Carnegie Mellon University
MELLON COLLEGE OF SCIENCE

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF Doctor of Philosophy

TITLE Some Asymptotic Results for Phase Transition Models

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ACCEPTED BY THE DEPARTMENT OF Mathematical Sciences

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Submitted in partial fulfillment
of the requirements
for the degree of
Doctor of Philosophy
in Mathematical Sciences
at Carnegie Mellon University

April 2016

Abstract

This thesis analyzes two types of phase transition models, namely the Cahn–Hilliard model and the Becker–Döring model. In the Cahn–Hilliard setting, this thesis establishes a second-order Γ-convergence result for the mass-constrained Cahn–Hilliard energy. This is obtained using a new variant of the Pòlya–Szegő inequality, along with some new regularity results for the isoperimetric function. For the Becker–Döring model, decay rates towards equilibrium are proved for certain broad classes of subcritical data. This is obtained by using new linear stability estimates and semigroup extension results, along with some classical interpolation inequalities.

Acknowledgments

I would like to thank my advisors, Giovanni Leoni and Bob Pego, for their patience, attention, and assistance in working on these projects. Without their help this thesis would certainly never have come to be.

I would like to acknowledge the support of the NSF PIRE grant, which supported me throughout much of my time at CMU. This grant gave me many valuable opportunities to interact with European mathematicians, which greatly enriched my education. In particular, the opportunity to visit SISSA, and work with Professor Dal Maso, was an invaluable part of my formation as a mathematician.

I also have benefited enormously from the supportive environment in the Math Department at CMU. Special thanks goes to Matteo Rinaldi, Slav Kirov, Nicolás García Trillos, Daniel Rodriguez, Brian Swenson, Ben Tengelsen, Dejan Slepčev, and Irene Fonseca for discussions and helpful advice.

Finally, and most importantly, I am incredibly grateful to my supportive family, Andrea, Lucy, and Emily. You make it all worth it.

Contents

Chapter 1

Introduction

This thesis consists of the study of two (very different) phase transition problems. Accordingly the thesis is divided into two discrete parts.

The first part studies the Cahn–Hilliard energy, which represents a microscopic theory for the formation of phase boundaries. This will be studied primarily using variational methods. The work given here is mostly contained in the two papers [73] and [83], although some results have been streamlined and improved here compared to the versions given in those papers.

The second part studies the Becker-Döring model, which represents a mean field theory of the nucleation of a phase transition. This was studied using semigroup theory and PDE methods. Some of the results presented here are contained in the paper [81].

1.1 Cahn–Hilliard Theory of Phase Transitions

The first part of this thesis will be concerned with the asymptotic expansion by Γ-convergence of the Cahn–Hilliard or Modica–Mortola functional, and some applications of the same. This functional is given by (see [63, 78, 101])

$$F_\varepsilon(u) := \int_\Omega W(u) + \varepsilon^2 |\nabla u|^2 \, dx, \qquad u \in H^1(\Omega), \tag{1.1.1}$$

subject to the mass constraint

$$\int_\Omega u \, dx = m. \tag{1.1.2}$$

Here $\Omega \subset \mathbb{R}^n$ is an open, bounded set, $W : \mathbb{R} \to [0, \infty)$ is a double-well potential and $\varepsilon > 0$.

The Cahn–Hilliard functional is one mathematical representation of the "energetic" cost of a phase transition in a material. Here Ω represents some physical domain (i.e. the limits of our material), and ε is a regularizing parameter, which turns out to be the approximate width of transition layers. The phase is represented by u, and W represents the potential energy of a given phase. In some cases u is called an *Order Parameter*, because it represents the relative order of a given phase. This model has been used to represent certain simple phase transitions, such as liquid-liquid phase transitions [108] [28] and antiphase boundaries [6]. The mass constraint is particularly relevant in the case of certain liquid phase transition problems, while other types of boundary conditions are more relevant in other situations.

Oftentimes phase transition energies are more appropriately modeled by considering vector-valued u [55], anisotropic gradient terms [90], higher-order terms [53] or contact energies [79]. With the exception of a few simple preliminary results for the anisotropic case, this thesis does not attempt to address these issues. However, the energy considered here is still a relevant toy model that gives reasonable intuition towards the more complicated cases.

As $\varepsilon \to 0$, minimizers of this energy approach sharp transition layers. One appropriate way to study this convergence is through Γ-convergence (see Section 2.4). In the interest of proving such a Γ-convergence result, define $\mathcal{F}_\varepsilon : L^1(\Omega) \to (-\infty, \infty]$ by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} F_\varepsilon(u) & \text{if } u \in H^1(\Omega) \text{ and (1.1.2) holds,} \\ \infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{1.1.3}$$

An asymptotic expansion by Γ-convergence essentially seeks to find an appropriate sort of Taylor expansion for the energy, namely

$$\mathcal{F}_\varepsilon \approx \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \varepsilon^2 \mathcal{F}^{(2)} + \dots$$

The notion of Γ-convergence only requires that this expansion hold in an appropriate limiting sense; for precise definitions see Section 2.4.

The Γ-limit $\mathcal{F}^{(1)}$ of order 1 (see (2.4.1) and (2.4.2)), which in this case is simply the Γ-limit of $\varepsilon^{-1}\mathcal{F}_\varepsilon$, has been characterized by Carr, Gurtin and Slemrod [31] for $n = 1$ and by Modica [78] and Sternberg [101] for $n \geq 2$ (see also [62], [80]), and is known to be, under appropriate assumptions on Ω and W,

$$\mathcal{F}^{(1)}(u) := \begin{cases} 2c_W \, \mathrm{P}(\{u = a\}; \Omega) & \text{if } u \in BV(\Omega; \{a, b\}) \text{ and (1.1.2) holds,} \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.1.4}$$

where $\mathrm{P}(\cdot; \Omega)$ is the perimeter in Ω (see Section 2.1), a, b are the wells of W and the constant c_W is given by

$$c_W := \int_a^b W^{1/2}(s)\, ds. \tag{1.1.5}$$

The recovery sequence used to obtain this result is given by functions of the form

$$u_\varepsilon(x) = z\left(\frac{d_E(x)}{\varepsilon}\right),$$

where z is the solution to the Cauchy problem

$$\begin{cases} z'(t) = \sqrt{W(z(t))} & \text{for } t \in \mathbb{R}, \\ z(0) = c, & z(t) \in [a, b], \end{cases} \tag{1.1.6}$$

with c being the central zero of W'. The function z solving this Cauchy problem will also play a crucial role in the analysis performed in this thesis. It is easy to see that $u_\varepsilon \to \operatorname{sgn}_{a,b} \circ d_E$, where

$$\operatorname{sgn}_{a,b}(t) := \begin{cases} a & \text{if } t \leq 0, \\ b & \text{if } t > 0. \end{cases} \tag{1.1.7}$$

In light of this Γ-convergence result, it is natural to study the family $\mathcal{U}_1$ of minimizers of the functional $\mathcal{F}^{(1)}$. Observe that u belongs to $\mathcal{U}_1$ if and only if $u \in BV(\Omega; \{a, b\})$ and the set $\{u = a\}$ is a solution of the classical *partition problem*, namely, if it solves

$$\min\{\mathrm{P}(E; \Omega) : \; E \subset \Omega \text{ Borel}, \; \mathcal{L}^n(E) = \mathfrak{v}_m\}, \tag{1.1.8}$$

where

$$\mathfrak{v}_m := \frac{b\mathcal{L}^n(\Omega) - m}{b-a}. \tag{1.1.9}$$

The properties of minimizers of (1.1.8) have been studied by Grüter [60] (see also [58, 75, 103]), who showed that when Ω is bounded and of class C^2, minimizers E of (1.1.8) exist, have constant generalized mean curvature κ_E, intersect the boundary of Ω orthogonally, and their singular set is empty if $n \le 7$, and has dimension of at most $n-8$ if $n \ge 8$. By way of convention, here κ_E *is the average of the principal curvatures taken with respect to the outward unit normal to* ∂E.

Furthermore, in studying the partition problem, which is closely linked to the problem of minimizing $\mathcal{F}_\varepsilon$, a natural construct is the *isoperimetric function* or *isoperimetric profile* (see, e.g., [96]), given by

$$\mathcal{I}_\Omega(\mathfrak{v}) := \inf\{\mathrm{P}(E;\Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = \mathfrak{v}\}, \quad \mathfrak{v} \in [0, \mathcal{L}^n(\Omega)]. \tag{1.1.10}$$

Throughout this work it will be helpful to consider an L^1-localized version of this function. Namely, given a measurable set $E_0 \subset \Omega$ with mass $\mathfrak{v}_m$ (see (1.1.8) and (1.1.9)) and $\delta > 0$, we define (see (6.1.3))

$$\mathcal{I}_\Omega^{\delta,E_0}(r) := \inf\{P(E,\Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = r, \alpha(E,E_0) \le \delta\}, \tag{1.1.11}$$

where

$$\alpha(E,E_0) := \min\{\mathcal{L}^n(E \setminus E_0), \mathcal{L}^n(E_0 \setminus E)\}. \tag{1.1.12}$$

A natural question, and really the starting point of the work of this thesis, is how to appropriately characterize the Γ-limit of order 2, written $\mathcal{F}^{(2)}$, of $\mathcal{F}_\varepsilon$. The first example of asymptotic development by Γ-convergence of order 2 for functionals of the type (1.1.1) was studied by Anzellotti and Baldo in [13], who considered the case in which $n=1$, the wells of W are not points but non-degenerate intervals and the mass constraint (1.1.2) is replaced by a Dirichlet condition. Subsequently Anzellotti, Baldo and Orlandi [14] studied (1.1.1) in arbitrary dimension, in the case in which W has only one well ($W(s) = s^2$) and again with Dirichlet boundary conditions in place of (1.1.2).

In dimension $n=1$, this problem has been extensively studied by a variety of authors, see e.g. [31],[59], [18]. Prior to the work in this thesis, the only work in the case $n \ge 2$ was given by Dal Maso, Fonseca, and Leoni in [41]. In that work, for a potential W satisfying

$$W(s) = W(-s)$$

for all $s \in \mathbb{R}$ and

$$W(s) = C|1-s|^{1+q} \tag{1.1.13}$$

near $s=1$, for some $q \in (0,1)$, and *under the assumption* that

$$u = 1 \text{ on } \partial\Omega, \tag{1.1.14}$$

in addition to (1.1.2), it was shown that $\mathcal{F}^{(2)} = 0$. More generally, this was proved in the case in which $\varepsilon^2 \int_\Omega |\nabla u|^2\,dx$ is replaced by $\varepsilon^2 \int_\Omega \Phi^2(\nabla u)\,dx$, with $\Phi : \mathbb{R}^n \to [0,\infty)$ an arbitrary norm. The Dirichlet condition (1.1.14) played a crucial role in the proof in [41] since it permitted the use of classical symmetrization techniques in $H_0^1(\Omega)$ to reduce the problem to the radial case. Moreover, the behavior of W near the wells (see (1.1.13)) did not allow for C^2 potentials W. The work of [41] left open several important questions, namely the characterization of $\mathcal{F}^{(2)}$ when

- the Dirichlet condition (1.1.14) is not imposed,

- W is of class C^2,
- W is not even.

The first part of this thesis addresses all of these questions, by characterizing the second order Γ-limit under fairly general conditions. In particular, in the case where W is C^2, the following theorem is given in Chapter 6 (see Theorems 6.1.2, 6.1.3).

Theorem 1.1.1. *Assume that* Ω *satisfies* (6.1.1), m *satisfies* (6.1.2) *and* $W \in C^2$ *satisfies hypotheses* (5.1.4)-(5.1.7). *Assume that* u *is an* $L^1(\Omega)$*-local minimizer of the functional* $\mathcal{F}^{(1)}$ *(see* (1.1.4)*). Finally, assume that, for some* $\delta > 0$, $\mathcal{I}_\Omega^{\delta,E_0}$ *is differentiable at* $\mathfrak{v}_m$, *with* $E_0 = \{u = a\}$*. Then*

$$\begin{aligned}\Gamma\text{-}\liminf \tilde{\mathcal{F}}_\varepsilon(u) &= \Gamma\text{-}\limsup \tilde{\mathcal{F}}_\varepsilon(u)\\ &= \frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + 2(c_{\text{sym}} + c_W\tau_u)(n-1)\kappa_u\,\mathrm{P}(\{u=a\};\Omega),\end{aligned}$$

where

$$\tilde{\mathcal{F}}_\varepsilon(w) := \frac{\mathcal{F}_\varepsilon^{(1)}(w) - \mathcal{F}^{(1)}(u)}{\varepsilon}$$

and

$$\mathcal{F}_\varepsilon^{(1)}(w) = \frac{\mathcal{F}_\varepsilon(w)}{\varepsilon}.$$

In particular, if $\mathcal{I}_\Omega$ *is differentiable at* $\mathfrak{v}_m$ *then*

$$\mathcal{F}^{(2)}(u) = \frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + 2(c_{\text{sym}} + c_W\tau_u)(n-1)\kappa_u\,\mathrm{P}(\{u=a\};\Omega)$$

if u *is a global minimizer of* $\mathcal{F}^{(1)}$ *and* $\mathcal{F}^{(2)}(u) = \infty$ *otherwise in* $L^1(\Omega)$*.*

In this theorem, κ_u is the constant mean curvature of the set $\{u = a\}$,

$$c_{\text{sym}} := \int_{\mathbb{R}} W(z(t))t\,dt,$$

where z is the solution to the Cauchy problem (1.1.6), and $\tau_u \in \mathbb{R}$ is a constant such that

$$\mathrm{P}(\{u=a\};\Omega)\int_{\mathbb{R}} z(t-\tau_u) - \mathrm{sgn}_{a,b}(t)\,dt = \frac{2c_W(n-1)}{W''(a)(b-a)}\kappa_u,$$

with $\mathrm{sgn}_{a,b}$ as defined in (1.1.7).

The previous theorem assumes that $\mathcal{I}_\Omega$ or $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathfrak{v}_m$. The validity of this assumption has only been previously considered in the case where Ω is convex. In that case, it is known that $\mathcal{I}_\Omega$ is concave [103]. However, many of the techniques in [103] generalize to the present setting. In particular, in Chapter 4, it is proven that

- $\mathcal{I}_\Omega$ is differentiable at all but countably many points.
- $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathfrak{v}_m$ if E_0 is an isolated local volume-constrained perimeter minimizer, for δ small enough.

The proof of Theorem 1.1.1 uses an adaptation of the Polyà–Szegő inequality, applicable to functions irrespective of boundary conditions, namely Theorem 3.3.4. The techniques used in the proof of this theorem are largely standard, but are included in Chapter 3 for clarity. A specific form of this inequality was previously used to study optimal constants for certain classes of Poincaré inequalities [34].

Using this rearrangement inequality, the problem of proving theorem 1.1.1 is reduced to the careful analysis of a one dimensional problem. This is conducted in Chapter 5. Much of the analysis here leans on classical tools, such as those used in [41] and [102].

Finally, these tools are combined in Chapter 6 to prove the main theorems.

One of the primary motivations for studying the asymptotic expansion of $\mathcal{F}_\varepsilon$ is to understand the motion of solutions of the underlying gradient flow.

In particular, one may study the slow motion of solutions to the nonlocal Allen–Cahn equation with Neumann boundary conditions, namely,

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon) + \varepsilon\lambda_\varepsilon & \text{in } \Omega \times [0,\infty), \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u_\varepsilon = u_{0,\varepsilon} & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.1.15}$$

Here $u_{0,\varepsilon}$ is the initial datum, and λ_ε is a Lagrange multiplier that renders solutions mass–preserving, to be precise

$$\lambda_\varepsilon = \frac{1}{\varepsilon \mathcal{L}^n(\Omega)} \int_\Omega W'(u_\varepsilon)\, dx.$$

In some references this is also called the mass-conserving Allen–Cahn equation.

This equation is precisely the L^2 mass-constrained gradient flow of the energy (1.1.3). It was introduced by Rubinstein and Sternberg [97] to model phase separation after quenching of homogeneous binary systems (e.g., glasses or polymers). An important property of this equation is that the total mass $\int_\Omega u_\varepsilon(x,t)\, dx$ is preserved in time. It can be shown that when $\varepsilon \to 0^+$ the domain Ω is divided into regions in which u_ε is close to a and to b, and that the interfaces between these regions as $\varepsilon \to 0^+$ evolve according to a nonlocal volume–preserving mean curvature flow.

In the past thirty years a significant effort has been given to the study of the asymptotic slow motion of solutions of the Allen–Cahn equation

$$\partial_t u_\varepsilon = \varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon) \tag{1.1.16}$$

and the Cahn–Hilliard equation

$$\partial_t u_\varepsilon = -\Delta(\varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon)). \tag{1.1.17}$$

These equations are precisely the rescaled gradient flows of the unconstrained energy (1.1.1). In dimension $n = 1$ the theory of slow motion was first developed in the seminal papers of Carr and Pego [32], [33] and Fusco and Hale [56]. In particular, Carr and Pego [32] studied the slow evolution of solutions of (1.1.16) when $n = 1$, using center manifold theory. They provided a system of differential equations which precisely describes the motion of the position of the transition layers (cf. Section 3 in [32]); such a result was formally derived by Neu [84], see also [33]. A similar approach has been recently adopted by several authors to extend these ideas to a more general setting, by studying the slow manifolds inherent to the dynamics of these equations, see [89] and the references therein.

Subsequently, Bronsard and Kohn [25] introduced a new variational method to study the behavior of solutions of the Allen–Cahn equation (1.1.16). They observed that the motion of solutions of this equation, subject to either Neumann or Dirichlet boundary conditions in an open, bounded interval $\Omega \subset \mathbb{R}$, could be studied by exploiting the gradient flow structure of (1.1.16) . The key tool in their paper is a careful analysis of the asymptotic behavior of the unconstrained energy

$$F_\varepsilon^{(1)}(u) := \int_\Omega \frac{1}{\varepsilon}W(u) + \frac{\varepsilon}{2}|\nabla u|^2 dx, \quad u \in H^1(\Omega).$$

Specifically, they prove that if $\{v_\varepsilon\}$ converges in $L^1(\Omega)$ to a function $v \in BV(\Omega; \{a, b\})$ with exactly N jumps, then, for any $k > 0$,

$$F_\varepsilon^{(1)}(v_\varepsilon) \geq Nc_W - C_1\varepsilon^k \tag{1.1.18}$$

for ε sufficiently small and some $C_1 > 0$. They then applied (1.1.18) to prove that (cf. Theorem 4.1 in [25]) if the initial data $u_{0,\varepsilon}$ of the equation (1.1.16) converges in $L^1(\Omega)$ to the jump function v, and $u_{0,\varepsilon}$ are energetically "well–prepared", that is,

$$F_\varepsilon^{(1)}(u_{0,\varepsilon}) \leq Nc_W + C_2\varepsilon^k$$

for some $C_2 > 0$, then for any $M > 0$,

$$\sup_{0\leq t\leq M\varepsilon^{-k}} ||u_\varepsilon(t) - v||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+.$$

Subsequently, Grant [59] improved the estimate (1.1.18) to

$$F_\varepsilon^{(1)}(v_\varepsilon) \geq Nc_W - C_1 e^{-C_2\varepsilon^{-1}} \tag{1.1.19}$$

for ε small, and some $C_1, C_2 > 0$, which in turn gives the more accurate slow motion estimate

$$\sup_{0\leq t\leq Me^{C\varepsilon^{-1}}} ||u_\varepsilon(t) - v||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+$$

for some $C > 0$. Finally, Bellettini, Nayam and Novaga [19] gave a sharp version of Grant's second–order estimate by proving

$$\begin{aligned} F_\varepsilon^{(1)}(v_\varepsilon) \geq{}& Nc_W - 2\alpha_+\kappa_+^2 \sum_{k=1}^N e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}} - 2\alpha_-\kappa_-^2 \sum_{k=1}^N e^{-\alpha_-\frac{d_k^\varepsilon}{\varepsilon}} \\ &+ \kappa_+^3\beta_+ \sum_{k=1}^N e^{-\frac{3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}} + \kappa_-^3\beta_- \sum_{k=1}^N e^{-\frac{3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}} \\ &+ o\left(\sum_{k=1}^N e^{-\frac{3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}}\right) + o\left(\sum_{k=1}^N e^{-\frac{3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}}\right) \end{aligned}$$

as $\varepsilon \to 0^+$, where $\alpha_\pm, \kappa_\pm, \beta_\pm$ are constants depending on the potential W and d_k^ε is the distance between the k–th and the $(k+1)$–th transitions of v_ε. This last work gives a variational validation of [32], [33]. Indeed, the sharp energy estimate allows the authors to (formally) recover the ODE describing the motion of transition points.

The situation in higher dimensions is not as clearly understood. This is due to the possibility of curvature effects. One still suspects that if initial data $u_{0,\varepsilon}$ approximates the function $u = a\chi_{E_0} + b\chi_{E_0^c}$, with E_0 a local minimizer of $\mathcal{F}^{(1)}$, then the solutions to (1.1.15) will still exhibit slow motion. However, it is generally not clear at what time scale curvature effects, which are absent when $n = 1$, may come into play. Much of the work in this setting has addressed the motion of phase "bubbles", namely solutions approximating a spherical interface compactly contained in Ω. For example, Bronsard and Kohn [26] utilize variational techniques to analyze radial solutions $u_{\varepsilon,\mathrm{rad}}$ of the Allen–Cahn equation. They prove that $u_{\varepsilon,\mathrm{rad}}$ separates Ω into two regions where $u_{\varepsilon,\mathrm{rad}} \approx +1$ and $u_{\varepsilon,\mathrm{rad}} \approx -1$ and that the interface moves with normal velocity equal to the sum of its principal curvatures. In [44], Ei and Yanagida investigate the dynamics of interfaces for the Allen–Cahn equation, where Ω is a strip–like domain in $\mathbb{R}^2$. They show that the evolution is slower than the mean curvature flow, *but* faster than exponentially slow. This suggests that estimates of the type (1.1.19) cannot be expected to hold in higher dimensions. In the Cahn–Hilliard case, Alikakos, Bronsard and Fusco [3] use energy methods and detailed spectral estimates to show the existence of solutions of (1.1.17) supporting almost spherical interfaces, which evolve by drifting towards the boundary with exponentially small velocity. Other related works include [2], [4] and [5]. Most of these works require significant machinery, and often focus only on the existence of slowly moving solutions.

Using Theorem 1.1.1, it is possible to give precise asymptotics for the energy (1.1.3). In particular, estimates of the form (1.1.18) can be obtained in the case $k = 1$. The techniques from [25] can then be applied to obtain the following result, see Theorem 7.0.1.

Theorem 1.1.2. *Assume that Ω satisfies* (6.1.1)*, m satisfies* (6.1.2) *and W satisfies hypotheses* (5.1.4)-(5.1.7)*. Assume that u is an $L^1(\Omega)$-local minimizer of the functional $\mathcal{F}^{(1)}$ (see* (1.1.4)*). Finally, assume that, for some $\delta > 0$, $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathfrak{v}_m$, with $E_0 = \{u = a\}$. Assume that $u_{0,\varepsilon} \in L^\infty(\Omega)$ satisfy*

$$u_{0,\varepsilon} \to u \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0^+$$

and

$$\mathcal{F}_\varepsilon^{(1)}(u_{0,\varepsilon}) \leq \mathcal{F}^{(1)}(u) + C\varepsilon$$

for some $C > 0$. Let u_ε be a solution to (1.1.15)*. Then, for any $M > 0$*

$$\sup_{0 \leq t \leq M\varepsilon^{-1}} ||u_\varepsilon(t) - u||_{L^2} \to 0 \text{ as } \varepsilon \to 0^+.$$

The proof of this theorem, which uses exactly the same techniques as those in [25], are found in Chapter 7.

1.2 Becker–Döring Equations

The second part of this thesis considers the Becker–Döring equations, namely the following (infinite) system of differential equations

$$\begin{aligned} \frac{d}{dt}c_i(t) &= J_{i-1}(t) - J_i(t), \quad i = 2, 3, \dots, \\ \frac{d}{dt}c_1(t) &= -J_1(t) - \sum_{i=1}^{\infty} J_i(t), \end{aligned} \tag{1.2.1}$$

where the J_i can be written as

$$J_i(t) = a_i c_1(t) c_i(t) - b_{i+1} c_{i+1}(t), \tag{1.2.2}$$

and where $\{a_i\}, \{b_i\}$ are fixed, positive sequences, known as the coagulation and fragmentation coefficients respectively.

Becker–Döring systems form a subclass of the more general *coagulation fragmentation* equations. In typical physical applications the c_i represent the discrete distribution function of particles of size i, and the evolution given by (1.2.1) represents the mean field approximation of the evolution of the distribution function c_i. In particular, $J_i(t)$ represents the net rate that particles of size i and size 1 either join to form particles of size $i+1$, or conversely are emitted by spontaneous breakup. Thus we are primarily interested in positive solutions, whose first moment is preserved in time, meaning that

$$c_i \geq 0, \qquad \sum_{i=1}^{\infty} ic_i(t) = \tilde{m}(t) \equiv \tilde{m} \quad \text{for all } t \geq 0. \tag{1.2.3}$$

The Becker–Döring equations are used to model reactions in various physical settings, such as vapor condensation, phase separation in alloys, and crystallization. This model was first proposed in [17], and was modified to the form we are considering in [27],[93]. A good mathematically-oriented review can be found in [99].

The well-posedness and convergence properties of the Becker–Döring equations have been well-studied. In particular, Ball, Carr and Penrose [16] demonstrated the existence of "mass"-preserving, non-negative solutions to this system, namely solutions of (1.2.1) satisfying (1.2.3). A later work [71] established well-posedness (including uniqueness) for any initial data with finite first moment, namely the space where the "mass" is well-defined. Ball et al. [16] also demonstrated that as $t \to \infty$ solutions must converge to some equilibrium $\{Q_i\}$, where $\{Q_i\}$ is uniquely determined by $\tilde{m}$. Furthermore, they prove the existence of a value $\tilde{m}_s$ such that if $\tilde{m} < \tilde{m}_s$ then the convergence to $\{Q_i\}$ is strong. On the other hand, if $\tilde{m} > \tilde{m}_s$ then there is a loss of mass to ∞, and the convergence is only weak. Any initial data satisfying $\tilde{m} < \tilde{m}_s$ is called *subcritical*, while data satisfying $\tilde{m} > \tilde{m}_s$ is *supercritical*.

The second part of this thesis seeks to quantify the trend to equilibrium in the subcritical case ($\tilde{m} < \tilde{m}_s$). Specifically, the goal is to establish uniform, local rates of convergence to equilibrium in spaces with polynomial moments.

To begin, define the detailed balance coefficients, a sequence $\{\tilde{Q}_i\}$, by the equations

$$\tilde{Q}_1 = 1, \qquad \tilde{Q}_i a_i = \tilde{Q}_{i+1} b_{i+1}, \quad i = 1, 2, \ldots \tag{1.2.4}$$

The equilibrium solution Q_i of (1.2.1) can be written as

$$Q_i = \tilde{Q}_i \zeta^i, \tag{1.2.5}$$

where the parameter ζ is related to the mass $\tilde{m}$ in the subcritical regime through the equation

$$\sum_{i=1}^{\infty} iQ_i = \tilde{m}.$$

It is straightforward to show that $\tilde{m}_s$ is linked to the radius of convergence ζ_s of the power series with coefficients $\tilde{Q}_i$.

One motivation for studying the Becker–Döring equations is that they serve as a suitable prototype of more general coagulation-fragmentation equations with detailed balance. Indeed, one suspects that many of the interesting phenomenon that occur for the Becker–Döring equations may be typical of other systems with detailed balance.

Convergence to equilibrium was proven by Ball, Carr and Penrose [16] using an entropy functional. Specifically, they prove that the quantity

$$\tilde{V}(c) := \sum_{i=1}^{\infty} c_i \left(\log \frac{c_i}{\tilde{Q}_i} - 1 \right) \tag{1.2.6}$$

is weak-$*$ continuous and that $\tilde{V}(c(t))$ is strictly decreasing.

Later, Jabin and Niethammer [65] proved an entropy dissipation inequality which gives a uniform dissipation rate for regular data. In particular, they proved that if the initial data decays exponentially fast, then the solution converges to equilibrium with a rate bounded by $e^{-Ct^{1/3}}$ in the mass-weighted space.

In a recent work, Cañizo and Lods [30] improved this bound to e^{-Ct}. They do so by observing that the Becker–Döring equations (1.2.1) have a type of symmetric structure. In particular, if one writes the Becker–Döring equations in terms of a perturbation of the equilibrium solution

$$c_i = Q_i(1 + h_i), \tag{1.2.7}$$

then the mass constraint (1.2.3) may be expressed as

$$\sum_{i=1}^{\infty} Q_i i h_i = 0, \tag{1.2.8}$$

and the original equation (1.2.1) in the abstract quasilinear form

$$\frac{d}{dt} h = \Theta(h_1(t))h.$$

Following Cañizo and Lods, the linear operator $\Theta(g)$ may be expressed as

$$\Theta(g) = L + g\Xi, \tag{1.2.9}$$

where L and Ξ are both linear operators, given in weak form by requiring that for all $\{\phi_i\}$ in a suitable space of test sequences,

$$\sum_{i=1}^{\infty} Q_i(Lh)_i\phi_i = \sum_{i=1}^{\infty} a_iQ_iQ_1(h_1 + h_i - h_{i+1})(\phi_{i+1} - \phi_i - \phi_1), \tag{1.2.10}$$
$$\sum_{i=1}^{\infty} Q_i(\Xi h)_i\phi_i = \sum_{i=1}^{\infty} a_iQ_iQ_1\, h_i(\phi_{i+1} - \phi_i - \phi_1).$$

If one considers an ℓ^2 space weighted by Q_i then L is clearly symmetric. Additionally, if $\{c_i\}$ is a solution of (1.2.1) and $\{h_i\}$ is determined by (1.2.7) it follows that $h_i \in [-1, \infty)$ and that $\sum Q_i i h_i = 0$. It is then natural to define the Hilbert space H by

$$H := \left\{ \{h_i\} : \|h\|_{\ell^2(Q_i)} := \left(\sum_{i=1}^{\infty} Q_i h_i^2 \right)^{1/2} < \infty, \quad \sum Q_i i h_i = 0 \right\}.$$

with the induced norm $\|\cdot\|_H = \|\cdot\|_{\ell^2(Q_i)}$ and inner product $\langle\cdot,\cdot\rangle_H$. Cañizo and Lods demonstrated that the linear part (L) of the Becker–Döring equations has a good spectral gap in H, or precisely that for some constant $\lambda_c > 0$ the following holds, independent of h:

$$\langle h, Lh\rangle_H = -\sum_{i=1}^{\infty} a_iQ_iQ_1(h_1 + h_i - h_{i+1})^2 \leq -\lambda_c\langle h, h\rangle_H. \tag{1.2.11}$$

A key point is that the mass constraint (1.2.8) precludes the null vector $h_i = i$. Detailed quantitative estimates of λ_c can then be obtained using Hardy's inequality—see [30] for details.

Cañizo and Lods then utilized a priori bounds from [65] to control the non-linear term and establish a rate of convergence to equilibrium. More precisely, defining the Banach space

$$Y_\eta := \left\{ \{h_i\} : \|h\|_{\ell^1(Q_i e^{\eta i})} := \sum_{i=1}^{\infty} Q_i e^{\eta i} |h_i| < \infty, \quad \sum Q_i i h_i = 0 \right\}, \quad 0 < \eta < 1,$$

with the induced norm $\|\cdot\|_{Y_\eta} = \|\cdot\|_{\ell^1(Q_i e^{\eta i})}$, they prove that for $0 < \eta < \bar{\eta}$, given initial data in $Y_{\bar{\eta}}$ then the solution must converge at a uniform exponential rate in Y_η. A key technical aspect of their proof was an operator decomposition technique from [61], which permits an extension of the spectral gap of L from H to Y_η. It is important here to recall that the space H is continuously embedded in Y_η for $\eta > 0$ sufficiently small, precisely because the Q_i are exponentially decaying, see Proposition 8.1.2.

The goal here is to study the trend to equilibrium in spaces with only polynomial moments. To this end, define the Banach spaces

$$X_k := \left\{ \{h_i\} : \|h\|_{\ell^1(Q_i i^k)} := \sum_{i=1}^{\infty} Q_i i^k |h_i| < \infty, \quad \sum Q_i i h_i = 0 \right\}, \quad k \geq 1, \tag{1.2.12}$$

with norm $\|\cdot\|_{X_k} = \|\cdot\|_{\ell^1(Q_i i^k)}$. The main result of the second part of the thesis is as follows:

Theorem 1.2.1. *Let $(h_i(t))$ defined by* (1.2.7) *represent the deviation from equilibrium of a solution $(c_i(t))$ to the Becker–Döring equations (see Definition 8.1.1). Assume that the model coefficients in* (1.2.2) *satisfy* (8.1.1)-(8.1.4) *below. Let k_1 and k_2 be real numbers satisfying $k_1 > 0$ and $k_2 > k_1 + 2$. Then there exist positive constants $\delta_{k_1,k_2}, C_{k_1,k_2}$ so that if $\|h(0)\|_{X_{1+k_2}} < \delta_{k_1,k_2}$ then we have that*

$$\|h(t)\|_{X_{1+k_1}} \leq C_{k_1,k_2} (1+t)^{-(k_2-k_1-1)} \|h(0)\|_{X_{1+k_2}} \quad \text{for all } t \geq 0.$$

This result is proven by first obtaining detailed estimates on the semigroup generated by L in the spaces X_k by using new dissipation estimates, together with the spectral gap estimate (1.2.11), the operator decomposition result from [61] and interpolation techniques from Engler's work on travelling wave stability [46]. This is the subject of Chapter 8.

Subsequently, Chapter 9 addresses the question of non-linear stability and convergence rates. The issue of non-linear stability is addressed using evolution families and an extension of the operator decomposition result. Subsequently, convergence rates are obtained by combining the linear decay results with the non-linear stability results, and using Duhamel's formula.

Chapter 2

Preliminaries

This chapter collects many of the necessary preliminaries for the results of this thesis. The results in this chapter are for the most part classical, and are not the original work of the author. They are included here in the interest of making this thesis self-contained, with citations to sources where proofs may readily be found.

By way of notation, given a non-empty set $E \subset \mathbb{R}^m$, E°, $\bar{E}$ and E^c will represent the interior, closure and complement of E respectively. Also, $\mathcal{L}^m$ and $\mathcal{H}^m$ are the m-dimensional Lebesgue and Hausdorff measures, respectively, see [51] for appropriate definitions. The constant $\omega_n := \mathcal{L}^n(B(0,1))$. Also, given two Banach spaces Y, Z, let $\mathcal{L}(Y,Z)$ denote the space of bounded linear operators from Y to Z and $\mathcal{L}(Y) = \mathcal{L}(Y,Y)$.

2.1 Geometric Measure Theory and Isoperimetric Problems

This section deals with a variety of standard definitions and results from geometric measure theory. Standard sources for this material include [12, 48, 109].

This section begins by recalling the definition of functions of bounded variation.

Definition 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. The space of functions of* bounded variation $BV(\Omega)$ *is the space of all functions $u \in L^1(\Omega)$ such that for all $i = 1, \dots, n$ there exist finite signed Radon measures $D_i u : \mathcal{B}(\Omega) \to \mathbb{R}$ such that*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega \phi dD_i u$$

for all $\phi \in C_0^\infty(\Omega)$. The measure $D_i u$ is called the weak, *or* distributional, partial derivative of u *with respect to x_i. In addition, for any function $u \in BV(\Omega)$, the total variation $|Du|$ of the measure Du, which is also called the* variation measure of u, *is a finite measure and satisfies the formula*

$$|Du|(\Omega) = \sup\left\{\int_\Omega u\mathrm{div}\phi\, dx : \quad \phi \in C_0(\Omega;\mathbb{R}^n),\ ||\phi||_{C_0(\Omega;\mathbb{R}^n)} \leq 1\right\} < \infty.$$

The measure Du turns out to have additional structural properties (see, e.g. [48]). Specifically, one can decompose

$$Du = \nabla u \mathcal{L}^n + Ju + Cu,$$

where ∇u is an $L^1(\Omega)$ function, where Ju takes support on a set of dimension $(n-1)$ and Cu is singular with respect to $\mathcal{L}^n$ and has support on a set of dimension greater than $(n-1)$. Furthermore, the measure Ju can be written as

$$Ju = (u_+ - u_-)\nu_u \mathcal{H}^{n-1} \lfloor S_u, \tag{2.1.1}$$

where

$$\begin{aligned}
u_+(x) &:= \inf\{t \in [-\infty, \infty] : \{x \in \Omega : u(x) > t\} \text{ has 0 density at } x\}, \\
u_-(x) &:= \sup\{t \in [-\infty, \infty] : \{x \in \Omega : u(x) < t\} \text{ has 0 density at } x\}, \\
\nu_u(x) &:= \lim_{r \to 0} \frac{Du(B(x,r))}{|Du|(B(x,r))} \quad \text{for } x \text{ in } supp(Du),
\end{aligned}$$

and where S_u is precisely the set where $u_+ \neq u_-$. The set S_u is called the *jump set* of u. The existence of the function ν_u is guaranteed Du a.e. by the Besicovitch derivation theorem (see e.g. [48]).

The first important property of BV functions is that they form a compact subset of L^1. This can be found in, e.g., Section 5.2 in [48] or Theorem 13.35 in [72].

Proposition 2.1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Assume that $u_k \in BV(\Omega)$, and that*

$$\sup_k \|u_k\|_{BV(\Omega)} < \infty.$$

Then there exist a subsequence u_{k_j} and a function $u \in BV(\Omega)$ satisfying

$$u_{k_j} \to u \text{ in } L^1(\Omega) \qquad Du_{k_j} \overset{*}{\rightharpoonup} Du.$$

Another important property is the fact that the total variation is lower semicontinuous.

Proposition 2.1.3 (Proposition 4.29 and 4.30 [75])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Given a sequence of Radon measures $\mu_k \overset{*}{\rightharpoonup} \mu$ supported on Ω, then the following inequality holds for any open $A \subset \Omega$:*

$$|\mu|(A) \leq \liminf_k |\mu_k|(A).$$

On the other hand, if $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k|(\Omega) \to |\mu|(\Omega) < \infty$ then $|\mu_k| \overset{*}{\rightharpoonup} |\mu|$.

Certain standard calculus rules apply for functions in $BV(\Omega)$. For example, the following chain rule is a special case of a more general chain rule given in Proposition 1.2 in [10] , see also [11].

Proposition 2.1.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Given a function $u \in BV(\Omega)$ and a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ then the function $v := f \circ u$ is an element of $BV(\Omega)$ and*

$$\begin{gathered}
Jv = (f(u_+) - f(u_-))\nu_u \lfloor S_u, \\
Cv = f'(u)Cu, \quad \nabla v = f'(u)\nabla u,
\end{gathered}$$

where u here is an appropriately chosen representative (namely u must coincide with u_+ at any point where u_+ and u_- coincide).

Remark 2.1.5. *The previous properties of BV functions continue to hold when all of the integrals in the norm are modified with a continuous weighting factor η. Some useful details in this regard can be derived from results in [100].*

One natural application of the total variation is to give a suitable definition of the perimeter of a wide class of sets.

Definition 2.1.6. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $\Omega \subset \mathbb{R}^n$ be an open set. The* perimeter *of E in Ω, denoted $\mathrm{P}(E;\Omega)$, is the variation of χ_E in Ω, that is,*

$$\mathrm{P}(E;\Omega) := |D\chi_E|(\Omega) = \sup\left\{ \sum_{i=1}^{n} \int_\Omega \phi_i dD_i u : \quad \phi \in C_0(\Omega;\mathbb{R}^n),\ ||\phi||_{C_0(\Omega;\mathbb{R}^n)} \leq 1 \right\}.$$

The set E is said to have finite perimeter in Ω *if $\mathrm{P}(E;\Omega) < \infty$, or in other words if $\chi_E \in BV(\Omega)$. If $\Omega = \mathbb{R}^n$, it is standard to write $\mathrm{P}(E) := \mathrm{P}(E;\mathbb{R}^n)$.*

Given a set E of finite perimeter we may naturally define a normal vector via

$$\nu_E(x) := -\nu_{\chi_E}(x) = -\frac{D\chi_E}{|D\chi_E|}(x) = -\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}, \quad x \in supp(D_{\chi_E}). \tag{2.1.2}$$

Again, by the Besicovitch theorem this object is well-defined for $|D_{\chi_E}|$ a.e. x.

Definition 2.1.7. *The* reduced boundary *of E, denoted by $\partial^* E$, is the set of all points in supp$(|D\chi_E|)$ where equation* (2.1.2) *holds.*

Moreover, by the *structure theorem for sets of finite perimeter*, (see, e.g., [48], Theorem 2, (iii), page 205), if E has finite perimeter in $\mathbb{R}^n$, then for any Borel set $F \subset \mathbb{R}^n$,

$$\mathrm{P}(E;F) = \mathcal{H}^{n-1}(\partial^* E \cap F).$$

This is somewhat natural in light of (2.1.1).

The next theorem presents the *coarea formula*, which is a cornerstone of geometric measure theory. A proof for Lipschitz functions can be found in [48], while a proof for Sobolev functions can be found in [76], and was originally given by Federer [49].

Theorem 2.1.8. *Let $u \in W^{1,p}(\Omega)$, with $p \geq 1$, and $\Omega \subset \mathbb{R}^n$ an open set. Then for any $g \in L^1(\Omega)$, we have that*

$$\int_\Omega g(x)|\nabla u(x)|\, dx = \int_\mathbb{R} \int_{\{u=s\}} g(x)\, d\mathcal{H}^{n-1}(x)\, ds.$$

The next theorem is the *isoperimetric inequality.* This problem has a very old history (dating back to the Greeks), but was first proved up to modern standards by Steiner. His proof can be found in [75], Chapter 14.

Theorem 2.1.9. *Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a set of finite perimeter. Then either E or $\mathbb{R}^n \setminus E$ has finite Lebesgue measure and*

$$\min\{\mathcal{L}^n(E),\ \mathcal{L}^n(\mathbb{R}^n \setminus E)\}^{\frac{n-1}{n}} \leq \frac{\omega_n^{-1/n}}{n} \mathrm{P}(E), \tag{2.1.3}$$

where equality holds if and only if E is a ball.

A similar inequality holds in bounded domains, and can be found in [77], Corollary 3.2.1 and Lemma 3.2.4, see also [37] and [1].

Proposition 2.1.10. *Let $\Omega \subset \mathbb{R}^n$ be bounded, connected and Lipschitz. Then there exists a constant $C > 0$ such that for any $E \subset \Omega$*

$$\mathrm{P}(E;\Omega) \geq C \min\{\mathcal{L}^n(E), \mathcal{L}^n(\Omega\backslash E)\}^{\frac{n-1}{n}}.$$

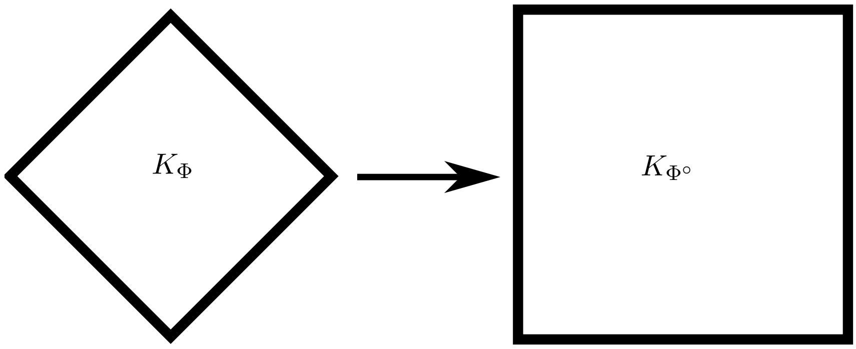

Figure 2.1: An example of K_Ψ and K_{Ψ°.

2.2 Anisotropic Extensions of the Perimeter Function

This section will extend the results of the previous section, namely the central results of geometric measure theory, to the anisotropic case. Anisotropic energies are common in materials science problems, particularly in relation to crystals [64, 107]. Most of these results correspond very closely to those in the classical, isotropic case, albeit with more involved proofs. Because these results are not as well-known, this section will give precise references wherever possible.

Throughout this section $\Psi : \mathbb{R}^n \to [0, \infty)$ will be a convex function which is positively 1-homogeneous, meaning that, for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\Psi(tx) = |t|\Psi(x). \tag{2.2.1}$$

Furthermore, for simplicity this work will assume that Ψ satisfies

$$C_1|x| \le \Psi(x) \le C_2|x| \tag{2.2.2}$$

and that Ψ is scaled so that the set $K_\Psi := \{x : \Psi(x) \le 1\}$ satisfies

$$\mathcal{L}^n(K_\Psi) = \omega_n.$$

Some references call Ψ the *gauge* of the set K. The support function of K, which is denoted by $\Psi^\circ(x)$ is given by

$$\Psi^\circ(x) := \sup_{\xi \in K_\Psi} \langle \xi, x \rangle.$$

It is straightforward to show that Ψ° is also a convex, 1-homogeneous function and that Ψ and Ψ° are polar to each other. It is then natural to define

$$K_{\Psi^\circ} := \{x : \Psi^\circ(x) \le 1\}.$$

The convex sets K_Ψ and K_{Ψ° are in fact polar to each other. The study of support functions and polars is central to convex analysis, see Sections 13-15 in [94] for a complete treatment.

Example 2.2.1. *Suppose that $\Psi(x) = \frac{1}{\sqrt{n}} \sum_i |x_i|$, namely Ψ is a rescaled ℓ^1 norm. Then K_Ψ is the rescaled unit ball, Ψ° is the ℓ^∞ norm and K_{Ψ° is the ℓ^∞ unit ball (see Figure 2.1).*

With these definitions in hand it is possible to define an anisotropic version of the BV norm.

Definition 2.2.2. *For any open set $\Omega \subset \mathbb{R}^n$, given $u \in BV(\Omega)$, we define the* total variation with respect to the gauge Ψ *by*

$$|Du|_\Psi(\Omega) := \sup \left\{ \int_\Omega u \, \mathrm{div}\phi \, dx : \phi \in C_0^1(\Omega;\mathbb{R}^n),\ \phi(x) \in K_{\Psi^\circ} \text{ for all } x \in \Omega \right\}.$$

Similarly, given a set with finite perimeter we define the perimeter with respect to the gauge Ψ *via*

$$\mathrm{P}_\Psi(E;\Omega) := |D\chi_E|_\Psi(\Omega).$$

When $\Psi(x) = |x|$ it is clear that these definitions coincide with the usual total variation and perimeter. If a function $u \in BV(\Omega)$ then, due to equation (2.2.2), $|Du|_\Psi(\Omega) < \infty$. Similarly if $u \in L^1(\Omega)$ and $|Du|_\Psi(\Omega) < \infty$ then $u \in BV(\Omega)$.

The following theorem can be found in [9].

Theorem 2.2.3. *Given a function $u \in BV(\Omega)$, the total variation with respect to the gauge Ψ permits the following integral representation:*

$$\int_\Omega \Psi\left(\frac{Du}{|Du|}\right) dDu(x) = |Du|_\Psi(\Omega).$$

Furthermore, a type of coarea formula holds, namely

$$|Du|_\Psi(\Omega) = \int_\mathbb{R} \mathrm{P}_\Psi(\{u > s\};\Omega)\, ds,$$

and a version of the structure theorem holds, specifically

$$\mathrm{P}_\Psi(E;\Omega) = \int_{\partial^* E} \Psi(\nu_E)\, d\mathcal{H}^{n-1}.$$

Remark 2.2.4. *If $u \in W^{1,1}(\Omega)$ then in fact we have that*

$$\int_\Omega \Psi(Du)\, dx = |Du|_\Psi(\Omega).$$

An appropriate version of the isoperimetric inequality also holds. This is known as the Wulff problem, and was completely treated in the setting of sets of finite perimeter by Fonseca [52] and Fonseca and Müller [54], see also [106] for earlier work.

2.3 Properties of Perimeter Minimizers and First and Second Variation Formulas

This section reviews some of the classical theory of volume-constrained perimeter minimizers. The definitions here are mostly classical, and all of them can be found in Chapter 17 of [75]. The first step is to define a suitable class of variations of sets.

Definition 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open. A* one-parameter family $\{f_t\}_t$ of diffeomorphisms of $\mathbb{R}^n$ *is a smooth function*

$$(x,t) \in \mathbb{R}^n \times (-\epsilon,\epsilon) \mapsto f(t,x) =: f_t(x) \in \mathbb{R}^n,\ \epsilon > 0,$$

such that $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ for each fixed $|t| < \epsilon$. In particular, $\{f_t\}_{|t|<\epsilon}$ is called a local variation in Ω *if it defines a one-parameter family of diffeomorphisms such that*

$$f_0(x) = x \quad \text{for all } x \in \mathbb{R}^n,$$
$$\{x \in \mathbb{R}^n : f_t(x) \neq x\} \subset\subset \Omega \quad \text{for all } 0 < |t| < \epsilon.$$

It follows from the previous definition that given a local variation $\{f_t\}_{|t|<\epsilon}$ in Ω, then

$$E\Delta f_t(E) \subset\subset \Omega \quad \text{for all } E \subset \mathbb{R}^n.$$

Moreover, one can show that there exists a compactly supported smooth vector field $V \in C_c^\infty(\Omega;\mathbb{R}^n)$ such that the following expansions hold uniformly on $\mathbb{R}^n$,

$$f_t(x) = x + V(x) + O(t^2), \quad \nabla f_t(x) = \mathrm{Id} + t\nabla V(x) + O(t^2), \tag{2.3.1}$$

and V satisfies

$$V(x) = \frac{\partial f_t}{\partial t}(x,0) \quad x \in \mathbb{R}^n.$$

Definition 2.3.2. *The smooth vector field V in* (2.3.1) *is called* initial velocity of $\{f_t\}_{|t|<\epsilon}$.

The following result establishes an explicit expression, given in terms of the initial velocity V, for the *first variation of the perimeter* of a set E, with respect to local variations $\{f_t\}_{|t|<\epsilon}$ in Ω, that is, a formula for

$$\frac{d}{dt}\Big|_{t=0} \mathrm{P}(f_t(E);\Omega).$$

Theorem 2.3.3 (First Variation of Perimeter)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, E a set of locally finite perimeter and $\{f_t\}_{|t|<\epsilon}$ a local variation in Ω. Then*

$$\mathrm{P}(f_t(E);\Omega) = \mathrm{P}(E;\Omega) + t\int_{\partial^* E} \mathrm{div}_E V d\mathcal{H}^{n-1} + O(t^2), \tag{2.3.2}$$

where V is the initial velocity of $\{f_t\}_{|t|<\epsilon}$ and $\mathrm{div}_E V : \partial^* E \to \mathbb{R}$, defined by

$$\mathrm{div}_E V(x) := \mathrm{div} V - \nu_E(x)\cdot\nabla V(x)\nu_E(x),\ x \in \partial^* E, \tag{2.3.3}$$

is a Borel function called the boundary divergence *or* tangential divergence *of V on E.*

In light of the form of the first variation, it is natural to seek a suitable version of the divergence theorem. The version given here requires that surfaces possess some classical regularity, and can be found in [75], Theorem 11.8 and equation 11.14.

