Spectral and Homogenization Problems

Rita Alexandra Gonçalves Ferreira

Dissertation for the Degree of Doctor of Philosophy
in Mathematics

Supervisors:
Professor Irene Fonseca (MCS/CMU)
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To my parents Rosa Maria and Manuel

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Abstract

In this dissertation we will address two types of homogenization problems. The first one is a spectral problem in the realm of lower dimensional theories, whose physical motivation is the study of waves propagation in a domain of very small thickness and where it is introduced a very thin net of heterogeneities. Precisely, we consider an elliptic operator with ε-periodic coefficients and the corresponding Dirichlet spectral problem in a three-dimensional bounded domain of small thickness δ. We study the asymptotic behavior of the spectrum as ε and δ tend to zero. This asymptotic behavior depends crucially on whether ε and δ are of the same order ($\delta \approx \varepsilon$), or ε is of order smaller than that of δ ($\delta = \varepsilon^\tau,\ \tau < 1$), or ε is of order greater than that of δ ($\delta = \varepsilon^\tau,\ \tau > 1$). We consider all three cases.

The second problem concerns the study of multiscale homogenization problems with linear growth, aimed at the identification of effective energies for composite materials in the presence of fracture or cracks. Precisely, we characterize $(n+1)$-scale limit pairs (u, U) of sequences $\{(u_\varepsilon \mathcal{L}^N{\lfloor_\Omega}, Du_\varepsilon{\lfloor_\Omega})\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^d) \times \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV(\Omega;\mathbb{R}^d)$. Using this characterization, we study the asymptotic behavior of periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation and described by $n \in \mathbb{N}$ microscales.

Key Words: Periodic homogenization, spectral analysis, dimension reduction, Γ-convergence, asymptotic expansions, space BV of functions of bounded variation, BV-valued measures, multiscale convergence, linear growth

Resumo

Nesta dissertação serão tratados dois problemas no âmbito da teoria da homogeneização. O primeiro refere-se a um problema espectral no domínio das teorias de baixa dimensão, que tem como motivação o estudo de propagação de ondas em domínios de pequena espessura e onde é introduzida uma fina rede de heterogeneidades. Mais precisamente, consideramos um problema espectral definido num domínio tridimensional de espessura δ, com condições de Dirichlet nulas, associado a um operador elíptico com coeficientes ε-periódicos. Apresentamos o comportamento assimptótico do espectro quando ε e δ tendem para zero, distinguindo três casos: o caso em que a frequência das oscilações e a espessura do domínio são da mesma ordem de grandeza ($\varepsilon \approx \delta$), o caso em que a frequência das oscilações é muito maior do que a espessura do domínio ($\delta = \varepsilon^\tau,\ \tau < 1$) e, finalmente, o caso em que a espessura do domínio é muito menor do que a frequência das oscilações ($\delta = \varepsilon^\tau,\ \tau > 1$).

O segundo problema aqui tratado reporta-se ao estudo de problemas de homogeneização caracterizados por múltiplas escalas microscópicas e condições de crescimento lineares, que têm em vista a identificação da energia efectiva de compósitos com fracturas ou rachas. Mais precisamente, caracterizamos os pares limite a $(n+1)$-escalas (u, U) de sucessões $\{(u_\varepsilon \mathcal{L}^N{}_{\lfloor \Omega}, Du_{\varepsilon \lfloor \Omega})\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^d) \times \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ em que $\{u_\varepsilon\}_{\varepsilon>0}$ é limitada. Usando esta caracterização, estudamos o comportamento assimptótico de funcionais periodicamente oscilantes com condições de crescimento lineares, definidos no espaço BV das funções de variação limitada e caracterizados por $n \in \mathbb{N}$ escalas microscópicas.

Termos Chave: homogeneização periódica, análise espectral, redução dimensional, Γ-convergência, expansões assimptóticas, espaço BV das funções de variação limitada, medias com valores em BV, convergência a múltiplas escalas, crescimento linear

Notation and List of Symbols

Symbol/Expression	
$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$:	set of natural, integer, and real numbers
$\mathbb{N}_0$:	$\{0\} \cup \mathbb{N}$
$\mathbb{R}^+$:	set of positive real numbers
$\mathbb{R}^m$:	m-dimensional Euclidean space
$[x]$ and $\langle x \rangle$ $(x \in \mathbb{R}^m)$:	integer and fractional parts of x componentwise, respectively
$\xi \cdot \zeta$ or $(\xi\|\zeta)$ $(\xi, \zeta \in \mathbb{R}^m)$:	$\sum_{i=1}^m \xi_i \zeta_i$
$\mathbb{Z}^m$:	$\big\{(z_1, \cdots, z_m)\colon z_i \in \mathbb{Z} \text{ for all } i \in \{1, \cdots, m\}\big\}$
$\mathbb{M}^{d\times N}$:	space of $d \times N$-dimensional matrices
$A\xi\zeta$ $(A \in \mathbb{M}^{N\times N}, \eta, \zeta \in \mathbb{R}^N)$:	$(A\xi\|\zeta)$
$\xi \otimes \zeta$ $(\xi \in \mathbb{R}^d, \zeta \in \mathbb{R}^N)$:	$(\xi_i\zeta_j)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant d} \in \mathbb{M}^{d\times N}$
$\mathbb{R}^{d\times N}$:	space of $d \times N$-dimensional matrices identified with $\mathbb{R}^{dN}$
$\xi : \zeta$ $(\xi, \zeta \in \mathbb{R}^{d\times N})$:	$\sum_{i=1}^d \sum_{j=1}^N \xi_{ij}\zeta_{ij}$
$\|\xi\|$ $(\xi \in \mathbb{R}^{d\times N})$:	$\sqrt{\xi : \xi}$
$(\,\cdot\,\|\,\cdot\,)$ or $(\,\cdot\,\|\,\cdot\,)_H$:	inner product in a Hilbert space H
$\\|\cdot\\|$ or $\\|\cdot\\|_X$:	norm in a Banach space X
$\langle\cdot,\cdot\rangle$:	duality pairing
δ_{ij} :	Kronecker symbol
Δ :	Laplacian
∇ :	gradient
div :	divergence
∇_i or $\frac{\partial}{\partial x_i}$:	first order partial derivative with respect to the variable x_i
Δ_i or $\frac{\partial^2}{\partial x_i^2}$:	second order partial derivative with respect to the variable x_i
Y (reference cell) :	$(0,1)^N$ or $[0,1]^N$
Y_i with $i \in \mathbb{N}$:	copy of Y
subscript # :	$Y_1 \times \cdots \times Y_n$-periodic functions (or measures) w.r.t. the variables $(y_1, \cdots, y_n)$
$\overline{\Omega}$ $(\Omega \subset \mathbb{R}^N)$:	closure of Ω

$\partial\Omega \;\; (\Omega \subset \mathbb{R}^N)$: boundary of Ω

$\Omega' \subset\subset \Omega \;\; (\Omega \subset \mathbb{R}^N)$: $\overline{\Omega'}$ compact with $\overline{\Omega'} \subset \Omega$

Ω domain $(\Omega \subset \mathbb{R}^N)$: Ω open and connected

$\operatorname{supp} f$, $\operatorname{Lip} f$: support and Lipschitz constant, respectively, of a function f

f^*, f^{**} : polar (or conjugate) and bipolar functions, respectively, of a function f

$\mathcal{C}f$, $\mathcal{Q}f$: convex and quasiconvex envelopes, respectively, of a function f

$C(\Omega)$, $C(\Omega;\mathbb{R}^d)$: space of real- and $\mathbb{R}^d$-valued continuous functions in Ω, respectively

$C_c(\Omega)$, $C_c(\Omega;\mathbb{R}^d)$: space of functions in $C(\Omega)$ and $C(\Omega;\mathbb{R}^d)$, respectively, with compact support

$C_0(\Omega)$, $C_0(\Omega;\mathbb{R}^d)$: closure of $C_c(\Omega)$ and $C_c(\Omega;\mathbb{R}^d)$, respectively, w.r.t. the supremum norm

$C_\#(Y_1 \times \cdots \times Y_n)$: $Y_1 \times \cdots \times Y_n$-periodic, real-valued continuous functions in $\mathbb{R}^N$

$C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d)$: $Y_1 \times \cdots \times Y_n$-periodic, $\mathbb{R}^d$-valued continuous functions in $\mathbb{R}^N$

$f \in C_c(\Omega; C_\#(Y_1 \times \cdots \times Y_n))$: for all $x \in \Omega$, $f(x,\cdot) \in C_\#(Y_1 \times \cdots \times Y_n)$ and for all $y_1,...,y_n \in \mathbb{R}^N$, $f(\cdot, y_1, \dots, y_n) \in C_c(\Omega)$

$f \in C_c(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$: for all $x \in \Omega$, $f(x,\cdot) \in C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d)$ and for all $y_1,...,y_n \in \mathbb{R}^N$, $f(\cdot, y_1, \dots, y_n) \in C_c(\Omega;\mathbb{R}^d)$

$C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n))$: closure of $C_c(\Omega; C_\#(Y_1 \times \cdots \times Y_n))$ w.r.t. the supremum norm

$C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$: closure of $C_c(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$ w.r.t. the supremum norm

$C^k(\Omega)$, $C^k(\Omega;\mathbb{R}^d)$: space of all functions in $C(\Omega)$ and $C(\Omega;\mathbb{R}^d)$, respectively, whose i^{th}-partial derivatives are continuous functions in Ω for all $i \in \{1, \cdots, k\}$; the spaces $C^k_c(\Omega;\mathbb{R}^d)$, $C^k_0(\Omega;\mathbb{R}^d)$, $C^k_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d)$, $C^k_c(\Omega; C^k_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$, and $C^k_0(\Omega; C^k_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$, with the co-domain omitted if $d = 1$, are defined in an obvious way

$C^\infty(\Omega)$, $C^\infty(\Omega;\mathbb{R}^d)$: space of all functions in $C^k(\Omega)$ and $C^k(\Omega;\mathbb{R}^d)$, respectively, for all $k \in \mathbb{N}$; the spaces $C^\infty_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$, $C^\infty_c(\Omega; C^\infty_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$, and $C^\infty_0(\Omega; C^\infty_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$, with the co-domain omitted if $d = 1$, are defined in an obvious way

$L^p(\Omega)$, $L^p(\Omega;\mathbb{R}^d)$: usual Lebesgue spaces

$W^{1,p}(\Omega)$, $W^{1,p}(\Omega;\mathbb{R}^d)$: usual Sobolev spaces

$BV(\Omega)$, $BV(\Omega;\mathbb{R}^d)$: space of functions of bounded variation

$\rightharpoonup$, $\overset{\star}{\rightharpoonup}$: weak and weak-$\star$ convergence, respectively

$\mathcal{B}(X)$: σ-algebra of the Borel subsets of a topological space X

$\mathcal{M}(X;\mathcal{Z})$ ($\mathcal{Z}$ Banach space) : space of $\mathcal{Z}$-valued Radon measures

$\mathcal{L}^d$: d-dimensional Lebesgue measure

a.e. in $\mathbb{R}^d$: everywhere in $\mathbb{R}^d$ except in a set of zero d-dimensional Lebesgue measure

$M(\zeta, \eta, \Omega)$: set of all $N \times N$ real matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant N} \in [L^\infty(\Omega)]^{N \times N}$ bounded and coercive a.e. in Ω

$M_S(\zeta, \eta, \Omega)$: set of all $N \times N$ real, symmetric matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant N} \in [L^\infty(\Omega)]^{N \times N}$ bounded and coercive a.e. in Ω

$D(\mathcal{A}), R(\mathcal{A}), N(\mathcal{A}), G(\mathcal{A})$: domain, range, kernel, and graph, respectively, of an operator $\mathcal{A}$

$\rho(\mathcal{A}), \sigma(\mathcal{A}), \sigma_p(\mathcal{A})$: resolvent, spectrum, and point spectrum, respectively, of an operator $\mathcal{A}$

$\mathcal{A}^*$: adjoint operator of an operator $\mathcal{A}$

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Chapter 1

Introduction

This dissertation is devoted to the study of mathematical problems within the framework of homogenization theory, which addresses the description of the macroscopic or effective behavior of a microscopically heterogeneous system. There are multiple applications in the fields of physics, mechanics and engineering sciences, from which we emphasize problems aimed at the modeling of composites, stratified or porous media, finely damaged materials, or materials with many holes or cracks.

From the mathematical point of view, homogenization is often associated to the study of the asymptotic behavior of oscillating partial differential equations, or of minimization problems yielding from certain oscillating functionals, depending on one or more small-scale parameters. Several approaches have been proposed to handle this type of problems, such as the method of asymptotic expansions (see the books of Bensoussan, Lions and Papanicolaou [13], Jikov, Kozlov and Oleĭnik [53], Bakhvalov and Panasenko [12], and Sanchez-Palencia [69]) and methods using the concepts of G-convergence due to Spagnolo (see Spagnolo [71] and De Giorgi and Spagnolo [35]), H-convergence due to Murat and Tartar (see Murat and Tartar [63], Tartar [73] and Murat [62]), Γ-convergence due to De Giorgi (See De Giorgi and Dal Maso [36] and De Giorgi and Letta [37]), and two-scale convergence due to Nguetseng (see Nguetseng [64]), further developed by Allaire [1] and Allaire and Briane [2]. For a comprehensive introduction to the theory of homogenization and for an overview of the different homogenization methods, we refer to the book of Cioranescu and Donato [28].

As a simple illustration of a homogenization problem, we briefly describe the problem regarding the study of the thermal conductivity of a periodic composite material. Composites are structures constituted by two or more finely mixed materials that, depending on the performance we are looking for, in general exhibit a better behavior than the average of its components, and for this reason they may have an impact in industrial applications. Loosely speaking, the smaller the heterogeneities, the better the mixture, which then seems homogeneous (see Fig. 1.0.1).

Assume that we are given two isotropic, homogeneous materials, one of thermal conductivity k_1 and the other of thermal conductivity k_2. Consider a three-dimensional body occupying a certain region $\Omega \subset \mathbb{R}^3$ made of a heterogeneous material, which is a mixture of the two given materials such that the material of thermal conductivity k_1 occupies a certain portion Ω_1 of Ω, and the material of thermal conductivity k_2 occupies its complement $\Omega_2 := \Omega \backslash \Omega_1$. Assume further that both Ω_1 and Ω_2 are the union of many subregions whose size is much smaller when compared to the size of the body, so that they *seem* evenly distributed and therefore may be modeled by a periodic distribution characterized by a small parameter ε. Precisely, assume that the thermal conductivity at each point $x \in \Omega$ is given by

$$k_\varepsilon(x) := k\Big(\frac{x}{\varepsilon}\Big),$$

where $\varepsilon > 0$ is a small parameter, and k is the Y-periodic function, being $Y := [0,1]^3$ the *reference cell* [1.1], defined for all $y \in Y$ by

$$k(y) := \begin{cases} k_1 & \text{if } y \in Y_1, \\ k_2 & \text{if } y \in Y \backslash Y_1, \end{cases}$$

where Y_1 is a measurable subset of Y.

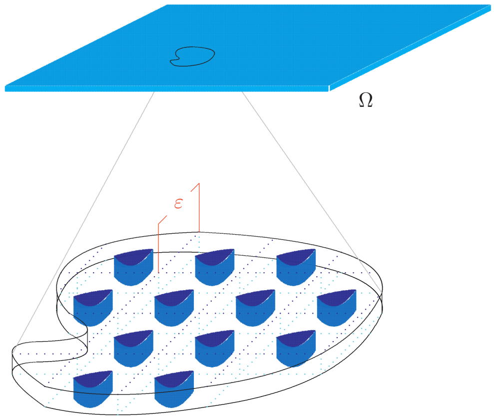

Fig. 1.0.1. Microscopically heterogeneous material

Note that

$$\Omega_1 := \Big(\bigcup_{z\in\mathbb{Z}^3} \varepsilon(z+Y_1)\Big) \cap \Omega \quad \text{and} \quad \Omega_2 := \Big(\bigcup_{z\in\mathbb{Z}^3} \varepsilon(z+Y\backslash Y_1)\Big) \cap \Omega.$$

Assuming without loss of generality that the temperature on the surface $\partial\Omega$ of the body is zero and representing by f the heat source, then the temperature $u_\varepsilon = u_\varepsilon(x)$ at each point $x \in \Omega$ satisfies the Dirichlet problem

$$\begin{cases} -\operatorname{div}(k_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.0.1}$$

We observe that two scales characterize problem (1.0.1): the macroscopic one, x, which indicates the position in Ω, and the microscopic or fast-oscillating scale, $\frac{x}{\varepsilon}$, which assigns the position in the reference cell, in the sense that there exists a unique $y \in Y$ such that $\frac{x}{\varepsilon} = y + z$ for some $z \in \mathbb{Z}^3$.

It is commonly agreed in the engineer and physics communities that the bigger is the ratio between the size of the body and the size of each of its separated components, that is, the smaller ε is, the more stable are the physical properties (in this case, the heat transfer) of the mixture. Moreover, the global or effective behavior of the mixture generally differs from the average of its components.

Heuristically, we seek to replace the heterogeneous material by a "fictitious" homogeneous material whose global characteristics are dictated by the effective properties. From the mathematical point of view, this reduces to the study of problem (1.0.1) in the limit as $\varepsilon \to 0^+$. Precisely, we want to investigate whether $\{u_\varepsilon\}_{\varepsilon>0}$ converges is some sense to some function u_0 as $\varepsilon \to 0^+$, and, if so, we aim

[1.1] For simplicity, we take here as reference cell the unit cube in $\mathbb{R}^3$, but we could have taken any bounded interval in $\mathbb{R}^3$, as suggested by Fig. 1.0.1.

at describing the limit problem of (1.0.1) that admits u_0 as solution. Under some mild hypotheses on f (see, for example, Cioranescu and Donato [28] for the details), the answer to these questions is affirmative: $\{u_\varepsilon\}_{\varepsilon>0}$ converges weakly in $H_0^1(\Omega)$, as $\varepsilon \to 0^+$, to a function u_0, solution of the *homogenized* Dirichlet problem

$$\begin{cases} -\operatorname{div}(A^h \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.0.2}$$

where $A^h := (a^h_{ij})_{1\leqslant i,j\leqslant 3} \in \mathbb{M}^{3\times 3}$ is the constant matrix whose coefficients are given by

$$a^h_{ij} := \int_Y k(y)\Big(\delta_{ij} - \frac{\partial w_j}{\partial y_i}\Big)\,\mathrm{d}y, \quad i,j \in \{1,2,3\},$$

where δ_{ij} is the Kronecker symbol, and w_j is the solution of the *cell problem*

$$\begin{cases} -\operatorname{div}(k\nabla w_j) = -\dfrac{\partial k}{\partial y_j} & \text{in } Y, \\ w_j \text{ Y-periodic}, & \displaystyle\int_Y w_j(y)\,\mathrm{d}y = 0. \end{cases} \tag{1.0.3}$$

The matrix A^h encodes the overall characteristics of the original mixture, and since it cannot be written in the form aI, with $a > 0$ and I the identity matrix, we conclude that the homogeneous limit material is not isotropic. We also observe that the considerations above are still valid in any dimension $N \in \mathbb{N}$ (and not just $N = 3$).

In this work we will sometimes adopt the variational point of view, i.e., instead of looking for solutions of boundary problems of the type (1.0.1) we will be interested in solutions of minimization problems associated with the energy functional corresponding to the physical system under study. For instance, in the case of problem (1.0.1), we would be led to the study of the minimization problem

$$\min\Big\{\int_\Omega k_\varepsilon(x)|\nabla u(x)|^2\,\mathrm{d}x - 2\int_\Omega f(x)u(x)\,\mathrm{d}x \colon\ u \in H_0^1(\Omega)\Big\}.$$

These minimization problems often assume the general form

$$\min\Big\{\int_\Omega f_\varepsilon(x, Du(x))\,\mathrm{d}x \colon\ u \in \mathcal{A}\Big\},$$

where $\mathcal{A}$ is the class of admissible u's. In the limit as $\varepsilon \to 0^+$ we expect a minimization *homogenized* problem of the form

$$\min\Big\{\int_\Omega f_{\mathrm{hom}}(Du(x))\,\mathrm{d}x \colon\ u \in \mathcal{A}\Big\},$$

where f_{hom} plays the role of the matrix A^h in (1.0.2), and it is given by *asymptotic homogenization formulas* or *cell problem formulas*, which correspond to the variational formulation of the cell problems (1.0.3).

In this dissertation we will address two types of homogenization problems. The first one, briefly described in Subsection 1.1, is a spectral problem within the realm of lower dimensional theories, whose physical motivation is the study of waves propagation in a domain of very small thickness and where a very thin net of heterogeneities is introduced. The second problem, outlined in Subsection 1.2, concerns the study of multiscale homogenization problems with linear growth, aimed at the identification of effective energies for composite materials in the presence of fracture or cracks.

1.1. Spectral Analysis in a Thin Domain with Periodically Oscillating Characteristics.

Within the framework of quantum mechanics, Schrödinger's equation for the time-independent wave function ψ associated to a particle in a three-dimensional space is given by:

$$-\frac{\hbar}{2m}\Delta\psi + V\psi = E\psi,$$

where $\hbar := h/2\pi$, h being Plank's constant, m is the mass of the particle, Δ is the usual Laplace's operator, V is the potential energy and E is the energy of the system with wave function ψ. When we consider the particle to be confined to a certain domain $\Omega \subset \mathbb{R}^3$, but otherwise free, precisely, when the potential function is of the form $V(x) := 0$ if $x \in \Omega$, and $V(x) := +\infty$ if $x \notin \Omega$, then the problem of finding the spacial wave function ψ and the energy levels E reduces to solving the following eigenvalue problem for the Laplace's operator:

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where, using standard mathematical notations, we identified $\psi \equiv v$ and $\lambda \equiv \frac{2m}{\hbar}E$. In the joint works with Mascarenhas [45] and with Mascarenhas and Piatnitski [46], we addressed this eigenvalue problem in the case in which the domain has a very small thickness δ and the material presents very small ε-periodic heterogeneities. We proved that the energy levels depend strongly on both small parameters δ and ε and on their ratio.

Precisely, let $\varepsilon, \delta > 0$ be small parameters, and consider the thin domain $\Omega_\delta := \omega \times \delta I$, where $\omega \subset \mathbb{R}^2$ is a bounded domain, and $I := (-1/2, 1/2)$. Our goal is to study the asymptotic behavior as $\varepsilon \to 0^+$ and $\delta \to 0^+$ of the spectral problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon^\delta) = \lambda_\varepsilon^\delta v_\varepsilon^\delta & \text{a.e. in } \Omega_\delta, \\ v_\varepsilon^\delta \in H_0^1(\Omega_\delta), \end{cases} \tag{1.1.1}$$

where $A_\varepsilon(\bar{x}) := A(\frac{\bar{x}}{\varepsilon})$, $\bar{x} \in \mathbb{R}^2$, with $A = (a_{ij})_{1\leqslant i,j\leqslant 3} \in [L^\infty(\mathbb{R}^2)]^{3\times 3}$ a real, symmetric and Y-periodic matrix, where $Y := (0,1)^2$, satisfying appropriate boundedness and coercivity hypotheses. We assume further that $a_{\alpha 3} = 0$ a.e. in Y, $\alpha \in \{1,2\}$. We refer to Chapter 3 for the details.

The spectrum $\sigma_\varepsilon^\delta$ of problem (1.1.1) is discrete and can be written as $\sigma_\varepsilon^\delta := \{\lambda_{\varepsilon,i}^\delta \in \mathbb{R}^+ : i \in \mathbb{N}\}$, where $0 < \lambda_{\varepsilon,i}^\delta \leqslant \lambda_{\varepsilon,i+1}^\delta$ for all $i \in \mathbb{N}$, and $\lambda_{\varepsilon,i}^\delta \to +\infty$ as $i \to +\infty$. For fixed $\varepsilon > 0$, as the thickness of the domain goes to zero ($\delta \to 0^+$) all the eigenvalues go to infinity. For fixed $\delta > 0$, a classical result in the theory of homogenization asserts that as $\varepsilon \to 0^+$ the eigenvalues converge to eigenvalues associated with the corresponding homogenized problem of (1.1.1). As we mentioned before, in the case in which the small parameters ε and δ converge to zero simultaneously, the asymptotic behavior of the spectrum $\sigma_\varepsilon^\delta$ depends crucially on whether ε and δ are of the same order ($\varepsilon \approx \delta$), or ε is of order smaller than that of δ ($\varepsilon \ll \delta$), or ε is of order greater than that of δ ($\varepsilon \gg \delta$). The results corresponding to the cases $\varepsilon \approx \delta$ and $\varepsilon \ll \delta$ were announced in Ferreira and Mascarenhas [45]. In Ferreira, Mascarenhas and Piatnitski [46] detailed proofs of the statements formulated in Ferreira and Mascarenhas [45] were provided, and the case $\varepsilon \gg \delta$ was studied. Our main tools are Γ-convergence and asymptotic expansion techniques.

The homogenization of spectral problems, supported by a large bibliography, was first treated in Vanninathan [75] and Kesavan [54], [55]. The methods of analysis of spectral problems in terms of operator convergence have been introduced in Oleĭnik, Shamaev and Yosifian [65] and Attouch [8]. Other homogenization approaches in spectral problems and related topics have been proposed by Allaire and Conca [3], Allaire and Malige [4]. The homogenization of singularly perturbed operators has been considered in Kozlov and Piatnitski [56], [57] and some other works. The novelty in the homogenization spectral problem treated here is its study in the realm of lower dimensional theories. We refer to Bouchitté, Mascarenhas and Trabucho [19], Gaudiello and Sili [50], Krejčiřík [59], and to the references therein, for other spectral problems within lower dimensional theories.

A brief description of the case $\varepsilon \approx \delta$. Let $\lambda_{\varepsilon,k}$ be a k^{th} eigenvalue associated with problem (1.1.1) for $\delta = \varepsilon$. Then (see Theorem 3.1.1)

$$\lambda_{\varepsilon,k} = \frac{\mu_0}{\varepsilon^2} + \nu_{\varepsilon,k},$$

where $\mu_0 > 0$ is the first eigenvalue associated with a certain bidimensional periodic spectral problem with nonzero potential. Moreover, $\nu_{\varepsilon,k} \to \nu_k$ as $\varepsilon \to 0^+$, with ν_k a k^{th} eigenvalue associated with the bidimensional homogenized spectral problem in the cross section ω

$$\begin{cases} -\operatorname{div}(\bar{B}^h \bar{\nabla}\varphi) = \nu\varphi \quad \text{a.e. in } \omega, \\ \varphi \in H_0^1(\omega), \end{cases} \tag{1.1.2}$$

where $\bar{B}^h$ is a certain 2×2 constant matrix. Loosely speaking, the term $\frac{\mu_0}{\varepsilon^2}$ provides information on how the eigenvalues $\lambda_{\varepsilon,k}$ diverge, and also on the precise shift of the spectrum in order to retain the macroscopic behavior of the physical problem under study, which is given by the limit problem (1.1.2). As expected this is a two-dimensional problem of the same type of the original three-dimensional one, but with constant coefficients. Note that $\left\{\lambda_{\varepsilon,k} - \frac{\mu_0}{\varepsilon^2}\right\}_{k\in\mathbb{N}} = \{\nu_{\varepsilon,k}\}_{k\in\mathbb{N}}$ is the spectrum of the shifted operator $-\operatorname{div}(A_\varepsilon \nabla) - \frac{\mu_0}{\varepsilon^2} I$ with zero Dirichlet boundary conditions, where I represents the identity operator.

