then if

$$|x - \bar{x}| < \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)}$$

then

$$\left|\dot{\Phi}_t(x)\right| \leq e^{-\lambda t} L_f \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)}$$

and

$$|\Phi_t(x) - \bar{x}| < \varepsilon$$

for all time t .

PROOF. First note that, for as long as $|\Phi_t(x) - \bar{x}| < \varepsilon$ is true for $t \in [0, T]$, the positive definite condition of the Hessian given by the previous lemma as mentioned in the hypothesis gives that

$$\begin{aligned} \frac{d}{dt} \left| \dot{\Phi}_t(x) \right|^2 &= \frac{d}{dt} |\nabla f(\Phi_t(x))|^2 \\ &= 2\nabla f(\Phi_t(x))^T \text{Hess}f(\Phi_t(x)) \dot{\Phi}_t(x) \\ &\leq -2\lambda' |\nabla f(\Phi_t(x))|^2 \\ &= -2\lambda' \left| \dot{\Phi}_t(x) \right|^2 \end{aligned}$$

and so Gronwall's inequality gives

$$\left| \dot{\Phi}_t(x) \right| \leq e^{-\lambda't} \left| \dot{\Phi}_0(x) \right|$$

for $t \in [0, T]$.

Furthermore,

$$\begin{aligned} \left| \dot{\Phi}_0(x) \right| &= |\nabla f(x)| \\ &\leq |\nabla f(\bar{x})| + |\nabla f(x) - \nabla f(\bar{x})| \\ &\leq |\nabla f(\bar{x})| + L_f |x - \bar{x}| \\ &= L_f |x - \bar{x}| \\ &< L_f \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)} \end{aligned}$$

so

$$(3.2.1) \quad \left| \dot{\Phi}_t(x) \right| < e^{-\lambda't} L_f \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)}$$

for $t \in [0, T]$

Now suppose for the sake of contradiction that there exists a time T such that $|\Phi_T(x) - \bar{x}| = \varepsilon$. Considering the hypothesis, this means that

$$\begin{aligned} \varepsilon - \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)} &< |\Phi_T(x) - x| \\ &\leq \int_0^T \left| \dot{\Phi}_t(x) \right| dt \end{aligned}$$

Then by (3.2.1),

$$\begin{aligned} \varepsilon - \frac{\varepsilon}{\left(\frac{L_f}{\lambda} + 1\right)} &< \int_0^T \left| \dot{\Phi}_t(x) \right| dt \\ &< \left(\int_0^T e^{-\lambda t} dt \right) L_f \frac{\varepsilon}{\left(\frac{L_f}{\lambda} + 1\right)} \\ &< \frac{L_f}{\lambda} \frac{\varepsilon}{\left(\frac{L_f}{\lambda} + 1\right)} \end{aligned}$$

so that

$$\varepsilon < \frac{\varepsilon}{\left(\frac{L_f}{\lambda} + 1\right)} \left(\frac{L_f}{\lambda} + 1 \right) = \varepsilon$$

a contradiction. So

$$|\Phi_t(x) - \bar{x}| < \varepsilon$$

and furthermore (3.2.1) holds for all time t . \square

THEOREM 12. *If \bar{x} is a steady-state such that for some $\lambda > 0$,*

$$\bar{v}^T \text{Hess } f(\bar{x}) \bar{v} \geq \lambda \bar{v}^T \bar{v}$$

for $\bar{v} \perp C(\bar{x})$, then there is some $\varepsilon' > 0$ such that for all x such that $|x - \bar{x}| < \varepsilon'$, a point

$$\hat{x} := \lim_{t \rightarrow \infty} \Phi_t(x) = \varphi_s(\bar{x})$$

exists, there is an $s \in \mathbb{R}$ such that

$$\hat{x} = \varphi_s(\bar{x})$$

(i.e. \hat{x} sits on a trajectory of \bar{x} along the vector field C), and there is a $\lambda' > 0$ such that

$$|\Phi_t(x) - \hat{x}| \leq e^{-\lambda' t} \frac{L_f \varepsilon'}{\lambda'}.$$

PROOF. Let ε and λ' come from Lemma 2.1. Then let

$$\begin{cases} \varepsilon' = \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)} & , \text{ if } \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)} < \frac{\lambda'}{c_f} \\ 0 < \varepsilon' < \frac{\lambda'}{c_f} & , \text{ if } \frac{\lambda'}{c_f} < \frac{\varepsilon}{\left(\frac{L_f}{\lambda'} + 1\right)} \end{cases}$$

which allows one to use the conclusions of Lemma 2.2.

The existence of \hat{x} comes from the fact that \mathbb{R}^d is complete and $\Phi_t(x)$ converges due to the Cauchy property since there is exponential decay of the velocity of $\Phi_t(x)$. To see this, fix $\eta > 0$, then for

$$\frac{\ln\left(\frac{\eta \lambda'}{L_f \varepsilon'}\right)}{-\lambda'} < s < t$$

one has

$$\begin{aligned}
|\Phi_t(x) - \Phi_s(x)| &\leq \int_s^t |\dot{\Phi}_r(x)| dr \\
&\leq \left(\int_s^t e^{-\lambda' r} dr \right) L_f \varepsilon' \\
&= \left(e^{-\lambda' s} - e^{-\lambda' t} \right) \frac{L_f \varepsilon'}{\lambda'} \\
&< e^{-\lambda' s} \frac{L_f \varepsilon'}{\lambda'} \\
&< \eta.
\end{aligned}$$

Further, this same decay implies exponential convergence to \hat{x} . To see that the rate is correct, note

$$\begin{aligned}
|\Phi_t(x) - \hat{x}| &\leq \int_t^\infty |\dot{\Phi}_s(x)| ds \\
&\leq \left(\int_t^\infty e^{-\lambda' s} ds \right) L_f \varepsilon' \\
&= e^{-\lambda' t} \frac{L_f \varepsilon'}{\lambda'}.
\end{aligned}$$

All that is left to see is that there is an $s \in \mathbb{R}$ such that

$$\hat{x} = \varphi_s(\bar{x})$$

(i.e. \hat{x} sits on a trajectory of \bar{x} along the vector field C). To show this, first define s^* such that

$$s^* := \arg \min_s |\varphi_s(\bar{x}) - \hat{x}|.$$

s^* is a critical point of $\frac{1}{2} |\varphi_s(\bar{x}) - \hat{x}|^2$, so it must solve the following condition:

$$0 = \frac{d}{ds} \left(\frac{1}{2} |\varphi_s(\bar{x}) - \hat{x}|^2 \right) = (\varphi_s(\bar{x}) - \hat{x}) \cdot \dot{\varphi}_s(\bar{x}) = (\varphi_s(\bar{x}) - \hat{x}) \cdot C(\varphi_s(\bar{x})).$$

Define

$$\bar{z} := \varphi_{s^*}(\bar{x})$$

then the above computation shows

$$\hat{x} - \bar{z} \perp C(\bar{z}).$$

Now by Taylor's theorem there exists an $r \in [0, 1]$ such that

$$|\nabla f(\hat{x})| = |\nabla f(\bar{z}) + \text{Hess}f((1-r)\hat{x} + r\bar{z})(\hat{x} - \bar{z})|$$

then since $\text{Hess}f$ is Lipschitz with constant c_f and $\nabla f(\varphi_{s^*}(\bar{x})) = 0$,

$$|\nabla f(\hat{x})| \geq |\text{Hess}f(\bar{z})(\hat{x} - \bar{z})| - c_f(1-r)|\hat{x} - \bar{z}|^2.$$

Now multiply the first summand on the right hand side by $\frac{|\hat{x}-\bar{z}|}{|\hat{x}-\bar{z}|}$ and apply the Cauchy-Schwartz inequality to show

$$\begin{aligned} |\nabla f(\hat{x})| &\geq \frac{|\hat{x}-\bar{z}|}{|\hat{x}-\bar{z}|} |\text{Hess}f(\bar{z})(\hat{x}-\bar{z})| - c_f(1-r)|\hat{x}-\bar{z}|^2 \\ &\geq \frac{1}{|\hat{x}-\bar{z}|} \left| (\hat{x}-\bar{z})^T \text{Hess}f(\bar{z})(\hat{x}-\bar{z}) \right| - c_f(1-r)|\hat{x}-\bar{z}|^2 \end{aligned}$$

and since before it was shown that $(\hat{x}-\bar{z}) \perp C(\bar{z})$ one can use the positive definiteness of the Hessian to show that

$$\begin{aligned} |\nabla f(\hat{x})| &\geq \frac{1}{|\hat{x}-\bar{z}|} \left| (\hat{x}-\bar{z})^T \text{Hess}f(\bar{z})(\hat{x}-\bar{z}) \right| - c_f(1-r)|\hat{x}-\bar{z}|^2 \\ &\geq \frac{1}{|\hat{x}-\bar{z}|} \lambda' |\hat{x}-\bar{z}|^2 - c_f(1-r)|\hat{x}-\bar{z}|^2 \\ &= \lambda' |\hat{x}-\bar{z}| - c_f(1-r)|\hat{x}-\bar{z}|^2. \end{aligned}$$

Now note by Lemma 2.2 that

$$|\hat{x}-\bar{z}| \leq |\hat{x}-\bar{x}| = \lim_{t \rightarrow \infty} |\Phi_t(x) - \bar{x}| \leq \varepsilon'$$

so

$$\begin{aligned} |\nabla f(\hat{x})| &\geq \lambda' |\hat{x}-\bar{z}| - c_f(1-r)|\hat{x}-\bar{z}|^2 \\ &\geq |\hat{x}-\bar{z}| (\lambda' - c_f(1-r)\varepsilon') \\ &\geq |\hat{x}-\bar{z}| (\lambda' - c_f\varepsilon'). \end{aligned}$$

Since ε' is chosen such that $\varepsilon' > \frac{\lambda'}{c_f}$,

$$\begin{aligned} |\nabla f(\hat{x})| &= 0 \\ \implies \\ \hat{x} &= \bar{z} \end{aligned}$$

and note that indeed by Lemma 2.2 that

$$0 \leq |\nabla f(\hat{x})| = \left| \nabla f \left(\lim_{t \rightarrow \infty} \Phi_t(x) \right) \right| = \left| \lim_{t \rightarrow \infty} \nabla f(\Phi_t(x)) \right| = \lim_{t \rightarrow \infty} \left| \dot{\Phi}_t(x) \right| \leq \lim_{t \rightarrow \infty} e^{-\lambda' t} L_f \varepsilon' = 0$$

so $\hat{x} = \bar{z} = \varphi_{s^*}(\bar{x})$, as was to be shown. \square

The Full Tangent Plane and Full Linear Stability in the Space of Probability Measures

4.1. Introduction to the Chapter

In chapter 3 section 2, the first lemma showed that having positive definiteness of the potential's Hessian along tangent vectors perpendicular to the energy's level curve going through the steady-state was enough to show positive definiteness of the Hessian along the gradient flow vectors at nearby states. In doing so, the gradient vector at points near the steady-state are pulled-back to vectors in the tangent plane of the steady-state. However, in infinite dimensions, it can be that all maps from the tangent plane at one state $\mu \in \mathcal{P}_2$ to another state $\bar{\mu} \in \mathcal{P}_2$ can be degenerate. For example, if $\bar{\mu}$ is a delta mass and μ is not, then no injective map from the tangent plane at μ to the tangent plane at $\bar{\mu}$ exists. To fix this problem, the goal of this chapter is to introduce the notion of "labeled tangent vectors". Along these a tool termed here "the full Hessian" will be computed, which allows one to check geodesic convexity even along geodesics that are not induced by velocity fields.

4.2. Admissible Vector Fields and Labeled Tangent Vectors

4.2.1. Admissible Velocity Vector Fields. Recall that for any configuration μ , rotations and translations of μ are level sets of the energy. However, the velocities on the configurations induced by the gradient flow are in fact non-translating and non-rotating, and thus it makes sense to restrict the analysis of the convexity of the energy at μ solely to those neighbors of μ that are not rotations or translations of it, thus restricting the velocity vector fields on which the Hessian will be calculated.

In particular, it is important to define which vector fields are "admissible vector fields at μ ", denoted by $v \in \mathcal{Adm}(\mu)$. These are the vector fields $v \in \mathcal{T}_\mu$ that are orthogonal to rotations and translations, the former requirement meaning that for any skew-symmetric matrix A

$$(4.2.1) \quad \int v(x)^T A x d\mu(x) = 0$$

and the latter requirement meaning that

$$(4.2.2) \quad \int v(x) d\mu(x) = 0.$$

This restriction is necessary because the interaction energy is invariant under rigid motions.

4.2.2. Labeled Tangent Vectors. Recall that in infinite dimensions, it can be that all maps from the tangent plane at one state $\mu \in \mathcal{P}_2$ to another state $\bar{\mu} \in \mathcal{P}_2$ are degenerate. This subsection will define “labeled vector fields”, which permits the analysis of the Hessian of the energy along gradient flow vector fields to be analyzed by pulling this vector fields back to a steady-state and using a generalization of the standard Hessian discussed in chapter 2, which will be called here the “full Hessian”.

Labeled vector fields are necessary because sometimes there are no transportation maps between two measures; for example there are no transportation maps that push the support of a single delta mass forward to the support of two delta masses. So to introduce this notion of labeled vector fields with an example: suppose one wanted to describe the McCann displacement interpolation starting at $\mu = \delta_0$ and ending at $\nu = \frac{1}{3}\delta_A + \frac{2}{3}\delta_B$ for two disjoint points $A, B \in \mathbb{R}^d$ as a “pushforward” of the measure μ . The delta mass would need to split into two distinct pieces to be pushed forward to A and B . To describe this “pushforward of μ ” it is necessary to keep track of how the mass is split as well as the trajectory of each piece. So for this example define the *splitting plan* π , a product measure where $\pi(\{0\}, \{A\}) = \frac{1}{3}$ and $\pi(\{0\}, \{B\}) = \frac{2}{3}$. And define

$$\begin{aligned}\Phi_t(0, A) &= tA \\ \Phi_t(0, B) &= tB.\end{aligned}$$

Then $\Phi_{t\#}\pi$ is just McCann’s displacement interpolation between μ and ν .

Here are the concepts used to talk about labeled vector fields more generally:

A set of labels S -which are labels among which mass can be split- which has an associated measurable space (S, \mathcal{S}) called the “label space”. (In the previous example $S = \{A, B\}$ and \mathcal{S} is the discrete σ -algebra.)

Then, to describe how mass in a measure μ is to be split among the labels, one needs:

A “splitting plan for μ (associated with a label space S)” π is a product probability measure on $\mathbb{R}^d \times S$ with μ as its first marginal. Note that the Disintegration Theorem implies that for μ -a.e. x there is a probability measure $\pi_x(s)$ such that $d\pi(x, s)$ can be decomposed into $d\pi_x(s) d\mu(x)$. Here π_x is a probability measure on the label space, and describes how the mass of μ at x is split between the labels in S . π could be defined conversely from π_x .

Using a splitting plan, then one can define a “labeled tangent vectors at μ ” which is a pairing $(v(x, s), \pi)$ where $v : \mathbb{R}^d \times S \rightarrow \mathbb{R}^d$.

The set is called the “full tangent plane at μ ”, defined as

$$(4.2.3) \quad \mathcal{FT}_\mu : = \left\{ (v, \pi) \mid \pi \text{ is a splitting plan for } \mu \right.$$

$$(4.2.4) \quad \left. \text{associated with any label space } S \right.$$

$$(4.2.5) \quad \left. \text{and } v(x, s) : \mathbb{R}^d \times S \rightarrow \mathbb{R}^d, v \in L^2(d\pi) \right\}.$$

REMARK 13. Note that the full tangent plane contains the usual tangent plane in it. For example, if the label space S consists of only a single element s , then for all $v(x) \in \mathcal{T}_\mu$, one can define $v(x, s) = v(x)$ and define π for Borel sets $X \subset \mathbb{R}^d$ such that $\pi(X, \{s\}) = 0$ and $\pi(X, \{s\}) = \mu(X)$ so that (v, π) is in the full tangent plane of μ , and $v(x) = v(x, s)$ is in the usual tangent plane at μ .

4.3. The Full Hessian

This section discusses a way to estimate the Hessian along a gradient vector field by evaluating an expression at a nearby steady-state. Recall for this section all the assumptions and definitions in chapter 1 section 7 hold.

4.3.1. Decomposing the Usual Hessian Using Labeled Tangent Vectors. Given a measure $\mu \in \mathcal{P}_2$ and a vector field $v(x) \in \mathcal{T}_\mu$ admissible in the sense of satisfying (4.2.1) and (4.2.2), recall from (1.5.4) that

$$(4.3.1) \quad \text{Hess}\mathcal{W}_\mu[v, v] = \int \int (v(x) - v(y))^T \text{Hess}W(x - y) (v(x) - v(y)) d\mu(x) d\mu(y).$$

Now when there is a steady-state $\bar{\mu} \in \mathcal{P}_2$ that is near to μ in the ∞ -Wasserstein sense, there is a labeled tangent vector $(v(x, \bar{x}), \pi) \in \mathcal{FT}_\mu$ that is naturally associated with $v(x) \in \mathcal{T}_\mu$. Namely

$$(4.3.2) \quad \begin{aligned} v(x, \bar{x}) &= v(x) \\ \pi &\in \Gamma_{opt, \infty}(\mu, \bar{\mu}). \end{aligned}$$

This labeled tangent vector can then be decomposed to give a “pullback” of the tangent vector $v(x) \in \mathcal{T}_\mu$ and a remainder labeled tangent vector that represents the spreading of the mass at $\bar{\mu}$. Namely define $\bar{v} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, the pullback of $v(x) \in \mathcal{T}_\mu$, using the disintegration $d\pi_{\bar{x}}$ defined by $d\pi(x, \bar{x}) = d\pi_{\bar{x}}(x) d\bar{\mu}(x)$, as

$$(4.3.3) \quad \bar{v}(\bar{x}) = \int v(x, \bar{x}) d\pi_{\bar{x}}(x)$$

and the remainder labeled tangent vector representing spreading is $(\tilde{v}(x, \bar{x}), \pi) \in \mathcal{FT}_\mu$ where \tilde{v} is defined as

$$(4.3.4) \quad \tilde{v}(x, \bar{x}) = v(x, \bar{x}) - \bar{v}(\bar{x}).$$

Note combining (4.3.2), (4.3.3) and (4.3.4) gives the pointwise decomposition of $v(x)$ as

$$(4.3.5) \quad v(x) = v(x, \bar{x}) = \bar{v}(\bar{x}) + \tilde{v}(x, \bar{x}).$$

Plugging this decomposition into (4.3.1) gives a useful decomposition of the Hessian as well:

$$\begin{aligned} \text{Hess}W_\mu[v(x), v(x)] &= \int \int (\bar{v}(\bar{x}) + \tilde{v}(x, \bar{x}) - \bar{v}(\bar{y}) - \tilde{v}(y, \bar{y}))^T \text{Hess}W(x-y) \\ &\quad (\bar{v}(\bar{x}) + \tilde{v}(x, \bar{x}) - \bar{v}(\bar{y}) - \tilde{v}(y, \bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}). \end{aligned}$$

Expanding this expression gives

$$\begin{aligned} &= \int \int (\bar{v}(\bar{x}) - \bar{v}(\bar{y}))^T \text{Hess}W(x-y) (\bar{v}(\bar{x}) - \bar{v}(\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad + 2 \int \int (\bar{v}(\bar{x}) - \bar{v}(\bar{y}))^T \text{Hess}W(x-y) (\tilde{v}(x, \bar{x}) - \tilde{v}(y, \bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad + \int \int (\tilde{v}(x, \bar{x}) - \tilde{v}(y, \bar{y}))^T \text{Hess}W(x-y) (\tilde{v}(x, \bar{x}) - \tilde{v}(y, \bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \end{aligned}$$

Further note that by definition,

$$(4.3.6) \quad \int \tilde{v}(x, \bar{x}) d\pi_{\bar{x}}(x) = 0$$

so these computations show the Hessian in (4.3.1) decomposes into:

$$(4.3.7) \quad \begin{aligned} &\int \int (\bar{v}(\bar{x}) - \bar{v}(\bar{y}))^T \text{Hess}W(x-y) (\bar{v}(\bar{x}) - \bar{v}(\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad + 2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(x-y) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \end{aligned}$$

4.3.2. Estimating the Hessian at μ by Expressions Evaluated at the Steady-State $\bar{\mu}$. Recall in the previous subsection, $\bar{\mu} \in \mathcal{P}_2$ denoted a steady-state of the evolution and μ is near to it in the ∞ -Wasserstein sense such that $d_\infty(\bar{\mu}, \mu) < \varepsilon$ for some $\varepsilon > 0$. Furthermore, recall that $\text{Hess}W$ is Lipschitz with Lipschitz constant c_W . This allows the Hessian decomposition (4.3.7) to be estimated below such that

$$\begin{aligned} &\int \int (\bar{v}(\bar{x}) - \bar{v}(\bar{y}))^T \text{Hess}W(x-y) (\bar{v}(\bar{x}) - \bar{v}(\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad + 2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(x-y) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \end{aligned}$$

$$\begin{aligned}
&\geq \int \int (\bar{v}(\bar{x}) - \bar{v}(\bar{y}))^T \text{Hess}W(\bar{x} - \bar{y}) (\bar{v}(\bar{x}) - \bar{v}(\bar{y})) d\bar{\mu}(\bar{x}) d\bar{\mu}(\bar{y}) \\
&\quad + 2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(\bar{x} - \bar{y}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\bar{\mu}(\bar{y}) \\
(4.3.8) \quad &\quad - 2c_W \varepsilon \int |\bar{v}(\bar{x})|^2 d\bar{\mu}(\bar{x}) - 2c_W \varepsilon \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}).
\end{aligned}$$

4.3.3. The Controlled Pullback Condition. Now part of the goal of this chapter is to discuss the positive definiteness of the Hessian, but as mentioned in Chapter 4 Section 2, this can only happen on admissible vector fields that satisfy (4.2.1) and (4.2.2), and one issue that has not been addressed yet is whether the pullback, \bar{v} , is admissible in this sense.