Theorem 2.3.4. *Let $M \subset \mathbb{R}^n$ be a C^2–hypersurface with boundary Γ. Then there exists a normal vector field $H_M \in C(M;\mathbb{R}^n)$ to M and a normal vector field $\nu_\Gamma^M \in C^1(\Gamma;S^{n-1})$ to Γ such that for every $V \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$*

$$\int_M \mathrm{div}_M V d\mathcal{H}^{n-1} = \int_M V\cdot H_M d\mathcal{H}^{n-1} + \int_\Gamma (V\cdot\nu_\Gamma^M) d\mathcal{H}^{n-2},$$

where H_M is the mean curvature vector *to M and $\mathrm{div}_M V$ is the* tangential divergence *of V on M, defined by* (2.3.3)*. Furthermore, $\nu_\Gamma^M \cdot \nu_M = 0$.*

In light of this divergence theorem, the formula (2.3.2) suggests that volume-constrained perimeter minimizers will necessarily have constant mean curvature. That is precisely the content of the next theorem.

Theorem 2.3.5 (Constant Mean Curvature)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E_0 \subset \Omega$ be a volume-constrained perimeter minimizer in the open set Ω. Then there exists $\lambda_0 \in \mathbb{R}$ such that*

$$\int_{\partial^* E} \mathrm{div}_E V d\mathcal{H}^{n-1} = \lambda_0 \int_{\partial^* E} (V\cdot\nu_E) d\mathcal{H}^{n-1} \quad \text{for all } V \in C_c^\infty(\Omega;\mathbb{R}^n).$$

In particular, E_0 has distributional mean curvature in Ω constantly equal to $\frac{\lambda_0}{n-1}$.

It turns out that surfaces with constant mean curvature enjoy regularity properties much like those of minimal surfaces. In particular, the following theorem holds, see e.g. [58], [60].

Theorem 2.3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2,\bar{\alpha}}$, and let $E_0 \subset \Omega$ be a volume-constrained local perimeter minimizer. Then the set $\overline{\partial E_0 \cap \Omega}$ can be decomposed into two sets $\overline{\partial E_0 \cap \Omega} = Reg(\partial E_0) \cup Sing(\partial E_0)$ such that*

- *The set $Sing(\partial E_0)$ is empty for $n \leq 7$, it is finite for $n = 8$ and has dimension of at most $n-8$ for $n > 8$.*
- *The set $Reg(\partial E_0) \cap \Omega$ can be locally represented as an analytic surface of constant mean curvature κ_{E_0}.*
- *The set $Reg(\partial E_0) \cap \partial\Omega$ can be locally represented as a $C^{2,\bar{\alpha}}$ surface of constant mean curvature κ_{E_0} which intersects $\partial\Omega$ orthogonally.*

The next goal will be to characterize the second variation. In order to do so, it is necessary to consider the *signed distance function* of a set E.

Proposition 2.3.7. *Let $\Omega \subset \mathbb{R}^n$ be open and $E \subset \Omega$ open with C^2 boundary. Then there exists an open set Ω' with $\Omega \cap \partial E \subset \Omega' \subset \Omega$ such that the signed distance function $d_E : \mathbb{R}^n \to \mathbb{R}$ of E,*

$$d_E(x) := \begin{cases} \operatorname{dist}(x, \partial E) & \text{if } x \in \mathbb{R}^n \setminus E, \\ -\operatorname{dist}(x, \partial E) & \text{if } x \in E, \end{cases} \tag{2.3.4}$$

satisfies $d_E \in C^2(\Omega')$.

The previous result allows one to define a vector field $N_E \in C^1(\Omega'; \mathbb{R}^n)$ and a tensor field $A_E \in C^0(\Omega'; \mathrm{Sym}(n))$ via

$$N_E := \nabla d_E, \quad A_E := \Delta d_E \ \text{ on } \Omega'.$$

In particular, one can show that for every $x \in \Omega \cap \partial E$ there exist $r > 0$, vector fields $\{\tau_h\}_{h=1}^{n-1} \subset C^1(B_r(x); S^{n-1})$, and functions $\{\kappa_h\}_{h=1}^{n-1} \subset C^0(B_r(x))$ such that $\{\tau_h\}_{h=1}^{n-1}$ is an orthonormal basis of $T_y\partial E$ for every $y \in B_r(x) \cap \partial E$, $\{\tau_h\}_{h=1}^{n-1} \cup \{N_E(y)\}$ is an orthonormal basis of $\mathbb{R}^n$ for every $y \in B_r(x)$, and

$$A_E(y) = \sum_{h=1}^{n-1} \kappa_h(y) \tau_h(y) \otimes \tau_h(y) \ \text{ for all } y \in B_r(x).$$

Definition 2.3.8. *Let $\Omega \subset \mathbb{R}^n$ be open and let $E \subset \Omega$ with C^2 boundary. For any $y \in B_r(x) \cap \partial E$, then $A_E(y)$ seen as symmetric tensor on $T_y\partial E \otimes T_y\partial E$ is called* second fundamental form of ∂E at y, *while $\{\tau_h\}_{h=1}^{n-1} \subset S^{n-1} \cap T_y\partial E$ and $\{\kappa_h\}_{h=1}^{n-1}$ are called the* principal directions *and the* principal curvatures *of ∂E at y.*

For any matrix $\mathfrak{M}$ the *Frobenius norm*, which will be denoted here by $|\mathfrak{M}|$, is defined via

$$|\mathfrak{M}| := \sqrt{\sum_i \sum_j |\mathfrak{M}_{ij}|^2}. \tag{2.3.5}$$

Proposition 2.3.9. *Let $\Omega \subset \mathbb{R}^n$ be open and let $E \subset \Omega$ with C^2 boundary. The* scalar mean curvature *κ_E of the C^2–hypersurface $\Omega \cap \partial E$ is locally representable as*

$$\kappa_E(y) = \frac{1}{(n-1)} \sum_{h=1}^{n-1} \kappa_h(y) \ \text{ for all } y \in B_r(x) \cap \partial E,$$

while the second fundamental form satisfies

$$|A_E(y)|^2 = \sum_{h=1}^{n-1} (\kappa_h(y))^2 \quad \text{for all } y \in B_r(x) \cap \partial E.$$

We are now in the position to state the following.

Theorem 2.3.10 (Second Variation of Perimeter). *Let $\Omega \subset \mathbb{R}^n$ be open, let E be an open set such that $\partial E \cap \Omega$ is C^2, $\zeta \in C_c^\infty(\Omega)$, and let $\{f_t\}_{|t|<\epsilon}$ be a local variation associated with the normal vector field $V = \zeta N_E \in C_c^1(\Omega; \mathbb{R}^n)$. Then*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{P}(f_t(E); \Omega) = \int_{\partial E} |\nabla_E \zeta|^2 + \left((n-1)^2 \kappa_E^2 - |A_E|^2\right) \zeta^2 d\mathcal{H}^{n-1},$$

where $\nabla_E \zeta := \nabla \zeta - (\nu_E \cdot \nabla \zeta)\nu_E$ denotes the tangential gradient of ζ with respect to the boundary of E.

Using the characterization of the first and second variation, it is possible to obtain the following estimate on level sets of the distance function.

Lemma 2.3.11. *Suppose that $E_0 \subset \Omega$ is a volume-constrained perimeter minimizer in Ω. Define the function $\eta(s) := \mathcal{H}^{n-1}(\{d_E(x) = s\})$, where d_E is the signed distance function (see (2.3.4)). Then η is twice differentiable at zero and satisfies*

$$\begin{aligned} \eta(0) &= \mathrm{P}(E; \Omega), \\ \eta'(0) &= (n-1)\kappa_E \, \mathrm{P}(E; \Omega), \\ \eta''(0) &= (n-1)^2 \kappa_E \, \mathrm{P}(E; \Omega) \\ &\quad - \int_{\partial E_0} |A_{E_0}|^2 \, d\mathcal{H}^{n-1} - \int_{\partial E_0 \cap \partial \Omega} \nu_{\partial E_0} \cdot A_\Omega \nu_{\partial E_0} \, d\mathcal{H}^{n-2}, \end{aligned}$$

where κ_E is the mean curvature of E. Furthermore, the function η is bounded.

Remark 2.3.12. *A careful proof of the fact that this function is twice differentiable at 0 can be found in [73]. The formulas given here can be found in [103]. The fact that η is bounded comes from [88].*

Remark 2.3.13. *If one instead considers*

$$\phi(r) := P(\{d_{E_0} \le s(r)\}; \Omega) \text{ where } \mathcal{L}^n(\{d_{E_0} \le s(r)\}) = r,$$

and sets $r_0 = \mathcal{L}^n(E_0)$ then the previous formulas become

$$\begin{aligned} \phi(r_0) &= P(E_0; \Omega), \\ \phi'(r_0) &= \kappa_{E_0}(n-1), \\ \phi''(r_0) &= -\frac{\int_{\partial E_0} |A_{E_0}|^2 \, d\mathcal{H}^{n-1} + \int_{\partial E_0 \cap \partial \Omega} \nu_{\partial E_0} \cdot A_\Omega \nu_{\partial E_0} \, d\mathcal{H}^{n-2}}{P(E_0; \Omega)^2}. \end{aligned}$$

This computation can be found, for example, in [103].

Finally, there is a significant rigidity in constant mean curvature surfaces. One way to study this is to consider the following definition:

Definition 2.3.14. *A set $E_0 \subset \Omega$ is called a (Λ, ρ_0) perimeter minimizer in Ω if*

$$P(E_0; B_\rho(x_0)) \le P(E; B_\rho(x)) + \Lambda \mathcal{L}^n(E_0 \Delta E),$$

for all $\rho < \rho_0$ and all measurable E satisfying

$$E_0 \Delta E \subset\subset B_\rho(x) \cap \Omega. \tag{2.3.6}$$

In particular, any volume-constrained perimeter minimizer is a (Λ, ρ_0) minimizer for Λ chosen appropriately (see Example 21.3 in [75]). The following result characterizes a sort of rigidity of a family of (Λ, ρ_0) perimeter minimizers, see Theorem 26.6 in [75].

Theorem 2.3.15. *Suppose that a sequence $\{E_k\}$ of (Λ, ρ_0) minimizers in Ω converges in $L^1(\Omega)$ to a (Λ, ρ_0) minimizer E_0. Then the sets in fact converge in $C^{1,\gamma}$, for any $\gamma < 1/2$.*

2.4 Γ-Convergence and Asymptotic Expansion

This section reviews the well-established theory of Γ-convergence and asymptotic expansion by Γ-Convergence.

First, the following definition of Γ-convergence is standard and can be found in [40], [21].

Definition 2.4.1. *Let X be a metric space and let $\{\mathcal{F}_\varepsilon\}$ be a family of functions, where $\mathcal{F}_\varepsilon : X \to \overline{\mathbb{R}}$ for any $\varepsilon > 0$. The family of functions $\{\mathcal{F}_\varepsilon\}$ is said to Γ-converge to $\mathcal{F}_0 : X \to \overline{\mathbb{R}}$ if the following two criteria are satisfied:*

- *For any $x_\varepsilon \to x$ in X it follows that $\mathcal{F}_0(x) \le \liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon)$.*
- *For any $x \in X$ there exists a sequence $x_\varepsilon \to x$ so that $\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \le \mathcal{F}_0(x)$.*

By way of notation, Γ-convergence will sometimes be written $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}_0$.

Remark 2.4.2. *The notion of Γ-convergence in a metric space can be stated equivalently in terms of the functions*

$$\Gamma\text{-}\liminf \mathcal{F}_\varepsilon(x) := \sup_{r>0} \liminf_{\varepsilon\to 0^+} \inf_{y\in B(x,r)} \mathcal{F}_\varepsilon(y)$$
$$\Gamma\text{-}\limsup \mathcal{F}_\varepsilon(x) := \sup_{r>0} \limsup_{\varepsilon\to 0^+} \inf_{y\in B(x,r)} \mathcal{F}_\varepsilon(y).$$

These two functions are always lower semicontinuous (see Proposition 6.8 in [40]). It is also clear that $\Gamma\text{-}\liminf \mathcal{F}_\varepsilon \le \Gamma\text{-}\limsup \mathcal{F}_\varepsilon$, with equality of the two functions precisely when $\mathcal{F}_\varepsilon$ Γ-converges.

This definition was first given by De Giorgi in [42]. This definition is primarily motivated by seeking minimal conditions which guarantee the convergence of minima and minimizers of a family of functionals. This notion will be made more precise by Theorem 2.4.5, which is sometimes called the fundamental theorem of Γ-convergence. In stating that theorem, the following definitions are used.

Definition 2.4.3. *A function $\mathcal{F} : X \to \overline{\mathbb{R}}$ is called* coercive *if the closure of the set $\{\mathcal{F} \le t\}$ is compact in X for any $t \in \mathbb{R}$.*

Definition 2.4.4. *A family of functions $\{\mathcal{F}_\varepsilon\}$, with $\mathcal{F}_\varepsilon : X \to \overline{\mathbb{R}}$, is called equicoercive if the following holds for any family $\{x_\varepsilon\}$:*

$$\sup_\varepsilon \mathcal{F}_\varepsilon(x_\varepsilon) < \infty \implies \{x_\varepsilon\} \textit{ is precompact in } X.$$

Theorem 2.4.5. *Let X be a metric space and let $\{\mathcal{F}_\varepsilon\}$ be a family of functions, where $\mathcal{F}_\varepsilon : X \to \overline{\mathbb{R}}$ for any $\varepsilon > 0$. Suppose that the family $\{\mathcal{F}_\varepsilon\}$ is equicoercive and that $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}_0$ (see Definition 2.4.4). Then the following two properties hold:*

- *$\mathcal{F}_0$ attains its infimum and satisfies* $\min_X \mathcal{F}_0 = \lim_{\varepsilon\to 0} \inf_X \mathcal{F}_\varepsilon$.
- *If, for $\varepsilon_k \to 0^+$, the sequence x_k satisfies $\mathcal{F}_{\varepsilon_k}(x_k) = \inf_X \mathcal{F}_{\varepsilon_k} + o(1)$, then up to a subsequence (not relabeled) x_k converges to some x^* which is a minimizer of $\mathcal{F}_0$.*

One useful point of view is that $\mathcal{F}_0$ provides a type of selection criteria on minimizers for the functionals $\mathcal{F}_\varepsilon$, or in other words by studying the minimizers of $\mathcal{F}_0$ it is possible to deduce information about the minimizers of $\mathcal{F}_\varepsilon$ (if they exist), in at least an asymptotic sense. In other words, any minimizing sequences of the $\mathcal{F}_\varepsilon$ that converges must converge to a minimizer of $\mathcal{F}_0$.

It is, however, important to note that minimizers of $\mathcal{F}_0$ do not necessarily correspond to limits of *minimizers* of the $\mathcal{F}_\varepsilon$. A simple example is instructive.

Example 2.4.6 ([22] Remark 2.6)**.** *Let $X = [0, 1]$ and $\mathcal{F}_\varepsilon = \varepsilon x^2$. Then $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} 0$, which is minimized at any $x \in X$, but $\mathcal{F}_\varepsilon$ is only minimized at $x = 0$.*

The following very specific case provides a framework where this phenomenon cannot occur, and was first given in [104], see also [69] and [22].

Proposition 2.4.7. *Let X be a metric space and let $\{\mathcal{F}_\varepsilon\}$ be a family of functions, where $\mathcal{F}_\varepsilon : X \to \overline{\mathbb{R}}$ for any $\varepsilon > 0$. Suppose that, for all $\varepsilon > 0$, $\mathcal{F}_\varepsilon$ is coercive (see 2.4.3) and lower semicontinuous. Also suppose that the family $\{\mathcal{F}_\varepsilon\}$ is equicoercive, and that $\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \mathcal{F}_0$ (see Definitions 2.4.1 and 2.4.4) . Suppose furthermore that $\tilde{x} \in X$ is a strict local minimizer of $\mathcal{F}_0$. Then there exists a sequence $x_\varepsilon \to \tilde{x}$ which are local minimizers of $\mathcal{F}_\varepsilon$ for all ε sufficiently small.*

It is clear that Example 2.4.6 is somewhat artificial: if one divides by ε (which does not affect the minimization problem) then all the confusion disappears. This suggests the need to derive a sort of expansion of the functionals in terms of Γ-convergence.

One method for producing such an expansion is known as the asymptotic development by Γ-convergence. This was first introduced in [13].

Definition 2.4.8. *Let X be a metric space and let $\{F_\varepsilon\}$ be a family of functions, where $F_\varepsilon : X \to \overline{\mathbb{R}}$ for any $\varepsilon > 0$. We say that an* asymptotic development *of* order k

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k)$$

holds if there exist functions $\mathcal{F}^{(i)} : X \to \overline{\mathbb{R}}$, $i = 0, 1, \ldots, k$, such that the functions

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\varepsilon} \tag{2.4.1}$$

are well-defined and

$$\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)}, \tag{2.4.2}$$

where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$.

One major aim of carrying out such an asymptotic expansion is that it may provide additional selection criteria for limits of minimizers. This is summarized in the following proposition.

Proposition 2.4.9. *Let $\mathcal{F}^{(i)}$ be an asymptotic development of order k of a family of functions $\{\mathcal{F}_\varepsilon\}$. Define*

$$\mathcal{U}_i := \{\textit{minimizers of } \mathcal{F}^{(i)}\}.$$

It then follows that

$$\mathcal{F}^{(i)} \equiv \infty \textit{ in } X \backslash \mathcal{U}_{i-1},$$

and that

$$\{\textit{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} \subset \mathcal{U}_k \subset \cdots \subset \mathcal{U}_0, \tag{2.4.3}$$

with

$$\inf \mathcal{F}_{\varepsilon_m} = \inf \mathcal{F}^{(0)} + \varepsilon_m \inf \mathcal{F}^{(1)} + \cdots + \varepsilon_m^k \inf \mathcal{F}^{(k)} + o(\varepsilon_m^k)$$

for every sequence $\varepsilon_m \to 0^+$, provided $\inf \mathcal{F}^{(i)} < \infty$ *for all $i = 0, \ldots, k$.*

Simple examples show that each of the inclusions in (2.4.3) may be strict (see [13]). Thus asymptotic development by Γ-convergence provides a selection criteria for minimizers of $\mathcal{F}^{(0)}$. Some other works that describe asymptotic development via Γ-convergence include [23], [50].

2.5 Semigroups and Evolution Families

This section outlines some classical results for "solving" linear problems of the form

$$\frac{d}{dt}u = A(t)u, \quad u(0) = u^0 \tag{2.5.1}$$

when u takes values in some Banach space X and $A(t)$ is an unbounded linear operator. These results mostly come from [91].

The first step is to consider the problem when A does not depend on t. This case is the subject of *semigroup theory.*

Definition 2.5.1. *A family $\{S(t)\}_{t\in[0,\infty)}$ with elements in $\mathcal{L}(X)$, with X a Banach space, is called a strongly continuous semigroup if it satisfies*

$$\begin{aligned} S(0) &= I, \\ S(t)S(s) &= S(t+s) \quad \textit{for all } t, s \geq 0, \\ \lim_{t\to 0} S(t)x &= x \textit{ for all } x \in X. \end{aligned}$$

A linear operator $A : D(A) \to \mathbb{R}$ is called the generator of S if

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t},$$

where $D(A) \subset X$ is the subspace of X for which the limit exists.

Another name for a strongly continuous semigroup is a $C0$ semigroup. This work will use the word "semigroup" in place of "$C0$ semigroup" for brevity.

A semigroup will satisfy equation (2.5.1) in the sense that

$$\frac{d}{dt}S(t)x = AS(t)x$$

for all $x \in D(A)$ (see, e.g., Theorem 2.4 in [91]). When A is the generator of a semigroup $S(t)$, it is customary to write $S(t) = e^{At}$.

The following proposition gives a characterization of generators of semigroups, see Theorem 1.5.3 in [91]. By way of definition, the *resolvent set* of a linear operator A, namely the set of $\lambda \in \mathbb{C}$ such that $(A - \lambda I)$ has a bounded inverse $R(\lambda; A)$, will be denoted by $\tilde{\rho}(A)$.

Proposition 2.5.2. *A linear operator $A : \mathrm{dom}(A) \subset X \to X$, with domain of definition $\mathrm{dom}(A)$, is the infinitesimal generator of a semigroup e^{At} satisfying $\|e^{At}\| \leq Me^{\omega t}$ if and only if*

- *A is closed and $\mathrm{dom}(A)$ is dense in X.*
- *The set $\tilde{\rho}(A)$ contains the ray (ω, ∞) and*

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \qquad \textit{for } \lambda > \omega.$$

In general the previous condition is difficult to verify. One particular case where this is possible is when $\|S(t)\| \leq 1$ for all t. In this case the semigroup is called a semigroup of contractions. The following definition and proposition give a characterization of semigroups of contractions.

Definition 2.5.3. *Let $x \in X$, with X a Banach space. Define*

$$\mathcal{J}(x) := \left\{ x^* \in X^* : \langle x^*, x \rangle_{X^*,X} = \|x\|_X^2 = \|x^*\|_{X^*}^2 \right\}. \tag{2.5.2}$$

A linear operator A with domain of definition $\mathrm{dom}(A) \subset X$ is called dissipative *if for every $x \in \mathrm{dom}(A)$ there exists an $x^* \in \mathcal{J}(x)$ such that*

$$\langle x^*, Ax \rangle_{X^*,X} \leq 0.$$

The next result is known as the Lumer–Phillips Theorem, see e.g. [45] Theorem II.3.15. It links semigroups of contractions with dissipative operators.

Proposition 2.5.4. *The following are equivalent for a densely-defined, dissipative operator A:*

1. *The range of $(A - \lambda I)$ is dense for some $\lambda > 0$.*
2. *A is closable and its closure (also denoted by A) generates a contraction semigroup.*

The next result will provide a dissipation estimate in later analysis for two symmetric operators. It can be found in [66], Theorem 4.12.

Proposition 2.5.5. *Suppose that Λ is a self-adjoint operator on a Hilbert space X, with $\langle \Lambda x, x \rangle \leq 0$. Suppose that B is a symmetric operator on X with $\|Bx\| \leq \|\Lambda x\|$. Then*

$$\langle (\Lambda + B)x, x \rangle \leq 0.$$

Another common avenue for proving that a linear operator generates a semigroup is to use perturbation theory. The following perturbation result is given in Theorem 3.1.1 in [91].

Proposition 2.5.6. *Suppose that A is the generator of a semigroup satisfying $\|e^{At}\| \leq M_1 e^{\omega_1 t}$, and that B is a bounded operator. Then $A + B$ generates a semigroup satisfying $\|e^{(A+B)t}\| \leq M_2 e^{\omega_2 t}$.*

Finally, the following proposition gives some information on the inhomogeneous case, and can be found in Corollary 4.2.2 in [91].

Proposition 2.5.7. *Suppose that $f \in L^1(0,T,X)$, with X a Banach space. Suppose that A is the generator of a semigroup e^{At} on X. Then the initial value problem*

$$\frac{d}{dt}x = Ax + f, \quad x(0) = x_0 \in X$$

has at most one solution. If it has a solution, then

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s)\, ds.$$

In the previous proposition, the integral is naturally meant in the sense of Bochner integrals. Some basic references on Bochner integrals and their properties include [24] and [43].

All of the previous results are in the autonomous case, namely the case where A is independent of time in (2.5.1). The following definition treats the time dependent case.

Definition 2.5.8. *Given a Banach space X, a two-parameter family $\{U(t,s)\}_{0\le s\le t\le T}$, with $T \in (0,\infty]$, taking values in $\mathcal{L}(X)$, is called an evolution family if*

$$\begin{aligned} U(s,s) &= I, \\ U(t,r)U(r,s) &= U(t,s), \\ (t,s) &\mapsto U(t,s)x \text{ is continuous for all } x \in X. \end{aligned}$$

A family of linear operators $\{A(t)\}_{t\in[0,T]}$, satisfying $Y \subset \operatorname{dom}(A(t))$ for all $t \in [0,T]$ and for some dense $Y \subset X$, is said to generate an evolution family U if

$$\begin{aligned} \frac{\partial^+}{\partial t}U(t,s)x|_{t=s} &= A(s)x, \\ \frac{\partial}{\partial s}U(t,s)x &= -U(t,s)A(s)x, \end{aligned}$$

for all $x \in Y$.

The results for the construction of such operators and their properties generally have complicated statements, primarily because the domain of A may vary in time. For this reason, some of the results here are stated in terms of the spaces X_{1+k}, which were defined in (1.2.12), and which are the only spaces where these results will be used in this work.

To begin, it is important to understand how evolution families are related to the solution of (2.5.1). The following proposition answers this question in a classical context, see Theorem 5.4.2 in [91].

Proposition 2.5.9. *Suppose that, for some $k \ge 0$, $\{A(t)\}_{t\in I}$ is the generator of an evolution family U in X_{1+k} on the interval $I = [0,T)$, with $T = \infty$ permitted. Furthermore, suppose that for some $h \in C(I; X_{2+k}) \bigcap C^1(I; X_{1+k})$ we have that*

$$\frac{d}{dt}h = A(t)h(t)$$

is satisfied in X_{1+k}. Then it must be that $U(t,0)h(0) = h(t)$.

The next two propositions give specific situations where an evolution family can be constructed from a family of linear operators $\{A(t)\}$. The first proposition comes from Corollary 5.4.7 and 5.4.8 in [91].

Proposition 2.5.10. *Let X be a Banach space and let $I = [0, T)$, with $T = \infty$ permitted. Suppose that, for any fixed $t \in I$, $A(t)$ is the generator of a semigroup $\{S_{A(t)}(s)\}_{s \geq 0}$ which satisfies*

$$\|S_{A(t)}(s)\|_{\mathcal{L}(X)} \leq e^{-\lambda s} \quad \text{for all } s \geq 0,$$

where λ is independent of t. Also suppose that $\operatorname{dom}(A(t)) \equiv D$ is independent of t and that for all $x \in D$ we have that $A(t)x$ is C^1 in X. Then the family of operators $\{A(t)\}_{t \in I}$ generates an evolution family U on X which satisfies

$$\|U(t,s)\|_{\mathcal{L}(X)} \leq e^{-\lambda(t-s)} \qquad \text{for } 0 \leq s \leq t < T.$$

Furthermore for $x_0 \in D$ we have that $x(t) := U(t,0)x_0$ is the unique solution of the non-autonomous Cauchy problem

$$\frac{d}{dt}x(t) = A(t)x(t), \qquad x(0) = x_0.$$

The next proposition is a direct application of Theorem 5.3.1 in [91].

Proposition 2.5.11. *Let $I = [0, T)$, with $T = \infty$ permitted, and suppose that a family of linear operators $\{A(t)\}_{t \in I}$ satisfies the following for all $t \in I$.*

1. *$A(t)$ generates a contraction semigroup on X_{1+k}.*
2. *$A(t)$ generates a contraction semigroup on X_{2+k}.*
3. *$A(t)$ is a bounded operator from X_{2+k} to X_{1+k}, and the map $t \mapsto A(t)$ is continuous from I to $\mathcal{L}(X_{2+k}, X_{1+k})$.*

Then $\{A(t)\}_{t \in I}$ generates an evolution family $V_{X_{1+k}}$ satisfying $\|V_{X_{1+k}}(t,s)\|_{\mathcal{L}(X_{1+k})} \leq 1$.

The following is Lemma 5.4.5 in [91].

Proposition 2.5.12. *Let $U(t,s)$ be an evolution system on a Banach space X satisfying $\|U(t,s)\| \leq M$. Let $B(t)$ be a strongly continuous family of bounded linear operators on X. Then there exists a unique evolution family $V(t,s)$ of bounded linear operators on X such that*

$$V(t,x)x = U(t,s)x + \int_s^t V(t,r)B(r)U(r,s)x\,dr.$$

Remark 2.5.13. *Proposition 2.5.12 readily implies that if $A(t)$ is the generator of an evolution family U, then $A(t) + B(t)$ is the generator of an evolution family V.*

2.6 Other Preliminaries

The following lemma is a slight modification of Proposition 1 in [39]. This thesis will use this lemma in studying rearrangement operators. This lemma is particularly noteworthy because it does not make any assumptions about linearity or continuity. The proof is included here for convenience.

Lemma 2.6.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be measure spaces and let $C \subset L^1(\mathfrak{M})$ be a closed under $\vee$, meaning that if $f, g \in C$ then $f \vee g \in C$. Let $\mathfrak{Z}$ be a mapping from $C \to L^1(\mathfrak{N})$ which satisfies*

$$\int_{\mathfrak{M}} f = \int_{\mathfrak{N}} \mathfrak{Z}(f) \quad \text{for all } f \in C.$$

Then the following are equivalent:

(i) $f, g \in C$ *and* $f \leq g \implies \mathfrak{Z}(f) \leq \mathfrak{Z}(g)$.

(ii) $\int_{\mathfrak{N}} (\mathfrak{Z}(f) - \mathfrak{Z}(g))^+ \leq \int_{\mathfrak{M}} (f-g)^+$ *for all* $f, g \in C$.

(iii) $\int_{\mathfrak{N}} |\mathfrak{Z}(f) - \mathfrak{Z}(g)| \leq \int_{\mathfrak{M}} |f-g|$ *for all* $f, g \in C$.

Proof. If we have (i) then $\mathfrak{Z}(f) \leq \mathfrak{Z}(f \vee g)$, and thus

$$\begin{aligned} \int_{\mathfrak{N}} (\mathfrak{Z}(f) - \mathfrak{Z}(g))^+ &\leq \int_{\mathfrak{N}} \mathfrak{Z}(f \vee g) - \mathfrak{Z}(g) \\ &= \int_{\mathfrak{M}} (f \vee g) - g = \int_{\mathfrak{M}} (f-g)^+, \end{aligned}$$

which is (ii). If we have (ii) then

$$\begin{aligned} \int_{\mathfrak{N}} |\mathfrak{Z}(f) - \mathfrak{Z}(g)| &= \int_{\mathfrak{N}} (\mathfrak{Z}(f) - \mathfrak{Z}(g))^+ + \int_{\mathfrak{N}} (\mathfrak{Z}(g) - \mathfrak{Z}(f))^+ \\ &\leq \int_{\mathfrak{M}} (f-g)^+ + \int_{\mathfrak{M}} (g-f)^+ = \int_{\mathfrak{M}} |f-g|, \end{aligned}$$

which gives (iii). If we have (iii), and $f, g \in C$, with $g \leq f$, then we use the identity $2s^+ = |s| + s$ to show that

$$\begin{aligned} 2\int_{\mathfrak{N}} (\mathfrak{Z}(g) - \mathfrak{Z}(f))^+ &= \int_{\mathfrak{N}} |\mathfrak{Z}(g) - \mathfrak{Z}(f)| + \int_{\mathfrak{N}} \mathfrak{Z}(g) - \mathfrak{Z}(f) \\ &\leq \int_{\mathfrak{M}} |g-f| + \int_{\mathfrak{M}} g - f = 0, \end{aligned}$$

which in turn implies that $\mathfrak{Z}(g) \leq \mathfrak{Z}(f)$ a.e., which is (i). This concludes the proof. □

The next proposition is a C^1 touching result, which originated in the study of Hamilton-Jacobi equations. The statement and proof can be found in [47], p. 584.

Proposition 2.6.2. *Assume that $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, which is differentiable at x_0. Then there exists a function $v \in C^1(\mathbb{R}^d)$ such that $u(x_0) = v(x_0)$ and $u - v$ has a strict local maximum at x_0.*

Remark 2.6.3. *By considering $-u$ it is clear that maximum can be replaced with minimum in the statement of the previous lemma.*

The next result gives a sufficient condition for a function to be concave, and can be found in Lemma 2.7 in [103].

Proposition 2.6.4. *Let $f : I \to \mathbb{R}$ be a lower semicontinuous function defined on an interval I and suppose f is locally concave in the sense that its graph admits a local upper support line in a neighborhood of any point on the graph. Then f is concave.*

Part I

Cahn–Hilliard Energy Asymptotics and Slow Motion Bounds

Chapter 3

Generalized Rearrangement of Functions on a Bounded Domain

This chapter studies a novel type of rearrangement of a function $f : \Omega \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^n$. Before introducing this new type of rearrangement, it is useful to review the definition and properties of the classically-studied spherically decreasing rearrangement (see e.g. [67, 68]). The spherically decreasing rearrangement is defined as follows: Given any positive, L^1 function u, we define the distribution function $\varrho_u(s) := \mathcal{L}^n(\{u > s\})$. Then define

$$g_u(t) := \sup\{s \in \mathbb{R} : \varrho_u(s) > \omega_n t^n\},$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$, and define u^*, the spherically decreasing rearrangement, via

$$u^*(x) := g_u(|x|).$$

This rearrangement is constructed using a simple approach: level sets of of u are rearranged into balls centered at the origin.

The spherically decreasing rearrangement has several important properties. First, the very definition of u^* readily implies that u^* and u are *equimeasurable*, meaning that $\mathcal{L}^n(\{u^* > s\}) = \mathcal{L}^n(\{u > s\})$ for almost every s. From this property, it is straightforward to show that $\int \psi(u)\, dx = \int \psi(u^*)\, dx$, for any Borel function ψ.

Second, this rearrangement is order preserving, meaning that if $u \geq v$ then $u^* \geq v^*$. This property, along with equimeasurability, implies [39] that the rearrangement operator is a contraction on L^p spaces, meaning that

$$\|u^* - v^*\|_{L^p} \leq \|u - v\|_{L^p}. \tag{3.0.1}$$

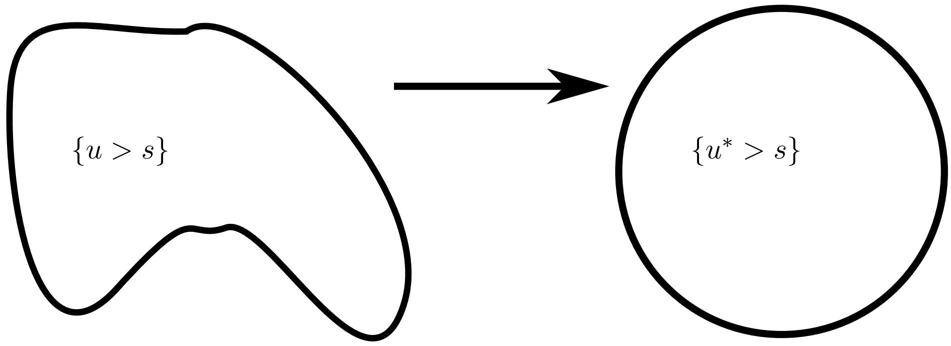

Figure 3.1: Rearranging the level sets of u.

Second, if $u \in W^{1,p}$, then u^* will be in $W^{1,p}$ and

$$\|u^*\|_{W^{1,p}} \leq \|u\|_{W^{1,p}} \tag{3.0.2}$$

This is known as the Pólya–Szegő inequality. The proof of this is classical, see e.g. [67] [72]. This inequality has been used to study the symmetries of solutions to certain elliptic problems[67], as well as to establish comparison principles [105]. The present interest lies in the fact that the Pólya–Szegő inequality permits the reduction of functional problems in n-dimensions to simpler weighted, one-dimensional problems.

For example, in [41], Dal Maso, Fonseca, and Leoni use the spherically decreasing rearrangement to study Γ-limits of the Cahn–Hilliard functional (1.1.1) in a domain when both a mass constraint and a Dirichlet condition are imposed. The Dirichlet condition is crucial in their analysis because it enables the use of the Pólya–Szegő inequality, which subsequently reduces the problem to a one-dimensional problem.

In light of equations (3.0.1) and (3.0.2), a natural question is the smoothness of the rearrangement operator. It turns out that the operator is not smooth on $W^{1,p}$ [7]. This is essentially due to the non-local nature of the rearrangement. However, the operator is actually continuous on fractional Sobolev spaces [7].

The proof of the Pólya Szegő inequality uses relatively simple tools. Specifically, it uses the coarea formula (2.1.8), the isoperimetric inequality (2.1.3), and some simple properties of the decreasing rearrangement in one dimension, namely (3.3.2).

The following section presents a natural extension of this proof to the setting of a bounded domain. This extension is independent of boundary conditions, and is hence well-suited to Neumann problems. In particular, the extension that we present here is very well-suited to studying sharp interface problems. A specialized version of the results presented here was used by Cianchi et. al. [34] [38] to study sharp bounds on a class of Poincaré constants.

3.1 Definition of the Rearrangement

This section assumes that

$$\Omega \subset \mathbb{R}^n \qquad \text{bounded and open with } \mathcal{L}^n(\Omega) = 1.$$

Furthermore, this section considers a continuous function $\mathcal{I} : \mathbb{R} \to \mathbb{R}$, which satisfies the following assumptions

$$\mathcal{I}(\mathfrak{v}) = 0 \quad \text{ for } \mathfrak{v} \in \mathbb{R}\backslash(0,1), \tag{3.1.1}$$

$$\mathcal{I}(\mathfrak{v}) \geq C \min\{\mathfrak{v}, 1-\mathfrak{v}\}^{\frac{n-1}{n}} \quad \text{ for } \mathfrak{v} \in (0,1). \tag{3.1.2}$$

Next, a measurable function $u : \Omega \to \mathbb{R}$ is said to have $\mathcal{I}$ comparable level sets if

$$\mathrm{P}(\{u > s\}; \Omega) \geq \mathcal{I}(\mathcal{L}^n(\{u > s\})).$$

In particular, if $\mathcal{I} = \mathcal{I}_\Omega$, where $\mathcal{I}_\Omega$ is the *Isoperimetric Function* of Ω, given by

$$\mathcal{I}_\Omega := \inf\{\mathrm{P}(E; \Omega) : E \subset \Omega, \mathcal{L}^n(E) = \mathfrak{v}\},$$

then any measurable function u will have $\mathcal{I}$ comparable level sets. Furthermore, if Ω is connected and Lipschitz then $\mathcal{I}_\Omega$ will satisfy (3.1.1) and (3.1.2) due to Proposition 2.1.10.

This section considers the general function $\mathcal{I}$ because in subsequent sections it will be necessary to consider certain modifications of the isoperimetric function $\mathcal{I}_\Omega$. For example, in some settings it will be necessary to consider either an L^1 localized version of $\mathcal{I}_\Omega$ or a smoothed version of the same.

Next, define a function V_Ω as a solution to the following Cauchy problem:

$$\frac{d}{dt}V_\Omega(t) = \mathcal{I}(V_\Omega(t)), \quad V_\Omega(0) = 1/2. \tag{3.1.3}$$

Since $\mathcal{I}$ is bounded and continuous, the Cauchy problem (3.1.3) admits a global solution $V_\Omega : \mathbb{R} \to [0,1]$. It follows from inequality (3.1.2) that there is a $T_1 > 0$ so that $0 < V_\Omega(t)$ for $-T_1 < t < 0$ and $V_\Omega(-T) = 0$. Similarly there exists a $T_2 > 0$ so that $V_\Omega(t) < 1$ for all $t < T_2$ and $V_\Omega(T_2) = 1$. Define

$$I := (-T_1, T_2). \tag{3.1.4}$$

In what follows for $y \in \mathbb{R}^n$ let $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Next, define a set $\Omega^* \subset \mathbb{R}^n$, which will be a type of rearrangement of Ω, via

$$\Omega^* := \left\{y : \ \ y_n \in I, \ \ y' \in B_{n-1}(0, r(y_n))\right\},$$

where for $t \in I$,

$$r(t) := \left(\frac{\mathcal{I}(V_\Omega(t))}{\omega_{n-1}}\right)^{1/(n-1)} \quad \text{and } \omega_{n-1} := \mathcal{L}^{n-1}(B_{n-1}(0,1)).$$

Note that the definition of $r(t)$ implies that

$$\mathcal{L}^{n-1}(B_{n-1}(0, r(t))) = \mathcal{I}(V_\Omega(t)) \tag{3.1.5}$$

for all $t \in \overline{I}$.

The following lemma motivates the choice of the Cauchy problem (3.1.3).

Lemma 3.1.1. *For any $t \in \overline{I}$ the following equalities hold:*

$$V_\Omega(t) = \mathcal{L}^n(\Omega^* \cap \{y_n < t\}), \tag{3.1.6}$$

$$\mathcal{I}(V_\Omega(t)) = \mathrm{P}(\{y_n < t\}; \Omega^*). \tag{3.1.7}$$

Proof. Equation (3.1.6)) is proved by using Fubini's theorem, equation (3.1.5), the Cauchy problem (3.1.3), the fundamental theorem of calculus, and the fact that $V_\Omega(-T_1) = 0$, in that order:

$$\begin{aligned}
\mathcal{L}^n(\Omega^* \cap \{y_n < t\}) &= \int_{-T_1}^{t} \mathcal{H}^{n-1}(\Omega^* \cap \{y_n = s\})\, ds \\
&= \int_{-T_1}^{t} \mathcal{I}(V_\Omega(s))\, ds \\
&= V_\Omega(t) - V_\Omega(-T_1) = V_\Omega(t).
\end{aligned}$$

Equality (3.1.7) follows immediately from equation (3.1.5) and Definition 3.1. □

Now given any measurable function $u : \Omega \to \mathbb{R}$, define the distribution function $\varrho_u(s) := \mathcal{L}^n(\{u > s\})$ and the following function:

$$g_u(t) := \sup\{s \in \mathbb{R} : \varrho_u(s) > V_\Omega(t)\}.$$

Here g_u is essentially an inverse of ϱ_u with respect to V_Ω. Next, define a function $u^* : \Omega^* \to \mathbb{R}$ as follows:

$$u^*(y', y_n) := g_u(y_n). \tag{3.1.8}$$

3.2 Fundamental Properties of the Rearrangement

The first important property of the rearranged function u^* is that it is equimeasurable with u.

Lemma 3.2.1. *Let $u : \Omega \to \mathbb{R}$ be a measurable function. Then the functions u^* and u are equimeasurable, meaning that $\varrho_u = \varrho_{u^*}$. This implies that for any Borel function $\psi : \mathbb{R} \to \mathbb{R}$,*

$$\int_\Omega \psi(u)\,dx = \int_{\Omega^*} \psi(u^*)\,dy = \int_I \psi(g_u)\mathcal{I}(V_\Omega)\,dt,$$

assuming that the previous integrals are well-defined. In particular the L^p norms of u and u^* are preserved.

Proof. First note that, by standard arguments, ϱ_u is decreasing and right continuous and that g_u is decreasing and left continuous (see, e.g., [72], p. 478).

Let $h(t) := \sup\{s : g_u(s) > t\}$. Since g_u is decreasing it follows that

$$\begin{aligned} \varrho_{u^*}(t) &= \mathcal{L}^n(\{y \in \Omega^* : g_u(y_n) > t\}) \\ &= \mathcal{L}^n(\{y \in \Omega^* : y_n < h(t)\}) = V_\Omega(h(t)), \end{aligned}$$

where the last equality uses Lemma 3.1.1.

We then claim that $V_\Omega(h(t)) = \varrho_u(t)$. To see this observe that since $\mathcal{I} > 0$ in $(0, 1)$, by (3.1.3) we have that V_Ω is strictly increasing and of class C^1 in I. Hence:

$$\begin{aligned} V_\Omega(h(t)) &= V_\Omega(\sup\{s : g_u(s) > t\}) = \sup\{V_\Omega(s) : g_u(s) > t\} \\ &= \sup\{V_\Omega(s) : \sup\{\tau : \varrho_u(\tau) > V_\Omega(s)\} > t\} \\ &= \sup\{\rho : \sup\{\tau : \varrho_u(\tau) > \rho\} > t\}. \end{aligned}$$

For every ρ such that $\sup\{\tau : \varrho_u(\tau) > \rho\} > t$, there exists $\tau > t$ such that $\varrho_u(\tau) > \rho$. But since ϱ_u is decreasing we have that $\varrho_u(t) \geq \varrho_u(\tau) > \rho$, which then shows that

$$V_\Omega(h(t)) \leq \varrho_u(t).$$

Now if $V_\Omega(h(t)) < \varrho_u(t)$, then $V_\Omega(h(t) < \varrho_u(t) - \epsilon$ for some $\epsilon > 0$. By equation (3.2) this implies that

$$\sup_s\{s : \varrho_u(s) > \varrho_u(t) - \varepsilon\} \leq t.$$

By the right continuity of ϱ_u for some $\delta > 0$ we have that $\varrho_u(t + \delta) > \varrho_u(t) - \epsilon$, which violates the previous inequality. This then implies that $\varrho_u(t) = \varrho_{u^*}(t)$ for all t, which is the desired conclusion.

To see the integral equality stated, we note that (see, e.g., Theorem B.61 in [72]):

$$\int_\Omega \psi(u(x))\,dx = \int_\mathbb{R} \psi(s)\,d\varrho_u(s) = \int_\mathbb{R} \psi(s)\,d\varrho_{u^*}(s) = \int_{\Omega^*} \psi(u^*(y))\,dy.$$

This concludes the proof.

□

The next proposition states that the rearrangement is a type of contraction, and in particular is a contraction on L^p spaces. The proof of this theorem is a straightforward adaptation of a similar result from [39]. There are several other possible proofs, using either simple functions or the Reisz rearrangement inequality, see e.g. Chapter 6 in [72].

Proposition 3.2.2. *Suppose that $j : [0,\infty) \to [0,\infty)$ is convex with $j(0) = 0$. Suppose that*

$$\int_\Omega j(|u_1|)\,dx, \int_\Omega j(|u_2|)\,dx < \infty, \quad u_1, u_2 \in L^1(\Omega).$$

Then

$$\int_{\Omega^*} j(|u_1^* - u_2^*|)\,dy \le \int_\Omega j(|u_1 - u_2|)\,dx.$$

In particular, the rearrangement operator is a contraction on L^p, meaning that

$$\|u_1^* - u_2^*\|_{L^p(\Omega^*)} \le \|u_1 - u_2\|_{L^p(\Omega)}.$$

Proof. First, since j' is a function of bounded variation, we may write, for $r > 0$,

$$\begin{aligned} j(r) &= \int_0^r j'(s)\,ds = \int_0^r \int_0^s dj'(t) + j'(0^+)\,ds \\ &= rj'(0^+) + \int_0^r \int_t^r ds\,dj'(t) = rj'(0^+) + \int_0^\infty (r-t)^+\,dj'(t). \end{aligned} \tag{3.2.1}$$

Next, for $\eta \in L^1(\Omega)$, define $K(\eta) := (\eta + u_2)^* - u_2^*$. Since $u \le v$ implies that $u^* \le v^*$, we immediately have that if $u \le v$ then $K(u) \le K(v)$. We also deduce, using Lemma 3.2.1, that

$$\int_{\Omega^*} K(\eta)\,dy = \int_\Omega (\eta + u_2) - u_2\,dx = \int_\Omega \eta\,dx. \tag{3.2.2}$$

By Lemma 2.6.1 we then have that

$$\int_{\Omega^*} (K(\eta_1) - K(\eta_2))^+\,dy \le \int_{\Omega^*} (\eta_1 - \eta_2)^+\,dx.$$

Now, we note that $K(t) = t$ for any $t \in \mathbb{R}$. Thus, for any $t > 0$,

$$\int_{\Omega^*} [K(\eta) - t]^+\,dy \le \int_\Omega [\eta - t]^+\,dy$$

Since j is convex, $dj'(t)$ is a positive measure. Thus after integrating with respect to $dj'(t)$, and using (3.2.1) and (3.2.2), we have that

$$\int_{\Omega^*} j(K(\eta))\,dy \le \int_\Omega j(\eta)\,dx,$$

for any $\eta \in L^1(\Omega)$ such that the right hand side is finite. If we set $\eta = u_1 \vee u_2 - u_2 = (u_1 - u_2)^+$ this implies that

$$\int_{\Omega^*} j((u_1 \vee u_2)^* - u_2^*)\,dy \le \int_\Omega j((u_1 - u_2)^+)\,dx.$$

Hence, by using monotonicity of the rearrangement, $(\cdot)^+$ and j, we find that

$$\int_{\Omega^*} j((u_1^* - u_2^*)^+)\,dy \le \int_{\Omega^*} j((u_1 \vee u_2)^* - u_2^*)\,dy \le \int_\Omega j((u_1 - u_2)^+)\,dx.$$

Switching u_1 and u_2 and summing then completes the proof.

∎

Corollary 3.2.3 (Hardy-Littlewood Inequality)**.** *Let* $u, v \in L^2(\Omega)$*. Then*

$$\int_\Omega uv\,dx \le \int_{\Omega^*} u^* v^*\,dy.$$

Proof. By Proposition 3.2.2 we have that

$$\int_{\Omega^*} [u^*]^2 + [v^*]^2 - u^*v^*\,dy = \int_{\Omega^*} [u^* - v^*]^2\,dy \le \int_\Omega [u-v]^2\,dx = \int_\Omega u^2 + v^2 - uv\,dx.$$

By then using Lemma 3.2.1 on the function $\psi(s) = s^2$ we thus have that

$$\int_{\Omega^*} u^*v^*\,dy \ge \int_\Omega uv\,dx,$$

as desired. □

The next lemma states a basic property of the rearrangement operator: namely that it commutes with increasing functions. This will later be used to prove that the rearrangement operator preserves absolute continuity.