The asymptotic behavior of the eigenfunctions associated with $\lambda_{\varepsilon,k}$ will also be provided. We refer to Chapter 3 for the details.

A brief description of the case $\varepsilon \ll \delta$. Assume that $a_{\alpha\beta}$ are uniformly Lipschitz continuous in Y. Let $\lambda_{\varepsilon,k}$ be a k^{th} eigenvalue associated with problem (1.1.1) for $\delta = \varepsilon^\tau$, with some $\tau \in (0,1)$, and let $i \in \mathbb{N}$ be such that $\frac{i-1}{i} < \tau \leqslant \frac{i}{i+1}$. Then (see Theorem 3.1.2)

$$\lambda_{\varepsilon,k} = \sum_{j=0}^{i} \frac{\varrho_j}{\varepsilon^{\tau(2j+2)-2j}} + \rho_\varepsilon^\tau + \nu_{\varepsilon,k},$$

where $\varrho_0 = \pi^2 \int_Y a_{33}(\bar{y})\, \mathrm{d}\bar{y} > 0$, and for $j \in \mathbb{N}$, ϱ_j are well-determined constants. Furthermore, $\rho_\varepsilon^\tau \to 0$ as $\varepsilon \to 0^+$, $\nu_{\varepsilon,k} \to \nu_k$ as $\varepsilon \to 0^+$, with ν_k a k^{th} eigenvalue associated with a certain bidimensional homogenized spectral problem in the cross section ω, of the same type as (1.1.2). Here, the sum $\sum_{j=0}^{i} \frac{\varrho_j}{\varepsilon^{\tau(2j+2)-2j}}$ plays the role of $\frac{\mu_0}{\varepsilon^2}$ in the above case $\varepsilon \approx \delta$. The term ρ_ε^τ, which is innocuous in the limit as it converges to zero, is related to this sum and we may think of it as a remainder. The asymptotic behavior of the eigenfunctions associated with $\lambda_{\varepsilon,k}$ will also be provided.

A brief description of the case $\varepsilon \gg \delta$. *i)* This case is considerable more difficult to handle than the previous ones, and it depends strongly on the behavior of the potential a_{33}. An interesting case in applications is when the potential a_{33} oscillates between two different values (which is the case of composites). In that direction new hypotheses on a_{33} are introduced: Assume that $a_{\alpha\beta}$ are smooth functions and that there exists an open and smooth subdomain Q of Y, $Q \subset\subset Y$, such that a_{33} coincides with its minimum, $a_{\min}$, in Q and is a smooth function strictly greater than $a_{\min}$ in $Y\backslash Q$. Let $\lambda_{\varepsilon,1}$ be the first eigenvalue of problem (1.1.1) with $\delta = \varepsilon^{\tau}$ for some $\tau \in (1, +\infty)$. Then (see Theorem 3.1.4)

$$\lambda_{\varepsilon,1} = \frac{a_{\min}\pi^2}{\varepsilon^{2\tau}} + \frac{\nu_0}{\varepsilon^2} + \varepsilon^{\tau-3}\mu_1 + \cdots + \varepsilon^{k(\tau-1)-2}\mu_k + \rho_\varepsilon^\tau + \nu_{\varepsilon,1}^\tau,$$

where k is the first integer greater or equal than $2/(\tau-1)$, $\nu_0 > 0$ is the first eigenvalue associated with a certain bidimensional spectral problem on Q, and μ_i, $i \in \{1, \cdots, k\}$, are well-determined constants, $|\rho_\varepsilon^\tau| \leqslant C\varepsilon^{(k+\frac{1}{2})\tau-(k+\frac{5}{2})} \to 0$ as $\varepsilon \to 0^+$, for some constant C independent of ε, and $\nu_{\varepsilon,1}^\tau$ vanishes as $\varepsilon \to 0^+$. In this case, the limit problem degenerates.

ii) Finally, under quite more general hypotheses than those above, we are able to characterize the limit spectrum in the sense of Kuratowsky: Assume that a_{33} attains a minimum value, $a_{\min}$, at some $\bar{y}_0 \in \mathbb{R}^2$ such that $a_{\alpha\beta}$ and a_{33} are continuous in some neighborhood of $\bar{y}_0$. Then (see Theorem 3.1.7)

$$\lim_{\varepsilon\to 0^+} \left(\varepsilon^{2\tau}\sigma_\varepsilon\right) = \left[a_{\min}\pi^2, +\infty\right], \tag{1.1.3}$$

where σ_ε represents the spectrum of problem (3.1.2) with $\delta = \varepsilon^\tau$ for some $\tau \in (1, +\infty)$, and the limit in (1.1.3) is to be understood in the sense of Kuratowsky, that is, $\left[a_{\min}\pi^2, +\infty\right]$ is the set of all cluster points of sequences $\{\lambda_\varepsilon\}_{\varepsilon>0}$, $\lambda_\varepsilon \in \varepsilon^{2\tau}\sigma_\varepsilon$.

1.2. Reiterated Homogenization in BV via Multiscale Convergence.

Within the framework of nonlinear elasticity, the elastic energy associated with an N-dimensional composite materials is of the form

$$\int_\Omega f_\varepsilon(x, \nabla u(x))\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^N$ denotes the reference configuration, $u : \Omega \to \mathbb{R}^d$ is the deformation of the body, and f_ε stands for the elastic stored density energy. We will assume that f_ε satisfies linear growth conditions, which is the natural setting for composite materials in the presence of fractures or cracks. In order to allow for jump-type discontinuities, we consider the space of admissible deformations to be the space $BV(\Omega;\mathbb{R}^d)$ of functions of bounded variation. In the presence of n microscales, or fast-oscillating variables, we seek to characterize the asymptotic behavior as $\varepsilon \to 0^+$ of energy functionals of the form

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}, \nabla u(x)\Big)\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}, \frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\mathrm{d}\|D^s u\|(x) \tag{1.2.1}$$

for $u \in BV(\Omega;\mathbb{R}^d)$ with $Du = \nabla u\mathcal{L}^N_{\lfloor\Omega} + D^s u$ the Lebesgue decomposition of Du with respect to $\mathcal{L}^N_{\lfloor\Omega}$, where

$$f^\infty(y_1, \cdots, y_n, \xi) := \limsup_{t\to+\infty} \frac{f(y_1, \cdots, y_n, t\xi)}{t}$$

is the recession function of a certain function $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$, separately periodic in the first n variables and satisfying linear growth conditions, and $\varrho_1, ..., \varrho_n$ are positive functions in $(0, \infty)$ such that for all $i \in \{1, \cdots, n\}$ and for all $j \in \{2, \cdots, n\}$,

$$\lim_{\varepsilon\to 0^+} \varrho_i(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0^+} \frac{\varrho_j(\varepsilon)}{\varrho_{j-1}(\varepsilon)} = 0. \tag{1.2.2}$$

In the context of multiscale composites, the functions $\varrho_1, ..., \varrho_n$ stand for the *length scales* or *scales of oscillation*. The second condition in (1.2.2) is known as *a separation of scales hypothesis*. A simple example of such functions ϱ_i is the case in which $\varrho_i(\varepsilon) := \varepsilon^i$, $i \in \{1, \cdots, n\}$.

We observe that for fixed $\varepsilon > 0$, and under some hypotheses on f, the functional in (1.2.1) is the relaxed functional in $BV(\Omega; \mathbb{R}^d)$ of

$$u \mapsto \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}, \nabla u(x)\Big) \mathrm{d}x$$

with respect to the $L^1(\Omega; \mathbb{R}^d)$ topology (see Fonseca and Müller [49]).

We now briefly describe the methodology adopted to carry out the aforementioned asymptotic characterization. In a joint work with Fonseca [43] we generalized the notion of two-scale convergence for sequences of Radon measures with finite total variation obtained in Amar [5] to the case of multiple periodic length scales of oscillations. The main result in Ferreira and Fonseca [43] concerns the characterization of the multiple-scale limit of $\{(u_\varepsilon \mathcal{L}^N\lfloor_\Omega, Du_\varepsilon\lfloor_\Omega)\}_{\varepsilon>0} \subset \mathcal{M}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV(\Omega; \mathbb{R}^d)$. Using this characterization and the periodic unfolding method (see, for example, Cioranescu, Damlamian and De Arcangelis [26] and Fonseca and Krömer [47]), in a subsequent joint work with Fonseca [44] we treated multiscale homogenized problems in the space BV of functions of bounded variation of the form (1.2.1). In the case of one microscale we recovered Amar's result [5] under more general conditions, as well as Bouchitté's result [16] and De Arcangelis and Gargiulo's result [34]; for two or more microscales the results we obtained are new.

Precisely, let $\Omega \subset \mathbb{R}^N$ be an open set and let $Y := (0,1)^N$. For $i \in \mathbb{N}$, Y_i is a copy of Y. We use the subscript $\#$ to represent $Y_1 \times \cdots \times Y_n$-periodic functions (or measures) with respect to the variables $(y_1, \cdots, y_n)$. We say that a sequence $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ of Radon measures with finite total variation in Ω, $(n+1)$-scale converges to a Radon measure with finite total variation $\mu_0 \in \big(C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))\big)' \simeq \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$ in the product space $\Omega \times Y_1 \times \cdots \times Y_n$, if for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$ we have

$$\lim_{\varepsilon\to 0^+} \int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) \cdot \mathrm{d}\mu_\varepsilon(x) = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}\mu_0(x, y_1, \cdots, y_n),$$

in which case we write $\mu_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \mu_0$.

This notion of convergence is justified by a compactness result, which asserts that every bounded sequence in $\mathcal{M}(\Omega; \mathbb{R}^m)$ admits a $(n+1)$-scale converging subsequence (see Theorem 4.1.3). Furthermore, the usual weak-$\star$ limit in $\mathcal{M}(\Omega; \mathbb{R}^m)$ is the projection on Ω of the $(n+1)$-scale limit, so that the latter captures more information on the oscillatory behavior of a bounded sequence in $\mathcal{M}(\Omega; \mathbb{R}^m)$ than the former. This leads us to study the asymptotic behavior with respect to the $(n+1)$-scale convergence of first order derivatives functionals with linear growth of the form (1.2.1).

In that direction, the first step is the characterization of the $(n+1)$-scale limit associated with $\big\{\big(u_\varepsilon \mathcal{L}^N\lfloor_\Omega, Du_\varepsilon\lfloor_\Omega\big)\big\}_\varepsilon \subset \mathcal{M}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$, $\{u_\varepsilon\}_{\varepsilon>0}$ being a bounded sequence in $BV(\Omega; \mathbb{R}^d)$. This was established in Ferreira and Fonseca [44] and may be summarized as follows (see Chapter 4 for the details). Assume that the length scales $\varrho_1, ..., \varrho_n$ are, in addition, well separated (c.f. Allaire and Briane [2]), i.e., there exists $m \in \mathbb{N}$ such that for all $i \in \{2, \cdots, n\}$, we have

$$\lim_{\varepsilon\to 0^+} \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)} = 0.$$

Then, up to a not relabeled subsequence,

$$u_\varepsilon \mathcal{L}^N{\lfloor_\Omega} \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \tau_u,$$

where $\tau_u \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^d)$ is a certain measure only depending on u, and

$$Du_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n},$$

where $\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n} \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^{d\times N})$ is a certain measure depending on u and on n measures $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$, i.e., measures $\boldsymbol{\mu}_i \in \mathcal{M}(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$ for which there exists a measure $\lambda_i \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^{d\times N})$ such that for all $B \in \mathcal{B}(\Omega \times Y_1 \times \cdots Y_{i-1})$, $E \in \mathcal{B}(Y_i)$, we have

$$\big(D_{y_i}(\boldsymbol{\mu}_i(B))\big)(E) = \lambda_i(B \times E).$$

The measures τ_u and $\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}$ admit an explicit characterization, whose proof is not a simple generalization of the case $n = 1$ treated in Amar [5]. In fact, considerable modifications are required when $n \geqslant 2$, similar to those in Allaire and Briane [2] in the Sobolev setting. Moreover, we found out that fully developing the underlying measure-theoretical background was not straightforward.

Using the main results in Ferreira and Fonseca [43], in Ferreira and Fonseca [44] we characterized and related the functionals

$$F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) := \inf\left\{ \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in BV(\Omega;\mathbb{R}^d),\ Du_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n} \right\}$$

and

$$F^{\mathrm{hom}}(u) := \inf\left\{ \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in BV(\Omega;\mathbb{R}^d),\ u_\varepsilon \overset{\star}{\rightharpoonup}_\varepsilon u \text{ weakly-}\star \text{ in } BV(\Omega;\mathbb{R}^d) \right\}$$

for $u \in BV(\Omega;\mathbb{R}^d)$ and $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$, $i \in \{1, \cdots, n\}$, where F_ε is given by (1.2.1).

Precisely, under certain hypotheses on the function f (see Chapter 5 for the details), for all $(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \in BV(\Omega;\mathbb{R}^d) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1;\mathbb{R}^d)) \times \cdots \times \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{n-1}; BV_\#(Y_n;\mathbb{R}^d))$ we have that (see Theorem 5.1.3)

$$\begin{aligned} F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) &= \int_{\Omega\times Y_1\times\cdots\times Y_n} f\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x, y_1, \cdots, y_n)\Big)\, \mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &+ \int_{\Omega\times Y_1\times\cdots\times Y_n} f^\infty\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|}(x, y_1, \cdots, y_n)\Big)\, \mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(x, y_1, \cdots, y_n). \end{aligned} \tag{1.2.3}$$

Moreover, for all $u \in BV(\Omega;\mathbb{R}^d)$,

$$\begin{aligned} F^{\mathrm{hom}}(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d)),\ldots,\\ \boldsymbol{\mu}_n\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{n-1};BV_\#(Y_n;\mathbb{R}^d))}} F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \\ &= \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\, \mathrm{d}x + \int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\, \mathrm{d}\|D^s u\|(x), \end{aligned} \tag{1.2.4}$$

where f_{hom} is given by a cell problem formula (see (5.1.4)).

For simplicity, in (1.2.4) we provided the integral representation concerning the case in which a coercivity hypothesis on f is assumed. However, one of our main contributions in Ferreira and Fonseca

[44] was proving a similar result without assuming coercivity, or boundedness from below, of f. Such weak hypotheses are often useful to deal with degenerate media.

The main ingredients we will use to establish (1.2.3) and (1.2.4) are the unfolding operator (see Cioranescu, Damlamian and De Arcangelis [25], Cioranescu, Damlamian and Griso [27]; see also Fonseca and Krömer [47]) and Reshetnyak's continuity- and lower semicontinuity-type results. The approach via the unfolding operator, in connection with the notion of two-scale convergence and in the framework of homogenization problems, sometimes referred as periodic unfolding method, has already been adopted by other authors in the Sobolev setting (see, for example, Cioranescu, Damlamian and De Arcangelis [25], Cioranescu, Damlamian and De Arcangelis [26], Fonseca and Krömer [47]).

This dissertation is organized as follows. In Chapter 2, we collect the basic notations and background results that are used in the subsequent chapters. In Chapter 3, we prove the results announced in Section 1.1 above concerning the asymptotic behavior as $\varepsilon \to 0^+$ and $\delta \to 0^+$ of the spectrum of an elliptic operator with ε-periodic coefficients in a three-dimensional bounded domain of small thickness δ. The aim of Chapter 4 is to prove the characterization of $(n+1)$-scale limit pairs (u, U) of sequences $\{(u_\varepsilon \mathcal{L}^N{}_{\lfloor\Omega}, Du_{\varepsilon\lfloor\Omega})\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^d) \times \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV(\Omega;\mathbb{R}^d)$ referred in the Section 1.2. Finally, in Chapter 5, we treat multiple-scale homogenization problems in the space BV of functions of bounded variation, using the notion of multiple-scale convergence developed in Chapter 4; in particular, we prove the integral representations (1.2.3) and (1.2.4) claimed in Section 1.2.

Chapter 2

Preliminaries

The aim of this chapter is to provide a survey of the concepts and known results used throughout this dissertation. At the beginning of each section we will give references where proofs of the results therein and further considerations on the corresponding topic may be found.

2.1. <u>Measure Theory.</u>

In this section we briefly overview properties of measures. We refer to the books Fonseca and Leoni [48], Rudin [68], Evans and Gariepy [40], Ambrosio, Fusco and Pallara [7], and to the references therein.

2.1.1. <u>Positive Measures</u>

Definition 2.1.1. (σ-algebra, measurable space, measurable set) *Let X be a nonempty set. We say that a collection $\mathfrak{M} \subset 2^X$ is a σ-algebra (in X) if $\emptyset \in \mathfrak{M}$, $X\backslash E \in \mathfrak{M}$ whenever $E \in \mathfrak{M}$, and $\mathfrak{M}$ is closed under countable unions. If $\mathfrak{M} \subset 2^X$ is a σ-algebra, we call the pair $(X, \mathfrak{M})$ a measurable space, and a set $E \subset X$ is said to be measurable if $E \in \mathfrak{M}$.*

Definition 2.1.2. (Borel σ-algebra, Borel set) *Let X be a topological space. The smallest σ-algebra in X that contains all open subsets of X is called the Borel σ-algebra (in X) and is represented by $\mathcal{B}(X)$. A set $E \in \mathcal{B}(X)$ is said to be a Borel set.*

Definition 2.1.3. (Positive measure, measure space) *Let $(X, \mathfrak{M})$ be a measurable space. We say that a set map $\mu : \mathfrak{M} \to [0, +\infty]$ is a positive measure on $\mathfrak{M}$ if $\mu(\emptyset) = 0$ and μ is countably additive, i.e.,*

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=1}^{+\infty} \mu(E_j) \tag{2.1.1}$$

whenever $\{E_j\}_{j\in\mathbb{N}}$ is a countable collection of mutually disjoint measurable sets. The triple $(X, \mathfrak{M}, \mu)$ is called a measure space.

Definition 2.1.4. (Restriction of a measure) *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $E \in \mathfrak{M}$. The measure $\mu\lfloor_E : \mathfrak{M} \to [0, +\infty]$ defined by*

$$\mu\lfloor_E(F) := \mu(F \cap E), \quad F \in \mathfrak{M},$$

is called the restriction of μ to the measurable set E.

Definition 2.1.5. (σ-finite set, σ-finite and finite measures) *Let $(X, \mathfrak{M}, \mu)$ be a measure space. A set $E \in \mathfrak{M}$ is said to have σ-finite μ measure if it can be written as a countable union of measurable sets of finite μ measure. In the case in which X has σ-finite μ measure we say that μ is σ-finite. If $\mu(X) < +\infty$ we say that μ is finite.*

The next result concerns monotone convergence properties of measures.

Proposition 2.1.6. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. If $\{E_j\}_{j\in\mathbb{N}} \subset \mathfrak{M}$ is an nondecreasing sequence, then*

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \lim_{j\to+\infty} \mu(E_j).$$

If $\{E_j\}_{j\in\mathbb{N}} \subset \mathfrak{M}$ is an nonincreasing sequence with $\mu(E_1) < +\infty$, then

$$\mu\left(\bigcap_{j=1}^{+\infty} E_j\right) = \lim_{j\to+\infty} \mu(E_j).$$

Definition 2.1.7. (Borel, Borel regular and Radon measures; inner and outer regular sets) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, with X a topological space. We say that*

(i) μ is a Borel measure if $\mathcal{B}(X) \subset \mathfrak{M}$;

(ii) μ is a Borel regular measure if it is a Borel measure and for every set $E \in \mathfrak{M}$ there exists a set $F \in \mathcal{B}(X)$ such that $F \supset E$ and $\mu(E) = \mu(F)$;

(iii) μ is a Radon measure if it is a Borel measure satisfying the following conditions:

(a) $\mu(K) < +\infty$ for every compact set $K \subset X$,

(b) every open set $A \subset X$ is inner regular, i.e $\mu(A) = \sup\{\mu(K)\colon K \subset A,\ K$ compact$\}$,

(c) every set $E \in \mathfrak{M}$ is outer regular, i.e $\mu(E) = \inf\{\mu(A)\colon A \supset E,\ A$ open$\}$.

Definition 2.1.8. (Support of a Borel measure) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, with X a topological space and μ a Borel measure. The support of μ is the set*

$$\operatorname{supp}\mu := \big\{x \in X\colon\ \mu(\mathcal{O}) > 0 \text{ for every open neighborhood } \mathcal{O} \text{ of } x\big\}.$$

Definition 2.1.9. (Negligible set) *Let $(X, \mathfrak{M}, \mu)$ be a measure space. We say that a set $M \subset X$ is μ-negligible if there is a measurable set $E \in \mathfrak{M}$ such that $E \supset M$ and $\mu(E) = 0$. A property $P(x)$ depending on $x \in X$ is said to hold μ-almost everywhere in X (in short, to hold μ-a.e. in X or to hold for μ-a.e. $x \in X$)*[2.1] *if the set $\{x \in X\colon\ P(x)$ does not hold$\}$ is μ-negligible.*

In applications it is often very important to guarantee that subsets of sets of zero measure are still measurable.

Definition 2.1.10. (Complete measure) *Let $(X, \mathfrak{M}, \mu)$ be a measure space. We say that μ is complete if given any set $E \in \mathfrak{M}$ with $\mu(E) = 0$, then every subset of E belongs to $\mathfrak{M}$.*

[2.1] If μ is the l-dimensional Lebesgue measure $\mathcal{L}^l$, $l \in \mathbb{N}$, then its dependence is often omitted, and we simply write *a.e. in X* or *for a.e. $x \in X$*.

It is always possible to complete a measure. Precisely,

Proposition 2.1.11. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\mathfrak{M}_c$ be the collection of all sets $E \subset X$ for which there exist $F, G \in \mathfrak{M}$ with $F \subset E \subset G$ and such that $\mu(G \backslash F) = 0$. Define $\mu_c(E) := \mu(F)$. Then $\mathfrak{M}_c$ is a σ-algebra that contains $\mathfrak{M}$ and $\mu_c : \mathfrak{M}_c \to [0, +\infty]$ is a complete measure, which are called the μ completion of $\mathfrak{M}$ and the completion of μ, respectively.*

One of the most important examples of completion of a measure is the completion of the Lebesgue measure on the Borel σ-algebra.

Notation 2.1.12. *Let $l \in \mathbb{N}$. We will represent by $\mathcal{L}^l$ both the l-dimensional Lebesgue measure on the Borel σ-algebra and its completion.*

2.1.2. *Measurable Functions*

Definition 2.1.13. (Measurable and Borel functions) *Let $(X, \mathfrak{M})$ and $(Z, \mathfrak{N})$ be two measurable spaces. We say that a function $u : X \to Z$ is measurable if $u^{-1}(F) \in \mathfrak{M}$ for all $F \in \mathfrak{N}$. In the case in which X and Z are topological spaces, $\mathfrak{M} = \mathcal{B}(X)$ and $\mathfrak{N} = \mathcal{B}(Z)$ we say that u is a Borel function.*

Remark 2.1.14. *If X and Z are topological spaces, a function $u : X \to Z$ is Borel if, and only if, for every open set $A \subset Z$ we have $u^{-1}(A) \in \mathcal{B}(X)$.*

We now extend the notion of measurability to functions defined everywhere except in a set of zero measure.

Definition 2.1.15. (Generalization of the notion of measurable function) *Let $(X, \mathfrak{M}, \mu)$ and $(Z, \mathfrak{N})$ be a measure and a measurable space, respectively, and let $E \in \mathfrak{M}$ be such that $\mu(X \backslash E) = 0$. We say that $u : E \to Z$ is measurable over X if $u^{-1}(F) \in \mathfrak{M}$ for all $F \in \mathfrak{N}$.*

Remark 2.1.16. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $u : E \to [-\infty, +\infty]$ be a measurable function over X, where $E \in \mathfrak{M}$ is such that $\mu(X \backslash E) = 0$. Then the function $\tilde{u} : X \to [-\infty, +\infty]$ defined by*

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in E, \\ 0 & \text{if } x \in X \backslash E, \end{cases}$$

is measurable and $\int_X \tilde{u} \, \mathrm{d}\mu = \int_E u \, \mathrm{d}\mu$, where $\int \mathrm{d}\mu$ is the usual Lebesgue integral with respect to the measure μ.

Definition 2.1.17. (Simple function) *Let $(X, \mathfrak{M})$ be a measurable space. We say that a function $s : X \to \mathbb{R}$ is a simple function if it is measurable and if it takes finitely many values. If c_1, ..., c_m are the distinct values of s, then we write*

$$s = \sum_{i=1}^{m} c_i \chi_{E_i},$$

where χ_{E_i} is the characteristic function of the measurable set $E_i := s^{-1}(\{c_i\})$.

Theorem 2.1.18. *Let $(X, \mathfrak{M})$ be a measurable space and $u : X \to [-\infty, +\infty]$ a measurable function. Then there exists a sequence $\{s_j\}_{j \in \mathbb{N}}$ of simple functions such that*

$$\lim_{j \to +\infty} s_j(x) = u(x)$$

for all $x \in X$. Moreover, the convergence is uniform in every set in which u is bounded.

Lemma 2.1.19. (Fatou's Lemma) *Let $(X, \mathfrak{M}, \mu)$ be a measure space. The following statements hold:*

(i) If $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of nonnegative measurable functions $u_j : X \to [0, +\infty]$, then the function $u := \liminf_{j\to+\infty} u_j$ is measurable and

$$\int_X u \,\mathrm{d}\mu \leqslant \liminf_{j\to+\infty} \int_X u_j \,\mathrm{d}\mu;$$

(ii) If $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of measurable functions $u_j : X \to [-\infty, +\infty]$ for which there exists a measurable function $v : X \to [0, +\infty]$ such that $u_j \leqslant v$ for all $j \in \mathbb{N}$, and $\int_X v \,\mathrm{d}\mu < +\infty$, then the function $u := \limsup_{j\to+\infty} u_j$ is measurable and

$$\int_X u \,\mathrm{d}\mu \geqslant \limsup_{j\to+\infty} \int_X u_j \,\mathrm{d}\mu.$$

Theorem 2.1.20. (Lebesgue Dominated Convergence Theorem) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions $u_j : X \to [-\infty, +\infty]$ such that*

$$\lim_{j\to+\infty} u_j(x) = u(x)$$

for μ-a.e. $x \in X$. If there exists a Lebesgue integrable function[2.2] $v : X \to [0, +\infty]$ such that $|u_j(x)| \leqslant v(x)$ for μ-a.e. $x \in X$ and for all $j \in \mathbb{N}$, then u is Lebesgue integrable and

$$\lim_{j\to+\infty} \int_X |u_j - u| \,\mathrm{d}\mu = 0.$$

In particular,

$$\lim_{j\to+\infty} \int_X u_j \,\mathrm{d}\mu = \int_X u \,\mathrm{d}\mu.$$

Corollary 2.1.21. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions $u_j : X \to [-\infty, +\infty]$. If*

$$\sum_{j=1}^{+\infty} \int_X |u_j| \,\mathrm{d}\mu < +\infty,$$

then the series $\sum_{j=1}^{+\infty} u_j(x)$ converges for μ-a.e. $x \in X$, the function $u(x) := \sum_{j=1}^{+\infty} u_j(x)$, defined for μ-a.e. $x \in X$, is Lebesgue integrable, and

$$\sum_{j=1}^{+\infty} \int_X u_j \,\mathrm{d}\mu = \int_X \sum_{j=1}^{+\infty} u_j \,\mathrm{d}\mu.$$

We now state a measurable selection criterion (see Fonseca and Krömer [47, Lemma 3.10]; see also Castaing and Valadier [24]) and we recall Lusin's Theorem, which will be useful results for our analysis in Chapter 5.