To see that (4.2.2) is satisfied for \bar{v} is a straightforward computation using the definition of \bar{v} , (4.3.3), and the fact that $v \in \mathcal{T}_\mu$ is admissible. Compute that

$$\begin{aligned}
\int \bar{v}(\bar{x}) d\bar{\mu}(\bar{x}) &= \int \left(\int v(x, \bar{x}) d\pi_{\bar{x}}(x) \right) d\bar{\mu}(\bar{x}) \\
&= \int v(x, \bar{x}) d\pi(x, \bar{x}) \\
&= \int v(x) d\mu(x) \\
(4.3.9) \quad &= 0.
\end{aligned}$$

However, in general \bar{v} does not satisfy (4.2.1). However, \bar{v} can be decomposed into the sum

$$(4.3.10) \quad \bar{v} = \bar{v}_{\mathcal{R}} + \bar{v}_{\mathcal{R}^\perp}$$

where $\bar{v}_{\mathcal{R}}$ is the rotational part of \bar{v} , i.e.

$$\bar{v}_{\mathcal{R}}(x) = \arg \max_{Ax, A \in so(d)} \int Ax \cdot \bar{v}(x).$$

and $\bar{v}_{\mathcal{R}^\perp}$ is orthogonal to the rotational part so that it satisfies (4.2.1). Previous works have circumvented this issue by restricting attention to a limited class of admissible perturbations. In [29, 31] the authors characterized stability for particle steady-states for the 1-D system. In [56] the authors studied how certain instabilities of the d -sphere lead to evolution towards “soccer ball” patterns that depend on the mode of the instabilities of the Fourier transform of the perturbations on the ball.

The Hessian acting on the rotational component $\bar{v}_{\mathcal{R}}$ is zero, i.e.

$$(4.3.11) \quad \text{Hess}\mathcal{W}_\mu[\bar{v}, \bar{v}] = \text{Hess}\mathcal{W}_\mu[\bar{v}_{\mathcal{R}^\perp}, \bar{v}_{\mathcal{R}^\perp}]$$

but when $\bar{\mu}$ is a finite particle steady-state, i.e.

$$(4.3.12) \quad \bar{\mu} = \sum_{i=1}^N m_i \delta_{\bar{x}_i}$$

for N particles with positions $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N\}$ and masses $\{m_1, m_2, \dots, m_N\}$ such that

$$(4.3.13) \quad \sum_{i=1}^N m_i = 1$$

then the magnitude of $\bar{v}_{\mathcal{R}}$ can be controlled. Furthermore in this case $\bar{v}_{\mathcal{R}^\perp} \in \mathcal{T}_{\bar{\mu}}$ since it is orthogonal to rotations and a function $\phi \in C_c^\infty$ can be found such that at $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N\}$, $\nabla \phi(x_i) = \bar{v}_{\mathcal{R}^\perp}$.

One says that the particles are in “general position” if $\{x_1, \dots, x_N\}$ are such that there exists a constant $c > 0$ (depending on $\bar{\mu}$) such that for all $A \in so(d)$ (the skew-symmetric matrices) the inequality

$$(4.3.14) \quad \max_{i \in \{1, 2, \dots, N\}} |A \bar{x}_i| \geq c \|A\|$$

holds. Then the magnitude of $v_{\mathcal{R}}$ can be controlled in the following way:

THEOREM 14. (*The Controlled Pullback Condition*) *Let $\mu \in \mathcal{P}_2$ be near the finite particle state $\bar{\mu} \in \mathcal{P}_2$, in the sense that $d_\infty(\mu, \bar{\mu}) < \varepsilon$, where $\bar{\mu}$ is defined by (4.3.12) and is in general position as defined by (4.3.14). Let $c > 0$ be defined by (4.3.14) and let $m = \min\{m_1, m_2, \dots, m_N\}$. Let $v \in \mathcal{T}_\mu$, and let $\bar{v} \in \mathcal{T}_{\bar{\mu}}$ be the pullback of v defined by equations (4.3.2) and (4.3.3). Then, defining $\bar{v}_{\mathcal{R}}$ as the rotational component in the decomposition of \bar{v} defined by (4.3.10), and where g_μ and $g_{\bar{\mu}}$ are the Riemannian metrics defined by (1.3.2), the following holds:*

$$(4.3.15) \quad \int |\bar{v}_{\mathcal{R}}(x)|^2 d\bar{\mu}(x) \leq \frac{1}{cm} g_\mu(v, v) \varepsilon^2$$

PROOF. Since $\bar{v}_{\mathcal{R}}$ is a rotational vector field, there exists a skew-symmetric matrix $A \in so(d)$ (that depends on μ) such that

$$\bar{v}_{\mathcal{R}}(\bar{x}_i) = A \bar{x}_i.$$

for all $i \in \{1, 2, \dots, N\}$, so one can take

$$(4.3.16) \quad \bar{v}_{\mathcal{R}}(\bar{x}) = A \bar{x}, \text{ for } \bar{x} \in \text{supp } \bar{\mu}.$$

Now $g_{\bar{\mu}}(\bar{v}_{\mathcal{R}}, \bar{v}_{\mathcal{R}})$ can be bounded below by

$$g_{\bar{\mu}}(\bar{v}_{\mathcal{R}}, \bar{v}_{\mathcal{R}}) = \int \bar{v}_{\mathcal{R}} \cdot \bar{v}_{\mathcal{R}} d\bar{\mu} = \sum_{i=1}^N m_i |A \bar{x}_i|^2 \geq m c^2 \|A\|^2$$

so that

$$(4.3.17) \quad \|A\| \leq \left(\frac{1}{mc} g_{\bar{\mu}}(\bar{v}_{\mathcal{R}}, \bar{v}_{\mathcal{R}}) \right)^{\frac{1}{2}}.$$

Note that since $v(x) \in \mathcal{T}_{\mu}$ is admissible in the sense of (4.2.1),

$$\int v(x) \cdot A x d\mu(x) = 0$$

so

$$\begin{aligned} \int |\bar{v}_{\mathcal{R}}(x)|^2 d\bar{\mu}(x) &= \int \bar{v}_{\mathcal{R}}(x) \cdot \bar{v}(x) d\bar{\mu}(x) \\ &= \int \bar{v}_{\mathcal{R}}(x) \cdot \bar{v}(x) d\bar{\mu}(x) - \int v(x) \cdot A x d\mu(x) \\ &= \int \left(\int v(x, \bar{x}) d\pi_{\bar{x}}(x) \right) \cdot A \bar{x} d\bar{\mu}(\bar{x}) - \int v(x) \cdot A x d\mu(x) \\ &= \int v(x) \cdot A(\bar{x} - x) d\pi(x, \bar{x}) \\ &\leq \left(\int |v(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int |A(\bar{x} - x)|^2 d\pi(x, \bar{x}) \right)^{\frac{1}{2}} \\ &\leq \left(\int |v(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \|A\| \left(\int |(\bar{x} - x)|^2 d\pi(x, \bar{x}) \right)^{\frac{1}{2}} \\ &\leq g_{\mu}(v, v)^{\frac{1}{2}} \left(\frac{1}{mc} \int |\bar{v}_{\mathcal{R}}(x)|^2 d\bar{\mu}(x) \right)^{\frac{1}{2}} \varepsilon. \end{aligned}$$

This gives the controlled pullback condition which was to be shown. \square

4.3.4. Full Linear Stability. The controlled pullback condition allows some meaningful positive definiteness conditions to be shown for $\text{Hess}\mathcal{W}_{\mu}[v, v]$ for admissible v . Indeed, as will be shown in a moment, part of bounding $\text{Hess}\mathcal{W}_{\mu}[v, v]$ below will only require checking the positive definiteness of $\text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}_{\mathcal{R}^{\perp}}, \bar{v}_{\mathcal{R}^{\perp}}]$. Before showing this in its entirety, one more lemma is required.

LEMMA 15. (*Characterization of Spreading Stability*) Let $\mu \in \mathcal{P}_2$ and $\bar{\mu} \in \mathcal{P}_2$. For some $v \in \mathcal{T}_{\mu}$, let $(\tilde{v}(x, \bar{x}), \pi) \in \mathcal{FT}_{\mu}$, where $\pi \in \Gamma_{\text{opt}, \infty}(\mu, \bar{\mu})$, be defined by (4.3.2) and (4.3.4). Then for $\lambda > 0$,

$$2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(\bar{x} - \bar{y}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\bar{\mu}(\bar{y}) \geq \lambda \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x})$$

if, and only if,

$$(4.3.18) \quad \min \text{eigenvalue of } \text{Hess}W * \bar{\mu}(\bar{x}) \geq \frac{\lambda}{2}$$

for $\bar{\mu}$ -a.e. \bar{x} .

PROOF. Since

$$2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(\bar{x} - \bar{y}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\bar{\mu}(\bar{y}) = 2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W * \bar{\mu}(\bar{x}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x})$$

sufficiency is obvious. To show necessity, assume for the sake of contradiction that the statement is false, i.e. that

$$2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W * \bar{\mu}(\bar{x}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) \geq \lambda \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x})$$

but there is a set A of positive measure where for $\bar{x} \in A$

$$\text{min eigenvalue of Hess}W * \bar{\mu}(\bar{x}) < \frac{\lambda}{2}$$

and denote the corresponding eigenvectors as $w(\bar{x})$.

Now a transport plan will be constructed that leads to the contradiction. Fix $\varepsilon > 0$ and define the maps

$$\begin{aligned} \varphi_{1,\varepsilon}(\bar{z}) &:= \begin{cases} \bar{z} & \text{if } \bar{z} \notin A \\ \bar{z} + \frac{\varepsilon}{2} \frac{w(\bar{z})}{|w(\bar{z})|} & \text{if } \bar{z} \in A \end{cases} \\ \varphi_{2,\varepsilon}(\bar{z}) &:= \begin{cases} \bar{z} & \text{if } \bar{z} \notin A \\ \bar{z} - \frac{\varepsilon}{2} \frac{w(\bar{z})}{|w(\bar{z})|} & \text{if } \bar{z} \in A \end{cases} \end{aligned}$$

and use these to define the transport plan

$$\pi = (\varphi_{1,\varepsilon} \times Id)_{\#} \left(\frac{1}{2} \bar{\mu} \right) + (\varphi_{2,\varepsilon} \times Id)_{\#} \left(\frac{1}{2} \bar{\mu} \right).$$

Note that $\|x - \bar{x}\|_{L^\infty(d\pi(x, \bar{x}))} < \varepsilon$.

Also, define

$$\tilde{v}(z, \bar{z}) := \begin{cases} \frac{w(\bar{z})}{|w(\bar{z})|} & \text{if } (z, \bar{z}) \in (\varphi_{1,\varepsilon} \times Id)(A) \\ -\frac{w(\bar{z})}{|w(\bar{z})|} & \text{if } (z, \bar{z}) \in (\varphi_{2,\varepsilon} \times Id)(A) \\ 0 & \text{else} \end{cases}$$

this ensures for all $\bar{\mu}$ -a.e. \bar{z} that

$$\begin{aligned} \int \tilde{v}(z, \bar{z}) d\pi_{\bar{z}}(z) &= \frac{1}{2} \int \tilde{v}(z, \bar{z}) d((\varphi_{1,\varepsilon})_{\#} \bar{\mu})(z) + \frac{1}{2} \int \tilde{v}(z, \bar{z}) d((\varphi_{2,\varepsilon})_{\#} \bar{\mu})(z) \\ &= \frac{1}{2} \frac{w(\bar{z})}{|w(\bar{z})|} - \frac{1}{2} \frac{w(\bar{z})}{|w(\bar{z})|} \\ &= 0. \end{aligned}$$

Now using \tilde{v} and π one sees

$$\frac{\lambda}{2} \bar{\mu}(A) = \frac{\lambda}{2} \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}) \leq \int \tilde{v}(x, \bar{x})^T \text{Hess}W * \bar{\mu}(\bar{x}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) =$$

$$= \int_{x \in A} \frac{w(\bar{x})}{|w(\bar{x})|} \text{Hess}W * \bar{\mu}(\bar{x}) \frac{w(\bar{x})}{|w(\bar{x})|} d\pi(x, \bar{x}) < \frac{\lambda}{2} \bar{\mu}(A)$$

where the last inequality comes from the assumption for the sake of contradiction. Thus the statement is proven. \square

These statements finally motivate the two conditions which together is called here “full Linear stability (with constant $\lambda > 0$)”:

One says the steady-state $\bar{\mu}$ is a “**displacement stable** (with constant $\lambda > 0$) configuration” of the interaction energy if for $\bar{v}_{\mathcal{R}^\perp} \in \mathcal{T}_{\bar{\mu}}$ that are admissible (i.e. satisfy (4.2.1) and (4.2.2)) when

$$(4.3.19) \quad \int \int (\bar{v}_{\mathcal{R}^\perp}(\bar{x}) - \bar{v}_{\mathcal{R}^\perp}(\bar{y}))^T \text{Hess}W(\bar{x} - \bar{y}) (\bar{v}_{\mathcal{R}^\perp}(\bar{x}) - \bar{v}_{\mathcal{R}^\perp}(\bar{y})) d\bar{\mu}(\bar{x}) d\bar{\mu}(\bar{y}) \geq \lambda \int |\bar{v}_{\mathcal{R}^\perp}(\bar{x})|^2 d\bar{\mu}(\bar{x}).$$

or equivalently

$$\text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}_{\mathcal{R}^\perp}, \bar{v}_{\mathcal{R}^\perp}] \geq \lambda g_{\bar{\mu}}(\bar{v}_{\mathcal{R}^\perp}, \bar{v}_{\mathcal{R}^\perp}).$$

One says the steady-state $\bar{\mu}$ is a “**spreading stable** (with constant λ) configuration” of the interaction energy if

$$(4.3.20) \quad \min \text{eigenvalue of } \text{Hess}W * \bar{\mu}(\bar{x}) \geq \frac{\lambda}{2}.$$

THEOREM 16. (*Full Linear Stability at the steady-state implies positivity of the Hessian along admissible vector fields at nearby configurations*) Let $\bar{\mu} \in \mathcal{P}_2$ be a finite particle state as defined by (4.3.12) such that $\bar{\mu}$ is fully linear stable with constant $\lambda > 0$, meaning that both (4.3.19) and (4.3.20) hold with constant $\lambda > 0$. Let $c > 0$ be defined by (4.3.14) and let $m = \min\{m_1, m_2, \dots, m_N\}$.

Then for any $\mu \in \mathcal{P}_2$ such that $d_\infty(\mu, \bar{\mu}) < \varepsilon$, and for all $v \in \mathcal{T}_\mu$ that are admissible in the sense of (4.2.1) and (4.2.2), it holds that

$$(4.3.21) \quad \text{Hess}\mathcal{W}_\mu[v, v] \geq \left(\lambda - 2c_W\varepsilon - \frac{\lambda}{cm}\varepsilon^2 \right) g_\mu(v, v)$$

Furthermore, for any $r \in (0, 1)$ if $\varepsilon > 0$ is such that

$$\varepsilon < -\frac{c_W cm}{\lambda} + \sqrt{\left(\frac{c_W cm}{\lambda}\right)^2 + (1-r)cm}$$

then

$$(4.3.22) \quad \text{Hess}\mathcal{W}_\mu[v, v] \geq r\lambda g_\mu(v, v).$$

PROOF. Recall that given the transportation plan $\pi \in \Gamma_{opt, \infty}(\mu, \bar{\mu})$ that $v(x) \in \mathcal{T}_\mu$ can be decomposed into the pullback $\bar{v} \in \mathcal{T}_{\bar{\mu}}$ and the labeled tangent vector called the spreading remainder $(\tilde{v}(x, \bar{x}), \pi)$ as defined by (4.3.2), (4.3.3), (4.3.4) and (4.3.5). Then using the definition

of the Hessian (1.5.4) and the lower bound (4.3.8) it holds that

$$\begin{aligned}
\text{Hess}\mathcal{W}_\mu[v, v] &\geq \text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}, \bar{v}] \\
&\quad + 2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(\bar{x} - \bar{y}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\bar{\mu}(\bar{y}) \\
(4.3.23) \quad &\quad - 2c_W\varepsilon \int |\bar{v}(\bar{x})|^2 d\bar{\mu}(\bar{x}) - 2c_W\varepsilon \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}).
\end{aligned}$$

Note that (4.3.6) implies that

$$\int \bar{v}(\bar{x}) \cdot \tilde{v}(x, \bar{x}) d\pi = 0$$

and thus by (4.3.5) one sees

$$\begin{aligned}
\int |\bar{v}(\bar{x})|^2 d\bar{\mu}(\bar{x}) + \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}) &= \int |\bar{v}(\bar{x}) + \tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}) \\
(4.3.24) \quad &= \int |v(x)|^2 d\mu(x)
\end{aligned}$$

so (4.3.23) can be rewritten as

$$\begin{aligned}
\text{Hess}\mathcal{W}_\mu[v, v] &\geq \text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}, \bar{v}] \\
&\quad + 2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(\bar{x} - \bar{y}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\bar{\mu}(\bar{y}) \\
(4.3.25) \quad &\quad - 2c_W\varepsilon \int |v(x)|^2 d\mu(x).
\end{aligned}$$