Lemma 3.2.4. *Let* $u : \Omega \to \mathbb{R}$ *be measurable. Let* $H : \mathbb{R} \to \mathbb{R}$ *be an increasing function. Then the following holds* $\mathcal{L}^n$ *a.e.:*

$$H(u^*) = [H(u)]^*.$$

In particular, given $s_1 < s_2$*, let* $\mathrm{Tr}_{s_1,s_2}(s) := (s \wedge s_1) \vee s_2, s \in \mathbb{R}$*. Then the following equality holds* $\mathcal{L}^n$ *a.e.:*

$$\mathrm{Tr}_{s_1,s_2}(u^*) = (\mathrm{Tr}_{s_1,s_2}(u))^*.$$

Proof. Fix any $t \in \mathbb{R}$, and let $Q = \{s : H(s) > t\}$. Since H is an increasing function the set Q will either take the form $[A, \infty)$ or (A, ∞). Thus we may write

$$\mathcal{L}^n(\{H(u^*) > a\}) = \mathcal{L}^n(\{u^* \in Q\}).$$

Due to Lemma 3.2.1 we have that

$$\mathcal{L}^n(\{u^* \in Q\}) = \mathcal{L}^n(\{u \in Q\}).$$

In turn by the definition of Q,

$$\mathcal{L}^n(\{u \in Q\}) = \mathcal{L}^n(\{H(u) > a\}).$$

Again applying Lemma 3.2.1 we have that

$$\mathcal{L}^n(\{H(u) > a\}) = \mathcal{L}^n(\{[H(u)]^* > a\}).$$

This implies that $H(u^*)$ and $[H(u)]^*$ are equimeasurable. By the definition of the rearrangement and since H is increasing, it is evident that both functions are only functions of y_n, and are decreasing in y_n. We will let $u_1(s) := H(u^*)(0, s)$ and $u_2(s) := [H(u)]^*(0, s)$. It suffices to show that u_1 and u_2 are equal $\mathcal{L}^1$ a.e.. Suppose that they are not. Then, since monotone functions are differentiable a.e., there exists a value s^* at which both u_1 and u_2 are continuous and so that $u_1(s^*) \ne u_2(s^*)$. Since both functions are monotone, this implies that $\mathcal{L}^1(\{s \in I : u_1(s) \ge u_2(s^*)\}) \ne \mathcal{L}^1(\{s \in I : u_2(s) \ge u_2(s^*)\})$. However, this contradicts the fact that $H(u^*)$ and $[H(u)]^*$ are equimeasurable. This concludes the proof.

□

Remark 3.2.5. *Lemmas 3.1.1 - 3.2.4 notably do not assume any special properties on u. They are simple consequences of the construction of Ω^* and u^*. In particular, these lemmas do not require that u have $\mathcal{I}$ comparable level sets. This fact will be used later in studying the anisotropic case.*

The next lemma is a straightforward analog of the isoperimetric inequality.

Lemma 3.2.6. *Given $u \in BV(\Omega)$ with $\mathcal{I}$ comparable level sets, for any $t \in \mathbb{R}$ the following must hold:*
$$\mathrm{P}(\{u^* > t\}; \Omega^*) \le \mathrm{P}(\{u > t\}; \Omega).$$

Proof. As g_u is a decreasing function (see (3.1)), we note that the set $\{u^* > t\}$ is actually a set of the form $\{y_n < s\}$. By Lemma 3.1.1 we have that $V_\Omega(s) = \mathcal{L}^n(\Omega^* \cap \{y_n < s\}) = \varrho_{u^*}(t)$. By then recalling that u and u^* are equimeasurable (see Lemma 3.2.1) and by Lemma 3.1.1 we have the following:

$$\begin{aligned}\mathrm{P}(\{u^* > t\}; \Omega^*) &= \mathcal{I}(\varrho_{u^*}(t)) \\ &= \mathcal{I}(\varrho_u(t)) \le \mathrm{P}(\{u > t\}; \Omega),\end{aligned}$$

where we have used the fact that u has $\mathcal{I}$ comparable level sets. This concludes the proof. □

3.3 A Pólya–Szegő Inequality

This section proves an analog of the Pólya–Szegő inequality. The first two lemmas, which are of independent interest, are preliminary to that goal.

Lemma 3.3.1. *Suppose that $u \in BV(\Omega)$ has $\mathcal{I}$ comparable level sets. Then $u^* \in BV(\Omega^*)$ and the following inequality holds:*

$$\int_I \mathcal{I}(V_\Omega(s)) d|Dg_u|(s) = |Du^*|(\Omega^*) \le |Du|(\Omega).$$

Proof. By Lemma 3.2.1 we have that $u^* \in L^1(\Omega^*)$. By (3.1) and by the fact that g_u is decreasing, it follows that $g_u \in BV_{\mathrm{loc}}(I)$ (see, e.g., Theorem 7.2 in [72]).

Moreover by the definition of u^* (see (3.1), (3.1.5), (3.1), and Lemma 3.2.1) we can write the following:

$$\begin{aligned}&|Du^*|(\Omega^*) = \sup\left\{\int_{\Omega^*} \phi(y', y_n) d(Dg_u)(y_n) : \phi \in C_0(\Omega^*), \|\phi\|_{C_0} \le 1\right\} \\ &= \sup\left\{\int_I \left(\int_{B_{n-1}(0, r(y_n))} \phi(y', y_n)\, dy'\right) d(Dg_u)(y_n) : \phi \in C_0(\Omega^*), \|\phi\|_{C_0} \le 1\right\} \\ &= \sup\left\{\int_I \mathcal{I}(V_\Omega(y_n))\psi(y_n)\, d(Dg_u)(y_n) : \psi \in C_0(-T, T), \|\psi\|_{C_0} \le 1\right\} \\ &= \int_I \mathcal{I}(V_\Omega(y_n))\, d|Dg_u|(y_n).\end{aligned}$$

Next we utilize the coarea formula and Lemma 3.2.6 as follows:

$$|Du^*|(\Omega^*) = \int_{\mathbb{R}} \mathrm{P}(\{u^* > t\}; \Omega^*)\, dt \le \int_{\mathbb{R}} \mathrm{P}(\{u > t\}; \Omega)\, dt = |Du|(\Omega).$$

This proves the desired lemma. □

Lemma 3.3.2. *Suppose that $u \in W^{1,1}(\Omega)$ has $\mathcal{I}$ comparable level sets. Then $u^* \in W^{1,1}(\Omega^*)$.*

Proof. By (3.1.8) it suffices to show that g_u is absolutely continuous on any subinterval $[t_0, t_1]$ compactly contained in I. Fix $\epsilon > 0$, and let δ be small enough such that for any measurable $E \subset \Omega$ with $\mathcal{L}^n(E) < \delta$ the following holds (see (3.1.2)):

$$\int_E |\nabla u| \, dx \leq \epsilon \min_{t \in [t_0, t_1]} \mathcal{I}(V_\Omega(t)).$$

Now consider any finite collection of non-overlapping subintervals (a_k, b_k) of $[t_0, t_1]$, satisfying

$$\sum_{k=1}^{N} (b_k - a_k) \leq \frac{\delta}{\max_{t \in [t_0, t_1]} \mathcal{I}(V_\Omega(t))}.$$

The following estimate holds by (3.1.3), (3.1.6), (3.1), (3.1.8), Lemma 3.2.1 and (3.3):

$$\begin{aligned}
&\mathcal{L}^n \left(\bigcup_{k=1}^{N} \{x \in \Omega : g_u(b_k) < u(x) < g_u(a_k)\} \right) \\
&= \sum_{k=1}^{N} \mathcal{L}^n(\{y \in \Omega^* : g_u(b_k) < u^*(y) < g_u(a_k)\}) \\
&\leq \sum_{k=1}^{N} (V_\Omega(b_k) - V_\Omega(a_k)) \leq \max_{t \in [t_0, t_1]} \mathcal{I}(V_\Omega(t)) \sum_{k=1}^{N} (b_k - a_k) \leq \delta.
\end{aligned}$$

Next, set $s_1 := g_u(b_k)$ and $s_2 := g_u(a_k)$ and let $v := \mathrm{Tr}_{s_1, s_2} u$. By applying Lemma 3.2.4, Lemma 3.3.1 above and the fact that the pointwise variation of a monotone function is bounded by its total variation (see Theorem 7.2 in [72]) we obtain

$$\begin{aligned}
&\min_{t \in [t_0, t_1]} \mathcal{I}(V_\Omega(t)) |g_u(a_k) - g_u(b_k)| \\
&\leq \int_{a_k}^{b_k} \mathcal{I}(V_\Omega(t)) \, d|Dg_u|(t) = \int_I \mathcal{I}(V_\Omega(t)) \, d|D(\mathrm{Tr}_{s_1, s_2} g_u)|(t) \\
&= |Dv^*|(\Omega^*) \leq |Dv|(\Omega) = \int_{\{g_u(b_k) < u < g_u(a_k)\}} |\nabla u| \, dx.
\end{aligned}$$

We then find the following:

$$\begin{aligned}
&\min_{t \in [t_0, t_1]} \mathcal{I}(V_\Omega(t)) \sum |g_u(a_k) - g_u(b_k)| \leq \int_{\bigcup_k \{g_u(b_k) < u < g_u(a_k)\}} |\nabla u| \, dx \\
&\leq \min_{t \in [t_0, t_1]} (\mathcal{I}(V_\Omega(t))) \epsilon,
\end{aligned}$$

where we have used (3.3) and (3.3). This implies that g_u is absolutely continuous on $[t_0, t_1]$, as claimed.

□

The next lemma gives an identity relating to the level sets of functions. It can be found in [35]; the proof is included here for completeness.

Lemma 3.3.3. *For $u \in W^{1,1}(\Omega)$ there exists a representative of u such that the following equality holds for all $s_1 < s_2$:*

$$\int_{s_1}^{s_2} \int_{u^{-1}(s)} |\nabla u(x)|^{-1} \, d\mathcal{H}^{n-1} \, ds = \mathcal{L}^n(\{x \in \Omega : u(x) \in (s_1, s_2), \nabla u(x) \neq 0\}).$$

Proof. Let $H_\varepsilon := (\varepsilon + |\nabla u|)^{-1}$. By the coarea formula, Theorem 2.1.8, we find that

$$\begin{aligned}\int_{\{s_1<u<s_2,\ \nabla u\neq 0\}} H_\varepsilon |\nabla u|\, dx &= \int_{\{s_1<u<s_2\}} H_\varepsilon |\nabla u|\, dx \\ &= \int_{s_1}^{s_2} \int_{u^{-1}(s)} H_\varepsilon\, d\mathcal{H}^{n-1}\, ds.\end{aligned}$$

By noting that $H_\varepsilon \to |\nabla u|^{-1}$ monotonically in the set $\{\nabla u \neq 0\}$, we find that (3.3.3) holds. $\square$

The following theorem is the main result of this section, namely an analog of the Pólya–Szegő inequality.

Theorem 3.3.4. *Suppose that $u \in W^{1,p}(\Omega)$ for $1 \le p \le \infty$, and that u has $\mathcal{I}$ comparable level sets. Then $u^* \in W^{1,p}(\Omega^*)$ and furthermore:*

$$\int_I |g_u'|^p \mathcal{I}(V_\Omega)\, ds = \int_{\Omega^*} |\nabla u^*|^p\, dy \le \int_\Omega |\nabla u|^p\, dx.$$

Proof. Lemmas 3.3.1 and 3.3.2 immediately give this inequality if $p = 1$. For $p > 1$ we can still apply the previous lemmas to show that $u^* \in W^{1,1}(\Omega^*)$, because Ω has finite measure.

Next we note that the following equality holds (by using the coarea formula):

$$\varrho_u(t) = \mathcal{L}^n(\{u > t\} \cap \{\nabla u = 0\}) + \int_t^\infty \int_{\{u=s,\ \nabla u \neq 0\}} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\, ds =: f_1^u(t) + f_2^u(t). \tag{3.3.1}$$

Clearly f_2^u is absolutely continuous, and f_1^u is decreasing. Thus ϱ_u is differentiable for a.e. t, with:

$$\varrho_u'(t) \le -\int_{\{u=t, \nabla u\neq 0\}} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}. \tag{3.3.2}$$

Next we claim that (following [35]) for a.e. t:

$$\frac{d}{dt} f_1^{u^*}(t) = \frac{d}{dt}\mathcal{L}^n(\{u^* > t\} \cap \{\nabla u^* = 0\}) = 0.$$

To establish this claim, we first note that for any open interval J we have the following:

$$\mathcal{L}^1(g_u(J)) \le \int_J |g_u'|\, ds.$$

By approximating measurable sets with disjoint open intervals we can then establish that

$$\mathcal{L}^1(g_u(\{g_u' = 0\})) \le \int_{\{g_u'=0\}} |g_u'|\, ds = 0.$$

Following [36] we then find that:

$$\mathcal{L}^1(u^*(\{\nabla u^* = 0\})) = \mathcal{L}^1(g_u(\{g_u' = 0\})) = 0.$$

Thus there exists a Borel set F_0 in $\mathbb{R}$ so that $\mathcal{L}^1(F_0) = 0$ and so that $u^*(\{\nabla u^* = 0\}) \subset F_0$.

We then claim that for any Borel set B in $\mathbb{R}$ we have that

$$|Df_1^{u^*}|(B) = \mathcal{L}^n((u^*)^{-1}(B) \cap \{\nabla u^* = 0\}).$$

To see this, we first note that $f_1^{u^*}$ is right continuous and decreasing. We then have that

$$\begin{aligned}|Df_1^{u^*}|((t_1,t_2)) &= f_1^{u^*}(t_1) - \lim_{t\to t_2^-} f_1^{u^*}(t_2)\\ &= \mathcal{L}^n(\{u^*>t_1\}\cap\{\nabla u^*=0\}) - \lim_{t\to t_2^-}\mathcal{L}^n(\{u^*>t\}\cap\{\nabla u^*=0\})\\ &= \mathcal{L}^n(\{u^*>t_1\}\cap\{\nabla u^*=0\}) - \mathcal{L}^n(\{u^*\geq t_2\}\cap\{\nabla u^*=0\})\\ &= \mathcal{L}^n((u^*)^{-1}((t_1,t_2))\cap\{\nabla u^*=0\}).\end{aligned}$$

As both $|Df_1^{u^*}|$ and $\mathcal{L}^n((u^*)^{-1}(\cdot)\cap\{\nabla u^*=0\})$ are Borel measures, and as they are equal on open intervals, they must be equal on all Borel sets. This and the fact that $u^*(\{\nabla u^*=0\})\subset F_0$ immediately give that

$$|Df_1^{u^*}|(\mathbb{R}\backslash F_0) = \mathcal{L}^n((u^*)^{-1}(\mathbb{R}\backslash F_0)\cap\{\nabla u^*=0\}) = \mathcal{L}^n(\emptyset)=0,$$

which proves (3.3). Utilizing (3.3.1) this then immediately implies that for a.e. t,

$$\varrho'_{u^*}(t) = -\int_{\{u^*=t,\nabla u^*\neq 0\}} |\nabla u^*|^{-1}\, d\mathcal{H}^{n-1}. \tag{3.3.3}$$

By the coarea formula we can write the following:

$$\begin{aligned}\int_{\Omega^*}|\nabla u^*|^p\,dy &= \int_{\Omega^*\cap\{\nabla u^*\neq 0\}}|\nabla u^*|^p\,dy\\ &= \int_{\mathbb{R}}\int_{\{u^*=t\}\cap\{\nabla u^*\neq 0\}}|\nabla u^*|^{p-1}\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

By (3.1.8) we know that $\nabla u^*(y) = (0, g'_u(y_n))\in\mathbb{R}^{n-1}\times\mathbb{R}$. Since g_u is decreasing we have that the set $\{u^*=t\}$ is a set of the form $\{y:\ y_n\in[t_1,t_2],\ y'\in B_{n-1}(0,r(y_n))\}$, for some $t_1\leq t_2$ with possibly $t_1=t_2$. If $t_1=t_2$ then clearly ∇u^* is constant on the set $\{u^*=t\}$. If $t_1\neq t_2$ then g'_u is zero on the set (t_1,t_2), and is either zero at t_1,t_2 or is undefined. Since ∇u^* is constant on level sets of u^* (where it's defined) we can then write

$$\int_{\Omega^*}|\nabla u^*|^p\,dy = \int_{\mathbb{R}}\frac{\left(\mathcal{H}^{n-1}(\{u^*=t\}\cap\{\nabla u^*\neq 0\})\right)^p}{\left(\int_{\{u^*=t\}\cap\{\nabla u^*\neq 0\}}|\nabla u^*|^{-1}\,d\mathcal{H}^{n-1}\right)^{p-1}}\,dt.$$

By (3.3.3) we have that

$$\int_{\Omega^*}|\nabla u^*|^p\,dy = \int_{\mathbb{R}}\frac{\mathrm{P}(\{u^*>t\};\Omega^*)^p}{(-\varrho'_{u^*}(t))^{p-1}}\,dt.$$

Next we utilize Lemma 3.2.1 and Lemma 3.2.6 to find that

$$\int_{\Omega^*}|\nabla u^*|^p\,dy \leq \int_{\mathbb{R}}\frac{\mathrm{P}(\{u>t\};\Omega)^p}{(-\varrho'_{u}(t))^{p-1}}\,dt.$$

Next (3.3.2) gives

$$\int_{\Omega^*}|\nabla u^*|^p\,dy \leq \int_{\mathbb{R}}\frac{\mathrm{P}(\{u>t\};\Omega)^p}{\left(\int_{\{u=t\}}|\nabla u|^{-1}\,d\mathcal{H}^{n-1}\right)^{p-1}}\,dt.$$

Jensen's inequality on $f(s)=s^{-(p-1)}$ then implies that

$$\int_{\Omega^*}|\nabla u^*|^p\,dy \leq \int_{\mathbb{R}}\int_{\{u=t\}}|\nabla u|^{p-1}\,d\mathcal{H}^{n-1}\,dt,$$

which after applying the coarea formula gives the desired result.

$\square$

Remark 3.3.5. *This section has considered a rearrangement of the function* u*, via the decreasing function* $g_u : I \to \mathbb{R}$*. However, all of the arguments would hold for an increasing rearrangement* f_u*. Indeed, in the case when* $\mathcal{I}$ *is symmetric, e.g.* $\mathcal{I} = \mathcal{I}_\Omega$*, it is straightforward to show that* $f_u(t) := g_u(-t)$*. In any case, for the increasing rearrangement* f_u *the following relations still hold:*

$$\int_I \psi(f_u(t))\mathcal{I}(V_\Omega(t))\, dt = \int_\Omega \psi(u)\, dx,$$
$$\int_I |f_u'(t)|^p \mathcal{I}(V_\Omega(t))\, dt \leq \int_\Omega |\nabla u|^p\, dx.$$

This section focuses on the decreasing rearrangement because that is the standard convention chosen in the literature involving rearrangement. However, subsequent chapters will use the increasing rearrangement f_u *of* u *in because of the conventions in the literature on phase transitions.*

The following corollary is the motivation for our development of the rearrangement in this section, and is a simple application of Lemma 3.2.1 and Theorem 3.3.4.

Corollary 3.3.6. *Let* $u \in H^1(\Omega)$*, and let* u *have* $\mathcal{I}$ *comparable level sets. Then the following inequality holds:*

$$\int_\Omega W(u) + \varepsilon^2 |\nabla u|^2\, dx \geq \int_I (W(f_u) + \varepsilon^2 (f_u')^2)\mathcal{I}(V_\Omega)\, dt.$$

Moreover

$$\int_\Omega u\, dx = \int_I f_u \mathcal{I}(V_\Omega)\, dt.$$

3.4 Anisotropic Extension

This section briefly considers an extension of the previous result to the anisotropic case. In the case where $\Omega = \mathbb{R}^n$ this problem was previously considered in [8]. For the most part, the proofs for the anisotropic case are identical to the isotropic case covered in the previous sections, with only minor modifications. Abbreviated versions of the proofs are included for completeness.

In this section, let $\Psi : \mathbb{R}^n \to [0, \infty)$ be a convex function that is positively homogeneous of degree one (see (2.2.1)). A measurable function $u : \Omega \to \mathbb{R}$ is said to have $(\mathcal{I}, \Psi)$ comparable level sets if

$$\mathrm{P}_\Psi(\{u > s\}; \Omega) \geq \mathcal{I}(\mathcal{L}^n(\{u > s\})),$$

where the definition of P_Ψ is given in (2.2.2). Next, define u^* and Ω^* as in Section 3.1. By Remark 3.2.5 we have that Lemmas 3.1.1 - 3.2.4 still hold. The main question in the anisotropic case is now whether an appropriate extension of the Pólya–Szegő inequality still holds. The first step is to establish the relevant isoperimetric inequality. The following proposition is a consequence of the definition of a function having $(\mathcal{I}, \Psi)$ comparable level sets.

Proposition 3.4.1. *Given* $u \in BV(\Omega)$ *with* $(\mathcal{I}, \Psi)$ *comparable level sets, for any* t *the following must hold:*

$$\mathrm{P}(\{u^* > t\}; \Omega^*) \leq \mathrm{P}_\Psi(\{u > t\}; \Omega).$$

Proof. As in the proof of Lemma 3.2.6, we remark that the set $\{u^* > t\}$ is actually a set of the form $\{y_n < s\}$. By Lemma 3.1.1 (see Remark 3.2.5), we have that $V_\Omega(s) = \mathcal{L}^n(\Omega^* \cap \{y_n < s\}) = \varrho_{u^*}(t)$. As u and u^* are equimeasurable (see Lemma 3.2.1) and by Lemma 3.1.1, which both apply due to Remark 3.2.5, we have the following:

$$\begin{aligned}\mathrm{P}(\{u^* > t\}; \Omega^*) &= \mathcal{I}(\varrho_{u^*}(t)) \\ &= \mathcal{I}(\varrho_u(t)) \leq \mathrm{P}_\Psi(\{u > t\}; \Omega),\end{aligned}$$

where we have used the fact that u has $(\mathcal{I}, \Psi)$ comparable level sets. This concludes the proof. □

Following the isotropic case, it is possible to compare the BV norm of u and u^* by using the coarea formula and Proposition 3.4.1.

Proposition 3.4.2. *Suppose that $u \in BV(\Omega)$ and that u has $(\mathcal{I}, \Psi)$ comparable level sets. Then $u^* \in BV(\Omega^*)$ and*

$$\int_I \mathcal{I}(V_\Omega(s))\, d|Dg_u|(s) = |Du^*|(\Omega^*) \leq |Du|_\Psi(\Omega).$$

Proof. As in the proof of Lemma 3.3.1, we have that

$$|Du^*|(\Omega^*) = \int_I \mathcal{I}(V_\Omega(y_n))\, d|Dg_u|(y_n).$$

Then by using the coarea formula, see Theorems 2.1.8 and 2.2.3, and Proposition 3.4.1 it follows that

$$|Du^*|(\Omega^*) = \int_{\mathbb{R}} \mathrm{P}(\{u^* > t\}; \Omega^*)\, dt \leq \int_{\mathbb{R}} \mathrm{P}_\Psi(\{u > t\}; \Omega)\, dt = |Du|_\Psi(\Omega).$$

□

Proposition 3.4.3. *Given $u \in W^{1,1}(\Omega)$ with $(\mathcal{I}, \Psi)$ comparable level sets, it follows that $u^* \in W^{1,1}(\Omega^*)$.*

Proof. Following the proof of Lemma 3.3.2, it suffices to show that g_u is absolutely continuous. Using the same notation as in the proof of Lemma 3.3.2, we find that

$$\begin{aligned}&\min_{t\in[t_0,t_1]} \mathcal{I}(V_\Omega(t))|g_u(a_k) - g_u(b_k)| \\ &\leq \int_{a_k}^{b_k} \mathcal{I}(V_\Omega(t))\, d|Dg_u|(t) = \int_I \mathcal{I}(V_\Omega(t))\, d|D(\mathrm{Tr}_{s_1,s_2}\, g_u)|(t) \\ &= |Dv^*|(\Omega^*) \leq |Dv|_\Psi(\Omega) \leq C \int_{\{g_u(b_k)<u<g_u(a_k)\}} |\nabla u|\, dx.\end{aligned}$$

where we have used Proposition 3.4.2 and the fact that Ψ is bounded, see equation (2.2.2). The result then follows as in the proof of Lemma 3.3.2. □

With these tools in hand it is now possible to give the anisotropic version of the Pòlya–Szegő inequality.

Theorem 3.4.4. *Suppose that $u \in W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$ and that u has $(\mathcal{I}, \Psi)$ comparable level sets. Then $u^* \in W^{1,p}(\Omega^*)$ and furthermore:*

$$\int_I |g_u'|^p \mathcal{I}(V_\Omega)\, ds = \int_{\Omega^*} |\nabla u^*|^p\, dy \leq \int_\Omega \Psi(|\nabla u|)^p\, dx. \tag{3.4.1}$$

Proof. As in the proof of Theorem 3.3.4, by applying Lemma 3.4.3 it is clear that $u^* \in W^{1,p}(\Omega^*)$. It only remains to prove the inequality (3.4.1).

To prove the inequality (3.4.1), we first remark that the argument between equations (3.3.1) and (3.3.3) still holds in the present case. This is because the argument only relies on equimeasurability and properties of monotone functions. This then implies that, for a.e. t,

$$\int_{\{u^*=t,\nabla u^*\neq 0\}} |\nabla u^*|^{-1}\, d\mathcal{H}^{n-1} \geq \int_{\{u=t,\nabla u\neq 0\}} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}. \tag{3.4.2}$$

By the coarea formula and the fact that u^* has constant gradient along level sets we may write

$$\int_{\Omega^*} |\nabla u^*|^p\, dy = \int_{\mathbb{R}} \frac{\mathrm{P}(\{u^* > t\};\Omega)^p}{(\int_{\{u^*=t\}\cap\{\nabla u^*\neq 0\}} |\nabla u^*|^{-1}\, d\mathcal{H}^{n-1})^{p-1}}\, dt.$$

This, along with Proposition 3.4.1 and Equation (3.4.2) implies that

$$\int_{\Omega^*} |\nabla u|^p\, dy \leq \int_{\mathbb{R}} \frac{\mathrm{P}_\Psi(\{u > t\};\Omega)^p}{(\int_{\{u=t\}\cap\{\nabla u\neq 0\}} |\nabla u|^{-1}\, d\mathcal{H}^{n-1})^{p-1}}\, dt.$$

By Hölder's inequality we have that

$$\int_{u=t} \Psi\left(\frac{\nabla u}{|\nabla u|}\right) d\mathcal{H}^{n-1} \leq \left(\int_{u=t} \frac{\Psi(\nabla u)^p}{|\nabla u|}\, d\mathcal{H}^{n-1}\right)^{1/p} \left(\int_{u=t} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right)^{1-1/p}.$$

Next, by Theorem 2.2.3 we have that

$$\int_{u=t} \Psi\left(\frac{\nabla u}{|\nabla u|}\right) d\mathcal{H}^{n-1} = \mathrm{P}_\Psi(\{u > t\};\Omega).$$

Thus by combining the previous three equations, and after applying the coarea formula, the desired inequality is established, namely

$$\int_{\Omega^*} |\nabla u|^p\, dy \leq \int_{\mathbb{R}} \int_{u=t} \frac{\Psi(\nabla u)^p}{|\nabla u|}\, dt = \int_{\Omega} \Psi(\nabla u)^p\, dx.$$

$\square$

Chapter 4

Properties of the Isoperimetric Function

The main results of the first part of this thesis require that the isoperimetric function, or perhaps a localized version of the same, be differentiable at some point of interest. This chapter will establish the validity of such a statement in a variety of situations.

The first natural question is whether the function $\mathcal{I}_\Omega^{\delta,E_0}$ (defined by (1.1.11)) is continuous. This is answered affirmatively by the following proposition.

Proposition 4.0.1. *Let Ω satisfy* (6.1.1)*, and let $E_0 \subset \Omega$ be a volume-constrained local perimeter minimizer in Ω with $r_0 := \mathcal{L}^n(E_0)$. Then for any $\delta > 0$ the function $\mathcal{I}_\Omega^{\delta,E_0}$ is continuous.*

Proof. By the lower semicontinuity of the perimeter function, BV compactness, and the fact that the constraint $\alpha(E, E_0) \leq \delta$ is closed in L^1, it is clear that for any $r \in (0, 1)$ there exists a minimizer of the minimization problem,

$$\min\{P(E; \Omega) : \alpha(E, E_0) \leq \delta, \mathcal{L}^n(E) = r\}, \tag{4.0.1}$$

which defines $\mathcal{I}_\Omega^{\delta,E_0}$ (see (1.1.11) and (1.1.12)). Again, by the lower semicontinuity of the perimeter function, we have that $\mathcal{I}_\Omega^{\delta,E_0}$ must be lower semicontinuous.

Now for any fixed $r \in (0, 1)$, a minimizer E_r of (4.0.1) must be a volume-constrained perimeter minimizer inside $E_0 \cap \Omega$ and $\Omega \backslash E_0$, and thus ∂E_r must be be a.e. smooth inside those sets (see Theorem 2.3.6). Suppose that $\alpha(E_0, E_r) = \mathcal{L}^n(E_0 \backslash E_r)$. Then pick any smooth vector field V compactly supported in $\Omega \backslash E_0$ which satisfies $\int_{\partial E_r} V \cdot \nu_{E_r} \, d\mathcal{H}^{n-1} \neq 0$ (such a vector field clearly exists given the smoothness of E_r). Perturbations with initial velocity V will still satisfy the $\alpha(\cdot, E_0) \leq \delta$, because $V \equiv 0$ in E_0. Furthermore, the perimeter will vary smoothly along these perturbations, and the volume will not be stationary (because $\int_{\partial E_r} V \cdot \nu_{E_r} \, d\mathcal{H}^{n-1} \neq 0$). Hence, by considering the the perimeter of perturbations along V we have that $\mathcal{I}_\Omega^{\delta,E_0}$ is touched from above by a smooth function near r. This readily implies that $\mathcal{I}_\Omega^{\delta,E_0}$ is continuous at r. A similar argument holds if $\alpha(E_0, E_r) = \mathcal{L}^n(E_r \backslash E_0)$. As r was arbitrary the proposition is proved. $\square$

In order to prove differentiability, one needs more precise arguments. The following lemma is a straightforward combination of Theorem 2.3.6 and Remark 2.3.13.

Lemma 4.0.2. *Let Ω satisfy* (6.1.1)*, and let $E_0 \subset \Omega$ be a volume-constrained local perimeter minimizer in Ω with $r_0 := \mathcal{L}^n(E_0)$. Then ∂E_0 is a surface of constant mean curvature κ_{E_0}, which intersects the boundary of Ω orthogonally. Moreover,* *there exists a neighborhood I of r_0 and a family of sets $\{\hat{E}_r\}_r$ constructed via a normal perturbation of E_0 (see Theorem 2.3.10), satisfying*

$$\mathcal{L}^n(\hat{E}_r) = r, \qquad \lim_{r\to r_0} |\hat{E}_r \Delta E_0| = 0,$$

and such that the function

$$r \mapsto \phi(r) := P(\hat{E}_r;\Omega), \quad \text{for } r \in I,$$

is smooth. Moreover, the function ϕ satisfies

$$\phi(r_0) = P(E_0;\Omega), \quad \frac{d\phi(r)}{dr}\Big|_{r=r_0} = \kappa_{E_0}(n-1), \tag{4.0.2}$$

and

$$\frac{d^2\phi(r)}{dr^2}\Big|_{r=r_0} = -\frac{\int_{\partial E_0} |A_{E_0}|^2 \, d\mathcal{H}^{n-1} + \int_{\partial E_0\cap\partial\Omega} \nu_{\partial E_0}\cdot A_\Omega \nu_{\partial E_0}\, d\mathcal{H}^{n-2}}{P(E_0;\Omega)^2},$$

where A_{E_0} and A_Ω are the second fundamental forms, see Definition 2.3.8.

The first step is to prove that $\mathcal{I}_\Omega^{\delta,E_0}$ is semi-concave under appropriate conditions.

Lemma 4.0.3. *Let Ω satisfy* (6.1.1)*, and let $E_0 \subset \Omega$ be a volume-constrained local perimeter minimizer in Ω with $r_0 := \mathcal{L}^n(E_0)$. Let $\delta > 0$, and let $I_{r_0} \subset\subset [0, \mathcal{L}^n(\Omega)]$ be an open interval containing r_0. Suppose that for every $r \in I_{r_0}$ at least one minimizer E_r of the problem*

$$\min\{P(E;\Omega) : \mathcal{L}^n(E) = r,\ \alpha(E,E_0) \le \delta\}$$

satisfies

$$\alpha(E_r, E_0) < \delta. \tag{4.0.3}$$

Then the local isoperimetric function $\mathcal{I}_\Omega^{\delta,E_0}$ is semi–concave in I_{r_0}, that is, there exists a constant $C > 0$ such that

$$r \mapsto \mathcal{I}_\Omega^{\delta,E_0}(r) - Cr^2 \tag{4.0.4}$$

is a concave function in I_{r_0}.

Proof. By Proposition 4.0.1, we have that $\mathcal{I}_\Omega^{\delta,E_0}$ is continuous. By (4.0.3) we have that E_r must be a local volume-constrained perimeter minimizer for every $r \in I_{r_0}$. Thus by Lemma 4.0.2 applied to E_r, for any $r \in I_{r_0}$ there exists a smooth function ϕ_r and a constant $\delta_r > 0$ depending on r such that

$$\phi_r(s) \ge \mathcal{I}_\Omega^{\delta,E_0}(s) \text{ for all } s \in (r-\delta_r, r+\delta_r), \quad \phi_r(r) = P(E_r;\Omega) = \mathcal{I}_\Omega^{\delta,E_0}(r), \tag{4.0.5}$$

and

$$\frac{d^2\phi_r(s)}{ds^2}\Big|_{s=r} = -\frac{\int_{\partial E_r} |A_{E_r}|^2 \, d\mathcal{H}^{n-1} + \int_{\partial E_r\cap\partial\Omega} \nu_{E_r}\cdot A_\Omega \nu_{E_r}\, d\mathcal{H}^{n-2}}{P(E_r;\Omega)^2}, \tag{4.0.6}$$

where we recall that $|A_{E_r}|$ is the Frobenius norm, see equation (2.3.5).

Let $C_\Omega := \max_{x\in\partial\Omega} |A_\Omega(x)|$. Then we have

$$\left|\int_{\partial E_r\cap\partial\Omega} \nu_{E_r}\cdot A_\Omega \nu_{E_r}\, d\mathcal{H}^{n-2}\right| \le C_\Omega \int_{\partial E_r\cap\partial\Omega} \nu_\Omega\cdot\nu_\Omega, d\mathcal{H}^{n-2}. \tag{4.0.7}$$

Since Ω is of class $C^{2,\alpha}$, we can locally express $\partial\Omega$ as the graph of a function of class $C^{2,\alpha}$ and, in turn, we can locally extend the normal to the boundary ν_Ω to a $C^{1,\alpha}$ vector field. Thus, using a partition of unity, we may extend the vector field $C_\Omega \nu_\Omega$ to a vector field $V \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ satisfying

$$\|V\|_\infty \le C, \qquad \|\nabla V\|_\infty \le C \tag{4.0.8}$$

for some constant $C > 0$. We then apply the divergence theorem (see Theorem 2.3.4) with $M = \overline{(\partial E_r) \cap \Omega}$ and $\Gamma = \partial E_r \cap \partial\Omega$ to find that

$$\begin{aligned} C_\Omega \int_{\partial E_r \cap \partial\Omega} \nu_\Omega \cdot \nu_\Omega \, d\mathcal{H}^{n-2} &= \int_{\partial E_r} \operatorname{div}_{E_r} V \, d\mathcal{H}^{n-1} - \int_{\partial E_r} V \cdot \kappa_{E_r} \nu_\Omega \, d\mathcal{H}^{n-1} \\ &\le C P(E_r;\Omega) + C \int_{\partial E_r} |\kappa_{E_r}| \, d\mathcal{H}^{n-1}, \end{aligned} \tag{4.0.9}$$

where in the last inequality we have used (2.3.3) and (4.0.8). Moreover, we recall that (see Proposition 2.3.9) for every $x \in \Omega \cap \partial E_r$,

$$|A_{E_r}(y)|^2 = \sum_{h=1}^{n-1} \kappa_{h,E_r}(y)^2, \qquad \kappa_{E_r}(y) = \sum_{h=1}^{n-1} \kappa_{h,E_r}(y) \;\text{ for all } y \in B_r(x) \cap \partial E_r \tag{4.0.10}$$

where κ_{h,E_r} are the principal curvatures of E_r. Thus, using (4.0.10), if we consider the principal curvatures κ_{h,E_r} as a vector in $\mathbb{R}^{n-1}$ then we have that

$$C|\kappa_{E_r}| \le \sqrt{n-1} C |A_{E_r}| \le \max\{(n-1)C^2, |A_{E_r}|^2\}. \tag{4.0.11}$$

In turn, putting together (4.0.6), (4.0.7), (4.0.9) and (4.0.11), we get

$$\begin{aligned} \frac{d^2\phi_r(s)}{ds^2}\Big|_{s=r} &\le \frac{-\int_{\partial E_r} |A_{E_r}|^2 \, d\mathcal{H}^{n-1} + C P(E_r;\Omega) + \int_{\partial E_r} \max\{(n-1)C^2, |A_{E_r}|^2\} \, d\mathcal{H}^{n-1}}{P(E_r;\Omega)^2} \\ &\le \frac{C P(E_r;\Omega) + (n-1)C^2 P(E_r;\Omega)}{P(E_r;\Omega)^2}. \end{aligned}$$

Denote

$$m_1 := \min_{s \in \overline{I_{r_0}}} \mathcal{I}_\Omega^{\delta,E_0}(s), \;\; m_2 := C + (n-1)C^2 < \infty,$$

and notice that

$$\min_{s \in \overline{I_{r_0}}} \mathcal{I}_\Omega^{\delta,E_0}(s) \ge \min_{s \in \overline{I_{r_0}}} \mathcal{I}_\Omega(s) > 0$$

where the last inequality follows from Proposition 2.1.10. From (4.0.6) we have that

$$\frac{d^2\phi_r(s)}{ds^2}\Big|_{s=r} \le \frac{m_2}{m_1}. \tag{4.0.12}$$

Thus by (4.0.5) for any r we can find a $\delta_r > 0$ so that for $s \in (r - \delta_r, r + \delta_r)$,

$$\begin{aligned} \mathcal{I}_\Omega^{\delta,E_0}(s) - \frac{m_2}{m_1} s^2 &\le \phi_r(s) - \frac{m_2}{m_1} s^2 \\ &= \phi_r(s) - \frac{m_2}{m_1}((s-r)^2 + 2sr - r^2) \\ &=: \psi(s) - \frac{m_2}{m_1}(2sr - r^2), \end{aligned} \tag{4.0.13}$$

where $\psi(s) = \phi_r(s) - \frac{m_1}{m_2}(s-r)^2$ is a concave function on $(r - \delta_r, r + \delta_r)$ by (4.0.12). The estimate (4.0.13) allows us to apply Proposition 2.6.4 and conclude that $\mathcal{I}_\Omega^{\delta,E_0}(s) - \frac{m_2}{m_1} s^2$ is a concave function on I_{r_0}. In turn, $\mathcal{I}_\Omega^{\delta,E_0}$ is semi–concave on I_{r_0}. $\square$

Corollary 4.0.4. *Under the assumption* (6.1.1)*, the function $\mathcal{I}_\Omega$ is differentiable at all but countably many points in* $[0,1]$*.*

Proof. By setting δ large enough we have that $\mathcal{I}_\Omega^{\delta,E_0} = \mathcal{I}_\Omega$, and that (4.0.3) is always satisfied. Thus $\mathcal{I}_\Omega$ is semi-concave on any $I_1 \subset\subset [0,1]$. Since convex functions are differentiable at all but countably many points, $\mathcal{I}_\Omega$ is as well. □

Corollary 4.0.5. *Under the assumptions of Lemma 4.0.3, the local isoperimetric function $\mathcal{I}_\Omega^{\delta,E_0}$ is locally Lipschitz in I_{r_0}. Furthermore, for all $J_{r_0} \subset\subset I_{r_0}$, for all $r \in J_{r_0}$, the values $\kappa_{E_r}(n-1)$ belong to the supergradient of $\mathcal{I}_\Omega^{\delta,E_0}$, and hence*

$$|\kappa_{E_r}| \leq L, \tag{4.0.14}$$

where L is the Lipschitz constant of $\mathcal{I}_\Omega^{\delta,E_0}$ in J_{r_0}.

Proof. Thanks to (4.0.3) in Lemma 4.3, for any $r \in I_{r_0}$ there exists a volume–constrained local perimeter minimizer E_r such that

$$\mathcal{I}_\Omega^{\delta,E_0}(r) = P(E_r;\Omega),\ \mathcal{L}^n(E_r) = r,\ \alpha(E_r, E_0) < \delta.$$

By Lemma 4.0.2 applied to E_r, in particular from (4.0.2), we have that $\kappa_{E_r}(n-1)$ belongs to the supergradient of $\mathcal{I}_\Omega^{\delta,E_0}$. From (4.0.4) we know that the mapping $r \mapsto \mathcal{I}_\Omega^{\delta,E_0}(r) - Cr^2$ is concave, and hence locally Lipschitz. In turn, $\mathcal{I}_\Omega^{\delta,E_0}$ is locally Lipschitz in I_{r_0}. Finally, as $\kappa_{E_r}(n-1)$ is in the supergradient of a locally Lipschitz function, there exists a constant $L > 0$ so that (4.0.14) holds on J_{r_0} (see Theorem 9.13 in [95]). □

We can now state one of the main results of this chapter.

Theorem 4.0.6. *Let $E_0 \subset \Omega$ be an isolated local volume-constrained perimeter minimizer in E_0. Then, for δ small enough, $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathcal{L}^n(E_0)$.*

Proof. By assumption, E_0 is the unique minimizer of the problem

$$\min\left\{P(E;\Omega):\ E \subset \Omega \text{ Borel},\ \mathcal{L}^n(E) = r,\ \alpha(E, E_0) \leq \delta\right\}, \tag{4.0.15}$$

for $r = r_0$ and for some fixed $0 < \delta$ small enough.

Let I be a neighborhood of r_0 (to be fixed later) and consider a sequence $\{r_k\}$ satisfying $r_k \to r_0$ as $k \to \infty$. Let E_{r_k} be a minimizer of the problem (4.0.15) for $r = r_k$.

Step 1. Lemma 2.3.11, along with the definition of $\mathcal{I}_\Omega^{\delta,E_0}$ naturally implies that

$$\mathcal{I}_\Omega^{\delta,E_0} \leq C \tag{4.0.16}$$

for some $C > 0$ and, in turn, by BV compactness, there exists a subsequence of $\{E_{r_k}\}$ (not relabeled) such that

$$E_{r_k} \to E^* \text{ in } L^1(\Omega) \text{ as } k \to \infty, \tag{4.0.17}$$

for some measurable set E^* such that $\chi_{E^*} \in BV(\Omega)$ and $\mathcal{L}^n(E^*) = r_0$.

We notice that since $\alpha(E^*, E_0) \leq \delta$ and $\mathcal{L}^n(E^*) = r_0$, by lower semi-continuity of the perimeter (see [48]), and Proposition 4.0.1, we have that

$$\begin{aligned} P(E^*;\Omega) &\leq \liminf_{k\to\infty} P(E_{r_k};\Omega) = \liminf_{k\to\infty} \mathcal{I}_\Omega^{\delta,E_0}(r_k) \leq \limsup_{k\to\infty} \mathcal{I}_\Omega^{\delta,E_0}(r_k) \\ &\leq \mathcal{I}_\Omega^{\delta,E_0}(r_0) = P(E_0;\Omega) \leq P(E^*;\Omega). \end{aligned}$$

By uniqueness of (4.0.15) for $r = r_0$, $E^* = E_0$, and so (4.0.17) reads

$$E_{r_k} \to E_0 \text{ in } L^1(\Omega) \text{ as } k \to \infty. \tag{4.0.18}$$

Thanks to (4.0.18), we obtain

$$\alpha(E_{r_k}, E_0) < \delta,$$

for k big enough. In turn, this implies that there exists an open neighborhood I_{r_0} of r_0 as in Lemma 4.0.3. Hence, $\mathcal{I}_\Omega^{\delta,E_0}$ is semiconcave on I_{r_0}, and by Corollary 4.0.5, we have that $\mathcal{I}_\Omega^{\delta,E_0}$ is locally Lipschitz in I_{r_0}.

Step 2. Fix an open neighborhood $J_{r_0} := (r_0 - R, r_0 + R) \subset\subset I_{r_0}$ of r_0, and let L be the associated Lipschitz constant of $\mathcal{I}_\Omega^{\delta,E_0}$ in J_{r_0} (see Corollary 4.0.5). Let k be large enough so that $r_k \in J_{r_0}$. Let $x_0 \in \Omega$, $\rho_0 > 0$. We claim that E_{r_k} is a (Λ, ρ_0)–perimeter minimizer (see Definition 2.3.14) with

$$\Lambda = \max\left\{L, \frac{2C}{\delta}, \frac{2C}{R}\right\},$$

where L is the Lipschitz constant in Corollary 4.0.5 and $C > 0$ is as in Step 1. Because of (2.3.6), we know that $P(E_{r_k}; B_\rho(x_0)) - P(E; B_\rho(x_0)) = P(E_{r_k}; \Omega) - P(E; \Omega)$, and thus it suffices to prove that

$$P(E_{r_k}; \Omega) \le P(E; \Omega) + \Lambda \mathcal{L}^n(E_{r_k} \Delta E). \tag{4.0.19}$$

We divide the proof of (4.0.19) into three cases. If

$$\alpha(E_0, E) \le \delta \text{ and } \mathcal{L}^n(E) \in J_{r_0},$$

then by our choice of L (see Corollary 4.0.5), we have

$$\begin{aligned}
P(E_{r_k}; \Omega) = \mathcal{I}_\Omega^{\delta,E_0}(E_{r_k}) &\le \mathcal{I}_\Omega^{\delta,E_0}(\mathcal{L}^n(E)) + L\,|\mathcal{L}^n(E_{r_k}) - \mathcal{L}^n(E))| \\
&\le P(E; \Omega) + L\,|\mathcal{L}^n(E_{r_k}) - \mathcal{L}^n(E))| \\
&\le P(E; \Omega) + L\mathcal{L}^n(E_{r_k} \Delta E),
\end{aligned}$$

and (4.0.19) is proved in this case.

If instead E is such that

$$\alpha(E_0, E) > \delta,$$

then by (4.0.18),

$$\mathcal{L}^n(E_{r_k} \Delta E) \ge \mathcal{L}^n(E_0 \Delta E) - \mathcal{L}^n(E_{r_k} \Delta E_0) \ge \frac{\delta}{2}, \tag{4.0.20}$$

for k sufficiently large. Moreover, by (4.0.16) and (4.0.20),

$$P(E_{r_k}; \Omega) \le C \le \frac{2C}{\delta} \mathcal{L}^n(E_{r_k} \Delta E) \le \frac{2C}{\delta} \mathcal{L}^n(E_{r_k} \Delta E) + P(E; \Omega),$$

so that (4.0.19) follows from our choice of Λ.

Finally, if

$$\mathcal{L}^n(E) \notin J_{r_0},$$

then for $r_k \in (r_0 - R/2, r_0 + R/2)$ we have that

$$\mathcal{L}^n(E_{r_k} \Delta E) \geq \frac{R}{2},$$

and so (4.0.19) follows as in the previous case.

Step 3. Fix $\boldsymbol{z}_0 \in \Omega \cap \partial E_0$, and choose $\mathfrak{r} > 0$ such that $B_{\mathfrak{r}}(\boldsymbol{z}_0) \subset\subset \Omega$ and

$$\partial E_0 \cap B_{\mathfrak{r}}(\boldsymbol{z}_0) = \text{graph}(u_0),$$

for some regular function u_0. By the theory of (Λ, ρ_0) minimizers (see Theorem 26.6 in [75]), choosing ρ_0 smaller if needed, it follows that for any sequence of points $\boldsymbol{z}_k \in \partial E_{r_k}$ such that $\boldsymbol{z}_k \to \boldsymbol{z}_0 \in \Omega \cap \partial E_0$, then for k large enough $\boldsymbol{z}_k \in \Omega \cap \partial^* E_{r_k}$ and

$$\lim_{k \to \infty} \nu_{E_{r_k}}(\boldsymbol{z}_k) = \nu_{E_0}(\boldsymbol{z}_0), \tag{4.0.21}$$

uniformly on $B_{\mathfrak{r}}(\boldsymbol{z}_0)$. In turn, by (4.0.18), for k big enough

$$\partial E_{r_k} \cap B_{\mathfrak{r}}(\boldsymbol{z}_0) = \text{graph}(u_k),$$

for some functions u_k. In particular, by equation (26.52) in [75], we obtain

$$\nabla u_k \to \nabla u_0, \text{ in } C^{0,\gamma}(\Omega),$$

for all $\gamma \in (0, 1/2)$.

Step 4. Since ∂E_{r_k} is a surface of constant mean curvature, u_k solves

$$\operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}}\right) = \kappa_k \text{ in } B_{\mathfrak{r}}(\boldsymbol{z}_0),$$

where κ_k is the mean curvature of ∂E_{r_k}. By standard Schauder estimates (see e.g. [57]) and (4.0.21), it follows that

$$||u_k||_{C^{2,\gamma}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))} \leq c_1 |\kappa_k| \leq C, \tag{4.0.22}$$

where $B'_{\mathfrak{r}/2}(\boldsymbol{z}_0)$ is the $(n-1)$–dimensional ball and the uniform bound on the curvatures comes from Corollary 4.0.5.

Step 5. By Rellich–Kondrachov compactness theorem and by a bootstrapping argument on (4.0.22), we deduce that there exists a subsequence of $\{r_k\}$, not relabeled, and $\tilde{u} \in W^{m,2}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))$ such that

$$u_{r_j} \to \tilde{u} \text{ in } W^{m,2}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))$$

for all $m > 0$. It follows from (4.0.18), that necessarily $\tilde{u} = u_0$.

Step 6. By properties of concave functions, $(\mathcal{I}_\Omega^{\delta, E_0})'(r) = Lr + \mathfrak{z}(r)$, where $\mathfrak{z}$ is a decreasing function. In particular, $\mathcal{I}_\Omega^{\delta, E_0}$ must have a left and right derivative at r_0, and if $r_k \uparrow r_0$ then $\kappa_{r_k} \to (\mathcal{I}_\Omega^{\delta, E_0})'_-$ (with an analogous result for $r_k \downarrow r_0$). The convergence result from Step 5 implies that the left and right derivatives of $\mathcal{I}_\Omega^{\delta, E_0}$ at r_0 must be equal to κ_0. This implies that $\mathcal{I}_\Omega^{\delta, E_0}$ is differentiable at r_0, which completes the proof.

□

Remark 4.0.7. *This chapter has proved that the differentiability assumption holds in two important cases: For global volume-constrained perimeter minimizers up to a.e. mass* m*, and for isolated volume-constrained perimeter minimizers. It is also possible to prove differentiability in certain other cases, for example when* E_0 *is a ball compactly contained in* Ω*, see [83] for details.*

Chapter 5

Weighted 1D Functional Problem

5.1 Assumptions and Notation

This chapter will be concerned with a weighted, one-dimensional functional problem. By way of notation, L^p_η will represent the space $L^p(I;\mathbb{R},\eta)$, where $p \geq 1$ and $\eta \geq 0$ is some measurable function on I. Here, and throughout this chapter,

$$I := (-T, T)$$

for some positive T.