[2.2] We recall that a function $v : X \to [-\infty, +\infty]$ is said to be Lebesgue integrable (in a measure space $(X, \mathfrak{M}, \mu)$) if it is measurable and $\int_X |v| \,\mathrm{d}\mu < +\infty$.

Lemma 2.1.22. *Let $(X, \mathfrak{M})$ and $(Z, \mathfrak{N})$ be two measurable spaces, with Z a separable metric space. Let $\Gamma : X \to 2^Z$ be a multifunction such that for every $x \in X$, $\Gamma(x) \subset Z$ is nonempty and open, and for every $z \in Z$, $\{x \in X : z \in \Gamma(x)\}$ is measurable. Then Γ admits a measurable selection, i.e., there exists a measurable function $\gamma : X \to Z$ such that for all $x \in X$, $\gamma(x) \in \Gamma(x)$.*

Theorem 2.1.23. (Lusin's Theorem) *Let $(X, \mathfrak{M}, \mu)$ and $(Z, \mathfrak{N})$ be a measure and a measurable space, respectively, with X a finite dimensional normed vector space, Z a separable metric space, and μ a finite Radon measure on $\mathfrak{M}$. Let $u : X \to Z$ be a measurable function. Then for all $\delta > 0$ there exists a compact set $K \subset X$ with $\mu(X \backslash K) < \delta$ such that $u_{|K}$ is continuous.*

2.1.3. Decomposition and Differentiation of Measures

Definition 2.1.24. (Absolutely continuous and mutually singular measures) *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, +\infty]$ be two positive measures on $\mathfrak{M}$. We say that*

(i) ν is absolutely continuous with respect to μ if $\nu(E) = 0$ whenever $E \in \mathfrak{M}$ is such that $\mu(E) = 0$, in which case we write $\nu \ll \mu$;

(ii) μ and ν are mutually singular if there exist two disjoint sets $X_\mu, X_\nu \in \mathfrak{M}$ such that $X = X_\mu \cup X_\nu$ and for all $E \in \mathfrak{M}$ one has

$$\mu(E) = \mu(E \cap X_\mu) \quad \text{and} \quad \nu(E) = \nu(E \cap X_\nu),$$

in which case we write $\mu \perp \nu$.

Theorem 2.1.25. (Radon–Nikodym Theorem) *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, +\infty]$ be two positive measures on $\mathfrak{M}$ such that μ is σ-finite and ν is absolutely continuous with respect to μ. Then there exists a measurable function $u : X \to [0, +\infty]$, unique up to a set of μ measure zero, such that $\nu = u\,\mu$, that is,*

$$\nu(E) = \int_E u \, \mathrm{d}\mu$$

for all $E \in \mathfrak{M}$.

Definition 2.1.26. (Radon–Nikodym derivative) *The function u in Theorem 2.1.25 is said to be the Radon–Nikodym derivative of ν with respect to μ, and we write $u = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.*

Theorem 2.1.27. (Lebesgue Decomposition Theorem) *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, +\infty]$ be two positive measures on $\mathfrak{M}$ being μ σ-finite. Then there exist two measures ν^{ac}, $\nu^s : \mathfrak{M} \to [0, +\infty]$ such that*

$$\nu = \nu^{ac} + \nu^s \tag{2.1.2}$$

and $\nu^{ac} \ll \mu$. Moreover, if ν is σ-finite, then $\nu^s \perp \mu$ and the decomposition (2.1.2) is unique, i.e., if $\tilde{\nu}^{ac}$ and $\tilde{\nu}^s$ are two positive measures on $\mathfrak{M}$ such that $\tilde{\nu}^{ac} \ll \mu$, $\tilde{\nu}^s \perp \mu$ and $\nu = \tilde{\nu}^{ac} + \tilde{\nu}^s$, then $\nu^{ac} = \tilde{\nu}^{ac}$ and $\nu^s = \tilde{\nu}^s$.

Definition 2.1.28. (Lebesgue decomposition of a measure, absolutely continuous part, singular part) *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, +\infty]$ be two σ-finite measures. We say that*

$$\nu = \nu^{ac} + \nu^s = \frac{\mathrm{d}\nu^{ac}}{\mathrm{d}\mu}\mu + \nu^s$$

is the Lebesgue decomposition of ν with respect to μ, where ν^{ac} and ν^s are the measures given by Theorem 2.1.27. The measures ν^{ac} and ν^s are called, respectively, the absolutely continuous part and the singular part of ν with respect to μ.

We finish this subsection by stating a local version of the Besicovitch Derivation Theorem.

Theorem 2.1.29. (Besicovitch Derivation Theorem) *Let $E \subset \mathbb{R}^l$ be a Borel set and let μ, $\nu : \mathcal{B}(E) \to [0, +\infty]$ be two Radon measures. Then*

$$\nu = \nu^{ac} + \nu^s, \quad \nu^{ac} \ll \mu, \quad \nu^s \perp \mu,$$

and there exists a Borel set $M \in \mathcal{B}(E)$ such that $\mu(M) = 0$ and for all $x \in E \backslash M$ it holds

$$\frac{\mathrm{d}\nu^{ac}}{\mathrm{d}\mu}(x) = \lim_{r \to 0^+} \frac{\nu((x + rC) \cap E)}{\mu((x + rC) \cap E)} \in \mathbb{R}, \quad \lim_{r \to 0^+} \frac{\nu^s((x + rC) \cap E)}{\mu((x + rC) \cap E)} = 0,$$

where C is an arbitrary bounded, convex closed set containing the origin in its interior.

2.1.4. Signed Measures

Definition 2.1.30. (Signed measure) *Let $(X, \mathfrak{M})$ be a measurable space. We say that a set map $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ is a signed measure on $\mathfrak{M}$ if $\lambda(\emptyset) = 0$, the range of λ is either contained in $[-\infty, +\infty)$ or in $(-\infty, +\infty]$, and λ is countably additive (i.e., (2.1.1) holds with μ replaced by λ).*

In particular, every positive measure is a signed measure.

Theorem 2.1.31. (Jordan Decomposition Theorem) *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ be a signed measure. Then there exists a unique pair (λ^-, λ^+) of mutually singular positive measures, one of which is finite, such that $\lambda = \lambda^+ - \lambda^-$.*[2.3]

Definition 2.1.32. (Total variation of a signed measure) *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ be a signed measure. The positive measure $\|\lambda\| : \mathfrak{M} \to [0, +\infty]$ defined for each $E \in \mathfrak{M}$ by*

$$\|\lambda\|(E) := \lambda^+(E) + \lambda^-(E)$$

is called the total variation of λ.

Proposition 2.1.33. *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ be a signed measure. Then*

$$\|\lambda\|(E) = \sup \left\{ \sum_{j=1}^{+\infty} |\lambda(E_j)| \colon \{E_j\}_{j \in \mathbb{N}} \subset \mathfrak{M} \text{ is a partition of } E \right\}$$

for all $E \in \mathfrak{M}$.

Definition 2.1.34. (σ-finite, absolutely continuous and mutually disjoint signed measures) *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda, \tau : \mathfrak{M} \to [-\infty, +\infty]$ be two signed measures. We say that λ is σ-finite if its total variation $\|\lambda\|$ is σ-finite. The measure τ is said to be absolutely continuous with respect to*

[2.3] This equality is often called the Jordan or Hahn decomposition of λ.

λ, and we write $\tau \ll \lambda$, if $\|\tau\|$ is absolutely continuous with respect to $\|\lambda\|$. We say that λ and τ are mutually singular, and we write $\lambda \perp \tau$, if $\|\lambda\|$ and $\|\tau\|$ are mutually singular.

Remark 2.1.35. *Let $(X, \mathfrak{M})$ be a measurable space, $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ a signed measure, and $\mu : \mathfrak{M} \to [0, +\infty]$ a σ- finite positive measure. Applying Lebesgue Decomposition Theorem to the two pairs $(\lambda^{\pm}, \mu)$ we can find positive measures $(\lambda^{ac})^{\pm}$, $(\lambda^{s})^{\pm}$ on $\mathfrak{M}$ such that $(\lambda^{ac})^{\pm} \ll \mu$ and*

$$\lambda^{\pm} = (\lambda^{ac})^{\pm} + (\lambda^{s})^{\pm}.$$

Moreover, in view of the Radon–Nikodym Theorem, there exist measurable functions $u^{\pm} : X \to [0, +\infty]$, unique up to a set of μ measure zero, such that $(\lambda^{ac})^{\pm} = u^{\pm}\mu$ (in other words, $u^{\pm} = \mathrm{d}(\lambda^{ac})^{\pm}/\mathrm{d}\mu$). Since at least one of the measures $\lambda^{\pm}$ is finite, we may define

$$\lambda^{ac} := (\lambda^{ac})^{+} - (\lambda^{ac})^{-}, \quad \lambda^{s} := (\lambda^{s})^{+} - (\lambda^{s})^{-}, \quad u := u^{+} - u^{-}.$$

Then λ^{ac} is a signed measure with $\lambda^{ac} \ll \mu$ and $\lambda^{ac} = u\,\mu$. Furthermore, if λ is in addition σ-finite, then $\lambda^{s} \perp \mu$ and the decomposition

$$\lambda = \lambda^{ac} + \lambda^{s} \tag{2.1.3}$$

is unique (in the sense of Lebesgue Decomposition Theorem). As in the positive case, (2.1.3), and the measures λ^{ac} and λ^{s} are called, respectively, the Lebesgue decomposition, the absolutely continuous part and the singular part of λ with respect to μ. We also write $u = \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mu}$.

Proposition 2.1.36. (Polar Decomposition Theorem) *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ be a σ-finite signed measure. Then there exists a measurable function $u : X \to [-\infty, +\infty]$ such that $|u(x)| = 1$ for $\|\lambda\|$-a.e. $x \in X$ and $u = \frac{\mathrm{d}\lambda}{\mathrm{d}\|\lambda\|}$*[2.4]*.*

Definition 2.1.37. (Signed Radon measure) *Let $(X, \mathfrak{M})$ be a measurable space with X a topological space. A signed measure $\lambda : \mathfrak{M} \to [-\infty, +\infty]$ is said to be a Radon measure if $\|\lambda\| : \mathfrak{M} \to [0, +\infty]$ is a positive Radon measure.*

In this dissertation we will also be interested in vector-valued measures.

Definition 2.1.38. (Vector-valued measures and their total variation) *Let $(X, \mathfrak{M})$ be a measurable space. We say that a set map $\lambda = (\lambda_1, \cdots, \lambda_m) : \mathfrak{M} \to \mathbb{R}^m$ is a vectorial measure on $\mathfrak{M}$*[2.5] *if each component $\lambda_i : \mathfrak{M} \to \mathbb{R}$ is a signed measure, $i \in \{1, \cdots, m\}$. The total variation of λ is the finite positive measure $\|\lambda\| : \mathfrak{M} \to [0, +\infty)$ on $\mathfrak{M}$ defined by*

$$\|\lambda\|(E) = \sup \left\{ \sum_{j=1}^{+\infty} |\lambda(E_j)| : \{E_j\}_{j\in\mathbb{N}} \subset \mathfrak{M} \text{ is a partition of } E \right\}^{2.6} \tag{2.1.4}$$

for all $E \in \mathfrak{M}$.

Definition 2.1.39. (Vectorial Radon measure, space $\mathcal{M}(X; \mathbb{R}^m)$) *Let $(X, \mathfrak{M})$ be a measurable space with X a topological space. A vectorial measure $\lambda = (\lambda_1, \cdots, \lambda_m) : \mathfrak{M} \to \mathbb{R}^m$ is said to be a Radon*

[2.4] This equation is often called the polar decomposition of λ.

[2.5] If $m = 1$, it is said to be a *real measure on* $\mathfrak{M}$.

[2.6] In fact, it can be checked that $\|\lambda\|(\cdot)$ given by this supremum defines a finite measure on $\mathfrak{M}$ (see, for example, Rudin [68]) *measure if each component $\lambda_i : \mathfrak{M} \to \mathbb{R}$ is a signed Radon measure, $i \in \{1, \cdots, m\}$. We represent by $\mathcal{M}(X; \mathbb{R}^m)$ the space of all vectorial Radon measures $\lambda : \mathcal{B}(X) \to \mathbb{R}^m$ on $\mathcal{B}(X)$, endowed with the total variation norm $\|\cdot\|$ given by (2.1.4).*

Remark 2.1.40. *It can be checked that $\mathcal{M}(X; \mathbb{R}^m)$ is a Banach space.*

The notions of σ-finite vectorial measure, absolutely continuous and mutually singular vectorial measures are defined in a similar way as in Definition 2.1.34. In particular, every vectorial measure is σ-finite. Moreover, arguing componentwise and in view of Remark 2.1.35, if $(X, \mathfrak{M})$ is a measurable space, $\lambda : \mathfrak{M} \to \mathbb{R}^m$ a vectorial measure and $\mu : \mathfrak{M} \to [0, +\infty]$ a σ-finite positive measure, then there exists a unique pair $(\lambda^{ac}, \lambda^s)$ of $\mathbb{R}^m$-valued measures on $\mathfrak{M}$ such that

$$\lambda = \lambda^{ac} + \lambda^s, \quad \lambda^{ac} \ll \mu, \quad \lambda^s \perp \mu,$$

and, up to a set of μ measure zero, there exits a unique measurable function $u : X \to \mathbb{R}^m$ such that $\lambda^{ac} = u\,\mu$. As before, u is known as the Radon–Nikodym derivative of λ^{ac} with respect to μ, and we write $u = \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mu}$.

Equality $\lambda = \lambda^{ac} + \lambda^s (= \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mu}\,\mu + \lambda^s)$, and the measures λ^{ac} and λ^s are called, respectively, the Lebesgue decomposition, the absolutely continuous part and the singular part of λ with respect to μ. We further observe that the Polar Decomposition Theorem still holds for $\mathbb{R}^m$-valued measures with the obvious modifications.

2.1.5. Product Measures

Definition 2.1.41. (Product σ-algebra) *Let $(X, \mathfrak{M})$ and $(Z, \mathfrak{N})$ be two measurable spaces. The smallest σ-algebra that contains all sets of the form $E \times F$, where $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$, is represented by $\mathfrak{M} \otimes \mathfrak{N}$ and called the product σ-algebra of $\mathfrak{M}$ and $\mathfrak{N}$.*

Theorem 2.1.42. (Fubini's Theorem) *Let $(X, \mathfrak{M}, \mu)$ and $(Z, \mathfrak{N}, \nu)$ be two measure spaces. Then there exist a σ-algebra $\mathfrak{M} \times \mathfrak{N}$ containing $\mathfrak{M} \otimes \mathfrak{N}$ and a positive measure $\mu \times \nu : \mathfrak{M} \times \mathfrak{N} \to [0, +\infty]$ on $\mathfrak{M} \times \mathfrak{N}$ such that for all $E \in \mathfrak{M}$, $F \in \mathfrak{N}$, we have*

$$(\mu \times \nu)(E \times F) = \mu(E)\,\nu(F)^{2.7}.$$

Moreover, if μ and ν are complete measures and $u : X \times Z \to [-\infty, +\infty]$ is $\mu \times \nu$-integrable, then for μ-a.e. $x \in X$ the function $u(x, \cdot)$ is ν-integrable and for ν-a.e. $z \in Z$ the function $u(\cdot, z)$ is μ-integrable; furthermore, the functions $\int_Z u(\cdot, z)\,\mathrm{d}\nu(z)$ and $\int_X u(x, \cdot)\,\mathrm{d}\mu(x)$ are μ-integrable and ν-integrable, respectively, and

$$\int_{X \times Z} u(x, z)\,\mathrm{d}(\mu \times \nu)(x, z) = \int_X \left(\int_Z u(x, z)\,\mathrm{d}\nu(z) \right) \mathrm{d}\mu(x) = \int_Z \left(\int_X u(x, z)\,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(z).$$

Remark 2.1.43. *Fubini's Theorem still holds for measures μ and ν not necessarily complete provided $u : X \times Z \to [-\infty, +\infty]$ is assumed to be $\mathfrak{M} \otimes \mathfrak{N}$-measurable. This will often be our case.*

Definition 2.1.44. (Product measure) *Let $(X, \mathfrak{M}, \mu)$ and $(Z, \mathfrak{N}, \nu)$ be two measure spaces. The measure $\mu \times \nu$ given by Fubini's Theorem is called the product measure of μ and ν. We represent by $\mu \otimes \nu$ the restriction of $\mu \times \nu$ to the product σ-algebra $\mathfrak{M} \otimes \mathfrak{N}$, i.e., $\mu \otimes \nu = \mu \times \nu_{\lfloor \mathfrak{M} \otimes \mathfrak{N}}$.*

[2.7] With the convention $\mu(E)\,\nu(F) := 0$ whenever $\mu(E) = 0$ or $\nu(F) = 0$.

We will be particularly interested in the case in which X and Z are topological spaces and $\mathfrak{M} = \mathcal{B}(X)$ and $\mathfrak{N} = \mathcal{B}(Z)$. In this case we have that $\mathcal{B}(X) \otimes \mathcal{B}(Z) \subset \mathcal{B}(X \times Z)$, but equality may fail. However, if X and Z are separable metric spaces, then $\mathcal{B}(X) \otimes \mathcal{B}(Z) = \mathcal{B}(X \times Z)$. Moreover, if $\mu \in \mathcal{M}(X;\mathbb{R})$ and $\nu \in \mathcal{M}(Z;\mathbb{R})$ are nonnegative, then $\mu \otimes \nu$ is a nonnegative measure in $\mathcal{M}(X \times Z;\mathbb{R})$ satisfying

$$(\mu \otimes \nu)(E \times F) = \mu(E)\,\nu(F) \tag{2.1.5}$$

for all $E \in \mathcal{B}(X)$, $F \in \mathcal{B}(Z)$.

More generally, if $\lambda \in \mathcal{M}(X;\mathbb{R})$, $\tau \in \mathcal{M}(Z;\mathbb{R})$, with X and Z separable metric spaces, we define

$$\lambda \otimes \tau := \lambda^+ \otimes \tau^+ + \lambda^- \otimes \tau^- - \lambda^+ \otimes \tau^- - \lambda^- \otimes \tau^+,$$

where $\lambda = \lambda^+ - \lambda^-$ and $\tau = \tau^+ - \tau^-$ are the Jordan decompositions of λ and τ, respectively. Then $\lambda \otimes \tau \in \mathcal{M}(X \times Z;\mathbb{R})$ and (2.1.5) holds with μ and ν replaced by λ and τ, respectively. Similarly, in the case in which $\lambda \in \mathcal{M}(X;\mathbb{R})$ and $\tau = (\tau_1, \cdots, \tau_m) \in \mathcal{M}(Z;\mathbb{R}^m)$, $\lambda \otimes \tau$ is the measure in $\mathcal{M}(X \times Z;\mathbb{R}^m)$ satisfying (2.1.5) (with μ and ν replaced by λ and τ, respectively) defined by $\lambda \otimes \tau := (\lambda \otimes \tau_1, \cdots, \lambda \otimes \tau_m)$.

2.1.6. Space of Radon Measures as a Dual Space

Throughout this subsection, $Y := (0,1)^N$ is the unit cube in $\mathbb{R}^N$, and for each $i \in \mathbb{N}$, Y_i stands for a copy of Y.

Definition 2.1.45. (Q-periodic function) *Let $\varphi : \mathbb{R}^N \to \mathbb{R}^m$ be a function and $Q = \Pi_{i=1}^N (0, b_i)$*[2.8] *an interval in $\mathbb{R}^N$. We say that φ is Q-periodic if for all $\kappa \in \mathbb{Z}$ and for a.e. $x \in \mathbb{R}^N$ one has $\varphi(x + \kappa b_i e_i) = \varphi(x)$, where $\{e_i\}_{i=1,\cdots,N}$ is the canonical basis of $\mathbb{R}^N$. If $\varphi : \mathbb{R}^{nN} \to \mathbb{R}^m$, we say that φ is $Y_1 \times \cdots \times Y_n$-periodic if for all $i \in \{1, \cdots, n\}$ the function $\varphi(y_1, ..., y_{i-1}, \cdot, y_{i+1}, ..., y_n)$ is Y_i-periodic for a.e. $y_1, ..., y_{i-1}, y_{i+1}, ..., y_n \in \mathbb{R}^N$.*

Let X be a normal σ-compact metrizable space. We represent by $C(X;\mathbb{R}^m)$ the space of all continuous functions $\varphi : X \to \mathbb{R}^m$, while $C_c(X;\mathbb{R}^m)$ is the subspace of $C(X;\mathbb{R}^m)$ of functions with compact support. The closure of $C_c(X;\mathbb{R}^m)$ with respect to the supremum norm $\|\cdot\|_\infty$ is denoted by $C_0(X;\mathbb{R}^m)$. It is well known that $C_0(X;\mathbb{R}^m)$ is a separable Banach space, and that $\varphi \in C_0(X;\mathbb{R}^m)$ if, and only if, $\varphi \in C(X;\mathbb{R}^m)$ and for all $\eta > 0$ there exists a compact set $K_\eta \subset X$ such that for all $x \in X \backslash K_\eta$, $|\varphi(x)| \leqslant \eta$. Moreover, if $\Omega \subset \mathbb{R}^N$ is an open and bounded set, then $C_0(\Omega;\mathbb{R}^m)$ coincides with the space of continuous functions in $\overline{\Omega}$ vanishing on $\partial\Omega$.

We will also consider the Banach spaces

$$C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m) := \big\{\varphi \in C(\mathbb{R}^{nN};\mathbb{R}^m) \colon \varphi \text{ is } Y_1 \times \cdots \times Y_n\text{-periodic}\big\}^{2.9}$$

endowed with the supremum norm $\|\cdot\|_\infty$, and $C_0(X; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$, which is the closure with respect to the supremum norm $\|\cdot\|_\infty$ of $C_c(X; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$. The latter is the space of all functions $\varphi : X \times \mathbb{R}^{nN} \to \mathbb{R}^m$ such that for all $x \in X$, $\varphi(x, \cdot) \in C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ and for all $y_1, ..., y_n \in \mathbb{R}^N$, $\varphi(\cdot, y_1, \ldots, y_n) \in C_c(X;\mathbb{R}^m)$.

[2.8] We could as well consider the case in which Q is not necessarily open.

[2.9] The space $C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ can be identified with the space $C_0(T_1 \times \cdots \times T_n;\mathbb{R}^m)$, where each T_i is a copy of the N-dimensional torus T.

For our convenience, we introduce here some related Banach spaces that will be used in the sequel of this work. We write $C^k(X;\mathbb{R}^m)$ (respectively, $C^k_c(X;\mathbb{R}^m)$ and $C^k_0(X;\mathbb{R}^m)$), $k \in \mathbb{N}$, to denote the space of all functions in $C(X;\mathbb{R}^m)$ (respectively, $C_c(X;\mathbb{R}^m)$ and $C_0(X;\mathbb{R}^m)$) whose i^{th}-partial derivatives are continuous functions in X for all $i \in \{1,\cdots,k\}$. We say that $\varphi \in C^\infty(X;\mathbb{R}^m)$ (respectively, $C^\infty_c(X;\mathbb{R}^m)$ and $C^\infty_0(X;\mathbb{R}^m)$) if for all $k \in \mathbb{N}$, $\varphi \in C^k(X;\mathbb{R}^m)$ (respectively, $C^k_c(X;\mathbb{R}^m)$ and $C^k_0(X;\mathbb{R}^m)$).

The spaces $C^k_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$, $C^\infty_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$, $C^k_c(X;C^k_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$, $C^\infty_c(X;C^\infty_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$, $C^k_0(X;C^k_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$ and $C^\infty_0(X;C^\infty_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$ are now defined in an obvious way.

If $m = 1$ the co-domain will often be omitted (e.g., we write $C_0(X)$ instead of $C_0(X;\mathbb{R})$).

The next theorem shows that we can identify the dual of $C_0(X;\mathbb{R}^m)$ with the space $\mathcal{M}(X;\mathbb{R}^m)$.

Theorem 2.1.46. (Riesz Representation Theorem in $C_0(X;\mathbb{R}^m)$) *Let X be a locally compact Hausdorff space. Then every bounded linear functional $L : C_0(X;\mathbb{R}^m) \to \mathbb{R}$ is represented by a unique vectorial Radon measure $\lambda = (\lambda_1,\cdots,\lambda_m) \in \mathcal{M}(X;\mathbb{R}^m)$ in the sense that*

$$L(\varphi) = \int_X \varphi \cdot \mathrm{d}\lambda := \sum_{i=1}^m \int_X \varphi_i(x)\,\mathrm{d}\lambda_i(x)^{2.10} \tag{2.1.6}$$

for all $\varphi = (\varphi_1,\cdots,\varphi_m) \in C_0(X;\mathbb{R}^m)$. Moreover, the norm of L coincides with the total variation norm $\|\lambda\|(X)$. Conversely, every functional of the form (2.1.6), where $\lambda = (\lambda_1,\cdots,\lambda_m) \in \mathcal{M}(X;\mathbb{R}^m)$, is a bounded linear functional on $C_0(X;\mathbb{R}^m)$.