Furthermore the spreading stability condition (4.3.20) and Lemma 6 show that

$$2 \int \int \tilde{v}(x, \bar{x})^T \text{Hess}W(\bar{x} - \bar{y}) \tilde{v}(x, \bar{x}) d\pi(x, \bar{x}) d\bar{\mu}(\bar{y}) \geq \lambda \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}),$$

and recall (4.3.11) that

$$\text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}, \bar{v}] = \text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}_{\mathcal{R}^\perp}, \bar{v}_{\mathcal{R}^\perp}]$$

so with (4.3.25) these imply that

$$\begin{aligned}
\text{Hess}\mathcal{W}_\mu[v, v] &\geq \text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}_{\mathcal{R}^\perp}, \bar{v}_{\mathcal{R}^\perp}] \\
&\quad + \lambda \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}) \\
(4.3.26) \quad &\quad - 2c_W\varepsilon \int |v(x)|^2 d\mu(x).
\end{aligned}$$

and since $\bar{v}_{\mathcal{R}^\perp} \in \mathcal{T}_{\bar{\mu}}$ is admissible in the sense of (4.2.1) and (4.2.2) then the displacement stability condition (4.3.19) says that

$$\text{Hess}\mathcal{W}_{\bar{\mu}}[\bar{v}_{\mathcal{R}^\perp}, \bar{v}_{\mathcal{R}^\perp}] \geq \lambda \int |\bar{v}_{\mathcal{R}^\perp}(\bar{x})|^2 d\bar{\mu}(\bar{x})$$

so that

$$\begin{aligned}
\text{Hess}\mathcal{W}_\mu[v, v] &\geq \lambda \int |\bar{v}_{\mathcal{R}^\perp}(\bar{x})|^2 d\bar{\mu}(\bar{x}) \\
&\quad + \lambda \int |\tilde{v}(x, \bar{x})|^2 d\pi(x, \bar{x}) \\
(4.3.27) \quad &\quad - 2c_W\varepsilon \int |v(x)|^2 d\mu(x).
\end{aligned}$$

Now add and subtract to the top term in the following way,

$$\begin{aligned}
\lambda \int |\bar{v}_{\mathcal{R}^\perp}(\bar{x})|^2 d\bar{\mu}(\bar{x}) &= \lambda \int |\bar{v}_{\mathcal{R}^\perp}(\bar{x})|^2 d\bar{\mu}(\bar{x}) + \lambda \int |\bar{v}_{\mathcal{R}}(\bar{x})|^2 d\bar{\mu}(\bar{x}) \\
&\quad - \lambda \int |\bar{v}_{\mathcal{R}}(\bar{x})|^2 d\bar{\mu}(\bar{x}) \\
&= \lambda \int |\bar{v}(\bar{x})|^2 d\bar{\mu}(\bar{x}) - \lambda \int |\bar{v}_{\mathcal{R}}(\bar{x})|^2 d\bar{\mu}(\bar{x})
\end{aligned}$$

so using the identity (4.3.24) and the Riemannian metric notations (4.3.27) becomes

$$(4.3.28) \quad \text{Hess}\mathcal{W}_\mu[v, v] \geq (\lambda - 2c_W\varepsilon) g_\mu(v, v) - \lambda g_{\bar{\mu}}(v_{\mathcal{R}}, v_{\mathcal{R}}).$$

But recall that the controlled pullback condition (4.3.15) gives

$$g_{\bar{\mu}}(v_{\mathcal{R}}, v_{\mathcal{R}}) \leq \frac{1}{cm} g_\mu(v, v) \varepsilon^2$$

so that

$$(4.3.29) \quad \text{Hess}\mathcal{W}_\mu[v, v] \geq \left(\lambda - 2c_W\varepsilon - \frac{\lambda}{cm} \varepsilon^2 \right) g_\mu(v, v).$$

Furthermore, for any $r \in (0, 1)$ the $\varepsilon > 0$ such that

$$\left(\lambda - 2c_W\varepsilon - \frac{\lambda}{cm} \varepsilon^2 \right) > r\lambda$$

are, using the quadratic formula and simplifying, those such that

$$\varepsilon < -\frac{c_W cm}{\lambda} + \sqrt{\left(\frac{c_W cm}{\lambda}\right)^2 + (1-r)cm}$$

so if ε satisfies this inequality then

$$\text{Hess}\mathcal{W}_\mu[v, v] \geq r\lambda g_\mu(v, v).$$

□

Note that the following corollary immediately proceeds from the preceding:

COROLLARY 17. *If for all $t \in [0, T]$, the hypothesis of the previous theorem hold, i.e, that $d_\infty(\mu_t, \bar{\mu}) \leq \varepsilon$ for some sufficiently small $0 < \varepsilon < -\frac{c_W cm}{\lambda} + \sqrt{\left(\frac{c_W cm}{\lambda}\right)^2 + \frac{1}{2}cm}$, and $\bar{\mu}$ is fully*

linear stable with constant $\lambda > 0$. Then

$$\int |v_{\mu_t}(x)|^2 d\mu_t(x) \leq e^{-\frac{\lambda}{2}t} \int |v_{\mu_0}(x)|^2 d\mu_0(x)$$

for all $t \in [0, T]$.

PROOF. Note that

$$\begin{aligned} \frac{d\mathcal{W}[\mu_t]}{dt} &= - \int |v_{\mu_t}(x)|^2 d\mu_t(x) \\ &= -g_{\mu_t}(v_{\mu_t}, v_{\mu_t}) \end{aligned}$$

and

$$\begin{aligned} \frac{d^2\mathcal{W}[\mu_t]}{dt^2} &= \int \int (v_{\mu_t}(x) - v_{\mu_t}(y))^T \text{Hess}W(x - y) (v_{\mu_t}(x) - v_{\mu_t}(y)) d\mu_t(x) d\mu_t(y) \\ &= \text{Hess}\mathcal{W}_{\mu_t}[v_{\mu_t}, v_{\mu_t}]. \end{aligned}$$

Thus (for the given ε) the previous theorem shows that for all $t \in [0, T]$

$$-\frac{d(g_{\mu_t}(v_{\mu_t}, v_{\mu_t}))}{dt} \geq \frac{\lambda}{2} g_{\mu_t}(v_{\mu_t}, v_{\mu_t})$$

so by Gronwall's inequality it holds for all $t \in [0, T]$ that

$$g_{\mu_t}(v_{\mu_t}, v_{\mu_t}) \leq e^{-\frac{\lambda}{2}t} g_{\mu_0}(v_{\mu_0}, v_{\mu_0})$$

otherwise written as

$$\int |v_{\mu_t}(x)|^2 d\mu_t(x) \leq e^{-\frac{\lambda}{2}t} \int |v_{\mu_0}(x)|^2 d\mu_0(x)$$

which was to be shown. □

Proof of the Stability Result

5.1. Introduction to the 2-Wasserstein Projection

Let $\bar{\mu}$ be a steady-state of the evolution, then one can define the finite dimensional manifold \mathcal{M} which contains all rotations of $\bar{\mu}$.

$$(5.1.1) \quad \mathcal{M} = \{O_{\#}\bar{\mu} : O \in SO(d)\}.$$

If the steady-state is full linear stable, then one would expect convergence of the gradient flow evolution to some element of \mathcal{M} , which is the level set of the energy containing the steady-state, analogous to the finite dimensional case in chapter 3. As the gradient flow μ_t evolves in time, the member of \mathcal{M} that is closest in the 2-Wasserstein sense changes as well. Recall that the full linear stability of the steady-state only implies positive definiteness of the Hessian along gradient vector fields at measures that are close in the ∞ -Wasserstein sense. Thus it is important to see how the member of \mathcal{M} closest to μ_t moves in time too, so as to guarantee that ∞ -Wasserstein bound between μ_t and $\bar{\mu}$ holds for all time as well.

The discussion starts by noting that for each measure μ , there exists a minimizer of $d_2(\mu, \cdot)$.

LEMMA 18. *Assuming all the assumptions and definitions of 1.7, let $\bar{\mu}$ be a steady-state of the gradient flow evolution, let μ be another measure, then there exists a measure σ such that*

$$\sigma \in \arg \min_{\nu \in \mathcal{M}} d_2(\nu, \mu).$$

PROOF. The proof proceeds by the direct method of calculus of variations. Let $\alpha \geq 0$ be such that

$$\alpha := \inf_{\nu \in \{O_{\#}\bar{\mu} : O \in SO(d)\}} d_2(\nu, \mu).$$

Let $\{\sigma_n\}_{n \in \mathbb{N}} \subset \{O_{\#}\bar{\mu} : O \in SO(d)\}$ be a (monotonically) minimizing sequence of the above infimum. Then there exist $\{O_n\}_{n \in \mathbb{N}} \subset SO(d)$ such that $\sigma_n = O_n\bar{\mu}$. Since $SO(d)$ is a compact metric space, it is sequentially compact, and thus $\{O_n\}$ has a convergent subsequence $\{O_{n_j}\}_{n_j \in \mathbb{N}}$ that converges to an element $\bar{O} \in SO(d)$. Now

$$\alpha \leq d_2(\bar{O}_{\#}\bar{\mu}, \mu) \leq \lim_{j \rightarrow \infty} d_2(\bar{O}_{n_j\#}\bar{\mu}, \mu) = \alpha$$

and $\sigma := \bar{O}_{\#}\bar{\mu} \in \{O_{\#}\bar{\mu} : O \in SO(d)\}$, showing this set attains a minimum. \square

There may indeed be more than one minimizer of $d_2(\mu, \cdot)$, so define a “local minimizer” as any measure σ such that there is a $\delta > 0$ such that for all $\nu \in \mathcal{M}$ such that $d_2(\sigma, \nu) < \delta$ then

$$(5.1.2) \quad d_2(\sigma, \mu) \leq d_2(\nu, \mu).$$

Around each measure $\nu \in \mathcal{M}$, there exists a useful neighborhood to define, $U_\varepsilon(\nu)$, which is

$$(5.1.3) \quad U_\varepsilon(\nu) = \{e_{\#}^A \nu \mid A^T = -A, \|A\| < \varepsilon\}$$

whose closure is

$$(5.1.4) \quad \overline{U_\varepsilon(\nu)} = \{e_{\#}^A \nu \mid A^T = -A, \|A\| \leq \varepsilon\}.$$

Note that for A skew symmetric as here, e^A is an orthogonal matrix such that $\det e^A = 1$, i.e. it is a rotation matrix on \mathcal{M} .

The following lemma allows one to establish the existence of a local minimizer by checking $d_2(\mu, \cdot)$ at a measure $\nu \in \mathcal{M}$ and $\partial U_\varepsilon(\nu)$.

LEMMA 19. *Let $\mu \in \mathcal{P}_2$ and $\nu \in \mathcal{M}$ where \mathcal{M} is defined by (5.1.1) for some steady-state $\bar{\mu}$. If there exists an $\varepsilon > 0$ such that for all A skew-symmetric matrices such that $\|A\| = \varepsilon$ it holds that*

$$d_2(\mu, e_{\#}^A \nu) > d_2(\mu, \nu)$$

then there exists a σ , an interior point of $U_\varepsilon(\nu)$ such that σ is a local minimizer.

PROOF. Since $\overline{U_\varepsilon(\nu)}$ is compact there exists $\sigma \in \overline{U_\varepsilon(\nu)}$ such that

$$d_2(\sigma, \mu) \leq d_2(\tilde{\nu}, \mu)$$

for all $\tilde{\nu} \in \overline{U_\varepsilon(\nu)}$ and in particular

$$d_2(\sigma, \mu) \leq d_2(\nu, \mu)$$

so by hypothesis $\sigma \notin \partial U_\varepsilon(\nu)$. So $\sigma \in U_\varepsilon(\nu)$ and thus is the local minimizer desired for the lemma. \square

LEMMA 20. *Let $\bar{\mu}$ be a steady-state of the gradient flow evolution, let \mathcal{M} be as defined by (5.1.1), let $\mu \in \mathcal{P}_2$, and let σ be a local minimizer of $d_2(\mu, \cdot)$ as defined in (5.1.2). Let $\pi \in \Gamma_{opt,2}(\sigma, \mu)$. Then for all skew-symmetric matrices A ,*

$$(5.1.5) \quad \int (\bar{x} - x) \cdot A \bar{x} d\pi(\bar{x}, x) = 0$$

and let ν_s be the displacement interpolant between σ and μ ,

$$\nu_s = ((1-s)\bar{x} + sx)_{\#} \pi$$

and let $\pi_s \in \Gamma_{opt,2}(\nu_s, \mu)$, then for all skew-symmetric matrices A ,

$$(5.1.6) \quad \int x \cdot Azd\pi_s(z, x) = 0.$$

PROOF. To see this, let φ_s be the flow map that satisfies for an arbitrary $A \in so(d)$

$$\begin{cases} \dot{\varphi}_s(\bar{x}) &= A\bar{x} \\ \varphi_0(\bar{x}) &= \bar{x} \end{cases}$$

so $\varphi_{s\#}\sigma \in \mathcal{M}$ for all s . Note that $\varphi_s(\bar{x}) = e^{As}\bar{x}$ is the flow map of this differential equation.

Since σ is a local minimizer it must be true that

$$\begin{aligned} 0 &= \left. \frac{d}{ds} \right|_{s=0} d_2^2(\varphi_{s\#}\sigma, \hat{\mu}) \\ &= \left. \frac{d}{ds} \right|_{s=0} \int |\bar{x} - x|^2 d\pi(\bar{x}, x) \\ &= 2 \int (\bar{x} - x) \cdot A\bar{x} d\pi(\bar{x}, x) \end{aligned}$$

for the arbitrarily chosen skew-symmetric matrix A . This shows the first claim. Note that since $\bar{x}^T A\bar{x} = 0$ that this further implies that

$$\int x \cdot A\bar{x} d\pi(\bar{x}, x) = 0$$

To see the second claim note that it is known that π_s is unique and $\pi_s = (((1-s)\bar{x} + sx), x)_{\#} \pi$. Thus

$$\begin{aligned} \int x \cdot Azd\pi_s(z, x) &= \int x \cdot A((1-s)\bar{x} + sx) d\pi(\bar{x}, x) \\ &= \int x \cdot A\bar{x} d\pi(\bar{x}, x) - s \int x \cdot A(\bar{x} - x) d\pi(\bar{x}, x) \\ &= 0 \end{aligned}$$

as was to be shown. \square

LEMMA 21. *Assuming all the assumptions and definitions of 1.7, for some steady-state $\bar{\mu}$ let \mathcal{M} be defined by (5.1.1). If for all $t \in [0, T]$, $d_{\infty}(\mu_t, \mathcal{M}) \leq \varepsilon$ for some sufficiently small $\frac{\lambda}{4c_W} > \varepsilon > 0$, then full linear stability (with constant λ) of the gradient flow evolution at $\bar{\mu}$ implies that*

$$d_2(\mu_t, \mathcal{M}) \leq \left(\frac{8L_W}{\lambda - 4c_W\varepsilon} \right) d_2(\mu_0, \mathcal{M}) e^{-\frac{\lambda}{4}t}$$

for all $t \in [0, T]$.

PROOF. Let σ_t be a measure on \mathcal{M} which is closest to μ_t in d_2 metric, whose existence is guaranteed by (18) such that

$$d_2(\mu_t, \mathcal{M}) = d_2(\mu_t, \sigma_t).$$

Fix some time $\bar{t} \in [0, T)$.

Let $\pi \in \Gamma_{opt,2}(\mu_{\bar{t}}, \sigma)$ and define $J : [0, 1] \rightarrow \mathbb{R}$ as the interaction energy along the interpolant $\pi_r = (rx + (1-r)\bar{x})_{\#} \pi$, i.e.

$$J(r) := \frac{1}{2} \int \int W(r(x-y) + (1-r)(\bar{x}-\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}).$$

Then for all $r \in [0, 1]$

$$J'(r) = \frac{1}{2} \int \int \nabla W(r(x-y) + (1-r)(\bar{x}-\bar{y}))^T ((x-y) - (\bar{x}-\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}).$$

Furthermore, J'' can be bounded from below uniformly in r . To see this, recall that $\text{Hess}W$ is Lipschitz with constant c_W (as stated in chapter 1 section 7),

$$\begin{aligned} J''(r) &= \frac{1}{2} \int \int ((x-y) - (\bar{x}-\bar{y}))^T \text{Hess}W(r(x-y) + (1-r)(\bar{x}-\bar{y})) ((x-y) - (\bar{x}-\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\geq \frac{1}{2} \int \int ((x-y) - (\bar{x}-\bar{y}))^T \text{Hess}W(\bar{x}-\bar{y}) ((x-y) - (\bar{x}-\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad - \frac{1}{2} c_W \int \int |(x-y) - (\bar{x}-\bar{y})|^3 d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\geq \frac{1}{2} \int \int ((x-y) - (\bar{x}-\bar{y}))^T \text{Hess}W(\bar{x}-\bar{y}) ((x-y) - (\bar{x}-\bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ (5.1.7) \quad &- 2c_W \varepsilon d_2^2(\mu_{\bar{t}}, \sigma). \end{aligned}$$

Now define

$$\begin{aligned} u(x, \bar{x}) &= \bar{x} - x \\ \bar{u}(\bar{x}) &= \bar{x} - \int x d\pi_{\bar{x}}(x) \\ \tilde{u}(x, \bar{x}) &= u(x, \bar{x}) - \bar{u}(\bar{x}) \end{aligned}$$

then we have the same decoupling of (5.1.7) as in the previous chapter, namely

$$\begin{aligned} J''(r) &\geq \frac{1}{2} \int \int (u(x, \bar{x}) - u(y, \bar{y}))^T \text{Hess}W(\bar{x}-\bar{y}) (u(x, \bar{x}) - u(y, \bar{y})) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad - 2c_W \varepsilon d_2^2(\mu_{\bar{t}}, \sigma) \\ &= \frac{1}{2} \int \int (\bar{u}(\bar{x}) - \bar{u}(\bar{y}))^T \text{Hess}W(\bar{x}-\bar{y}) (\bar{u}(\bar{x}) - \bar{u}(\bar{y})) d\sigma(\bar{x}) d\sigma(\bar{y}) \\ &\quad + \int \int \tilde{u}(x, \bar{x})^T \text{Hess}W(\bar{x}-\bar{y}) \tilde{u}(x, \bar{x}) d\pi(x, \bar{x}) d\pi(y, \bar{y}) \\ &\quad - 2c_W \varepsilon d_2^2(\mu_{\bar{t}}, \sigma). \end{aligned}$$

Recalling by (5.1.5) that \bar{u} is admissible in the sense of (4.2.1) and (4.2.2), the fact that full linear stability is given at σ gives that

$$\begin{aligned} J''(r) &\geq \frac{\lambda}{2} \left(\int \int |\bar{u}(\bar{x})|^2 d\sigma(\bar{x}) d\sigma(\bar{y}) + \int \int |\tilde{u}(x, \bar{x})|^2 d\pi(x, \bar{x}) d\pi(y, \bar{y}) \right) - 2c_W \varepsilon d_2^2(\mu_{\bar{t}}, \sigma) \\ &= \frac{\lambda}{2} \left(\int |\bar{x} - x|^2 d\pi(x, \bar{x}) \right) - 2c_W \varepsilon d_2^2(\mu_{\bar{t}}, \sigma) \\ &= d_2^2(\mu_{\bar{t}}, \sigma) \left(\frac{\lambda}{2} - 2c_W \varepsilon \right). \end{aligned}$$

This gives a uniform lower bound for J'' .