Similarly, BV_η to be the space $BV(I;\mathbb{R},\eta)$ with weight η, meaning that

$$\|v\|_{BV_\eta} := \int_I |v(t)|\eta(t)\,dt + \int_I \eta(t)\,d|Dv|(t).$$

For $v \in BV_\eta$, the weighted total variation of the derivative will be denoted by

$$|Dv|_\eta(E) = \int_E \eta(t)\,d|Dv|(t). \tag{5.1.1}$$

Here H^1_η will be the analogous weighted version of H^1.

This chapter considers the mass-constrained Cahn–Hilliard functional in one dimension, with an integral weight η. Precisely, this chapter studies the functional

$$G_\varepsilon(v) := \int_I (W(v) + \varepsilon^2(v')^2)\eta\,dt, \quad v \in H^1_\eta, \tag{5.1.2}$$

subject to the constraint that

$$\int_I v\eta\,dt = m \in \left(a\int_I \eta\,dt, b\int_I \eta\,dt\right). \tag{5.1.3}$$

Here G_ε is extended to all of L^1_η by setting $G_\varepsilon(v) := \infty$ if $v \in L^1_\eta \backslash H^1_\eta$ or if (5.1.3) fails. Chapter 1.1 introduces the theory of the unweighted version of this functional in n dimensions, and the results and definitions from that chapter will be used freely throughout this chapter.

The results in this chapter, and accordingly in subsequent chapters, require the following assumptions on $W : \mathbb{R} \to [0, \infty)$:

$$W \text{ is of class } C^2(\mathbb{R}\backslash\{a,b\}) \text{ and has precisely two zeros at } a<b, \tag{5.1.4}$$

$$\lim_{s\to a}\frac{W''(s)}{|s-a|^{q-1}}=\lim_{s\to b}\frac{W''(s)}{|s-b|^{q-1}}:=\ell>0, \quad q\in(0,1], \tag{5.1.5}$$

$$W' \text{ has exactly 3 zeros at } a<c<b, \quad W''(c)<0, \tag{5.1.6}$$

$$\liminf_{|s|\to\infty}|W'(s)|>0. \tag{5.1.7}$$

Most of these assumptions are standard (see [63]). In the case where $q=1$ it is evident that ℓ is simply $W''(a)$. In particular, $q=1$ when $W(s)=\frac{1}{2}(s^2-1)^2$, which is the classical Cahn–Hilliard potential (see, e.g., [28]). While the analysis in this chapter does not require identical limits at a and b in (5.1.5), that case is not dealt with for clarity of presentation.

Remark 5.1.1. *In view of* (5.1.4)-(5.1.7), *there must exist an* $\hat{L}>0$ *and* $\hat{T}>0$ *so that*

$$W(s)\geq\hat{L}|s| \tag{5.1.8}$$

for all $|s|>\hat{T}$.

Remark 5.1.2. *In view of* (5.1.4) *and* (5.1.5) *if follows from de l'Hôpital's rule that*

$$\lim_{s\to a}\frac{W(s)}{|s-a|^{1+q}}=\lim_{s\to b}\frac{W(s)}{|s-b|^{1+q}}=\frac{\ell}{q(1+q)}, \tag{5.1.9}$$

$$\lim_{s\to a}\frac{W'(s)}{(s-a)|s-a|^{q-1}}=\lim_{s\to b}\frac{W'(s)}{(s-b)|s-b|^{q-1}}=\frac{\ell}{q}. \tag{5.1.10}$$

In turn, by (5.1.4), there exist $c_1,c_2>0$ such that $c_1^2(b-s)^{1+q}\leq W(s)\leq c_2^2(b-s)^{1+q}$ for all $s\in[\frac{a+b}{2},b]$. It follows that the solution z of the Cauchy problem (1.1.6) satisfies

$$\begin{aligned}\left[(b-z(t_0))^{\frac{1-q}{2}}-\frac{(1-q)c_2}{2}(t-t_0)\right]_+^{\frac{2}{1-q}}&\leq b-z(t)\\&\leq\left[(b-z(t_0))^{\frac{1-q}{2}}-\frac{(1-q)c_1}{2}(t-t_0)\right]_+^{\frac{2}{1-q}}\end{aligned}$$

for all $t\geq t_0\geq 0$ if $0<q<1$ and

$$(b-z(t_0))e^{-c_2(t-t_0)}\leq b-z(t)\leq(b-z(t_0))e^{-c_1(t-t_0)} \tag{5.1.11}$$

for all $t\geq t_0\geq 0$ for $q=1$, where $[\cdot]_+$ denotes the positive part. In particular, in the case $0<q<1$, since $z(0)=c$, there exists a constant

$$\left(\frac{b-a}{2}\right)^{\frac{1-q}{2}}\frac{2}{c_2(1-q)}\leq t_b\leq\left(\frac{b-a}{2}\right)^{\frac{1-q}{2}}\frac{2}{c_1(1-q)}$$

such that

$$z(t)\equiv b \quad \text{for all } t\geq t_b. \tag{5.1.12}$$

Similar estimates hold near a, so that $z(t)\equiv a$ for all $t\leq t_a<0$ when $0<q<1$.

Furthermore, the results in this chapter assume that η satisfies the following assumptions:

$$\eta \in C^1(I), \quad \eta > 0 \text{ in } I, \tag{5.1.13}$$

$$d_1(t+T)^{n_1-1} \le \eta(t) \le d_2(t+T)^{n_1-1} \text{ for } t \in (-T, -T+t^*], \tag{5.1.14}$$

$$d_3(T-t)^{n_2-1} \le \eta(t) \le d_4(T-t)^{n_2-1} \text{ for } t \in [T-t^*, T), \tag{5.1.15}$$

$$|\eta'(t)| \le \frac{d_5\eta(t)}{\min\{T-t, t+T\}} \quad \text{ for } t \in I, \tag{5.1.16}$$

for some constants $d_1, \ldots, d_5 > 0$, $n_1, n_2 \in \mathbb{N}$ and $t^* > 0$.

Remark 5.1.3. *Two important weights are covered in under these assumptions. The unweighted case $\eta \equiv 1$ can be recovered by taking $n_1 = n_2 = 1$ and $d_i = 1$ for $i = 1, \ldots, 4$, while the radial weight $\eta(t) = (T+t)^{n-1}$ can be obtained by taking $n_1 = n$, $n_2 = 1$, $d_1 = d_2 = 1$ and appropriate d_3 and d_4.*

Previously, this functional has been studied in a few special settings. When $\eta \equiv 1$ this is simply the one dimensional Cahn–Hilliard functional, which was studied in detail in [31], and was subsequently studied by [25, 59, 18]. The radial case, when $\eta = cr^{n-1}$ has been studied by a variety of authors, including [87, 26, 41]. Finally, the general weighted case was studied in [70]. In that work Kurata and Shibata studied a very different question, namely monotonicity properties of minimizers of the Cahn–Hilliard energy when the domain Ω is a curved strip in $\mathbb{R}^2$.

The aim in this chapter is to study second-order Γ-limits in the general weighted case. This is motivated by the generalized Pòlya–Szegő inequality established in Chapter 3. In that chapter the weight η is given by $\mathcal{I}_\Omega(V_\Omega)$, which does not typically have any closed form, but generally will satisfy assumptions (5.1.13)-(5.1.16). In Chapter 6 the Pòlya–Szegő result will be combined with the results from this chapter to establish a second-order Γ-limit result for the Cahn–Hilliard functional in n dimensions. This follows the framework used in the radial case in [41], and in many ways the analysis here is similar.

5.2 Zero and First-Order Γ-limit of G_ε

The first step is to establish the zeroth-order Γ-limit of the functional G_ϵ.

Theorem 5.2.1. *Assume that W satisfies hypotheses* (5.1.4)-(5.1.7) *and that η satisfies hypotheses* (5.1.13)-(5.1.16)*. Then the family $\{G_\varepsilon\}$ Γ-converges to $G^{(0)}$ in L^1_η, where*

$$G^{(0)}(v) := \begin{cases} \int_I W(v)\eta\,dt & \text{if } v \in L^1_\eta \text{ and } \int_I v\eta\,dt = m, \\ \infty & \text{otherwise in } L^1_\eta. \end{cases}$$

Proof. For the lim inf inequality assume that $v_\varepsilon \to v$ in L^1_η. By utilizing Fatou's lemma along with (5.1.4) we have that

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(v_\varepsilon) \ge \liminf_{\varepsilon\to 0^+} \int_I W(v_\varepsilon)\eta\,dt \ge \int_I W(v)\eta\,dt.$$

For the lim sup inequality, we begin by assuming that v is bounded and satisfies (5.1.3) (the case where v does not satisfy (5.1.3) is trivial). Let ϕ_δ be the standard mollifier, let $\tilde{v}$ be v extended to all of $\mathbb{R}$ by zero and consider $\tilde{v}_\varepsilon := \phi_{\delta_\varepsilon} * \tilde{v}$, where we select δ_ε so that $\|v - \tilde{v}_\varepsilon\|_{L^1_\eta} = o(1)$ and so that

$$\int_I (\tilde{v}_\varepsilon')^2\eta\,dt \le C\varepsilon^{-1}.$$

We then select $d_\varepsilon \in \mathbb{R}$ so that $v_\varepsilon := \tilde{v}_\varepsilon + d_\varepsilon$ satisfies (5.1.3). It is evident that $d_\varepsilon = o(1)$. Finally, by the Lebesgue dominated convergence theorem we have that

$$\lim_{\varepsilon \to 0^+} \int_I W(v_\varepsilon)\eta \, dt = \int_I W(v)\eta \, dt,$$

which gives the desired result for v bounded. Now if $v \in L^1_\eta$ and $\int_I v\eta \, dt = m$ we can construct a sequence $\{v_k\}$ of truncations of v, so that $W(v_k) \leq W(v_{k+1})$ (see (5.1.6)) and so that $\int_I v_k \eta \, dt = m$. Since the Γ-lim sup is lower semicontinuous (see Remark 2.4.2), by applying the Lebesgue monotone convergence theorem we have that

$$\Gamma\text{-}\limsup G_\varepsilon(v) \leq \liminf_{k\to\infty} \Gamma\text{-}\limsup G_\varepsilon(v_k) \leq \liminf_{k\to\infty} \int_I W(v_k)\eta \, dt = \int_I W(v)\eta \, dt, \tag{5.2.1}$$

which concludes the proof.

□

By considering a measurable function taking values at a, b and satisfying (5.1.3), it is clear that $\inf G^{(0)} = 0$, and thus

$$G^{(1)}_\varepsilon(v) = \varepsilon^{-1} G_\varepsilon(v) = \int_I \left(\frac{W(v)}{\varepsilon} + \varepsilon |v'|^2 \right) \eta \, dt \tag{5.2.2}$$

for all $v \in H^1_\eta$ satisfying (5.1.3), and $G^{(1)}_\varepsilon(v) = \infty$ otherwise in L^1_η. The next result deals with compactness, and utilizes arguments from [55].

Proposition 5.2.2. *Let $v_\varepsilon \in H^1_\eta$ be such that $\sup_\varepsilon G^{(1)}_\varepsilon(v_\varepsilon) < \infty$. Then up to a subsequence $v_\varepsilon \to v \in \mathcal{C}$ in L^1_η, where*

$$\mathcal{C} := \{v \in BV_\eta(I; \{a,b\}) : v \textit{ satisfies } (5.1.3)\}. \tag{5.2.3}$$

Proof. We first show that $\{v_\varepsilon\}$ is uniformly bounded in L^1_η and equi-integrable. This is since, by applying (5.1.8),

$$\int_{|v_\varepsilon| > \hat{T}} |v_\varepsilon| \eta \, dt \leq \hat{L}^{-1} \int_I W(v_\varepsilon)\eta \, dt \leq C\varepsilon G^{(1)}_\varepsilon(v_\varepsilon) \leq C\varepsilon,$$

which, in turn, implies that

$$\int_E |v_\varepsilon| \eta \, dt \leq \hat{T} \int_E \eta \, dt + C\varepsilon.$$

As $\int_I \eta \, dt < \infty$ and using the fact that any finite collections of L^1_η functions in L^1_η is equi-integrable, we obtain that the sequence $\{v_\varepsilon\}$ is bounded in L^1_η and equi-integrable.

Next, define

$$W_1(s) := \min\{W(s), K\}, \quad \Phi_1(t) := \int_a^t W_1^{1/2}(s) \, ds, \tag{5.2.4}$$

where $K := \max_{s \in [a,b]} W(s)$. Using Young's inequality, and the fact that $0 \leq W_1 \leq W$ we have that

$$2 \int_I W_1^{1/2}(v_\varepsilon) |v'_\varepsilon| \eta \, dt \leq G^{(1)}_\varepsilon(v_\varepsilon) \leq C.$$

Utilizing the chain rule (see Proposition 2.1.4), we find that

$$\int_I |(\Phi_1 \circ v_\varepsilon)'|\eta\, dt \le C.$$

Furthermore, as Φ_1 is Lipshitz and $\Phi_1(a) = 0$, we have that $\Phi_1 \circ v_\varepsilon$ is uniformly bounded in L^1_η. This then implies, by BV compactness, that, up to a subsequence, not relabeled,

$$\Phi_1 \circ v_\varepsilon \to \tilde{v} \quad \text{in } L^1_\eta$$

for some function $\tilde{v} \in BV_\eta$. It is easy to show, using (5.1.6), that Φ_1 has a continuous inverse. This implies that, up to a subsequence, v_ε must converge pointwise to $v := \Phi_1^{-1}(\tilde{v})$. Thus, up to a subsequence, the v_ε converge in L^1_η to v. Using Fatou's lemma and the fact that $\sup_\varepsilon G^{(1)}_\varepsilon(v_\varepsilon) < \infty$, it must be $W(v(t)) = 0$ for a.e. $t \in I$, or, in other words, that $v \in L^1_\eta(I; \{a, b\})$ by (5.1.4). As $\tilde{v} \in BV_\eta$, this implies that $v \in BV_\eta(I; \{a, b\})$. The L^1_η convergence of the v_ε then implies that v satisfies (5.1.3). This concludes the proof.

□

The first main theorem of this section characterizes the first-order Γ-limit of G_ε.

Theorem 5.2.3. *Assume that W satisfies* (5.1.4)-(5.1.7) *and that η satisfies* (5.1.13)-(5.1.16). *Then the family $\{G^{(1)}_\varepsilon\}$ Γ-converges to the functional*

$$G^{(1)}(v) = \begin{cases} \frac{2c_W}{b-a}|Dv|_\eta(I) & \text{if } v \in \mathcal{C}, \\ \infty & \text{otherwise in } L^1_\eta, \end{cases} \tag{5.2.5}$$

where c_W is the constant given in (1.1.5) *and $\mathcal{C}$ defined in* (5.2.3).

By definition (5.1.1), it is immediate that

$$|Dv|_\eta = (b-a)\sum \eta(t_i),$$

where t_i are the locations of jumps of the function v. It is not surprising that Proposition 5.2.2 and Theorem 5.2.3 are completely analogous to classical results (e. g. [78, 101]) in the unweighted, higher-dimensional case.

Proof. We first characterize the Γ-lim sup. Specifically, given a $v \in \mathcal{C}$, we construct a family of functions v_ε that converge in L^1_η to v satisfying

$$\limsup_{\varepsilon\to 0^+} G^{(1)}_\varepsilon(v_\varepsilon) \le G^{(1)}(v). \tag{5.2.6}$$

To begin with, we assume that v is of the form

$$v(t) = \begin{cases} a & \text{if } t \in [t_{2k}, t_{2k+1}), \\ b & \text{otherwise,} \end{cases}$$

where $-T = t_0 < t_1 < \cdots < t_{2N} = T$. Define

$$f(t) := \begin{cases} t - t_1 & \text{if } t \in [t_0, t_1), \\ -\min\{t - t_{2k}, t_{2k+1} - t\} & \text{if } t \in [t_{2k}, t_{2k+1}), \text{ and } k \ge 1, \\ \min\{t - t_{2k+1}, t_{2k+2} - t\} & \text{if } t \in (t_{2k+1}, t_{2k+2}], \text{ and } k < N-1, \\ t - t_{2N-1} & \text{if } t \in [t_{2N-1}, t_{2N}). \end{cases}$$

Observe that f is the signed distance function (see (2.3.4)) of the set $E := \{t \in I : v(t) = a\}$, where we naturally are considering ∂E relative to I, not $\mathbb{R}$. We note that $v(t) = \operatorname{sgn}_{a,b}(f(t))$, where $\operatorname{sgn}_{a,b}$ is the function given in (1.1.7). Thus the goal is to construct smooth approximations of the function $\operatorname{sgn}_{a,b}$ that make the energy $G_\varepsilon^{(1)}$ small.

One possible approximation comes from the construction in [78]. Although the argument is almost identical, it is included here for completeness. Consider the function

$$\tilde{\phi}_\varepsilon(s) := \int_a^s \left(\frac{\varepsilon^2}{\varepsilon + W(r)}\right)^{1/2} dr, \tag{5.2.7}$$

and define the constant

$$\xi_\varepsilon := \tilde{\phi}_\varepsilon(b).$$

Since $W \geq 0$, equation (5.2.7) gives

$$0 \leq \xi_\varepsilon \leq (b-a)\varepsilon^{1/2}.$$

Note that $\tilde{\phi}_\varepsilon$ is strictly increasing and differentiable. Now define $\phi_\varepsilon : [0, \xi_\varepsilon] \to [a, b]$ to be the inverse of $\tilde{\phi}_\varepsilon$ on the interval $[a, b]$. By the fundamental theorem of calculus and the inverse function theorem ϕ_ε will satisfy the equation

$$\varepsilon \phi_\varepsilon'(t) = (\varepsilon + W(\phi_\varepsilon(t)))^{1/2}.$$

Next, extend ϕ_ε to be equal to a for $t < 0$ and b for $t > \xi_\varepsilon$. Note that for all $t \in \mathbb{R}$ we have that $\phi_\varepsilon(t) \leq \operatorname{sgn}_{a,b}(t)$ and that $\phi_\varepsilon(t + \xi_\varepsilon) \geq \operatorname{sgn}_{a,b}(t)$. Thus as $v \in \mathcal{C}$ we can find a $\tau_\varepsilon \in (0, \xi_\varepsilon)$ that gives

$$\int_I \phi_\varepsilon(f(t) + \tau_\varepsilon)\eta(t)\, dt = m.$$

Define $v_\varepsilon(t) := \phi_\varepsilon(f(t) + \tau_\varepsilon)$. As $\{v_\varepsilon\}$ converges to v pointwise and $|v_\varepsilon| < C$ we have that $v_\varepsilon \to v$ in L^1_η. We then examine the energy associated with v_ε, when ε is sufficiently small that transition layers do not overlap or leave $\overline{I}$:

$$\begin{aligned}
G_\varepsilon^{(1)}(v_\varepsilon) &= \sum_{k=1}^{2N-1} \int_0^{\xi_\varepsilon} \left(\varepsilon(\phi_\varepsilon'(t))^2 + \varepsilon^{-1} W(\phi_\varepsilon(t))\right) \eta(t_k + (t - \tau_\varepsilon)(-1)^{k+1})\, dt \\
&\leq \sum_{k=1}^{2N-1} \int_0^{\xi_\varepsilon} 2(\varepsilon + W(\phi_\varepsilon(t)))^{1/2} \phi_\varepsilon'(t) \eta(t_k + (t - \tau_\varepsilon)(-1)^{k+1})\, dt \\
&\leq \sum_{k=1}^{2N-1} \sup\{\eta(t_k + (s - \tau_\varepsilon)(-1)^{k+1}) : s \in (0, \xi_\varepsilon)\} \int_0^{\xi_\varepsilon} 2(\varepsilon + W(\phi_\varepsilon(t)))^{1/2} \phi_\varepsilon'(t)\, dt \\
&= \sum_{k=1}^{2N-1} \sup\{\eta(t_k + (s - \tau_\varepsilon)(-1)^{k+1}) : s \in (0, \xi_\varepsilon)\} \int_a^b 2(\varepsilon + W(s))^{1/2}\, ds.
\end{aligned}$$

Thus taking the limit as $\varepsilon \to 0^+$ we find that

$$\limsup_{\varepsilon \to 0^+} G_\varepsilon^{(1)}(v_\varepsilon) \leq 2c_W \sum_{k=1}^{2N-1} \eta(t_k) = G^{(1)}(v).$$

The cases where v has a finite number of jump points, but starting or ending at different values than we assumed are analogous. Reasoning as in (5.2.1), by noting that functions with a finite number of jumps can approximate elements of $\mathcal{C}$ arbitrarily well in BV_η, and as the Γ-lim sup is lower semicontinuous, we then have (5.2.6).

Next we will establish our Γ-lim inf. Assume that $v_\varepsilon \to v$ in L^1_η. By Proposition 5.2.2 if $v \notin \mathcal{C}$ then $\liminf_{\varepsilon\to 0^+} G_\varepsilon^{(1)} = \infty$, and there is nothing to prove. We claim that for any sequence $\{v_\varepsilon\}$ that converges in L^1_η to some $v \in \mathcal{C}$ the following inequality holds:

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon^{(1)}(v_\varepsilon) \geq G^{(1)}(v). \tag{5.2.8}$$

To establish this inequality we use Young's inequality, the chain rule (see Proposition 2.1.4) and lower semicontinuity of $\|\cdot\|_{BV_\eta}$ (see Proposition 2.1.3 and Remark 2.1.5) and the definition (5.2.4) as follows:

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} G_\varepsilon^{(1)}(v_\varepsilon) &\geq \liminf_{\varepsilon\to 0^+} \int_I (\varepsilon^{-1}W_1(v_\varepsilon) + \varepsilon(v_\varepsilon')^2)\eta\, dt \\ &\geq \liminf_{\varepsilon\to 0^+} 2\int_I |(\Phi_1\circ v_\varepsilon)'|\eta\, dt \geq 2\int_I \eta\, d|D\Phi_1(v)| \\ &= 2\int_I \eta\, d|D\Phi(v)| = \frac{2c_W}{b-a}\int_I \eta\, d|Dv| = G^{(1)}(v_0).\end{aligned}$$

Here we have used the fact that $\Phi_1\circ v_\varepsilon$ converges to $\Phi_1\circ v$ in L^1_η (because Φ_1 is Lipschitz), and the fact that $\Phi_1\circ v = \Phi\circ v$, where $\Phi := \int_a^t W^{1/2}(s)\, ds$. This proves the claim. □

The fundamental theorem of Γ-convergence (Theorem 2.4.5), which applies due to Proposition 5.2.2, then establishes the following corollary.

Corollary 5.2.4. *Under the hypotheses of Theorem 5.2.3 if v_ε are minimizers of $G_\varepsilon^{(1)}$ then, up to a subsequence, they converge in L^1_η to v which is a minimizer of $G^{(1)}$. Furthermore the v_ε will satisfy the following*

$$\lim_{\varepsilon\to 0^+} G_\varepsilon^{(1)}(v_\varepsilon) = G^{(1)}(v).$$

The remainder of this section will be devoted to proving two theorems that will be important in later analysis. First, select t_0 so that

$$v_0(t) := \operatorname{sgn}_{a,b}(t-t_0)$$

satisfies (5.1.3). By (5.1.13) it is clear that t_0 is uniquely determined. In general, v_0 is *not a global minimizer* of $G^{(1)}$. However, it is the case that v_0 is an isolated local minimizer of $G^{(1)}$ in L^1_η.

Theorem 5.2.5. *Assume that W satisfies* (5.1.4)-(5.1.7) *and that η satisfies* (5.1.13)-(5.1.16)*. Then there exists $\delta > 0$ such that v_0 is an isolated δ-local minimizer of $G^{(1)}$ in L^1_η, that is, there is no $v_1 \in \mathcal{C}$ (see* (5.2.3)*), with $0 < \|v_1 - v_0\|_{L^1_\eta} \leq \delta$ such that*

$$G^{(1)}(v_1) \leq G^{(1)}(v_0).$$

Proof. Assume by contradiction that such v_1 exists. By continuity of η, for every $\epsilon > 0$ there is $\mathfrak{r}_\epsilon > 0$ such that

$$|\eta(t) - \eta(t_0)| \leq \epsilon \tag{5.2.9}$$

for all $t \in [t_0 - \mathfrak{r}_\epsilon, t_0 + \mathfrak{r}_\epsilon]$. Let $M_0 := \max|\eta'| + 1$ and fix

$$0 < \mathfrak{r}_0 < \min\left\{\frac{1}{2}t^*, T - t_0, T + t_0, \frac{d_1 n_1 \eta(t_0)}{2d_2 M_0}, \frac{d_3 n_2 \eta(t_0)}{2d_4 M_0}\right\}, \tag{5.2.10}$$

where t^*, n_1, n_2 and the constants $d_i, i = 1\ldots 4$ are given in (5.1.14) and (5.1.15). Then define

$$I_0 := [-T + \mathfrak{r}_0, T - \mathfrak{r}_0],$$

and fix

$$0 < \epsilon_1 < \min\{\min_{I_0} \eta, \eta(t_0)/2\}$$

in (5.2.9) and let $\mathfrak{r}_{\epsilon_1}$ be the corresponding $\mathfrak{r}_\epsilon$.

Step 1: We claim that v_1 has a jump at some $t_1 \in B(t_0, \mathfrak{r}_{\epsilon_1})$. If not, then either $v_1 \equiv a$ in $B(t_0, \mathfrak{r}_{\epsilon_1})$ or $v_1 \equiv b$ in $B(t_0, \mathfrak{r}_{\epsilon_1})$. Assume that $v_1 \equiv a$ in $B(t_0, \mathfrak{r}_{\epsilon_1})$. Then by (5.2.9),

$$\delta \geq \int_{B(t_0,\mathfrak{r}_{\epsilon_1})} |v_1 - v_0|\eta \, dt \geq (b-a)\frac{\eta(t_0)}{2}\mathfrak{r}_{\epsilon_1},$$

where we used the fact that $0 < \epsilon_1 < \eta(t_0)/2$. Since the case $v_1 \equiv b$ gives an identical estimate, the claim follows provided

$$0 < \delta < (b-a)\frac{\eta(t_0)}{2}\mathfrak{r}_{\epsilon_1}.$$

Step 2: We claim that v_1 has no jump other than t_1 in I_0. Indeed, assume that there is a second jump $t_2 \neq t_1$ in I_0. Then by (5.2.9) and Step 1,

$$\begin{aligned} G^{(1)}(v_1) &\geq 2c_W(\eta(t_1) + \eta(t_2)) \\ &\geq 2c_W(\eta(t_0) - \epsilon_1 + \min_{I_0} \eta) > 2c_W \eta(t_0) = G^{(1)}(v_0), \end{aligned}$$

where in the last inequality we used the fact that $0 < \epsilon_1 < \min_{I_0} \eta$. This is impossible since we are assuming that $G^{(1)}(v_1) \leq G^{(1)}(v_0)$.

Step 3: We claim that v_1 jumps from a to b at t_1. Suppose not, and suppose that $t_1 \leq t_0$. Then

$$\delta \geq \int_{B(t_0,\mathfrak{r}_{\epsilon_1})} |v_1 - v_0|\eta \, dt \geq (b-a)\frac{\eta(t_0)}{2}\mathfrak{r}_{\epsilon_1},$$

which again leads to a contradiction if δ is chosen small enough. The case $t_1 > t_0$ is analogous.

Step 4: We claim that $t_1 = t_0$. Indeed, if $t_1 > t_0$, then

$$0 = \int_I (v_1 - v_0)\eta \, dt = \int_{-T}^{-T+\mathfrak{r}_0} (v_1 - a)\eta \, dt + \int_{t_0}^{t_1} (a-b)\eta \, dt + \int_{T-\mathfrak{r}_0}^{T} (v_1 - b)\eta \, dt,$$

which implies, as the last two terms are negative, that there must be a jump t_3 that belongs to $(-T, -T + \mathfrak{r}_0)$, with

$$0 < \frac{\eta(t_0)}{2}(b-a)(t_1 - t_0) \leq \int_{t_0}^{t_1} (b-a)\eta \, dt \leq (b-a)\int_{-T}^{t_3} \eta \, dt \leq d_2(b-a)\frac{(T+t_3)^{n_1}}{n_1}, \tag{5.2.11}$$

where in the last equality we used (5.1.14), in conjunction with (5.2.10). By the mean value theorem and inequality (5.2.11), for some $\theta \in (t_0, t_1)$,

$$\begin{aligned} \eta(t_1) &= \eta(t_0) + \eta'(\theta)(t_1 - t_0) \geq \eta(t_0) - M_0|t_1 - t_0| \\ &\geq \eta(t_0) - \frac{2M_0 d_2}{n_1 \eta(t_0)}(T + t_3)^{n_1}. \end{aligned}$$

Hence by (5.2.10),

$$
\begin{aligned}
G^{(1)}(v_1) &\geq 2c_W(\eta(t_1)+\eta(t_3)) \\
&\geq 2c_W\eta(t_0) - 2c_W\frac{2M_0d_2}{n_1\eta(t_0)}(T+t_3)^{n_1} + 2c_Wd_1(T+t_3)^{n_1-1} \\
&> 2c_W\eta(t_0) = G^{(1)}(v_0),
\end{aligned}
$$

which violates our assumption. The case $t_1 < t_0$ is analogous. This proves that $t_1 = t_0$, and so $G^{(1)}(v_1) \geq 2c_W\eta(t_0) = G^{(1)}(v_0)$, which implies that $G^{(1)}(v_1) = G^{(1)}(v_0)$. In particular, v_1 has no jumps in $I\backslash I_0$. But then $v_1 = v_0$, which is a contradiction. This completes the proof. □

Although v_0 is a local minimizer for $G^{(1)}$, In general v_0 may not be a global minimizer without further assumptions on η (e.g., $\eta \equiv$ constant). However, in certain cases it will be important to study a type of second-order asymptotic development of G_ε where in the definition of $G_\varepsilon^{(2)}$ (see (2.4.1)) in place of $\inf G^{(1)}$ we take $G^{(1)}(v_0)$. This in fact corresponds to studying the second-order asymptotic development of the localized functional

$$
J_\varepsilon(v) := \begin{cases} G_\varepsilon(v) & \text{if } \|v-v_0\|_{L^1_\eta} \leq \delta, \\ \infty & \text{otherwise.} \end{cases} \tag{5.2.12}
$$

The following theorem gives a lim sup inequality. It also does not require the same regularity results on η as most of the other theorems in this chapter.

Theorem 5.2.6. *Assume that W satisfies* (5.1.4)-(5.1.7)*, and that $\eta : I \to [0,\infty)$ is measurable, bounded, differentiable at t_0, $\eta(t_0) > 0$ and*

$$
|\eta(t) - \eta(t_0) - \eta'(t_0)(t-t_0)| = o(|t-t_0|) \tag{5.2.13}
$$

for some constant $C > 0$ and for all t in a neighborhood of t_0. Then there exists a sequence $\{v_\varepsilon\}$ converging to v_0 in L^1_η so that

$$
\begin{aligned}
\limsup_{\varepsilon\to0^+} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} &\leq 2\eta'(t_0)(\tau_0c_W + c_{\mathrm{sym}}) \\
&\quad + \begin{cases} \frac{\lambda_0^2}{2W''(a)}\int_I \eta\,ds & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases}
\end{aligned} \tag{5.2.14}
$$

where c_W and c_{sym} are given by (1.1.5), (6.1.6)*, τ_0 is determined by the equation*

$$
\eta(t_0)\int_{\mathbb{R}} (z(s-\tau_0) - \mathrm{sgn}_{a,b})\,ds = \begin{cases} \frac{\lambda_0}{W''(a)}\int_I \eta\,dt & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases} \tag{5.2.15}
$$

where z is the solution to (1.1.6) *and λ_0 is defined by*

$$
\lambda_0 := \frac{2\eta'(t_0)c_W}{(b-a)\eta(t_0)}. \tag{5.2.16}
$$

Proof. **Step 1:** Assume $q = 1$. Define $z_\varepsilon(t) := z(\frac{t-t_0}{\varepsilon})$ and then define

$$
v_\varepsilon(t) := z_\varepsilon(t - \varepsilon\tau_\varepsilon) - \frac{\lambda_0\varepsilon}{W''(a)}, \tag{5.2.17}
$$

where τ_ε is selected so that (5.1.3) is satisfied. We first claim that

$$\lim_{\varepsilon\to 0^+} \tau_\varepsilon = \tau_0. \tag{5.2.18}$$

To this end, we can write, via (5.1.3),

$$\int_I v_\varepsilon \eta \, dt = \int_I v_0 \eta \, dt = m.$$

In turn this implies that

$$\begin{aligned}\int_I (z_\varepsilon(t-\varepsilon\tau_\varepsilon) - z_\varepsilon(t-\varepsilon\tau_0))\eta \, dt = &\int_I (\operatorname{sgn}_{a,b}(t-t_0) - z_\varepsilon(t-\varepsilon\tau_0))\eta \, dt \\ &+ \frac{\varepsilon\lambda_0}{W''(a)} \int_I \eta \, dt.\end{aligned} \tag{5.2.19}$$

After the change of variables $s = \frac{t-t_0}{\varepsilon}$ we can write the right-hand side as

$$\varepsilon \int_{\frac{-T-t_0}{\varepsilon}}^{\frac{T-t_0}{\varepsilon}} (\operatorname{sgn}_{a,b}(s) - z(s-\tau_0))\eta(\varepsilon s + t_0) \, ds + \frac{\varepsilon\lambda_0}{W''(a)} \int_I \eta \, dt. \tag{5.2.20}$$

By our choice of τ_0 (via (5.2.15)) and (1.1.7) this is equal to

$$\begin{aligned}&\varepsilon \int_{\frac{-T-t_0}{\varepsilon}}^{\frac{T-t_0}{\varepsilon}} (\operatorname{sgn}_{a,b}(s) - z(s-\tau_0))(\eta(\varepsilon s + t_0) - \eta(t_0)) \, ds \\ &- \varepsilon\eta(t_0) \int_{-\infty}^{\frac{-T-t_0}{\varepsilon}} (a - z(s-\tau_0)) \, ds - \varepsilon\eta(t_0) \int_{\frac{T-t_0}{\varepsilon}}^{\infty} (b - z(s-\tau_0)) \, ds.\end{aligned} \tag{5.2.21}$$

By (5.2.13) there exists a $R_0 > 0$ such that $|\eta(t) - \eta(t_0)| \leq (|\eta'(t_0)| + 1)|t - t_0|$ for all $t \in B(t_0, R_0)$. Since η is bounded by assumption, we thus have for all $t \in I \backslash B(t_0, R_0)$,

$$|\eta(t) - \eta(t_0)| \leq 2\|\eta\|_\infty \leq 2\frac{\|\eta\|_\infty}{R_0}|t - t_0|.$$

Hence for all $t \in I$ we have that $|\eta(t) - \eta(t_0)| \leq C_\eta |t - t_0|$ for some $C_\eta > 0$. Thus, using (5.1.11), the first term in (5.2.21) can be bounded by

$$2(b-a)\varepsilon \int_{\frac{-T-t_0}{\varepsilon}}^{\frac{T-t_0}{\varepsilon}} e^{-c_1|s|}|\eta(\varepsilon s + t_0) - \eta(t_0)| \, ds \leq 2(b-a)C_\eta\varepsilon^2 \int_{\mathbb{R}} e^{-c_1|s|}|s| \, ds.$$

By (5.1.11) we know that the last two terms of (5.2.21) are bounded from above by $\frac{(b-a)}{c_1}\|\eta\|_\infty \varepsilon^2 e^{-\frac{c_1 T_1}{\varepsilon}}$, where $T_1 := \min(T - t_0, T + t_0) > 0$. Hence, the right-hand side of (5.2.19) is bounded from above by $C\varepsilon^2$ for all $\varepsilon > 0$ sufficiently small.

Now assume that the τ_ε do not converge to τ_0. Assume without loss of generality that for some subsequence (not relabeled) the $\tau_\varepsilon \leq \tau_0 - k_0$ for some $k_0 > 0$ (the case where $\tau_\varepsilon \geq \tau_0 + k_0$ is similar). Since z is increasing (see (1.1.6)), by (5.2.19) and what we just proved,

$$\begin{aligned}C\varepsilon^2 &\geq \int_I (z_\varepsilon(t-\varepsilon\tau_\varepsilon) - z_\varepsilon(t-\varepsilon\tau_0))\eta(t) \, dt \geq \inf_{B(t_0+\varepsilon\tau_0, k_1\varepsilon)} \eta \int_{B(t_0+\varepsilon\tau_0, k_1\varepsilon)} \int_{t-\varepsilon\tau_0}^{t-\varepsilon\tau_\varepsilon} z'_\varepsilon(s) \, ds \, dt \\ &\geq \inf_{B(t_0+\varepsilon\tau_0, k_1\varepsilon)} \eta \int_{B(t_0+\varepsilon\tau_0, k_1\varepsilon)} \int_{t-\varepsilon\tau_0}^{t-\varepsilon(\tau_0-k_0)} \varepsilon^{-1}\sqrt{W(z(\varepsilon^{-1}(s-t_0))} \, ds \, dt \\ &\geq 2k_1 k_0 \varepsilon \inf_{t\in B(0, k_1+k_0)} \sqrt{W(z(t))} \inf_{B(t_0+\varepsilon\tau_0, k_1\varepsilon)} \eta,\end{aligned}$$

where $0 < k_1 < 1$ and where we have used the facts that η is continuous at t_0 and that $\eta(t_0) > 0$. Since $z(0) = c$, by taking k_0 and k_1 sufficiently small we can assume that $z(t) \in B(c, \min\{\frac{c-a}{2}, \frac{b-c}{2}\})$ for all $t \in B(0, k_0 + k_1)$. In turn the right-hand side of the previous inequality is bounded from below by $C_1\varepsilon$ for some $C_1 > 0$. This is a contradiction, which proves our claim.

Next we prove (5.2.14). We will write $R_\varepsilon := C_k\varepsilon|\log\varepsilon|$, with $C_k > 0$ to be chosen later. We then write

$$\begin{aligned}\frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} &= \varepsilon^{-1}\left(\int_{B(t_0,R_\varepsilon)}(\varepsilon^{-1}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2)\eta\,dt - 2c_W\eta(t_0)\right)\\ &\quad+\int_{I\setminus B(t_0,R_\varepsilon)}(\varepsilon^{-2}W(v_\varepsilon) + (v_\varepsilon')^2)\eta\,dt.\end{aligned} \tag{5.2.22}$$

First we examine the second term, namely the tail integral. We first note that by (5.1.11) and the fact that the $\tau_\varepsilon \to \tau_0$ we then have that

$$b - z_\varepsilon(t - \varepsilon\tau_\varepsilon) \le \frac{b-a}{2}e^{c_1(1+|\tau_0|)}\varepsilon^{c_1C_k} \le \varepsilon^k$$

for $t \in [t_0 + R_\varepsilon, T]$ and for ε small, provided $C_k \ge 2\frac{k}{c_1}$. Similarly, $z_\varepsilon(t-\varepsilon\tau_\varepsilon) - a < \varepsilon^k$ for $t \in [-T, t_0 - R_\varepsilon]$. Thus for $t \in I\setminus B(t_0, R_\varepsilon)$ we have that

$$|z_\varepsilon(t - \varepsilon\tau_\varepsilon) - v_0(t)| \le \varepsilon^k \tag{5.2.23}$$

which in turn implies, after recalling (5.2.17), that, for k large,

$$(v_\varepsilon(t) - v_0)^2 \le \frac{\lambda_0^2\varepsilon^2}{W''(a)^2} + C\varepsilon^{k+1} \tag{5.2.24}$$

for all $t \in I\setminus B(t_0, R_\varepsilon)$ and for some fixed $C > 0$.

We then fix $\gamma > 0$. By (5.1.9) there exists s_γ such that

$$W(s) \le \left(\frac{W''(a)}{2} + \gamma\right)(s-a)^2 \tag{5.2.25}$$

for all s with $|s - a| \le s_\gamma$, and

$$W(s) \le \left(\frac{W''(a)}{2} + \gamma\right)(s-b)^2 \tag{5.2.26}$$

for all s with $|s - b| \le s_\gamma$. By (5.2.24), (5.2.25) and (5.2.26) we then have for ε sufficiently small that

$$\int_{I\setminus B(t_0,R_\varepsilon)} W(v_\varepsilon)\eta\,dt \le \left(\frac{W''(a)}{2} + \gamma\right)\varepsilon^2\lambda_0^2W''(a)^{-2}\int_I\eta\,dt + O(\varepsilon^{k+1}).$$

On the other hand, using (1.1.6), (5.2.23), (5.2.25), and (5.2.26),

$$(v_\varepsilon'(t))^2 = \frac{1}{\varepsilon^2}W(z_\varepsilon(t+\varepsilon\tau_\varepsilon)) \le \frac{C}{\varepsilon^2}(z_\varepsilon(t+\varepsilon\tau_\varepsilon) - v_0(t))^2 \le C\varepsilon^{2k-2}$$

for $t \in I\setminus B(t_0, R_\varepsilon)$. After taking limits (first as $\varepsilon \to 0^+$ and then as $\gamma \to 0^+$) we thus find that

$$\limsup_{\varepsilon\to0^+}\int_{I\setminus B(t_0,R_\varepsilon)}(\varepsilon^{-2}W(v_\varepsilon) + (v_\varepsilon')^2)\eta\,dt \le \frac{\lambda_0^2}{2W''(a)}\int_I\eta\,dt. \tag{5.2.27}$$

Next we estimate the energy in the region $B(t_0, R_\varepsilon)$. We will define $s_1^\varepsilon := v_\varepsilon(t_0 - R_\varepsilon)$ and $s_2^\varepsilon := v_\varepsilon(t_0 + R_\varepsilon)$. Note that by (5.2.24), $s_1^\varepsilon = a + O(\varepsilon)$ and $s_2^\varepsilon = b + O(\varepsilon)$. Thus recalling the definition of c_W, (1.1.5), and (5.1.9), we find that

$$c_W = \int_{s_1^\varepsilon}^{s_2^\varepsilon} W^{1/2}(s)\,ds + O(\varepsilon^2) = \int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon)v_\varepsilon'\,dt + O(\varepsilon^2),$$

where we have used the change of variables $s = v_\varepsilon(t)$. Thus we have that

$$\begin{aligned}&\int_{B(t_0,R_\varepsilon)} (\varepsilon^{-1}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2)\eta\,dt - 2c_W\eta(t_0)\\ &= \int_{B(t_0,R_\varepsilon)} (\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}v_\varepsilon')^2\eta + W^{1/2}(v_\varepsilon)v_\varepsilon'(2\eta - 2\eta(t_0))\,dt + O(\varepsilon^2).\end{aligned} \tag{5.2.28}$$

We now estimate the terms on the right-hand side of (5.2.28). Recalling the fact that $|W^{1/2}(s_1) - W^{1/2}(s_2)| \le C|s_1 - s_2|$ for all $s_1, s_2 \in [a-1, b+1]$ (see (5.1.4) and (5.1.5)), it follows from (1.1.6), (5.2.17), and the boundedness of η, that

$$\begin{aligned}\int_{B(t_0,R_\varepsilon)} (\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}v_\varepsilon')^2\eta\,dt &\le \varepsilon^{-1}\int_{B(t_0,R_\varepsilon)} (W^{1/2}(v_\varepsilon(t)) - W^{1/2}(z_\varepsilon(t - \varepsilon\tau_\varepsilon)))^2\eta(t)\,dt\\ &\le C\varepsilon^{-1}\int_{B(t_0,R_\varepsilon)} \left(\frac{\varepsilon\lambda_0}{W''(a)}\right)^2\eta\,dt \le C\varepsilon^2|\log\varepsilon|.\end{aligned} \tag{5.2.29}$$

Next we will use (1.1.6), (5.2.13) and (5.2.17) to obtain:

$$\begin{aligned}&2\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon)v_\varepsilon'(\eta - \eta(t_0))\,dt\\ &= 2\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon(t))v_\varepsilon'(t)(\eta'(t_0)(t - t_0) + o(|t - t_0|))\,dt\\ &= 2\eta'(t_0)\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon(t))v_\varepsilon'(t)\left((t - t_0) + |t - t_0|o(1)\right)\,dt.\end{aligned}$$

Changing variables to $s = \frac{t - t_0 - \varepsilon\tau_\varepsilon}{\varepsilon}$ we can then write

$$\begin{aligned}&2\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon)v_\varepsilon'(\eta - \eta(t_0))\,dt\\ &= 2\eta'(t_0)\varepsilon\int_{B(\tau_\varepsilon,C_k|\log\varepsilon|)} W^{1/2}(z(s) - \lambda_0W''(a)^{-1}\varepsilon)z'(s)(\tau_\varepsilon + s)\,ds\\ &+ \varepsilon o(1)\int_{B(\tau_\varepsilon,C_k|\log\varepsilon|)} W^{1/2}(z(s) - \lambda_0W''(a)^{-1}\varepsilon)z'(s)|s + \tau_\varepsilon|\,ds \qquad (5.2.30)\\ &= 2\eta'(t_0)\varepsilon\int_{B(\tau_\varepsilon,C_k|\log\varepsilon|)} W^{1/2}(z(s) - \lambda_0W''(a)^{-1}\varepsilon)z'(s)(\tau_\varepsilon + s)\,ds + o(\varepsilon), \qquad (5.2.31)\end{aligned}$$

where in estimating (5.2.30) we have used that z' decays exponentially, and thus the integral on that line is uniformly bounded. We remark that, by (1.1.5) and (6.1.6) and (5.2.18), the integral on the right-hand side of (5.2.31) converges to

$$\int_{\mathbb{R}} W^{1/2}(z(s))z'(s)(\tau_0 + s)\,ds = \tau_0c_W + c_{sym}.$$

By then combining estimates (5.2.22), (5.2.27), (5.2.28), (5.2.29), (5.2.31), to find that

$$\limsup_{\varepsilon\to 0^+} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \le 2\eta'(t_0)\,(\tau_0 c_W + c_{\mathrm{sym}}) + \frac{\lambda_0^2}{2W''(a)} \int_I \eta\, dt,$$

which is the desired conclusion.

Step 2: The case $q < 1$ is simpler since by (5.1.12) the function z in (1.1.6) satisfies $z(t) \equiv b$ for $t \ge t_b$ and $z(t) \equiv a$ for $t \le t_a$. We define $v_\varepsilon(t) := z_\varepsilon(t - \varepsilon\tau_\varepsilon)$. Then the second term in the right-hand side of (5.2.19) should be replaced by 0, while (5.2.20) becomes

$$\varepsilon \int_{t_a+\tau_0}^{t_b+\tau_0} (\mathrm{sgn}_{a,b}(s) - z(s-\tau_0))\eta(\varepsilon s + t_0)\, ds.$$

In turn, in (5.2.21) the first integral is over $[t_a+\tau_0, t_b+\tau_0]$, while the other two integrals vanish. Using the regularity of η near t_0 we can bound the integral in the new (5.2.21) by $2(b-a)C_\eta\varepsilon^2(t_b - t_a)$. We can continue as before to conclude that $\tau_\varepsilon \to \tau_0$.

By (1.1.5) and (1.1.6), in place of (5.2.22) we now have

$$\frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} = \varepsilon^{-1} \int_{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_a}^{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_b} W^{1/2}(v_\varepsilon(t))v_\varepsilon'(t)(\eta(t) - \eta(t_0))\, dt.$$

Using (5.2.13) and the fact that $\tau_\varepsilon \to \tau_0$, the right-hand side can be bounded from above by

$$\begin{aligned} &\le 2\varepsilon^{-1}\eta'(t_0) \int_{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_a}^{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_b} W^{1/2}(v_\varepsilon(t))v_\varepsilon'(t)(t-t_0)\, dt + o(1) \\ &= 2\eta'(t_0) \int_{t_a}^{t_b} W^{1/2}(z(s))z'(s)(s+\tau_\varepsilon)\, ds + o(1), \end{aligned}$$

where we have used a change of variables $s = \frac{t-t_0-\varepsilon\tau_\varepsilon}{\varepsilon}$, and where the error term in the Taylor formula, namely (5.2.30), still has a uniformly bounded integral, this time because both the integrand and the interval of integration are bounded. It now suffices to let $\varepsilon \to 0^+$.

□

5.3 Local Minimizers of G_ε

This section proves the existence of certain types of local minimizers of G_ε and studies their qualitative properties. In the next subsection these properties will permit a characterization of the second-order asymptotic development of the family J_ε defined in (5.2.12). The following proposition is based on an argument from [69] (see also [22]). The proof is included for completeness.

Proposition 5.3.1. *Assume that W satisfies* (5.1.4)-(5.1.7) *and that η satisfies* (5.1.13)-(5.1.16). *Then for all $\varepsilon > 0$ there exists a global minimizer v_ε of the functional J_ε. Furthermore, the functions v_ε must converge to v_0 in L^1_η, and thus for ε small enough v_ε is a local minimizer of G_ε. Additionally, the following equality holds:*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon^{(1)}(v_\varepsilon) = G^{(1)}(v_0). \tag{5.3.1}$$

Proof. First we prove the existence of a global minimizer. Fix $\varepsilon > 0$ and suppose that $\{f_k\}$ is a minimizing sequence in the sense that

$$\lim_{k\to\infty} J_\varepsilon(f_k) = \inf_v J_\varepsilon(v) < \infty.$$

In particular, $\|f_k - v_0\|_{L^1_\eta} \le \delta$ for all k sufficiently large. By (5.2.2) and (5.2.12) it follows that $\{f_k'\}$ is bounded in L^2_η. Since $\{f_k\}$ is bounded in L^1_η, by (5.1.13) and a diagonal argument, we may find a function $v_\varepsilon \in H^1_{\eta,\text{loc}}$ such that $f_k' \rightharpoonup v_\varepsilon'$ in L^2_η and $f_k \to v_\varepsilon$ in $L^1_{\eta,\text{loc}}$, and pointwise a.e.. By Fatou's lemma and the weak lower semi-continuity of the L^2_η norm, we then have, provided that $v_\varepsilon \in H^1_\eta$ (see (5.1.2)), that

$$G_\varepsilon(v_\varepsilon) \le \liminf_{k\to\infty} G_\varepsilon(f_k) = \inf_v J_\varepsilon(v)$$

and that $\|v_\varepsilon - v_0\|_{L^1_\eta} \le \delta$. Thus it remains to show that $v_\varepsilon \in L^2_\eta$. Since v_ε is locally absolutely continuous, by Hölder's inequality, for $-T < t < -T + t^*$ we have

$$\begin{aligned}
v_\varepsilon^2(t)\eta(t) &= \eta(t)\left(v_\varepsilon(-T+t^*) - \int_t^{-T+t^*} v_\varepsilon'(s)\,ds\right)^2\\
&\le 2\eta(t)v_\varepsilon^2(-T+t^*) + 2\eta(t)\left(\int_t^{-T+t^*} v_\varepsilon'(s)\frac{\eta^{1/2}(s)}{\eta^{1/2}(s)}\,ds\right)^2\\
&\le 2\eta(t)v_\varepsilon^2(-T+t^*) + 2\eta(t)\int_t^{-T+t^*}\frac{1}{\eta(s)}\,ds\int_t^{-T+t^*}|v_\varepsilon'(s)|^2\eta(s)\,ds\\
&\le 2\eta(t)v_\varepsilon^2(-T+t^*) + 2\frac{d_2}{d_1}t^*\int_I |v_\varepsilon'(s)|^2\eta(s)\,ds,
\end{aligned}$$

where we have used the fact that if $t < s < -T+t^*$ then $\eta(s) \ge \frac{d_1}{d_2}\eta(t)$ (see (5.1.14)). By integrating in t over $(-T, -T+t^*)$ we observe that $v_\varepsilon \in L^2_\eta((-T,-T+t^*))$. A similar estimate can be obtained on the interval $(T-t^*, T)$. On the other hand, by (5.1.13), we have that $\eta \ge \eta_0 > 0$ in $[-T+t^*, T-t^*]$, and thus $v_\varepsilon \in L^2((-T+t^*, T-t^*))$, which then implies that $v_\varepsilon \in L^2_\eta$, as desired. This establishes the existence of a global minimizer, v_ε.