In view of the previous theorem, the norm of a vectorial Radon measure $\lambda \in \mathcal{M}(X;\mathbb{R}^m)$ is alternatively given by

$$\|\lambda\|(X) = \sup\left\{ \int_X \varphi(x) \cdot \mathrm{d}\lambda(x) \colon \varphi \in C_0(X;\mathbb{R}^m),\ \|\varphi\|_\infty \leqslant 1 \right\}.$$

Moreover,

Theorem 2.1.47. *Let X be a σ-compact metric space, and let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(X;\mathbb{R}^m)$ be a sequence of vectorial Radon measures such that*

$$\sup_{j\in\mathbb{N}} \|\lambda_j\|(X) < +\infty.$$

Then there exist a subsequence $\{\lambda_{j_k}\}_{k\in\mathbb{N}}$ of $\{\lambda_j\}_{j\in\mathbb{N}}$ and a vectorial Radon measure $\lambda \in \mathcal{M}(X;\mathbb{R}^m)$ such that

$$\lambda_{j_k} \overset{\star}{\rightharpoonup} \lambda \text{ weakly-}\star \text{ in } \mathcal{M}(X;\mathbb{R}^m) \text{ as } k \to +\infty,$$

that is,

$$\lim_{k\to+\infty} \int_X \varphi \cdot \mathrm{d}\lambda_{j_k} = \int_X \varphi \cdot \mathrm{d}\lambda$$

for all $\varphi \in C_0(X;\mathbb{R}^m)$.

Proposition 2.1.48. *Let X be a locally compact, separable metric space. Let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(X;\mathbb{R}^m)$, $\lambda \in \mathcal{M}(X;\mathbb{R}^m)$ be such that $\lambda_j \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}(X;\mathbb{R}^m)$ as $j \to +\infty$. Then*

$$\|\lambda\|(X) \leqslant \liminf_{j\to+\infty} \|\lambda_j\|(X).$$

2.10 Also written as the duality pairing $\langle \lambda, \varphi\rangle_{\mathcal{M}(X;\mathbb{R}^m),C_0(X;\mathbb{R}^m)}$.

Notation 2.1.49. *If $\varphi \in C_0(X)$ and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{M}(X;\mathbb{R}^m)$, then we set*

$$\int_X \varphi(z)\,\mathrm{d}\lambda(z) := \Big(\int_X \varphi(z)\,\mathrm{d}\lambda_1(z), \cdots, \int_X \varphi(z)\,\mathrm{d}\lambda_m(z)\Big).$$

If $\varphi = (\varphi_1, \cdots, \varphi_m) \in C_0(X;\mathbb{R}^m)$ and $\lambda \in \mathcal{M}(X;\mathbb{R})$, then we define

$$\int_X \varphi(z)\,\mathrm{d}\lambda(z) := \Big(\int_X \varphi_1(z)\,\mathrm{d}\lambda(z), \cdots, \int_X \varphi_m(z)\,\mathrm{d}\lambda(z)\Big).$$

Having in mind Theorem 2.1.46 and footnote 2.9, we write $\mathcal{M}_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ and $\mathcal{M}_{y\#}(X \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ to denote the duals of $C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ and $C_0(X;C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$, respectively.

2.1.7. Disintegration of Measures

In this subsection we recall a disintegration property of Radon measures in a product space. We refer to Evans [42] for the proof (see also Ambrosio, Fusco and Pallara [7]).

Theorem 2.1.50. *Let X and Z be two σ-compact, separable metric spaces, and let $\mu : \mathcal{B}(X \times Z) \to [0,+\infty)$ be a finite positive Radon measure. Represent by π_μ the canonical projection of μ onto Z, i.e., the measure defined by $\pi_\mu(F) := \mu(X \times F)$ for all $F \in \mathcal{B}(Z)$. Then for π_μ-a.e. $z \in Z$ there exists a finite positive Radon measure $\nu_z : X \to [0,+\infty)$ such that $\nu_z(X) = 1$, and such that for all bounded and continuous function $\varphi : X \times Z \to \mathbb{R}$ the mapping*

$$z \mapsto \int_X \varphi(x,z)\,\mathrm{d}\nu_z(x)$$

is π_μ-measurable and

$$\int_{X\times Z} \varphi(x,z)\,\mathrm{d}\mu(x,z) = \int_Z \Big(\int_X \varphi(x,z)\,\mathrm{d}\nu_z(x)\Big)\,\mathrm{d}\pi_\mu(z). \tag{2.1.7}$$

2.1.8. Reshetnyak's Continuity and Lower Semicontinuity Results

In this subsection we recall two results due to Reshetnyak [66] (see also Ambrosio, Fusco and Pallara [7], Spector [72]).

Theorem 2.1.51. (Reshetnyak's Continuity Theorem) *Let X be a locally compact, separable metric space, and let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(X;\mathbb{R}^m)$, $\lambda \in \mathcal{M}(X;\mathbb{R}^m)$ be such that $\lambda_j \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}(X;\mathbb{R}^m)$ and $\|\lambda_j\|(\Omega) \to \|\lambda\|(\Omega)$ as $j \to +\infty$. Then*

$$\lim_{j\to+\infty} \int_X \varphi\Big(x, \frac{\mathrm{d}\lambda_j}{\mathrm{d}\|\lambda_j\|}(x)\Big)\,\mathrm{d}\|\lambda_j\|(x) = \int_X \varphi\Big(x, \frac{\mathrm{d}\lambda}{\mathrm{d}\|\lambda\|}(x)\Big)\,\mathrm{d}\|\lambda\|(x)$$

for every continuous function $\varphi : X \times \mathbb{R}^m \to \mathbb{R}$ satisfying a growth condition of the type $|\varphi(x,z)| \leqslant C|z|$ for some $C > 0$ and for all $(x,z) \in X \times \mathbb{R}^m$.

Remark 2.1.52. *If we replace X by an open set $\Omega \subset \mathbb{R}^N$, then Reshetnyak's Continuity Theorem holds for every continuous and bounded function $\varphi : \Omega \times S^{m-1} \to \mathbb{R}$, where S^{m-1} denotes the unit sphere of $\mathbb{R}^m$.*

Theorem 2.1.53. (Reshetnyak's Lower Semicontinuity Theorem) *Let X be a locally compact, separable metric space, and let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(X;\mathbb{R}^m)$, $\lambda \in \mathcal{M}(X;\mathbb{R}^m)$ be such that $\lambda_j \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}(X;\mathbb{R}^m)$. Then*

$$\lim_{j\to+\infty} \int_X \varphi\Big(x, \frac{\mathrm{d}\lambda_j}{\mathrm{d}\|\lambda_j\|}(x)\Big)\,\mathrm{d}\|\lambda_j\|(x) \geqslant \int_X \varphi\Big(x, \frac{\mathrm{d}\lambda}{\mathrm{d}\|\lambda\|}(x)\Big)\,\mathrm{d}\|\lambda\|(x)$$

for every continuous function $\varphi : X\times\mathbb{R}^m \to \mathbb{R}$, positively 1-homogeneous and convex in the second variable, and satisfying a growth condition of the type $|\varphi(x,z)| \leqslant C|z|$ for some $C>0$ and for all $(x,z)\in X\times\mathbb{R}^m$.

Remark 2.1.54. *If we replace X by an open set $\Omega\subset\mathbb{R}^N$, then Reshetnyak's Lower Semicontinuity Theorem holds for every lower semicontinuous function $\varphi : \Omega\times\mathbb{R}^m \to [0,+\infty]$, positively 1-homogeneous and convex in the second variable.*

2.2. Lebesgue and Sobolev Spaces.

In this section we recall well known results concerning Lebesgue and Sobolev spaces that will be used in this work. We refer to the books Brezis [22], Evans and Gariepy [40], Fonseca and Leoni [48], Leoni [60], and to the references therein.

2.2.1. Lebesgue Spaces

Definition 2.2.1. (L^p Spaces) *Let $(X,\mathfrak{M},\mu)$ be a measure space and let $1\leqslant p\leqslant +\infty$. We define*

$$L^p(X,\mu) := \Big\{u : X\to[-\infty,+\infty]\colon\ u \text{ measurable and } \|u\|_{L^p(X,\mu)} < +\infty\Big\}^{2.11},$$

where

$$\|u\|_{L^p(X,\mu)} := \begin{cases} \left(\int_X |u|^p\,\mathrm{d}\mu\right)^{1/p} & \text{if } 1\leqslant p<+\infty,\\ \operatorname{esse\,sup}|u| := \inf\{C\in\mathbb{R}\colon |u(x)|\leqslant C \text{ for } \mu\text{-a.e. } x\in X\} & \text{if } p=+\infty.\end{cases}$$

Notation 2.2.2. *When there is no possibility of confusion, we will simply write $L^p(X)$ in place of $L^p(X,\mu)$, and $\|\cdot\|_{L^p(X)}$, $\|\cdot\|_{L^p}$ or $\|\cdot\|_p$ in place of the norm $\|\cdot\|_{L^p(X,\mu)}$. Moreover, if $p=+\infty$ then L^∞ stands for L^p. In the case in which $\mu=\mathcal{L}^l$, we define $\int_X u(x)\,\mathrm{d}x := \int_X u(x)\,\mathrm{d}\mu$.*

Remark 2.2.3. *Endowed with the norm $\|\cdot\|_{L^p(X,\mu)}$, $L^p(X,\mu)$ is a Banach space for every $1\leqslant p\leqslant+\infty$, and $L^2(X,\mu)$ is a Hilbert space. If $1<p<+\infty$, then $L^p(X,\mu)$ is reflexive and its dual may be identified with $L^{p/(p-1)}(X,\mu)$. In the case in which μ is σ-finite the dual of $L^1(X,\mu)$ may be identified with $L^\infty(X,\mu)$, and if in addition $(X,\mathfrak{M})$ is separable, then so is $L^p(X,\mu)$ for all $1\leqslant p<+\infty$.*

Definition 2.2.4. (Hölder conjugate exponent) *Let $1\leqslant p\leqslant +\infty$. The Hölder conjugate p' of p is given by*

$$p' := \begin{cases} \dfrac{p}{p-1} & \text{if } 1<p<+\infty,\\ +\infty & \text{if } p=1,\\ 1 & \text{if } p=+\infty,\end{cases}$$

[2.11] Underlined is the identification of a measurable function u with its equivalence class $[u]$, that is, the set of all measurable functions that coincide with u μ-a.e. in X.

so that (with an abuse of notation) $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Theorem 2.2.5. (Hölder's Inequality) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $1 \leqslant p \leqslant +\infty$. *If* u, $v : X \to [-\infty, +\infty]$ *are measurable functions, then*

$$\|uv\|_{L^1(X)} \leqslant \|u\|_{L^p(X)} \|v\|_{L^{p'}(X)}.$$

In particular, if $u \in L^p(X)$ *and* $v \in L^{p'}(X)$ *then* $uv \in L^1(X)$.

Definition 2.2.6. (Weak convergence in L^p) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $1 \leqslant p \leqslant +\infty$. *If* $p \in \{1, +\infty\}$ *assume in addition that* μ *is* σ*-finite. We say that a sequence* $\{u_j\}_{j\in\mathbb{N}} \subset L^p(X)$ *weakly (weakly-*$\star$ *if* $p = +\infty$*) converges to a function* $u \in L^p(X)$, *and we write* $u_j \rightharpoonup u$ ($\overset{\star}{\rightharpoonup}$ *if* $p = +\infty$*), if for all* $v \in L^{p'}(X)$ *we have*

$$\lim_{j\to+\infty} \int_X u_j \, v \, \mathrm{d}\mu = \int_X u \, v \, \mathrm{d}\mu.$$

Proposition 2.2.7. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $1 \leqslant p \leqslant +\infty$. *If* $p = +\infty$ *assume in addition that* μ *is* σ*-finite. Let* $\{u_j\}_{j\in\mathbb{N}} \subset L^p(X)$, $u \in L^p(X)$. *The following conditions hold:*

(i) If $u_j \rightharpoonup u$ *weakly in* $L^p(X)$ ($\overset{\star}{\rightharpoonup}$, *weakly-*$\star$ *if* $p = +\infty$*) as* $j \to +\infty$, *then*

$$\|u\|_{L^p(X)} \leqslant \liminf_{j\to+\infty} \|u_j\|_{L^p(X)} \leqslant \sup_{j\in\mathbb{N}} \|u_j\|_{L^p(X)} < +\infty.$$

(ii) If $1 < p < +\infty$, $u_j \rightharpoonup u$ *weakly in* $L^p(X)$ *as* $j \to +\infty$, *and* $\|u\|_{L^p(X)} = \liminf_{j\to+\infty} \|u_j\|_{L^p(X)}$, *then* $u_j \to u$ *in* $L^p(X)$ *as* $j \to +\infty$.

(iii) If $1 < p < +\infty$ *and* $\sup_{j\in\mathbb{N}} \|u_j\|_{L^p(X)} < +\infty$, *then there exists a subsequence* $\{u_{j_k}\}_{k\in\mathbb{N}}$ *of* $\{u_j\}_{j\in\mathbb{N}}$ *such that* $u_{j_k} \rightharpoonup v$ *weakly in* $L^p(X)$ *as* $k \to +\infty$ *for some* $v \in L^p(X)$. *If in addition* $(X, \mathfrak{M})$ *is separable, then this property also holds in* $L^\infty(X)$ *with respect to the weak-*$\star$ *convergence.*

Definition 2.2.8. (Vectorial L^p spaces) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $1 \leqslant p \leqslant +\infty$. *We define*

$$L^p(X, \mu; \mathbb{R}^d) := \Big\{ u = (u_1, \cdots, u_d) \colon X \to \mathbb{R}^d \colon \ u_i \in L^p(X, \mu) \text{ for all } i \in \{1, \cdots, d\} \Big\},$$

and we endow $L^p(X, \mu; \mathbb{R}^d)$ *with the norm* $\|\cdot\|_{L^p(X,\mu;\mathbb{R}^d)}$ *given by*

$$\|u\|_{L^p(X,\mu;\mathbb{R}^d)} := \begin{cases} \left(\displaystyle\sum_{i=1}^d \|u_i\|^p_{L^p(X,\mu)} \right)^{1/p} & \text{if } 1 \leqslant p < +\infty, \\ \displaystyle\sum_{i=1}^d \|u_i\|_{L^\infty(X,\mu)} & \text{if } p = +\infty. \end{cases}$$

When there is no possibility of confusion, we write $L^p(X; \mathbb{R}^d)$ *instead of* $L^p(X, \mu; \mathbb{R}^d)$.

Remark 2.2.9. *In some situations it will be more convenient to use the equivalent norm in* $L^p(X, \mu; \mathbb{R}^d)$ *defined by*

$$\|u\|_{L^p(X,\mu;\mathbb{R}^d)} := \begin{cases} \left(\displaystyle\int_X |u|^p \, \mathrm{d}\mu \right)^{1/p} & \text{if } 1 \leqslant p < +\infty, \\ \operatorname{esse\,sup}|u| := \inf\{C \in \mathbb{R} \colon |u(x)| \leqslant C \text{ for } \mu\text{-a.e. } x \in X\} & \text{if } p = +\infty, \end{cases}$$

where, we recall, $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^d$.

Definition 2.2.10. (Spaces $L^p_{\rm loc}$) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, with X a topological space and $\mu : \mathfrak{M} \to [0, +\infty]$ a Borel measure, and let $1 \leqslant p \leqslant +\infty$. A measurable function $u : X \to [-\infty, +\infty]$ (respectively $u : X \to \mathbb{R}^d$) is said to belong to $L^p_{\rm loc}(X)$ (respectively $L^p_{\rm loc}(X; \mathbb{R}^d)$) if $u \in L^p(K)$ (respectively $L^p(K; \mathbb{R}^d)$) for every compact set $K \subset X$.*

We finish this subsection by stating Riemann–Lebesgue's Lemma, which provides an example of bounded sequences in L^p whose weak (weak-$\star$ if $p = +\infty$) limit can be explicitly characterized.

Theorem 2.2.11. (Riemann–Lebesgue's Lemma)[2.12] *Let $1 \leqslant p \leqslant +\infty$ and let $u \in L^p_{\rm loc}(\mathbb{R}^N)$ be a Q-periodic function, with Q an arbitrary N-dimensional bounded interval. For $\varepsilon > 0$ and $x \in \mathbb{R}^N$, define*

$$u_\varepsilon(x) := u\Big(\frac{x}{\varepsilon}\Big).$$

Then $u_\varepsilon \rightharpoonup \bar{u}$ ($\overset{\star}{\rightharpoonup}$ if $p = +\infty$) in $L^p_{\rm loc}(\mathbb{R}^N)$ ($L^\infty(\mathbb{R}^N)$ if $p = +\infty$), where $\bar{u}$ is (the constant) given by

$$\bar{u} := \frac{1}{\mathcal{L}^N(Q)} \int_Q u(y)\,\mathrm{d}y.$$

Using Riemann-Lebesgue's Lemma, in Donato [39] it is shown that if $\varrho_1, ..., \varrho_n$, $n \in \mathbb{N}$, are positive functions in $(0, \infty)$ such that for all $i \in \{1, \cdots, n\}$ and $j \in \{2, \cdots, n\}$, $\lim_{\varepsilon \to 0^+} \varrho_i(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0^+} \varrho_j(\varepsilon)/\varrho_{j-1}(\varepsilon) = 0$, then given $\varphi \in C(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^d))$ we have that

$$\varphi\Big(\cdot, \frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_n(\varepsilon)}\Big) \overset{\star}{\rightharpoonup} \int_{Y_1 \times \cdots \times Y_n} \varphi(\cdot, y_1, \cdots, y_n)\,\mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{2.2.1}$$

weakly-$\star$ in $L^\infty_{\rm loc}(\Omega; \mathbb{R}^d)$. In particular, if $\varphi \in C_0\big(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^d)\big)$ then (2.2.1) holds weakly-$\star$ in $L^\infty(\Omega; \mathbb{R}^d)$.

Also, if $\varphi : \mathbb{R}^{nN} \to \mathbb{R}$ is a $Y_1 \times \cdots \times Y_n$-periodic function such that for some $1 \leqslant p \leqslant +\infty$ and for a.e. $y_n \in Y_n$ we have $\varphi(\cdot, y_n) \in C_\#(Y_1 \times \cdots \times Y_{n-1})$ and $\|\varphi(\cdot, y_n)\|_{C_\#(Y_1 \times \cdots \times Y_{n-1})} \in L^p(Y_n)$, then

$$\begin{cases} \varphi\Big(\frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_n(\varepsilon)}\Big) \rightharpoonup \bar{\varphi} \text{ weakly in } L^p_{\rm loc}(\mathbb{R}^N) & \text{if } 1 \leqslant p < +\infty, \\ \varphi\Big(\frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_n(\varepsilon)}\Big) \overset{\star}{\rightharpoonup} \bar{\varphi} \text{ weakly-}\star \text{ in } L^\infty_{\rm loc}(\mathbb{R}^N) & \text{if } p = +\infty, \end{cases} \tag{2.2.2}$$

where

$$\bar{\varphi} := \int_{Y_1 \times \cdots \times Y_n} \varphi(y_1, \cdots, y_n)\,\mathrm{d}y_1 \ldots \mathrm{d}y_n.$$

2.2.2. Sobolev Spaces

Throughout this subsection Ω denotes an open subset of $\mathbb{R}^N$ and we consider the L^p spaces with respect to N-dimensional Lebesgue measure. The space of $(d \times N)$-dimensional matrices will be identified with $\mathbb{R}^{dN}$, and we write $\mathbb{R}^{d \times N}$. If $\xi = (\xi_{kl})_{1 \leqslant k \leqslant d, 1 \leqslant l \leqslant N}$, $\zeta = (\zeta_{kl})_{1 \leqslant k \leqslant d, 1 \leqslant l \leqslant N} \in \mathbb{R}^{d \times N}$, then

$$\xi : \zeta := \sum_{k=1}^{d} \sum_{l=1}^{N} \xi_{kl} \zeta_{kl}$$

[2.12] Here, and in the sequel, ε is a small parameter taking values on an arbitrary sequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$ of positive numbers converging to zero. We write ε, $\{u_\varepsilon\}_{\varepsilon > 0}$ and $\varepsilon \to 0^+$ in place of ε_j, $\{u_{\varepsilon_j}\}_{j \in \mathbb{N}}$ and $\varepsilon_j \to 0^+$ as $j \to \infty$, respectively. Moreover, ε' represents a subsequence of ε, and we write $\varepsilon' \prec \varepsilon$.

represents the inner product of ξ and ζ, while $|\xi| := \sqrt{\xi : \xi}$ denotes the norm of ξ.

Definition 2.2.12. (Weak derivatives) *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and let $i \in \{1, \cdots, N\}$. A function $g \in L^1_{\mathrm{loc}}(\Omega)$ satisfying for all $\phi \in C_c^\infty(\Omega)$ the equality*

$$\int_\Omega u \, \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x = -\int_\Omega \phi \, g \, \mathrm{d}x$$

is said to be the i^{th} derivative of u, and we write $g = \nabla_i u$ or $\partial u / \partial x_i$.

Remark 2.2.13. *If it exists, the i-th derivative of an $L^1_{\mathrm{loc}}(\Omega)$ function is unique. Moreover, it coincides with the classical one in the case in which $u \in C^1(\Omega)$, the reason why the same notation is used for both.*

Definition 2.2.14. (Spaces $W^{1,p}$) *Let $1 \leqslant p \leqslant +\infty$. We define*

$$W^{1,p}(\Omega) := \big\{ u \in L^p(\Omega) \colon \nabla_i u \in L^p(\Omega) \text{ for all } i \in \{1, \cdots, N\} \big\},$$

and if $u \in W^{1,p}(\Omega)$ we set $\nabla u := (\nabla_1 u, \cdots, \nabla_N u) \in \mathbb{R}^N$.

Similarly, we define

$$W^{1,p}(\Omega; \mathbb{R}^d) := \big\{ u = (u_1, \cdots, u_d) \colon \Omega \to \mathbb{R}^d \colon \; u_j \in W^{1,p}(\Omega) \text{ for all } j \in \{1, \cdots, d\} \big\},$$

and if $u = (u_1, \cdots, u_d) \in W^{1,p}(\Omega; \mathbb{R}^d)$ we set $\nabla u := (\nabla_i u_j)_{\substack{1 \leqslant j \leqslant d \\ 1 \leqslant i \leqslant N}} \in \mathbb{R}^{d \times N}$.

If $p = +\infty$, we simply write $W^{1,\infty}(\Omega) := W^{1,p}(\Omega)$ and $W^{1,\infty}(\Omega; \mathbb{R}^d) := W^{1,p}(\Omega; \mathbb{R}^d)$.

Remark 2.2.15. *When endowed with the norms*

$$\|u\|_{W^{1,p}(\Omega)} := \begin{cases} \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)} \right)^{1/p} & \text{if } 1 \leqslant p < +\infty, \\ \|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^N)} & \text{if } p = +\infty, \end{cases}$$

and

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^d)} := \begin{cases} \left(\|u\|^p_{L^p(\Omega;\mathbb{R}^d)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^{d\times N})} \right)^{1/p} & \text{if } 1 \leqslant p < +\infty, \\ \|u\|_{L^\infty(\Omega;\mathbb{R}^d)} + \|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^{d\times N})} & \text{if } p = +\infty, \end{cases}$$

the spaces $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega; \mathbb{R}^d)$, respectively, are:

(i) Banach spaces if $1 \leqslant p \leqslant +\infty$,

(ii) Hilbert spaces if $p = 2$,

(iii) separable if $1 \leqslant p < +\infty$,

(iv) reflexive if $1 < p < +\infty$.

Definition 2.2.16. (Weak convergence in $W^{1,p}$) *Let $1 \leqslant p \leqslant +\infty$. We say that a sequence $\{u_j\}_{j \in \mathbb{N}} \subset W^{1,p}(\Omega)$ weakly (weakly-$\star$ if $p = +\infty$) converges to a function $u \in W^{1,p}(\Omega)$, and we write $u_j \rightharpoonup u$ ($\overset{\star}{\rightharpoonup}$ if $p = +\infty$), if $u_j \rightharpoonup u$ and $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$ ($\overset{\star}{\rightharpoonup}$, weakly-$\star$ if $p = +\infty$) as $j \to +\infty$.*

Proposition 2.2.17. *Let $1 < p \leqslant +\infty$. If $\{u_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega)$ then there exist a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ of $\{u_j\}_{j\in\mathbb{N}}$ and a function $u \in W^{1,p}(\Omega)$ such that $u_{j_k} \rightharpoonup u$ ($\overset{\star}{\rightharpoonup}$ if $p = +\infty$) weakly (weakly-$\star$ if $p = +\infty$) in $W^{1,p}(\Omega)$ as $k \to +\infty$.*

Definition 2.2.18. (Higher order Sobolev spaces) *Let $k \in \mathbb{N}$ with $k \geqslant 2$, and let $1 \leqslant p \leqslant +\infty$. We define by induction the Sobolev space $W^{k,p}(\Omega)$ as*

$$W^{k,p}(\Omega) := \Big\{u \in L^p(\Omega)\colon\ \nabla u \in W^{k-1,p}(\Omega;\mathbb{R}^N)\Big\}.$$

Remark 2.2.19. *Alternatively, $W^{k,p}(\Omega)$ is given by*

$$W^{k,p}(\Omega) = \Big\{u \in L^p(\Omega)\colon\ D^\alpha u \in L^p(\Omega),\ 1 \leqslant |\alpha| \leqslant k\Big\},$$

where for $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}_0^N$ we put

$$D^\alpha u := \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_N^{\alpha_N}} \quad \text{and} \quad |\alpha| = \sum_{i=1}^N \alpha_i.$$

When endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \Big(\|u\|^p_{L^p(\Omega)} + \displaystyle\sum_{1\,\leqslant\,|\alpha|\,\leqslant\,k} \|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{1/p} & \text{if } 1 \leqslant p < +\infty,\\ \|u\|_{L^\infty(\Omega)} + \displaystyle\sum_{1\,\leqslant\,|\alpha|\,\leqslant\,k} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{if } p = +\infty, \end{cases}$$

properties (i)–(iv) of Remark 2.2.15 hold for $W^{k,p}(\Omega)$.

We now state two results concerning density of smooth functions in $W^{k,p}(\Omega)$.

Theorem 2.2.20. (Meyers–Serrin) *Let $k \in \mathbb{N}$ and $1 \leqslant p < +\infty$. Then the space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Theorem 2.2.21. *Let $1 \leqslant p < +\infty$ and assume that $\partial\Omega$ is Lipschitz. Then the space $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*[2.13]

2.2.3. Poincaré-Type Inequalities and Embeddings

Definition 2.2.22. (Spaces $W_0^{1,p}$) *Let $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$. We represent by $W_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ (with respect to the topology of $W^{k,p}(\Omega)$).*

Notation 2.2.23. *We set $H^k(\Omega) := W^{k,2}(\Omega)$ and $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.*

Theorem 2.2.24. (Poincaré's Inequality) *Let $1 \leqslant p < +\infty$. The following statements hold:*

(i) (Poincaré in $W_0^{1,p}(\Omega)$) Assume that the open set Ω has finite width, i.e., it lies between two parallel hyperplanes. Then for all $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |u(x)|^p\,\mathrm{d}x \leqslant \frac{d^p}{p}\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x,$$

[2.13] This result is valid for more general open sets Ω, precisely, those having *the segment property*; we refer to Leoni [60] for the details.

where d is the distance between the two hyperplanes.