Now, by the mean value theorem one has that there is an $s \in [0, 1]$ such that

$$J'(1) - J'(0) = J''(s) \geq d_2^2(\mu_{\bar{t}}, \sigma) \left(\frac{\lambda}{2} - 2c_W \varepsilon \right)$$

and recalling that $\int \nabla W(\bar{x} - \bar{y}) d\sigma(\bar{y})$ for $\bar{x} \in \text{supp}\sigma$ one sees that,

$$J'(0) = \int (x - \bar{x})^T \left(\int \nabla W(\bar{x} - \bar{y}) d\sigma(\bar{y}) \right) d\pi(x, \bar{x}) + \int (y - \bar{y})^T \left(\int -\nabla W(\bar{y} - \bar{x}) d\sigma(\bar{x}) \right) d\pi(y, \bar{y}) = 0$$

and $J'(1)$ is

$$\begin{aligned} J'(1) &= \int (x - \bar{x})^T \left(\int \nabla W(x - y) d\mu_{\bar{t}}(y) \right) d\pi(x, \bar{x}) + \int (y - \bar{y})^T \left(\int \nabla W(y - x) d\mu_{\bar{t}}(x) \right) d\pi(y, \bar{y}) \\ &= 2 \int (\bar{x} - x)^T v_{\mu_{\bar{t}}}(x) d\pi(x, \bar{x}) \end{aligned}$$

so that

$$d_2^2(\mu_{\bar{t}}, \sigma) \left(\frac{\lambda}{2} - 2c_W \varepsilon \right) \leq 2 \int (\bar{x} - x)^T v_{\mu_{\bar{t}}}(x) d\pi(x, \bar{x}).$$

Using the Cauchy-Schwartz inequality one can further estimate the right hand side with

$$\begin{aligned} 2 \int (\bar{x} - x)^T v_{\mu_{\bar{t}}}(x) d\pi(x, \bar{x}) &\leq 2 \sqrt{\int |\bar{x} - x|^2 d\pi(x, \bar{x})} \sqrt{\int |v_{\mu_{\bar{t}}}(x)|^2 d\mu_{\bar{t}}(x)} \\ &= 2d_2(\mu_{\bar{t}}, \sigma) \sqrt{\int |v_{\mu_{\bar{t}}}(x)|^2 d\mu_{\bar{t}}(x)} \end{aligned}$$

so that

$$d_2(\mu_{\bar{t}}, \sigma) \left(\frac{\lambda}{2} - 2c_W \varepsilon \right) \leq 2 \sqrt{\int |v_{\mu_{\bar{t}}}(x)|^2 d\mu_{\bar{t}}(x)}.$$

Now it follows for sufficiently small $\varepsilon > 0$ from 17 that

$$\sqrt{\int |v_{\mu_{\bar{t}}}(x)|^2 d\mu_{\bar{t}}(x)} \leq e^{-\frac{\lambda}{4}\bar{t}} \sqrt{\int |v_{\mu_0}(x)|^2 d\mu_0(x)}$$

and by defining σ_0 to be the closest element of \mathcal{M} to μ_0 , and π_0 a 2-Wasserstein optimal transport plan between μ_0 and σ_0 , one sees that

$$\begin{aligned}
& \sqrt{\int |v_{\mu_0}(x)|^2 d\mu_0(x)} = \\
&= \sqrt{\int \left| \int \nabla W(x-y) d\mu_0(y) \right|^2 d\mu_0(x)} \\
&= \sqrt{\int \left| \int \nabla W(x-y) \pm \nabla W(x-\bar{y}) - \nabla W(\bar{x}-\bar{y}) d\pi_0(y, \bar{y}) \right|^2 d\mu_0(x, \bar{x})} \\
&\leq \sqrt{\int \left| \int \nabla W(x-y) - \nabla W(x-\bar{y}) d\pi_0(y, \bar{y}) \right|^2 d\mu_0(x, \bar{x})} \\
&\quad + \sqrt{\int \left| \int \nabla W(x-\bar{y}) - \nabla W(\bar{x}-\bar{y}) d\pi_0(y, \bar{y}) \right|^2 d\mu_0(x, \bar{x})} \\
&\leq L_W d_2(\mu_0, \sigma_0) + L_W d_2(\mu_0, \sigma_0) \\
&= 2L_W d_2(\mu_0, \sigma_0)
\end{aligned}$$

where the first inequality is the Minkowski inequality.

Therefore,

$$d_2(\mu_{\bar{t}}, \sigma) \left(\frac{\lambda}{2} - 2c_W \varepsilon \right) \leq 4e^{-\frac{\lambda}{4}\bar{t}} L_W d_2(\mu_0, \sigma_0)$$

or by recalling the definitions of σ and σ_0 :

$$d_2(\mu_{\bar{t}}, \mathcal{M}) \leq \left(\frac{8L_W}{\lambda - 4c_W \varepsilon} \right) d_2(\mu_0, \mathcal{M}) e^{-\frac{\lambda}{4}\bar{t}}$$

for $\bar{t} \in [0, T]$. □

5.2. Long Time Control of the $d_\infty(\mu_t, \mathcal{M})$

The previous section showed how a path μ_t that starts at a μ_0 that is near in an ∞ -Wasserstein sense to a full linear stable steady-state $\bar{\mu}$, will experience 2-Wasserstein contraction from μ_t and the manifold \mathcal{M} generated by $\bar{\mu}$. This contraction holds as long as μ_t stays sufficiently close in the ∞ -Wasserstein sense to \mathcal{M} . Thus if one can control this distance for all time, then one can get 2-Wasserstein contraction of μ_t to \mathcal{M} .

In order to do this one can first bound the speed at which the 2-Wasserstein projection of μ_t on to the manifold \mathcal{M} moves, and then use that to bound the speed at which μ_t might be distancing itself from \mathcal{M} in the ∞ -Wasserstein sense, such that $d_\infty(\mu_t, \mathcal{M})$ always stays bounded.

First note that the spreading stability of the steady-state implies that the velocity vector field generated by the steady-state is controlled in the sense that all vectors at points near to the steady-state's particles are pointing toward the steady-state.

LEMMA 22. *Assuming all the assumptions and definitions of 1.7, let $\bar{\mu}$ be a finite particle steady-state (as defined by (4.3.12)) that is spreading stable with constant $\lambda > 0$ (in the sense that it satisfies (4.3.20)). Then when \bar{x}_i is in the support of $\bar{\mu}$ and x is such that $|x - \bar{x}_i| < \frac{\lambda}{2c_W}$, it holds that*

$$(5.2.1) \quad v_{\bar{\mu}}(x) \cdot (x - \bar{x}_i) \leq -\frac{\lambda}{2} |x - \bar{x}_i|^2.$$

PROOF. Let f be defined as

$$f(s) = -\nabla W * \bar{\mu}(sx + (1-s)\bar{x}_i).$$

Since $\bar{\mu}$ is a steady-state, $f(0) = 0$. And by definition $f(1) = v_{\bar{\mu}}(x)$. Computing directly, by Taylor's theorem there exists some $\bar{s} \in [0, 1]$ such that

$$\begin{aligned} v_{\bar{\mu}}(x) \cdot (x - \bar{x}_i) &= f(1) \cdot (x - \bar{x}_i) \\ &= (f(0) + f'(\bar{s})) \cdot (x - \bar{x}_i) \\ &= -(x - \bar{x}_i)^T \text{Hess}W * \bar{\mu}(\bar{s}x + (1-\bar{s})\bar{x}_i)(x - \bar{x}_i) \end{aligned}$$

and then recalling that $\text{Hess}W$ being Lipschitz with constant c_W gives that

$$\left| (x - \bar{x}_i)^T (\text{Hess}W * \bar{\mu}(\bar{x}_i) - \text{Hess}W * \bar{\mu}(\bar{s}x + (1-\bar{s})\bar{x}_i))(x - \bar{x}_i) \right| \leq c_W \bar{s} |x - \bar{x}_i|^3$$

so then

$$v_{\bar{\mu}}(x) \cdot (x - \bar{x}_i) \leq -(x - \bar{x}_i)^T \text{Hess}W * \bar{\mu}(\bar{x}_i)(x - \bar{x}_i) + c_W \bar{s} |x - \bar{x}_i|^3$$

and since $s \leq 1$ and applying the spreading stability with constant $\lambda > 0$ at $\bar{\mu}$ one can further state that

$$v_{\bar{\mu}}(x) \cdot (x - \bar{x}_i) \leq (-\lambda + c_W |x - \bar{x}_i|) |x - \bar{x}_i|^2$$

so that since $|x - \bar{x}_i| < \frac{\lambda}{2c_W}$ it holds that

$$v_{\bar{\mu}}(x) \cdot (x - \bar{x}_i) \leq -\frac{\lambda}{2} |x - \bar{x}_i|^2$$

which is what was to be shown. □

Now the goal is to show that one can bound for all time the ∞ -Wasserstein distance from some gradient flow evolution that starts sufficiently near (in an ∞ -Wasserstein sense) to a fully linear stable steady-state. This then allows someone to apply (21) above for all time, implying the 2-Wasserstein convergence to (a rotation of) the steady-state.

As stated at the beginning of this section, the first goal is to bound the speed at which the d_2 projection onto \mathcal{M} of a path μ_t moves by a function of the speed at which μ_t moves and its distance from \mathcal{M} .

In order to do this, suppose one has a general path μ_t parametrized by t in \mathcal{P}_2 , not necessarily a gradient flow. One more thing to do is track how quickly the d_2 projection onto a manifold \mathcal{M} generated as in (5.1.1) by a finite particle steady-state $\bar{\mu}$ (as defined by (4.3.12)) moves in time. Let $\delta > 0$ be such that

$$\min_{i \in \{1, \dots, N\}} |x_i - x_j| > 10\delta.$$

Thus define σ_2 to be a local minimizer of $d_2(\mu_0, \cdot)$ on to the manifold \mathcal{M} . The tangent space to \mathcal{M} at σ_2 is given by vector fields generated by skew-symmetric matrices, i.e.

$$\mathcal{T}_{\sigma_2}\mathcal{M} = \{Ax \mid A^T = -A\} \cong H = \{A \in \mathbb{R}^{d \times d} \mid A^T = -A\}.$$

The inner product is

$$g_{\sigma_2}(A, \tilde{A}) = \int (Ax) \cdot \tilde{A}x \, d\sigma_2(x).$$

Let

$$B_H = \{A_1, \dots, A_K\}$$

where $K = \frac{d(d-1)}{2}$ be an orthonormal basis of H . Then for $i \in \{1, \dots, K\}$ and $j \neq i$,

$$(5.2.2) \quad \begin{aligned} \int |A_i x|^2 \, d\sigma_2 &= 1 \\ \int (A_i x) \cdot A_j x \, d\sigma_2(x) &= 0. \end{aligned}$$

Note that the exponential mapping from a neighborhood of 0 in H to a neighborhood of σ_2 in \mathcal{M} is given by

$$(s_1 A_1 + \dots + s_K A_K) \mapsto e_{\#}^{s_1 A_1 + \dots + s_K A_K} \sigma_2.$$

Consider the mapping $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$f(t, s_1, \dots, s_K) = d_2^2 \left(e_{\#}^{\sum_{i=1}^K s_i A_i} \sigma_2, \mu_t \right).$$

Let $G : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$

$$G(t, s_1, \dots, s_K) = D_s f = \left[\frac{\partial f}{\partial s_1}, \dots, \frac{\partial f}{\partial s_K} \right].$$

Note that if $G = 0$ then $e_{\#}^{\sum_{i=1}^K s_i A_i} \sigma_2$ is a critical point of $d_2^2(\cdot, \mu_t)$ and thus a candidate for a local minimizer. The goal is to prove using the implicit function theorem that there exists a C^1 function $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ such that $G(t, \gamma(t)) = 0$.

Let $\pi \in \Gamma_{opt,2}(\sigma_2, \mu_0)$. Assume that $d_\infty(\mu_0, \sigma_2) < \delta$. Then where φ_t is the flow map of the path μ_t , for t, s_1, \dots, s_K small it holds that

$$\left(e^{\sum_{i=1}^K s_i A_i} \times \varphi_t \right)_\# \pi \in \Gamma_{opt,2} \left(e^{\sum_{i=1}^K s_i A_i} \sigma_2, \mu_t \right).$$

This implies that

$$f(t, s_1, \dots, s_K) = d_2^2 \left(e^{\sum_{i=1}^K s_i A_i} \sigma_2, \mu_t \right) = \int \left| e^{\sum_{i=1}^K s_i A_i} x - \varphi_t(y) \right|^2 d\pi(x, y).$$

Thus f is a C^1 function. So

$$(5.2.3) \quad \begin{aligned} \frac{\partial f}{\partial s_i} &= 2 \int \left(\frac{\partial}{\partial s_i} e^{\sum_{i=1}^K s_i A_i} x \right) \cdot \left(e^{\sum_{i=1}^K s_i A_i} x - \varphi_t(y) \right) d\pi(x, y) \\ \frac{\partial f}{\partial s_i} \Big|_{s_j=0, j=1, \dots, K, t=0} &= 2 \int (A_i x) \cdot (x - y) d\pi(x, y) \\ &= 0 \end{aligned}$$

where the last equality is by (20). So $G(0, \dots, 0) = 0$.

To show that there exists the desired γ such that $G(t, \gamma(t)) = 0$ for all t small one uses the implicit function theorem, which means it only needs to be shown that $DG(0, \dots, 0)$ is invertible. Note that

$$DG(0, \dots, 0) = \left[\frac{\partial^2 f}{\partial s_i \partial s_j} \right]_{(0, \dots, 0)}.$$

From (5.2.3) it follows that (using $t = 0$)

$$\begin{aligned} \frac{\partial^2 f}{\partial s_j \partial s_k} &= 2 \int \left(\frac{\partial^2}{\partial s_j \partial s_k} e^{\sum_{i=1}^K s_i A_i} x \right) \cdot \left(e^{\sum_{i=1}^K s_i A_i} x - \varphi_t(y) \right) d\pi(x, y) \\ &\quad + \left(\frac{\partial}{\partial s_k} e^{\sum_{i=1}^K s_i A_i} x \right) \cdot \left(\frac{\partial}{\partial s_j} e^{\sum_{i=1}^K s_i A_i} x \right) d\pi(x, y). \end{aligned}$$

So so by plugging in $\vec{0} = (0, \dots, 0)$ one gets

$$(5.2.4) \quad \frac{\partial^2 f}{\partial s_j \partial s_k} \Big|_{\vec{0}} = 2 \int \left(\frac{A_k A_j + A_j A_k}{2} x \right) \cdot (x - y) + (A_k x) \cdot (A_j x) d\pi(x, y)$$

where $\frac{\partial^2}{\partial s_j \partial s_k} e^{\sum_{i=1}^K s_i A_i} \Big|_{\vec{0}}$ is found using the formula

$$e^{\sum_{i=1}^K s_i A_i} = I + \sum_{k=1}^K s_k A_k + \frac{1}{2} \sum_{k=1}^K \sum_{j=1}^K s_k s_j A_k A_j + o \left(s_j^2 + s_k^2 + 2s_k s_j + \sum_{i \notin \{j, k\}} |s_i| \right).$$

Now f can be estimated since

$$\begin{aligned}
 \left| \int (A_k A_j x) \cdot (x - y) d\pi(x, y) \right| &= \left| \int (A_j x) \cdot A_k(x - y) d\pi(x, y) \right| \\
 &\leq \sqrt{\int |A_j x|^2 d\sigma_2(x)} \sqrt{\|A_k\|^2 \int |x - y|^2 d\pi(x, y)} \\
 (5.2.5) \qquad &= \|A_k\| d_2(\sigma_2, \mu_0)
 \end{aligned}$$

and note that

$$\left(\frac{\partial^2}{\partial s_j \partial s_k} e^{\sum_{i=1}^K s_i A_i x} \right)$$

$$\int (A_k x) \cdot A_j x d\pi(x, y) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{else} \end{cases}.$$

Now since $\bar{\mu}$ is in general position, there exists a $c_1 > 0$ such that for all $i \in \{1, \dots, K\}$

$$1 = \int |A_i x|^2 d\sigma_2(x) \geq c_1 \|A_i\|^2$$

so

$$\|A_i\| \leq \frac{1}{\sqrt{c_1}}.$$

Assume that δ is such that

$$\delta < \frac{\sqrt{c_1}}{3K}.$$

So from (5.2.5)

$$\left| \int (A_k A_j x) \cdot (x - y) d\pi(x, y) \right| < \frac{1}{\sqrt{c_1}} \delta < \frac{1}{3K}.$$

Thus from (5.2.4)

$$\begin{aligned}
 \left. \frac{\partial^2 f}{\partial s_k^2} \right|_{\vec{0}} &> 1 - \frac{1}{3K} \\
 \left. \frac{\partial^2 f}{\partial s_j \partial s_k} \right|_{\vec{0}, j \neq k} &< \frac{2}{3K}.
 \end{aligned}$$

Thus $\left[\frac{\partial^2 f}{\partial s_i \partial s_j} \right]_{(0, \dots, 0)}$ is diagonally dominant, therefore positive definite, and therefore invertible.

So by the implicit function theorem there exists $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^d$ for some $\varepsilon > 0$ such that $G(t, \gamma(t)) = 0$. Let

$$\gamma(t) = (s_1(t), \dots, s_K(t))$$

so one can define

$$\sigma_2(t) = e^{\sum_{i=1}^K s_i(t) A_i} \sigma_2$$

which, since $G(t, \gamma(t)) = 0$, is a critical point of $d_2^2(\cdot, \mu_t)$. Moreover, since

$$\left[\frac{\partial^2 f}{\partial s_i \partial s_j} \right]_{(0, \dots, 0)} > \frac{1}{4} I$$

one can conclude by continuity that

$$\left[\frac{\partial^2 f}{\partial s_i \partial s_j} \right]_{(t, \gamma(t))} > \frac{1}{8} I$$

for t small. Therefore $\sigma_2(t)$ is a local minimizer of $d_2^2(\cdot, \mu_t)$. So $t \mapsto \sigma_2(t)$ is a continuous curve of local minimizers of $d_2(\cdot, \mu_t)$.

Furthermore, one can take the derivative in time of $\sigma_2(t)$: By the definition of G it follows that

$$\begin{aligned} \left(\frac{\partial G}{\partial t} \right) - 2 \int (A_i x) \cdot v_{\mu_0}(y) d\pi(x, y) &= -2 \int A_i(x-y) \cdot v_{\mu_0}(y) d\pi(x, y) \\ &\leq 2 \sqrt{\int |A_i(x-y)|^2 d\pi(x, y)} \sqrt{\int |v_{\mu_0}(y)| d\mu_0(y)} \\ &\leq 2 \|v_{\mu_0}\|_{L^2(d\mu_0)} \|A_i\| d_2(\mu_0, \sigma_2(0)) \\ &\leq \frac{2}{c_1} \|v_{\mu_0}\|_{L^2(d\mu_0)} d_2(\mu_0, \sigma_2(0)). \end{aligned}$$

By implicit function theorem ($G(t, \gamma(t)) = 0$) it follows that

$$\gamma'(0) = (DG(0))^{-1} \frac{\partial G}{\partial t}.$$

Since $DG(0) \geq \frac{1}{4} I$, it holds that $(DG(0))^{-1} \leq 4I$. Thus

$$|\gamma'(0)| \leq 4 \left| \frac{\partial G}{\partial t} \right| \leq 8 \frac{K}{c_1} \|v_{\mu_0}\|_{L^2(d\mu_0)}.$$

Thus

$$\left. \frac{d}{dt} \right|_{t=0} d_2(\sigma_2(t), \sigma_2(0)) = |\gamma'(0)| \leq 8 \frac{K}{c_1} \|v_{\mu_0}\|_{L^2(d\mu_0)}$$

. Since this can be carried out for any t ,

$$(5.2.6) \quad \frac{d}{dt} (d_2(\sigma_2(t), \sigma_2(0))) \leq 8 \frac{K}{c_1} \|v_{\mu_0}\|_{L^2(d\mu_0)}$$

Now that the speed of the d_2 projection of μ_t onto \mathcal{M} can be bounded by (5.2.6), it is time to show that $d_\infty(\mu, \sigma_2)$ can be controlled by $d_\infty(\mu, \bar{\mu})$. To do this, for the finite particle steady-state $\bar{\mu}$ define

$$(5.2.7) \quad \ell = \min_{i \neq j} |\bar{x}_i - \bar{x}_j|$$

$$(5.2.8) \quad L = \max_i |\bar{x}_i|$$

LEMMA 23. *Assuming all the assumptions and definitions of 1.7, let $\bar{\mu}$ be a finite particle steady-state (as defined by (4.3.12)). Let ℓ and L be defined by (5.2.7) and (5.2.8), respectively, and note $\ell \leq 2L$ since 0 is center of mass. Let c_1 be defined as above and let $m = \min_i \{m_1, \dots, m_N\}$ and assume $0 < \delta < \frac{1}{8}m\sqrt{c_1}$.*

Assume μ is such that $d_\infty(\mu, \bar{\mu}) < \delta$. Then there exists $\sigma_2 \in \mathcal{M}$ a local minimizer of the distance to μ such that

$$d_\infty(\mu, \sigma_2) < \left(\frac{4L}{m\sqrt{c_1}} + 1 \right) \delta$$

PROOF. Let A be a skew-symmetric matrix such that $\|A\| \leq \frac{1}{2}$. Note that

$$\begin{aligned} |e^A x - x| &\geq \frac{1}{2} |Ax| \\ |e^A x - x| &\leq \|A\| |x|. \end{aligned}$$

Thus if

$$(5.2.9) \quad \|A\| \leq \frac{\ell}{2L}$$

then

$$\|A\| \leq \frac{1}{2}$$

and

$$|e^A \bar{x}_i - \bar{x}_i| \leq \frac{\ell}{2L} |\bar{x}_i|$$

so

$$\min_j |e^A \bar{x}_i - \bar{x}_j| = |e^A \bar{x}_i - \bar{x}_i|.$$

Thus $x \mapsto e^A x$ is an optimal transportation mapping (both for the 2-Wasserstein and ∞ -Wasserstein distances) between $\bar{\mu}$ and $e^A_{\#}\bar{\mu}$.