By Theorem 5.2.3 we know that there exists a sequence $\{\tilde v_\varepsilon\}$ converging to v_0 in L^1_η with $G^{(1)}_\varepsilon(\tilde v_\varepsilon) \to G^{(1)}(v_0)$. In particular $\|\tilde v_\varepsilon - v_0\|_{L^1_\eta} \le \delta$ for ε sufficiently small. Since v_ε is a global minimizer of J_ε we then know that $G_\varepsilon(v_\varepsilon) \le G_\varepsilon(\tilde v_\varepsilon)$ for ε small. Thus

$$\limsup_{\varepsilon\to0^+} G^{(1)}_\varepsilon(v_\varepsilon) \le \limsup_{\varepsilon\to0^+} G^{(1)}_\varepsilon(\tilde v_\varepsilon) \le G^{(1)}(v_0).$$

By Proposition 5.2.2 we then have that (up to a subsequence, not relabeled), $v_\varepsilon \to \tilde v$ in L^1_η, with $\tilde v \in \mathcal{C}$ and with $\|\tilde v - v_0\|_{L^1_\eta} \le \delta$. By again applying Theorem 5.2.3 we find that

$$G^{(1)}(\tilde v) \le \liminf_{\varepsilon\to0^+} G^{(1)}_\varepsilon(v_\varepsilon) \le \limsup_{\varepsilon\to0^+} G^{(1)}_\varepsilon(v_\varepsilon) \le G^{(1)}(v_0). \tag{5.3.2}$$

Theorem 5.2.5 then implies that $\tilde v = v_0$, which along with (5.3.2) implies (5.3.1). As $v_\varepsilon \to v_0$ in L^1_η we then have that the v_ε must be local minimizers of G_ε, for ε sufficiently small. This completes the proof. □

In light of the fact that the global minimizers of J_ε are local minimizers of G_ε for ε sufficiently small it is possible to identify the Euler–Lagrange equations.

Theorem 5.3.2. *Under the hypotheses of Proposition 5.3.1 the sequence $\{v_\varepsilon\}$ of global minimizers of the functionals J_ε will satisfy the following Euler–Lagrange equations (for ε sufficiently small):*

$$2\varepsilon^2(v_\varepsilon'(t)\eta(t))' - W'(v_\varepsilon(t))\eta(t) = \varepsilon\lambda_\varepsilon\eta(t), \tag{5.3.3}$$

where $\lambda_\varepsilon \in \mathbb{R}$. Moreover the Lagrange multipliers λ_ε satisfy

$$\lim_{\varepsilon\to 0^+} \lambda_\varepsilon = \lambda_0, \tag{5.3.4}$$

where λ_0 is the number given in (5.2.16).

Proof. Reasoning somewhat as in the proof of step 4 in [41] we have that $v_\varepsilon \in C^2(I)$ and satisfies (5.3.3). Next, we will prove (5.3.4), namely the limit of the Lagrange multipliers λ_ε. The argument here follows [74], with the necessary adaptations to the weighted setting.

To prove (5.3.4), fix some $\psi \in C_c^\infty(I)$. We multiply the Euler–Lagrange equations (5.3.3) by $\psi v_\varepsilon'$ and integrate to obtain

$$\varepsilon\lambda_\varepsilon \int_I \psi v_\varepsilon'\eta\,dt = \int_I (2\varepsilon^2(v_\varepsilon''\eta + v_\varepsilon'\eta') - W'(v_\varepsilon)\eta)\psi v_\varepsilon'\,dt.$$

Integrating by parts, we find that

$$\varepsilon\lambda_\varepsilon \int_I \psi v_\varepsilon'\eta\,dt = \int_I (W(v_\varepsilon) - \varepsilon^2 v_\varepsilon'^2)(\eta\psi)' + 2\varepsilon^2(v_\varepsilon')^2\eta'\psi\,dt. \tag{5.3.5}$$

By Theorem 5.2.3 and Proposition 5.3.1 we know that

$$\lim_{\varepsilon\to 0^+} \int_I (\varepsilon^{-1}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2)\eta\,dt = 2c_W\eta(t_0).$$

Furthermore, as in the proof of (5.2.8), by lower semicontinuity

$$\liminf_{\varepsilon\to 0^+} 2\int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\,dt = \liminf_{\varepsilon\to 0^+} 2\int_I |(\Phi(v_\varepsilon))'|\eta\,dt \geq 2c_W\eta(t_0), \tag{5.3.6}$$

where we recall that $\Phi(t) := \int_a^t W^{1/2}(s)\,ds$. These together give the following:

$$\begin{aligned} 0 &\leq \limsup_{\varepsilon\to 0^+} \int_I (\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}(v_\varepsilon'))^2\eta\,dt \\ &= \limsup_{\varepsilon\to 0^+} \int_I (\varepsilon^{-1}W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2 - 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|)\eta\,dt \leq 0. \end{aligned}$$

We thus have that $\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'|$ goes to zero in L^2_η. Moreover, the liminf in (5.3.6) is actually a limit and equality holds, so that

$$\lim_{\varepsilon\to 0^+} \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\,dt = c_W\eta(t_0). \tag{5.3.7}$$

Additionally, we can write the following:

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \int_I |\varepsilon^{-1}W(v_\varepsilon) - \varepsilon(v_\varepsilon')^2|\eta\,dt \\ &= \lim_{\varepsilon\to 0^+} \int_I \left|\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'|\right| \left|\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) + \varepsilon^{1/2}|v_\varepsilon'|\right| \eta\,dt \\ &\leq \lim_{\varepsilon\to 0^+} \left(\int_I \left(\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'|\right)^2 \eta\,dt\right)^{1/2} \\ &\quad\times \left(\int_I \left(\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) + \varepsilon^{1/2}|v_\varepsilon'|\right)^2 \eta\,dt\right)^{1/2} \\ &\leq \lim_{\varepsilon\to 0^+} C\left(\int_I \left(\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'|\right)^2 \eta\,dt\right)^{1/2} = 0, \end{aligned}$$

where we have used Hölder's inequality in the first inequality, Young's inequality and the boundedness of $G_\varepsilon^{(1)}(v_\varepsilon)$ in the second. By (5.1.13) we can deduce that $\varepsilon^{-1}W(v_\varepsilon) - \varepsilon(v_\varepsilon')^2$ goes to zero in $L^1_{\mathrm{loc}}(I)$. Thus by dividing (5.3.5) by ε, and recalling that ψ is compactly supported in I, we obtain

$$\lim_{\varepsilon\to 0^+} \lambda_\varepsilon \int_I \psi v_\varepsilon' \eta\, dt = \lim_{\varepsilon\to 0^+} 2\int_I \varepsilon(v_\varepsilon')^2 \eta' \psi\, dt.$$

We then use the L^2 convergence shown above to estimate the following

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \left| \int_I (\varepsilon(v_\varepsilon')^2 - W^{1/2}(v_\varepsilon)|v_\varepsilon'|)\eta'\psi\, dt \right| \\
&= \lim_{\varepsilon\to 0^+} \left| \int_I \varepsilon^{1/2}|v_\varepsilon'|(\varepsilon^{1/2}|v_\varepsilon'| - \varepsilon^{-1/2}W^{1/2}(v_\varepsilon))\eta'\psi\, dt \right| \\
&\le \lim_{\varepsilon\to 0^+} \left(\int_I \varepsilon(v_\varepsilon')^2 \left(\frac{\eta'\psi}{\eta}\right)^2 \eta\, dt \right)^{1/2} \left(\int_I (\varepsilon^{1/2}|v_\varepsilon'| - \varepsilon^{-1/2}W^{1/2}(v_\varepsilon))^2 \eta\, dt \right)^{1/2} = 0,
\end{aligned}$$

where we have used the fact that $\frac{\psi\eta'}{\eta}$ is uniformly bounded, since ψ has compact support in I.

Thus we can write the following:

$$\lim_{\varepsilon\to 0^+} \lambda_\varepsilon \int_I \psi v_\varepsilon' \eta\, dt = \lim_{\varepsilon\to 0^+} 2\int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta'\psi\, dt. \tag{5.3.8}$$

We know that $v_\varepsilon'\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} Dv_0 = (b-a)\delta_{t_0}$ and $W^{1/2}(v_\varepsilon)v_\varepsilon'\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} D(\Phi\circ v_0) = c_W\delta_{t_0}$, both in $(C_0(\overline{I}))'$. In turn, $W^{1/2}(v_\varepsilon)v_\varepsilon'\eta\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} c_W\eta(t_0)\delta_{t_0}$. In view of (5.3.7), it follows from Proposition 4.30 in [75] that $W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} c_W\eta(t_0)\delta_{t_0}$. Hence,

$$\lim_{\varepsilon\to 0^+} \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta'\psi\, dt = \lim_{\varepsilon\to 0^+} \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\frac{\eta'}{\eta}\psi\, dt = c_W\eta(t_0)\frac{\eta'(t_0)}{\eta(t_0)}\psi(t_0).$$

We thus take limits in (5.3.8) to find that

$$\lim_{\varepsilon\to 0^+} \lambda_\varepsilon(b-a)\psi(t_0)\eta(t_0) = 2\eta'(t_0)c_W\psi(t_0).$$

This then gives the desired conclusion, namely that (5.3.4) holds.

□

The next step is to establish tight bounds on the functions v_ε, as well as a Neumann condition.

Theorem 5.3.3. *Under the hypotheses of Proposition 5.3.1, for all $\varepsilon > 0$ sufficiently small the minimizers v_ε of J_ε satisfy*

$$a_\varepsilon \le v_\varepsilon(t) \le b_\varepsilon, \quad t \in I, \tag{5.3.9}$$
$$v_\varepsilon'(-T) = v_\varepsilon'(T) = 0, \tag{5.3.10}$$

where $a_\varepsilon < c_\varepsilon < b_\varepsilon$ are the only zeros of $W' + \lambda_\varepsilon\varepsilon$. Moreover

$$a_\varepsilon = a - \lambda_\varepsilon|\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q} + o(\varepsilon^{1/q}), \tag{5.3.11}$$
$$c_\varepsilon = c - \lambda_\varepsilon W''(c)^{-1}\varepsilon + o(\varepsilon), \tag{5.3.12}$$
$$b_\varepsilon = b - \lambda_\varepsilon|\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q} + o(\varepsilon^{1/q}), \tag{5.3.13}$$

where ℓ is given in (5.1.5).

Proof. By hypothesis (5.1.7), $|W'(s)| \geq w_0 > 0$ for all $|s| \geq C$. Since W' has only three zeros at a, b, c and is strictly monotonic in a ball centered at each of these points with radius $\zeta_0 > 0$ (see (5.1.5) and (5.1.6)), by taking w_0 smaller we can assume that $|W'(s)| \geq w_0$ for all $s \in \mathbb{R} \setminus (B(a,\zeta_0) \cup B(c,\zeta_0) \cup B(b,\zeta_0))$. By (5.3.4), $|\varepsilon\lambda_\varepsilon| \leq w_0/2$ for all $\varepsilon > 0$ small. Hence $W' + \varepsilon\lambda_\varepsilon$ has only three zeros

$$a_\varepsilon < b_\varepsilon < c_\varepsilon, \tag{5.3.14}$$

for all $\varepsilon > 0$ small. Furthermore by (5.1.6) and (5.1.10) we can derive the explicit forms in (5.3.11)-(5.3.13).

Next, consider the open set $U_\varepsilon := \{t \in I : v_\varepsilon(t) < a_\varepsilon\}$. We claim that U_ε is empty. Indeed, if not, let I_ε be a maximal subinterval of U_ε, and since $W'(v_\varepsilon) + \varepsilon\lambda_\varepsilon < 0$ for all $t \in I_\varepsilon$ by (5.3.3) we have that $(v_\varepsilon'(t)\eta(t))' < 0$ for all $t \in I_\varepsilon$. Since $\eta > 0$ on I by (5.1.13), this implies that v_ε' has at most one zero in $\overline{I_\varepsilon}$. Hence there exist $\lim_{t\to t_\varepsilon^+} v_\varepsilon(t) = \ell_\varepsilon$ and $\lim_{t\to T_\varepsilon^-} v_\varepsilon(t) = L_\varepsilon$, where $t_\varepsilon, T_\varepsilon$ are the left and right endpoints of I_ε, respectively. Note that $\ell_\varepsilon, L_\varepsilon$ could be infinite if one of the endpoints is $-T$ or T. Consider $\inf_{I_\varepsilon} v_\varepsilon$. If there exists $s_\varepsilon \in I_\varepsilon^\circ$ such that $v_\varepsilon(s_\varepsilon) = \inf_{I_\varepsilon} v_\varepsilon$, then $v_\varepsilon'(s_\varepsilon) = 0$ and $v_\varepsilon''(s_\varepsilon) \geq 0$. This is impossible, as $(v_\varepsilon'\eta)' < 0$ on I_ε. Thus it follows that $\inf_{I_\varepsilon} v_\varepsilon$ is either ℓ_ε or L_ε. Assume first that $\inf_{I_\varepsilon} v_\varepsilon = \ell_\varepsilon$. By the definition of I_ε it cannot be that $\ell_\varepsilon = a_\varepsilon$, but then, by the maximality of I_ε, necessarily $t_\varepsilon = -T$. By (5.3.3) for all $t_1, t_2 \in I_\varepsilon$, with $t_1 < t_2$:

$$2\varepsilon^2 v_\varepsilon'(t_2)\eta(t_2) - 2\varepsilon^2 v_\varepsilon'(t_1)\eta(t_1) = \int_{t_1}^{t_2} (W'(v_\varepsilon(s)) + \varepsilon\lambda_\varepsilon)\eta(s)\,ds. \tag{5.3.15}$$

Since $W'(v_\varepsilon(t)) + \varepsilon\lambda_\varepsilon < 0$ for all $t \in I_\varepsilon$, the integral $\int_{-T}^{t_2}(W'(v_\varepsilon(s)) + \varepsilon\lambda_\varepsilon)\eta(s)\,ds$ is well-defined in $\mathbb{R} \cup \{-\infty\}$. Hence, letting $t_1 \to -T^+$ in (5.3.15), it follows that there exists

$$\lim_{t\to -T^+} v_\varepsilon'(t)\eta(t) = M_\varepsilon \in \mathbb{R} \cup \{\infty\}. \tag{5.3.16}$$

Assume, for the sake of contradiction, that $M_\varepsilon \neq 0$. Then by (5.1.14) and (5.3.16), $|v_\varepsilon'(t)| \geq C_0(T+t)^{-n_1+1}$ for all $t \in (-T, -T+\delta_\varepsilon)$, for some $\delta_\varepsilon > 0$. It would then follow that

$$\int_{-T}^{-T+\delta_\varepsilon} |v_\varepsilon'|^2 \eta\,dt \geq d_1 \int_{-T}^{-T+\delta_\varepsilon} C_0^2 (T+t)^{-n_1+1}\,dt = \infty$$

if $n_1 \geq 2$. On the other hand, if $n_1 = 1$ then $v_\varepsilon'(-T) = 0$, since v_ε is a minimizer. Thus in both cases we must have that $M_\varepsilon = 0$. In turn, letting $t_1 \to -T^+$ in (5.3.15) it follows that $v_\varepsilon'(t) < 0$ for all $t \in I_\varepsilon$, which contradicts the fact that $\ell_\varepsilon = \inf_{I_\varepsilon} v_\varepsilon$. Using a similar argument we can exclude the case that $L_\varepsilon = \inf_{I_\varepsilon} v_\varepsilon$. This proves that I_ε, and in turn U_ε, is empty. Thus $v_\varepsilon \geq a_\varepsilon$ in I. Similarly, we can show that $v_\varepsilon \leq b_\varepsilon$ in I.

It remains to prove the Neumann boundary condition (5.3.10). If $n_i = 1$ then this comes from the minimality of v_ε. When $n_i \geq 2$, since v_ε is bounded by what we just proved, it follows that the integral on the right-hand side of (5.3.15) is bounded for all $t \in I$. Hence as in the first part of the proof we can conclude that the limit M_ε in (5.3.16) exists and must be zero. Hence letting $t_1 \to -T^+$ in (5.3.15) we obtain

$$2\varepsilon^2 v_\varepsilon'(t)\eta(t) = \int_{-T}^{t} (W'(v_\varepsilon) + \lambda_\varepsilon\varepsilon)\eta(s)\,ds.$$

Using again the fact that v_ε is bounded, along with (5.1.4) and (5.1.14), we have that

$$0 \leq 2\varepsilon^2 |v_\varepsilon'(t)| \leq \frac{C}{d_1(T+t)^{n_1-1}} \int_{-T}^{t} d_2(T+s)^{n_1-1}\,ds = \frac{Cd_2}{d_1 n_1}(T+t) \to 0$$

as $t \to -T^+$. A similar estimate holds near T. This completes the proof. □

The following theorem specifies the qualitative behavior of v_ε, which are global minimizers of J_ε. Despite the fact that $v_\varepsilon \to v_0 \in L^1_\eta$ by Proposition 5.3.1, v_ε need not be increasing. Indeed in the radial case $\eta(t) \equiv (t+T)^{n-1}$, on an unbounded domain and for n large, Ni [86] has shown that all positive solutions of (5.3.3) approach b_ε as $t \to \infty$ in an oscillatory way. The presence of possible oscillations makes the analysis significantly more involved. However, the overall idea of the proof is the same as the proof of Theorem 5.2.5.

Fix

$$\theta_i \in \left(\frac{1}{n_i}, \frac{1}{n_i - 1}\right), \quad i = 1, 2, \tag{5.3.17}$$

where n_i are the exponents given in (5.1.14) and (5.1.15). Let $k \in \mathbb{N}$ and define

$$O_\varepsilon := \{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : \ a_\varepsilon + \varepsilon^k \le v_\varepsilon(t) \le b_\varepsilon - \varepsilon^k\}, \tag{5.3.18}$$

with $c(n_i) := 0$ if $n_i = 1$ and 1 otherwise.

Theorem 5.3.4. *Assume that W satisfies (5.1.4)-(5.1.7), and that η satisfies (5.1.13)-(5.1.16). Let v_ε be a minimizer of J_ε. Write $I_0 := [-T + \mathfrak{r}_0, T - \mathfrak{r}_0]$, with $\mathfrak{r}_0 > 0$ a constant to be defined. Then for δ sufficiently small in (5.2.12) and for all $\varepsilon > 0$ sufficiently small the following properties hold:*

1. *$\Gamma_\varepsilon := O_\varepsilon \cap I_0$ has exactly one component $[T_1^\varepsilon, T_2^\varepsilon]$, with $v_\varepsilon(T_1^\varepsilon) = a_\varepsilon + \varepsilon^k$ and $v_\varepsilon(T_2^\varepsilon) = b_\varepsilon - \varepsilon^k$. Moreover, there exists $0 < \mathfrak{r}_1 < \mathfrak{r}_0$ so that $\Gamma_\varepsilon \subset B(t_0, \mathfrak{r}_1)$.*
2. *For every fixed ε, the points in Γ_ε where $v_\varepsilon = c_\varepsilon$ are at most distance $C\varepsilon$ apart, for some $C > 0$ independent of ε.*
3. *For $t \in (-T, T_1^\varepsilon)$ we have that $v_\varepsilon(t) \in [a_\varepsilon, a_\varepsilon + \varepsilon^k)$ except on a set of $\eta\mathcal{L}^1$ measure $o(\varepsilon)$. Similarly for $t \in (T_2^\varepsilon, T)$ we have that $v_\varepsilon(t) \in (b_\varepsilon - \varepsilon^k, b_\varepsilon]$ except on a set of $\eta\mathcal{L}^1$ measure $o(\varepsilon)$.*

The proof of this theorem requires a number of preliminary results. Let $\mathfrak{r}_0 > 0$ be chosen as in (5.2.10). As $v_\varepsilon \to v_0$ in L^1_η, by selecting a subsequence, it is safe to assume that $v_\varepsilon(t) \to v_0(t)$ for $\mathcal{L}^1$ a.e. $t \in I$. Hence, given

$$0 < \rho < \frac{1}{2}\min\{c - a, b - c\}, \tag{5.3.19}$$

there exists $\varepsilon_\rho > 0$ such that

$$|v_\varepsilon(T_1) - a| < \rho, \quad |v_\varepsilon(T_2) - a| < \rho, \quad |v_\varepsilon(T_3) - b| < \rho, \quad |v_\varepsilon(T_4) - b| < \rho \tag{5.3.20}$$

for all $0 < \varepsilon \le \varepsilon_\rho$ sufficiently small and some $T_1 \in (-T, -T + \mathfrak{r}_0)$, $T_2 \in (-T + 2\mathfrak{r}_0, t_0 - \mathfrak{r}_0)$, $T_3 \in (t_0 + \mathfrak{r}_0, T - 2\mathfrak{r}_0)$ and $T_4 \in (T - \mathfrak{r}_0, T)$. Fix $\varepsilon > 0$ sufficiently small so that (5.3.20) holds.

The first two lemmas are adapted from [102].

Lemma 5.3.5. *Let $s_0, s_1 > 0$ be such that $a_\varepsilon + s_0 < c_\varepsilon < b_\varepsilon - s_1$ for all $\varepsilon > 0$ sufficiently small. Fix any such ε. Let $I_\varepsilon \subseteq I$ be a non-empty maximal interval such that $a_\varepsilon + s_0 < v_\varepsilon(t) < b_\varepsilon - s_1$ for all $t \in I_\varepsilon$. Then there exists $t_\varepsilon \in \overline{I_\varepsilon}$ such that $v_\varepsilon(t_\varepsilon) = c_\varepsilon$.*

Proof. If not, then either $a_\varepsilon + s_0 \le v_\varepsilon(t) < c_\varepsilon$ for all $t \in \overline{I_\varepsilon}$ or $c_\varepsilon < v_\varepsilon(t) \le b_\varepsilon - s_1$ for all $t \in \overline{I_\varepsilon}$. Consider the second case. Then $W'(v_\varepsilon(t)) + \varepsilon\lambda_\varepsilon < 0$ for all $t \in I_\varepsilon$, and so by (5.3.3) we have that $(v_\varepsilon'(t)\eta(t))' < 0$ for all $t \in I_\varepsilon$. Let $\tilde{t} \in \overline{I_\varepsilon}$ be the point of minimum of v_ε in $\overline{I_\varepsilon}$. Reasoning as in the proof of (5.3.9), we have that $\tilde{t}$ cannot belong to I_ε, and so $\tilde{t} \in \partial I_\varepsilon$. If $\tilde{t} \in I$, then necessarily, $v_\varepsilon(\tilde{t}) = c_\varepsilon$, which contradicts the fact that $c_\varepsilon < v_\varepsilon(t) < b_\varepsilon - s_1$ for all $t \in \overline{I_\varepsilon}$. it follows that $\tilde{t} \in \{-T, T\}$. We can now continue as in the proof of (5.3.9) to exclude this possibility. $\square$

Lemma 5.3.6. *Let ρ be as in* (5.3.19) *and suppose that I_ε is a maximal subinterval of the set $\{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : v_\varepsilon(t) \ge c + \rho\}$. Then there exists a $\mu > 0$ such that we have the following estimate for all $t \in I_\varepsilon$:*

$$b_\varepsilon - v_\varepsilon(t) \le 2(b_\varepsilon - c - \rho)e^{-\mu d(t, I_\varepsilon^c)\varepsilon^{-1}}.$$

In addition an analogous bound holds for the set $\{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : v_\varepsilon(t) \le c - \rho\}$.

Here $d(t, E)$ is the distance from t to the set E and E^c is the complement of E (see Section 2.1).

Proof. First, we claim that there exists a μ such that for any $s \in [c + \rho, b_\varepsilon]$ the following inequality holds

$$-(W'(s) + \varepsilon\lambda_\varepsilon) \ge 2\mu^2(b_\varepsilon - s). \tag{5.3.21}$$

If $q = 1$ in (5.1.5), then also by (5.1.4) we have that $W \in C^2(\mathbb{R})$. Since $W''(b) > 0$ by continuity we have that $W''(s) \ge 2\mu^2 > 0$ for all $s \in B(b, R_1)$, for some $\mu \ne 0$, and $R_1 > 0$. It follows from (5.3.14) that

$$W'(s) + \varepsilon\lambda_\varepsilon = -\int_s^{b_\varepsilon} W''(r)\,dr \le -2\mu^2(b_\varepsilon - s)$$

for all $s \in B(b, R_1)$, with $s < b_\varepsilon$. Using the fact that $W' + \varepsilon\lambda_\varepsilon < 0$ in $(c_\varepsilon, b_\varepsilon)$ (see Theorem 5.3.3), and by taking μ smaller, if necessary, we can assume that

$$W'(s) + \varepsilon\lambda_\varepsilon \le -2\mu^2(b_\varepsilon - s)$$

for all $s \in [c + \rho, b_\varepsilon]$. Note that μ depends upon ρ but not on ε. On the other hand, if $0 < q < 1$ then since $\lim_{s\to b} W''(s) = \infty$ by (5.1.5), we can still assume that $W''(s) \ge \mu^2 > 0$ near b. Hence we can continue as before to conclude that (5.3.21) holds even in this case. This proves the claim.

Write $I_\varepsilon = [t_1, t_2]$ and define

$$\phi(t) := (b_\varepsilon - v_\varepsilon(t_1))e^{-\mu(t - t_1)\varepsilon^{-1}} + (b_\varepsilon - v_\varepsilon(t_2))e^{-\mu(t_2 - t)\varepsilon^{-1}} \tag{5.3.22}$$

with μ fixed by (5.3.21). We note that ϕ satisfies the following differential inequality:

$$\begin{aligned}(\phi'\eta)' &= \frac{\mu^2}{\varepsilon^2}\phi\eta + \frac{\mu}{\varepsilon}\eta'\left(-(b_\varepsilon - v_\varepsilon(t_1))e^{-\mu(t - t_1)\varepsilon^{-1}} + (b_\varepsilon - v_\varepsilon(t_2))e^{-\mu(t_2 - t)\varepsilon^{-1}}\right)\\ &\le \frac{1}{\varepsilon^2}\left(\mu^2 + \varepsilon\frac{|\eta'|}{\eta}\mu\right)\phi\eta.\end{aligned}$$

If $n_1 > 1$ in (5.1.14), then $c(n_1) = 1$ in (5.3.18) and so by (5.1.16),

$$\varepsilon\frac{|\eta'(t)|}{\eta(t)} \le \frac{\varepsilon d_5}{t + T} \le d_5\varepsilon^{1-\theta_1} \le \mu$$

for all $t \in [-T+\varepsilon^{\theta_1}, 0]$ and all ε sufficiently small. On the other hand, if $n_1 = 1$ in (5.1.14), then $c(n_1) = 0$ in (5.3.18) and so by (5.1.13) and (5.1.15), $\eta(t) \geq \eta_0 > 0$ for all $t \in [-T, 0]$. Thus,

$$\varepsilon \frac{|\eta'(t)|}{\eta(t)} \leq \varepsilon \frac{\max |\eta'|}{\eta_0} \leq \mu$$

for all $t \in [-T, 0]$ and all ε sufficiently small. Similar inequalities hold in $[0, T - c(n_2)\varepsilon^{\theta_2}]$. Thus in I_ε,

$$(\phi'\eta)' \leq 2\varepsilon^{-2}\mu^2\phi\eta. \tag{5.3.23}$$

We then set $g(t) := b_\varepsilon - v_\varepsilon(t)$ and using (5.3.3) and (5.3.21) we have that

$$(g'\eta)' = -\varepsilon^{-2}(W'(v_\varepsilon) + \varepsilon\lambda_\varepsilon)\eta \geq 2\varepsilon^{-2}\mu^2 g\eta. \tag{5.3.24}$$

We define $U := g - \phi$. By (5.3.22), (5.3.23) and (5.3.24), for ε small we have the following:

$$\begin{aligned} &(U'\eta)' \geq 2\varepsilon^{-2}\mu^2 U\eta, \\ &U(t_1) \leq 0, \quad U(t_2) \leq 0. \end{aligned}$$

The maximum principle implies that $U \leq 0$ for all $t \in I_\varepsilon$. Thus

$$b_\varepsilon - v_\varepsilon(t) \leq (b_\varepsilon - v_\varepsilon(t_1))e^{-\mu(t-t_1)\varepsilon^{-1}} + (b_\varepsilon - v_\varepsilon(t_2))e^{-\mu(t_2-t)\varepsilon^{-1}} \leq 2(b_\varepsilon - c - \rho))e^{-\mu\varepsilon^{-1}d(t, I_\varepsilon^c)}, \tag{5.3.25}$$

which is the desired result. □

Corollary 5.3.7. *Let ρ be as in* (5.3.19) *and let*

$$\begin{aligned} A_\varepsilon &:= \{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : \ a_\varepsilon + \varepsilon^k \leq v_\varepsilon(t) \leq c - \rho\}, \\ B_\varepsilon &:= \{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : \ c + \rho \leq v_\varepsilon(t) \leq b_\varepsilon - \varepsilon^k\}. \end{aligned}$$

Then for any maximal interval I_ε contained in $A_\varepsilon \cup B_\varepsilon$,

$$\operatorname{diam} I_\varepsilon \leq C\varepsilon|\log\varepsilon|$$

for all $\varepsilon > 0$ sufficiently small and for some constant $C > 0$ depending only on W, k, μ, ρ, where μ is given in Lemma 5.3.6.

Proof. Assume $(t_1, t_2) = I_\varepsilon^\circ \subset B_\varepsilon$. By Lemma 5.3.6 we have that for $t = \frac{t_1+t_2}{2}$:

$$\varepsilon^k \leq b_\varepsilon - v_\varepsilon(t) \leq 2(b_\varepsilon - c - \rho)e^{-\mu 2^{-1}(t_2-t_1)\varepsilon^{-1}},$$

which implies that $-\frac{\mu}{2}(t_2 - t_1)\varepsilon^{-1} \geq k\log\varepsilon - \log 2(b_\varepsilon - c - \rho)$, that is,

$$0 \leq t_2 - t_1 \leq 2\mu^{-1}k\varepsilon|\log\varepsilon| + 2\mu^{-1}\varepsilon\log 2(b_\varepsilon - c - \rho).$$

This shows that $\operatorname{diam} I_\varepsilon \leq C\varepsilon|\log\varepsilon|$. The proof for the case $I_\varepsilon \subset A_\varepsilon$ is similar, and we omit it. □

The next lemma is quoted from [102], which gives estimates on the size of certain sets. In what follows given a set E and $s > 0$ define the set

$$E^s := \{x \in \mathbb{R}^n : d(x, E) \leq s\} \tag{5.3.26}$$

Lemma 5.3.8. *Given a measurable set $A \subset \mathbb{R}^n$, for all numbers $0 < s_1 < s_2$ we have that*

$$\frac{\mathcal{L}^n(A^{s_2})}{\mathcal{L}^n(A^{s_1})} \leq C_n \left(\frac{s_2}{s_1}\right)^n,$$

where we are using the notation (5.3.26).

The next step is to establish an estimate on the derivative of v_ε.

Lemma 5.3.9. *There exists a constant $C > 0$ such that*

$$|v_\varepsilon'(t)| \leq C\varepsilon^{-1}$$

for all $t \in I$.

Proof. By (5.3.3) and the fact that $v_\varepsilon'(-T) = 0$,

$$2\varepsilon^2 v_\varepsilon'(t)\eta(t) = \int_{-T}^{t} (W'(v_\varepsilon(s)) + \varepsilon\lambda_\varepsilon)\eta(s)\, ds$$

for every $t \in \overline{I}$. In light of (5.1.13)-(5.1.14) we know that that there exist constants $c_1, c_2 > 0$ so that $c_1(T+t)^{n_1-1} \leq \eta(t) \leq c_2(T+t)^{n_1-1}$ for all $t \in [-T, T-t^*]$. Since v_ε is bounded by (5.3.9), this implies that

$$\begin{aligned} 2\varepsilon^2|v_\varepsilon'(t)| &\leq \frac{C}{\eta(t)}\int_{-T}^{t}\eta(s)\, ds \leq \frac{C}{c_1(T+t)^{n_1-1}}\int_{-T}^{t} c_2(T+s)^{n_1-1}\, ds \\ &= \frac{Cc_2}{c_1 n_1}(T+t) \end{aligned}$$

for all $t \in (-T, T-t^*)$. Using a similar argument in $(-T+t^*, T)$, we conclude that

$$\varepsilon^2|v_\varepsilon'(t)| \leq C\min\{T+t, T-t\}$$

for all $t \in I$. By (5.3.3), v_ε satisfies

$$2\varepsilon^2 v_\varepsilon''(t) + 2\varepsilon^2\frac{\eta'(t)}{\eta(t)}v_\varepsilon'(t) = W'(v_\varepsilon(t)) + \varepsilon\lambda_\varepsilon.$$

Using (5.1.16), (5.3.9) and the previous inequality we get

$$2\varepsilon^2|v_\varepsilon''(t)| \leq \left|\frac{\eta'(t)}{\eta(t)}\right| 2\varepsilon^2|v_\varepsilon'(t)| + C \leq C.$$

Next we use a classical interpolation result. Let $t \in I$ and consider $t_1 \in I$ with $|t - t_1| = \varepsilon$. By the mean value theorem $v_\varepsilon(t) - v_\varepsilon(t_1) = v_\varepsilon'(\theta)(t - t_1)$ and so by the fundamental theorem of calculus

$$v_\varepsilon'(t) = v_\varepsilon'(\theta) + \int_\theta^t v_\varepsilon''(s)\, ds = \frac{v_\varepsilon(t) - v_\varepsilon(t_1)}{t - t_1} + \int_\theta^t v_\varepsilon''(s)\, ds.$$

Again by (5.3.9) it follows that

$$|v_\varepsilon'(t)| \leq \frac{C}{\varepsilon} + \sup|v_\varepsilon''||t-\theta| \leq \frac{C}{\varepsilon} + \frac{C}{\varepsilon^2}\varepsilon.$$

This concludes the proof. □

With these lemmas it is now possible to prove Theorem 5.3.4. By way of notation, for every measurable subset $E \subset I$ and for every $v \in H_\eta^1$ satisfying $\|v - v_0\|_{L_\eta^1} \leq \delta$ and (5.1.3) we define the localized energy

$$J_\varepsilon^{(1)}(v; E) := \int_E \left(\frac{1}{\varepsilon}W(v) + \varepsilon(v')^2\right)\eta\, dt. \tag{5.3.27}$$

Figure 5.1 gives a visual representation of the notation used in the following proof.

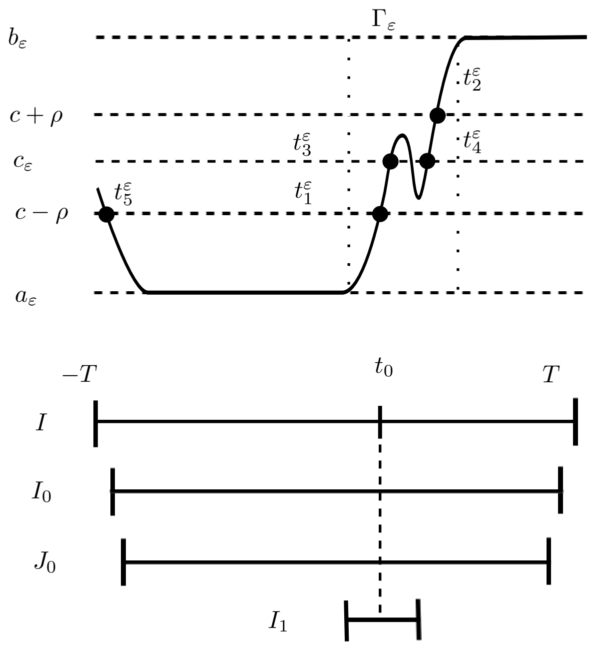

Figure 5.1: Important intervals and points for the proof of Theorem 5.3.4

Symbol	Definition	Characteristics
O_ε	(5.3.18)	Step 1 proves that $\mathcal{L}^1(O_\varepsilon) = o(1)$.
I_0	$[-T + \mathfrak{r}_0, T - \mathfrak{r}_0]$ (see statement of Theorem 5.3.4)	
J_0	$[-T + 2\mathfrak{r}_0, T - 2\mathfrak{r}_0]$ (see Step 2)	
I_1	$[t_0 - \hat{\mathfrak{r}}, t_0 + \hat{\mathfrak{r}}]$ (see (5.3.32))	
Γ_ε	A maximal subinterval of O_ε which intersects $\overline{B(t_0, \mathfrak{r}_1/2)}$	Existence proved in Step 3, uniqueness, endpoint values and width estimate in Step 4.
$t_1^\varepsilon, t_2^\varepsilon$	(5.3.39)	
$t_3^\varepsilon, t_4^\varepsilon$	The first and last time in Γ_ε where $v_\varepsilon = c_\varepsilon$ (see Step 3)	Step 3 proves that these are $O(\varepsilon)$ distance apart.
t_5^ε	The last point to the left of Γ_ε where $v_\varepsilon(t_5^\varepsilon) = c - \rho$	Step 5 proves that t_5^ε, if it exists, must be in $[-T, -T + c(n_1)\varepsilon^{\theta_1}]$.

Figure 5.2: Explanations of some of the notation in the proof of Theorem 5.3.4.

Proof of Theorem 5.3.4. By Theorem 5.2.6 there exists $\tilde{v}_\varepsilon$ converging to v_0 in L^1_η such that

$$G^{(1)}_\varepsilon(v_\varepsilon) = J^{(1)}_\varepsilon(v_\varepsilon) \le J^{(1)}_\varepsilon(\tilde{v}_\varepsilon) \le G^{(1)}_\varepsilon(\tilde{v}_\varepsilon) \le G^{(1)}(v_0) + C\varepsilon = 2c_W\eta(t_0) + C\varepsilon, \tag{5.3.28}$$

where we have used the fact that v_ε is a minimizer of J_ε. We fix

$$0 < \epsilon_1 < \min\left\{\frac{\eta(t_0)}{2}, \frac{\eta(t_0)}{2c_W}\int_c^{c+\rho} W^{1/2}(s)\,ds, \frac{\min\{c_-, c_+\}}{2c_W}\min_{I_0}\eta\right\}, \tag{5.3.29}$$

where

$$c_- := \int_a^c W^{1/2}(s)\ ds, \quad c_+ := \int_c^b W^{1/2}(s)\ ds. \tag{5.3.30}$$

By the continuity of η there exists $\mathfrak{r}_{\epsilon_1} > 0$ so that

$$|\eta(t) - \eta(t_0)| \le \epsilon_1 \tag{5.3.31}$$

for all $t \in [t_0 - \mathfrak{r}_{\epsilon_1}, t_0 + \mathfrak{r}_{\epsilon_1}]$. Pick $\hat{\mathfrak{r}} > 0$ so that

$$I_1 := [t_0 - \hat{\mathfrak{r}}, t_0 + \hat{\mathfrak{r}}] \subset I, \tag{5.3.32}$$

and let

$$\eta_1 := \min_{I_1}\eta > 0. \tag{5.3.33}$$

Choose $\mathfrak{r}_1$ so that

$$0 < \mathfrak{r}_1 < \min\{\mathfrak{r}_{\epsilon_1}, \hat{\mathfrak{r}}\}. \tag{5.3.34}$$

Fix δ so that

$$0 < \delta < (c - a - \rho)\frac{\eta(t_0)}{2}\mathfrak{r}_1. \tag{5.3.35}$$

Step 1: We claim that $\mathcal{L}^1(O_\varepsilon) = o(1)$ (see (5.3.18)). Define the set

$$D_\varepsilon := O_\varepsilon \cap v_\varepsilon^{-1}([c-\rho, c+\rho]\}).$$

By Lemma 5.3.9, $|v'_\varepsilon| \le C_0\varepsilon^{-1}$, and so, using the notation in (5.3.26), $(D_\varepsilon)^{l\varepsilon} \subset v_\varepsilon^{-1}([c-2\rho, c+2\rho])$, provided $0 < l \le \rho C_0^{-1}$. In turn

$$\begin{aligned}
\mathcal{L}^1((D_\varepsilon)^{l\varepsilon}) &\le \int_{\{c-2\rho\le v_\varepsilon\le c+2\rho\}} 1\,dt \\
&\le \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + \left(\min_{[c-2\rho,c+2\rho]} W\right)^{-1}\int_{-T+\varepsilon^{\theta_1}}^{T-\varepsilon^{\theta_2}} W(v_\varepsilon)\,dt \\
&\le \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\left(\varepsilon^{-\theta_1(n_1-1)} + \varepsilon^{-\theta_2(n_2-1)}\right)\int_{-T+\varepsilon^{\theta_1}}^{T-\varepsilon^{\theta_2}} W(v_\varepsilon)\eta\,dt \\
&\le \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\left(\varepsilon^{1-\theta_1(n_1-1)} + \varepsilon^{1-\theta_2(n_2-1)}\right),
\end{aligned} \tag{5.3.36}$$

where we have used (5.1.4), (5.1.13)-(5.1.15), (5.3.19) and (5.3.28).

Next we claim that

$$O_\varepsilon \subset (D_\varepsilon)^{C\varepsilon|\log\varepsilon|} \cup [-T, -T + c(n_1)\varepsilon^{\theta_1} + C\varepsilon|\log\varepsilon|] \cup [T - c(n_2)\varepsilon^{\theta_2} - C\varepsilon|\log\varepsilon|, T]. \tag{5.3.37}$$

Indeed, as $O_\varepsilon = A_\varepsilon \cup B_\varepsilon \cup D_\varepsilon$, it suffices to consider $\tilde{t} \in A_\varepsilon$, as the case $\tilde{t} \in B_\varepsilon$ is analogous. Let I_ε be the maximal subinterval of A_ε containing $\tilde{t}$. By Corollary 5.3.7, $\operatorname{diam} I_\varepsilon \le C\varepsilon|\log\varepsilon|$. If I_ε intersects D_ε, then $d(\tilde{t}, D_\varepsilon) \le \operatorname{diam} I_\varepsilon \le C\varepsilon|\log\varepsilon|$. Otherwise, since reasoning as in the proof of (5.3.9) and Lemma 5.3.5 it cannot happen that v_ε takes the value $b_\varepsilon - \varepsilon^k$ at both endpoints of I_ε, it follows that one of the endpoints of I_ε is $-T + c(n_1)\varepsilon^{\theta_1}$ or $T - c(n_2)\varepsilon^{\theta_2}$, say, $-T + c(n_1)\varepsilon^{\theta_1}$. Thus

$$d(\tilde{t}, [-T, -T + c(n_1)\varepsilon^{\theta_1}]) \leq C\varepsilon|\log\varepsilon|.$$

This proves (5.3.37).

By Lemma 5.3.8 and (5.3.36) we have that

$$\mathcal{L}^1((D_\varepsilon)^{C\varepsilon|\log\varepsilon|}) \leq C|\log\varepsilon|\mathcal{L}^1((D_\varepsilon)^{l\varepsilon}) \leq C|\log\varepsilon|\left(\varepsilon^{\theta_1} + \varepsilon^{\theta_2} + \varepsilon^{1-\theta_1(n_1-1)} + \varepsilon^{1-\theta_2(n_2-1)}\right).$$

Hence by (5.3.37) we have that

$$\begin{aligned}\mathcal{L}^1(O_\varepsilon) &\leq \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\varepsilon|\log\varepsilon| + \mathcal{L}^1((D_\varepsilon)^{C\varepsilon|\log\varepsilon|}) \\ &\leq C_1|\log\varepsilon|\left(\varepsilon^{\theta_1} + \varepsilon^{\theta_2} + \varepsilon^{1-\theta_1(n_1-1)} + \varepsilon^{1-\theta_2(n_2-1)}\right),\end{aligned}$$

where $C_1 > 0$ is *independent of* $\mathfrak{r}_0$.

Step 2: We claim if I_ε is a maximal subinterval of the set O_ε (see (5.3.18)) that intersects the interval $J_0 := [-T + 2\mathfrak{r}_0, T - 2\mathfrak{r}_0]$, then I_ε is contained in I_0 for all $\varepsilon > 0$ sufficiently small, with

$$\mathcal{L}^1(I_\varepsilon) \leq C\varepsilon|\log\varepsilon|. \tag{5.3.38}$$

The first part of the claim, namely, that $I_\varepsilon \subset I_0$, follows immediately from Step 1. Lemma 5.3.5 then implies that $I_\varepsilon \cap D_\varepsilon \neq \emptyset$. Reasoning as in the proof of (5.3.36) but using the fact that $\eta \geq \eta_0 > 0$ in I_0 we find that $\mathcal{L}^1((I_\varepsilon \cap D_\varepsilon)^{C\varepsilon}) < C\varepsilon$. Again due to the fact that $I_\varepsilon \subset I_0$, reasoning as in the proof of (5.3.37) we can show that $I_\varepsilon \subset (I_\varepsilon \cap D_\varepsilon)^{C\varepsilon|\log\varepsilon|}$. Using Lemma 5.3.8 once more gives (5.3.38).

Step 3: We claim that there exist t_1^ε, $t_2^\varepsilon \in \overline{B(t_0, \mathfrak{r}_1/2)}$ such that

$$v_\varepsilon(t_1^\varepsilon) \leq c - \rho, \quad v_\varepsilon(t_2^\varepsilon) \geq c + \rho \tag{5.3.39}$$

provided $\varepsilon > 0$ is sufficiently small. Indeed, if t_1^ε does not exist, then $c - \rho < v_\varepsilon$ in $\overline{B(t_0, \mathfrak{r}_1/2)}$, and so by (5.2.9),

$$\delta \geq \int_{\overline{B(t_0,\mathfrak{r}_1/2)}} |v_\varepsilon - v_0|\eta \, dt \geq (c - a - \rho)\frac{\eta(t_0)}{2}\mathfrak{r}_1,$$

where we used (5.3.29). This contradicts (5.3.35). Hence the t_1^ε in (5.3.39) exists, and with a similar argument we can prove the existence of t_2^ε.

Since v_ε is continuous, by the intermediate value theorem it will take all values between $c - \rho$ and $c + \rho$ in $\overline{B(t_0, \mathfrak{r}_1/2)}$. Let Γ_ε^- be a maximal subinterval of O_ε intersecting $\overline{B(t_0, \mathfrak{r}_1/2)}$ such that $v_\varepsilon(\Gamma_\varepsilon^-) \supset [c - \rho, c]$ and let Γ_ε^+ be a maximal subinterval of O_ε intersecting $\overline{B(t_0, \mathfrak{r}_1/2)}$ such that $v_\varepsilon(\Gamma_\varepsilon^+) \supset [c, c + \rho]$. By Step 1, for ε small enough, both intervals are contained in the interval I_1 given by (5.3.32).

We claim that either $v_\varepsilon(\Gamma_\varepsilon^-) = [a_\varepsilon + \varepsilon^k, b_\varepsilon - \varepsilon^k]$ or $v_\varepsilon(\Gamma_\varepsilon^+) = [a_\varepsilon + \varepsilon^k, b_\varepsilon - \varepsilon^k]$. Indeed, if this is not the case, then by the maximality of Γ_ε^- and Γ_ε^+, Lemma 5.3.5 and the definition of O_ε (see (5.3.18)) $v_\varepsilon = a_\varepsilon + \varepsilon^k$ at both endpoints of Γ_ε^- and $v_\varepsilon = b_\varepsilon - \varepsilon^k$ at both endpoints of Γ_ε^+. Let $t_\varepsilon \in \Gamma_\varepsilon^-$ be such that $v_\varepsilon(t_\varepsilon) = c$. Hence, by (5.3.27), (5.3.33), Young's inequality and a change of variables,

$$\begin{aligned}J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon^-) &\geq 2\eta_1 \int_{\Gamma_\varepsilon^-} W^{1/2}(v_\varepsilon)|v_\varepsilon'| \, dt \\ &= 2\eta_1 \int_{\Gamma_\varepsilon^- \cap (-T, t_\varepsilon]} W^{1/2}(v_\varepsilon)|v_\varepsilon'| \, dt + 2\eta_1 \int_{\Gamma_\varepsilon^- \cap (t_\varepsilon, T)} W^{1/2}(v_\varepsilon)|v_\varepsilon'| \, dt \\ &\geq 4\eta_1 \int_{a_\varepsilon + \varepsilon^k}^{c} W^{1/2}(s) \, ds \geq 4c_-\eta_1 - C\varepsilon^{(q+3)/2q},\end{aligned} \tag{5.3.40}$$

where we have used (5.3.30) and the fact that

$$\int_a^{a_\varepsilon+\varepsilon^k} W^{1/2}(s)\ ds \leq C|a-a_\varepsilon-\varepsilon^k|^{(q+3)/2} \leq C\varepsilon^{(q+3)/2q}$$

by (5.1.9) and (5.3.9) where here C is independent of $\mathfrak{r}_0$. A similar inequality holds for $J_\varepsilon^{(1)}(v_\varepsilon;\Gamma_\varepsilon^+)$ with the only difference that c_- should be replaced by c_+. Hence, also by (5.2.9) and (5.3.28),

$$2c_W\eta(t_0)+C\varepsilon \geq J_\varepsilon^{(1)}(v_\varepsilon;\Gamma_\varepsilon^-)+J_\varepsilon^{(1)}(v_\varepsilon;\Gamma_\varepsilon^+) \geq 4c_W(\eta(t_0)-\epsilon_1)-C\varepsilon^{(q+3)/2q},$$

which gives

$$C\varepsilon \geq 2(\eta(t_0)-2\epsilon_1)c_W.$$

This contradicts (5.3.29) provided ε is sufficiently small. This proves the claim. We denote by Γ_ε a maximal subinterval of O_ε intersecting $\overline{B(t_0,\mathfrak{r}_1/2)}$ such that $v_\varepsilon(\Gamma_\varepsilon)=[a_\varepsilon+\varepsilon^k, b_\varepsilon-\varepsilon^k]$.