(ii) (Poincaré in $W^{1,p}(\Omega)$) Assume that the open set Ω is also bounded, connected, and with $\partial\Omega$ Lipschitz. Let $E \subset \Omega$ be a Lebesgue measurable set with positive measure. Then there exists a positive constant $C = C(p, \Omega, E)$ such that for all $u \in W^{1,p}(\Omega)$,

$$\int_\Omega |u(x) - u_E|^p \,\mathrm{d}x \leqslant C \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x,$$

where $u_E := \dfrac{1}{\mathcal{L}^N(E)} \displaystyle\int_E u(x)\,\mathrm{d}x$.

Theorem 2.2.25. (Rellich–Kondrachov Theorem) *Assume that Ω is bounded and that $\partial\Omega$ is locally Lipschitz. Let $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$. Then $W^{k,p}(\Omega)$ is compactly embedded in:*

(i) $L^q(\Omega)$ if $kp \leqslant N$, $p \leqslant q < p^*$,

(ii) $C(\overline{\Omega})$ if $kp > N$,

(iii) $C^{0,s}(\overline{\Omega})$ if $0 < s < k - \frac{N}{p}$,

where $p^* := Np/(N - kp)$ if $kp < N$ and $p^* := +\infty$ if $kp = N$, and $C^{0,s}(\overline{\Omega})$ is the space of all functions $u \in C(\overline{\Omega})$ such that

$$\sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^s} < +\infty.$$

Remark 2.2.26. *The previous theorem holds for an arbitrary open and bounded set Ω if we replace $W^{k,p}(\Omega)$ by $W^{k,p}_0(\Omega)$.*

2.3. Integration with Respect to Functions of Bounded Variation-Valued Radon Measures.

In this section we start by recalling some well known properties of functions of bounded variation, and we refer to the books Ambrosio, Fusco and Pallara [7], Evans and Gariepy [40], Ziemer [77], and to the references therein. We also collect properties of integration with respect to certain Banach-valued measures, which seems to be hard to find in literature and which will play an important role in this dissertation.

As in subsection 2.2.2, throughout this section Ω denotes an open subset of $\mathbb{R}^N$ and we consider the L^1 space with respect to N-dimensional Lebesgue measure. The space of $(d \times N)$-dimensional matrices will be identified with $\mathbb{R}^{dN}$, and we write $\mathbb{R}^{d\times N}$.

2.3.1. Space of Functions of Bounded Variation

Definition 2.3.1. (Function of bounded variation, spaces $BV(\Omega;\mathbb{R}^d)$ and $BV_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$) *A function $u : \Omega \to \mathbb{R}^d$ is said to be of bounded variation if $u \in L^1(\Omega;\mathbb{R}^d)$ and its distributional derivative Du belongs to $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$, that is, if there exists a measure $Du \in \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ such that for all $\phi \in C_c(\Omega)$, $j \in \{1, \cdots, d\}$ and $i \in \{1, \cdots, N\}$ one has*

$$\int_\Omega u_j(x) \frac{\partial \phi}{\partial x_i}(x)\,\mathrm{d}x = -\int_\Omega \phi(x)\,\mathrm{d}D_i u_j(x),$$

where $u = (u_1, \cdots, u_d)$ *and* $Du_j = (D_1 u_j, \cdots, D_N u_j)$. *The space of all such functions* u *is denoted by* $BV(\Omega;\mathbb{R}^d)$. *We say that* $u \in BV_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ *if* $u \in BV(\Omega';\mathbb{R}^d)$ *for every open set* Ω' *compactly contained in* Ω *(briefly* $\Omega' \subset\subset \Omega$*).*

Remark 2.3.2. *The space* $BV(\Omega;\mathbb{R}^d)$ *is a Banach space when endowed with the norm*

$$\|u\|_{BV(\Omega;\mathbb{R}^d)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|Du\|(\Omega).$$

Remark 2.3.3. *If* $u \in W^{1,1}(\Omega;\mathbb{R}^d)$, *then* $u \in BV(\Omega;\mathbb{R}^d)$ *with* $Du = \nabla u \mathcal{L}^N_{\lfloor\Omega}$ *and* $\|Du\|(\Omega) = \int_\Omega |\nabla u|\,\mathrm{d}x$.

Notation 2.3.4. *For a function* $u \in BV(\Omega;\mathbb{R}^d)$ *the Radon–Nikodym derivative of the absolutely continuous part of* Du *with respect to* $\mathcal{L}^N_{\lfloor\Omega}$ *is represented by* ∇u, *and the singular part of* Du *with respect to* $\mathcal{L}^N_{\lfloor\Omega}$ *is denoted by* $D^s u$. *With this convention, the Lebesgue decomposition of* Du *with respect to* $\mathcal{L}^N_{\lfloor\Omega}$ *becomes*

$$Du = \nabla u \mathcal{L}^N_{\lfloor\Omega} + D^s u.$$

Theorem 2.3.5. (Lower semicontinuity in L^1_{loc} of the total variation) *Let* $\{u_j\}_{j\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^d)$, $u \in BV(\Omega;\mathbb{R}^d)$ *be such that* $u_j \to u$ *in* $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. *Then*

$$\|Du\|(\Omega) \leqslant \liminf_{j\to+\infty} \|Du_j\|(\Omega).$$

Theorem 2.3.6. (Approximation by smooth functions) *Let* $u \in BV(\Omega;\mathbb{R}^d)$. *Then there exists a sequence* $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\Omega;\mathbb{R}^d) \cap BV(\Omega;\mathbb{R}^d)$ *such that*

$$\lim_{j\to+\infty} \|u_j - u\|_{L^1(\Omega;\mathbb{R}^d)} = 0, \qquad \lim_{j\to+\infty} \int_\Omega |\nabla u_j(x)|\,\mathrm{d}x = \|Du\|(\Omega).$$

The norm topology is too strong for our purposes, which motivates the introduction of a weaker notion of convergence in $BV(\Omega;\mathbb{R}^d)$. The usefulness of the latter is justified by a compactness result.

Definition 2.3.7. (Weak-$\star$ convergence in BV) *We say that* $\{u_j\}_{j\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^d)$ *weakly-*$\star$ *converges to a function* $u \in BV(\Omega;\mathbb{R}^d)$ *in* $BV(\Omega;\mathbb{R}^d)$, *and we write* $u_j \overset{\star}{\rightharpoonup} u$, *if* $u_j \to u$ *(strongly) in* $L^1(\Omega;\mathbb{R}^d)$ *and* $Du_j \overset{\star}{\rightharpoonup} Du$ *weakly-*$\star$ *in* $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ *as* $j \to +\infty$.

Theorem 2.3.8. *From every bounded sequence in* $BV(\Omega;\mathbb{R}^d)$ *we can extract a weakly-*$\star$ *convergent subsequence in* $BV(\Omega;\mathbb{R}^d)$.

We now state the BV version of Theorems 2.2.24 and 2.2.25.

Theorem 2.3.9. (Embedding Theorem) *Assume that* Ω *is bounded and that* $\partial\Omega$ *is Lipschitz. Then for all* $1 \leqslant p < 1^*$[2.14] *we have that* $BV(\Omega;\mathbb{R}^d)$ *is compactly embedded in* $L^p(\Omega;\mathbb{R}^d)$. *Moreover,* $BV(\Omega;\mathbb{R}^d)$ *is continuously embedded in* $L^{1^*}(\Omega;\mathbb{R}^d)$.

Theorem 2.3.10. (Poincaré's Inequality) *Assume that* Ω *is bounded and that* $\partial\Omega$ *is Lipschitz. Then there exists a positive constant* $C = C(\Omega)$ *such that for all* $u \in BV(\Omega;\mathbb{R}^d)$ *and* $1 \leqslant p \leqslant 1^*$, *we have*

$$\|u - u_\Omega\|_{L^p(\Omega;\mathbb{R}^d)} \leqslant C\|Du\|(\Omega).$$

[2.14] We recall that $1^* = +\infty$ if $N = 1$, and $1^* = N/(N-1)$ if $N > 1$.

In this dissertation the functions of bounded variation that are periodic assume an important role.

Definition 2.3.11. (Space $BV_{\#}(Y;\mathbb{R}^d)$) *We define*

$$BV_{\#}(Y;\mathbb{R}^d) := \big\{u \in BV_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^d)\colon\ u \text{ is } Y\text{-periodic}\big\},$$

endowed with the norm of $BV(Y;\mathbb{R}^d)$, where $Y := (0,1)^N$.

Remark 2.3.12. *We have that $BV_{\#}(Y;\mathbb{R}^d)$ is a Banach space, and if $u \in BV_{\#}(Y;\mathbb{R}^d)$, then $Du \in \mathcal{M}_{\#}(Y;\mathbb{R}^{d\times N})$.*

2.3.2. Integration with respect to $BV_{\#}(Y;\mathbb{R}^d)$-valued Radon measures

In this subsection, X denotes a σ-compact separable metric space, Ω an open subset of $\mathbb{R}^N$, and for each $i \in \mathbb{N}$, Y_i stands for a copy of $Y := (0,1)^N$.

Integration with respect to Banach-valued measures seems to be hard to find in literature. Here we collect properties of integration with respect to $BV_{\#}(Y;\mathbb{R}^d)$-valued Radon measures, which will play an important role in Chapters 4 and 5. The considerations in this subsection may also be found in Ferreira and Fonseca [43].

We start by recalling the notion of Banach space-valued measures. For a more detailed exposition see, for example, Diestel and Uhl [38].

Definition 2.3.13. (Borel and Radon Banach-valued measures) *Let $\mathcal{Z}$ be a Banach space. We say that $\boldsymbol{\mu} : \mathcal{B}(X) \to \mathcal{Z}$ is a ($\mathcal{Z}$-valued) Borel measure if the following conditions are satisfied:*

i) $\boldsymbol{\mu}(\emptyset) = 0$,

ii) Given any countable family $\{B_j\}_{j\in\mathbb{N}}$ of mutually disjoint Borel subsets of X, the series $\sum_{j=1}^{+\infty} \boldsymbol{\mu}(B_j)$ converges (in $\mathcal{Z}$) and

$$\boldsymbol{\mu}\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \boldsymbol{\mu}(B_j).$$

If, in addition, the condition

iii) The total variation of $\boldsymbol{\mu}$,

$$\|\boldsymbol{\mu}\|(X) := \sup\Big\{\sum_{j=1}^{+\infty} \|\boldsymbol{\mu}(B_j)\|_{\mathcal{Z}}\colon\ \{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(X) \text{ is a partition of } X\Big\},$$

is finite,

is satisfied, then we say that $\boldsymbol{\mu}$ is a ($\mathcal{Z}$-valued) Radon measure with finite total variation, and we write $\boldsymbol{\mu} \in \mathcal{M}(X;\mathcal{Z})$.

Notice that if $\boldsymbol{\mu} \in \mathcal{M}(X;\mathcal{Z})$, then $\|\boldsymbol{\mu}\| : \mathcal{B}(X) \to [0,\infty)$ defined by

$$\|\boldsymbol{\mu}\|(B) := \sup\Big\{\sum_{j=1}^{+\infty} \|\boldsymbol{\mu}(B_j)\|_{\mathcal{Z}}\colon\ \{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(X) \text{ is a partition of } B\Big\}, \quad B \in \mathcal{B}(X),$$

is a finite positive Radon measure on $\mathcal{B}(X)$.

We will be particularly interested in the case in which $X = \Omega \times Y_1 \times \cdots \times Y_{i-1}$ for some $i \in \mathbb{N}$, where

$$\Omega \times Y_1 \times \cdots \times Y_{i-1} := \Omega \quad \text{if } i = 1,$$

and $\mathcal{Z} = BV_\#(Y_i; \mathbb{R}^d)$.

Let $\boldsymbol{\mu} \in \mathcal{M}(X; BV_\#(Y; \mathbb{R}^d))$ and $B \in \mathcal{B}(X)$. Then $\boldsymbol{\mu}(B) \in BV_\#(Y; \mathbb{R}^d)$, and so $D_y(\boldsymbol{\mu}(B)) \in \mathcal{M}_\#(Y; \mathbb{R}^{d\times N})$. Moreover, it can be checked that the mapping $D_y\boldsymbol{\mu} : B \in \mathcal{B}(X) \mapsto D_y\boldsymbol{\mu}(B) := D_y(\boldsymbol{\mu}(B))$ belongs to $\mathcal{M}(X; \mathcal{M}_\#(Y; \mathbb{R}^{d\times N}))$ in the sense of Definition 2.3.13.

In fact, since $\boldsymbol{\mu}(\emptyset) = 0$, we have

$$D_y\boldsymbol{\mu}(\emptyset) = D_y(\boldsymbol{\mu}(\emptyset)) = 0. \tag{2.3.1}$$

Let $\{B_j\}_{j\in\mathbb{N}}$ be a countable family of mutually disjoint Borel subsets of X. Define the functions v_k and v_∞ by

$$v_k := \sum_{j=1}^{k} \boldsymbol{\mu}(B_j) \ \text{ and } \ v_\infty := \sum_{j=1}^{+\infty} \boldsymbol{\mu}(B_j),$$

respectively. Since $\boldsymbol{\mu} \in \mathcal{M}(X; BV_\#(Y; \mathbb{R}^d))$, we have v_k, $v_\infty \in BV_\#(Y; \mathbb{R}^d)$. Moreover, given any $w \in C^\infty_\#(Y; \mathbb{R}^d)$,

$$\sum_{j=1}^{+\infty} \int_Y |w(y) \cdot \boldsymbol{\mu}(B_j)(y)| \,\mathrm{d}y \leqslant \|w\|_\infty \sum_{j=1}^{+\infty} \|\boldsymbol{\mu}(B_j)\|_{BV_\#(Y;\mathbb{R}^d)} \leqslant \|w\|_\infty \|\boldsymbol{\mu}\|(X) < \infty.$$

Consequently (see Corollary 2.1.21),

$$\int_Y (w \cdot v_\infty) \,\mathrm{d}y = \int_Y \sum_{j=1}^{+\infty} \big(w \cdot \boldsymbol{\mu}(B_j)\big) \,\mathrm{d}y = \sum_{j=1}^{+\infty} \int_Y \big(w \cdot \boldsymbol{\mu}(B_j)\big) \,\mathrm{d}y,$$

and

$$\sum_{j=1}^{+\infty} \int_Y \big(w \cdot \boldsymbol{\mu}(B_j)\big) \,\mathrm{d}y = \lim_{k\to+\infty} \sum_{j=1}^{k} \int_Y \big(w \cdot \boldsymbol{\mu}(B_j)\big) \,\mathrm{d}y = \lim_{k\to+\infty} \int_Y (w \cdot v_k) \,\mathrm{d}y.$$

Therefore,

$$\int_Y (w \cdot v_\infty) \,\mathrm{d}y = \lim_{k\to+\infty} \int_Y (w \cdot v_k) \,\mathrm{d}y, \tag{2.3.2}$$

and since $w \in C^\infty_\#(Y; \mathbb{R}^d)$ was taken arbitrarily, we conclude that $v_k \to v_\infty$ as $k \to +\infty$, in the sense of distributions. Thus, $D_y v_k \to D_y v_\infty$ as $k \to +\infty$, in the sense of distributions, and so

$$\begin{aligned}\lim_{k\to+\infty} \sum_{j=1}^{k} D_y(\boldsymbol{\mu}(B_j)) &= \lim_{k\to+\infty} D_y\bigg(\sum_{j=1}^{k} \boldsymbol{\mu}(B_j)\bigg) = \lim_{k\to+\infty} D_y v_k = D_y v_\infty \\ &= D_y\bigg(\sum_{j=1}^{+\infty} \boldsymbol{\mu}(B_j)\bigg) = D_y\bigg(\boldsymbol{\mu}\bigg(\bigcup_{j=1}^{+\infty} B_j\bigg)\bigg) = D_y\boldsymbol{\mu}\bigg(\bigcup_{j=1}^{+\infty} B_j\bigg),\end{aligned}$$

in the sense of distributions, where we have used the fact that $\boldsymbol{\mu} \in \mathcal{M}(X; BV_\#(Y; \mathbb{R}^d))$. Since $B^\infty := \cup_{j=1}^{+\infty} B_j \in \mathcal{B}(X)$, we have $D_y\boldsymbol{\mu}(B^\infty) = D_y(\boldsymbol{\mu}(B^\infty)) \in \mathcal{M}_\#(Y; \mathbb{R}^{d\times N})$, and thus we proved that $\sum_{j=1}^{+\infty} D_y\boldsymbol{\mu}(B_j)$ converges and it is equal to $D_y\boldsymbol{\mu}(B^\infty) \in \mathcal{M}_\#(Y; \mathbb{R}^{d\times N})$, so that

$$D_y\boldsymbol{\mu}\bigg(\bigcup_{j=1}^{+\infty} B_j\bigg) = \sum_{j=1}^{+\infty} D_y\boldsymbol{\mu}(B_j). \tag{2.3.3}$$

Moreover, the total variation of $D_y\boldsymbol{\mu}$,

$$\|D_y\boldsymbol{\mu}\|(X) := \sup\left\{\sum_{j=1}^{+\infty}\|D_y\boldsymbol{\mu}(B_j)\|_{\mathcal{M}_\#(Y;\mathbb{R}^{d\times N})}\colon\ \{B_j\}_{j\in\mathbb{N}}\subset\mathcal{B}(X) \text{ is a partition of } X\right\},$$

is finite due to the inequality $\|D_y(\boldsymbol{\mu}(B))\|_{\mathcal{M}_\#(Y;\mathbb{R}^{d\times N})}\leqslant\|\boldsymbol{\mu}(B)\|_{BV_\#(Y;\mathbb{R}^d)}$, for all $B\in\mathcal{B}(X)$, and to the fact that $\boldsymbol{\mu}$ has finite total variation. This, (2.3.1) and (2.3.3) yield $D_y\boldsymbol{\mu}\in\mathcal{M}(X;\mathcal{M}_\#(Y;\mathbb{R}^{d\times N}))$.

As we will see in Chapter 4, the measures $\boldsymbol{\mu}\in\mathcal{M}(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$ whose associated mapping $D_y\boldsymbol{\mu}$ may be identified with an element of $\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})$ will play an important role in the characterization of the multiscale limit of the sequence of distributional derivatives of a bounded sequence in $BV(\Omega;\mathbb{R}^d)$. This motivates the following definition.

Definition 2.3.14. (Space $\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$) *We represent by $\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$ the space of all $BV_\#(Y_i;\mathbb{R}^d)$-valued Radon measures $\boldsymbol{\mu}\in\mathcal{M}(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$ for which there exists a $\mathbb{R}^{d\times N}$-valued Radon measure $\lambda\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})$ such that for all $B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$, $E\in\mathcal{B}(Y_i)$,*

$$\big(D_{y_i}(\boldsymbol{\mu}(B))\big)(E)=\lambda(B\times E). \tag{2.3.4}$$

We say that λ is the measure associated with $D_{y_i}\boldsymbol{\mu}$.

Note that since $\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})\otimes\mathcal{B}(Y_i)=\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_i)$, it follows that if $\boldsymbol{\mu}\in\mathcal{M}(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$, then there exists at most one measure $\lambda\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})$ satisfying (2.3.4).

Example 2.3.15. *Fix $i\in\mathbb{N}$, let $\tau\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_{i-1};\mathbb{R})$, and let $v\in BV_\#(Y_i;\mathbb{R}^d)$. Then the mapping*

$$\boldsymbol{\mu}: B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})\mapsto\boldsymbol{\mu}(B):=\tau(B\times Y_1\times\cdots\times Y_{i-1})\,v$$

belongs to $\mathcal{M}(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$, with

$$\|\boldsymbol{\mu}\|(\Omega\times Y_1\times\cdots\times Y_{i-1})=\|\tau\|(\Omega\times Y_1\times\cdots\times Y_{i-1})\|v\|_{BV(Y_i;\mathbb{R}^d)}.$$

Observe also that for all $B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$, $(D_{y_i}\boldsymbol{\mu})(B)=D_{y_i}(\boldsymbol{\mu}(B))=\tau(B)Dv$. Moreover, defining $\lambda:=\tau\otimes Dv$, we have that $\lambda\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})$ and (2.3.4) holds. Thus, $\boldsymbol{\mu}\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$.

Our goal now is to give sense to the expression

$$\int_{\Omega\times Y_1\times\cdots\times Y_i}\varphi(x,y_1,\cdots,y_i)\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i, \tag{2.3.5}$$

whenever $\varphi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_i))$ and $\boldsymbol{\mu}\in\mathcal{M}(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$.

Step 1. We start by assuming that $i=1$, and we write Y in place of Y_1. As it is usual when defining an integral, we will start by giving meaning to (2.3.5) for simple functions and then, using approximation arguments, we will extend such notion to more general functions. Let $s:\Omega\to\mathbb{R}$ be a Borel, simple function, with

$$s:=\sum_{i=1}^{m}c_i\chi_{B_i}, \tag{2.3.6}$$

where $m \in \mathbb{N}$, $c_1, ..., c_m \in \mathbb{R}$ are distinct and $B_1, ..., B_m \in \mathcal{B}(\Omega)$ are mutually disjoint. If $B \in \mathcal{B}(\Omega)$, then we define the integral of s over B with respect to $\boldsymbol{\mu}$, and we write $\int_B s(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, as the function in $BV_\#(Y;\mathbb{R}^d)$ given by

$$\int_B s(x)\,\mathrm{d}\boldsymbol{\mu}(x) := \sum_{i=1}^{m} c_i \boldsymbol{\mu}(B_i \cap B). \tag{2.3.7}$$

Let $\phi : \Omega \to \mathbb{R}$ be a bounded, Borel function, and let $\{s_j\}_{j\in\mathbb{N}}$ be a sequence of Borel, simple functions converging uniformly in Ω to ϕ, with $s_j := \sum_{i=1}^{m_j} c_i^{(j)} \chi_{B_i^{(j)}}$ as in (2.3.6). We have that

$$\int_Y \left| \int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y = \int_Y \left| \sum_{i=1}^{m_j} c_i^{(j)} \boldsymbol{\mu}\Big(B_i^{(j)}\Big) \right| \mathrm{d}y \leqslant \sum_{i=1}^{m_j} |c_i^{(j)}| \Big\| \boldsymbol{\mu}\Big(B_i^{(j)}\Big) \Big\|_{L^1(Y;\mathbb{R}^d)}$$

and

$$\left\| D_y \left(\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right) \right\| (Y) \leqslant \sum_{i=1}^{m_j} |c_i^{(j)}| \Big\| D_y \Big(\boldsymbol{\mu}\Big(B_i^{(j)}\Big)\Big) \Big\| (Y),$$

where we used (2.3.7). Consequently, using the definition of the total variation of $\boldsymbol{\mu}$,

$$\int_Y \left| \int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y + \left\| D_y \left(\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right) \right\| (Y) \leqslant \|s_j\|_\infty \|\boldsymbol{\mu}\|(\Omega) \tag{2.3.8}$$

and also

$$\int_Y \left| \int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y \leqslant \sum_{i=1}^{m_j} |c_i^{(j)}| \|\boldsymbol{\mu}\| \Big(B_i^{(j)}\Big) = \int_\Omega |s_j(x)|\,\mathrm{d}\|\boldsymbol{\mu}\|(x). \tag{2.3.9}$$

Since $\sup_j \|s_j\|_\infty < \infty$ and $\boldsymbol{\mu}$ has finite total variation, we deduce from (2.3.8) that the sequence

$$\left\{ \int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right\}_{j\in\mathbb{N}}$$

is uniformly bounded in $BV_\#(Y;\mathbb{R}^d)$. Thus, up to a (not relabeled) subsequence, we may find $u \in BV_\#(Y;\mathbb{R}^d)$ such that

$$\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \overset{\star}{\rightharpoonup} u \ \text{ weakly-}\star \text{ in } BV_\#(Y;\mathbb{R}^d) \text{ as } j \to +\infty.$$

Assume now that $\{t_j\}_{j\in\mathbb{N}}$ is another sequence of Borel, simple functions converging uniformly in Ω to ϕ, and such that

$$\int_\Omega t_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \overset{\star}{\rightharpoonup} v \ \text{ weakly-}\star \text{ in } BV_\#(Y;\mathbb{R}^d) \text{ as } j \to +\infty,$$

for some $v \in BV_\#(Y;\mathbb{R}^d)$. Then $\{s_j - t_j\}_{j\in\mathbb{N}}$ is a sequence of Borel, simple functions converging uniformly in Ω to 0, and so (2.3.8) ensures that $u = v$ for $\mathcal{L}^N$-a.e. $y \in \mathbb{R}^N$. This gives sense to the following definition.

Definition 2.3.16. (Integral with respect to $\boldsymbol{\mu}$) *Let $\phi : \Omega \to \mathbb{R}$ be a bounded, Borel measurable function. If $B \in \mathcal{B}(\Omega)$ and $\boldsymbol{\mu} \in \mathcal{M}(\Omega; BV_\#(Y;\mathbb{R}^d))$, then we define the integral of ϕ over B with respect to $\boldsymbol{\mu}$, and we write $\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, as the function in $BV_\#(Y;\mathbb{R}^d)$ given by*

$$\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) := \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \int_B s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x),$$

where $\{s_j\}_{j\in\mathbb{N}}$ is a sequence of Borel, simple functions converging uniformly in Ω to ϕ.

The following lemma will be useful in the sequel. Its proof uses (2.3.8), (2.3.9), Definition 2.3.16, Lebesgue Dominated Convergence Theorem and the lower semicontinuity of the total variation.

Lemma 2.3.17. *Let $\phi : \Omega \to \mathbb{R}$ be a bounded, Borel measurable function and $\boldsymbol{\mu} \in \mathcal{M}(\Omega; BV_\#(Y;\mathbb{R}^d))$. The following hold:*

i) $\displaystyle\int_Y \left| \int_\Omega \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y \leqslant \int_\Omega |\phi(x)|\,\mathrm{d}\|\boldsymbol{\mu}\|(x);$

ii) If $\boldsymbol{\nu}$ is the set application given by $\boldsymbol{\nu}(B) := \displaystyle\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, $B \in \mathcal{B}(\Omega)$, then $\boldsymbol{\nu} \in \mathcal{M}(\Omega; BV_\#(Y;\mathbb{R}^d))$, and $\|\boldsymbol{\nu}\|(B) \leqslant \|\phi\|_\infty \|\boldsymbol{\mu}\|(B)$ for all $B \in \mathcal{B}(\Omega)$.