Thus

$$d_2(e^A_{\#}\bar{\mu}, \bar{\mu}) \geq \frac{1}{2} m \max_i |Ax_i| \geq \frac{\sqrt{c_1}m}{2} \|A\|.$$

On the other hand

$$d_\infty(e^A_{\#}\bar{\mu}, \bar{\mu}) \leq \max_i \|A\| |\bar{x}_i| \leq \|A\| L.$$

Consider A such that $\|A\| = \frac{4\delta}{m\sqrt{c_1}}$. Then by the assumption on δ it holds that

$$\|A\| < \frac{4}{m\sqrt{c_1}} \cdot \frac{1}{8} m\sqrt{c_1} \frac{\ell}{L} = \frac{\ell}{2L}.$$

So this A satisfies (5.2.9).

Therefore

$$d_2(e^A_{\#}\bar{\mu}, \bar{\mu}) \geq \frac{\sqrt{c_1}m}{2} \cdot \frac{4\delta}{m\sqrt{c_1}} = 2\delta$$

and for any $\|\tilde{A}\| < \frac{4\delta}{m\sqrt{c_1}}$, \tilde{A} skew-symmetric, it holds that

$$d_\infty\left(e_{\#}^{\tilde{A}}\bar{\mu}, \bar{\mu}\right) \leq \frac{4\delta}{m\sqrt{c_1}} \cdot L$$

Note that

$$d_2\left(e_{\#}^A\bar{\mu}, \sigma_2\right) \geq d_2\left(e_{\#}^A\bar{\mu}, \bar{\mu}\right) - d_2\left(\bar{\mu}, \sigma_2\right) > 2\delta - \delta = \delta \geq d_2\left(\bar{\mu}, \sigma_2\right).$$

Thus by (19) there exists a skew-symmetric \tilde{A} such that $\|\tilde{A}\| < \frac{4\delta}{m\sqrt{c_1}} \leq \frac{1}{2}$ such that $\sigma_2 = e_{\#}^{\tilde{A}}\bar{\mu}$ is a local minimizer of the 2-Wasserstein distance to μ . Furthermore

$$d_\infty(\mu, \sigma_2) \leq d_\infty(\mu, \bar{\mu}) + d_\infty(\bar{\mu}, \sigma_2) \leq \delta + \frac{4L}{m\sqrt{c_1}}\delta$$

□

This next lemma uses the previous one to bound the speed at which the ∞ -Wasserstein distance between μ_t and \mathcal{M} grows. The idea here is to use the d_∞ distance between the μ_t and the d_2 projection of μ_t onto \mathcal{M} as a way to control $d_\infty(\mu_t, \mathcal{M})$.

LEMMA 24. *Assuming all the assumptions and definitions of 1.7, let $\bar{\mu}$ be a finite particle steady-state (as defined by (4.3.12)). Let ℓ and L be defined by (5.2.7) and (5.2.8), respectively, and note $\ell \leq 2L$ since 0 is center of mass. Let c_1 be defined as above and let $m = \min_i \{m_1, \dots, m_N\}$ and assume $0 < \delta < \frac{1}{8}m\sqrt{c_1}$.*

Assume

$$d_\infty(\mu_0, \mathcal{M}) < \delta_0 = \min \left\{ \frac{1}{2\left(\frac{4L}{m\sqrt{c_1}} + 1\right)}, \frac{\lambda}{4\left(L_W + \frac{8K}{mc_1}2L_W\right)} \right\} \frac{\delta}{2}.$$

Then for all t such that $d_\infty(\mu_s, \mathcal{M}) < \delta$ for all $s \in [0, t]$ it holds that

$$\frac{d}{dt}(d_\infty(\mu_t, \mathcal{M})) < \frac{3\delta}{4}.$$

PROOF. First note that

$$d_\infty(\mu_0, \sigma_2) < \frac{\delta}{2\left(\frac{4L}{m\sqrt{c_1}} + 1\right)} \cdot \left(\frac{4L}{m\sqrt{c_1}} + 1\right) = \frac{\delta}{2}.$$

Also note by the triangle inequality that

$$d_\infty(\mu_t, \sigma_2(t)) \leq d_\infty(\mu_t, \sigma_2(s)) + d_\infty(\sigma_2(s), \sigma_2(t))$$

and since $\sigma_2(s)$ and $\sigma_2(t)$ are just rotations of the same particle steady-states in holds that

$$d_\infty(\sigma_2(s), \sigma_2(t)) \leq \frac{1}{m}d_2(\sigma_2(s), \sigma_2(t))$$

so that

$$d_\infty(\mu_t, \sigma_2(t)) \leq d_\infty(\mu_t, \sigma_2(s)) + \frac{1}{m}d_2(\sigma_2(s), \sigma_2(t)).$$

Now since it holds that the righthand side of the above inequality is increasing at time $t = s$ then one can say

$$\frac{d}{dt} \Big|_{t=s} d_\infty(\mu_t, \sigma_2(t)) \leq \frac{d}{dt} \Big|_{t=s} d_\infty(\mu_t, \sigma_2(s)) + \frac{d}{dt} \Big|_{t=s} d_2(\mu_s, \sigma_2(t)).$$

So then by definition of d_∞ it holds that

$$\frac{d}{dt} \Big|_{t=s} d_\infty(\mu_t, \sigma_2(s)) \leq \sup_{(y,x) \in \text{supp} \pi \in \Gamma_{opt,\infty}(\mu_t, \sigma_2(s))} v_{\mu_t}(y) \cdot \frac{y-x}{|y-x|}.$$

And by (5.2.6)

$$\frac{1}{m} \frac{d}{dt} \Big|_{t=s} d_2(\sigma_2(s), \sigma_2(t)) \leq \frac{8K}{mc_1} \|v_{\mu_t}\|_{L^2(d\mu_t)} d_2(\mu_s, \sigma_2(s))$$

which, since

$$\begin{aligned} \|v_{\mu_t}\|_{L^2(d\mu_t)} &= \sqrt{\int \int |\nabla W(x-y)|^2 d\mu_t(y) d\mu_t(x)} \\ \|v_{\mu_t}\|_{L^2(d\mu_t)} &= \sqrt{\int |v_{\mu_t}(x)|^2 d\mu_t(x)} \\ &= \sqrt{\int \left| \int \nabla W(x-y) d\mu_t(y) \right|^2 d\mu_t(x)} \\ &= \sqrt{\int \left| \int \nabla W(x-y) \pm \nabla W(x-\bar{y}) - \nabla W(\bar{x}-\bar{y}) d\pi_t(y, \bar{y}) \right|^2 d\mu_t(x, \bar{x})} \\ &\leq \sqrt{\int \left| \int \nabla W(x-y) - \nabla W(x-\bar{y}) d\pi_t(y, \bar{y}) \right|^2 d\mu_t(x, \bar{x})} \\ &\quad + \sqrt{\int \left| \int \nabla W(x-\bar{y}) - \nabla W(\bar{x}-\bar{y}) d\pi_t(y, \bar{y}) \right|^2 d\mu_t(x, \bar{x})} \\ &\leq L_W d_2(\mu_t, \sigma_2(t)) + L_W d_2(\mu_t, \sigma_2(t)) \\ &= 2L_W d_2(\mu_t, \sigma_2(t)). \end{aligned}$$

Therefore it holds that

$$\frac{d}{dt} \Big|_{t=s} d_\infty(\mu_t, \sigma_2(t)) \leq \sup_{(y,x) \in \text{supp} \pi \in \Gamma_{opt,\infty}(\mu_t, \sigma_2(s))} v_{\mu_t}(y) \cdot \frac{y-x}{|y-x|} + \frac{8K}{mc_1} \lambda d_\infty(\mu_t, \mathcal{M}) d_2(\mu_s, \sigma_2(s)).$$

Now consider that for any $(y, x) \in \text{supp}\pi \in \Gamma_{opt, \infty}(\mu_t, \sigma_2(t))$, then

$$\begin{aligned} v_{\mu_t}(y) \cdot \frac{y-x}{|y-x|} &= (-\nabla W * \mu_s(y) + \nabla W * \sigma_2(s)(y)) \cdot \frac{y-x}{|y-x|} \\ &\quad - \nabla W * \sigma_2(s)(y) \cdot \frac{y-x}{|y-x|} \\ &\leq L_W d_2(\mu_s, \sigma_2(s)) - \frac{\lambda}{2} |y-x|. \end{aligned}$$

Applying (5.2.1) one sees

$$-\nabla W * \sigma_2(s)(y) \cdot \frac{y-x}{|y-x|} \leq -\frac{\lambda}{2} |y-x|$$

and by the Lipschitz continuity of ∇W with Lipschitz constant L_W , let $\pi \in \Gamma_{opt, 2}(\mu_s, \sigma_2(s))$ then one sees

$$\begin{aligned} &\left| (-\nabla W * \mu_s(y) + \nabla W * \sigma_2(s)(y)) \cdot \frac{y-x}{|y-x|} \right| \\ &\leq |-\nabla W * \mu_s(y) + \nabla W * \sigma_2(s)(y)| \left| \frac{y-x}{|y-x|} \right| \\ &= \left| -\int \nabla W(y-\tilde{x}) d\mu_s(\tilde{x}) + \int \nabla W(y-\tilde{x}) d\sigma_2(s)(\tilde{x}) \right| \\ &= \left| \int -\nabla W(y-\tilde{x}) + \nabla W(y-\tilde{x}) d\pi(\tilde{x}, \tilde{x}) \right| \\ &\leq \int |\nabla W(y-\tilde{x}) - \nabla W(y-\tilde{x})| d\pi(\tilde{x}, \tilde{x}) \\ &\leq L_W \int |\tilde{x} - \tilde{x}| d\pi(\tilde{x}, \tilde{x}) \\ &\leq L_W \sqrt{\int |\tilde{x} - \tilde{x}|^2 d\pi(\tilde{x}, \tilde{x})} \\ &= L_W d_2(\mu_s, \sigma_2(s)). \end{aligned}$$

Therefore

$$v_{\mu_t}(y) \cdot \frac{y-x}{|y-x|} \leq L_W d_2(\mu_s, \sigma_2(s)) - \frac{\lambda}{2} |y-x|.$$

Putting that with the estimate above gives that

$$\begin{aligned} \frac{d}{dt}(d_\infty(\mu_t, \sigma_2(t))) &\leq \left(L_W + \frac{8K}{m} \lambda \delta \right) d_2(\mu_t, \sigma_2(t)) \\ &\leq \left(L_W + \frac{8K}{m} \lambda \delta \right) e^{-\frac{\lambda}{2} t}. \end{aligned}$$

So

$$\begin{aligned} \frac{d}{dt}(d_\infty(\mu_t, \sigma_2(t))) &\leq d_\infty(\mu_0, \sigma_2) + \left(L_W + \frac{8K}{mc_1}\lambda\delta\right)\delta_0\frac{2}{\lambda} \\ &< \frac{\delta}{2} + \left(L_W + \frac{8K}{mc_1}\lambda\delta\right)\delta_0\frac{2}{\lambda} \\ &< \frac{3\delta}{4}. \end{aligned}$$

□

Using the previous lemmas is now enough to show that $d_\infty(\mu_t, \mathcal{M})$ can be controlled in time and therefore the d_2 contraction holds.

THEOREM 25. *Assuming all the assumptions and definitions of 1.7, let $\bar{\mu}$ be a finite particle steady-state (as defined by (4.3.12)). Let ℓ and L be defined by (5.2.7) and (5.2.8), respectively, and note $\ell \leq 2L$ since 0 is center of mass. Let c_1 be defined as above and let $m = \min_i \{m_1, \dots, m_N\}$ and assume $0 < \delta < \frac{1}{8}m\sqrt{c_1}$.*

Assume

$$d_\infty(\mu_0, \mathcal{M}) < \delta_0 = \min \left\{ \frac{1}{2\left(\frac{4L}{m\sqrt{c_1}} + 1\right)}, \frac{\lambda}{4\left(L_W + \frac{8K}{m}\lambda\right)} \right\} \frac{\delta}{2}.$$

Then for all $t > 0$ it holds that

$$d_\infty(\mu_t, \mathcal{M}) < \frac{3\delta}{4}$$

and therefore

$$d_2(\mu_t, \mathcal{M}) \leq \delta d_2(\mu_0, \mathcal{M}) e^{-\frac{\lambda}{4}t}$$

PROOF. Assume there is a t such that $d_\infty(\mu_t, \mathcal{M}) \geq \frac{3\delta}{4}$. Let T be such that

$$T = \inf_{s \geq 0} d_\infty(\mu_s, \mathcal{M}) \geq \frac{3\delta}{4}$$

then

$$d_\infty(\mu_T, \mathcal{M}) = \frac{3\delta}{4}.$$

Thus there is an $\varepsilon > 0$ such that $d_\infty(\mu_s, \mathcal{M}) < \delta$ on $[0, t + \varepsilon]$. Then by the previous lemma, $d_\infty(\mu_s, \mathcal{M}) < \frac{3\delta}{4}$ for s in $[0, t + \varepsilon]$ which contradicts that $d_\infty(\mu_T, \mathcal{M}) = \frac{3\delta}{4}$. Since

$$d_\infty(\mu_0, \sigma_2(0)) \leq \left(\frac{4L}{m\sqrt{c_1}} + 1\right)\delta_0 < \delta$$

then

$$d_2(\mu_t, \mathcal{M}) \leq d_2(\mu_0, \sigma_2(0)) e^{-\frac{\lambda}{4}t}.$$

□

REMARK 26. From (21) one could state there was d_2 contraction of a gradient flow towards the manifold that held at least for some finite time greater than zero. The only reason that contraction might stop is that the estimate on the Hessian (4.3.8) may not hold if the d_∞ distance between the gradient flow evolution and the manifold \mathcal{M} generated by $\bar{\mu}$ (as defined by (5.1.1)) grows sufficiently large. Thus (25) shows that the d_∞ distance between the gradient flow evolution and the manifold \mathcal{M} can be bounded arbitrary small for all time if the initial data of the gradient flow evolution starts close enough to the fully linear stable steady-state $\bar{\mu}$.

CHAPTER 6

Applications

6.1. Preliminary Lemmas

In order for the stability results to hold, the example configurations that are explicitly computed here need to be finite particle configurations in general position. Recall that if x_i for $i \in \{1, \dots, N\}$ means for all $A \in so(d)$ (the skew-symmetric matrices) then being in general position means

$$\max_{i \in \{1, 2, \dots, N\}} |Ax_i| \geq c \|A\|.$$

Fortunately, in \mathbb{R}^2 all finite particle configurations with at least 2 at distinct positions are in general position, which is what the next lemma shows.

LEMMA 27. *In \mathbb{R}^2 all finite particle configurations with at least 2 particles at distinct positions are in general position.*

PROOF. For all $A \in so(2)$, there exists an $a \in \mathbb{R}$ such that

$$A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$$

so that

$$|Ax_i| = a |x_i|.$$

Since there are at least 2 particles at distinct positions at least 1 of them is not at the origin, so there exists a $c > 0$ such that

$$c \leq \max_{i \in \{1, 2, \dots, N\}} |x_i|$$

then it holds that as was to be shown. □

It is also useful to know that when dealing with radially symmetric configurations it is only necessary to check the spreading stability condition at one point.

LEMMA 28. *If $\bar{\mu}$ has a radial symmetry meaning that there is a rotation matrix R such that $\bar{\mu}(x) = \bar{\mu}(Rx)$, and for notational convenience define*

$$M(x) := \text{Hess}W * \bar{\mu}(x)$$

then

$$w^T M(Rx) w = (R^T w)^T M(x) (R^T w)$$

which implies that it is only necessary to check the spreading stability condition at one point in the configuration.

PROOF. Using the definition of $M(Rx)$ one sees

$$\begin{aligned} w^T M(Rx) w &= w^T \int F'(|Rx - y|) \frac{Rx - y}{|Rx - y|} \left(\frac{Rx - y}{|Rx - y|} \right)^T \\ &\quad + \frac{F(|Rx - y|)}{|Rx - y|} \left(I - \frac{Rx - y}{|Rx - y|} \left(\frac{Rx - y}{|Rx - y|} \right)^T \right) d\bar{\mu}(y) w \end{aligned}$$

By the radial symmetry of $\bar{\mu}$ one gets

$$\begin{aligned} w^T M(Rx) w &= w^T \int F'(|Rx - Ry|) \frac{Rx - Ry}{|Rx - Ry|} \left(\frac{Rx - Ry}{|Rx - Ry|} \right)^T \\ &\quad + \frac{F(|Rx - Ry|)}{|Rx - Ry|} \left(RR^T - \frac{Rx - Ry}{|Rx - Ry|} \left(\frac{Rx - Ry}{|Rx - Ry|} \right)^T \right) d\bar{\mu}(y) w \\ &= w^T \int F'(|x - y|) R \frac{x - y}{|x - y|} \left(\frac{x - y}{|x - y|} \right)^T R^T \\ &\quad + \frac{F(|x - y|)}{|x - y|} \left(RR^T - R \frac{x - y}{|x - y|} \left(\frac{x - y}{|x - y|} \right)^T R^T \right) d\bar{\mu}(y) w \\ &= (R^T w)^T \int F'(|x - y|) \frac{x - y}{|x - y|} \left(\frac{x - y}{|x - y|} \right)^T \\ &\quad + \frac{F(|x - y|)}{|x - y|} \left(I - \frac{x - y}{|x - y|} \left(\frac{x - y}{|x - y|} \right)^T \right) d\bar{\mu}(y) R^T w \\ &= (R^T w)^T M(x) (R^T w) \end{aligned}$$

which completes the proof. □

6.2. Stability of 3-particle rings

Consider a configuration made of a triangle of 3 particles (i.e. delta masses) with locations at $x_1, x_2, x_3 \in \mathbb{R}^2$ with corresponding masses

$$\begin{aligned} m_1, m_2, m_3 &> 0 \\ m_1 + m_2 + m_3 &= 1 \end{aligned}$$

By assuming that the center of mass is 0, then, without loss of generality, one can suppose that for an $i \in \{1, 2, 3\}$

$$x_i = \frac{L\sqrt{3}}{3}e_1$$

where L is defined for F as described in the introductory chapter.

The positions x_2 and x_3 are

$$\begin{aligned} x_2 &= R_{\frac{2\pi}{3}}x_1 \\ x_3 &= R_{\frac{2\pi}{3}}x_2 = R_{\frac{4\pi}{3}}x_1 \end{aligned}$$

so the steady-state is

$$\bar{\mu} = m_1\delta_{x_1} + m_2\delta_{x_2} + m_3\delta_{x_3}$$

6.2.1. Spreading Stability of the Triangular Steady State.

6.2.1.1. *The Equal-mass Triangle.* As a motivational and more easily computed first case, consider when the masses are equal, i.e.

$$m_1 = m_2 = m_3 = \frac{1}{3}$$

and recall that to show spreading stability, one must show that

$$\text{Hess}W * \bar{\mu}(x)$$

is uniformly λ -positive definite $\bar{\mu}$ -a.e. for some $\lambda > 0$. Again allow for notational convenience the definition

$$M(x) = \text{Hess}W * \bar{\mu}(x).$$

Then checking the spreading stability is equivalent to checking the λ -positive definiteness of

$$M(x_i) := \sum_{j=1}^3 \frac{1}{3} \text{Hess}W(x_i - x_j)$$

for all $i = 1, 2, 3$. However, this configuration has rotational symmetries of $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, such that by all of these matrices are positive definite iff one of them is. Here will be checked the positive definiteness of $M(x_1)$.