First we claim that v_ε takes the values $a_\varepsilon+\varepsilon^k$ and $b_\varepsilon-\varepsilon^k$ on the endpoints of Γ_ε. If not then reasoning as in (5.3.40) we would have

$$J_\varepsilon^{(1)}(v_\varepsilon;\Gamma_\varepsilon) \geq 4c_W\eta_1 - C\varepsilon^{(q+3)/2}$$

which is a contradiction. Next let t_3^ε and t_4^ε be the first time and last time in Γ_ε that v_ε equals c_ε. We claim that

$$t_4^\varepsilon - t_3^\varepsilon \leq C_2\varepsilon, \tag{5.3.41}$$

for some constant $C_2>0$ *independent of* $\mathfrak{r}_0$, for all ε sufficiently small. Indeed, if $v_\varepsilon(t)\in[c-\rho,c+\rho]$ for all $t\in[t_3^\varepsilon,t_4^\varepsilon]$, then by (5.2.9),

$$J_\varepsilon^{(1)}(v_\varepsilon;[t_3^\varepsilon,t_4^\varepsilon]) \geq \varepsilon^{-1}\frac{\eta(t_0)}{2}(t_4^\varepsilon-t_3^\varepsilon)\min_{[c-\rho,c+\rho]} W,$$

and so (5.3.41) follows from (5.3.28), where all the constants appearing are independent of $\mathfrak{r}_0$. On the other hand if there exists $\tilde{t}^\varepsilon\in[t_3^\varepsilon,t_4^\varepsilon]$ such that $|v_\varepsilon(\tilde{t}^\varepsilon)-c|\geq\rho$, say, $v_\varepsilon(\tilde{t}^\varepsilon)\geq c+\rho$, then by Young's inequality, Step 1, (5.3.29), (5.3.31) and a change of variables we get

$$J_\varepsilon^{(1)}(v_\varepsilon;[t_3^\varepsilon,t_4^\varepsilon]) \geq 2\frac{\eta(t_0)}{2}\int_c^{c+\rho} W^{1/2}(s)\ ds - C\varepsilon^{(q+3)/2q}.$$

Furthermore, by again reasoning as in (5.3.40), and using the fact that v_ε takes the values $a_\varepsilon+\varepsilon^k$ and $b_\varepsilon-\varepsilon^k$ on the endpoints of Γ_ε we have that

$$J_\varepsilon^{(1)}(v_\varepsilon;\Gamma_\varepsilon\backslash[t_3^\varepsilon,t_4^\varepsilon]) \geq 2\eta_1\int_{a_\varepsilon+\varepsilon^k}^{b_\varepsilon-\varepsilon^k} W^{1/2}(s)\ ds \geq 2c_W\eta_1 - C\varepsilon^{(q+3)/2q}, \tag{5.3.42}$$

with C independent of $\mathfrak{r}_0$.

Hence, by (5.2.9), (5.3.28), and (5.3.42),

$$\begin{aligned} 2c_W\eta(t_0)+C\varepsilon &\geq J_\varepsilon^{(1)}(v_\varepsilon;\Gamma_\varepsilon\backslash[t_3^\varepsilon,t_4^\varepsilon])+J_\varepsilon^{(1)}(v_\varepsilon;[t_3^\varepsilon,t_4^\varepsilon]) \\ &\geq 2c_W(\eta(t_0)-\epsilon_1)+\eta(t_0)\int_c^{c+\rho} W^{1/2}(s)\ ds - C\varepsilon^{(q+3)/2q}, \end{aligned}$$

which gives

$$C\varepsilon \geq \eta(t_0)\int_c^{c+\rho} W^{1/2}(s)\ ds - 2c_W\epsilon_1,$$

which contradicts (5.3.29), provided ε is sufficiently small. The case where $v_\varepsilon(\tilde{t}^\varepsilon) \le c - \rho$ is analogous.
Step 4: We claim that for all $\varepsilon > 0$ sufficiently small, Γ_ε is the only maximal subinterval of the set O_ε that intersects the interval J_0 defined in Step 2. Indeed, assume that there exists another maximal subinterval I_ε of O_ε that intersects J_0. By Step 1, $I_\varepsilon \subset I_0$ and (5.3.38) holds. In view of Lemma 5.3.5 there exists $t_\varepsilon \in I_\varepsilon$ such that $v_\varepsilon(t_\varepsilon) = c_\varepsilon$. Since I_ε is a maximal interval of O_ε at one of the endpoints it attains either the value $a_\varepsilon + \varepsilon^k$ or $b_\varepsilon - \varepsilon^k$. In the first case, reasoning as in (5.3.40), we get

$$\begin{aligned} J_\varepsilon^{(1)}(v_\varepsilon; I_\varepsilon) &\ge 2 \min_{I_\varepsilon} \eta \int_{I_\varepsilon} W^{1/2}(v_\varepsilon)|v_\varepsilon'|\, dt \ge 2 \min_{I_\varepsilon} \eta \int_{a_\varepsilon+\varepsilon^k}^{c_\varepsilon} W^{1/2}(s)\, ds \\ &\ge 2c_- \min_{I_\varepsilon} \eta - C|c - c_\varepsilon| - C\varepsilon^{(q+3)/2q}. \end{aligned}$$

A similar inequality holds in the second case, with c_+ in place of c_-. Hence, by (5.2.9), (5.3.28), and by (5.3.42),

$$\begin{aligned} 2c_W\eta(t_0) + C\varepsilon &\ge J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) + J_\varepsilon^{(1)}(v_\varepsilon; I_\varepsilon) \\ &\ge 2c_W \min_{\Gamma_\varepsilon} \eta + 2\min\{c_-, c_+\} \min_{I_\varepsilon} \eta - C\varepsilon \\ &\ge 2c_W(\eta(t_0) - \epsilon_1) + 2\min\{c_-, c_+\} \min_{I_0} \eta - C\varepsilon, \end{aligned}$$

which gives

$$C\varepsilon \ge 2\min\{c_-, c_+\} \min_{I_0} \eta - 2c_W\epsilon_1,$$

which contradicts (5.3.29) provided ε is sufficiently small.

This proves that Γ_ε is the only maximal subinterval of O_ε that intersects J_0. In view of (5.3.20) it follows that v_ε takes the value $a_\varepsilon + \varepsilon^k$ on its left endpoint of Γ_ε and $b_\varepsilon - \varepsilon^k$ on the right endpoint. Indeed, if v_ε takes the value $b_\varepsilon - \varepsilon^k$ at the left endpoint of Γ_ε then since $v_\varepsilon(T_2) < a + \rho$ by (5.3.20), then Γ_ε could not be the only maximal subinterval of O_ε intersecting J_0. At this point we have established parts (i) and (ii) of our theorem.

Next we show that

$$\mathcal{L}^1(\Gamma_\varepsilon) \le C_3\varepsilon|\log\varepsilon|, \tag{5.3.43}$$

for some constant $C_3 > 0$ *independent of* $\mathfrak{r}_0$. By Step 1, and the fact that Γ_ε intersects $\overline{B(t_0, \mathfrak{r}_1/2)}$, we have that $\Gamma_\varepsilon \subset B(t_0, \mathfrak{r}_1)$ for ε sufficiently small, where $\mathfrak{r}_1$ is given in (5.3.34). By (5.3.33) and (5.3.34), we have that $\eta \ge \eta_1 > 0$ on Γ_ε, with η_1 independent of $\mathfrak{r}_0$. The argument in Step 2 then implies (5.3.43).
Step 5: We claim that $v_\varepsilon < c - \rho$ in $[-T + c(n_1)\varepsilon^{\theta_1}, -T + 2\mathfrak{r}_0]$. We first consider the case where $n_1 > 1$ in (5.1.14). Suppose the claim does not hold. By (5.3.20), $v_\varepsilon(T_1) < a + \rho$ for ε sufficiently small and where $T_1 \in (-T, -T + \mathfrak{r}_0)$. By the intermediate value theorem there exists a point in $(T_1, -T + 2\mathfrak{r}_0)$ where v_ε takes the value $c - \rho$. Since $-T + \varepsilon^{\theta_1} < T_1$ for ε sufficiently small, we have that v_ε takes the value $c - \rho$ in $[-T + \varepsilon^{\theta_1}, -T + 2\mathfrak{r}_0]$. Let t_5^ε be the last time in $[-T + \varepsilon^{\theta_1}, -T + 2\mathfrak{r}_0]$ such that $v_\varepsilon(t_5^\varepsilon) = c - \rho$. We claim that

$$|t_3^\varepsilon - t_0| \le C_4(\varepsilon|\log\varepsilon| + (T + t_5^\varepsilon)^{n_1}), \tag{5.3.44}$$

for some $C_4 > 0$ independent of $\mathfrak{r}_0$, where we recall that t_3^ε and t_4^ε are the first time and last time in Γ_ε that v_ε equals c_ε. If $t_3^\varepsilon \le t_0 \le t_4^\varepsilon$, then this follows from (5.3.41). Assume next that $t_0 < t_3^\varepsilon$. Then from (5.1.3),

$$0 = \int_I (v_\varepsilon - v_0)\eta\, dt = \int_{-T}^{t_0} (v_\varepsilon - a)\eta\, dt + \int_{t_0}^{t_3^\varepsilon} (v_\varepsilon - b)\eta\, dt + \int_{t_3^\varepsilon}^{T} (v_\varepsilon - b)\eta\, dt. \tag{5.3.45}$$

By (5.2.9),

$$0 < \frac{\eta(t_0)}{2}(b - c_\varepsilon)(t_3^\varepsilon - t_0) \le \int_{t_0}^{t_3^\varepsilon} (b - v_\varepsilon)\eta \, dt \tag{5.3.46}$$
$$= \int_{-T}^{t_0} (v_\varepsilon - a)\eta \, dt + \int_{t_3^\varepsilon}^{T} (v_\varepsilon - b)\eta \, dt.$$

We now estimate the two terms on the right-hand side of (5.3.46). By (5.3.9) and (5.3.13),

$$\int_{t_3^\varepsilon}^{T} (v_\varepsilon - b)\eta \, dt \le |b_\varepsilon - b| 2T \max \eta \le C\varepsilon^{1/q}, \tag{5.3.47}$$

where C is independent of $\mathfrak{r}_0$. We decompose the interval $[-T, t_0]$ as follows

$$[-T, t_0] = [-T, t_5^\varepsilon] \cup [t_5^\varepsilon, -T+2\mathfrak{r}_0] \cup ([-T+2\mathfrak{r}_0, t_0] \setminus \Gamma_\varepsilon) \cup ([-T+2\mathfrak{r}_0, t_0] \cap \Gamma_\varepsilon), \tag{5.3.48}$$

and estimate the integrals over each of these subintervals. By (5.1.14), (5.3.9), and (5.3.13),

$$\int_{-T}^{t_5^\varepsilon} (v_\varepsilon - a)\eta \, dt \le (b_\varepsilon - a) d_2 \int_{-T}^{t_5^\varepsilon} (T + t)^{n_1 - 1} \, dt \le 2(b - a) d_2 (T + t_5^\varepsilon)^{n_1}. \tag{5.3.49}$$

Let $Q_\varepsilon := [t_5^\varepsilon, -T + 2\mathfrak{r}_0] \cap O_\varepsilon$. Since $v_\varepsilon(t_5^\varepsilon) = c - \rho$, we have that $t_5^\varepsilon \in Q_\varepsilon$. Since t_5^ε is the last time in $[-T + \varepsilon^{\theta_1}, -T + 2\mathfrak{r}_0]$ such that v_ε takes the value $c - \rho$, and since, by Step 4, $v_\varepsilon(-T + 2\mathfrak{r}_0) \le a_\varepsilon + \varepsilon^k$ for ε small, it must be that $v_\varepsilon < c - \rho$ in $(t_5^\varepsilon, -T + 2\mathfrak{r}_0]$. By Corollary 5.3.7, we get that

$$\mathcal{L}^1(Q_\varepsilon) \le C\varepsilon |\log \varepsilon|, \tag{5.3.50}$$

with C independent of $\mathfrak{r}_0$. Thus by (5.1.13) and (5.3.9),

$$\int_{Q_\varepsilon} (v_\varepsilon - a)\eta \, dt \le C\varepsilon |\log \varepsilon| \tag{5.3.51}$$

with C independent of $\mathfrak{r}_0$. On the other hand, since $v_\varepsilon \le a_\varepsilon + \varepsilon^k$ in $[t_5^\varepsilon, -T+2\mathfrak{r}_0] \setminus Q_\varepsilon$, by (5.3.9) and (5.3.11),

$$\int_{[t_5^\varepsilon, -T+2\mathfrak{r}_0] \setminus Q_\varepsilon} (v_\varepsilon - a)\eta \, dt \le |a_\varepsilon + \varepsilon^k - a| d_2 \int_{-T}^{-T+2\mathfrak{r}_0} (T + t)^{n_1 - 1} \, dt \le C\mathfrak{r}_0^{n_1} \varepsilon^{1/q}, \tag{5.3.52}$$

with C independent of $\mathfrak{r}_0$. Since the set O_ε intersects the interval J_0 only in Γ_ε by Step 3, and as $t_0 < t_3^\varepsilon$, we have that $v_\varepsilon \le a_\varepsilon + \varepsilon^k$ in $[-T + 2\mathfrak{r}_0, t_0] \setminus \Gamma_\varepsilon$. Hence, by (5.3.9) and (5.3.11),

$$\int_{[-T+2\mathfrak{r}_0, t_0] \setminus \Gamma_\varepsilon} (v_\varepsilon - a)\eta \, dt \le |a_\varepsilon + \varepsilon^k - a| 2T \max \eta \le C\varepsilon^{1/q}, \tag{5.3.53}$$

with C again independent of $\mathfrak{r}_0$. Again by Step 3, $[-T+2\mathfrak{r}_0, t_0] \cap \Gamma_\varepsilon = [t_0 - \mathfrak{r}_1, t_0] \cap \Gamma_\varepsilon$. Hence, by (5.3.9) and (5.3.43),

$$\int_{[t_0 - \mathfrak{r}_1, t_0] \cap \Gamma_\varepsilon} (v_\varepsilon - a)\eta \, dt \le C\varepsilon |\log \varepsilon|, \tag{5.3.54}$$

for C independent of $\mathfrak{r}_0$. Combining the inequalities (5.3.46), (5.3.47), (5.3.48), (5.3.49), (5.3.50), (5.3.51), (5.3.52), (5.3.53) and (5.3.54) gives

$$\frac{\eta(t_0)}{2}(b - c_\varepsilon)(t_3^\varepsilon - t_0) \le C\varepsilon |\log \varepsilon| + 2(b - a) d_2 (T + t_5^\varepsilon)^{n_1},$$

with C independent of $\mathfrak{r}_0$, which implies (5.3.44) in the case $t_0 < t_3^\varepsilon$.

It remains to prove (5.3.44) in the case $t_4^\varepsilon < t_0$. Then (5.3.45) should be replaced by

$$0 = \int_{-T}^{T} (v_\varepsilon - v_0)\eta \; dt = \int_{-T}^{t_4^\varepsilon} (v_\varepsilon - a)\eta \; dt + \int_{t_4^\varepsilon}^{t_0} (v_\varepsilon - a)\eta \; dt + \int_{t_0}^{T} (v_\varepsilon - b)\eta \; dt$$

and (5.3.46) by

$$0 < \frac{\eta(t_0)}{2}(c_\varepsilon - a)(t_0 - t_4^\varepsilon) \le \int_{t_4^\varepsilon}^{t_0} (v_\varepsilon - a)\eta \; dt \le \int_{t_0}^{T} (b - v_\varepsilon)\eta \; dt + \int_{-T}^{t_4^\varepsilon} (a - v_\varepsilon)\eta \; dt.$$

By (5.3.9) and (5.3.11),

$$\int_{-T}^{t_4^\varepsilon} (a - v_\varepsilon)\eta \; dt \le |a - a_\varepsilon| 2T \le C\varepsilon^{1/q},$$

with C independent of $\mathfrak{r}_0$. The integral $\int_{t_0}^{T} (b - v_\varepsilon)\eta \; dt$ can be estimated as in the case $t_0 < t_3^\varepsilon$. We omit the details. Hence, we have shown that (5.3.44) holds in all cases.

Since $t_3^\varepsilon \in \Gamma_\varepsilon$, by (5.3.43) and (5.3.44), it follows that for any $t \in \Gamma_\varepsilon$,

$$|t - t_0| \le |t - t_3^\varepsilon| + |t_3^\varepsilon - t_0| \le C_5(\varepsilon|\log\varepsilon| + (T + t_5^\varepsilon)^{n_1}),$$

where $C_5 > 0$ is independent of $\mathfrak{r}_0$. In turn, by the mean value theorem

$$\begin{aligned} \eta(t) &= \eta(t_0) + \eta'(\theta)(t - t_0) \ge \eta(t_0) - M_0|t - t_0| \\ &\ge \eta(t_0) - C_5 M_0(\varepsilon|\log\varepsilon| + (T + t_5^\varepsilon)^{n_1}), \end{aligned}$$

where we recall that $M_0 = \max|\eta'| + 1$. Hence, also by (5.3.42) we get

$$J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) \ge 2c_W \min_{\Gamma_\varepsilon} \eta - C\varepsilon^{(q+3)/2q} \ge 2c_W\eta(t_0) - C_6(\varepsilon|\log\varepsilon| + (T + t_5^\varepsilon)^{n_1})$$

with $C_6 > 0$ independent of $\mathfrak{r}_0$. On the other hand, since $v_\varepsilon(t_5^\varepsilon) = c - \rho$, there exists a maximal subinterval S_ε of Q_ε that contains t_5^ε. As argued just before (5.3.50), it must be that $v_\varepsilon(S_\varepsilon) \supset [a_\varepsilon + \varepsilon^k, c - \rho]$, and so reasoning as in (5.3.40), by (5.1.14), which can be applied since $2\mathfrak{r}_0 < t^*$ by (5.2.10) and (5.3.50) holds,

$$\begin{aligned} J_\varepsilon^{(1)}(v_\varepsilon; S_\varepsilon) &\ge 2 \min_{S_\varepsilon} \eta \int_{a_\varepsilon + \varepsilon^k}^{c-\rho} W^{1/2}(s) \; ds \\ &\ge 2d_1(T + t_5^\varepsilon)^{n_1 - 1} \int_{a+\rho}^{c-\rho} W^{1/2}(s) \; ds, \end{aligned}$$

for $\varepsilon > 0$ small enough. Combining these last two estimates, it follows from (5.3.28) that

$$\begin{aligned} 2c_W\eta(t_0) + C\varepsilon \ge J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) + J_\varepsilon^{(1)}(v_\varepsilon; S_\varepsilon) &\ge 2c_W\eta(t_0) - C_6(\varepsilon|\log\varepsilon| + (T + t_5^\varepsilon)^{n_1}) \\ &\quad + 2d_1(T + t_5^\varepsilon)^{n_1 - 1} \int_{a+\rho}^{c-\rho} W^{1/2}(s) \; ds, \end{aligned}$$

which gives

$$C\varepsilon|\log\varepsilon| \ge (T + t_5^\varepsilon)^{n_1 - 1} \left(2d_1 \int_{a+\rho}^{c-\rho} W^{1/2}(s) \; ds - C_6(T + t_5^\varepsilon) \right).$$

Since $-T + \varepsilon^{\theta_1} \le t_5^\varepsilon \le -T + 2\mathfrak{r}_0$, by taking

$$0 < \mathfrak{r}_0 < \frac{d_1}{C_6} \int_{a+\rho}^{c-\rho} W^{1/2}(s)\; ds,$$

we get a contradiction, since $\theta_1(n_1 - 1) < 1$ by (5.3.17).

Finally we consider the case where $n_1 = 1$. In this case we can use energy estimates, as in Step 4, the fact that $\eta \ge C > 0$ on $[-T, -T + 2\mathfrak{r}_0]$, and Lemma 5.3.5 to show that $v_\varepsilon(t) < a_\varepsilon + \varepsilon^k$ on the interval $[-T, -T + 2\mathfrak{r}_0]$. We omit the details.

Step 6: Finally, we prove the last claim in our theorem. We write $\Gamma_\varepsilon = [T_1^\varepsilon, T_2^\varepsilon]$. By the remark at the end of Step 5, in the case $n_1 = 1$ we are already done, so we only need to consider the case $n_1 > 1$. In view of Step 5 we can use the barrier method in Lemma 5.3.6 to show that for $t \in [-T + \varepsilon^{\theta_1}, T_1^\varepsilon]$

$$|v_\varepsilon(t) - a_\varepsilon| \le Ce^{-\mu\varepsilon^{-1} d(t, \{-T+\varepsilon^{\theta_1}, T_1^\varepsilon\})}$$

This clearly implies that $v_\varepsilon(t) \in [a_\varepsilon, a_\varepsilon + \varepsilon^k)$ for all $t \in (-T + \varepsilon^{\theta_1} + 2k\mu^{-1}\varepsilon|\log\varepsilon|, T_1^\varepsilon)$. Using (5.1.14) we then estimate the η measure of the remaining set as follows:

$$\int_{-T}^{-T+\varepsilon^{\theta_1}+2k\mu^{-1}\varepsilon|\log\varepsilon|} \eta\, dt \le \frac{d_2}{n_1}(\varepsilon^{\theta_1} + C\varepsilon|\log\varepsilon|)^{n_1} \le C\varepsilon^{n_1\theta_1}$$

Since $n_1\theta_1 > 1$ by (5.3.17), then we have the desired estimate. Thus the result holds to the left of T_1^ε. We can use the same argument to the right of T_2^ε to obtain the desired result. □

5.4 Second-Order Γ-limit

This section is devoted to proving the lim inf counterpart of Theorem 5.2.6.

Theorem 5.4.1. *Assume that W satisfies* (5.1.4)-(5.1.7) *and that η satisfies* (5.1.13)-(5.1.16) *and let v_0 and v_ε be given in Theorems 5.2.5 and 5.3.1 respectively. Then*

$$\liminf_{\varepsilon\to 0^+} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \ge 2\eta'(t_0)(\tau_0 c_W + c_{\mathrm{sym}}) + \begin{cases} \frac{\lambda_0^2}{2W''(a)}\int_I \eta\, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases} \tag{5.4.1}$$

Note that Theorems 5.2.6 and 5.4.1 together provide a second-order asymptotic development by Γ-convergence for the functionals J_ε defined in (5.2.12). To prove Theorem 5.4.1 it is convenient to rescale the functionals G_ε. Define

$$H_\varepsilon(w) := \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (W(w(s)) + (w'(s))^2)\eta_\varepsilon(s)\, ds \tag{5.4.2}$$

for all $w \in H^1_{\eta_\varepsilon}((A\varepsilon^{-1}, B\varepsilon^{-1}))$ such that

$$\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} |w(s) - \mathrm{sgn}_{a,b}(s)|\eta_\varepsilon(s)\, ds \le \frac{\delta}{\varepsilon}, \quad \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w(s) - \mathrm{sgn}_{a,b}(s))\eta_\varepsilon(s)\, ds = 0, \tag{5.4.3}$$

where $A = -T - t_0$, $B = T - t_0$ and

$$\eta_\varepsilon(s) := \eta(t_0 + \varepsilon s). \tag{5.4.4}$$

Observe that s is obtained by shifting our variables so that t_0 moves to zero and scaling by ε^{-1}, which in view of (5.4.3) implies that minimizers of H_ε are precisely rescaled versions of minimizers of J_ε. Thus it is natural to study the behavior of minimizers w_ε of H_ε. The first step is to prove a bound on the locations where $w_\varepsilon = c_\varepsilon$, in the region close to $t = 0$.

Lemma 5.4.2. *Let w_ε be a minimizer of H_ε, and let $\tau_\varepsilon \in B(0, \mathfrak{r}_1\varepsilon^{-1})$ satisfy $w_\varepsilon(\tau_\varepsilon) = c_\varepsilon$, with $\mathfrak{r}_1$ as in Theorem 5.3.4 (i). Then we have that*

$$|\tau_\varepsilon| \leq C$$

for all $\varepsilon > 0$ sufficiently small and for some constant $C > 0$ independent of ε.

Proof. This proof essentially combines the mass constraint with the exponential decay to obtain the desired bounds.

Let s_1^ε be the first time in $[-\mathfrak{r}_1\varepsilon^{-1}, \mathfrak{r}_1\varepsilon^{-1}]$ so that $w_\varepsilon(s_1^\varepsilon) = c - \rho$, and s_4^ε be the last time in $[-\mathfrak{r}_1\varepsilon^{-1}, \mathfrak{r}_1\varepsilon^{-1}]$ so that $w_\varepsilon(s_4^\varepsilon) = c + \rho$. Then let s_2^ε and s_3^ε be the first and last times in $[-\mathfrak{r}_1\varepsilon^{-1}, \mathfrak{r}_1\varepsilon^{-1}]$ where w_ε takes the value c_ε. We note that such points exist by Theorem 5.3.4 (i). Furthermore, by Theorem 5.3.4 (ii) we know that $s_3^\varepsilon - s_2^\varepsilon \leq C$ and that $-\mathfrak{r}_1\varepsilon^{-1} < s_1^\varepsilon < s_2^\varepsilon \leq s_3^\varepsilon < s_4^\varepsilon < \mathfrak{r}_1\varepsilon^{-1}$. Furthermore, using the same argument from the proof of (5.3.9) we know that $w_\varepsilon([s_1^\varepsilon, s_2^\varepsilon]) = [c - \rho, c_\varepsilon]$, and that $w_\varepsilon([s_3^\varepsilon, s_4^\varepsilon]) = [c_\varepsilon, c + \rho]$. We can then estimate the following:

$$(s_2^\varepsilon - s_1^\varepsilon) \inf_{B(t_0,\mathfrak{r}_1)} \eta \inf_{(c-\rho,c+\rho)} W \leq \int_{s_1^\varepsilon}^{s_2^\varepsilon} W(w_\varepsilon)\eta_\varepsilon \, ds \leq C.$$

This, along with a similar estimate for $s_4^\varepsilon - s_3^\varepsilon$, then implies that $s_4^\varepsilon - s_1^\varepsilon \leq C$. Thus if we can prove that the s_1^ε are bounded above and that the s_4^ε are bounded below then we are done.

Suppose, for the sake of contradiction that the s_1^ε are not bounded above. By taking a subsequence as necessary we may assume that $s_1^\varepsilon \to \infty$.

By (5.3.9) and Lemma 5.3.6 we have the following bounds

$$0 < w_\varepsilon(s) - a_\varepsilon \leq 2(c - \rho - a_\varepsilon)e^{-\mu|s - s_1^\varepsilon|} \quad \text{for } s \in [-\mathfrak{r}_1\varepsilon^{-1}, s_1^\varepsilon], \tag{5.4.5}$$

$$0 < b_\varepsilon - w_\varepsilon(s) \leq 2(b_\varepsilon - c - \rho)e^{-\mu(s - s_4^\varepsilon)} \quad \text{for } s \in [s_4^\varepsilon, \mathfrak{r}_1\varepsilon^{-1}]. \tag{5.4.6}$$

By our mass constraint (5.4.3) we can write:

$$\begin{aligned} 0 = \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds &= \int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \\ + \int_{s_1^\varepsilon}^{s_4^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds &+ \int_{s_4^\varepsilon}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds. \end{aligned} \tag{5.4.7}$$

We will estimate these terms to obtain a contradiction. By (5.3.9) and the fact that $0 < s_4^\varepsilon - s_1^\varepsilon \leq C$ we have that

$$\left| \int_{s_1^\varepsilon}^{s_4^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| \leq C.$$

We can also calculate

$$\begin{aligned} &\int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \\ &= \int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds + \int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds. \end{aligned}$$

By (5.4.5) we have that

$$0 \le \int_{-\mathfrak{r}_1\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds \le 2(c - \rho - a_\varepsilon) \max \eta \int_{-\mathfrak{r}_1\varepsilon^{-1}}^{s_1^\varepsilon} e^{-\mu|s - s_1^\varepsilon|} \, ds \le C,$$

whereas by Theorem 5.3.4 (iii) and (5.3.9) we know that

$$\left| \int_{A\varepsilon^{-1}}^{-\mathfrak{r}_1\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds \right| \le C\varepsilon^{k-1} + o(1).$$

Furthermore as $a_\varepsilon = a + O(\varepsilon^{1/q})$ by Theorem 5.3.3, we may estimate that

$$\left| \int_{A\varepsilon^{-1}}^{0} (a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| \le C\varepsilon^{\frac{1-q}{q}}.$$

A similar argument, and the fact that $0 < s_1^\varepsilon < s_4^\varepsilon$ shows that

$$\left| \int_{s_4^\varepsilon}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| \le C.$$

Now as $s_1^\varepsilon \to \infty$ we then have that

$$\lim_{\varepsilon \to 0^+} \left| \int_0^{s_1^\varepsilon} (a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| \ge \lim_{\varepsilon \to 0^+} \inf_{B(t_0, \mathfrak{r}_1)} \eta \left| \int_0^{s_1^\varepsilon} (a_\varepsilon - b) \, ds \right| = \infty. \tag{5.4.8}$$

Combining (5.4.7)–(5.4.8) gives

$$\lim_{\varepsilon \to 0^+} \left| \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| = \infty.$$

This violates the mass constraint. Thus we must have that the s_1^ε are bounded above.

A similar argument shows that s_4^ε is bounded below. As $\tau_\varepsilon \in (s_1^\varepsilon, s_4^\varepsilon)$ and $s_4^\varepsilon - s_1^\varepsilon \le C$, we then have that $|\tau_\varepsilon| \le C$, which is the desired conclusion.

□

The next step is to prove that the functions w_ε necessarily converge.

Lemma 5.4.3. *Let w_ε be as in Lemma 5.4.2. Then (up to a subsequence, not relabeled) $\{w_\varepsilon\}$ converges weakly in $H^1((-l,l))$ for every $l \in \mathbb{N}$ to the profile $w_0(s) := z(s - \tau_0)$, where τ_0 is determined by (5.2.15). Moreover, the family $\{w_\varepsilon'\}$ is bounded in $L^\infty((A\varepsilon^{-1}, B\varepsilon^{-1}))$.*

Proof. Throughout this proof we let w_ε be associated with its extension by constants outside of $[A\varepsilon^{-1}, B\varepsilon^{-1}]$. The fact that the family $\{w_\varepsilon'\}$ is uniformly bounded in $L^\infty(\mathbb{R})$ follows immediately from Lemma 5.3.9. Furthermore, we have that the w_ε are bounded in $L^\infty(\mathbb{R})$ by (5.3.9). After a diagonalization argument, this implies that for some $w_0 \in H^1_{\mathrm{loc}}(\mathbb{R})$,

$$w_\varepsilon \rightharpoonup w_0 \text{ in } H^1_{\mathrm{loc}}(\mathbb{R}). \tag{5.4.9}$$

By (5.3.3) and (5.3.10) we have that

$$\begin{cases} 2(w_\varepsilon'\eta_\varepsilon)' - W'(w_\varepsilon)\eta_\varepsilon = \varepsilon\lambda_\varepsilon\eta_\varepsilon & \text{on } (A\varepsilon^{-1}, B\varepsilon^{-1}), \\ w_\varepsilon'(A\varepsilon^{-1}) = w_\varepsilon'(B\varepsilon^{-1}) = 0. \end{cases}$$

Hence for every $\phi \in C_c^\infty(\mathbb{R})$ for ε small enough we find that

$$\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} 2w_\varepsilon' \eta_\varepsilon \phi' + W'(w_\varepsilon)\eta_\varepsilon \phi \, ds = -\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} \varepsilon\lambda_\varepsilon \eta_\varepsilon \phi \, ds.$$

Letting $\varepsilon \to 0$ and using (5.4.4) and (5.4.9) gives

$$\int_{\mathbb{R}} 2w_0'\eta(t_0)\phi' + W'(w_0)\eta(t_0)\phi \, ds = 0,$$

which then shows that w_0 satisfies the differential equation

$$2w_0'' = W'(w_0). \tag{5.4.10}$$

Furthermore, by (5.3.9) we know that $a \le w_0 \le b$, which by (5.4.10) implies that $|w_0''| \le C$. Also, by (5.3.1) and the fact that $H_\varepsilon(w_\varepsilon) = J_\varepsilon(v_\varepsilon)$, where v_ε is a minimizer of J_ε,

$$\eta(t_0)\int_{-l}^{l} (w_0')^2 + W(w_0)\, ds \le \lim_{\varepsilon\to 0}\int_{-l}^{l} ((w_\varepsilon')^2 + W(w_\varepsilon))\eta_\varepsilon \, ds \le \lim_{\varepsilon \to 0^+} H_\varepsilon(w_\varepsilon) = 2c_W\eta(t_0)$$

for every $l \in \mathbb{N}$, and thus

$$\eta(t_0)\int_{\mathbb{R}} (w_0')^2 + W(w_0)\, ds \le 2c_W\eta(t_0). \tag{5.4.11}$$

This combined with the fact that $|w_0''| \le C$ (by (5.4.10)) implies that $\lim_{s\to\pm\infty} w_0'(s) = 0$. By then using (5.4.5) and (5.4.6) along with Lemma 5.4.2 we have that $\lim_{s\to-\infty} w_0(s) = a$, and that $\lim_{s\to\infty} w_0(s) = b$. Thus by integrating (5.4.10) we find that

$$(w_0')^2 = W(w_0). \tag{5.4.12}$$

We next claim that w_0 is increasing. Suppose not. Then by (5.4.12) there exists critical points $t_1 < t_2$ of w_0, with $w_0(t_1) = b$ and $w_0(t_2) = a$. This then implies, by Young's inequality, (5.4.11) and a change of variables that

$$6c_W\eta(t_0) \le 2c_W\eta(t_0).$$

This is impossible and thus w_0 is increasing. Moreover, by (5.3.12), (5.4.9), and Lemma 5.4.2, up to a subsequence, $\tau_\varepsilon \to \tau_0$ with $w_0(\tau_0) = c$. This then implies that $w_0(s) = z(s - \tau_0)$, where z is the solution of the Cauchy problem (1.1.6).

The only thing left to prove is that τ_0 is determined by equation (5.2.15). To this end, fix l large enough that $(s_1^\varepsilon, s_4^\varepsilon) \subset (-l, l)$ for all ε, where s_1^ε and s_4^ε are as in the proof of Lemma 5.4.2. Then by the mass constraint (5.4.3) we have that

$$\begin{aligned} 0 = \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds &= \int_{-l}^{l} (w_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds \\ &+ \int_{-\mathfrak{r}_1\varepsilon^{-1}}^{-l} (w_\varepsilon - a_\varepsilon + a_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds + \int_{l}^{\mathfrak{r}_1\varepsilon^{-1}} (w_\varepsilon - b_\varepsilon + b_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds \\ &+ \int_{A\varepsilon^{-1}}^{-\mathfrak{r}_1\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon + a_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds + \int_{\mathfrak{r}_1\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - b_\varepsilon + b_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds. \end{aligned}$$

By the definitions of s_1^ε and s_4^ε it must be that $v_\varepsilon \le c - \rho$ in the interval $[-\mathfrak{r}_1\varepsilon^{-1}, -l]$ and $v_\varepsilon \ge c + \rho$ in the interval $[l, \mathfrak{r}_1\varepsilon^{-1}]$. Hence by (5.3.9) and (5.3.25) we have that

$$\begin{aligned} 0 \le \int_{l}^{\mathfrak{r}_1\varepsilon^{-1}} (b_\varepsilon - w_\varepsilon)\eta_\varepsilon \, ds &\le 2\left((b_\varepsilon - w_\varepsilon(l)) + (b_\varepsilon - w_\varepsilon(\mathfrak{r}_1\varepsilon^{-1}))\right) \max \eta \int_0^\infty e^{-\mu s}\, ds \\ &\le C(b_\varepsilon - w_\varepsilon(l) + \varepsilon^k), \end{aligned}$$

where in the last inequality we have used (5.3.18) and Theorem 5.3.4. Similarly, we have

$$0 \leq \int_{-\mathfrak{r}_1\varepsilon^{-1}}^{-l} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds \leq C(w_\varepsilon(-l) - a_\varepsilon + \varepsilon^k).$$

By (5.3.9) we can write:

$$\int_{A\varepsilon^{-1}}^{-l} (a_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds = -\lambda_\varepsilon|\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q-1}\int_{-T}^{t_0} \eta \, dt + o(\varepsilon^{1/q-1}),$$
$$\int_{l}^{B\varepsilon^{-1}} (b_\varepsilon - \operatorname{sgn}_{a,b})\eta_\varepsilon \, ds = -\lambda_\varepsilon|\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q-1}\int_{t_0}^{T} \eta \, dt + o(\varepsilon^{1/q-1}).$$

Furthermore by Theorem 5.3.4 along with (5.3.9) we have that

$$\int_{A\varepsilon^{-1}}^{-\mathfrak{r}_1\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds = o(1),$$
$$\int_{\mathfrak{r}_1\varepsilon^{-1}}^{B\varepsilon^{-1}} (b_\varepsilon - w_\varepsilon)\eta_\varepsilon \, ds = o(1).$$

Utilizing these estimates, and taking $\varepsilon \to 0$ we find that

$$0 = \eta(t_0)\int_{-l}^{l} w_0 - \operatorname{sgn}_{a,b} \, ds - \lambda_0|\lambda_0|^{1/q-1}(q/\ell)^{1/q}\lim_{\varepsilon\to 0^+}\varepsilon^{1/q-1}\int_I \eta \, dt$$
$$+ O(|a - w_0(-l)|) + O(|b - w_0(l)|).$$

Taking l to infinity, and using (5.1.5) then implies that

$$\eta(t_0)\int_{\mathbb{R}} w_0 - \operatorname{sgn}_{a,b} \, ds = \begin{cases} \frac{\lambda_0}{W''(a)}\int_I \eta \, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases}$$

which then implies that τ_0 has the desired form. This completes the proof. □

Using the previous lemmas it is possible to derive a second-order liminf inequality, which immediately implies Theorem 5.4.1.

Lemma 5.4.4. *Let $\{w_\varepsilon\}$ be minimizers of the functionals $\{H_\varepsilon\}$. Then we have the following:*

$$\liminf_{\varepsilon\to 0^+} \frac{H_\varepsilon(w_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \geq 2\eta'(t_0)(\tau_0 c_W + c_{\mathrm{sym}}) \tag{5.4.13}$$
$$+ \begin{cases} \frac{\lambda_0^2}{2W''(a)}\int_I \eta(s)\, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases}$$

where c_W, c_{sym}, τ_0, λ_0 are given by (1.1.5), (6.1.6), (5.2.15) *and* (5.3.4) *respectively.*

Proof. Fix k to be a large integer. By (5.4.5) and (5.4.6) and the fact that s_1^ε and s_4^ε are bounded we can find $l_\varepsilon \in (s_2^\varepsilon, \mathfrak{r}_1\varepsilon^{-1})$ such that $b_\varepsilon - w_\varepsilon(l_\varepsilon) < \varepsilon^k$ and $w_\varepsilon(-l_\varepsilon) - a_\varepsilon < \varepsilon^k$ for $\varepsilon > 0$ sufficiently small. Recall that by Corollary 5.3.7 we can take

$$l_\varepsilon < C|\log\varepsilon|. \tag{5.4.14}$$

By (5.4.2) we can compute

$$\begin{aligned}
&\frac{H_\varepsilon(w_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon}\\
&= \varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} (W^{1/2}(w_\varepsilon) - w_\varepsilon')^2\eta_\varepsilon\,ds + 2\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(t_0))\,ds\\
&\quad + \varepsilon^{-1}\int_{[A\varepsilon^{-1},B\varepsilon^{-1}]\setminus(-l_\varepsilon,l_\varepsilon)} \left(W(w_\varepsilon) + (w_\varepsilon')^2\right)\eta_\varepsilon\,ds + \varepsilon^{-1}2\eta(t_0)\left(\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds - c_W\right)\\
&\geq 2\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(t_0))\,ds\\
&\quad + \varepsilon^{-1}\int_{[A\varepsilon^{-1},B\varepsilon^{-1}]\setminus(-l_\varepsilon,l_\varepsilon)} W(w_\varepsilon)\eta_\varepsilon\,ds + \varepsilon^{-1}2\eta(t_0)\left(\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\,ds - c_W\right).
\end{aligned}$$

We will examine the individual terms. The last term goes to zero as

$$\begin{aligned}
\varepsilon^{-1}\left|\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'\ ds - c_W\right| &\leq \varepsilon^{-1}\left|\int_{w_\varepsilon(-l_\varepsilon)}^{w_\varepsilon(l_\varepsilon)} W^{1/2}(r)\,dr - \int_a^b W^{1/2}(r)\,dr\right|\\
&\leq \varepsilon^{-1}\left|\int_{a_\varepsilon}^{b_\varepsilon} W^{1/2}(r)\,dr - \int_a^b W^{1/2}(r)\,dr\right| + C\varepsilon^{k-1}\\
&\leq C\varepsilon^{-1}\int_0^{\varepsilon^{1/q}} t^{\frac{1+q}{2}}\,dt + C\varepsilon^{k-1} = o(1), \qquad (5.4.15)
\end{aligned}$$

where we have used (1.1.5), (5.1.9) and (5.3.9).

For $s \in [l_\varepsilon, B\varepsilon^{-1}] \cap \{w_\varepsilon \geq b_\varepsilon - \varepsilon^k\}$ by the mean value theorem we can write

$$W(w_\varepsilon(s)) = W(b_\varepsilon) + W'(\zeta_\varepsilon)(w_\varepsilon(s) - b_\varepsilon),$$

where $\zeta_\varepsilon \in [w_\varepsilon(s), b_\varepsilon]$. By (5.1.10) and (5.3.13) for such s we then have that

$$\begin{aligned}
|W'(\zeta_\varepsilon)|(b_\varepsilon - w_\varepsilon(s)) &\leq C|\zeta_\varepsilon - b|^q(b_\varepsilon - w_\varepsilon(s))\\
&\leq C(|\zeta_\varepsilon - b_\varepsilon|^q + |b_\varepsilon - b|^q)(b_\varepsilon - w_\varepsilon(s))\\
&\leq C(\varepsilon^{qk} + \varepsilon)\varepsilon^k \leq C\varepsilon^{k+1}.
\end{aligned}$$

Thus we can write, after applying (5.1.9), part (iii) of Theorem 5.3.4, (5.3.13), and (5.4.14),

$$\begin{aligned}
&\varepsilon^{-1}\int_{l_\varepsilon}^{B\varepsilon^{-1}} W(w_\varepsilon)\eta_\varepsilon\,ds \geq \varepsilon^{-1}W(b_\varepsilon)\int_{l_\varepsilon}^{B\varepsilon^{-1}} \eta_\varepsilon\,ds + O(\varepsilon^{k-1})\\
&= \varepsilon^{-1}\left(\frac{\ell}{q(1+q)}|b_\varepsilon - b|^{1+q} + o(|b_\varepsilon - b|^{1+q})\right)\left(\varepsilon^{-1}\int_{t_0}^T \eta\,dt + O(|\log\varepsilon|)\right) + O(\varepsilon^{k-1})\\
&= \left(\frac{q^{1/q}|\lambda_\varepsilon|^{1+1/q}}{(1+q)\ell^{1/q}} + o(1)\right)\left(\varepsilon^{1/q-1}\int_{t_0}^T \eta\,dt + O(\varepsilon^{1/q}|\log\varepsilon|)\right) + O(\varepsilon^{k-1}).
\end{aligned}$$

An analogous bound will hold on the interval $[A\varepsilon^{-1}, -l_\varepsilon]$. Hence

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1}\int_{[A\varepsilon^{-1},B\varepsilon^{-1}]\setminus(-l_\varepsilon,l_\varepsilon)} W(w_\varepsilon)\eta_\varepsilon\,ds = \begin{cases} \frac{\lambda_0^2}{2W''(a)}\int_I \eta\,dt & \text{if } q = 1,\\ 0 & \text{if } q < 1. \end{cases} \qquad (5.4.16)$$

In considering the first term, by using (5.4.5), and for M large enough, on the interval $[-l_\varepsilon, -M]$ it follows that

$$
\begin{aligned}
|W^{1/2}(w_\varepsilon)| &\leq |W^{1/2}(w_\varepsilon) - W^{1/2}(a_\varepsilon)| + |W^{1/2}(a_\varepsilon)| \\
&\leq |W(w_\varepsilon) - W(a_\varepsilon)|^{1/2} + |W^{1/2}(a_\varepsilon)| \leq Ce^{-C|s+M|} + C\varepsilon^{\frac{(1+q)}{2q}}.
\end{aligned}
$$

A similar bound holds on $[M, l_\varepsilon]$. Then using (5.1.13), along with Lemma 5.3.9 and Theorem 5.3.3, it follows that

$$
\begin{aligned}
&\left|\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(t_0) - \eta'(t_0)\varepsilon s)\,ds\right| \\
&\leq o(1)\left(C\int_{-l_\varepsilon}^{-M}|s|\left(e^{-C|s+M|} + \varepsilon^{\frac{1-q}{2q}}\right)ds + C\int_{-M}^{M}|s|\,ds + C\int_{M}^{l_\varepsilon}|s|\left(e^{-C|s-M|} + \varepsilon^{\frac{1-q}{2q}}\right)ds\right) \\
&= o(1).
\end{aligned}
$$

Thus we find that:

$$
\lim_{\varepsilon\to 0^+} 2\varepsilon^{-1}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(t_0))\,ds = 2\eta'(t_0)\lim_{\varepsilon\to 0^+}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds.
$$

Now for any fixed l by (5.4.9) and the fact that $w_0(s) = z(s - \tau_0)$, we can write

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{-l}^{l} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds &= \int_{-l}^{l} W^{1/2}(w_0)w_0' s\,ds \\
&= \int_{-l-\tau_0}^{l-\tau_0} W^{1/2}(z(t))z'(t)(t+\tau_0)\,dt \\
&= \tau_0\Phi(z(l-\tau_0)) - \tau_0\Phi(z(-l-\tau_0)) + \int_{-l-\tau_0}^{l-\tau_0} W^{1/2}(z(t))z'(t)t\,dt,
\end{aligned}
$$

where we recall that $\Phi(s) = \int_a^s W^{1/2}(r)\,dr$. Furthermore we can establish the following bound using (5.1.9), (5.4.6) and Lemma 5.4.3:

$$
\begin{aligned}
&\left|\int_{l}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds\right| \leq C\int_{l}^{l_\varepsilon}|b - w_\varepsilon|^{\frac{1+q}{2}} s\,ds \\
&\leq C(|b_\varepsilon - c - \rho|^{\frac{1+q}{2}} + |b_\varepsilon - b|^{\frac{1+q}{2}})\int_{l}^{\infty} e^{-\frac{1+q}{2}\mu(s - s_4^\varepsilon)} s\,ds,
\end{aligned}
$$

provided $l > s_4^\varepsilon$. Thus we can write

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s\,ds &= \tau_0\Phi(z(l-\tau_0)) - \tau_0\Phi(z(-l-\tau_0)) \\
&\quad + \int_{-l-\tau_0}^{l-\tau_0} W^{1/2}(z(s))z'(s)s\,ds + O(le^{-\frac{1+q}{2}\mu l}).
\end{aligned}
$$

Taking l to ∞, combined with (5.4.15) and (5.4.16) gives the desired claim, namely, (5.4.13). □

The proof of Theorem 5.4.1 is now straightforward.

Proof of Theorem 5.4.1. By changing variables it is immediate that $H(w_\varepsilon) = G_\varepsilon^{(1)}(v_\varepsilon)$. Lemma 5.4.4 then immediately implies (5.4.1). This concludes the proof. □

Chapter 6

Characterization of a Second-Order Γ-Limit

6.1 Main Results

This chapter uses tools from the previous two chapters to prove an asymptotic expansion of order 2 by Γ-convergence of the functionals (1.1.3). In particular, the goal is to prove Theorems 6.1.2 and 6.1.3.

These theorems are proven under the same assumptions on the potential W that were given in Chapter 5, namely (5.1.4)-(5.1.7). Some remarks regarding the consequences of those assumptions can be found in Chapter 5.

The theorems in this chapter also assume that $\Omega \subset \mathbb{R}^n, n \leq 7$, is an open, connected, bounded set with

$$\mathcal{L}^n(\Omega) = 1 \quad \text{ and } \quad \partial\Omega \text{ is of class } C^{2,\hat{\alpha}}, \quad \hat{\alpha} \in (0,1]. \tag{6.1.1}$$

The restriction to $n \leq 7$ is necessary to guarantee classical regularity of minimizers of the problem (1.1.8) [58, 60, 75, 103], while the assumption that $\mathcal{L}^n(\Omega) = 1$ is for simplicity (the general case follows by a scaling argument). It is likely that the results would still hold in dimension $n > 7$, with appropriate modifications to accomodate the loss of classical regularity, but for simplicity this thesis only focuses on the classical setting.

Another assumption is that the mass m in (1.1.2) satisfies

$$a < m < b, \tag{6.1.2}$$

where a, b are the wells of W. This assumption is natural because it imposes a phase transition, while other choices of mass would not.

Finally, given a measurable set $E_0 \subset \Omega$ with mass $\mathfrak{v}_m$ (see (1.1.8) and (1.1.9)) and $\delta > 0$, define the *local isoperimetric function* of parameter δ about the set E_0 to be

$$\mathcal{I}_\Omega^{\delta,E_0}(r) := \inf\{P(E,\Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = r, \alpha(E_0,E) \leq \delta\}, \tag{6.1.3}$$

where

$$\alpha(E_1,E_2) := \min\{\mathcal{L}^n(E_1 \setminus E_2), \mathcal{L}^n(E_2 \setminus E_1)\} \tag{6.1.4}$$

for all Borel sets $E_1, E_2 \subset \Omega$.