Proof. Let $\{s_j\}_{j\in\mathbb{N}}$, $s_j(x) := \sum_{i=1}^{m_j} c_i^{(j)} \chi_{B_i^{(j)}}(x)$, be a sequence of Borel, simple functions converging uniformly in Ω to ϕ.

i) It suffices to pass (2.3.9) to the limit as $j \to +\infty$, using Definition 2.3.16 and Lebesgue Dominated Convergence Theorem.

ii) Using the fact that $\boldsymbol{\mu}(\emptyset) = 0$, from (2.3.7) we deduce that

$$\boldsymbol{\nu}(\emptyset) = \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \int_\emptyset s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) = 0.$$

Let $\{B_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(\Omega)$ be a countable family of mutually disjoint sets. Define $B^\infty := \cup_{k=1}^{+\infty} B_k$ and, for $M \in \mathbb{N}$, $B^M := \cup_{k=1}^{M} B_k$. We want to show that $\sum_{k=1}^{+\infty} \boldsymbol{\nu}(B_k) \in BV_\#(Y;\mathbb{R}^d)$ and $\boldsymbol{\nu}(B^\infty) = \sum_{k=1}^{+\infty} \boldsymbol{\nu}(B_k)$. By (2.3.7), we have that

$$\int_{B_M} s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) = \sum_{k=1}^{M} \int_{B_k} s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x).$$

Thus,

$$\begin{aligned}
\boldsymbol{\nu}(B^M) &= \int_{B^M} \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) = \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \int_{B^M} s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \\
&= \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \left\{ \sum_{k=1}^{M} \int_{B_k} s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right\} \\
&= \sum_{k=1}^{M} \left\{ \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \int_{B_k} s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right\} \\
&= \sum_{k=1}^{M} \int_{B_k} \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) = \sum_{k=1}^{M} \boldsymbol{\nu}(B_k).
\end{aligned} \tag{2.3.10}$$

On the other hand, since $\chi_{B^\infty}\phi - \chi_{B^M}\phi$ is a bounded, Borel measurable function, by *i)* we get

$$\begin{aligned}
\big\|\boldsymbol{\nu}(B^\infty) - \boldsymbol{\nu}(B^M)\big\|_{L^1(Y;\mathbb{R}^d)} &= \int_Y \left| \int_\Omega \big[\chi_{B^\infty}(x) - \chi_{B^M}(x)\big]\phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y \\
&\leqslant \int_\Omega \big|\chi_{B^\infty}(x) - \chi_{B^M}(x)\big| \big|\phi(x)\big|\,\mathrm{d}\|\boldsymbol{\mu}\|(x) \\
&= \int_{B^\infty \setminus B^M} |\phi(x)|\,\mathrm{d}\|\boldsymbol{\mu}\|(x) \leqslant \|\phi\|_{L^\infty(\Omega)} \|\boldsymbol{\mu}\|\big(B^\infty \setminus B^M\big) \underset{M\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

where we have used the fact that $\|\boldsymbol{\mu}\|$ is a positive finite Radon measure on $\mathcal{B}(\Omega)$ (see Proposition 2.1.6). Hence,

$$\boldsymbol{\nu}(B^\infty) = \big(L^1_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{M\to+\infty} \boldsymbol{\nu}\big(B^M\big). \tag{2.3.11}$$

Using the fact that $\boldsymbol{\nu}(B^\infty) \in BV_\#(Y;\mathbb{R}^d)$, from (2.3.10) and (2.3.11) we conclude that $\sum_{k=1}^{+\infty} \boldsymbol{\nu}(B_k) \in BV_\#(Y;\mathbb{R}^d)$ and $\boldsymbol{\nu}(B^\infty) = \sum_{k=1}^{+\infty} \boldsymbol{\nu}(B_k)$.

Finally, since for all $B \in \mathcal{B}(\Omega)$, we have $\boldsymbol{\nu}(B) = \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \int_B s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, by the lower semicontinuity of the total variation and using (2.3.8) we get

$$\|\boldsymbol{\nu}(B)\|_{BV_\#(Y;\mathbb{R}^d)} \leqslant \lim_{j\to+\infty} \int_Y \left| \int_B s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y + \liminf_{j\to+\infty} \left\| D_y \left(\int_B s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right) \right\| (Y) \leqslant \|\phi\|_\infty \|\boldsymbol{\mu}\|(B).$$

Hence,

$$\|\boldsymbol{\nu}\|(B) := \sup \left\{ \sum_{j=1}^{+\infty} \|\boldsymbol{\nu}(B_j)\|_{BV_\#(Y;\mathbb{R}^d)} \colon \{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } B \right\} \leqslant \|\phi\|_\infty \|\boldsymbol{\mu}\|(B),$$

which concludes the proof of *ii)*. □

Note that if $\phi : \Omega \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ are bounded, Borel functions, then given $\boldsymbol{\mu} \in \mathcal{M}(\Omega; BV_\#(Y;\mathbb{R}^d))$ and $B \in \mathcal{B}(\Omega)$, the integral

$$\int_{B\times Y} \phi(x)\psi(y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y := \int_Y \left(\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right)(y)\,\psi(y)\,\mathrm{d}y \tag{2.3.12}$$

is well defined in $\mathbb{R}^d$.

By considering first bounded, Borel simple functions, one can show that

$$\left| \sum_{i=1}^m \int_Y \left(\int_\Omega \phi_i(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right)(y)\,\psi_i(y)\,\mathrm{d}y \right| \leqslant \left\| \sum_{i=1}^m \phi_i\psi_i \right\|_\infty \|\boldsymbol{\mu}\|(\Omega), \tag{2.3.13}$$

whenever $\phi_i : \Omega \to \mathbb{R}$, $\psi_i : Y \to \mathbb{R}$, $i \in \{1, \cdots, m\}$, are bounded, Borel functions.

In fact, for simplicity, assume that $m = 2$. Let s_1, s_2, t_1, t_2 be simple functions, and write

$$s_1 = \sum_{i=1}^{m_1} a_i \chi_{A_i}, \quad s_2 = \sum_{i=1}^{m_2} b_i \chi_{B_i}, \quad t_1 = \sum_{i=1}^{l_1} c_i \chi_{C_i}, \quad t_2 = \sum_{i=1}^{l_2} d_i \chi_{D_i},$$

with $m_1, m_2, l_1, l_2 \in \mathbb{N}$, $\{a_i\}_{i=1}^{m_1}$, $\{b_i\}_{i=1}^{m_2}$, $\{c_i\}_{i=1}^{l_1}$, $\{d_i\}_{i=1}^{l_2}$ finite collections of distinct real numbers, $\{A_i\}_{i=1}^{m_1}$, $\{B_i\}_{i=1}^{m_2} \subset \mathcal{B}(\Omega)$, and $\{C_i\}_{i=1}^{l_1}$, $\{D_i\}_{i=1}^{l_2} \subset \mathcal{B}(Y)$ finite collections of mutually disjoint sets.

It can be shown that

$$s_1t_1 + s_2t_2 = \sum_{i=1}^{\bar{m}} \kappa_i \chi_{E_i} \chi_{F_i},$$

where for all $i \in \{1, \cdots, \bar{m}\}$, $\kappa_i \in \mathbb{R}$ and $|\kappa_i| \leqslant \|s_1t_1 + s_2t_2\|_\infty$, $\{E_i\}_{i=1}^{\bar{m}}$ is a family of mutually disjoint Borel subsets of Ω, and for all $i \in \{1, \cdots, \bar{m}\}$, $F_i \in \mathcal{B}(Y)$.

Thus,

$$\begin{aligned}
&\left| \int_Y \left(\int_\Omega s_1(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right)(y)\,t_1(y)\,\mathrm{d}y + \int_Y \left(\int_\Omega s_2(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right)(y)\,t_2(y)\,\mathrm{d}y \right| \\
&\quad = \left| \int_Y \left(\sum_{i=1}^{m_1} a_i \big(\boldsymbol{\mu}(A_i)\big)(y) \right) \left(\sum_{i=1}^{l_1} c_i \chi_{C_i}(y) \right) + \left(\sum_{i=1}^{m_2} b_i \big(\boldsymbol{\mu}(B_i)\big)(y) \right) \left(\sum_{i=1}^{l_2} d_i \chi_{D_i}(y) \right) \mathrm{d}y \right| \\
&\quad = \left| \int_Y \sum_{i=1}^{\bar{m}} \kappa_i \big(\boldsymbol{\mu}(E_i)\big)(y)\,\chi_{F_i}(y)\,\mathrm{d}y \right| \leqslant \|s_1t_1 + s_2t_2\|_\infty \sum_{i=1}^{\bar{m}} \int_{F_i} \big|\big(\boldsymbol{\mu}(E_i)\big)(y)\big|\,\mathrm{d}y \\
&\quad \leqslant \|s_1t_1 + s_2t_2\|_\infty \sum_{i=1}^{\bar{m}} \int_Y \big|\big(\boldsymbol{\mu}(E_i)\big)(y)\big|\,\mathrm{d}y \leqslant \|s_1t_1 + s_2t_2\|_\infty \|\boldsymbol{\mu}\|(\Omega),
\end{aligned}$$

from which we deduce (2.3.13) for simple functions. To prove the general case, if $\phi_i : \Omega \to \mathbb{R}$, $\psi_i : Y \to \mathbb{R}$, $i \in \{1, \cdots, m\}$, are bounded, Borel functions, then for each $j \in \mathbb{N}$ we can find $s_j^{(i)} : \Omega \to \mathbb{R}$ and $t_j^{(i)} : Y \to \mathbb{R}$, Borel simple functions, such that $s_j^{(i)} \to \phi_i$ uniformly in Ω as $j \to +\infty$, and $t_j^{(i)} \to \psi_i$ uniformly in Y as $j \to +\infty$. By definition,

$$\int_\Omega \phi_i(x)\,\mathrm{d}\boldsymbol{\mu}(x) = \big(w\star\text{-}BV_\#(Y;\mathbb{R}^d)\big)\text{-}\lim_{j\to+\infty} \int_\Omega s_j^{(i)}(x)\,\mathrm{d}\boldsymbol{\mu}(x),$$

so that the uniform convergence $t_j^{(i)} \to \psi_i$ in Y as $j \to +\infty$ entails

$$\lim_{j\to+\infty} \int_Y \left(\int_\Omega s_j^{(i)}(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\, t_j^{(i)}(y)\,\mathrm{d}y = \int_Y \left(\int_\Omega \phi_i(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\, \psi_i(y)\,\mathrm{d}y,$$

for all $i \in \{1, \cdots, m\}$. To conclude, it suffices to pass to the limit as $j \to +\infty$ the inequality

$$\left|\sum_{i=1}^m \int_Y \left(\int_\Omega s_j^{(i)}(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\, t_j^{(i)}(y)\,\mathrm{d}y\right| \leqslant \left\|\sum_{i=1}^m s_j^{(i)} t_j^{(i)}\right\|_\infty \|\boldsymbol{\mu}\|(\Omega)$$

established above for simple functions.

We are finally in position to give sense to (2.3.5) (for $i = 1$).

Definition 2.3.18. (Integral with respect to "$\boldsymbol{\mu} \otimes \mathcal{L}^N_{\lfloor Y}$") *Let $\boldsymbol{\mu} \in \mathcal{M}(\Omega; BV_\#(Y;\mathbb{R}^d))$ and let $\varphi \in C_0(\Omega; C_\#(Y))$. We define*

$$\int_{\Omega\times Y} \varphi(x,y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y := \lim_{j\to+\infty} \left\{\sum_{i=1}^{m_j} \int_Y \left(\int_\Omega \phi_i^{(j)}(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\, \psi_i^{(j)}(y)\,\mathrm{d}y\right\}, \tag{2.3.14}$$

where for each $j \in \mathbb{N}$, $m_j \in \mathbb{N}$, and for all $i \in \{1, ..., m_j\}$, $\phi_i^{(j)} \in C_0(\Omega)$, $\psi_i^{(j)} \in C_\#(Y)$, and $\{\varphi_j\}_{j\in\mathbb{N}}$, with $\varphi_j := \sum_{i=1}^{m_j} \phi_i^{(j)} \psi_i^{(j)}$, converges to φ in $C_0(\Omega; C_\#(Y))$.

Remark 2.3.19. *(i) Given $\varphi \in C_0(\Omega; C_\#(Y))$, the existence of a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ as in Definition 2.3.18 is a consequence of the Stone-Weierstrass Theorem.*

(ii) Note that (2.3.14) reduces to (2.3.12) when $\varphi(x,y) = \phi(x)\psi(y)$ with $\phi \in C_0(\Omega)$, $\psi \in C_\#(Y)$.

(iii) Estimate (2.3.13) ensures that the limit in the Definition 2.3.18 exists and does not depend on the approximating sequence. Moreover,

$$\left|\int_{\Omega\times Y} \varphi(x,y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y\right| \leqslant \|\varphi\|_\infty \|\boldsymbol{\mu}\|(\Omega), \tag{2.3.15}$$

for all $\varphi \in C_0(\Omega; C_\#(Y))$, and

$$\varphi \in C_0(\Omega; C_\#(Y)) \mapsto \int_{\Omega\times Y} \varphi(x,y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y$$

defines a linear continuous functional.

(iv) *We could have considered the more general setting in which $\varphi \in C(\Omega; C_\#(Y)) \cap L^\infty(\Omega\times Y)$. In this case, (iii) above still holds with "$\varphi \in C_0(\Omega; C_\#(Y))$" replaced by "$\varphi \in C(\Omega; C_\#(Y)) \cap L^\infty(\Omega \times Y)$".*

Next we prove an integration by parts formula for measures in $\mathcal{M}_\star(\Omega; BV_\#(Y;\mathbb{R}^d))$. We first introduce some notations. If $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^N$, then $a \otimes b$ stands for the $(d \times N)$-dimensional rank-one matrix defined by $a \otimes b := (a_i b_j)_{1 \leqslant i \leqslant d, 1 \leqslant j \leqslant N} \in \mathbb{R}^{d \times N}$.

Lemma 2.3.20. *Let $\boldsymbol{\mu} \in \mathcal{M}_\star(\Omega; BV_\#(Y;\mathbb{R}^d))$, $\phi \in C_0(\Omega)$ and $\psi \in C^1_\#(Y)$ be given. Then*

$$\int_Y \left(\int_\Omega \phi(x)\, \mathrm{d}\boldsymbol{\mu}(x) \right)(y) \otimes \nabla\psi(y)\, \mathrm{d}y = -\int_{\Omega \times Y} \phi(x)\, \psi(y)\, \mathrm{d}\lambda(x,y), \tag{2.3.16}$$

where $\lambda \in \mathcal{M}_{y\#}(\Omega \times Y; \mathbb{R}^{d \times N})$ is the measure associated with $D_y \boldsymbol{\mu}$.

Proof. Fix $B \in \mathcal{B}(\Omega)$, and let $\lambda_B \in \mathcal{M}_\#(Y;\mathbb{R})$ be the (projection) measure defined by $\lambda_B(\cdot) := \lambda(B \times \cdot)$. We have that

$$\begin{aligned} \int_Y \left(\int_\Omega \chi_B(x)\, \mathrm{d}\boldsymbol{\mu}(x) \right)(y) \otimes \nabla\psi(y)\, \mathrm{d}y &= \int_Y (\boldsymbol{\mu}(B))(y) \otimes \nabla\psi(y)\, \mathrm{d}y = -\int_Y \psi(y)\, \mathrm{d}D_y(\boldsymbol{\mu}(B))(y) \\ = -\int_Y \psi(y)\, \mathrm{d}\lambda_B(y) &= -\int_{B \times Y} \psi(y)\, \mathrm{d}\lambda(x,y) = -\int_{\Omega \times Y} \chi_B(x)\, \psi(y)\, \mathrm{d}\lambda(x,y), \end{aligned} \tag{2.3.17}$$

where we have used the fact that $\boldsymbol{\mu}(B) \in BV_\#(Y;\mathbb{R}^d)$ and the disintegration property of a Radon measure (see (2.1.7)) applied to $\lambda^\pm_{\lfloor B \times Y}$.

Since any function in $C_0(\Omega)$ can be approximated with respect to the uniform convergence in Ω by Borel, simple functions, (2.3.16) follows from (2.3.17) and Definition 2.3.16. $\square$

<u>*Step 2.*</u> We define (2.3.5) recursively for an arbitrary $i \in \mathbb{N}$. Fix $i \geqslant 2$, and let $\vartheta \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_{i-1}))$ and $\boldsymbol{\mu} \in \mathcal{M}(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i, \mathbb{R}^d))$.

Proceeding as before (see (2.3.7) and Definition 2.3.16), we define the integral of ϑ over $B \in \mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_{i-1})$ with respect to $\boldsymbol{\mu}$, and we write $\int_B \vartheta(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})$, as the function in $BV_\#(Y_i;\mathbb{R}^d)$ given by

$$\begin{aligned} &\int_B \phi(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \\ &\qquad := \left(w\star\text{-}BV_\#(Y_i;\mathbb{R}^d) \right)\text{-} \lim_{j \to +\infty} \int_B s_j(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}), \end{aligned}$$

where $\{s_j\}_{j \in \mathbb{N}}$ is a sequence of Borel simple functions $s_j : \Omega \times \mathbb{R}^{(i-1)N} \to \mathbb{R}$, $Y_1 \times \cdots \times Y_{i-1}$-periodic in the variables $(y_1, \cdots, y_{i-1})$, converging uniformly in $\Omega \times Y_1 \times \cdots \times Y_{i-1}$ to ϑ.

Let $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_i))$, and take a sequence $\{\varphi_j\}_{j \in \mathbb{N}}$ converging to φ in $C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_i))$, where each φ_j is of the form $\varphi_j(x, y_1, \cdots, y_{i-1}, y_i) = \sum_{k=1}^{m_j} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1}) \psi_k^{(j)}(y_i)$ with $m_j \in \mathbb{N}$, and for all $k \in \{1, \cdots, m_j\}$, $\vartheta_k^{(j)} \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_{i-1}))$, $\psi_k^{(j)} \in C_\#(Y_i)$. Once again proceeding as before (see (2.3.12) and Definition 2.3.18) we can give sense to the expression

$$\sum_{k=1}^{m_j} \int_{Y_i} \left(\int_{\Omega \times Y_1 \times \cdots \times Y_{i-1}} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \right)(y_i)\, \psi_k^{(j)}(y_i)\, \mathrm{d}y_i \tag{2.3.18}$$

in $\mathbb{R}^d$, and prove that the limit of (2.3.18) as $j \to +\infty$ exists and is independent of the approximating sequence. We then define

$$\begin{aligned} &\int_{\Omega \times Y_1 \times \cdots \times Y_i} \varphi(x, y_1, \cdots, y_i)\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \mathrm{d}y_i \\ &\quad := \lim_{j \to +\infty} \sum_{k=1}^{m_j} \int_{Y_i} \left(\int_{\Omega \times Y_1 \times \cdots \times Y_{i-1}} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \right)(y_i)\, \psi_k^{(j)}(y_i)\, \mathrm{d}y_i. \end{aligned} \tag{2.3.19}$$

Similarly, if $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_i; \mathbb{R}^d))$, then we set

$$\begin{aligned}&\int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi(x, y_1, \cdots, y_i) \cdot \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})\mathrm{d}y_i \\ &\quad := \lim_{j\to+\infty} \sum_{k=1}^{m_j} \int_{Y_i} \bigg(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \bigg)(y_i) \cdot \psi_k^{(j)}(y_i)\, \mathrm{d}y_i,\end{aligned} \tag{2.3.20}$$

where $\varphi_j(x, y_1, \cdots, y_{i-1}, y_i) := \sum_{k=1}^{m_j} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\psi_k^{(j)}(y_i)$ with $m_j \in \mathbb{N}$, and for all $k \in \{1, \cdots, m_j\}$, $\vartheta_k^{(j)} \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_{i-1}))$, $\psi_k^{(j)} \in C_\#(Y_i, \mathbb{R}^d)$, converges to φ in $C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_i; \mathbb{R}^d))$ as $j \to +\infty$.

If, in particular, $\boldsymbol{\mu} \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i, \mathbb{R}^d))$ then similar arguments to those of Lemma 2.3.20 ensure that for all $\vartheta \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_{i-1}))$, $\psi \in C^1_\#(Y_i)$ and $\theta \in C^1_\#(Y_i; \mathbb{R}^N)$ one has

$$\begin{aligned}&\int_{Y_i} \bigg(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \vartheta(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \bigg)(y_i) \otimes \nabla\psi(y_i)\, \mathrm{d}y_i \\ &\qquad = -\int_{\Omega\times Y_1\times\cdots\times Y_i} \vartheta(x, y_1, \cdots, y_{i-1})\, \psi(y_i)\, \mathrm{d}\lambda(x, y_1, \cdots, y_i),\end{aligned} \tag{2.3.21}$$

where $\lambda \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^{d\times N})$ is the measure associated with $D_{y_i}\boldsymbol{\mu}$, and for all $k \in \{1, \cdots, d\}$,

$$\begin{aligned}&\int_{Y_i} \bigg(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \vartheta(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}_k(x, y_1, \cdots, y_{i-1}) \bigg)(y_i)\, \operatorname{div}\theta(y_i)\, \mathrm{d}y_i \\ &\qquad = -\int_{\Omega\times Y_1\times\cdots\times Y_i} \vartheta(x, y_1, \cdots, y_{i-1})\, \theta(y_i) \cdot \mathrm{d}\lambda_{(k)}(x, y_1, \cdots, y_i),\end{aligned} \tag{2.3.22}$$

where $\lambda_{(k)}$ denotes the k^{th} row of λ and $\boldsymbol{\mu}_k$ denotes the k^{th} component of $\boldsymbol{\mu}$.

Remark 2.3.21. *As observed in Remark 2.3.19 (iv), in (2.3.20) we may consider the more general setting in which* $\varphi \in C(\Omega; C_\#(Y_1 \times \cdots \times Y_i; \mathbb{R}^d)) \cap L^\infty(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^d)$. *In this case, the functions* $\vartheta_k^{(j)}$ *are to be taken in* $C(\Omega; C_\#(Y_1 \times \cdots \times Y_i)) \cap L^\infty(\Omega \times Y_1 \times \cdots \times Y_i)$, *and, as before, the corresponding limit in (2.3.20) is independent of the approximating sequence (with respect to the supremum norm* $\|\cdot\|_\infty$ *in* $\Omega \times Y_1 \times \cdots \times Y_i$*).*

Moreover,

$$F(\varphi) := \int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi(x, y_1, \cdots, y_i) \cdot \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})\mathrm{d}y_i$$

for $\varphi \in C(\Omega; C_\#(Y_1 \times \cdots \times Y_i; \mathbb{R}^d)) \cap L^\infty(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^d)$, *defines a linear continuous functional, and we have*

$$\big|F(\varphi)\big| \leqslant \|\varphi\|_\infty \|\boldsymbol{\mu}\|(\Omega \times Y_1 \times \cdots \times Y_{i-1}).$$

Furthermore, proceeding as in Lemma 2.3.17 and (2.3.19), in the particular case in which φ *is scalar and does not depend on* y_i, *then*

$$\begin{aligned}&\int_{Y_i} \bigg| \int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \varphi(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \bigg| \mathrm{d}y_i \\ &\qquad \leqslant \int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} |\varphi(x, y_1, \cdots, y_{i-1})|\, \mathrm{d}\|\boldsymbol{\mu}\|(x, y_1, \cdots, y_{i-1}),\end{aligned}$$

and if we define for all $B \in \mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_{i-1})$,

$$\boldsymbol{\nu}(B) := \int_B \varphi(x, y_1, \cdots, y_{i-1}) \, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}),$$

then we have that $\boldsymbol{\nu} \in \mathcal{M}(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i; \mathbb{R}^d))$, and $\|\boldsymbol{\nu}\|(B) \leqslant \|\varphi\|_\infty \|\boldsymbol{\mu}\|(B)$.

2.4. Unbounded Linear Operators in Hilbert Spaces: Spectral Theory.

The purpose of this section is to recall some results regarding spectral properties of unbounded linear operators defined in Hilbert spaces. We refer to the books Brezis [22], Dal Maso [31], Dautray and Lions [32], Gilbarg and Trudinger [52], Oleĭnik, Shamaev and Yosifian [65], and to the references therein.

We start by recalling certain definitions concerning unbounded linear operators in Hilbert spaces. Let H be a real Hilbert Space, endowed with a scalar product $(\cdot|\cdot)$ and the associated norm $\|\cdot\|$.

Definition 2.4.1. (Unbounded linear operator (u.l.o.); domain, range and kernel of an u.l.o.) *A linear map $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ defined in a linear subspace $D(\mathcal{A})$ of H with values in H is said to be an unbounded linear operator (briefly u.l.o.) in H. The set $D(\mathcal{A})$ is called the domain of $\mathcal{A}$. The sets $R(\mathcal{A})$ and $N(\mathcal{A})$ given by*

$$R(\mathcal{A}) := \{\mathcal{A}u \colon u \in D(\mathcal{A})\} \quad \text{and} \quad N(\mathcal{A}) := \{u \in D(\mathcal{A}) \colon \mathcal{A}u = 0\},$$

respectively, are named the range of $\mathcal{A}$ and the kernel (or null space) of $\mathcal{A}$, respectively.

Definition 2.4.2. (Densely defined, closed, coercive and bounded u.l.o.) *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be an unbounded linear operator in H. We say that $\mathcal{A}$ is*

(i) densely defined if $D(\mathcal{A})$ is dense in H;

(ii) closed if the graph of $\mathcal{A}$, that is, the set $G(\mathcal{A})$ defined by

$$G(\mathcal{A}) := \big\{(u, \mathcal{A}u) \colon u \in D(\mathcal{A})\big\},$$

is a closed subset of $H \times H$;

(iii) bounded (or continuous) if $D(\mathcal{A}) = H$ and there exists a constant $c \geqslant 0$ such that for all $u \in H$ one has

$$\|\mathcal{A}u\| \leqslant c\|u\|.$$

We represent by $\mathfrak{L}(H)$ the set of all bounded linear operators in H.

Remark 2.4.3. *When endowed with the norm*

$$\|\mathcal{A}\|_{\mathfrak{L}(H)} := \sup_{u \in H \setminus \{0\}} \frac{\|\mathcal{A}u\|}{\|u\|}, \quad \mathcal{A} \in \mathfrak{L}(H),$$

$\mathfrak{L}(H)$ is a Banach space.

Notation 2.4.4. *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be an unbounded linear operator in H, and let E be a subset of $D(\mathcal{A})$. We represent by $\mathcal{A}(E)$ the image of E through $\mathcal{A}$, that is, the set $\mathcal{A}(E) := \{\mathcal{A}u \colon u \in E\}$.*

Definition 2.4.5. (Compact u.l.o.) *We say that a bounded operator $\mathcal{A} \in \mathfrak{L}(H)$ in H is compact if $\mathcal{A}(B_H)$ has compact closure in H, where $B_H := \{u \in H \colon \|u\| \leqslant 1\}$ is the closed unit ball in H.*

Notation 2.4.6. *We represent by $\mathcal{I}$ the identity operator in H.*

Definition 2.4.7. (Resolvent of an u.l.o., resolvent operator) *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a closed unbounded linear operator in H. The resolvent $\rho(\mathcal{A})$ of $\mathcal{A}$ is the set defined by*

$$\rho(\mathcal{A}) := \{\lambda \in \mathbb{R} \colon (\mathcal{A} - \lambda\mathcal{I}) \text{ is bijective from } H \text{ onto } H\}.$$

For each $\lambda \in \rho(\mathcal{A})$ we call the resolvent operator associated with λ to the bounded linear operator in H defined by $\mathcal{R}_\lambda := (\mathcal{A} - \lambda\mathcal{I})^{-1}$, the inverse map of $(\mathcal{A} - \lambda\mathcal{I})$.