Furthermore, wlog, let x_1 be such that

$$x_1 = \frac{L\sqrt{3}}{3}$$

Recall that

$$\text{Hess}W(x_i - x_j) = F'(|x_i - x_j|) \widehat{x_i - x_j}^{\otimes 2} + \frac{F(|x_i - x_j|)}{|x_i - x_j|} \left(I - \widehat{x_i - x_j}^{\otimes 2} \right)$$

So recalling that $F(0) = F(L) = 0$,

$$\begin{aligned} M(x_1) &= \frac{1}{3} \left\{ F'(0) I \right. \\ &\quad + F'(L) \frac{\left(R_{\frac{2\pi}{3}} - I \right) x_1 x_1^T \left(R_{\frac{2\pi}{3}} - I \right)^T}{L^2} \\ &\quad \left. + F'(L) \frac{\left(R_{\frac{4\pi}{3}} - I \right) x_1 x_1^T \left(R_{\frac{4\pi}{3}} - I \right)^T}{L^2} \right\} \end{aligned}$$

and by using the assumption that $x_1 = \begin{bmatrix} L\frac{\sqrt{3}}{3} \\ 0 \end{bmatrix}$ then

$$\begin{aligned} M(x_1) &= \frac{1}{3} \left\{ F'(0) I \right. \\ &\quad + F'(L) \frac{\begin{bmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} L^2 \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}}{L^2} \\ &\quad \left. + F'(L) \frac{\begin{bmatrix} -\frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} L^2 \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}}{L^2} \right\} \\ &= \frac{1}{3} \left\{ F'(0) I \right. \\ &\quad + F'(L) \begin{bmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{6} \\ 0 & 0 \end{bmatrix} \\ &\quad \left. + F'(L) \begin{bmatrix} -\frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{6} \\ 0 & 0 \end{bmatrix} \right\} \\ &= \frac{1}{3} \left\{ F'(0) I \right. \\ &\quad + F'(L) \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \\ &\quad \left. + F'(L) \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \right\} \\ &= \frac{1}{3} \begin{bmatrix} F'(0) + \frac{3}{4} F'(L) & 0 \\ 0 & F'(0) + \frac{1}{2} F'(L) \end{bmatrix} \end{aligned}$$

From this it is enough to check that the smallest eigenvalue, λ_{\min} , is positive where

$$\lambda_{\min} = \frac{1}{3} \left(F'(0) + \frac{1}{2} F'(L) \right).$$

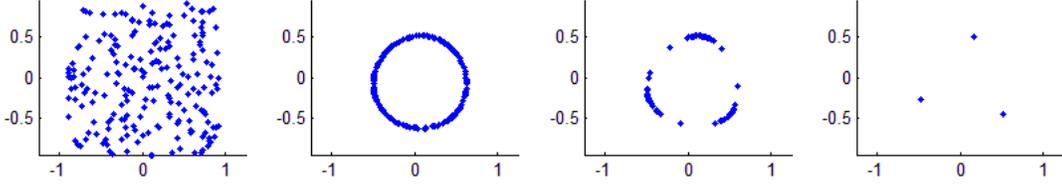


FIGURE 6.2.1. An example evolution where the initial distribution is uniformly distributed in a box and then evolves to a 3 particle steady-state.

6.2.1.2. *The General-mass Triangle configuration.* The main difference between this case and the equal-mass triangle is that it is less obvious how to take advantage of the rotational symmetry. However, since the positions are still rotation invariant, even if the masses aren't, and therefore the minimum eigenvalue of each $M(x_i)$ should be the same up to a permutation of the masses.

Consider $\{i, j, k\}$ a permutation of $\{1, 2, 3\}$, then this subsection will compute

$$M(x_i) := \sum_{j=1}^3 m_j \text{Hess}W(x_i - x_j)$$

and check the positivity of its minimum eigenvalue. Recall that

$$\text{Hess}W(x_i - x_j) = F'(|x_i - x_j|) \widehat{x_i - x_j}^{\otimes 2} + \frac{F(|x_i - x_j|)}{|x_i - x_j|} \left(I - \widehat{x_i - x_j}^{\otimes 2} \right)$$

So recalling that $F(0) = F(L) = 0$, one computes

$$\begin{aligned} M(x_i) &= F'(0) m_i I \\ &+ m_j F'(L) \frac{\left(R_{\frac{2\pi}{3}} - I \right) x_i x_i^T \left(R_{\frac{2\pi}{3}} - I \right)^T}{L^2} \\ &+ m_k F'(L) \frac{\left(R_{\frac{4\pi}{3}} - I \right) x_i x_i^T \left(R_{\frac{4\pi}{3}} - I \right)^T}{L^2} \end{aligned}$$

Again, without loss of generality, assume

$$x_i = \begin{bmatrix} L \frac{\sqrt{3}}{3} \\ 0 \end{bmatrix}$$

So

$$\begin{aligned}
M(x_i) &= F'(0) m_i I \\
&+ m_j F'(L) \frac{\begin{bmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} L^2 \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}}{L^2} \\
&+ m_k F'(L) \frac{\begin{bmatrix} -\frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} L^2 \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}}{L^2} \\
&= F'(0) m_i I \\
&+ m_j F'(L) \begin{bmatrix} -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{6} \\ 0 & 0 \end{bmatrix} \\
&+ m_k F'(L) \begin{bmatrix} -\frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{6} \\ 0 & 0 \end{bmatrix} \\
&= F'(0) m_i I \\
&+ m_j F'(L) \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \\
&+ m_k F'(L) \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \\
&= \begin{bmatrix} m_i F'(0) + \frac{3}{4} (m_j + m_k) F'(L) & \frac{\sqrt{3}}{4} (m_j - m_k) F'(L) \\ \frac{\sqrt{3}}{4} (m_j - m_k) F'(L) & m_i F'(0) + \frac{1}{4} (m_j + m_k) F'(L) \end{bmatrix}
\end{aligned}$$

Now the explicit computation of this matrix's minimum eigenvalue, λ_{\min} , is

$$\lambda_{\min} = F'(0) m_i + F'(L) \frac{1}{2} \left\{ (m_j + m_k) - \sqrt{m_j^2 - m_j m_k + m_k^2} \right\}$$

which is strictly increasing in m_j and m_k and strictly decreasing in m_i , so to calculate the minimum eigenvalue over all the $\{M(x_i)\}_{i=1}^3$ it is enough to consider the λ_{\min} for the i where $m_i \geq m_j, m_k$.

6.2.2. Displacement stability of the three particle triangle. If the three masses are equal, than the computations in the section on the N-particle ring will give an explicit computation for the maximal λ for the displacement stability of the 3 particle triangle with equal masses. The rest of this section will demonstrate that the 3 particle triangle is displacement stable no matter how the weights are allocated, but it does not give an explicit computation for the maximal λ by which the triangle is displacement stable.

To check the linear stability for the 3 particle triangle for any allocation of the mass one needs to show, for all nonzero admissible velocity vector fields \bar{v} , that

$$\sum_{i=1}^3 \sum_{j=1}^3 m_i m_j (\bar{v}(x_i) - \bar{v}(x_j))^T \text{Hess}W(x_i - x_j) (\bar{v}(x_i) - \bar{v}(x_j)) > 0$$

Note that for summands where $j = i$ are zero. Further, when $j \neq i$, $F(L) = 0$. So for $j \neq i$,

$$\text{Hess}W(x_i - x_j) = F'(L) \widehat{x_i - x_j}^{\otimes 2}$$

so

$$\begin{aligned} & \sum_{i=1}^3 \sum_{j=1}^3 m_i m_j (\bar{v}(x_i) - \bar{v}(x_j))^T \text{Hess}W(x_i - x_j) (\bar{v}(x_i) - \bar{v}(x_j)) = \\ & 2F'(L) \left\{ m_1 m_2 \left[(\bar{v}(x_1) - \bar{v}(x_2)) \cdot \widehat{x_1 - x_2} \right]^2 \right. \\ & + m_1 m_3 \left[(\bar{v}(x_1) - \bar{v}(x_3)) \cdot \widehat{x_1 - x_3} \right]^2 \\ & \left. + m_2 m_3 \left[(\bar{v}(x_2) - \bar{v}(x_3)) \cdot \widehat{x_2 - x_3} \right]^2 \right\} \end{aligned}$$

Clearly, this is always positive except in the case when all of the following are true

$$\begin{aligned} (\bar{v}(x_1) - \bar{v}(x_2)) \cdot \widehat{x_1 - x_2} &= 0 \\ (\bar{v}(x_1) - \bar{v}(x_3)) \cdot \widehat{x_1 - x_3} &= 0 \\ (\bar{v}(x_2) - \bar{v}(x_3)) \cdot \widehat{x_2 - x_3} &= 0 \end{aligned}$$

The proceeding will check that the only admissible vector field for which this happens is the zero vector field, and this the linearized hessian above is positive definite on all admissible vector fields:

Let a_i and b_i respectively be the first and second co-ordinates of $v(x_i)$, then this is a set of linear equations that can be solved:

$$\begin{aligned} \frac{1}{2}(a_1 - a_2) + \frac{\sqrt{3}}{2}(b_1 - b_2) &= 0 \\ \frac{-1}{2}(a_1 - a_3) + \frac{\sqrt{3}}{2}(b_1 - b_3) &= 0 \\ a_2 - a_3 &= 0 \end{aligned}$$

Furthermore, since these need to be *admissible* as defined in the previous chapter, they must satisfy the conservation of center of mass and the conservation of angular momentum. For these it will be needed the precise co-ordinates of the particles. Recall here that 0 is the center of

mass, so

$$\begin{aligned}
0 &= m_1x_1 + m_2x_2 + m_3x_3 \\
&= m_1x_1 + m_2x_1 + m_2L \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} + m_3x_1 + m_3L \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \\
&= x_1 + \frac{L}{2} \begin{bmatrix} m_2 - m_3 \\ \sqrt{3}(m_2 + m_3) \end{bmatrix} \\
&= x_1 + \frac{L}{2} \begin{bmatrix} m_2 - m_3 \\ \sqrt{3}(1 - m_1) \end{bmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
x_1 &= \frac{L}{2} \begin{bmatrix} m_3 - m_2 \\ -\sqrt{3}(m_2 + m_3) \end{bmatrix} \\
x_2 &= \frac{L}{2} \begin{bmatrix} m_3 - m_2 + 1 \\ m_1\sqrt{3} \end{bmatrix} \\
x_3 &= \frac{L}{2} \begin{bmatrix} m_3 - m_2 - 1 \\ m_1\sqrt{3} \end{bmatrix}
\end{aligned}$$

. Then in the chosen co-ordinates the admissibility requirements are:

$$\begin{aligned}
m_1a_1 + m_2a_2 + m_3a_3 &= 0 \\
m_1b_1 + m_2b_2 + m_3b_3 &= 0 \\
\frac{L}{2} \left[m_1 \left(a_1\sqrt{3}(m_2 + m_3) + b_1(m_3 - m_2) \right) + m_2 \left(-a_2m_1\sqrt{3} + b_2(m_3 - m_2 + 1) \right) \right. \\
&\quad \left. + m_3 \left(-a_3m_1\sqrt{3} + b_3(m_3 - m_2 - 1) \right) \right] = 0
\end{aligned}$$

So finding solutions of the above equations for $a_1, b_1, a_2, b_2, a_3, b_3$ is equivalent to finding solutions to

$$\left[\begin{array}{cccccc|c}
1 & \sqrt{3} & -1 & -\sqrt{3} & 0 & 0 & 0 \\
-1 & \sqrt{3} & 0 & 0 & 1 & -\sqrt{3} & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 \\
m_1 & 0 & m_2 & 0 & m_3 & 0 & 0 \\
0 & m_1 & 0 & m_2 & 0 & m_3 & 0 \\
\frac{\sqrt{3}m_1(m_2+m_3)}{1} & \frac{m_1(m_3-m_2)}{1} & \frac{-m_1m_2\sqrt{3}}{1} & \frac{(m_3-m_2+1)}{1} & \frac{-m_1m_3\sqrt{3}}{1} & \frac{(m_3-m_2-1)}{1} & 0
\end{array} \right]$$

This coefficient matrix can be row reduced in six steps:

1. Use row 1 to reduce rows 2, 4, and 6

$$\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{3} & -1 & -\sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -m_1\sqrt{3} & m_2 + 2m_1 & m_1\sqrt{3} & m_3 & 0 \\ 0 & m_1 & 0 & m_2 & 0 & m_3 \\ 0 & m_1(2m_3 - 4m_2) & \sqrt{3}m_1m_3 & \frac{(m_3 - m_2 + 1) + m_1(m_2 + m_3)}{1} & -m_1m_3\sqrt{3} & (m_3 - m_2 - 1) \end{bmatrix}$$

2. Use row 2 to reduce rows 4, 5, and 6

$$\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{3} & -1 & -\sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & m_2 + \frac{3m_1}{2} & \frac{m_1\sqrt{3}}{2} & m_3 + \frac{m_1}{2} & -\frac{m_1\sqrt{3}}{2} \\ 0 & 0 & \frac{m_1}{2\sqrt{3}} & m_2 + \frac{m_1}{2} & \frac{-m_1}{2\sqrt{3}} & m_3 + \frac{m_1}{2} \\ 0 & 0 & \frac{m_1\sqrt{3}(4m_3 - 2m_2)}{3} & \frac{m_3 - m_2 + 1 - m_1m_2 + 2m_1m_3}{1} & \frac{-m_1\sqrt{3}(4m_3 - 2m_2)}{3} & \frac{m_3 - m_2 - 1 + m_1(m_3 - 2m_2)}{1} \end{bmatrix}$$

3. Use row 3 to reduce rows 4, 5, and 6

$$\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{3} & -1 & -\sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{m_1\sqrt{3}}{2} & 1 + m_1 & -\frac{m_1\sqrt{3}}{2} \\ 0 & 0 & 0 & m_2 + \frac{m_1}{2} & 0 & m_3 + \frac{m_1}{2} \\ 0 & 0 & 0 & m_3 - m_2 + 1 - m_1m_2 + 2m_1m_3 & 0 & m_3 - m_2 - 1 + m_1(m_3 - 2m_2) \end{bmatrix}$$

4. Switch rows 4 and 5

$$\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{3} & -1 & -\sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & m_2 + \frac{m_1}{2} & 0 & m_3 + \frac{m_1}{2} \\ 0 & 0 & 0 & \frac{m_1\sqrt{3}}{2} & 1 + m_1 & -\frac{m_1\sqrt{3}}{2} \\ 0 & 0 & 0 & m_3 - m_2 + 1 - m_1m_2 + 2m_1m_3 & 0 & m_3 - m_2 - 1 + m_1(m_3 - 2m_2) \end{bmatrix}$$

5. Use row 4 to reduce rows 5 and 6

$$\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{3} & -1 & -\sqrt{3} & 1 & -\sqrt{3} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & m_2 + \frac{m_1}{2} & 0 & m_3 + \frac{m_1}{2} \\ 0 & 0 & 0 & 0 & 1 + m_1 & \frac{-m_1\sqrt{3}}{2} \left(1 + \frac{2m_3+m_1}{2m_2+m_1}\right) \\ 0 & 0 & 0 & 0 & 0 & \frac{m_3-m_2-1+m_1(m_3-2m_2)}{1} - 2 \frac{m_3-m_2+1-m_1m_2+2m_1m_3}{2m_2+m_1} \end{bmatrix}$$

Now this matrix is full rank if all of its diagonal entries are nonzero. The only nontrivial one to check is the last one. Note

$$\begin{aligned} & m_3 - m_2 - 1 + m_1(m_3 - 2m_2) - 2 \frac{m_3 - m_2 + 1 - m_1m_2 + 2m_1m_3}{2m_2 + m_1} = \\ & = [m_3 - m_2 - 1 + m_1(1 - m_1 - 3m_2)] + \left[-2 \frac{m_3 - 1 + m_1 + m_3 + 1 - m_1 + m_1^2 + m_1m_3}{2m_2 + m_1} \right] \end{aligned}$$

Now if both of these summands are negative then the sum must be negative (thus nonzero). First conserve that the first summand:

$$\begin{aligned} [m_3 - m_2 - 1 + m_1(1 - m_1 - 3m_2)] &= m_3 - m_2 - m_1 - m_2 - m_3 + m_1 - m_1^2 - 3m_1m_2 \\ &= -2m_2 - m_1^2 - 3m_1m_2 \end{aligned}$$

so that summand is negative. Now consider the numerator of the other summand,

$$-2 [m_3 - 1 + m_1 + m_3 + 1 - m_1 + m_1^2 + m_1m_3] = -2 [m_3 + m_3 + m_1^2 + m_1m_3]$$

which is negative, so that summand is negative. Thus the last diagonal entry of that row reduced coefficient matrix is nonzero, so as was stated before it is full rank.

Since it is full rank, this shows that only the zero vector field is an admissible vector field such that the linearized hessian equals zero. So for all nonzero admissible vector fields, the linearized hessian is positive definite for *any* 3 particle triangle.

6.3. Stability of N-particle rings

The N-particle ring discussed here is the “circle” of N particles evenly spaced around a circle and with equal mass. The displacement stability of the N-particle ring was first analyzed in [39] and later [12].

6.3.1. The Displacement Stability of the N-particle ring. To understand the *displacement stability* of the ring it is necessary to analyze the eigenvalues and eigenvectors of the

matrix that represents it:

$$H := \begin{bmatrix} \sum_{j \neq 1} \text{Hess}W(x_1 - x_j) & -\text{Hess}W(x_2 - x_1) & \cdots & -\text{Hess}W(x_N - x_1) \\ -\text{Hess}W(x_2 - x_1) & \sum_{j \neq 2} \text{Hess}W(x_2 - x_j) & \cdots & -\text{Hess}W(x_N - x_2) \\ \vdots & \vdots & \ddots & \vdots \\ -\text{Hess}W(x_N - x_1) & -\text{Hess}W(x_N - x_2) & \cdots & \sum_{j \neq N} \text{Hess}W(x_N - x_j) \end{bmatrix}$$

In the ring case of interest here the $\{x_i\}_{i=1}^N$ are the positions of particles of a steady N-particle ring, that are a radius r_N away from the center of mass such that the particle ring is a steady state. Here (up to rotation and translation) the particles have the positions

$$x_k = r_N R_k e_1$$

where

$$R_k := \begin{bmatrix} \cos \frac{2\pi}{N} k & -\sin \frac{2\pi}{N} k \\ \sin \frac{2\pi}{N} k & \cos \frac{2\pi}{N} k \end{bmatrix}$$

and

$$\begin{aligned} \text{Hess}W(x_i - x_j) &= F'(|x_i - x_j|) \widehat{x_i - x_j}^{\otimes 2} + \frac{F(|x_i - x_j|)}{|x_i - x_j|} (I - \widehat{x_i - x_j}^{\otimes 2}) \\ (6.3.1) \quad &= F'(\ell_{ij}) \widehat{x_i - x_j}^{\otimes 2} + \frac{F(\ell_{ij})}{\ell_{ij}} (I - \widehat{x_i - x_j}^{\otimes 2}) \end{aligned}$$

where

$$\ell_{ij} := |x_i - x_j| = 2r_N \left| \sin \frac{\pi(i-j)}{N} \right|.$$

The goal here in analyzing H is to find an orthonormal change-of-coordinates for the velocity vector fields that diagonalize H , thus giving an explicit representation of the eigenvalues and eigenvectors of H . Note that this means the eigenvectors of H are eigen-velocity vector fields on the positions of the particles on the ring.