Remark 6.1.1. *When δ is sufficiently large then $\mathcal{I}_\Omega^{\delta,E_0}(r) = \mathcal{I}_\Omega$. Thus in the theorems one may safely replace $\mathcal{I}_\Omega^{\delta,E_0}$ with $\mathcal{I}_\Omega$, which is precisely the case considered in [73].*

The main technical assumption in the theorems given here is that $\mathcal{I}_\Omega^{\delta,E_0}$ be differentiable at $\mathfrak{v}_m = \frac{b-m}{b-a}$ (see (1.1.9)). In Chapter 4 it was demonstrated that this assumption is rather generic, in the sense that it will hold for all but countably many m, see Corollary 4.0.4. It was also demonstrated that the assumption holds for isolated local volume-constrained perimeter minimizers, see Theorem 4.0.6.

After giving these assumptions, it is now possible to state the two main results.

Theorem 6.1.2. *Assume that Ω satisfies* (6.1.1)*, m satisfies* (6.1.2) *and W satisfies hypotheses* (5.1.4)-(5.1.7) *with $q = 1$. Assume that u is an $L^1(\Omega)$-local minimizer of the functional $\mathcal{F}^{(1)}$ (see* (1.1.4)*). Finally, assume that, for some $\delta > 0$, $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathfrak{v}_m$, with $E_0 = \{u = a\}$. Then*

$$\begin{aligned}\Gamma\text{-}\liminf \tilde{\mathcal{F}}_\varepsilon(u) &= \Gamma\text{-}\limsup \tilde{\mathcal{F}}_\varepsilon(u)\\ &= \frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + 2(c_{\mathrm{sym}} + c_W\tau_u)(n-1)\kappa_u \,\mathrm{P}(\{u=a\};\Omega),\end{aligned} \tag{6.1.5}$$

where

$$\tilde{\mathcal{F}}_\varepsilon(w) := \frac{\mathcal{F}_\varepsilon^{(1)}(w) - \mathcal{F}^{(1)}(u)}{\varepsilon}$$

and

$$\mathcal{F}_\varepsilon^{(1)}(w) = \frac{\mathcal{F}_\varepsilon(w)}{\varepsilon}.$$

In particular, if $\mathcal{I}_\Omega$ is differentiable at $\mathfrak{v}_m$ then

$$\mathcal{F}^{(2)}(u) = \frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + 2(c_{\mathrm{sym}} + c_W\tau_u)(n-1)\kappa_u \,\mathrm{P}(\{u=a\};\Omega)$$

if u is a global minimizer of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.

In this theorem, κ_u is the constant mean curvature of the set $\{u = a\}$,

$$c_{\mathrm{sym}} := \int_{\mathbb{R}} W(z(t))t\,dt, \tag{6.1.6}$$

where z is the solution to the Cauchy problem (1.1.6), and $\tau_u \in \mathbb{R}$ is a constant such that

$$\mathrm{P}(\{u=a\};\Omega)\int_{\mathbb{R}} z(t-\tau_u) - \mathrm{sgn}_{a,b}(t)\,dt = \frac{2c_W(n-1)}{W''(a)(b-a)}\kappa_u, \tag{6.1.7}$$

with $\mathrm{sgn}_{a,b}$ as defined in (1.1.7).

In the case $q = 1$, W is approximately quadratic near the wells, and thus the solution of the Cauchy problem (1.1.6) approaches a and b as $t \to -\infty$ and ∞ respectively, see (5.1.11). On the other hand, when W is subquadratic near the wells, that is, when $q < 1$ in (5.1.10), then the solution reaches a and b in finite time, see (5.1.12). The analysis is thus somewhat difference in this case, but a similar theorem still holds.

Theorem 6.1.3. *Assume that Ω satisfies* (6.1.1)*, m satisfies* (6.1.2) *and W satisfies hypotheses* (5.1.4)-(5.1.7) *with $q \in (0,1)$. Assume that u is an $L^1(\Omega)$-local minimizer of the functional $\mathcal{F}^{(1)}$ (see* (1.1.4)*). Finally, for some $\delta > 0$, assume that $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathfrak{v}_m$, with $E_0 = \{u = a\}$. Then*

$$\begin{aligned}\Gamma\text{-}\liminf \tilde{\mathcal{F}}_\varepsilon(u) &= \Gamma\text{-}\limsup \tilde{\mathcal{F}}_\varepsilon(u)\\ &= 2(c_{\mathrm{sym}} + c_W\tau_u)(n-1)\kappa_u \,\mathrm{P}(\{u=a\};\Omega).\end{aligned} \tag{6.1.8}$$

where

$$\tilde{\mathcal{F}}_\varepsilon(w) := \frac{\mathcal{F}^{(1)}_\varepsilon(w) - \mathcal{F}^{(1)}(u)}{\varepsilon}$$

and

$$\mathcal{F}^{(1)}_\varepsilon(w) = \frac{\mathcal{F}_\varepsilon(w)}{\varepsilon}.$$

In particular, if $\mathcal{I}_\Omega$ is differentiable at $\mathfrak{v}_m$ then

$$\mathcal{F}^{(2)}(u) = 2(c_{\mathrm{sym}} + c_W \tau_u)(n-1)\kappa_u \,\mathrm{P}(\{u = a\}; \Omega)$$

if u is a global minimizer of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.

Here now τ_u is a constant such that

$$\int_{\mathbb{R}} z(t - \tau_u) - \mathrm{sgn}_{a,b}(t)\, dt = 0. \tag{6.1.9}$$

Note that (6.1.8) and (6.1.9) correspond to the case $W''(a) = \infty$ in (6.1.5) and (6.1.7) respectively.

Remark 6.1.4. *In both of these theorems the fact that $\mathcal{F}^{(2)}(u) = \infty$ for u that are not global minimizers of $\mathcal{F}^{(1)}$ is trivial given* (2.4.1) *and* (2.4.2)*. This fact is summarized in Proposition 2.4.9.*

A crucial hypothesis in these results is that the local isoperimetric function (see definition (1.1.10)) be differentiable at the point $\mathfrak{v}_m$ given by (1.1.9). In particular the differentiability of $\mathcal{I}^{\delta,E_0}_\Omega$ at $\mathfrak{v}_m$ implies that (see [75])

$$(\mathcal{I}^{\delta,E_0}_\Omega)'(\mathfrak{v}_m) = (n-1)\kappa_{E_0}. \tag{6.1.10}$$

However, differentiability of $\mathcal{I}^{\delta,E_0}_\Omega$ must fail whenever the mean curvature of minimizers of the L^1-restricted partition problem (6.1.3) is not uniquely determined. For example, if Ω is a square in $\mathbb{R}^2$, it can be shown that there exists a value of $\mathfrak{v}_m$ for which there are two minimizers of (1.1.8), one being a line segment and the other being an arc of a circle. This implies that $\mathcal{I}_\Omega$ is not differentiable at an appropriately chosen value, see Figure 6.1. However, the competing minimizers given in Figure 6.1 are actually L^1 isolated minimizers, and thus the theorems of this section should still apply, by using $\mathcal{I}^{\delta,E_0}_\Omega$ instead of $\mathcal{I}_\Omega$. Discussion of various cases where the assumption of differentiability can be proven were given in chapter 4.

Without assuming the differentiability of the local isoperimetric function $\mathcal{I}^{\delta,E_0}_\Omega$ at $\mathfrak{v}_m$ one can only conclude that $(n-1)\kappa_u \in [(\mathcal{I}^{\delta,E_0}_\Omega)'_-(\mathfrak{v}_m), (\mathcal{I}^{\delta,E_0}_\Omega)'_+(\mathfrak{v}_m)]$, where $(\mathcal{I}^{\delta,E_0}_\Omega)'_-$, $(\mathcal{I}^{\delta,E_0}_\Omega)'_+$ are the left and right derivatives of $\mathcal{I}^{\delta,E_0}_\Omega$, which must exist as $\mathcal{I}^{\delta,E_0}_\Omega$ is semi-concave, see Chapter 4. Whether this situation can possibly persist as $\delta \to 0$ is not clear. One could hope that the rigidity of constant mean curvature surfaces gives some traction on the problem, but so far no results have been obtained.

If this theorem continues to hold in the case where $\mathcal{I}^{\delta,E_0}_\Omega$ is not differentiable, then this theorem gives a new selection criteria on limits of minimizers of $\mathcal{F}_\varepsilon$. In particular, when W is symmetric about $\frac{a+b}{2}$ then surfaces with larger magnitude mean curvature are energetically favored (see Corollary 6.1.5 below).

A heuristic explanation for the terms in (6.1.5) may prove helpful. Critical points u_ε of (1.1.1) subject to (1.1.2) satisfy the Neumann problem

$$\begin{cases} 2\varepsilon\Delta u_\varepsilon = \frac{1}{\varepsilon}W'(u_\varepsilon) + \Lambda_\varepsilon & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

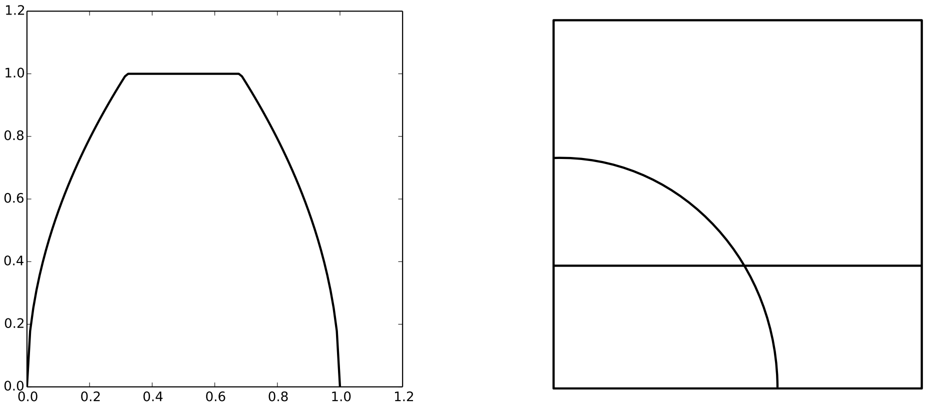

Figure 6.1: $\mathcal{I}_\Omega$ for the domain $\Omega = Q_2$, the cube in $\mathbb{R}^2$. When $\mathcal{I}_\Omega$ is not differentiable there are two competing sets minimizing the perimeter, as shown.

where ν is the outward unit normal to $\partial\Omega$ and Λ_ε is a Lagrange multiplier that accounts for the constraint (1.1.2). In [74], Luckhaus and Modica proved that if $0 < a < b$ and $\{u_\varepsilon\}$ is a sequence of non-negative minimizers of (1.1.1), (1.1.2), uniformly bounded in $L^\infty(\Omega)$ and converging in $L^1(\Omega)$ to a minimizer of $\mathcal{F}^{(1)}$, then

$$\Lambda_\varepsilon \to \Lambda_u := \frac{2c_W(n-1)}{b-a}\kappa_u. \tag{6.1.11}$$

Thus the first term in equation (6.1.5) can be written as $\frac{\Lambda_u^2}{2W''(a)}$. Our proofs suggest (see (5.2.17)) that minimizers u_ε of the energy E_ε will in fact be of the form

$$u_\varepsilon(x) \approx z\left(\frac{d(x,\{u=a\}) - \varepsilon\tau_u}{\varepsilon}\right) - \frac{\Lambda_u\varepsilon}{W''(a)}. \tag{6.1.12}$$

It turns out that the first term in equation (6.1.5) is linked to a small vertical shift in the bulk values of minimizers, namely the second term in (6.1.12). The τ_u term in (6.1.5) is caused by the shift inside z in the first term of (6.1.12), which essentially pushes the transition layer "outward" along curved surfaces. We note that the horizontal shift caused by τ_u and the vertical shift in the bulk must be in some sense balanced so that the mass constraint is satisfied.

The term involving $c_{\rm sym}$ may be thought of as a penalty for directional asymmetry. If the profiles are symmetric this term disappears entirely. This term is of order ε for any q that we consider.

In the case where W is symmetric about $(b+a)/2$, then the function z in (1.1.6) is odd, and so the constants $c_{\rm sym}$ and τ_u simplify to give the following:

Corollary 6.1.5. *In addition to the assumptions above, suppose that W is symmetric about $(b+a)/2$, and that $\mathcal{I}_\Omega$ is differentiable at $\mathfrak{v}_m$. Then for u minimizing $\mathcal{F}^{(1)}$ we have that*

$$\mathcal{F}^{(2)}(u) = \begin{cases} -\frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases}$$

Remark 6.1.6. *A straightforward calculation shows that in the case of the Cahn–Hilliard potential $W(s) = \frac{1}{2}(1-s^2)^2$ the second-order Γ-limit takes the form*

$$\mathcal{F}^{(2)}(u) = -\frac{(n-1)^2}{9}\kappa_u^2.$$

Following the approach of [41], the next section will prove the main results. Of course, much of the work has already been done in Chapters 3 and 5.

6.2 Proof of Main Results

The first step, is to connect the definition of the local isoperimetric function (6.1.3) with the topology of L^1 convergence. This in turn will connect the L^1 topology used for the Γ-convergence results with the notion of $\mathcal{I}$-comparable level sets from Chapter 3.

Proposition 6.2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $E_0 \subset \Omega$ be a Borel set and let $v_{E_0} = a\chi_{E_0} + b\chi_{E_0{}^c}$. Then*

$$\alpha(E_0, \{u \le s\}) \le \delta \tag{6.2.1}$$

for all $u \in L^1(\Omega)$ such that

$$\|u - v_{E_0}\|_{L^1} \le (b-a)\delta, \tag{6.2.2}$$

and for every $s \in \mathbb{R}$, where α is the number given in (6.1.4).

Proof. Fix $\delta > 0$ and for $s \in \mathbb{R}$ define $F_s := \{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) \le s\}$. If $s \in (-1, 1)$, then by (6.2.2),

$$\begin{aligned} 2\delta &\ge \int_{F_s \setminus E_0} |u - v_{E_0}|\, dx + \int_{E_0 \setminus F_s} |u - v_{E_0}| d\boldsymbol{x} \\ &\ge (1-s)\mathcal{L}^n(F_s \setminus E_0) + (1+s)\mathcal{L}^n(E_0 \setminus F_s) \ge 2\alpha(E_0, F_s), \end{aligned}$$

so that (6.2.1) is proved in this case. If $s \ge 1$, again by (6.2.2),

$$2\delta \ge \int_{E_0 \setminus F_s} |u - v_{E_0}| d\boldsymbol{x} \ge (1+s)\mathcal{L}^n(E_0 \setminus F_s) \ge 2\alpha(E_0, F_s).$$

The case $s \le -1$ is analogous. □

Corollary 6.2.2. *Fix $\delta > 0$ and $E_0 \subset \Omega$ Borel. Given a family of functions $u_\varepsilon \xrightarrow{L^1(\Omega)} u_0 = a\chi_{E_0} + b\chi_{E_0}$ then for ε sufficiently small the inequality*

$$\alpha(E_0, \{u \le s\}) \le \delta$$

is satisfied. In particular, if $\mathcal{I} = \mathcal{I}_\Omega^{\delta, E_0}$, then for ε sufficiently small the function u has $\mathcal{I}$ comparable level sets.

Finally, the next result is an elementary result about touching a function from below.

Proposition 6.2.3. *Suppose that $\hat{\mathcal{I}} : [0,1] \to [0, \infty)$ is a continuous function, which is differentiable at $\mathfrak{v}_m$ and which satisfies*

$$\hat{\mathcal{I}}(\mathfrak{v}) \ge C_1 \min\{\mathfrak{v}, 1-\mathfrak{v}\}^{\frac{n-1}{n}} \quad \textit{for all } \mathfrak{v} \in [0,1]. \tag{6.2.3}$$

Then there exists a function $\mathcal{I}^* \in C^1((0,1))$ satisfying:

$$\hat{\mathcal{I}} \ge \mathcal{I}^* > 0 \quad \textit{in } (0,1), \tag{6.2.4}$$

$$\hat{\mathcal{I}}(\mathfrak{v}_m) = \mathcal{I}^*(\mathfrak{v}_m), \quad (\hat{\mathcal{I}})'(\mathfrak{v}_m) = (\mathcal{I}^*)'(\mathfrak{v}_m), \tag{6.2.5}$$

$$\mathcal{I}^*(\mathfrak{v}) = C_0 \mathfrak{v}^{\frac{n-1}{n}} \quad \textit{for all } \mathfrak{v} \in (0, \delta) \tag{6.2.6}$$

$$\mathcal{I}^*(\mathfrak{v}) = C_0 (1-\mathfrak{v})^{\frac{n-1}{n}} \quad \textit{for all } \mathfrak{v} \in (1-\delta, 1)$$

for some $C_0 > 0$ and $0 < \delta < 1$.

Proof. Proposition 2.6.2 gives the construction of such a function in a neighborhood of $\mathfrak{v}_m$. By then using the functions $\frac{C_1}{2}\mathfrak{v}^{\frac{n-1}{n}}$ and $\frac{C_1}{2}(1-\mathfrak{v})^{\frac{n-1}{n}}$, and patching appropriately the result follows. □

These lemmas are then applied to obtain the main results of this chapter.

Proof of main results: Theorem 6.1.2 and 6.1.3. **Step 1: limsup inequality.** Let u be a local minimizer of $\mathcal{F}^{(1)}$. Then u must be of the form $a\chi_E + b\chi_{E^c}$. Define

$$\eta(t) := \mathcal{H}^{n-1}(\{x : d_E(x) = t\}). \tag{6.2.7}$$

By Lemma 2.3.11 we have that η satisfies the assumptions of Theorem 5.2.6. Let v_ε be the one-dimensional function constructed in Theorem 5.2.6, using η chosen via (6.2.7). Define $u_\varepsilon(x) := v_\varepsilon(d_E(x))$. By the coarea formula for Lipschitz functions we have that

$$F_\varepsilon^{(2)}(u_\varepsilon) = \frac{1}{\varepsilon}\left(\int_{\mathbb{R}} (\varepsilon^{-1}W(v_\varepsilon(t)) + \varepsilon(v_\varepsilon')^2)\mathcal{H}^{n-1}(\{x : d_E(x) = t\})\,dt - 2c_W\eta(0)\right).$$

Applying Theorem 5.2.6 then proves that the Γ-lim sup has the desired form.

Step 2: liminf inequality. Let $u = a\chi_{E_0} + b\chi_{E_0^c}$ be a local minimizer of $\mathcal{F}^{(1)}$, and let $u_\varepsilon \to u$ in $L^1(\Omega)$. We claim that $\hat{\mathcal{I}} = \mathcal{I}_\Omega^{\delta,E_0}$ satisfies the assumptions of Lemma 6.2.3. The fact that $\hat{\mathcal{I}}$ satisfies (6.2.3) follows from the fact that $\mathcal{I}_\Omega^{\delta,E_0} \geq \mathcal{I}_\Omega$ and Proposition 2.1.10. By assumption, $\hat{\mathcal{I}}$ is differentiable at $\mathfrak{v}_m$, and fact that $\hat{\mathcal{I}}$ is continuous will be proved in Proposition 4.0.1, and thus the claim holds.

Now, set $\mathcal{I} = \mathcal{I}^*$, with $\mathcal{I}^*$ as in Lemma 6.2.3. Note that u_ε has $\mathcal{I}$ comparable level sets by Corollary 6.2.2 and the fact that $\mathcal{I} \leq \mathcal{I}_\Omega^{\delta,E_0}$. Thus, applying corollary 3.3.6 implies that, for ε sufficiently small,

$$\mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_I (W(f_{u_\varepsilon}) + \varepsilon^2(f_{u_\varepsilon}')^2)\mathcal{I}^*(V_\Omega)\,dt, \quad m = \int_\Omega u_\varepsilon\,dx = \int_I f_{u_\varepsilon}\mathcal{I}^*(V_\Omega)\,dt,$$

where V_Ω and f_u are defined in Section 3 (see (3.1.3), (3.1.8) and Remark 3.3.5) and where I is defined by the support of $\mathcal{I}(V_\Omega)$, see (3.1.4). By making an appropriate shift in coordinates, from this point forward we will assume that $I = (-T, T)$.

We then set $\eta := \mathcal{I}^*(V_\Omega)$. This η will satisfy all of the assumptions in Section 4. Indeed, since $V_\Omega > 0$ in $(-T, \infty)$ and $V_\Omega(-T) = 0$, by (6.2.6) and (3.1.3), $V_\Omega(t) = [C_0/n(t+T)]^n$ near $-T$, and so $\eta = C_0^n[\frac{1}{n}(t+T)]^{n-1}$, which shows that (5.1.14) and (5.1.16) hold for t close to $-T$. Using similar reasoning, we have that $\eta(t) = C_0^n[\frac{1}{n}(T-t)]^{n-1}$ and thus (5.1.15) and (5.1.16) hold close to T. Since $\mathcal{I}^* \in C^1_{\text{loc}}(0,1)$, by (3.1.3) we have that $V_\Omega \in C^2_{\text{loc}}(I)$, and in turn $\eta \in C^1_{\text{loc}}(I)$. Thus (5.1.13) is satisfied. Finally, since $\mathcal{I}^* > 0$ in $(0,1)$ we have by (6.2.4) that $\eta > 0$ in I, and thus (5.1.16) holds on any compact subset of I by uniform continuity.

Next observe that since $u \in BV(\Omega, \{a, b\})$ and (1.1.2) holds, by Lemma 3.2.1 we have that f_u only takes the values a and b and $\int_I f_u\eta\,dt = \int_\Omega u\,dx = m$. Since f_u is increasing, this implies that $f_u(t) = \operatorname{sgn}_{a,b}(t - t_0)$ for some $t_0 \in I$ and all $t \in I$. It follows from Theorem 5.2.5 that f_u is a local minimizer of the functional $G^{(1)}$ defined in (5.2.5). Moreover, by Lemma 3.2.2 we have that $u_\varepsilon \to u$ in $L^1(\Omega)$ implies that $f_{u_\varepsilon} \to f_u$ in $L^1_\eta(I)$. Hence, $\|f_{u_\varepsilon} - f_u\|_{L^1_\eta} \leq \delta$ for all ε sufficiently small, where $\delta > 0$ is the number given in Theorem 5.2.5 (with $v_0 = f_u$). In turn choosing v_ε to be minimizers of the function J_ε defined in (5.2.12), by Corollary 3.3.6 we have that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \geq G_\varepsilon(f_{u_\varepsilon}) = J_\varepsilon(f_{u_\varepsilon}) \geq J_\varepsilon(v_\varepsilon). \tag{6.2.8}$$

Since $\int_I f_u \eta \, dt = m$, it follows from the fact that (see (6.1.1) and Lemma 3.2.1)

$$1 = \mathcal{L}^n(\Omega) = \int_I \eta \, dt \tag{6.2.9}$$

and (3.1.3) that

$$\mathfrak{v}_m = \frac{b-m}{b-a} = \mathcal{L}^n(\{u=a\}) = \int_{-T}^{t_0} \eta \, dt = \int_{-T}^{t_0} \frac{d}{dt} V_\Omega \, dt = V_\Omega(t_0). \tag{6.2.10}$$

In turn, by (6.2.5),

$$\eta(t_0) = \mathcal{I}_\Omega^*(\mathfrak{v}_m) = \mathcal{I}_\Omega(\mathfrak{v}_m) = \mathrm{P}(\{u=a\};\Omega),$$

which shows that $\mathcal{F}^{(1)}(u) = G^{(1)}(f_u)$. Hence by (6.2.8) we have that

$$\mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) = \frac{\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon) - \mathcal{F}^{(1)}(u)}{\varepsilon} \geq \frac{J_\varepsilon^{(1)}(v_\varepsilon) - J^{(1)}(f_u)}{\varepsilon} = J_\varepsilon^{(2)}(v_\varepsilon).$$

By applying Lemma 5.4.4 we thus have that

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq 2\eta'(t_0)(\tau_0 c_W + c_{\mathrm{sym}}) + \begin{cases} \frac{\lambda_0^2}{2W''(a)} & \text{if } q=1, \\ 0 & \text{if } q<1, \end{cases} \tag{6.2.11}$$

where we have used (6.2.9). By (3.1.3) we have that $\eta'(t) = (\mathcal{I}_\Omega^*)'(V_\Omega(t))\eta(t)$, and so by (6.1.10), (6.2.5) and (6.2.10),

$$\eta'(t_0) = \mathcal{I}_\Omega'(\mathfrak{v}_m)\mathcal{I}_\Omega(\mathfrak{v}_m) = (n-1)\kappa_u \, \mathrm{P}(\{u=a\};\Omega).$$

In turn by (6.1.11) and (5.2.16),

$$\lambda_0 = \frac{2(n-1)c_W}{(b-a)}\kappa_u = \Lambda_u, \tag{6.2.12}$$

and so by (5.2.15) the number τ_0 coincides with the number τ_u in (6.1.7). Combining (6.2.11)-(6.2.12) gives

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq 2(\tau_u c_W + c_{\mathrm{sym}})(n-1)\kappa_u \, \mathrm{P}(\{u=a\};\Omega) + \begin{cases} \frac{\Lambda_u^2}{2W''(a)} & \text{if } q=1, \\ 0 & \text{if } q<1. \end{cases}$$

This completes the proof. □

Remark 6.2.4. *The analysis for the liminf problem (ie using the rearrangement induced by $\mathcal{I}_\Omega^{\delta,E_0}$) in fact implies that for any u_ε satisfying $\|u_\varepsilon - u\| \leq (b-a)\delta$ then the following bound holds*

$$\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon) \geq \mathcal{F}^{(1)}(u) - C\varepsilon.$$

Remark 6.2.5. *In many settings in materials science it is natural to consider an anisotropic energy of the form*

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_\Omega W(u) + \varepsilon^2 \Psi^2(\nabla u) \, dx & \text{for } u \in H^1(\Omega), \int_\Omega u \, dx = m \\ \infty & \text{otherwise.} \end{cases}$$

Here Ψ is a non-negative convex, 1-homogeneous function, and W is a double-well potential. It is well-known [90] that

$$\varepsilon^{-1}\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \begin{cases} c_W P_\Psi(\{u=a\}) & \text{if } u \in BV(\Omega;\{a,b\}), \int_\Omega u\,dx = m \\ \infty & \text{otherwise.} \end{cases}$$

In light of Theorem 3.4.4 the rearrangement techniques used in this thesis are still valid in this case. However, some of the other aspects of the present work, such as the differentiability of the isoperimetric function and the construction of appropriate recovery sequences, are not as obviously extendable to the anisotropic case. This is the subject of current investigations.

Chapter 7

Slow Motion for Non-Local Allen–Cahn Equation

This chapter utilizes the energy asymptotics from the previous chapter to obtain slow motion bounds on the gradient flows associated with the energy (1.1.3). Recall that the L^2-constrained gradient flow of (1.1.3) is the non-local Allen–Cahn equation, which is given by

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon) + \varepsilon \lambda_\varepsilon & \text{in } \Omega \times [0,\infty), \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u_\varepsilon = u_{0,\varepsilon} & \text{on } \Omega \times \{0\}. \end{cases} \tag{7.0.1}$$

Here $u_{0,\varepsilon}$ is the initial datum, and λ_ε is a Lagrange multiplier that renders solutions mass–preserving, to be precise

$$\lambda_\varepsilon = \frac{1}{\varepsilon \mathcal{L}^n(\Omega)} \int_\Omega W'(u_\varepsilon)\, dx.$$

The main goal of this chapter is to prove the following main result.

Theorem 7.0.1. *Assume that Ω satisfies (6.1.1), m satisfies (6.1.2) and W satisfies hypotheses (5.1.4)-(5.1.7). Assume that u is an $L^1(\Omega)$-local minimizer of the functional $\mathcal{F}^{(1)}$ (see (1.1.4)). Finally, assume that, for some $\delta > 0$, $\mathcal{I}_\Omega^{\delta,E_0}$ is differentiable at $\mathfrak{v}_m$, with $E_0 = \{u = a\}$. Assume that $u_{0,\varepsilon} \in L^\infty(\Omega)$ satisfy*

$$u_{0,\varepsilon} \to u \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0^+ \tag{7.0.2}$$

and

$$\mathcal{F}_\varepsilon^{(1)}(u_{0,\varepsilon}) \le \mathcal{F}^{(1)}(u) + C\varepsilon \tag{7.0.3}$$

for some $C > 0$. Let u_ε be a solution non-local Allen–Cahn equation, namely (7.0.1). Then, for any $M > 0$

$$\sup_{0 \le t \le M\varepsilon^{-1}} ||u_\varepsilon(t) - u||_{L^2} \to 0 \text{ as } \varepsilon \to 0^+. \tag{7.0.4}$$

Remark 7.0.2. *The assumption that $u_{0,\varepsilon} \in L^\infty$ and the fact that $\mathcal{F}_\varepsilon^{(1)}(u_{0,\varepsilon}) < \infty$ is sufficient to guarantee that solutions to the equation (7.0.1) exists. Results to this effect can be found in Theorem 1.1.1 of [85].*

Remark 7.0.3. *The assumption (7.0.3) is a standard assumption in this theory, and such initial data is sometimes called "energetically well-prepared." The assumption on $\mathcal{I}_\Omega^{\delta,E_0}$ is the non-standard assumption in this case, and was at least partially addressed in Chapter 4.*

The proof for this theorem is largely identical to that in [25]. It is included for completeness. The first step is to prove the following auxiliary result.

Proposition 7.0.4. *Under the assumptions of Theorem 7.0.1, there exist two positive constants k_1 and k_2, not depending on ε, such that*

$$\int_0^{k_1\varepsilon^{-2}} ||\partial_t u_\varepsilon(t)||_{L^2}^2\,dt \le k_2\varepsilon^2,$$

where u_ε is the solution of (7.0.1).

Proof. Since u_ε is a solution to the gradient flow, for any $T > 0$ we have

$$\mathcal{F}_\varepsilon^{(1)}(u_{0,\varepsilon}) - \mathcal{F}_\varepsilon^{(1)}(u_\varepsilon(T)) = \varepsilon^{-1}\int_0^T ||\partial_t u_\varepsilon(s)||_{L^2}^2\,ds, \tag{7.0.5}$$

which shows that $t \mapsto \mathcal{F}_\varepsilon^{(1)}(u_\varepsilon)(t)$ is decreasing and $||\partial_t u_\varepsilon||_{L^2}^2$ is integrable. Given δ as in the assumptions, then by (7.0.2),

$$||u_{0,\varepsilon} - u||_{L^1} \le \delta$$

for ε sufficiently small. Now suppose that there exists $T_\varepsilon > 0$ small enough that

$$\int_0^{T_\varepsilon} ||\partial_t u_\varepsilon(t)||_{L^1}\,dt \le \delta. \tag{7.0.6}$$

Then,

$$\delta \ge \int_0^{T_\varepsilon} ||\partial_t u_\varepsilon(t)||_{L^1}\,dt \ge \left|\left|\int_0^{T_\varepsilon} \partial_t u_\varepsilon(t)\,dt\right|\right|_{L^1} = ||u_\varepsilon(T_\varepsilon) - u_{0,\varepsilon}||_{L^1},$$

so that

$$||u_\varepsilon(T_\varepsilon) - u||_{L^1} \le ||u_\varepsilon(T_\varepsilon) - u_{0,\varepsilon}||_{L^1} + ||u_{0,\varepsilon} - u||_{L^1} \le 2\delta$$

and, in particular, if δ is small enough then by Theorems 6.1.2 and 6.1.3 (see also Remark 6.2.4),

$$\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon(T_\varepsilon)) \ge \mathcal{F}_0^{(1)}(u) - C(\kappa)\varepsilon. \tag{7.0.7}$$

By (7.0.3) and (7.0.7) together with (7.0.5),

$$\begin{aligned}\int_0^{T_\varepsilon} ||\partial_t u_\varepsilon(s)||_{L^2}^2\,ds &= \varepsilon\mathcal{F}_\varepsilon^{(1)}(u_{0,\varepsilon}) - \varepsilon\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon(T_\varepsilon))\\ &\le \varepsilon\mathcal{F}^{(1)}(u) + C\varepsilon^2 - \varepsilon\mathcal{F}^{(1)}(u) \le C\varepsilon^2.\end{aligned} \tag{7.0.8}$$

In turn, by Hölder's inequality we get

$$\left(\int_0^{T_\varepsilon} ||\partial_t u_\varepsilon(t)||_{L^1}\,dt\right)^2 \le CT_\varepsilon\varepsilon^2,$$

so that

$$T_\varepsilon \ge \frac{1}{C\varepsilon^2}\left(\int_0^{T_\varepsilon} ||\partial_t u_\varepsilon(t)||_{L^1}\,dt\right)^2. \tag{7.0.9}$$

In order to conclude the proof, we need to make sure that it is always possible to choose T_ε as in (7.0.6) and that $T_\varepsilon \ge k_1\varepsilon^{-2}$ for some $k_1 > 0$. We argue as follows: suppose first that

$$\int_0^\infty ||\partial_t u_\varepsilon(t)||_{L^1}\,dt > \delta.$$

Then by continuity we can choose $T_\varepsilon > 0$ such that

$$\int_0^{T_\varepsilon} ||\partial_t u_\varepsilon(t)||_{L^1}\, dt = \delta,$$

and for such a choice of T_ε, (7.0.9) gives

$$T_\varepsilon \geq \frac{\delta^2}{C\varepsilon^2}.$$

Thus, by (7.0.8),

$$\int_0^{k_1\varepsilon^{-2}} ||\partial_t u_\varepsilon(s)||_{L^2}^2\, ds \leq C\varepsilon^2 =: k_2\varepsilon^2, \tag{7.0.10}$$

for

$$k_1 := \frac{\delta^2}{C}.$$

On the other hand, if

$$\int_0^{\infty} ||\partial_t u_\varepsilon(t)||_{L^1}\, dt \leq \delta,$$

then (7.0.8) must hold for all $T_\varepsilon > 0$, and (7.0.10) holds true in this case as well. □

With this proposition in hand, the proof of the main result is relatively straightforward.

Proof of Theorem 7.0.1. Let k_1, k_2 be as in Proposition 7.0.4, and rescale u_ε by setting $\tilde{u}_\varepsilon(\boldsymbol{x}, t) = u_\varepsilon(\boldsymbol{x}, \varepsilon^{-1}t)$. Proposition 7.0.4 applied to $\tilde{u}_\varepsilon$ reads

$$\int_0^{k_1\varepsilon^{-1}} ||\partial_t \tilde{u}^\varepsilon(t)||_{L^2}^2\, dt \leq k_2\varepsilon,$$

and, in turn, by Hölder's inequality, for $0 < M < k_1\varepsilon^{-1}$,

$$\int_0^{M} ||\partial_t \tilde{u}_\varepsilon(t)||_{L^1}\, dt \leq M^{1/2}(k_2\varepsilon)^{1/2}. \tag{7.0.11}$$

For any $0 < s < M$, by the properties of the Bochner integral (see e.g. [43]) we have

$$\begin{aligned} ||\tilde{u}_\varepsilon(s) - u_{0,\varepsilon}||_{L^1} = \left|\left| \int_0^s \partial_t \tilde{u}_\varepsilon(t)\, dt \right|\right|_{L^1} &\leq \int_0^s ||\partial_t \tilde{u}_\varepsilon(t)||_{L^1}\, dt \\ &\leq \int_0^M ||\partial_t \tilde{u}_\varepsilon(t)||_{L^1}\, dt, \end{aligned}$$

and thus

$$\sup_{0\leq s\leq M} ||\tilde{u}_\varepsilon(s) - u_{0,\varepsilon}||_{L^1} \leq \int_0^M ||\partial_t \tilde{u}_\varepsilon(t)||_{L^1}\, dt. \tag{7.0.12}$$

On the other hand, by (7.0.2),

$$||\tilde{u}_{0,\varepsilon} - u_{E_0}||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+. \tag{7.0.13}$$

Putting together (7.0.11), (7.0.12) and (7.0.13) leads to

$$\sup_{0\leq s\leq M} ||\tilde{u}_\varepsilon(t) - u_{E_0}||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+,$$

which implies the desired result (7.0.4). □

Remark 7.0.5. *This result can also be extended to global minimizers of the Cahn–Hilliard energy, using a simpler argument. See [83] for details.*

Part II

Decay Estimates for the Becker–Döring Equations

Chapter 8

Stability Estimates for the Becker–Döring Equations

This chapter establishes various stability estimates for the Becker–Döring equations. These estimates will be stated in terms of sequence spaces with polynomial moments, and satisfying a zero mean condition, see Definition (1.2.12).

8.1 Definitions, Assumptions and Previous Results

This section states all of the necessary assumptions for the theorems of this part of the thesis. It also quotes all of the external results about the Becker–Döring equations that will be necessary for the results presented here.

It will be necessary to assume the following on the model coefficients:

$$a_i > C_1 > 0 \qquad \text{for all } i \geq 1, \tag{8.1.1}$$

$$\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1, \tag{8.1.2}$$

$$\lim_{i\to\infty} \frac{a_i}{b_i} =: \frac{1}{\zeta_s} \in (0,\infty) \tag{8.1.3}$$

$$a_i, b_i \leq C_2 i \qquad \text{for all } i \geq 1, \tag{8.1.4}$$

with a_i, b_i as in (1.2.1) and (1.2.2), and where C_1, C_2 are fixed constants, independent of i.

This part of the thesis will only consider solutions $(c_i(t))$ of the Becker–Döring equations (1.2.1) with some fixed, subcritical mass, meaning that for some $\zeta < \zeta_s$, the Q_i defined by (1.2.5) will satisfy

$$\sum_{i=1}^{\infty} Q_i i = \tilde{m} = \sum_{i=1}^{\infty} i c_i(t).$$

Using (1.2.4), (1.2.5), (8.1.2) and (8.1.3), it is immediate that

$$\lim_{i\to\infty} \frac{Q_{i+1}}{Q_i} = \frac{\zeta}{\zeta_s} < 1. \tag{8.1.5}$$

This naturally implies that the Q_i are exponentially decaying.

Also, by (8.1.3), it follows that

$$a_i(\zeta+\delta) = a_i(Q_1+\delta) \leq b_i, \qquad \text{for all } i > N_\zeta, \tag{8.1.6}$$

for some $\delta > 0$ and N_ζ that are fixed and independent of i, but possibly dependent on ζ. The assumptions given here are fairly standard, and versions of them can be found in [15, 30, 65]. Specifically, in [15] Ball and Carr made the assumption that

$$a_i \zeta \leq b_i$$

for $i > \hat{N}$, and for all $\zeta < \zeta_s$. In that work, this assumption was made in order to guarantee that $\tilde{V}(c(t_n))$ converges to the minimum value of $\tilde{V}$, where $\tilde{V}$ is given by (1.2.6). In their work, coefficients were required to be $O(i/\log(i))$, but this was subsequently relaxed in [98].

One of the primary advantages to the ℓ^1 estimates given here is that they connect convergence to equilibrium in a quantitative way with inequality (8.1.6). Specifically, inequality (8.1.6) arises naturally in attempting to establish dissipation estimates, thus motivating the analytical need for such assumptions. More importantly, (8.1.6) is satisfied by many of the relevant physical models. For example, one physically-motivated form of the model coefficients is (see [92])

$$a_i = i^\alpha, \quad b_i = a_i \left(\zeta_s + \frac{q}{i^{1-\mu}} \right), \quad \alpha \in (0, 1], \quad \mu \in [0, 1], \quad q > 0.$$

For this model we have

$$b_i - Q_1 a_i \geq (\zeta_s - \zeta) a_i,$$

which naturally implies that assumption (8.1.6) is only satisfied in the subcritical setting.

Following [16], a solution to the Becker–Döring equations is defined in the following way:

Definition 8.1.1. *A function $(c_i(t))$ is a solution to the Becker–Döring equations on $[0, T)$ if*

1. *$\sum_{i=1}^\infty i|c_i| < \infty$ for all $t \in [0, T)$.*
2. *For all i we have that $c_i(t)$ is continuous in time, and non-negative.*
3. *The following equations are satisfied (and well-defined)*

$$c_i(t) = c_i(0) + \int_0^t (J_{i-1}(s) - J_i(s))\, ds, \qquad i \geq 2,$$
$$c_1(t) = c_1(0) - \int_0^t \left(J_1(s) + \sum_{i=1}^\infty J_i(s) \right) ds.$$

The following well-posedness result gives a simplified version of Theorem 2.2 in [16] and Theorem 2.1 in [71].

Proposition 8.1.2. *Assume that $\{a_i\}, \{b_i\}$ satisfy assumptions* (8.1.1)-(8.1.4). *Let $\{c_i^0\}$ be a positive sequence with finite first moment. Then there exists a unique solution $\{c_i(t)\}$ to the Becker–Döring equations satisfying $c_i(0) = c_i^0$.*

The following stability estimate, which can be found in the proof of Theorem 2.2 and Proposition 2.4 in [16], will prove convenient later in the analysis.

Proposition 8.1.3. *Let $\{c_i\}$ be a solution to the Becker–Döring equations, and let $\{h_i\}$ be defined by* (1.2.7). *Suppose that $h(0) \in X_{1+k}$, with $k \geq 0$. Then $\|h(t)\|_{X_{1+k}} \leq \|h(0)\|_{X_{1+k}} C e^{Kt}$ for some C and K independent of h.*

Using the fact that the Q_i are exponentially decaying, the following result is straightforward to prove, and can be deduced from Equation (3.2) in [30]. The proof is included for convenience.

Proposition 8.1.4. *The space H is continuously embedded in Y_η for $\eta > 0$ sufficiently small.*

Proof. One can estimate using Cauchy-Schwarz

$$\sum_{i=1}^{\infty} Q_i e^{\eta i}|h_i| \le \left(\sum_{i=1}^{\infty} Q_i e^{2\eta i}\right)^{1/2} \left(\sum_{i=1}^{\infty} Q_i h_i^2\right)^{1/2}.$$

As long as $\frac{\zeta e^{2\eta}}{\zeta_s} < 1$ then by (8.1.5) it follows that

$$\|h\|_{Y_\eta} \le C\|h\|_H$$

□

The next result comes from [30] (Corollary 2.11 and Theorem 3.5), and concerns the semigroup generated by L, defined by (1.2.10)

Proposition 8.1.5. *For some $\lambda_c > 0$, the operator L generates a contraction semigroup e^{Lt} on H satisfying*

$$\|e^{Lt}\|_{\mathcal{L}(H)} \le e^{-\lambda_c t} \quad \text{for all } t \ge 0.$$

Furthermore, for $\eta > 0$ sufficiently small there exist constants M and $\lambda_\eta > 0$ so that the operator L generates a semigroup on Y_η satisfying

$$\|e^{Lt}\|_{\mathcal{L}(Y_\eta)} \le Me^{-\lambda_\eta t} \quad \text{for all } t \ge 0.$$

At one point some more fine estimates will be needed on the operator L in the space H. Given fixed N, define Λ to be a diagonal operator given by

$$(\Lambda h)_i = -\sigma_i h_i\,, \qquad \sigma_i := Q_1 a_i + b_i\,, \tag{8.1.7}$$

define S to be the operator

$$(Sh)_i := b_i h_{i-1}\mathbf{1}_{\{i>N+1\}} + a_i Q_1 h_{i+1}\mathbf{1}_{\{i>N\}}.$$

and $K := L - \Lambda - S$. In the proof of Lemma 9.1.2 we will use the following facts (see Proposition 2.10 and Corollary 2.11 in [30]).

Proposition 8.1.6. *Assuming* (8.1.1)-(8.1.4), *the operator L given by* (1.2.10) *satisfies the following properties:*

1. *L is self-adjoint in $\ell^2(Q_i)$, with $\mathrm{dom}_{\ell^2(Q_i)}(L) = \mathrm{dom}_{\ell^2(Q_i)}(\Lambda) = \ell^2(Q_i\sigma_i^2)$.*
2. *For some $\lambda_c > 0$ we have that $\langle h, Lh\rangle_H \le -\lambda_c\|h\|_H^2$ for all $h \in H \cap \ell^2(Q_i\sigma_i^2)$.*
3. *$L = \Lambda + S + K$, where K is compact on $\ell^2(Q_i)$, S is symmetric, and for N large enough S satisfies $\|Sh\|_{\ell^2(Q_i)} \le \theta\|\Lambda h\|_{\ell^2(Q_i)}$ for all $h \in \ell^2(Q_i\sigma_i^2)$, where $\theta < 1$.*

8.2 Linearized Stability Estimates in X_1

This section establishes stability estimates for the semigroup generated by the operator L, in the space X_1. As stated in the preliminaries, the reader is reminded that the term "semigroup" always refers to a strongly continuous semigroup of linear operators.

The goal will be to use some recent operator decomposition techniques to derive uniform bounds on e^{Lt} in X_1. This technique was first developed by Gualdani, Mischler and Mouhot [61] to study the Boltzmann equation, and was previously applied to the Becker–Döring equations by Canizo and Lods [30]. The following proposition is one instance of this technique, as given in [30]. The proof is much the same, with the natural extension to the non-autonomous case.

Proposition 8.2.1 (Extension Principle)**.** *Let $Z \subset Y$ be Banach spaces, with Z continuously embedded into Y. Let $I = [0, T)$ with $T = \infty$ permitted, and let $\{A(t)\}_{t\in I}$ and $\{B(t)\}_{t\in I}$ be families of linear operators on Y. Suppose that*

1. *$\{A(t) + B(t)\}_{t\in I}$ generates an evolution family U^Z on Z, satisfying*
$$\|U^Z(t,s)\|_{\mathcal{L}(Z)} \le M_Z e^{-\lambda_Z(t-s)} \qquad \text{for } 0 \le s \le t < T,$$
for some $\lambda_Z \in \mathbb{R}$.
2. *$B(t)$ is "regularizing," meaning that $B(\cdot) \in C(I; \mathcal{L}(Y, Z))$, and that $\|B(t)\|_{\mathcal{L}(Y,Z)} < M_B$, uniformly for $t \in I$.*
3. *$\{A(t)\}_{t\in I}$ generates an evolution family V on Y, satisfying*
$$\|V(t,s)\|_{\mathcal{L}(Y)} \le M_V e^{-\lambda_Y(t-s)} \qquad \text{for } 0 \le s \le t < T,$$
for some $\lambda_Y \in \mathbb{R}$, with $\lambda_Y < \lambda_Z$.

Then $\{A(t) + B(t)\}_{t\in I}$ generates an evolution family U^Y on Y with bound
$$\|U^Y(t,s)\|_{\mathcal{L}(Y)} \le M_Y e^{-\lambda_Y(t-s)} \qquad \text{for } 0 \le s \le t < T. \tag{8.2.1}$$

Proof. Since $B(t)$ is bounded and continuous in t, Remark 2.5.13 implies that $\{A(t)+B(t)\}_{t\in I}$ generates an evolution family on Y. Thus the goal is to prove (8.2.1).

Using Duhamel's formula, see Proposition 2.5.12 and Remark 2.5.13, we can write the evolution family generated by $A(t) + B(t)$ as follows:
$$U^Y(t,s)h(s) = V(t,s)h(s) + \int_s^t U^Y(t,r)(B(r)V(r,s)h(s))\,dr.$$

We then estimate
$$\|U^Y(t,s)h(s)\|_Y \le M_V e^{-\lambda_Y(t-s)}\|h(s)\|_Y + \int_s^t \|U^Y(t,r)B(r)V(r,s)h(s)\|_Y\,dr.$$

As B maps from Y to Z we can replace U^Y with U^Z inside the integral, and then estimate using the decay estimate in Z to infer
$$\|U(t,s)^Y h(s)\|_Y \le M_V e^{-\lambda_Y(t-s)}\|h(s)\|_Y + \int_s^t M_Z e^{-\lambda_Z(t-r)}\|B(r)V(r,s)h(s)\|_Z\,dr.$$

Using our bounds on B and V we obtain

$$\begin{aligned}\|U^Y(t,s)h(s)\|_Y \le M_V e^{-\lambda_Y(t-s)}\|h(s)\|_Y \\ + \|h(s)\|_Y M_V M_Z M_B e^{-\lambda_Y(t-s)} \int_s^t e^{-(\lambda_Z-\lambda_Y)(t-r)}\,dr \\ \le M_Y e^{-\lambda_Y(t-s)}\|h(s)\|_Y,\end{aligned}$$

which is the desired result. □

Remark 8.2.2. *When A and B are constant in time this reduces to a statement about semigroups, and indeed in that case the statement and proof are found in [30]. This section only uses the proposition to prove bounds on the semigroup e^{Lt}, but Section 9.1 uses it in the case of evolution families.*

It is important that the previous result is valid when $\lambda_Y = 0$, meaning that the result applies to semigroups which are only stable.

Next, recall that the operator L is determined by the weak form (1.2.10). Now write

$$L = A + B\,,$$

with the operator A determined via the weak form

$$\begin{aligned}\sum_{i=1}^{\infty} Q_i(Ah)_i\phi_i := \sum_{i=N}^{\infty} Q_iQ_1a_i(h_i-h_{i+1})(\phi_{i+1}-\phi_i-\phi_1) \\ - Q_{N-1}Q_1a_{N-1}h_N(\phi_N-\phi_{N-1}-\phi_1),\end{aligned} \tag{8.2.2}$$

where N is some fixed integer satisfying $N \ge N_\zeta + 1$, with N_ζ given in (8.1.6). The domain of definition for both A and L is initially taken to be the set of sequences with finite support that satisfy (1.2.8), namely having zero "mass".

Remark 8.2.3. *Note that if one sets $\phi_i = i$ then one gets zero, implying that A and B both map into the space of sequences with zero mass. In particular, the operators A, B and L all take values in the spaces Y_η and X_k which incorporate the zero-mass constraint.*

The first step is to give an elementary bound on L and Ξ, which indicates a minimal size for the domain of the closure of these operators. It will be shown that B is bounded, which in turn means that this also gives information about the domain of the closure of A.

Lemma 8.2.4. *For any $m \ge 0$, and for some constant C_m the following bound holds*

$$\|\Xi h\|_{X_{1+m}} \le C_m\|h\|_{X_{2+m}} \qquad \|Lh\|_{X_{1+m}} \le C_m\|h\|_{X_{2+m}}.$$

Proof. We only show the estimate for L, as the estimate for Ξ is essentially identical. We simply estimate

$$\begin{aligned}\|Lh\|_{X_{1+m}} &= \sum_{i=1}^{\infty} Q_i(Lh)_i i^{1+m}\,\mathrm{sgn}(Lh)_i \\ &\le \sum_{i=1}^{\infty} Q_i(a_iQ_1+b_i)|h_i|3(i+1)^{1+m} + 3|h_1|\sum_{i=1}^{\infty} Q_iQ_1a_i(i+1)^{1+m} \\ &\le C\sum_{i=1}^{\infty} Q_i i^{2+m}|h_i|,\end{aligned}$$

where we have used (8.1.4) and the exponential decay of the Q_i. This proves the lemma. □

In order to use the extension principle, Proposition 8.2.1, one must prove that B is "regularizing." (Recall $H \subset Y_\eta \subset X_1$.)