Definition 2.4.8. (U.l.o. with compact resolvent) *We say that a closed unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ in H is an operator with compact resolvent if there is $\lambda \in \rho(\mathcal{A})$ such that the associated resolvent operator $\mathcal{R}_\lambda(\mathcal{A})$ is compact.*

Remark 2.4.9. *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a closed unbounded linear operator in H. If there is a $\lambda \in \rho(\mathcal{A})$ for which $\mathcal{R}_\lambda(\mathcal{A})$ is compact, then for all $\mu \in \rho(\mathcal{A})$ the corresponding resolvent operator $\mathcal{R}_\mu(\mathcal{A})$ is compact.*

Definition 2.4.10. (Spectrum, point spectrum of an u.l.o.) *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a closed unbounded linear operator in H. The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is the complement in $\mathbb{R}$ of its resolvent, that is, $\sigma(\mathcal{A}) := \mathbb{R}\backslash\rho(\mathcal{A})$. The point spectrum $\sigma_p(\mathcal{A})$ of $\mathcal{A}$ is the set of all $\lambda \in \sigma(\mathcal{A})$ for which $N(\mathcal{A} - \lambda\mathcal{I}) \neq \{0\}$.*

Definition 2.4.11. (Eigenvalue, eigenspace, eigenfunction, multiplicity of an eigenvalue, simple eigenvalue, (normalized) eigenpair) *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a closed unbounded linear operator in H. Each $\lambda \in \sigma_p(\mathcal{A})$ is called an eigenvalue of $\mathcal{A}$, in which case the subspace $N(\mathcal{A} - \lambda\mathcal{I})$ of H is said to be the associated eigenspace, $u \in N(\mathcal{A} - \lambda\mathcal{I})\backslash\{0\}$ an associated eigenfunction, and the $\dim N(\mathcal{A} - \lambda\mathcal{I})$, i.e., the dimension of the space $N(\mathcal{A} - \lambda\mathcal{I})$, its multiplicity (or geometric multiplicity). If $\lambda \in \sigma_p(\mathcal{A})$ is such that $\dim N(\mathcal{A} - \lambda\mathcal{I}) = 1$, then λ is said to be simple. An eigenpair of $\mathcal{A}$ is a pair (λ, u), where $\lambda \in \sigma_p(\mathcal{A})$ and $u \in N(\mathcal{A} - \lambda\mathcal{I})\backslash\{0\}$; it is said to be normalized (in H) if $\|u\| = 1$.*

Definition 2.4.12. (Adjoint operator of an u.l.o., self-adjoint u.l.o.) *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a densely defined unbounded linear operator in H. The adjoint operator of $\mathcal{A}$ is the unbounded linear operator $\mathcal{A}^* : D(\mathcal{A}^*) \subset H \to H$ in H, where*

$$D(\mathcal{A}^*) := \{v \in H \colon \text{there exists } c_v \geqslant 0 \text{ s.t. } |(v|\mathcal{A}u)| \leqslant c_v\|u\| \text{ for all } u \in D(\mathcal{A})\}, \tag{2.4.1}$$

defined for each $v \in D(\mathcal{A}^*)$ by $\mathcal{A}^*v := w$ with $w \in H$ the unique element in H satisfying

$$(v|\mathcal{A}u) = (w|u) \text{ for all } u \in D(\mathcal{A}). \tag{2.4.2}$$

We say that $\mathcal{A}$ is self-adjoint if $\mathcal{A}^* = \mathcal{A}$.

Remark 2.4.13. *The above definition makes sense since the set $D(\mathcal{A}^*)$ in (2.4.1) comprises the elements $v \in H$ for which the linear map $u \in D(\mathcal{A}) \mapsto (v|\mathcal{A}u)$ is continuous with respect to the norm of H. Since $D(\mathcal{A})$ is dense in H, this linear map can be uniquely extended to a continuous linear* *map in H. In view of the Riesz Representation Theorem, there is a unique element $w \in H$ satisfying (2.4.2). We further observe that for all $u \in D(\mathcal{A})$ and $v \in D(\mathcal{A}^*)$ one has $(v|\mathcal{A}u) = (\mathcal{A}^*v|u)$.*

Proposition 2.4.14. *If $\mathcal{A}$ is an unbounded linear operator in H that is closed and densely defined, then so is its adjoint operator $\mathcal{A}^*$.*

The next result concerns the solvability of the equation $u - \mathcal{A}u = f$, where $\mathcal{A} \in \mathfrak{L}(H)$ is a compact operator and $f \in H$. In particular, it asserts that either for every $f \in H$ the equation $u - \mathcal{A}u = f$ has a unique solution, or the homogeneous equation $u - \mathcal{A}u = 0$ has l linearly independent solutions (for some $l \in \mathbb{N}$), in which case the inhomogeneous equation $u - \mathcal{A}u = f$ is solvable if, and only if, f satisfies l orthogonality conditions, namely, $f \in N(\mathcal{I} - \mathcal{A}^*)^{\perp}$. This dichotomy is known as the Fredholm Alternative.

Theorem 2.4.15. (Fredholm Alternative) *Let $\mathcal{A} \in \mathfrak{L}(H)$ be a compact operator. Then*

(i) $N(\mathcal{I} - \mathcal{A})$ *is finite dimensional;*

(ii) $R(\mathcal{I} - \mathcal{A})$ *is closed;*

(iii) $R(\mathcal{I} - \mathcal{A}) = N(\mathcal{I} - \mathcal{A}^*)^{\perp}$;

(iv) $N(\mathcal{I} - \mathcal{A}) = \{0\}$ *if, and only if,* $R(\mathcal{I} - \mathcal{A}) = H$;

(iv) $\dim N(\mathcal{I} - \mathcal{A}) = \dim N(\mathcal{I} - \mathcal{A}^*)$.

We now state two results concerning spectral properties of self-adjoint compact linear operators in H. In general, the operators that we will be dealing with in this dissertation are not compact but admit a compact inverse operator for which these results will apply.

Theorem 2.4.16. (Spectrum of a self-adjoint compact operator) *Let $\mathcal{A} \in \mathfrak{L}(H)$ be a self-adjoint compact operator in H. Then*

(i) $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \{0\}$, *and the set $\sigma_p(\mathcal{A})$ of eigenvalues of $\mathcal{A}$ is either finite or can be written as a sequence converging to zero; moreover, $0 \in \sigma_p(\mathcal{A})$ if $N(\mathcal{A}) \neq \{0\}$;*

(ii) *for each $\lambda \in \sigma_p(\mathcal{A})$ except perhaps for $\lambda = 0$, the dimension of the associated eigenspace $N(\mathcal{A} - \lambda\mathcal{I})$ is finite;*

(iii) *the spaces $N(\mathcal{A} - \lambda\mathcal{I})$, with $\lambda \in \sigma_p(\mathcal{A})$, are pairwise disjoint and H is the direct Hilbert sum of the eigenspaces $N(\mathcal{A} - \lambda\mathcal{I})$, i.e.,*

$$H = \bigoplus_{\lambda \in \sigma_p(\mathcal{A})} N(\mathcal{A} - \lambda\mathcal{I}).$$

Lemma 2.4.17 (Vishik-Lyusternik Lemma) *Let $\mathcal{A} \in \mathfrak{L}(H)$ be a compact self-adjoint operator in H. Suppose that there exist a real number $\gamma > 0$ and an element $f \in H$ with $\|f\| = 1$ such that $\|\mathcal{A}f - \gamma f\| \leqslant c$, for some constant $c > 0$. Then there is an eigenvalue $\lambda \in \sigma_p(\mathcal{A})$ of $\mathcal{A}$ such that $|\lambda - \gamma| \leqslant c$. Moreover, for any $C > c$ there exists $u \in H$, which is a linear combination of eigenfunctions associated with eigenvalues of $\mathcal{A}$ belonging to the interval $[\gamma - C, \gamma + C]$, and such that $\|u\| = 1$ and $\|u - f\| \leqslant 2c/C$.*

2.4.1. Quadratic Forms and Associated Unbounded Linear Operators

Definition 2.4.18. (Quadratic form) *A function $F : H \to [0, +\infty]$ is said to be a (nonnegative) quadratic form if there is a (unique) symmetric bilinear form $a : D(F) \times D(F) \to \mathbb{R}$, where $D(F) := \{u \in H\colon F(u) < +\infty\}$ is the domain of F, such that*

$$F(u) = \begin{cases} a(u,u) & \text{if } u \in D(F), \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 2.4.19. *Every quadratic form is convex.*

Definition 2.4.20. *Let $F : H \to [0, +\infty]$ be a quadratic form and $a : D(F) \times D(F) \to \mathbb{R}$ the associated symmetric bilinear form. Set $V := \overline{D(F)}$. The unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset V \to V$ in V associated with F is the operator defined by*

$$\begin{cases} D(\mathcal{A}) := \big\{u \in D(F)\colon \text{there is (a unique) } v \in V \text{ such that } a(u,w) = (v|w) \text{ for all } w \in D(F)\big\}, \\ \mathcal{A}u = v \text{ for all } u \in D(F). \end{cases} \tag{2.4.3}$$

Remark 2.4.21. *The uniqueness of $v \in V$ in (2.4.3) is a consequence of the density of $D(F)$ into V. Moreover, by the Riesz Representation Theorem, $u \in D(\mathcal{A})$ if, and only if, the linear application $w \mapsto a(u,w)$ is continuous in $D(F)$ with respect to the topology of H. We further observe that for every $u \in D(\mathcal{A})$ and $w \in D(F)$ we have $(\mathcal{A}u|w) = a(u,w)$. In particular, taking $w = u$ we obtain $(\mathcal{A}u|u) = a(u,u)$ for all $u \in D(\mathcal{A})$.*

Theorem 2.4.22. *Let $F : H \to [0, +\infty]$ be a quadratic form and let $\mathcal{A} : D(\mathcal{A}) \subset V \to V$ be the unbounded linear operator in $V := \overline{D(F)}$ associated with F. If F is lower semicontinuous in H*[2.15] *then $\mathcal{A}$ is self-adjoint in V.*

Definition 2.4.23. (Scalar product and norm in $D(F)$) *Let $F : H \to [0, +\infty]$ be a quadratic form and $a : D(F) \times D(F) \to \mathbb{R}$ the associated symmetric bilinear form. The scalar product $(\cdot|\cdot)_{D(F)}$ in $D(F)$ is defined by*

$$(u|v)_{D(F)} := a(u,v) + (u|v), \quad u, v \in D(F),$$

and the corresponding norm $\|\cdot\|_{D(F)}$ in $D(F)$ is given by

$$\|u\|_{D(F)} := \sqrt{F(u) + \|u\|^2}, \quad u \in D(F).$$

Proposition 2.4.24. *Let $F : H \to [0, +\infty]$ be a quadratic form. Then $D(F)$ is a Hilbert space when endowed with the scalar product $(\cdot|\cdot)_{D(F)}$ if, and only if, F is lower semicontinuous in H.*

Proposition 2.4.25. *Let $F : H \to [0, +\infty]$ be a lower semicontinuous quadratic form and let $\mathcal{A} : D(\mathcal{A}) \subset V \to V$ be the unbounded linear operator in $V := \overline{D(F)}$ associated with F. Then $D(\mathcal{A})$ is dense in $D(F)$ with respect to the norm $\|\cdot\|_{D(F)}$.*

Proposition 2.4.26. *Let $F : H \to [0, +\infty]$ be a lower semicontinuous quadratic form and let $\mathcal{A} : D(\mathcal{A}) \subset V \to V$ be the unbounded linear operator in $V := \overline{D(F)}$ associated with F. Consider the quadratic form $G : H \to [0, +\infty]$ defined by*

$$G(u) := \begin{cases} (\mathcal{A}u|u) & \text{if } u \in D(\mathcal{A}), \\ +\infty & \text{otherwise.} \end{cases}$$

[2.15] See Definition 2.5.3 in Subsection 2.5.1 below.

Then $F = sc^-G$, where sc^-G is the lower semicontinuous envelope of G in H[2.16].

2.4.2. The Case of Elliptic Partial Differential Operators

In Chapter 3 we will be particularly interested in spectral properties of a specific type of unbounded linear operators, namely, partial differential operators.

Definition 2.4.27. ($M(\zeta, \eta, \Omega)$, $M_S(\zeta, \eta, \Omega)$) *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $\zeta, \eta \in \mathbb{R}$ be such that $0 < \zeta < \eta$. We represent by $M(\zeta, \eta, \Omega)$ the set of all $N \times N$ real matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant N} \in [L^\infty(\Omega)]^{N\times N}$ such that for all $\xi \in \mathbb{R}^N$ and for $\mathcal{L}^N$-a.e. $x \in \Omega$,*

$$(A(x)\xi|\xi) \geqslant \zeta\|\xi\|^2, \tag{2.4.4}$$

$$\|A(x)\xi\| \leqslant \eta\|\xi\|. \tag{2.4.5}$$

We represent by $M_S(\zeta, \eta, \Omega)$ the set of all matrices in $M(\zeta, \eta, \Omega)$ that are symmetric, precisely, the set of all matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant N} \in M(\zeta, \eta, \Omega)$ such that for all $i, j \in \{1, \cdots, N\}$ and for $\mathcal{L}^N$-a.e. $x \in \Omega$,

$$a_{ij}(x) = a_{ji}(x).$$

Notation 2.4.28. *For simplicity, if A is an $N \times N$ matrix and $\xi_1, \xi_2 \in \mathbb{R}^N$, we often write $A\xi_1\xi_2$ in place of $(A\xi_1|\xi_2)$.*

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded and open set, let $b \in L^\infty(\Omega)$ be nonnegative and let $A \in M_S(\zeta, \eta, \Omega)$ for some $\zeta, \eta \in \mathbb{R}$ such that $0 < \zeta < \eta$. Define the continuous and symmetric bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ in $H_0^1(\Omega)$ by

$$a(u, v) := \int_\Omega A(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega b(x)u(x)v(x)\,\mathrm{d}x \tag{2.4.6}$$

for $u, v \in H_0^1(\Omega)$.

In view of (2.4.4) and Poincaré's Inequality, we have that $a(\cdot,\cdot)$ is *coercive* in $H_0^1(\Omega) \times H_0^1(\Omega)$, i.e., there exists $\bar{\zeta} > 0$ such that for all $u \in H_0^1(\Omega)$, one has

$$a(u, u) \geqslant \bar{\zeta}\|u\|^2_{H_0^1(\Omega)}. \tag{2.4.7}$$

We now introduce the densely defined self-adjoint unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ in $L^2(\Omega)$ defined by

$$\begin{cases} D(\mathcal{A}) := \big\{u \in H_0^1(\Omega) \colon v \mapsto a(u, v) \text{ is continuous in } H_0^1(\Omega) \text{ for the topology of } L^2(\Omega)\big\}, \\ a(u, v) = (\mathcal{A}u|v) \text{ for all } u \in D(\mathcal{A}) \text{ and } v \in H_0^1(\Omega), \end{cases} \tag{2.4.8}$$

where $(\cdot|\cdot)$ stands for the inner product in $L^2(\Omega)$.

Remark 2.4.29. *In other words,*

$$\mathcal{A}u = -\operatorname{div}(A\nabla u) + bu = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + bu \tag{2.4.9}$$

[2.16] See Definition 2.5.4 in Subsection 2.5.1 below.

for $u \in D(\mathcal{A})$, where

$$D(\mathcal{A}) = \left\{u \in H_0^1(\Omega)\colon \operatorname{div}(A\nabla u) \in L^2(\Omega)\right\}. \tag{2.4.10}$$

An operator operator of the form (2.4.9) is said to be a partial differential operator. Under condition (2.4.7) it is said to be elliptic.

Equipped with the graph norm

$$\|u\|_{D(\mathcal{A})} := \sqrt{\|u\|_{L^2(\Omega)} + \|\mathcal{A}u\|_{L^2(\Omega)}}, \quad u \in D(\mathcal{A}),$$

$D(\mathcal{A})$ is a Hilbert space embedded into $H_0^1(\Omega)$ with continuous injection. Since the injection of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, it follows that the injection of $D(\mathcal{A})$ into $H_0^1(\Omega)$ is also compact. In view of the Lax–Milgram Theorem, $\mathcal{A}$ is an isomorphism of $D(\mathcal{A})$ onto $L^2(\Omega)$, $\mathcal{A}^{-1} \in \mathfrak{L}(L^2(\Omega))$ is self-adjoint and compact. Therefore, we can apply the following general result with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ in order obtain spectral properties of $\mathcal{A}$.

Theorem 2.4.30. *Let V be a Hilbert space dense and compactly embedded in H. Let $a(\cdot,\cdot)$ be a continuous, symmetric and coercive bilinear form in $V \times V$, and let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be the self-adjoint unbounded linear operator in H defined by*

$$\begin{cases} D(\mathcal{A}) := \left\{u \in V\colon\ v \mapsto a(u,v) \text{ is continuous in } V \text{ for the topology of } H\right\}, \\ a(u,v) = (\mathcal{A}u|v) \text{ for all } u \in D(\mathcal{A}) \text{ and } v \in V. \end{cases} \tag{2.4.11}$$

Then

(i) $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ *and for each eigenvalue* $\lambda \in \sigma_p(\mathcal{A})$ *the associated eigenspace* $N(\mathcal{A} - \lambda\mathcal{I})$ *is finite dimensional;*

(ii) $\sigma_p(\mathcal{A})$ *can be written as a nondecreasing sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$*, where each eigenvalue is repeated according to its multiplicity, such that* $\lambda_k \to +\infty$ *as* $k \to +\infty$*;*

(iii) the eigenfunctions u_k *of the operator* $\mathcal{A}$ *normalized in* H *and associated with* λ_k *satisfy the variational formulation*

$$\begin{cases} a(u_k, v) = \lambda_k(u_k|v) \quad \text{for all } v \in V, \\ \|u_k\| = 1, \end{cases} \tag{2.4.12}$$

(iv) the vector subspace generated by the eigenfunctions u_k *(normalized in* H*) is dense in* V *and in* H*, with* $\{u_k\}_{k\in\mathbb{N}}$ *forming an orthonormal basis of* H*;*

(v) representing for each l *by* U_l *the subspace generated by* $\{u_1, \cdots, u_l\}$*, we have that*

$$\lambda_1 = \min_{u\in V, \|u\|=1} a(u,u), \quad \lambda_k = \min_{\substack{u\in V, \|u\|=1,\\ u\in U_{k-1}^{\perp}}} a(u,u), \ \ k \geqslant 2. \tag{2.4.13}$$

Remark 2.4.31. (A word on the nomenclature) *Consider the elliptic partial differential operator defined by (2.4.6) and (2.4.8). In the literature, the problem of finding the eigenvalues and the corresponding eigenfunctions of* $\mathcal{A}$ *is often referred as the spectral problem associated with the operator* $(-\operatorname{div}(A\nabla) + b)$ *with Dirichlet boundary conditions, and is posed as: find* (λ, u) *such that*

$$\begin{cases} -\operatorname{div}(A\nabla u) + bu = \lambda u \quad \text{a.e. in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{2.4.14}$$

Remark 2.4.32. (The periodic case) *Besides Dirichlet boundary conditions, we will also be interested in the case of periodic boundary conditions. Precisely, let* $Y := (0,1)^N$. *We represent by* $H^1_\#(Y)$ *the closure of* $C^\infty_\#(Y)$ *with respect to the* $H^1(Y)$*-norm. For* $1 \leqslant p \leqslant +\infty$, $L^p_\#(Y;\mathbb{R}^m)$ *stands for the space of all functions in* $L^p_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^m)$ *which are* Y*-periodic. We say that a matrix is* Y*-periodic if each of its components is a* Y*-periodic function.*

Let $b \in L^\infty_\#(Y)$ *be such that* $\inf_Y b > 0$, *and let* $A = (a_{ij})_{1\leqslant i,j\leqslant N} \in [L^\infty_\#(Y)]^{N\times N}$ *be a* Y*-periodic matrix in* $M_S(\zeta,\eta,Y)$. *Arguing as in the beginning of this subsection, Theorem 2.4.30 can be applied to the bilinear form* $a : H^1_\#(Y) \times H^1_\#(Y) \to \mathbb{R}$ *defined in* $H^1_\#(Y)$ *by*

$$a(u,v) := \int_Y A(y)\nabla u(y)\nabla v(y)\,\mathrm{d}y + \int_Y b(y)u(y)v(y)\,\mathrm{d}y \tag{2.4.15}$$

for $u,v \in H^1_\#(Y)$, *with* $V = H^1_\#(Y)$ *and* $H = L^2_\#(Y)$.

As in the Dirichlet case, the associated elliptic partial differential operator $\mathcal{A} : D(\mathcal{A}) \subset L^2_\#(Y) \to L^2_\#(Y)$ *defined by (2.4.11) can be alternatively given by*

$$\mathcal{A}u = -\operatorname{div}(A\nabla u) + bu = -\sum_{i,j=1}^N \frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial u}{\partial y_j}\Big) + bu$$

for $u \in D(\mathcal{A})$, *where*

$$D(\mathcal{A}) = \big\{u \in H^1_\#(Y) \colon \operatorname{div}(A\nabla u) \in L^2_\#(Y)\big\}.$$

In this setting, (2.4.14) takes the form: find (λ, u) *such that*

$$\begin{cases} -\operatorname{div}(A\nabla u) + bu = \lambda u & \text{a.e. in } Y,\\ u \in H^1_\#(Y), \end{cases}$$

and is said to be the periodic spectral problem associated with the operator $(-\operatorname{div}(A\nabla) + b)$.

We observe also that since Y *is connected and Lipschitz continuous, it can be proved that the first eigenvalue* λ_1,

$$\lambda_1 = \min_{\substack{u\in H^1_\#(Y)\\ \|u\|_{L^2(Y)}=1}} \left\{ \int_Y A(y)\nabla u(y)\nabla v(y)\,\mathrm{d}y + \int_\Omega b(y)u(y)v(y)\,dy \right\},$$

is simple and the associated eigenfunction u_1 *belongs to* $H^1_\#(Y) \cap C^{0,s}_\#(Y)$ *for some* $0 < s < 1$, *and can be chosen to be a strictly positive function (see Gilbarg and Trudinger [52]).*

2.5. Γ-Convergence and G-Convergence.

In this dissertation, we will often adopt variational methods to study the problems treated here. In particular, the notions of Γ-convergence and G-convergence will play an important role. In this section we collect some properties concerning these two concepts, and we refer to the books Dal Maso [31], Cioranescu and Donato [28], Jikov, Kozlov and Oleĭnik [53], and to the references therein.

2.5.1. Γ-Convergence

The notion of Γ-convergence may be introduced in an arbitrary topological space. However, for our purposes throughout this work, it suffices to consider topological spaces satisfying the first axiom of countability, in which case Γ-convergence acquires a sequential characterization.

In the sequel of this subsection, X is a topological space satisfying the first axiom of countability, i.e., every point has a countable neighborhood basis. We represent by $\{F_j\}_{j\in\mathbb{N}}$ and $\{F_\delta\}_{\delta>0}$ a sequence and a family of functions from X into $\overline{\mathbb{R}}$, respectively.

Definition 2.5.1. (Γ-limit inferior, Γ-limit superior, Γ-limit) *We say that $F' : X \to \overline{\mathbb{R}}$ and $F'' : X \to \overline{\mathbb{R}}$ are the Γ-limit inferior and the Γ-limit superior, respectively, of the sequence $\{F_j\}_{j\in\mathbb{N}}$ if for all $x \in X$ we have*

$$F'(x) = \min\left\{\liminf_{j\to+\infty} F_j(x_j) \colon x_j \in X,\, x_j \to x \text{ in } X \text{ as } j \to +\infty\right\}$$

and

$$F''(x) = \min\left\{\limsup_{j\to+\infty} F_j(x_j) \colon x_j \in X,\, x_j \to x \text{ in } X \text{ as } j \to +\infty\right\}.$$

We often represent F' by $\Gamma\text{-}\liminf_{j\to+\infty} F_j$ and F'' by $\Gamma\text{-}\limsup_{j\to+\infty} F_j$.
If there exists a function $F : X \to \overline{\mathbb{R}}$ such that

$$F' = F'' = F$$

then we write $F = \Gamma\text{-}\lim_{j\to+\infty} F_j$, and we say that the sequence $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to F in X or that F is the Γ-limit of $\{F_j\}_{j\in\mathbb{N}}$ in X.
We say that $F' : X \to \overline{\mathbb{R}}$ and $F'' : X \to \overline{\mathbb{R}}$ are the Γ-limit inferior and the Γ-limit superior, respectively, of the family $\{F_\delta\}_{\delta>0}$ if for every sequence of positive numbers $\{\delta_j\}_{j\in\mathbb{N}}$ converging to zero we have

$$F' = \Gamma\text{-}\liminf_{j\to+\infty} F_{\delta_j} \quad \text{and} \quad F'' = \Gamma\text{-}\limsup_{j\to+\infty} F_{\delta_j},$$

respectively. We say that the family $\{F_\delta\}_{\delta>0}$ Γ-converges to a function $F : X \to \overline{\mathbb{R}}$ if for every sequence of positive numbers $\{\delta_j\}_{j\in\mathbb{N}}$ converging to zero we have

$$F = \Gamma\text{-}\liminf_{j\to+\infty} F_{\delta_j} = \Gamma\text{-}\limsup_{j\to+\infty} F_{\delta_j}.$$

Remark 2.5.2. *As an immediate consequence of the definition, the sequence $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to F in X if, and only if, the following conditions are satisfied:*

(i) for all $x \in X$ and for every sequence $\{x_j\}_{j\in\mathbb{N}}$ converging to x in X, we have

$$F(x) \leqslant \liminf_{j\to+\infty} F_j(x_j);$$

(ii) for all $x \in X$ there is a sequence $\{x_j\}_{j\in\mathbb{N}}$ converging to x in X such that

$$F(x) = \lim_{j\to+\infty} F_j(x_j).$$

Definition 2.5.3. (Lower semicontinuous function, l.s.c.) *We say that a function $F : X \to \overline{\mathbb{R}}$ is lower semicontinuous (briefly l.s.c.) in X if for all $x \in X$ and for every sequence $\{x_j\}_{j\in\mathbb{N}}$ converging to x in X, we have*

$$F(x) \leqslant \liminf_{j\to+\infty} F(x_j).$$

Definition 2.5.4. (Lower semicontinuous envelope) *Let $F : X \to \overline{\mathbb{R}}$. The l.s.c. envelope of F is the function $sc^{-}F$ defined for all $x \in X$ by*

$$sc^{-}F(x) := \sup\big\{G(x)\colon\ G \text{ is l.s.c., } G \leqslant F\big\}.$$

Remark 2.5.5. *It can be checked that $sc^{-}F$ is the greatest l.s.c. function bounded from above by F.*

Examples 2.5.6. (Γ-limits) 1) If the functions F_j are independent of x, i.e., if for all $j \in \mathbb{N}$ there exits a constant $a_j \in \overline{\mathbb{R}}$ such that $F_j(x) = a_j$ for all $x \in X$, then $F' \equiv \liminf_{j\to+\infty} a_j$ and $F'' \equiv \limsup_{j\to+\infty} a_j$.