Now H is written for velocity vector fields whose co-ordinates are in euclidean form, but the analysis here and the previous analysis start their diagonalization by rewriting the velocity vector fields in terms of radial and tangential co-ordinates. To do this, define the matrix R which has the block entries such that the ij^{th} entry of R is $(R)_{ij}$ where

$$(R)_{ij} := \begin{cases} 0 & \text{if } i \neq j \\ R_i & \text{if } i = j \end{cases}$$

Thus R changes the co-ordinates of a velocity vector field from euclidean to radial/tangential co-ordinates.

The main result of for the displacement stability is 31, namely that the eigenvalues are

$$\begin{aligned}\lambda_1^{(1)} &= \sum_{k=1}^{N-1} F'(\ell_{Nk}) \left(2 \sin^2 \frac{\pi k}{N} \right) \\ \lambda_1^{(2)} &= 0\end{aligned}$$

and for $i \in \{2, \dots, N\}$

$$\lambda_i^{(1)} = \lambda_i^{(2)} = \sum_{k=1}^{N-1} \left(F'(\ell_{Nk}) + \frac{F(\ell_{Nk})}{\ell_{Nk}} \right) \left(\frac{1 - \cos \frac{\pi(4-2i)k}{N}}{2} \right).$$

In particular, note that

$$\begin{aligned}\lambda_1^{(2)} &= \lambda_2^{(1)} = \lambda_2^{(2)} = 0 \\ \lambda_3^{(1)} &= \lambda_3^{(2)} = \frac{1}{2} \lambda_1^{(1)}.\end{aligned}$$

The eigen-velocity vector fields corresponding to $\{\lambda_i^{(1)}\}_{i \in \mathbb{Z}_N}$ are the Fourier basis on the radial co-ordinates (of mode $i-1$), whereas the eigen-velocity vector fields corresponding to $\{\lambda_i^{(2)}\}_{i \in \mathbb{Z}_N}$ are the Fourier basis on the tangential co-ordinates (of mode $i-1$).

Attaining this result requires several computations using the change of coordinates for the Hessian discussed earlier in this section.

PROPOSITION 29. *Applying the change-of-coordinates R to H , it can be shown*

$$R^T H R = \begin{bmatrix} B & -C_1^T & \cdots & -C_{N-1}^T \\ -C_1 & B & \cdots & -C_{N-2}^T \\ \vdots & \vdots & \ddots & \vdots \\ -C_{N-1} & -C_{N-2} & \cdots & B \end{bmatrix}$$

where

$$\begin{aligned}
B &= \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} \sin^2 \left(\frac{\pi k}{N} \right) & 0 \\ 0 & \cos^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\
&\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi k}{N} \right) & 0 \\ 0 & \sin^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\
C_k &= F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} -\sin^2 \left(\frac{\pi k}{N} \right) & \sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) \\ -\sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) & \cos^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\
&\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi k}{N} \right) & \sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) \\ -\sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) & -\sin^2 \left(\frac{\pi k}{N} \right) \end{bmatrix}. \\
C_{N-k}^T &= C_k
\end{aligned}$$

and more explicitly the ij^{th} block entry of $R^T H R$ is

$$(R^T H R)_{ij} = \begin{cases} B & , \text{ if } i = j \\ -C_{i-j} & , \text{ if } i > j \\ -C_{j-i}^T & , \text{ if } i < j \end{cases}.$$

The proof of this proposition will use the following lemma:

LEMMA 30. For

$$x_k := r_N R_k e_1,$$

one can rewrite $\widehat{x_i - x_j}^{\otimes 2}$ either as

$$\widehat{x_i - x_j}^{\otimes 2} = R_i \begin{bmatrix} \sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} R_i^T$$

or

$$\widehat{x_i - x_j}^{\otimes 2} = R_i \begin{bmatrix} -\sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} R_j^T.$$

PROOF. (of lemma)

One can see by a geometrical diagram that

$$\widehat{x_i - x_j} = R_{\frac{i+j}{2}} e_2$$

which can be rewritten as

$$\widehat{x_i - x_j} = R_i R_{\frac{i-j}{2}} e_2$$

or as

$$\widehat{x_i - x_j} = R_j R_{\frac{i-j}{2}} e_2$$

Thus, $\widehat{x_i - x_j}^{\otimes 2}$ can be written as

$$\begin{aligned} \widehat{x_i - x_j}^{\otimes 2} &= \left(R_i R_{\frac{i-j}{2}} e_2 \right) \left(R_i R_{\frac{i-j}{2}} e_2 \right)^T \\ &= R_i \left(\begin{bmatrix} \cos \frac{\pi(j-i)}{N} & -\sin \frac{\pi(j-i)}{N} \\ \sin \frac{\pi(j-i)}{N} & \cos \frac{\pi(j-i)}{N} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \frac{\pi(j-i)}{N} & \sin \frac{\pi(j-i)}{N} \\ -\sin \frac{\pi(j-i)}{N} & \cos \frac{\pi(j-i)}{N} \end{bmatrix} \right) R_i^T \\ &= R_i \left(\begin{bmatrix} 0 & -\sin \frac{\pi(j-i)}{N} \\ 0 & \cos \frac{\pi(j-i)}{N} \end{bmatrix} \begin{bmatrix} \cos \frac{\pi(j-i)}{N} & \sin \frac{\pi(j-i)}{N} \\ -\sin \frac{\pi(j-i)}{N} & \cos \frac{\pi(j-i)}{N} \end{bmatrix} \right) R_i^T \\ &= R_i \begin{bmatrix} \sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} R_i^T \end{aligned}$$

or as

$$\begin{aligned} \widehat{x_i - x_j}^{\otimes 2} &= \left(R_i R_{\frac{i-j}{2}} e_2 \right) \left(R_j R_{\frac{i-j}{2}} e_2 \right)^T \\ &= R_i \left(\begin{bmatrix} \cos \frac{\pi(j-i)}{N} & -\sin \frac{\pi(j-i)}{N} \\ \sin \frac{\pi(j-i)}{N} & \cos \frac{\pi(j-i)}{N} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \frac{\pi(i-j)}{N} & \sin \frac{\pi(i-j)}{N} \\ -\sin \frac{\pi(i-j)}{N} & \cos \frac{\pi(i-j)}{N} \end{bmatrix} \right) R_j^T \\ &= R_i \left(\begin{bmatrix} 0 & -\sin \frac{\pi(j-i)}{N} \\ 0 & \cos \frac{\pi(j-i)}{N} \end{bmatrix} \begin{bmatrix} \cos \frac{\pi(i-j)}{N} & \sin \frac{\pi(i-j)}{N} \\ -\sin \frac{\pi(i-j)}{N} & \cos \frac{\pi(i-j)}{N} \end{bmatrix} \right) R_j^T \\ &= R_i \begin{bmatrix} -\sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} R_j^T \end{aligned}$$

which proves the lemma. \square

The previous lemma can now be used to prove the proposition stated above.

PROOF. (of Proposition)

Define

$$\begin{aligned} M_{ij} &:= R_i^T \text{Hess}W(x_i - x_j) R_i \\ N_{ij} &:= R_i^T \text{Hess}W(x_i - x_j) R_j \end{aligned}$$

then the ij^{th} entry of the matrix H can be rewritten as

$$H_{ij} = \begin{cases} R_i \left(\sum_{k \neq i} M_{ik} \right) R_i^T & , \text{ if } i = j \\ -R_i N_{ij} R_j^T & , \text{ if } i \neq j \end{cases}$$

and thus the ij^{th} entry of $R^T H R$ can be written as

$$(R^T H R)_{ij} = \begin{cases} \sum_{k \neq i} M_{ik} & , \text{ if } i = j \\ -N_{ij} & , \text{ if } i \neq j \end{cases}.$$

Now the proposition is true if for any $i, j \in \mathbb{Z}_N$

$$(6.3.2) \quad \sum_{k \neq i} M_{ik} = \sum_{k \neq i} M_{Nk} \quad \left(=: B \right)$$

and

$$(6.3.3) \quad N_{ij} = N_{N(i-j \pmod N)} \quad \left(=: C_{i-j} \right)$$

and

$$(6.3.4) \quad C_k = C_{N-k}^T.$$

To show (6.3.2), note from the definition of M_{ij} and (6.3.1) that

$$\begin{aligned} \sum_{k \neq i} M_{ik} &= R_i^T \left(\sum_{k \neq i} \text{Hess} W(x_i - x_k) \right) R_i \\ &= R_i^T \left(\sum_{k \neq i} F'(\ell_{ik}) \widehat{x_i - x_k}^{\otimes 2} + \frac{F(\ell_{ik})}{\ell_{ik}} \left(I - \widehat{x_i - x_k}^{\otimes 2} \right) \right) R_i. \end{aligned}$$

Then by applying the lemma one rewrites this as

$$\begin{aligned} \sum_{k \neq i} M_{ik} &= \sum_{k \neq i} F'(\ell_{ik}) \begin{bmatrix} \sin^2 \left(\frac{\pi(k-i)}{N} \right) & -\sin \left(\frac{\pi(k-i)}{N} \right) \cos \left(\frac{\pi(k-i)}{N} \right) \\ -\sin \left(\frac{\pi(k-i)}{N} \right) \cos \left(\frac{\pi(k-i)}{N} \right) & \cos^2 \left(\frac{\pi(k-i)}{N} \right) \end{bmatrix} \\ &\quad + \frac{F(\ell_{ik})}{\ell_{ik}} \begin{bmatrix} 1 - \sin^2 \left(\frac{\pi(k-i)}{N} \right) & \sin \left(\frac{\pi(k-i)}{N} \right) \cos \left(\frac{\pi(k-i)}{N} \right) \\ \sin \left(\frac{\pi(k-i)}{N} \right) \cos \left(\frac{\pi(k-i)}{N} \right) & 1 - \cos^2 \left(\frac{\pi(k-i)}{N} \right) \end{bmatrix} \\ &= \sum_{k \neq i} F' \left(2r_N \left| \sin \frac{\pi(k-i)}{N} \right| \right) \begin{bmatrix} \sin^2 \left(\frac{\pi(k-i)}{N} \right) & 0 \\ 0 & \cos^2 \left(\frac{\pi(k-i)}{N} \right) \end{bmatrix} \\ &\quad + \frac{F \left(2r_N \left| \sin \frac{\pi(k-i)}{N} \right| \right)}{2r_N \left| \sin \frac{\pi(k-i)}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi(k-i)}{N} \right) & 0 \\ 0 & \sin^2 \left(\frac{\pi(k-i)}{N} \right) \end{bmatrix} \end{aligned}$$

where the second equality comes from the fact that $F' \left(2r_N \left| \sin \frac{\pi(k-i)}{N} \right| \right)$ and $\frac{F \left(2r_N \left| \sin \frac{\pi(k-i)}{N} \right| \right)}{2r_N \left| \sin \frac{\pi(k-i)}{N} \right|}$ are even over $k \in \mathbb{Z}_N$, while $\sin \left(\frac{\pi(k-i)}{N} \right) \cos \left(\frac{\pi(k-i)}{N} \right)$ is odd over $k \in \mathbb{Z}_N$. This resulting expression's summands are even over $k \in \mathbb{Z}_N$ and this is enough symmetry to show that the sums are independent of the i chosen. Thus, one can calculate B by picking any $i \in \mathbb{Z}_N$, and here $i = N$ is used:

$$\begin{aligned} B &= \sum_{k \neq N} M_{Nk} \\ &= \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} \sin^2 \left(\frac{\pi k}{N} \right) & 0 \\ 0 & \cos^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\ &\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi k}{N} \right) & 0 \\ 0 & \sin^2 \left(\frac{\pi k}{N} \right) \end{bmatrix}. \end{aligned}$$

This completes the demonstration of (6.3.2).

Now to show (6.3.3) and (6.3.4), recall the definition of N_{ij} and (6.3.1) that

$$\begin{aligned} N_{ij} &= R_i^T \text{Hess}W(x_i - x_j) R_j \\ &= R_i^T \left(F'(\ell_{ij}) \widehat{x_i - x_j}^{\otimes 2} + \frac{F(\ell_{ij})}{\ell_{ij}} \left(I - \widehat{x_i - x_j}^{\otimes 2} \right) \right) R_j. \end{aligned}$$

Then by applying the lemma one rewrites this as

$$N_{ij} =$$

$$(6.3.5) \quad F' \left(2r_N \left| \sin \frac{\pi(j-i)}{N} \right| \right) \begin{bmatrix} -\sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} \\ + \frac{F \left(2r_N \left| \sin \frac{\pi(j-i)}{N} \right| \right)}{2r_N \left| \sin \frac{\pi(j-i)}{N} \right|} R_i^T R_j$$

$$(6.3.6) \quad - \frac{F \left(2r_N \left| \sin \frac{\pi(j-i)}{N} \right| \right)}{2r_N \left| \sin \frac{\pi(j-i)}{N} \right|} \begin{bmatrix} -\sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix}$$

and

$$\left(R_i^T R_j - \begin{bmatrix} -\sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} \right) = \\ \begin{bmatrix} \cos \frac{2\pi(j-i)}{N} & -\sin \frac{2\pi(j-i)}{N} \\ \sin \frac{2\pi(j-i)}{N} & \cos \frac{2\pi(j-i)}{N} \end{bmatrix} \\ - \frac{1}{2} \begin{bmatrix} \cos \frac{2\pi(j-i)}{N} - 1 & -\sin \frac{2\pi(j-i)}{N} \\ \sin \frac{2\pi(j-i)}{N} & \cos \frac{2\pi(j-i)}{N} + 1 \end{bmatrix} \\ = \frac{1}{2} \begin{bmatrix} \cos \frac{2\pi(j-i)}{N} + 1 & -\sin \frac{2\pi(j-i)}{N} \\ \sin \frac{2\pi(j-i)}{N} & \cos \frac{2\pi(j-i)}{N} - 1 \end{bmatrix} \\ = \begin{bmatrix} \cos^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & -\sin^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix}$$

so that

$$(6.3.7) \quad N_{ij} = F' \left(2r_N \left| \sin \frac{\pi(j-i)}{N} \right| \right) \begin{bmatrix} -\sin^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & \cos^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix} \\ + \frac{F \left(2r_N \left| \sin \frac{\pi(j-i)}{N} \right| \right)}{2r_N \left| \sin \frac{\pi(j-i)}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi(j-i)}{N} \right) & -\sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) \\ \sin \left(\frac{\pi(j-i)}{N} \right) \cos \left(\frac{\pi(j-i)}{N} \right) & -\sin^2 \left(\frac{\pi(j-i)}{N} \right) \end{bmatrix}.$$

This expression only depends on $i - j \pmod N$. Thus, for all i and j ,

$$N_{ij} = N_{N(i-j \pmod N)}.$$

Furthermore (6.3.6) shows

$$N_{ji} = N_{ij}^T.$$

So when C_k is defined such that

$$\begin{aligned} C_k &:= N_{Nk} \\ &= F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} -\sin^2 \left(\frac{\pi k}{N} \right) & \sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) \\ -\sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) & \cos^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\ &\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi k}{N} \right) & \sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) \\ -\sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) & -\sin^2 \left(\frac{\pi k}{N} \right) \end{bmatrix}. \end{aligned}$$

then

$$C_k = C_{N-k}^T.$$

Note that

$$\begin{aligned} N_{ij} &= \begin{cases} C_{i-j} & , \text{ if } i > j \\ C_{N-(i-j)}^T & , \text{ if } i < j \end{cases} \\ &= \begin{cases} C_{i-j} & , \text{ if } i > j \\ C_{j-i}^T & , \text{ if } i < j \end{cases} \end{aligned}$$

thus completing the proposition that

$$(R^T H R)_{ij} = \begin{cases} B & , \text{ if } i = j \\ -C_{i-j} & , \text{ if } i > j \\ -C_{j-i}^T & , \text{ if } i < j \end{cases}$$

with B and C as defined. □

Now for the diagonalization being presented here, define the Fourier matrix S whose the ij^{th} block entry is

$$S_{ij} = R_{(i-1)(j-1)}.$$

$\frac{1}{\sqrt{N}}S$ is an orthonormal change-of-coordinates that diagonalizes $R^T H R$.

THEOREM 31. *Using S and R , the diagonalization of H is achieved, in that the ij^{th} entry of $\frac{1}{N}S^T R^T HRS$ is*

$$\left(\frac{1}{N}S^T R^T HRS\right)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \begin{bmatrix} \lambda_i^{(1)} & 0 \\ 0 & \lambda_i^{(2)} \end{bmatrix} & \text{if } i = j \end{cases}$$

where the eigenvalues are

$$\begin{aligned} \lambda_1^{(1)} &= \sum_{k=1}^{N-1} F'(\ell_{Nk}) \left(2 \sin^2 \frac{\pi k}{N}\right) \\ \lambda_1^{(2)} &= 0 \end{aligned}$$

and for $i \in \{2, \dots, N\}$

$$\lambda_i^{(1)} = \lambda_i^{(2)} = \sum_{k=1}^{N-1} \left(F'(\ell_{Nk}) + \frac{F(\ell_{Nk})}{\ell_{Nk}} \right) \left(\frac{1 - \cos \frac{\pi(4-2i)k}{N}}{2} \right).$$

In particular, note that

$$\begin{aligned} \lambda_1^{(2)} &= \lambda_2^{(1)} = \lambda_2^{(2)} = 0 \\ \lambda_3^{(1)} &= \lambda_3^{(2)} = \frac{1}{2}\lambda_1^{(1)}. \end{aligned}$$

And furthermore, the eigen-velocity vector fields corresponding to $\{\lambda_i^{(1)}\}_{i \in \mathbb{Z}_N}$ are the Fourier basis on the radial co-ordinates (of mode $i-1$), whereas the eigen-velocity vector fields corresponding to $\{\lambda_i^{(2)}\}_{i \in \mathbb{Z}_N}$ are the Fourier basis on the tangential co-ordinates (of mode $i-1$).

PROOF. The proof is a computation of the individual ij^{th} entry of $\frac{1}{N}S^T R^T HRS$. To prove this takes several steps: \square

Step 1: Compute that

$$(S^T R^T HRS)_{ij} = \sum_{h=1}^N R_{(i-1)(h-1)}^T \left(B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)} \right) R_{(h-1)(j-1)}.$$

Step 2: Show that if Q_j is a symmetric matrix, then

$$\sum_{h=1}^N R_{(i-1)(h-1)}^T Q_j R_{(h-1)(j-1)} = \begin{cases} 0 & , \text{ if } i \neq j \\ NQ_j & , \text{ if } i = j = 1 \\ \left(N \frac{\text{tr} Q_j}{2} \right) I & , \text{ if } i = j \neq 1 \end{cases}$$

Step 3: Define and calculate $Q_j := \left(B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)} \right)$. It is a symmetric matrix, so step 2 applies to give an explicit computation of the expression in step 1.