Lemma 8.2.5. *The operator B is a bounded operator from X_1 to H.*

Proof. We compute in weak form:

$$\begin{aligned}\sum_{i=1}^{\infty} Q_i(Bh)_i\phi_i &= \sum_{i=1}^{N-2} Q_iQ_1a_i(h_i - h_{i+1})(\phi_{i+1} - \phi_i - \phi_1)\\ &\quad + \sum_{i=1}^{\infty} Q_iQ_1a_ih_1(\phi_{i+1} - \phi_i - \phi_1)\\ &\quad + Q_{N-1}Q_1a_{N-1}h_{N-1}(\phi_N - \phi_{N-1} - \phi_1)\\ &=: B_1(h,\phi) + B_2(h,\phi) + B_3(h,\phi).\end{aligned}$$

By the Cauchy-Schwarz inequality, the fact that $0 < c \le Q_i/Q_{i+1} \le C < \infty$ by (8.1.5), and the equivalence of finite dimensional norms,

$$|B_1(h,\phi)| \le C\left(\sum_{i=1}^{N-1} Q_i\phi_i^2\right)^{1/2}\left(\sum_{i=1}^{N-1} Q_ih_i^2\right)^{1/2} \le C\|\phi\|_H\|h\|_{X_1}.$$

Furthermore,

$$|B_2(h,\phi)| \le C|h_1|\left(\sum_{i=1}^{\infty} Q_ia_i^2\right)^{1/2}\left(\sum_{t=1}^{\infty} Q_i\phi_i^2\right)^{1/2} \le C\|h\|_{X_1}\|\phi\|_H.$$

Similarly, $|B_3(h,\phi)| \le C\|h\|_{X_1}\|\phi\|_H$. By taking the supremum over $\phi \in H$ with $\|\phi\|_H \le 1$, we obtain the desired result, $\|Bh\|_H \le C\|h\|_{X_1}$. □

The next step is to show that A, or more precisely its closure, generates a contraction semigroup on X_1. This will be proved by showing that A is dissipative and applying the Lumer–Phillips theorem, see Definition 2.5.3 and Proposition 2.5.4.

By way of notation, when $X = \ell^1(Q_iw_i)$ and $\|h\|_X = \sum_{i=1}^{\infty} Q_iw_i|h_i|$ define

$$\langle \mathrm{sgn}(h), \phi\rangle_{X^*,X} := \sum_{i=1}^{\infty} Q_iw_i\phi_i\mathrm{sgn}(h_i)\,.$$

By the definition of $\mathcal{J}(x)$, namely (2.5.2), it is clear that if $\langle \mathrm{sgn}(h), Ah\rangle_{X^*,X} \le 0$ for all h in the domain of definition of A then A is dissipative.

Proposition 8.2.6. *The operator A given by* (8.2.2) *is dissipative on X_1.*

Proof. Rearranging our sum and using (1.2.4) to say $Q_iQ_1a_i = Q_{i+1}b_{i+1}$, we find that

$$
\begin{aligned}
&\langle \mathrm{sgn}(h), Ah\rangle_{X_1^*,X_1} \\
&\quad = \sum_{i=N}^{\infty} Q_i Q_1 a_i h_i((i+1)\mathrm{sgn}(h_{i+1}) - i\,\mathrm{sgn}(h_i) - \mathrm{sgn}(h_1))) \\
&\qquad - \sum_{i=N}^{\infty} Q_i b_i h_i(i\,\mathrm{sgn}(h_i) - (i-1)\mathrm{sgn}(h_{i-1}) - \mathrm{sgn}(h_1)) \\
&\quad = \sum_{i=N}^{\infty} Q_i h_i \Big(Q_1 a_i (i+1)(\mathrm{sgn}(h_{i+1}) - \mathrm{sgn}(h_i)) + b_i(i-1)(\mathrm{sgn}(h_{i-1}) - \mathrm{sgn}(h_i))\Big) \\
&\qquad + \sum_{i=N}^{\infty} Q_i |h_i| (a_i Q_1 - b_i) + \mathrm{sgn}(h_1) \sum_{i=N}^{\infty} Q_i h_i (b_i - Q_1 a_i) \\
&\quad =: E_1 + E_2 + E_3,
\end{aligned}
$$

Because $h_i(\mathrm{sgn}(h_{i\pm1}) - \mathrm{sgn}(h_i)) \leq 0$, we see $E_1 \leq 0$. Furthermore, by (8.1.6) we have that

$$
E_2 + E_3 = 2 \sum_{\substack{i=N \\ \mathrm{sgn}(h_1)\neq \mathrm{sgn}(h_i)}}^{\infty} Q_i |h_i| (a_i Q_1 - b_i) \leq 0\,.
$$

This readily implies that A is dissipative (see Definition 2.5.3). $\square$

Remark 8.2.7. *In the case where $a_i \sim i$ and $a_i Q_1 - b_i > \bar{\lambda} i$ for all $i \geq 1$, the previous estimate with $N = 1$ gives*

$$
\begin{aligned}
\langle \mathrm{sgn}(h), Ah\rangle_{X^*,X} &\leq 2 \sum_{\substack{i=N \\ \mathrm{sgn}(h_1)\neq \mathrm{sgn}(h_i)}}^{\infty} Q_i |h_i| (a_i Q_1 - b_i) \\
&\leq -\bar{\lambda} \sum_{\substack{i=N \\ \mathrm{sgn}(h_1)\neq \mathrm{sgn}(h_i)}}^{\infty} Q_i |h_i| i = -\frac{\bar{\lambda}}{2} \sum_{i=1}^{\infty} Q_i |h_i| i \\
&\leq -\frac{\bar{\lambda}}{2} \|h\|_X,
\end{aligned}
$$

where we have used (1.2.8)*. This implies that A has a spectral gap in X_1, and hence, by using the operator decomposition result, that L has a spectral gap in X_1. This type of result, namely exponential decay in X_1 when $a_i \sim i$, was obtained using entropy dissipation estimates in [29].*

With the dissipation estimate in hand, it is now possible to show that the closure of A indeed generates a semigroup.

Lemma 8.2.8. *The closure of A (which we also denote by A), generates a contraction semigroup on X_1.*

Proof. We know that $H \subset X_1$, and that H is dense in X_1. By Proposition 8.1.5 we know that L generates a contraction semigroup on H. As B is bounded on H, we know that the closure in H of $A = L - B$ generates a semigroup on H with bound $Me^{\omega t}$, see Proposition 2.5.6. Proposition 2.5.2 implies that for $\lambda > 0$ large enough $A - \lambda I$ is invertible on H. Thus the range of (the closure in H of) $A - \lambda I$ contains H, and thus the range of $A - \lambda I$ is dense in X_1. Because A is dissipative by Proposition 8.2.6, the Lumer–Phillips theorem, namely Theorem 2.5.4, then implies that A generates a contraction semigroup on X_1. $\square$

By combining Proposition 8.1.5, Proposition 8.2.1, Lemma 8.2.5 and Lemma 8.2.8 the following is immediate.

Theorem 8.2.9. *The closure of L generates a semigroup e^{Lt} on X_1 uniformly bounded in time:*

$$\|e^{Lt}\|_{\mathcal{L}(X_1)} \leq M \qquad \text{for all } t \geq 0.$$

It is natural to question the sharpness of these dissipation bounds. The following theorem demonstrates a limited type of sharpness of the bounds from Theorem 8.2.9.

Theorem 8.2.10. *Suppose, in addition to* (8.1.1)-(8.1.4)*, that $\lim_{i\to\infty} \frac{a_i}{i^\alpha} = 0$, for some $\alpha \in (0,1)$, and that $a_i - a_{i-1} = o(1)$. Then the operator L has an approximate eigenvalue at 0 in X_1. In other words, there exists a sequence with $\|h_j\|_{X_1} = 1$ but $\|Lh_j\|_{X_1} \to 0$.*

Proof. Define

$$\tilde{h}_i = \begin{cases} 0 & \text{if } i < N_1, \\ \frac{1}{iQ_i} & \text{if } N_1 \leq i \leq N_2, \\ 0 & \text{if } N_2 < i, \end{cases}$$

where $N_1 < N_2$ are constants to be determined. Clearly

$$\sum_{i=1}^{\infty} Q_i i^k |\tilde{h}_i| = \sum_{i=N_1}^{N_2} i^{k-1}.$$

Furthermore, for $N_1 < i < N_2$

$$\begin{aligned} Q_i(L\tilde{h})_i i^k &= i^k Q_i \left(b_i(\tilde{h}_{i-1} - \tilde{h}_i) + a_i Q_1(\tilde{h}_{i+1} - \tilde{h}_i) \right) \\ &= i^{k-1} a_i \left(\frac{b_i}{a_i} \left(\frac{Q_i i}{Q_{i-1}(i-1)} - 1 \right) + Q_1 \left(\frac{Q_i i}{Q_{i+1}(i+1)} - 1 \right) \right) \\ &= i^{k-1} a_i \left(\frac{b_i}{a_i} \left(-1 + \frac{a_{i-1}Q_1}{b_i} \left(1 + \frac{1}{i-1} \right) \right) + Q_1 \left(\frac{b_{i+1}}{a_i Q_1} \left(1 - \frac{1}{i+1} \right) - 1 \right) \right) \\ &= i^{k-1} \left(a_{i-1}Q_1 - b_i + b_{i+1} - a_i Q_1 - \frac{b_{i+1}}{i+1} + \frac{a_{i-1}Q_1 a_i}{b_i(i-1)} \right) \end{aligned}$$

As $a_i - a_{i-1} = o(1)$ and $\frac{a_i}{i^\alpha} \to 0$, and by (8.1.3), for any $\delta > 0$, we can find an N_1 so that

$$Q_i(Lh)_i i^k \leq i^{k-1}\delta$$

for $N_1 < i < N_2$.

Next, for any $i > 1$,

$$\begin{aligned} Q_i |(L\tilde{h})_i| i^k &= i^k Q_i \left| b_i(\tilde{h}_{i-1} - \tilde{h}_i) + a_i Q_1(\tilde{h}_{i+1} - \tilde{h}_i) \right| \\ &= i^k \left| \frac{b_i}{i} \left(\frac{Q_i i}{Q_{i-1}(i-1)} - 1 \right) + \frac{a_i Q_1}{i} \left(\frac{Q_i i}{Q_{i+1}(i+1)} - 1 \right) \right| \\ &\leq C i^{k+\alpha-1}, \end{aligned}$$

where we have used the fact that $\frac{a_i}{i^\alpha} \to 0$ and (8.1.5), and where C is independent of i, N_1, and N_2.

Last, for $i = 1$,

$$\begin{aligned} |Q_1(L\tilde{h})_1| &= \left| \sum_{i=1}^{\infty} a_i Q_i Q_1 (\tilde{h}_{i+1} - \tilde{h}_i) \right| \\ &\leq C \sum_{i=N_1}^{N_2} \frac{a_i}{i} \leq C(N_2 - N_1), \end{aligned}$$

with C independent of N_1, N_2, and where we have used the fact that $\frac{a_i}{i^\alpha} \to 0$.

Thus we find that

$$\sum_{i=1}^{\infty} Q_i (L\tilde{h})_i i^k \leq \delta \sum_{i=N_1}^{N_2} i^{k-1} + CN_1^{k+\alpha-1} + CN_2^{k+\alpha-1},$$

where we have made C larger as necessary to absorb the $i = 1$ term.

Thus if we set $N_2 = 2N_1$, we find that

$$\lim_{N_1 \to \infty} \frac{\sum_{i=1}^{\infty} Q_i i^k |\tilde{h}_i|}{\sum_{i=1}^{\infty} Q_i i^k |(L\tilde{h})_i|} = \infty.$$

By constructing two of these pulses, one negative and one positive with non-overlapping support, and then adding together scaled versions of the same so that the mass constraint is satisfied, we obtain the desired result. This completes the proof.

□

Some sharper estimates for lower bounds on the decay are a subject of current investigation [82].

8.3 Algebraic Decay Estimates

This section proves algebraic decay estimates for e^{Lt}. The key tool is an interpolation result, which is a slight modification of Theorem 2.1 in [46]. In that case the result was used to study convergence properties of travelling waves.

Theorem 8.3.1. *Let $\eta \in (0,1)$ and $k_1, k_2 \in \mathbb{R}$ with $0 < k_1 < k_2$. Let $\{S(t)\}_{t \geq 0}$ be a family of linear operators on X_1 which for any $t > 0$ satisfies*

$$\|S(t)u\|_{X_1} \leq M\|u\|_{X_1}, \qquad \|S(t)u\|_{Y_\eta} \leq Me^{-\lambda_\eta t}\|u\|_{Y_\eta}, \tag{8.3.1}$$

where u is an arbitrary element of the appropriate spaces, M is a fixed positive constant and $\lambda_\eta > 0$. Then the operators $S(t)$ necessarily are bounded from X_{1+k_2} to X_{1+k_1} and satisfy

$$\|S(t)u\|_{X_{1+k_1}} \leq C(1+t)^{-(k_2-k_1)}\|u\|_{X_{1+k_2}} \quad \text{for all } u \in X_{1+k_2} \text{ and } t \geq 0,$$

where C depends on k_1, k_2, M and λ_η.

Proof. The proof is very similar to that found in [46], with modifications necessary, however, to handle the mass constraint and weighted norm on X_1.

1. Consider $K : \mathbb{R} \times X_1 \to \mathbb{R}$ defined by

$$K(s,u) = \inf_{v \in Y_\eta} (\|u - v\|_{X_1} + e^s\|v\|_{Y_\eta}).$$

In interpolation theory [20] this is known as a modified K-functional. For fixed s, $K(s,\cdot)$ is a norm. Clearly $K(s,u)$ is increasing in s and bounded above by $\|u\|_{X_1}$. Furthermore, we claim that K is absolutely continuous in s. Indeed, if we define $\tilde{K}(\tilde{s},u) := K(\log \tilde{s}, u)$, then $\tilde{K}(\cdot,u)$ can be written an the infimum of affine functions, and thus must be concave. This readily implies that $K(s,u)$ is absolutely continuous in s.

We begin by proving upper and lower bounds on K. First, we get the lower bound

$$K(s,u) \geq \sum_{i=1}^{\infty} Q_i \inf_{v \in \mathbb{R}} (|u_i - v|i + e^{s+\eta i}|v|) = \sum_{i=1}^{\infty} Q_i |u_i| (i \wedge e^{s+\eta i}) \,. \tag{8.3.2}$$

For the upper bound, observe $x \wedge e^{s+\eta x} = x$ for all $x \geq 0$ if and only if $s \geq s_\eta := -1 - \log \eta$. Thus for $s \geq s_\eta$,

$$K(s,u) \leq \|u\|_{X_1} = \sum_{i=1}^{\infty} Q_i|u_i|(i \wedge e^{s+\eta i}),$$

so that $K(s,u) = \|u\|_{X_1}$ in this case. Suppose now that $s < s_\eta$. Then $1/\eta \in \{x : e^{s+\eta x} \leq x\} = [z_-, z_+] \subset (0,\infty)$. Let $j(s)$ be the least integer greater than or equal to z_+, and define the sequence $v_s(u)$ by

$$v_s(u)_i := \begin{cases} u_i & \text{for } i < j(s), \\ (Q_i i)^{-1} \sum\limits_{k \geq j(s)} Q_k k u_k & \text{for } i = j(s), \\ 0 & \text{for } i > j(s). \end{cases}$$

In particular note that $\sum_{i=1}^\infty Q_i i v_s(u)_i = 0$, so $v_s(u) \in Y_\eta$. Writing $j = j(s)$, we then find

$$\begin{aligned} K(s,u) &\leq \|u - v_s(u)\|_{X_1} + e^s \|v_s(u)\|_{Y_\eta} \\ &= \left| \sum_{i>j} Q_i i u_i \right| + \sum_{i>j} Q_i i |u_i| + e^s \sum_{i=1}^{j-1} Q_i e^{\eta i}|u_i| + e^s \frac{Q_j e^{\eta j}}{Q_j j} \left| \sum_{i=j}^{\infty} Q_i i u_i \right| \\ &\leq \left(2 + \frac{e^{s+\eta j}}{j}\right) \sum_{i=j}^{\infty} Q_i i |u_i| + \sum_{i=1}^{j-1} Q_i e^{s+\eta i}|u_i|. \end{aligned}$$

Now, $j^{-1}e^{s+\eta j} \leq z_+^{-1} e^{s+\eta(z_+ +1)} = e^\eta$, and $i \geq j$ implies $i = i \wedge e^{s+\eta i}$. Furthermore, whenever $1 \leq i \leq z_-$ we have $e^{s+\eta i} \leq e^{s+\eta z_-} = z_- \leq 1/\eta \leq i/\eta = (i \wedge e^{s+\eta i})/\eta$. By these estimates we find that with $C = \max\{2 + e^\eta, 1/\eta\}$ we have that for any $s \in \mathbb{R}$,

$$K(s,u) \leq C \sum_{i=1}^{\infty} Q_i |u_i| (i \wedge e^{s+\eta i}). \tag{8.3.3}$$

2. In the next step, for $r > 0$ we set

$$h_r(s) := \begin{cases} e^{-s} & \text{for } s \geq 0, \\ (1-s)^{r-1} & \text{for } s \leq 0, \end{cases}$$

and define the norm

$$\|u\|_{*,r} := \int_{\mathbb{R}} K(s,u) h_r(s)\, ds.$$

We claim that $\|\cdot\|_{*,r}$ is equivalent to the norm in X_{1+r}. By (8.3.2) and (8.3.3), it suffices to show there exist $C_-, C_+ > 0$ independent of i such that

$$C_-(1+i)^{1+r} \leq \int_{\mathbb{R}} (i \wedge e^{s+\eta i}) h_r(s)\, ds \leq C_+(1+i)^{1+r} \quad \text{for } i \geq 1. \tag{8.3.4}$$

To show this, we first bound the part of the integral over $s \in [0,\infty)$, finding that

$$1 \leq \int_0^\infty (i \wedge e^{s+\eta i}) e^{-s}\, ds \leq i \leq (1+i)^{1+r}. \tag{8.3.5}$$

For the part over $s \in (-\infty, 0]$, after changing variables twice via $\tilde{s} = -s$, $\sigma = \tilde{s} - \eta i$, we have

$$\begin{aligned}\int_{-\infty}^{0} (i \wedge e^{s+\eta i})(1-s)^{r-1}\, ds &\le i \int_0^\infty (1 \wedge e^{-\tilde{s}+\eta i})(1+\tilde{s})^{r-1}\, d\tilde{s} \\ &= i \int_0^{\eta i} (1+\tilde{s})^{r-1}\, d\tilde{s} + i \int_0^\infty e^{-\sigma}(1+\eta i+\sigma)^{r-1}\, d\sigma \\ &\le Ci(1+\eta i)^r \le C(1+i)^{1+r}\,.\end{aligned}$$

This establishes the upper bound in (8.3.4).

To get the lower bound, choose I_η so large that $i > I_\eta$ implies $\eta i - \log i \ge \frac{1}{2}\eta i$. For $i \le I_\eta$ we have $(1+i)^{r+1} \le (1+I_\eta)^{r+1}$, hence we get the lower bound in (8.3.4) with $C_- = (1+I_\eta)^{-1-r}$ by using (8.3.5). For $i > I_\eta$, we find

$$\begin{aligned}\int_{-\infty}^{0} (i \wedge e^{s+\eta i})(1-s)^{r-1}\, ds &= i \int_0^\infty (1 \wedge e^{-\tilde{s}+\eta i-\log i})(1+\tilde{s})^{r-1}\, d\tilde{s} \\ &\ge i \int_0^{\eta i/2} \tilde{s}^{r-1}\, d\tilde{s} \ge C(1+i)^{1+r}\,.\end{aligned}$$

Thus $\|\cdot\|_{*,r}$ is equivalent to $\|\cdot\|_{X_{1+r}}$.

3. Now, let $H_r(t) := \int_t^\infty h_r(\tau)d\tau$. We claim that, for fixed $0 < k_1 < k_2$,

$$H_{k_1}(s+t) \le CH_{k_2}(s)(1+t)^{k_1-k_2}$$

for all $s \in \mathbb{R}$, and for $t \ge 0$. To prove the claim, we first note that

$$H_{k_1}(s) = \begin{cases} e^{-s} & \text{for } s \ge 0, \\ 1 + \frac{(1-s)^{k_1}-1}{k_1} & \text{for } s < 0, \end{cases}$$

and furthermore, for $s < 0$, we have that

$$\frac{(1-s)^{k_1}}{k_1+1} \le H_{k_1}(s) \le \frac{(k_1+1)(1-s)^{k_1}}{k_1}. \tag{8.3.6}$$

We then consider separate cases. First, if $s \ge 0$,

$$H_{k_1}(s+t) = e^{-(s+t)} \le Ce^{-s}(1+t)^{k_1-k_2} = CH_{k_2}(s)(1+t)^{k_1-k_2}.$$

Next suppose that $s < 0 \le s+t$. Then

$$\begin{aligned}H_{k_1}(s+t) = e^{-(s+t)} &\le C(1+s+t)^{-k_2} \\ &= C\frac{(1-s)^{k_2}}{(1+t-s(s+t))^{k_2}} \le C(1+t)^{-k_2}H_{k_2}(s)\,,\end{aligned}$$

where we have used (8.3.6). Finally, in the case that $t < -s$, we note that because $k_1 - k_2 < 0$,

$$(1-(s+t))^{k_1} \le (1-s)^{k_1} \le (1-s)^{k_2}(1+t)^{k_1-k_2}\,.$$

In light of (8.3.6) this proves the claim.

4. Next, we use assumption (8.3.1) to estimate

$$\begin{aligned}K(s, S(t)u) &\le \inf_{v\in Y_\eta} (\|S(t)u - S(t)v\|_{X_1} + e^s\|S(t)v\|_{Y_\eta}) \\ &\le M \inf_{v\in Y_\eta} (\|u-v\|_{X_1} + e^{s-\lambda_\eta t}\|v\|_{Y_\eta}) \\ &= MK(s-\lambda_\eta t, u).\end{aligned}$$

We remark that for $u \in Y_\eta$ we have that $0 \leq K(s,u) \leq \|u\|_{X_1} \wedge e^s \|u\|_{Y_\eta}$, and thus for $u \in Y_\eta$ we have that $H_r(s)K(s,u)$ goes to zero as $s \to \pm\infty$. Thus we may use integration by parts, and our previous estimates, to obtain the following for any $u \in Y_\eta$:

$$\begin{aligned}
\|S(t)u\|_{X_{1+k_1}} &\leq C \int_{\mathbb{R}} K(s, S(t)u) h_{k_1}(s)\, ds \\
&\leq C \int_{\mathbb{R}} K(s - \lambda_\eta t, u) h_{k_1}(s)\, ds \\
&= C \int_{\mathbb{R}} \frac{\partial K}{\partial s}(s,u) H_{k_1}(s + \lambda_\eta t)\, ds \\
&\leq C(1+t)^{k_1 - k_2} \int \frac{\partial K}{\partial s}(s,u) H_{k_2}(s)\, ds \\
&= C(1+t)^{k_1 - k_2} \int_{\mathbb{R}} K(s,u) h_{k_2}(s)\, ds \\
&= C(1+t)^{k_1 - k_2} \|u\|_{*,k_2} \leq C(1+t)^{k_1 - k_2} \|u\|_{X_{1+k_2}}.
\end{aligned}$$

Because Y_η is dense in X_{1+k_2}, we have the desired inequality. This completes the proof. □

It is natural to apply this theorem to the semigroup generated by e^{Lt}.

Corollary 8.3.2. *Provided $0 < k_1 < k_2$, the semigroup e^{Lt} generated by the operator L satisfies*

$$\|e^{Lt}u\|_{X_{1+k_1}} \leq C(1+t)^{-(k_2-k_1)} \|u\|_{X_{1+k_2}} \qquad \textit{for all } u \in X_{1+k_2},$$

where C depends on k_1 and k_2, but not on u or t.

Proof. This follows directly from Proposition 8.1.5, Theorem 8.2.9, Theorem 8.3.1. □

At this point in the analysis it is not clear whether the semigroup e^{Lt} can be defined on the space X_k. This is addressed by Corollary 9.1.8.

Chapter 9

Decay Rates for the Becker–Döring Equations

The goal of this chapter is to prove Theorem (1.2.1). First, non-linear stability results, namely Theorem 9.1.1, will be established using the theory of evolution families. This will be combined with the linearized decay rates of the previous chapter to prove Theorem (1.2.1).

9.1 Stability estimates in X_k

This section proves stability estimates for $\Theta(g)$, and some associated semigroup results. These estimates are very similar to those proved in the space X_1. These estimates are primarily technical in nature, in the sense that they are used to deduce existence of the necessary evolution families. It is probably possible to use these results to derive well-posedness and stability results like those given in Proposition 8.1.3, but that is not the aim of this work.

The main goal is to prove the following theorem.

Theorem 9.1.1. *Let $\{c_i\}$ be a solution to the Becker–Döring equations (see Definition 8.1.1), and let $\{h_i\}$ be determined by* (1.2.7)*. Assume that the model coefficients in* (1.2.2) *satisfy* (8.1.1)-(8.1.4)*. Fix $k > 2$. Then given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|h(0)\|_{X_{1+k}} < \delta$ then $\|h(t)\|_{X_{1+k}} < \varepsilon$ for all $t \geq 0$.*

The general strategy is to derive bounds on the evolution family $U(t,s)$ generated by $\Theta(h_1(t))$ when h_1 is small. The first step is to establish bounds in H directly using dissipation estimates. Consequently, it is possible to establish stability bounds on $U(t,s)$ in X_{1+k} by using the extension principle from Proposition 8.2.1. This then immediately implies Theorem 9.1.1.

9.1.1 Non-linear Stability in H

The following lemma gives a local, non-linear stability estimate in the space H.

Lemma 9.1.2. *Suppose that $g(t) \in C^1(I;\mathbb{R})$, for $I = [0,T)$ with T possibly infinite. Suppose furthermore that the model coefficients in* (1.2.2) *satisfy* (8.1.1)-(8.1.4)*. Then there exist δ_H and $\lambda > 0$ such that if $|g(t)| < \delta_H$ then $\{\Theta(g(t))\}_{t\in I}$ generates an evolution family U_H in H on the interval I with bound*

$$\|U_H(t,s)\|_{\mathcal{L}(H)} \leq e^{-\lambda(t-s)} \qquad \textit{for } 0 \leq s \leq t < T.$$

The central tools in proving this lemma are Propositions 2.5.10 and 8.1.6.

Proof. We first claim that the following spectral gap estimate holds as long as g is sufficiently small: For some $\lambda_H > 0$,

$$\langle \Theta(g)h, h\rangle_H \le -\lambda_H \|h\|_H^2 \qquad \text{for all } h \in H \cap \ell^2(Q_i\sigma_i).$$

To prove this inequality, we recall (1.2.9) and use Proposition 8.1.6 to estimate

$$\begin{aligned}\langle \Theta(g)h, h\rangle_H &= \langle (1-\varepsilon)Lh, h\rangle_H + \varepsilon\langle Kh, h\rangle_{\ell^2(Q_i)} + \langle (g\Xi + \varepsilon(\Lambda + S))h, h\rangle_{\ell^2(Q_i)}\\ &\le -(1-\varepsilon)\lambda_c\|h\|_H^2 + \varepsilon\|K\|_{\mathcal{L}(\ell^2(Q_i))}\|h\|_H^2 + \langle (g\Xi + \varepsilon(\Lambda + S))h, h\rangle_{\ell^2(Q_i)}.\end{aligned}$$

(Here, note that Λh, Sh and Kh belong to $\ell^2(Q_i)$ but perhaps not to the zero-mass subspace H.) We select ε small enough that $\frac{(1-\varepsilon)\lambda_c}{2} > \varepsilon\|K\|_{\mathcal{L}(\ell^2(Q_i))}$. As S is Λ-bounded with Λ-bound $\theta < 1$ we have that S is relatively bounded (with relative bound smaller than one) by $\frac{1+\theta}{2}\Lambda$. Because S is symmetric, Proposition 2.5.5 implies that

$$\left\langle \left(S + \left(\frac{1+\theta}{2}\right)\Lambda\right)h, h\right\rangle_{\ell^2(Q_i)} \le 0.$$

Thus we can estimate

$$\begin{aligned}\langle \Theta(g)h, h\rangle_H &\le -\frac{(1-\varepsilon)\lambda_c}{2}\|h\|_H^2 + \langle\left(\varepsilon\frac{1-\theta}{2}\Lambda + g\Xi\right)h, h\rangle_{\ell^2(Q_i)}\\ &= -\frac{(1-\varepsilon)\lambda_c}{2}\|h\|_H^2 + \sum_{i=1}^{\infty} Q_i\left(-\varepsilon\frac{1-\theta}{2}\sigma_i h_i^2 + Q_1 a_i g h_i(h_{i+1} - h_i - h_1)\right)\\ &\le -\frac{(1-\varepsilon)\lambda_c}{2}\|h\|_H^2 + \sum_{i=1}^{\infty} Q_i\left(-\varepsilon\frac{1-\theta}{2}\sigma_i h_i^2 + \frac{|Q_1 g|}{2}a_i(4h_i^2 + h_{i+1}^2 + h_1^2)\right)\\ &\le -\frac{(1-\varepsilon)\lambda_c}{2}\|h\|_H^2 + \sum_{i=1}^{\infty} Q_i\left(-\varepsilon\frac{1-\theta}{2}\sigma_i + a_i C|Q_1 g|\right)h_i^2,\end{aligned}$$

where we have used the assumptions (8.1.2) and (8.1.5) and the fact that $\sum_{i=1}^{\infty} Q_i a_i$ is finite. By (8.1.7) there exists a $\delta_H > 0$ so that if $|g| < \delta_H$ then $(a_i C|Q_1 g| - \varepsilon\frac{1-\theta}{2}\sigma_i) < 0$. Thus if $|g| < \delta_H$ we deduce that

$$\langle \Theta(g)h, h\rangle_H \le -\frac{(1-\varepsilon)\lambda_c}{2}\|h\|_H^2 =: -\lambda_H\|h\|_H^2,$$

which proves the claim.

We observe, from the previous estimates, that indeed $\|\Xi h\|_H = \|\Xi h\|_{\ell^2(Q_i)} \le C\|\Lambda h\|_{\ell^2(Q_i)}$. This implies that $S + g\Xi$ (and also $S + g\Xi + K$) is relatively bounded by Λ with relative bound strictly less than one, as long as $|g| < \delta_H$, where perhaps we have made δ_H smaller.

We then claim that this implies that $(\lambda - \Theta(g))$ is invertible on $\ell^2(Q_i)$ for some $\lambda > 0$. First, since Λ is diagonal, it is clear that $(\lambda - \Lambda)$ is invertible for any $\lambda > 0$ with $(\lambda - \Lambda)^{-1} = \operatorname{diag}(\lambda + \sigma_i)^{-1}$. We note that if $(I - (S + g\Xi + K)(\lambda - \Lambda)^{-1})$ is invertible for some $\lambda > 0$, then $(\lambda - \Theta(g))$ is invertible at that same λ, with

$$(\lambda - \Theta(g))^{-1} = (\lambda - \Lambda)^{-1}(I - (S + g\Xi + K)(\lambda - \Lambda)^{-1})^{-1}.$$

Recall that $I - W$ is invertible for any linear operator satisfying $\|W\| < 1$. Thus if we can prove that $\|(S + g\Xi + K)(\lambda - \Lambda)^{-1}\| < 1$, then the claim must hold true.

To prove this, we estimate, for $h \in \ell^2(Q_i)$, and for some $\theta < 1$,

$$\|(S + g\Xi + K)(\lambda - \Lambda)^{-1}h\|_{\ell^2(Q_i)} \le \theta\|\Lambda(\lambda - \Lambda)^{-1}h\| + C\|(\lambda - \Lambda)^{-1}h\|_{\ell^2(Q_i)},$$

where we have used the fact that $(S + g\Xi + K)$ is relatively bounded with constant less than one. We then remark that $\Lambda(\lambda - \Lambda)^{-1} = \mathrm{diag} - \frac{\sigma_i}{\lambda+\sigma_i}$. This implies that $\|\Lambda(\lambda - \Lambda)^{-1}\| \leq 1$ for all $\lambda > 0$. On the other hand, $\|(\lambda - \Lambda)^{-1}\| < \lambda^{-1}$ for $\lambda > 0$. This implies that $\|(S + g\Xi + K)(\lambda - \Lambda)^{-1}\| < 1$ for $\lambda > 0$ large enough. This proves the claim.

Now, since $\Theta(g)$ holds the zero mass subspace of $\ell^2(Q_i)$ invariant, we have that $(\lambda - \Theta(g))$ is invertible on H for some $\lambda > 0$. Thus by the Lumer-Phillips theorem, $\Theta(g)$ generates a semigroup in H as long as $|g| < \delta_H$. Furthermore, by the relative bound it is clear that $\mathrm{dom}_H(\Theta(g)) = \mathrm{dom}_{\ell^2(Q_i)}(\Theta(g)) \cap H = \mathrm{dom}_{\ell^2(Q_i)}(\Lambda) \cap H = \ell^2(Q_i\sigma_i^2) \cap H$.

Now, as $g(t)$ is C^1 it is clear that for $v \in \ell^2(Q_i\sigma_i^2)$ we have that $\Theta(g(t)))v$ is in $C^1(I; H)$. We then directly apply Proposition 2.5.10 to obtain the desired result. □

9.1.2 Non-linear Stability in X_{1+k}

The main goal of this subsection is to prove the following lemma.

Lemma 9.1.3. *Suppose that $g(t) \in C^1(I; \mathbb{R})$, for $I = [0, T)$ with T possibly infinite. Suppose furthermore that the model coefficients in* (1.2.2) *satisfy* (8.1.1)-(8.1.4) *and that $k > 0$. Then there exists a δ_k such that if $|g(t)| < \delta_k$ then $\{\Theta(g(t))\}_{t\in I}$ generates an evolution family $U_{X_{1+k}}(t, s)$ in X_{1+k} on the interval I with bound*

$$\|U_{X_{1+k}}(t, s)\|_{\mathcal{L}(X_{1+k})} \leq M_k,$$

where M_k is independent of s, t and the particular choice of g.

This lemma is proved using Proposition 8.2.1, in conjunction with the stability in H established in the previous subsection. To begin, define the operator $A(g)$ in weak form by

$$\begin{aligned}\sum_{i=1}^{\infty} Q_i(A(g)h)_i\phi_i := &\sum_{i=N}^{\infty} Q_iQ_1a_i(h_i - h_{i+1} + gh_i)(\phi_{i+1} - \phi_i - \phi_1) \\ &- Q_{N-1}Q_1a_{N-1}h_N(\phi_N - \phi_{N-1} - \phi_1)\,,\end{aligned} \tag{9.1.1}$$

where N is a constant, greater than $N_\zeta + 1$, to be determined. Define $B(g) := \Theta(g) - A(g)$.

The next proposition establishes the dissipativity of $A(g)$.

Proposition 9.1.4. *Under the assumptions of Lemma 9.1.3, and if N in* (9.1.1) *is chosen large enough, then there exists a constant δ_k so that if $|g| < \delta_k$ then*

$$\langle \mathrm{sgn}(h), A(g)h\rangle_{X^*_{1+k}, X_{1+k}} \leq 0 \quad \textit{for all } h \in X_{2+k}.$$

Proof. With $w_i = i^{1+k}$ and using $\phi_i = w_i\mathrm{sgn}(h_i)$ in (9.1.1), we compute, as in the proof of Proposition 8.2.6,

$$\begin{aligned}
&\langle \mathrm{sgn}(h), A(g)h\rangle_{X^*_{1+k},X_{1+k}} \\
&\quad= \sum_{i=N}^{\infty} Q_i h_i \Big(Q_1 a_i w_{i+1}(\mathrm{sgn}(h_{i+1}) - \mathrm{sgn}(h_i)) + b_i w_{i-1}(\mathrm{sgn}(h_{i-1}) - \mathrm{sgn}(h_i))\Big) \\
&\qquad + \sum_{i=N}^{\infty} Q_i |h_i| (a_i Q_1 (w_{i+1} - w_i) + b_i (w_{i-1} - w_i)) \\
&\qquad + \mathrm{sgn}(h_1) \sum_{i=N}^{\infty} Q_i h_i (b_i - Q_1 a_i) \\
&\qquad + g \sum_{i=N}^{\infty} Q_i h_i Q_1 a_i (w_{i+1}\mathrm{sgn}(h_{i+1}) - w_i \mathrm{sgn}(h_i) - \mathrm{sgn}(h_1)) \\
&\quad =: E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

First we estimate E_2, written as

$$E_2 = \sum_{i=N}^{\infty} Q_i |h_i| (w_{i+1} - w_i) \left(a_i Q_1 - b_i \frac{w_i - w_{i-1}}{w_{i+1} - w_i}\right).$$

By choosing N sufficiently large we can make the ratio $\frac{w_i - w_{i-1}}{w_{i+1} - w_i}$ arbitrarily close to 1. Thus we apply (8.1.6) to find that

$$E_2 \le -C \sum_{i=N}^{\infty} Q_i |h_i| a_i (w_{i+1} - w_i).$$

We next calculate

$$E_3 \le \sum_{i=N}^{\infty} Q_i |h_i| (b_i + Q_1 a_i).$$

Recalling (8.1.3), and using that $w_{i+1} - w_i \to \infty$ since $k > 0$, we thus have, for N sufficiently large,

$$E_2 + E_3 \le -C \sum_{i=N}^{\infty} Q_i |h_i| a_i (w_{i+1} - w_i).$$

Because $h_i(\mathrm{sgn}(h_{i\pm1}) - \mathrm{sgn}(h_i)) \le 0$, we infer $E_1 \le 0$. Thus, in the case $g \ge 0$ we estimate

$$\begin{aligned}
E_1 + E_4 \le E_4 &\le |g| \sum_{i=N}^{\infty} Q_i Q_1 a_i |h_i| (w_{i+1} - w_i + 1) \\
&\le C|g| \sum_{i=N}^{\infty} Q_i |h_i| a_i (w_{i+1} - w_i).
\end{aligned}$$

For $g < 0$ we find that

$$\begin{aligned}
E_1 + E_4 \le &\sum_{\substack{i \ge N \\ \mathrm{sgn}(h_i) \ne \mathrm{sgn}(h_{i+1})}} Q_i |h_i| Q_1 a_i (-2w_{i+1} - g(w_{i+1} + w_i)) \\
&+ |g| \sum_{i=N}^{\infty} Q_i |h_i| Q_1 a_i \qquad (9.1.2)
\end{aligned}$$

When $|g| < 1$ we have that the first term in (9.1.2) is negative. This then readily implies that for N sufficiently large and for $|g|$ sufficiently small we have that

$$\langle \operatorname{sgn}(h), A(g)h \rangle_{X^*_{1+k}, X_{1+k}} \leq -C \sum_{i=N}^{\infty} Q_i(w_{i+1} - w_i) a_i |h_i| \leq 0,$$

which completes the proof. □

The next step is to prove that $\{A(g(t))\}$ indeed generates an evolution family.

Lemma 9.1.5. *Suppose that the assumptions of Lemma 9.1.3 are satisfied. Suppose furthermore that*

$$|g(t)| < \min\{\delta_k, \delta_{k+1}, \delta_H\}, \tag{9.1.3}$$

where δ_k is given in Proposition 9.1.4 and δ_H in Lemma 9.1.2. Then for N chosen as in Proposition 9.1.4, the family $\{A(g(t))\}_{t \in I}$ generates an evolution family $V_{X_{1+k}}$ on the interval $I = [0, T)$ in the space X_{1+k}, which for $0 \leq s \leq t < T$ satisfies

$$\|V_{X_{1+k}}(t, s)\|_{\mathcal{L}(X_{1+k})} \leq 1.$$

Proof. We claim that $\{A(g(t))\}_{t \in I}$ satisfies the assumptions of Proposition 2.5.11. By (9.1.3) and Proposition 9.1.4 we have that $A(g(t))$ is dissipative on X_{1+k} and X_{2+k}. For fixed $t \in I$, by (9.1.3), $\Theta(g(t))$ generates a semigroup on H (as established in the proof of Lemma 9.1.2). As $B(g(t))$ is a bounded operator on H, it then must be that $A(g(t))$ generates a semigroup on H. This then implies that for some large, positive real λ we must have that the range of $A(g(t)) - \lambda$ contains H. Thus the range of $A(g(t)) - \lambda$ is dense in X_{1+k} and X_{2+k}. As in the proof of Lemma 8.2.8, this implies that $A(g(t))$ generates a semigroup on X_{1+k} and X_{2+k}, and thus the first two assumptions are satisfied.

Next, as $g(t)$ is C^1 and by (8.1.4), the third assumption is necessarily satisfied. Thus we may apply Proposition 2.5.11, which proves the lemma. □

The next result follows from a computation as in Lemma 8.2.5, the proof is omitted.

Lemma 9.1.6. *Under the assumptions of Lemma 9.1.5, the operator $B(g(t))$ is uniformly bounded from X_1 to H, with a bound that depends only on δ_k, and not on g or t.*

With these tools it is possible to prove Lemma 9.1.3.

Proof of Lemma 9.1.3. In light of Lemmas 9.1.5 and 9.1.6 this follows from Proposition 8.2.1. □

Remark 9.1.7. *We note that the bound M_k is not dependent on the particular function $g(t)$, and only on its bound δ_k. This is because of the independence on $g(t)$ in the bounds obtained in lemmas 9.1.5 and 9.1.6.*

It is important that the previous lemma is independent of the choice of $g(t)$. Using Lemma 9.1.3, the following is elementary.

Proposition 9.1.8. *The operator L generates a semigroup on the space X_{1+k}, for any $k \geq 0$.*

Proof. Applying Lemma 9.1.3 when $g \equiv 0$, that is for $F(g) = F(0) = L$, gives the desired result when $k > 0$. The result when $k = 0$ was already established in Theorem 8.2.9. □

It is now possible to prove Theorem 9.1.1.

Proof of Theorem 9.1.1. Let M_k be the uniform bound in the space X_{1+k} given in Lemma 9.1.3. Set

$$\delta = \frac{Q_1 \min\{\delta_{k-2}, \delta_{k-1}, \delta_k, \delta_{k+1}, \delta_H, \varepsilon Q_1^{-1}\}}{2M_k}.$$

Now, let $\{h_i\}$ correspond to a solution of the Becker–Döring equations, with $\|h(0)\|_{X_{1+k}} < \delta$. By Lemma 9.2.1 and as $k > 2$ we know that h_1 is C^1. By Lemma 9.1.3 we thus know that $\{\Theta(h_1(t))\}_{t\in I}$ generates an evolution family U on $X_{1+(k-2)}$ and X_{1+k} on the (non-empty) interval I such that $|h_1(t)| \le \min\{\delta_{k-2}, \delta_{k-1}, \delta_k, \delta_{k+1}, \delta_H\}$. As $k > 2$, by Lemma 9.2.1 we know that the conditions of Proposition 2.5.9 are satisfied in $X_{1+(k-2)}$, and thus $U(t,0)h(0) = h(t)$ for all $t \in I$.

The uniform bounds from Lemma 9.1.3 then imply that $\|h(t)\|_{X_{1+k}} \le M_k \|h(0)\|_{X_{1+k}}$ on the interval I. Our choice of δ immediately implies that $I = [0,\infty)$ and that $\|h(t)\|_{X_{1+k}} \le \varepsilon/2$, which completes the proof. □

9.2 Non-linear Decay Rates

This section will prove the main theorem. The first step is to justify the use of Duhamel's formula.

Lemma 9.2.1. *Assume that $(c_i(t))$ is a solution of the Becker–Döring equations and $(h_i(t))$ is defined by* (1.2.7)*, and let $k \ge 0$. If $h(0) \in X_{3+k}$ then the following is satisfied (strongly) in X_{1+k}:*

$$\frac{d}{dt} h = Lh + h_1 \Xi h. \tag{9.2.1}$$

In particular, if $h(0) \in X_{3+k}$ then we have that the following is satisfied in X_{1+k}:

$$h(t) = e^{Lt} h(0) + \int_0^t e^{L(t-s)} h_1(s) \Xi h(s)\, ds, \tag{9.2.2}$$

where e^{Lt} is the semigroup generated by L on X_{1+k} (see Proposition 9.1.8).

Proof. Because $h(0) \in X_{3+k}$ by Proposition 8.1.3 and Lemma 8.2.4 we have that $Lh + h_1 \Xi h$ is bounded in X_{2+k} on any finite interval. Because each h_i is continuous by definition (8.1.1), it must be that $Lh + h_1 \Xi h$ is measurable in X_{2+k}. We claim that in X_{2+k} we have that

$$h(t) = h(0) + \int_0^t Lh(s) + h_1(s) \Xi h(s)\, ds. \tag{9.2.3}$$

Indeed, the right hand side of the equation is well-defined, and must match the coordinate-wise integrals from definition 8.1.1. This implies that $h(t)$ is locally Lipschitz in X_{2+k}. As (9.2.3) also holds in X_{1+k} we thus have that $h(t)$ must be differentiable in X_{1+k}. This implies (9.2.1).

Again by Proposition 8.1.3 we know that $h_1 \Xi h \in L^1((0,T); X_{1+k})$. Proposition 2.5.7 then implies (9.2.2). □

Next, it is necessary to derive a specialized version of Gronwall's inequality.

Lemma 9.2.2. *Let $u(t)$ be a positive, continuous function on $[0, \infty)$. Suppose that u satisfies*

$$u(t) \leq \tilde{C}_2(1+t)^{-r} + \int_0^t \tilde{C}_1(1+t-s)^{-r}u(s)ds. \tag{9.2.4}$$

Furthermore, suppose that $r > 1$ and that C_1 is small enough that

$$\tilde{C}_1 \int_0^t (1+t-s)^{-r}(1+s)^{-r}\, ds \leq \theta(1+t)^{-r} \tag{9.2.5}$$

for some $\theta < 1$ and for all $t > 0$. Then we must have that

$$u(t) \leq \frac{\tilde{C}_2}{1-\theta}(1+t)^{-r}.$$

Proof. Let $v(t) = u(t)(1+t)^r$. Then we have that

$$v(t) \leq \tilde{C}_2 + (1+t)^r \int_0^t \tilde{C}_1(1+t-s)^{-r}(1+s)^{-r}v(s)\, ds.$$

This then readily implies that for any $T > 0$,

$$\|v\|_{C(0,T)} \leq \tilde{C}_2 + \theta\|v\|_{C(0,T)}.$$

Thus for all $t \geq 0$

$$v(t) \leq \frac{\tilde{C}_2}{1-\theta},$$

which establishes the desired result.

□

Remark 9.2.3. *Note that for any $r > 1$ one can find a $\tilde{C}_1 > 0$ such that* (9.2.5) *is satisfied. This is because*

$$\begin{aligned}\int_0^t (1+s)^{-r}(1+t-s)^{-r}\, ds &= 2\int_0^{t/2} (1+s)^{-r}(1+t-s)^{-r}\, ds \\ &\leq 2\left(1+\frac{t}{2}\right)^{-r} \int_0^{t/2} (1+s)^{-r}\, ds \\ &\leq \frac{2^{r+1}}{r-1}(1+t)^{-r}.\end{aligned}$$

Thus if $\tilde{C}_1 < (r-1)2^{-(r+1)}$ then we have that (9.2.5) *is satisfied.*

Remark 9.2.4. *The dependence on the constant $\tilde{C}_1$ is critical in the previous proof. Indeed, if $\int_0^\infty \tilde{C}_1(1+s)^{-r}ds > 1$ then it is possible to show that for some $u(t) \equiv c > 0$ the inequality* (9.2.4) *is satisfied. Thus decay estimates can only be obtained if $\tilde{C}_1$ is sufficiently small.*

It is now possible to prove the main result.

Proof of Theorem 1.2.1. Recall that we have assumed that $0 < k_1 < k_2 - 2$. By Lemma 9.2.1 we know that the equation

$$h(t) = e^{Lt}h(0) + \int_0^t e^{L(t-s)}h_1(s)\Gamma h(s)\, ds$$

is satisfied in X_{1+k_1}, where e^{Lt} is the semigroup generated by L. By Corollary 8.3.2 we can thus estimate

$$\begin{aligned}\|h(t)\|_{X_{1+k_1}} \le{}& C(1+t)^{-(k_2-k_1)}\|h(0)\|_{X_{1+k_2}}\\ &+C\int_0^t |h_1(s)|\|\Gamma h(s)\|_{X_{k_2}}(1+t-s)^{-(k_2-k_1-1)}\,ds.\end{aligned}$$

By Lemma 8.2.4 we know that Γ is bounded from X_{k_2+1} to X_{k_2}, and thus

$$\begin{aligned}\|h(t)\|_{X_{1+k_1}} \le{}& C(1+t)^{-(k_2-k_1)}\|h(0)\|_{X_{1+k_2}}\\ &+C\int_0^t |h_1(s)|\|h(s)\|_{X_{1+k_2}}(1+t-s)^{-(k_2-k_1-1)}\,ds.\end{aligned}$$

It is then immediate that

$$\begin{aligned}\|h(t)\|_{X_{1+k_1}} \le{}& C(1+t)^{-(k_2-k_1)}\|h(0)\|_{X_{1+k_2}}\\ &+C\sup_\tau\|h(\tau)\|_{X_{1+k_2}}\int_0^t (1+t-s)^{-(k_2-k_1-1)}|h_1(s)|\,ds.\end{aligned}$$

We then use a crude bound to obtain

$$\begin{aligned}\|h(t)\|_{X_{1+k_1}} \le{}& C(1+t)^{-(k_2-k_1)}\|h(0)\|_{X_{1+k_2}}\\ &+C\sup_\tau\|h(\tau)\|_{X_{1+k_2}}\int_0^t (1+t-s)^{-(k_2-k_1-1)}\|h(s)\|_{X_{1+k_1}}\,ds.\end{aligned}$$

By Theorem 9.1.1 for any $\varepsilon > 0$ we can choose δ_{k_1,k_2} small enough to guarantee that

$$\|h(t)\|_{X_{1+k_1}} \le C(1+t)^{-(k_2-k_1-1)}\|h(0)\|_{X_{1+k_2}}+\varepsilon\int_0^t (1+t-s)^{-(k_2-k_1-1)}\|h(s)\|_{X_{1+k_1}}\,ds,$$

where we have additionally used that $(1+t)^{-(k_2-k_1)} \le (1+t)^{-(k_2-k_1-1)}$. As $k_2 > k_1+2$, by applying Lemma 9.2.2 (whose conditions will be satisfied for ε small due to Remark 9.2.3), we then find that

$$\|h(t)\|_{X_{1+k_1}} \le C(1+t)^{-(k_2-k_1-1)}\|h(0)\|_{X_{1+k_2}},$$

which is the desired result. □

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