PROOF. To show the first step, that

$$(S^T R^T HRS)_{ij} = \sum_{h=1}^N R_{(i-1)(h-1)}^T \left(B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)} \right) R_{(h-1)(j-1)},$$

recall from the previous proposition that

$$(R^T HR)_{ij} = \begin{cases} B & , \text{ if } i = j \\ -C_{i-j} & , \text{ if } i > j \\ -C_{j-i}^T & , \text{ if } i < j \end{cases}$$

and that

$$C_{N-x}^T = C_x.$$

Then

$$\begin{aligned} (R^T HRS)_{ij} &= \sum_{k=1}^N (R^T HR)_{ik} S_{kj} \\ &= -\sum_{k=1}^{i-1} C_{i-k} R_{(k-1)(j-1)} + BR_{(i-1)(j-1)} - \sum_{k=i+1}^N C_{(k-i)}^T R_{(k-1)(j-1)} \\ &= -\sum_{k=1}^{i-1} C_{i-k} R_{(k-1)(j-1)} + BR_{(i-1)(j-1)} - \sum_{k=i+1}^N C_{N+i-k} R_{(k-1)(j-1)}. \end{aligned}$$

By using the change of variables $\tilde{k} := N + i - k$, one sees

$$\begin{aligned} \sum_{k=i+1}^N C_{N+i-k} R_{(k-1)(j-1)} &= \sum_{\tilde{k}=i}^{N-1} C_{\tilde{k}} R_{(N+i-\tilde{k}-1)(j-1)} \\ &= \sum_{\tilde{k}=i}^{N-1} C_{\tilde{k}} R_{(i-\tilde{k}-1)(j-1)} \end{aligned}$$

and by using the change of variables $\tilde{k} := i - k$, one sees

$$\sum_{k=1}^{i-1} C_{i-k} R_{(k-1)(j-1)} = \sum_{\tilde{k}=1}^{i-1} C_{\tilde{k}} R_{(i-\tilde{k}-1)(j-1)}$$

and so by relabeling

$$(R^T HRS)_{ij} = BR_{(i-1)(j-1)} - \sum_{k=1}^{i-1} C_k R_{(i-k-1)(j-1)}.$$

Then to complete the first step of the proof, calculate

$$\begin{aligned}
(S^T R^T HRS)_{ij} &= \sum_{h=1}^N S_{ih}^T (R^T HRS)_{hj} \\
&= \sum_{h=1}^N R_{(i-1)(h-1)}^T (BR_{(h-1)(j-1)} - \sum_{k=1}^{N-1} C_k R_{(h-k-1)(j-1)}) \\
&= \sum_{h=1}^N R_{(i-1)(h-1)}^T (B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)}) R_{(h-1)(j-1)}
\end{aligned}$$

Now the task is to show the second step of the proof, that if Q is a symmetric matrix, then

$$\sum_{h=1}^N R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} = \begin{cases} 0 & , \text{ if } i \neq j \\ NQ & , \text{ if } i = j = 1 . \\ \left(N \frac{\text{tr} Q}{2}\right) I & , \text{ if } i = j \neq 1 \end{cases}$$

To see this recall that by the spectral theorem there exists an orthogonal matrix E and eigenvalues of $\lambda^{(1)}$, $\lambda^{(2)}$, and v_1, v_2 orthogonal vectors such that

$$\begin{aligned}
Q &= E^T \begin{bmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{bmatrix} E \\
E &= \begin{bmatrix} v_1 & v_2 \end{bmatrix}
\end{aligned}$$

and without loss of generality suppose that

$$v_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v_1.$$

Now to compute the sum, rewrite it as

$$\sum_{h=1}^N R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} = E \left(\sum_{h=1}^N E^T R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} E \right) E^T$$

and start by computing the summands:

$$E^T R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} E = \left(R_{(i-1)(h-1)} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \right)^T Q \left(R_{(h-1)(j-1)} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \right)$$

To do this, note that

$$\begin{aligned}
\left(R_{(h-1)(j-1)} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \right)^T &= \begin{bmatrix} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_1 + \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_2 \\ \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_2 - \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_1 \end{bmatrix} \\
\left(R_{(h-1)(i-1)} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \right)^T &= \begin{bmatrix} \cos \left(\frac{2\pi}{N} (h-1)(i-1) \right) v_1^T + \sin \left(\frac{2\pi}{N} (h-1)(i-1) \right) v_2^T \\ \cos \left(\frac{2\pi}{N} (h-1)(i-1) \right) v_2^T - \sin \left(\frac{2\pi}{N} (h-1)(i-1) \right) v_1^T \end{bmatrix}
\end{aligned}$$

and because v_1 and v_2 are eigenvectors of Q with eigenvalues λ_1 and λ_2 , respectively, one gets

$$QR_{(h-1)(j-1)} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\ = \begin{bmatrix} \left(\begin{array}{c} \lambda^{(1)} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_1 \\ + \lambda^{(2)} \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_2 \end{array} \right) & \left(\begin{array}{c} \lambda^{(2)} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_2 \\ - \lambda^{(1)} \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) v_1 \end{array} \right) \end{bmatrix}.$$

Putting together the last two equations one can compute the components of $E^T R_{(i-1)(h-1)}^T QR_{(h-1)(j-1)} E$

$$\begin{aligned} \left(E^T R_{(i-1)(h-1)}^T QR_{(h-1)(j-1)} E \right)_{11} &= \lambda^{(1)} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) \cos \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ &\quad + \lambda^{(2)} \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) \sin \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ \left(E^T R_{(i-1)(h-1)}^T QR_{(h-1)(j-1)} E \right)_{12} &= \lambda^{(2)} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) \sin \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ &\quad - \lambda^{(1)} \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) \cos \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ \left(E^T R_{(i-1)(h-1)}^T QR_{(h-1)(j-1)} E \right)_{21} &= -\lambda^{(1)} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) \sin \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ &\quad + \lambda^{(2)} \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) \cos \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ \left(E^T R_{(i-1)(h-1)}^T QR_{(h-1)(j-1)} E \right)_{22} &= \lambda^{(2)} \cos \left(\frac{2\pi}{N} (h-1)(j-1) \right) \cos \left(\frac{2\pi}{N} (h-1)(i-1) \right) \\ &\quad + \lambda^{(1)} \sin \left(\frac{2\pi}{N} (h-1)(j-1) \right) \sin \left(\frac{2\pi}{N} (h-1)(i-1) \right). \end{aligned}$$

Note that for any i, j that

$$\begin{aligned} \sum_{h=1}^N \cos\left(\frac{2\pi}{N}(h-1)(j-1)\right) \sin\left(\frac{2\pi}{N}(h-1)(i-1)\right) &= 0 \\ \sum_{h=1}^N \cos\left(\frac{2\pi}{N}(h-1)(j-1)\right) \sin\left(\frac{2\pi}{N}(h-1)(i-1)\right) &= 0 \\ \sum_{h=1}^N \cos\left(\frac{2\pi}{N}(h-1)(j-1)\right) \cos\left(\frac{2\pi}{N}(h-1)(i-1)\right) &= \begin{cases} 0 & \text{if } i \neq j \\ N & \text{if } i = j = 1 \\ \frac{N}{2} & \text{if } i = j \neq 1 \end{cases} \\ \sum_{h=1}^N \sin\left(\frac{2\pi}{N}(h-1)(j-1)\right) \sin\left(\frac{2\pi}{N}(h-1)(i-1)\right) &= \begin{cases} 0 & \text{if } i \neq j \\ 0 & \text{if } i = j = 1 \\ \frac{N}{2} & \text{if } i = j \neq 1 \end{cases} \end{aligned}$$

and therefore

$$\sum_{h=1}^N E^T R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} E = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } i \neq j \\ N \begin{bmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{bmatrix} & \text{if } i = j = 1. \\ N \begin{bmatrix} \frac{\lambda^{(1)} + \lambda^{(2)}}{2} & 0 \\ 0 & \frac{\lambda^{(1)} + \lambda^{(2)}}{2} \end{bmatrix} & \text{if } i = j \neq 1 \end{cases}$$

Thus, noting that

$$\begin{aligned}
\sum_{h=1}^N R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} &= E \left(\sum_{h=1}^N E^T R_{(i-1)(h-1)}^T Q R_{(h-1)(j-1)} E \right) E^T \\
&= \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } i \neq j \\ N E^T \begin{bmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{bmatrix} E & \text{if } i = j = 1 \\ N E^T \begin{bmatrix} \frac{\lambda^{(1)} + \lambda^{(2)}}{2} & 0 \\ 0 & \frac{\lambda^{(1)} + \lambda^{(2)}}{2} \end{bmatrix} E & \text{if } i = j \neq 1 \end{cases} \\
&= \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } i \neq j \\ N Q & \text{if } i = j = 1 \\ \left(N \frac{\text{tr} Q}{2} \right) I & \text{if } i = j \neq 1 \end{cases}
\end{aligned}$$

which is what was to be shown for step 2.

The third step of the proof is to compute $\left(B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)} \right)$ and show it is symmetric, and therefore this computation and step 2 will give an explicit computation of $(S^T R^T H R S)_{ij}$ because of what was shown in the first step of the proof. Recall from 29 that

$$\begin{aligned}
B &= \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} \sin^2 \left(\frac{\pi k}{N} \right) & 0 \\ 0 & \cos^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\
&\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi k}{N} \right) & 0 \\ 0 & \sin^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\
C_k &= F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} -\sin^2 \left(\frac{\pi k}{N} \right) & \sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) \\ -\sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) & \cos^2 \left(\frac{\pi k}{N} \right) \end{bmatrix} \\
&\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos^2 \left(\frac{\pi k}{N} \right) & \sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) \\ -\sin \left(\frac{\pi k}{N} \right) \cos \left(\frac{\pi k}{N} \right) & -\sin^2 \left(\frac{\pi k}{N} \right) \end{bmatrix}.
\end{aligned}$$

To compute the sum it will be useful to have the following equivalent expression of C_k

$$C_k = \frac{1}{2}F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \cos \left(\frac{2\pi k}{N} \right) & \sin \left(\frac{2\pi k}{N} \right) \\ -\sin \left(\frac{2\pi k}{N} \right) & \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \right) \\ + \frac{1}{2} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} \cos \left(\frac{2\pi k}{N} \right) & \sin \left(\frac{2\pi k}{N} \right) \\ -\sin \left(\frac{2\pi k}{N} \right) & \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \right)$$

so

$$C_k R_{-k(j-1)} \\ = \frac{1}{2}F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ + \frac{1}{2}F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} \cos \left(\frac{2\pi k}{N} \right) & \sin \left(\frac{2\pi k}{N} \right) \\ -\sin \left(\frac{2\pi k}{N} \right) & \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \begin{bmatrix} \cos \frac{2\pi k(j-1)}{N} & \sin \frac{2\pi k(j-1)}{N} \\ -\sin \frac{2\pi k(j-1)}{N} & \cos \frac{2\pi k(j-1)}{N} \end{bmatrix} \\ + \frac{1}{2} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ + \frac{1}{2} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos \left(\frac{2\pi k}{N} \right) & \sin \left(\frac{2\pi k}{N} \right) \\ -\sin \left(\frac{2\pi k}{N} \right) & \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \begin{bmatrix} \cos \frac{2\pi k(j-1)}{N} & \sin \frac{2\pi k(j-1)}{N} \\ -\sin \frac{2\pi k(j-1)}{N} & \cos \frac{2\pi k(j-1)}{N} \end{bmatrix} \\ = \frac{1}{2}F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} -\cos \frac{2\pi k(j-1)}{N} + \cos \frac{2\pi k(j-2)}{N} & -\sin \frac{2\pi k(j-1)}{N} + \sin \frac{2\pi k(j-2)}{N} \\ -\sin \frac{2\pi k(j-1)}{N} - \sin \frac{2\pi k(j-2)}{N} & \cos \frac{2\pi k(j-1)}{N} + \cos \frac{2\pi k(j-2)}{N} \end{bmatrix} \\ + \frac{1}{2} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} \cos \frac{2\pi k(j-1)}{N} + \cos \frac{2\pi k(j-2)}{N} & \sin \frac{2\pi k(j-1)}{N} + \sin \frac{2\pi k(j-2)}{N} \\ \sin \frac{2\pi k(j-1)}{N} - \sin \frac{2\pi k(j-2)}{N} & -\cos \frac{2\pi k(j-1)}{N} + \cos \frac{2\pi k(j-2)}{N} \end{bmatrix}$$

Note that $F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)$ and $\frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|}$ are even over $k \in \mathbb{Z}_N$, while $\sin \left(\frac{2\pi k p}{N} \right)$ is odd over $k \in \mathbb{Z}_N$ for all $p \in \mathbb{Z}$. Thus, for all $p \in \mathbb{Z}$,

$$\sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \sin \left(\frac{2\pi k p}{N} \right) = \sum_{k=1}^{N-1} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \sin \left(\frac{2\pi k p}{N} \right) = 0$$

so all the off diagonal entries of $\sum_{k=1}^{N-1} C_k R_{-k(j-1)}$ are zero.

This means that if one also rewrites B as

$$\begin{aligned} B &= \frac{1}{2} \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} 1 - \cos \left(\frac{2\pi k}{N} \right) & 0 \\ 0 & 1 + \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \\ &\quad + \frac{1}{2} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} 1 + \cos \left(\frac{2\pi k}{N} \right) & 0 \\ 0 & 1 - \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \end{aligned}$$

then

$$\begin{aligned} &B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)} \\ &= \frac{1}{2} \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} 1 - \cos \left(\frac{2\pi k}{N} \right) + \cos \frac{2\pi k(j-1)}{N} - \cos \frac{2\pi k(j-2)}{N} & 0 \\ 0 & 1 + \cos \left(\frac{2\pi k}{N} \right) - \cos \frac{2\pi k(j-1)}{N} - \cos \frac{2\pi k(j-2)}{N} \end{bmatrix} \\ &\quad + \frac{1}{2} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} 1 + \cos \left(\frac{2\pi k}{N} \right) - \cos \frac{2\pi k(j-1)}{N} - \cos \frac{2\pi k(j-2)}{N} & 0 \\ 0 & 1 - \cos \left(\frac{2\pi k}{N} \right) + \cos \frac{2\pi k(j-1)}{N} - \cos \frac{2\pi k(j-2)}{N} \end{bmatrix}. \end{aligned}$$

Thus, Q_j is symmetric, which allows us to apply step 2 to step 1.

For simplicity, denote

$$Q_j := B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)}$$

then

$$\begin{aligned}
Q_1 &= \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \begin{bmatrix} 1 - \cos \left(\frac{2\pi k}{N} \right) & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \begin{bmatrix} 0 & 0 \\ 0 & 1 - \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \\
\text{tr} Q_j &= \sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right) \left(1 - \cos \left(\frac{2\pi k(j-2)}{N} \right) \right) \\
&\quad + \sum_{k=1}^{N-1} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right| \right)}{2r_N \left| \sin \frac{\pi k}{N} \right|} \left(1 - \cos \left(\frac{2\pi k(j-2)}{N} \right) \right).
\end{aligned}$$

This completes the proof because by step 1,

$$\begin{aligned}
(S^T R^T HRS)_{ij} &= \sum_{h=1}^N R_{(i-1)(h-1)}^T \left(B - \sum_{k=1}^{N-1} C_k R_{-k(j-1)} \right) R_{(h-1)(j-1)} \\
&= \sum_{h=1}^N R_{(i-1)(h-1)}^T Q_j R_{(h-1)(j-1)}
\end{aligned}$$

and because Q_j is symmetric (shown in step 3), step 2 shows that

$$\begin{aligned}
(S^T R^T HRS)_{ij} &= \sum_{h=1}^N R_{(i-1)(h-1)}^T Q_j R_{(h-1)(j-1)} \\
&= \begin{cases} 0 & \text{if } i \neq j \\ NQ_1 & \text{if } i = j = 1 \\ N \left(\frac{\text{tr} Q_j}{2} \right) I & \text{if } i = j \neq 1 \end{cases}
\end{aligned}$$

and the computations at the end of step three further show that

$$(S^T R^T HRS)_{ij}$$

$$\begin{aligned}
&= \begin{cases} 0 & \text{if } i \neq j \\ NQ_1 & \text{if } i = j = 1 \\ N \left(\frac{\text{tr} Q_j}{2} \right) I & \text{if } i = j \neq 1 \end{cases} \\
&= \begin{cases} 0 & \text{if } i \neq j \\ N \left\{ \begin{aligned} &\sum_{k=1}^{N-1} F' \left(2r_N \left| \sin \frac{\pi k}{N} \right. \right) \begin{bmatrix} 1 - \cos \left(\frac{2\pi k}{N} \right) & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right. \right)}{2r_N \left| \sin \frac{\pi k}{N} \right.} \begin{bmatrix} 0 & 0 \\ 0 & 1 - \cos \left(\frac{2\pi k}{N} \right) \end{bmatrix} \end{aligned} \right\} & \text{if } i = j = 1 \\ \frac{N}{2} \left(\sum_{k=1}^{N-1} \left(F' \left(2r_N \left| \sin \frac{\pi k}{N} \right. \right) + \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right. \right)}{2r_N \left| \sin \frac{\pi k}{N} \right.} \right) \left(1 - \cos \left(\frac{2\pi k(j-2)}{N} \right) \right) \right) I & \text{if } i = j \neq 1. \end{cases}
\end{aligned}$$

Further note that, when denoting

$$\frac{1}{N} (S^T R^T H R S)_{ii} = \begin{bmatrix} \lambda_i^{(1)} & 0 \\ 0 & \lambda_i^{(2)} \end{bmatrix}$$

that in fact, from the condition that the configuration is a steady N-particle ring,

$$\lambda_1^{(2)} = \sum_{k=1}^{N-1} \frac{F \left(2r_N \left| \sin \frac{\pi k}{N} \right. \right)}{2r_N \left| \sin \frac{\pi k}{N} \right.} \left(1 - \cos \left(\frac{2\pi k}{N} \right) \right) = 0.$$

Moreover, $\lambda_i^{(1)} = \lambda_i^{(2)}$ for all $i > 1$, and in particular:

$$\lambda_2^{(1)} = \lambda_2^{(2)} = 0.$$

This completes the proof. \square

REMARK 32. This diagonalization of $R^T H R$ is distinct from that in [39] and [12]. That is because they do not use the Fourier matrix S for their change-of-coordinates, but rather in coordinates represented by \tilde{S} where the ij^{th} entry of \tilde{S} is

$$\tilde{S}_{ij} = \begin{bmatrix} \cos \frac{2\pi ij}{N} & \cos \frac{2\pi ij}{N} \\ \sin \frac{2\pi ij}{N} & -\sin \frac{2\pi ij}{N} \end{bmatrix}.$$

\tilde{S} is neither invertible nor orthogonal, but most of its block entries invertible. What the authors compute there are matrices $M(m)$ such that, independently of j , satisfy

$$\left(R^T H R \tilde{S} \right)_{jm} = \tilde{S}_{jm} M(m).$$

Thus if \tilde{S} were invertible, then one could compute the ij^{th} entry of $\tilde{S}^{-1} R^T H R S$ as

$$\begin{aligned} \left(\tilde{S}^{-1} R^T H R S \right)_{ij} &= \sum_{k=1}^N \left(\tilde{S}^{-1} \right)_{ik} \left(R^T H R \tilde{S} \right)_{kj} \\ &= \left(\sum_{k=1}^N \left(\tilde{S}^{-1} \right)_{ik} \tilde{S}_{kj} \right) M(j) \\ &= \left(\tilde{S}^{-1} \tilde{S} \right)_{ij} M(j) \\ &= \begin{cases} M(j) & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases} \end{aligned}$$

thus giving a block-diagonalization of the matrix H .

6.3.2. Spreading Stability of N-particle ring. As stated in the definition of spreading stability, (4.3.20), it is necessary at at least one point $\bar{x} \in \text{supp} \bar{\mu}$ the values of $\text{Hess} W * \bar{\mu}(\bar{x})$.

Checking this at $\bar{x} = r_N e_1$ gives two entries on the diagonal of the matrix, λ_1 and λ_2 which are

$$\begin{aligned} \lambda_1 &= \frac{1}{N} \sum_{k=1}^N \frac{F'(2r_N \sin \frac{\pi k}{N})}{4 \sin^2 \frac{\pi k}{N}} \left(1 - 2 \cos \left(\frac{2\pi k}{N} \right) + \cos^2 \left(\frac{2\pi k}{N} \right) \right) \\ &\quad + \frac{1}{N} \sum_{k=1}^N \frac{F'(2r_N \sin \frac{\pi k}{N})}{8r_N \sin^3 \frac{\pi k}{N}} \left(2 \cos \left(\frac{2\pi k}{N} \right) - \cos^2 \left(\frac{2\pi k}{N} \right) \right) \\ \lambda_2 &= \frac{1}{N} \sum_{k=1}^N \frac{F'(2r_N \sin \frac{\pi k}{N})}{4 \sin^2 \frac{\pi k}{N}} \left(\sin^2 \left(\frac{2\pi k}{N} \right) \right) \\ &\quad + \frac{1}{N} \sum_{k=1}^N \frac{F'(2r_N \sin \frac{\pi k}{N})}{8r_N \sin^3 \frac{\pi k}{N}} \left(\cos^2 \left(\frac{2\pi k}{N} \right) \right) \end{aligned}$$

so if λ is the smaller of the two then the N-particle ring is 2λ spreading stable.

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