2) If the functions F_j are independent of $j \in \mathbb{N}$, i.e., if there exits a function $F : X \to \overline{\mathbb{R}}$ such that $F_j \equiv F$ for all $j \in \mathbb{N}$, then $F' = F'' = sc^{-}F$, that is, $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to $sc^{-}F$ in X.

We now state two important properties of Γ-limits.

Proposition 2.5.7. *If $\{F_{j_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{F_j\}_{j\in\mathbb{N}}$, then*

$$\Gamma\text{-}\liminf_{j\to+\infty} F_j \leqslant \Gamma\text{-}\liminf_{k\to+\infty} F_{j_k} \quad \text{and} \quad \Gamma\text{-}\limsup_{j\to+\infty} F_j \geqslant \Gamma\text{-}\limsup_{k\to+\infty} F_{j_k}.$$

In particular, if $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to F in X then also $\{F_{j_k}\}_{k\in\mathbb{N}}$ Γ-converges to F in X.

Proposition 2.5.8. *The functions F' and F'' are l.s.c. in X.*

The first of the next two results shows that the Γ-limits remain unchanged if we replace the functions F_j by their l.s.c. envelopes $sc^{-}F_j$. The second characterizes the behavior of the Γ-convergence under continuous perturbations.

Proposition 2.5.9. *We have that*

$$\Gamma\text{-}\liminf_{j\to+\infty} sc^{-}F_j = \Gamma\text{-}\liminf_{j\to+\infty} F_j \quad \text{and} \quad \Gamma\text{-}\limsup_{j\to+\infty} sc^{-}F_j = \Gamma\text{-}\limsup_{j\to+\infty} F_j.$$

In particular, $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to $F : X \to \overline{\mathbb{R}}$ in X if, and only if, $\{sc^{-}F_j\}_{j\in\mathbb{N}}$ Γ-converges to F in X .

Propositions 2.5.10. *Let $G : X \to \overline{\mathbb{R}}$ be a continuous function. Then*

$$\Gamma\text{-}\liminf_{j\to+\infty}(F_j + G) = \Big(\Gamma\text{-}\liminf_{j\to+\infty} F_j\Big) + G \quad \text{and} \quad \Gamma\text{-}\limsup_{j\to+\infty}(F_j + G) = \Big(\Gamma\text{-}\limsup_{j\to+\infty} F_j\Big) + G.$$

In particular, if $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to F in X, then $\{F_j + G\}_{j\in\mathbb{N}}$ Γ-converges to $F + G$ in X.

We will now see that, under some equi-coercivity hypotheses, the Γ-convergence of $\{F_j\}_{j\in\mathbb{N}}$ to a function F in X implies the convergence of the infima of F_j to the minimum of F.

Definition 2.5.11. (Coercive function) *We say that $F : X \to \overline{\mathbb{R}}$ is coercive if for all $t \in \mathbb{R}$, the closure of the set $\{F \leqslant t\} := \{x \in X\colon\ F(x) \leqslant t\}$ is a compact subset of X.*

Definition 2.5.12. (Equi-coercive sequence of functions) *The sequence $\{F_j\}_{j\in\mathbb{N}}$ is said to be equi-coercive (in X) if for all $t \in \mathbb{R}$ there exist a closed and compact subset K_t of X such that for all $j \in \mathbb{N}$ one has $\{F_j \leqslant t\} \subset K_t$.*

Proposition 2.5.13. *The sequence $\{F_j\}_{j\in\mathbb{N}}$ is equi-coercive if, and only if, there exists a l.s.c. and coercive function $G : X \to \overline{\mathbb{R}}$ such that $F_j \geqslant G$ in X for all $j \in \mathbb{N}$.*

Definition 2.5.14. (δ-minimizer of a function) *Let $F : X \to \overline{\mathbb{R}}$ and $\delta > 0$. A point $x \in X$ is said to be a δ-minimizer of F in X if*

$$F(x) \leqslant \max\left\{\inf_{y\in X} F(y) + \delta, -\frac{1}{\delta}\right\}.$$

Remark 2.5.15. *If $\inf_X F > -\infty$ and if $\delta > 0$ is small enough, then x is a δ-minimizer of F in X if, and only if,*

$$F(x) \leqslant \inf_{y\in X} F(y) + \delta.$$

The next result concerns the convergence of the infima of an equi-coercive sequence of functions.

Theorem 2.5.16. *Assume that the sequence $\{F_j\}_{j\in\mathbb{N}}$ is equi-coercive and that Γ-converges to a function $F : X \to \overline{\mathbb{R}}$ in X. Then F is coercive and*

$$\min_{x\in X} F(x) = \lim_{j\to+\infty} \inf_{x\in X} F_j(x).$$

Moreover, if for all $j \in \mathbb{N}$ x_j is a minimizer of F_j in X, or more generally a δ_j-minimizer, where $\{\delta_j\}_{j\in\mathbb{N}}$ is a sequence of positive numbers converging to zero, then every accumulation point x of $\{x_j\}_{j\in\mathbb{N}}$ is a minimizer of F in X and

$$F(x) = \lim_{j\to+\infty} F_j(x_j).$$

Finally, we state two results, the first of which shows that the Γ-convergence in X satisfies Urysohn's convergence property; the second one establishes a compactness property of the Γ-convergence.

Proposition 2.5.17. *The sequence $\{F_j\}_{j\in\mathbb{N}}$ Γ-converges to a function $F : X \to \overline{\mathbb{R}}$ in X if, and only if, from every subsequence of $\{F_j\}_{j\in\mathbb{N}}$ we can extract a further subsequence that Γ-converges to F in X.*

Theorem 2.5.18. *Assume that X admits a countable basis. Then from every sequence $\{F_j\}_{j\in\mathbb{N}}$ we can extract a Γ-convergent subsequence.*

Remark 2.5.19. *In some cases we will be interested in the study of the Γ-convergence with respect to the weak topology of a Banach space, which is not metrizable. Nevertheless, under some suitable hypotheses the previous definitions and results may be extended to this case. Precisely, assume that X is a Banach space whose dual is separable. Then there exists a metric d in X for which the weak topology in every bounded in norm subset B coincides with the topology induced in B by the metric d. Thus, restricted to B, the weak topology satisfies the second axiom of countability.*

Assume now that $\{F_j\}_{j\in\mathbb{N}}$ is a sequence of functions defined in X with values in $\overline{\mathbb{R}}$ such that $F_j \geqslant G$ for all $j \in \mathbb{N}$, where $G : X \to \overline{\mathbb{R}}$ is a function satisfying $\lim_{\|x\|\to+\infty} G(x) = +\infty$. In this setting, the study of the Γ-convergence of $\{F_j\}_{j\in\mathbb{N}}$ with respect to the weak topology of X reduces to study of the Γ-convergence of $\{F_j\}_{j\in\mathbb{N}}$ with respect to the weak topology of a bounded in norm subset B of X; we are thus confined to the study of the Γ-convergence in a topological space satisfying the second axiom of countability. As such, under these hypotheses, we can make use of the definitions and results *stated above, where in place of convergence in X one should read convergence with respect to the weak topology of X.*

2.5.2. G-Convergence

Definition 2.5.20. (Weak solution for the Dirichlet problem) *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $f \in H^{-1}(\Omega)$, and $A = (a_{ij})_{1\leqslant 1,j\leqslant N} \in [L^\infty(\Omega)]^{N\times N}$. We say that u is a weak solution of the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5.1}$$

if u belongs to $H_0^1(\Omega)$ and satisfies the variational equation

$$\int_\Omega A(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

for all $v \in H_0^1(\Omega)$.

Remark 2.5.21. *In view of Lax–Milgram Theorem, if we assume that $A \in M(\zeta, \eta, \Omega)$ for some $\zeta, \eta \in \mathbb{R}$ such that $0 < \zeta < \eta$ then for all $f \in H^{-1}(\Omega)$ there exists a unique weak solution u for the corresponding Dirichlet problem (2.5.1). Moreover, under additional regularity hypotheses on Ω, A and f, it can be proved that u is a weak solution if, and only if, it is a solution in the classical sense (see, for example, Gilbarg and Trudinger [52]).*

Definition 2.5.22. (G-convergence, G-limit) *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $\zeta, \eta \in \mathbb{R}$ be such that $0 < \zeta < \eta$. We say that a sequence of matrices $\{A_\varepsilon\}_{\varepsilon>0} \subset M_S(\zeta, \eta, \Omega)$ G-converges to a matrix $A_0 \in M_S(\zeta, \eta, \Omega)$ if for every $f \in H^{-1}(\Omega)$ the weak solution u_ε of the Dirichlet problem*

$$\begin{cases} -\operatorname{div}(A_\varepsilon\nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

is such that $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0^+$, where u_0 is the weak solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A_0\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, the matrix A_0 is said to be the G-limit of the sequence $\{A_\varepsilon\}_{\varepsilon>0}$.

The next theorem states the main properties of G-convergence.

Theorem 2.5.23. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $\zeta, \eta \in \mathbb{R}$ be such that $0 < \zeta < \eta$. Let $\{A_\varepsilon\}_{\varepsilon>0}$ be a sequence of matrices in $M_S(\zeta, \eta, \Omega)$. Then*

- *(i) (compactness) there exists a subsequence $\{A_{\varepsilon'}\}_{\varepsilon'>0}$ of $\{A_\varepsilon\}_{\varepsilon>0}$ that G-converges to some $A_0 \in M_S(\zeta, \eta, \Omega)$;*
- *(ii) (uniqueness) the sequence $\{A_\varepsilon\}_{\varepsilon>0}$ admits at most one G-limit;*
- *(iii) (locality) if $\{A_\varepsilon\}_{\varepsilon>0}$ G-converges to some $A_0 \in M_S(\zeta, \eta, \Omega)$ and if $\{B_\varepsilon\}_{\varepsilon>0} \subset M_S(\zeta, \eta, \Omega)$ is another sequence G-convergent to some $B_0 \in M_S(\zeta, \eta, \Omega)$ and for which there exists a set $\omega \subset \Omega$ such that $A_\varepsilon = B_\varepsilon$ for all $\varepsilon > 0$ and a.e. in ω, then $A_0 = B_0$;*
- *(iv) (Urysohn) $\{A_\varepsilon\}_{\varepsilon>0}$ G-converges to some $A_0 \in M_S(\zeta, \eta, \Omega)$ if, and only if, from every subsequence of $\{A_\varepsilon\}_{\varepsilon>0}$ we can extract a further subsequence that G-converges to A_0.*

The next theorem relates the G-convergence of a sequence of matrices and the Γ-convergence of a certain sequence of associated functionals.

Theorem 2.5.24. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $\zeta, \eta \in \mathbb{R}$ be such that $0 < \zeta < \eta$. Let $\{A_\varepsilon\}_{\varepsilon>0} \subset M_S(\zeta, \eta, \Omega)$ and $A_0 \in M_S(\zeta, \eta, \Omega)$. Then $\{A_\varepsilon\}_{\varepsilon>0}$ G-converges to A_0 if, and only if, the sequence of functionals $\{J_\varepsilon\}_{\varepsilon>0}$, where $J_\varepsilon : L^2(\Omega) \to [0, +\infty]$ is defined by*

$$J_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega A_\varepsilon(x)\nabla u(x)\nabla u(x)\,\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

for $u \in L^2(\Omega)$, Γ-converges in $L^2(\Omega)$ to the functional $J_0 : L^2(\Omega) \to [0, +\infty]$ given for $u \in L^2(\Omega)$ by

$$J_0(u) := \begin{cases} \displaystyle\int_\Omega A_0(x)\nabla u(x)\nabla u(x)\,\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 2.5.25. *The notion of G-convergence above may be generalized to the case in which the matrices in $M(\zeta, \eta, \Omega)$ are not necessarily symmetric. This generalization is called H-convergence and it was introduced by Tartar in [73], and further developed by Murat and Tartar in [63] (see also Murat [62]).*

In Chapter 3 we will mostly be interested in a particular type of sequences of matrices in $M_S(\zeta, \eta, \Omega)$. Precisely, let $Y := (0,1)^N$, let $\zeta, \eta \in \mathbb{R}$ be such that $0 < \zeta < \eta$, and let $A = (a_{ij})_{1\leqslant i,j\leqslant N} \in [L^\infty_\#(Y)]^{N\times N}$ be a Y-periodic matrix in $M_S(\zeta, \eta, Y)$. For each $\varepsilon > 0$, set

$$A_\varepsilon := (a^\varepsilon_{ij})_{1\leqslant i,j\leqslant N}, \text{ where } a^\varepsilon_{ij}(x) := a_{ij}\left(\frac{x}{\varepsilon}\right), \ \ i, j \in \{1, \cdots, N\}. \tag{2.5.2}$$

Observe that each matrix A_ε is εY-periodic and belongs to $M_S(\zeta, \eta, \mathcal{O})$ for every open subset $\mathcal{O}$ of $\mathbb{R}^N$. The next result is a classical one within homogenization theory.

Theorem 2.5.26. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, and let $\{A_\varepsilon\}_{\varepsilon>0}$ be the sequence of matrices defined in (2.5.2). Then there exists a constant matrix $A^h \in M_S(\zeta, \eta, \Omega)$ such that $\{A_\varepsilon\}_{\varepsilon>0}$ G-converges to A^h. Moreover, the sequence of functionals $\{J_\varepsilon\}_{\varepsilon>0}$, where $J_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$ is given by*

$$J_\varepsilon(u) := \int_\Omega A_\varepsilon(x)\nabla u(x)\nabla u(x)\,\mathrm{d}x$$

for $u \in H_0^1(\Omega)$, Γ-converges with respect to the weak topology of $H_0^1(\Omega)$ to the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \int_\omega A^h \nabla u(x)\nabla u(x)\,\mathrm{d}x$$

for $u \in H_0^1(\Omega)$. The matrix A^h is also called the homogenized limit of the sequence $\{A_\varepsilon\}_{\varepsilon>0}$.

2.5.3. Convergence of Eigenvalues of Sequences of Unbounded Operators

In this subsection we state two results in terms of G-convergence and Γ-convergence regarding convergence of eigenvalues, and of the associated eigenfunctions, of sequences of densely defined self-adjoint operators. The first one is a classic result within homogenization theory (see Kesavan [54], [55]; see also Boccardo and Marcellini [14]).

Theorem 2.5.27. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, $Y := (0,1)^N$, and $A = (a_{ij})_{1\leqslant i,j\leqslant N} \in [L^\infty_\#(Y)]^{N\times N}$ a Y-periodic matrix in $M_S(\zeta,\eta,Y)$ for some $\zeta,\eta \in \mathbb{R}$ such that $0 < \zeta < \eta$. For each $\varepsilon > 0$ consider the εY-periodic matrix in $M_S(\zeta,\eta,\Omega)$ defined by (2.5.2). Let A^h be the homogenized limit of $\{A_\varepsilon\}_{\varepsilon>0}$. Represent by $\{\lambda_{\varepsilon,k}\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ the nondecreasing sequences formed by the eigenvalues of the operators $-\operatorname{div}(A_\varepsilon\nabla)$ and $-\operatorname{div}(A^h\nabla)$ with Dirichlet boundary conditions, respectively, where each eigenvalue is repeated according to its multiplicity, and let $\{u_{\varepsilon,k}\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$ be the associated $L^2(\Omega)$-normalized sequences of eigenfunctions, respectively. Then for all $k \in \mathbb{N}$,*

$$\lambda_{\varepsilon,k} \to \lambda_k \text{ as } \varepsilon \to 0^+$$

and, up to a not relabeled subsequence,

$$u_{\varepsilon,k} \rightharpoonup u_k \text{ weakly in } H^1_0(\Omega) \text{ as } \varepsilon \to 0^+.$$

Moreover, if λ_k is simple then the whole sequence $\{u_{\varepsilon,k}\}_{\varepsilon>0}$ converges.

The next result, whose proof can be found in Bouchitté, Mascarenhas and Trabucho [19, Thm. 3.1], will play an important role in the study of the spectral problem addressed in Chapter 3 of this dissertation.

Proposition 2.5.28. *Let $\mathcal{A}_\varepsilon : H_\varepsilon \to H_\varepsilon$ be a sequence of densely defined self-adjoint operators, where H_ε coincides algebraically with H endowed with a scalar product $(\cdot|\cdot)_\varepsilon$ such that*

$$c_1\|u\|^2 \leqslant (u|u)_\varepsilon \leqslant c_2\|u\|^2, \text{ for suitable positive constants } c_1,\, c_2, \tag{2.5.3}$$

$$\lim_{\varepsilon\to 0^+}(u_\varepsilon|v_\varepsilon)_\varepsilon = (u|v) \text{ whenever } u_\varepsilon \to u \text{ and } v_\varepsilon \to v \text{ in } H \text{ as } \varepsilon \to 0^+. \tag{2.5.4}$$

Let $G_\varepsilon : H \to (-\infty,+\infty]$ be defined by $G_\varepsilon(u) := (\mathcal{A}_\varepsilon u|u)_\varepsilon$, if $u \in D(\mathcal{A}_\varepsilon)$, and $G_\varepsilon(u) := +\infty$, otherwise. Assume further that the three following conditions hold:

(i) $G_\varepsilon(u) \geqslant -c_0\|u\|^2$*, for a suitable constant $c_0 \geqslant 0$ independent of ε;*

(ii) If $\sup_{\varepsilon>0} G_\varepsilon(u_\varepsilon) < +\infty$ and $\sup_{\varepsilon>0}\|u_\varepsilon\| < +\infty$, then the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is strongly relatively compact in H;

(iii) $\{G_\varepsilon\}_{\varepsilon>0}$ Γ-converges to a certain functional G.

Then, the limit functional G determines a unique closed linear operator $\mathcal{A}_0 : H \to H$ with compact resolvent such that $G(u) = (\mathcal{A}_0 u|u)$, for all $u \in D(\mathcal{A}_0)$. Furthermore, the spectral problems associated with $\mathcal{A}_\varepsilon$ converge in the following sense: let $(\nu_{\varepsilon,k}, u_{\varepsilon,k})$ and (ν_k, u_k) be such that

$$\begin{array}{cccc} u_{\varepsilon,k} \in D(\mathcal{A}_\varepsilon), & \mathcal{A}_\varepsilon u_{\varepsilon,k} = \nu_{\varepsilon,k} u_{\varepsilon,k}, & \nu_{\varepsilon,1} \leqslant \nu_{\varepsilon,2} \leqslant \cdots \leqslant \nu_{\varepsilon,k} \leqslant \cdots, & (u_{\varepsilon,k}|u_{\varepsilon,l})_\varepsilon = \delta_{kl}, \\ u_k \in D(\mathcal{A}_0), & \mathcal{A}_0 u_k = \nu_k u_k, & \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_k \leqslant \cdots, & (u_k|u_l) = \delta_{kl}, \end{array}$$

where $k,l \in \mathbb{N}$ and δ_{kl} denotes the Kronecker symbol. Then $\nu_{\varepsilon,k} \to \nu_k$ as $\varepsilon \to 0^+$. Moreover, up to a subsequence that we will not relabel, $\{u_{\varepsilon,k}\}_{\varepsilon>0}$ converges as $\varepsilon \to 0^+$ to an eigenfunction associated to ν_k. Conversely, any eigenfunction u_k is the strong limit of a particular sequence of eigenfunctions of $\mathcal{A}_\varepsilon$ associated with $\nu_{\varepsilon,k}$.

Chapter 3

Spectral Analysis in a Thin and Periodically Oscillating Medium

Under the motivation mentioned in the Introduction (see Subsection 1.1), we consider an elliptic operator with ε-periodic coefficients and the corresponding Dirichlet spectral problem in a three-dimensional bounded domain of small thickness δ. We study the asymptotic behavior of the spectrum of this problem as both positive parameters ε and δ tend to zero. As we will see this asymptotic behavior depends crucially on the ratio between ε and δ.

The results corresponding to the cases $\varepsilon \approx \delta$ $(\delta = \varepsilon)$ and $\varepsilon \ll \delta$ $(\delta = \varepsilon^\tau,\ \tau < 1)$ were announced in Ferreira and Mascarenhas [45]. In Ferreira, Mascarenhas and Piatnitski [46] detailed proofs of the statements formulated in Ferreira and Mascarenhas [45] were provided, and the case $\varepsilon \gg \delta$ $(\delta = \varepsilon^\tau,\ \tau > 1)$ was studied.

Our analysis relies on Γ-convergence and asymptotic expansions techniques for spectral problems. Some of our arguments are based on the Vishik–Lyusternik Lemma.

3.1. Main Results.

Let ω be an open and bounded subset of $\mathbb{R}^2$ and let δ be a positive parameter. Consider the thin domain $\Omega_\delta := \omega \times \delta I$, where $I := (-1/2, 1/2)$. Throughout this chapter the Greek characters α and β take their values in the set $\{1,2\}$ and we will often write $\bar{x}$ instead of (x_1, x_2). Given a function $f : \mathbb{R}^d \to \mathbb{R}$, $d \in \{2,3\}$, $\bar{\nabla} f$ stands for the vector $(\partial f/\partial x_1, \partial f/\partial x_2)$, while $\nabla_3 f$ and $\Delta_3 f$ stand for $\partial f/\partial x_3$ and $\partial^2 f/\partial x_3^2$, respectively.

Let $Y := (0,1)^2$ and let $A = (a_{ij})_{1\leqslant i,j\leqslant N} \in [L^\infty_\#(Y)]^{3\times 3}$ be a 3×3 real, Y-periodic matrix in $M_S(\zeta, \eta, Y \times I)$ for some $\zeta, \eta \in \mathbb{R}$ such that $0 < \zeta < \eta$ (see Definition 2.4.27). Notice that in view of (2.4.4) and (2.4.5), we have that for all $\xi \in \mathbb{R}^3$ and for a.e. $\bar{y} \in Y$,

$$\zeta \|\xi\|^2 \leqslant (A(\bar{y})\xi|\xi) \leqslant \eta \|\xi\|^2. \tag{3.1.1}$$

In order to simplify the notations, we will often write $A\xi\xi$ in place of $(A\xi|\xi)$. For each $\varepsilon > 0$ define

$$A_\varepsilon := (a^\varepsilon_{ij})_{1\leqslant i,j\leqslant 3}, \text{ where } a^\varepsilon_{ij}(\bar{x}) := a_{ij}\left(\frac{\bar{x}}{\varepsilon}\right),\ i,j \in \{1,2,3\}.$$

We observe that each matrix A_ε is εY-periodic and belongs to $M_S(\zeta, \eta, \omega \times I)$; moreover, it satisfies (3.1.1) a.e. in ω. Our goal is to characterize the asymptotic behavior as ε and δ tend to zero of the eigenvalues $\lambda^\delta_\varepsilon$ associated with the spectral problem

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla \tilde{v}_\varepsilon^\delta) = \lambda_\varepsilon^\delta \, \tilde{v}_\varepsilon^\delta & \text{a.e. in } \Omega_\delta, \\ \tilde{v}_\varepsilon^\delta \in H_0^1(\Omega_\delta). \end{cases} \tag{3.1.2}$$

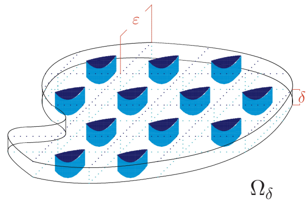

Fig. 3.1.1. Thin and periodically oscillating media

We also assume that $a_{\alpha 3} = 0$ a.e. in $\mathbb{R}^2$, thus we admit that the planar flux associated to the wave function depends exclusively on the behavior of this function in the cross-section ω. This hypothesis enables us to decouple the limit problem, simplifying a lot our computations. We denote by $\bar{A}$ and $\bar{A}_\varepsilon$ the 2×2 matrices $\bar{A} := (a_{\alpha\beta})$ and $\bar{A}_\varepsilon := (a_{\alpha\beta}^\varepsilon)$, respectively.

As we have seen in Subsection 2.4.2, by Theorem 2.4.30 the spectrum $\sigma_\varepsilon^\delta$ of the self-adjoint operator $-\mathrm{div}(A_\varepsilon \nabla)$ in $L^2(\Omega_\delta)$ with Dirichlet boundary conditions is discrete and can be written as a nondecreasing sequence $\{\lambda_{\varepsilon,k}^\delta\}_{k\in\mathbb{N}}$, where each eigenvalue is repeated according to its multiplicity, such that $\lambda_{\varepsilon,k}^\delta \to +\infty$ as $k \to +\infty$.

Moreover, in view of (2.4.13), (3.1.1) and Poincaré's Inequality, $\lambda_{\varepsilon,1}^\delta > 0$. We further observe that by Theorem 2.5.27, for each δ fixed we have that for all $k \in \mathbb{N}$, $\lambda_{\varepsilon,k}^\delta \to \lambda_k^\delta$ as $\varepsilon \to 0^+$, where $\{\lambda_k^\delta\}_{k\in\mathbb{N}}$ is the nondecreasing sequence formed by the eigenvalues of the operator $-\mathrm{div}(A^h \nabla)$ in $L^2(\Omega_\delta)$ with Dirichlet boundary conditions, being A^h the homogenized limit of $\{A_\varepsilon\}_{\varepsilon>0}$. On the other hand, using (2.4.13), it can be checked that for each ε fixed one has $\lambda_{\varepsilon,k}^\delta \to +\infty$ as $\delta \to 0^+$, for all $k \in \mathbb{N}$. Here we are interested in the case in which both parameters ε and δ converge to zero simultaneously.

A detailed characterization of the asymptotic behavior of $\sigma_\varepsilon^\delta$ is given in Theorem 3.1.1 for the case $\varepsilon \approx \delta$, in Theorem 3.1.2 for the case $\varepsilon \ll \delta$, and in Theorems 3.1.4 and 3.1.7 for the case $\varepsilon \gg \delta$. As we mentioned before, our analysis relies on Γ-convergence and asymptotic expansions techniques for spectral problems. Some of our arguments are based on the Vishik–Lyusternik Lemma (see Lemma 2.4.17).

Consider the quadratic energy $\widetilde{E}_\varepsilon^\delta : L^2(\omega \times \delta I) \to [0, +\infty]$ defined by

$$\widetilde{E}_\varepsilon^\delta(\tilde{v}) := \begin{cases} \displaystyle\int_{\omega\times\delta I} A_\varepsilon(\bar{x}^\delta) \nabla \tilde{v}(x^\delta) \nabla \tilde{v}(x^\delta) \, \mathrm{d}x^\delta & \text{if } \tilde{v} \in H_0^1(\omega \times \delta I), \\ +\infty & \text{otherwise}, \end{cases} \tag{3.1.3}$$

for $\tilde{v} \in L^2(\omega \times \delta I)$, whose associated operator (in the sense of Definition 2.4.20) is precisely the self-adjoint operator $-\mathrm{div}(A_\varepsilon \nabla)$ in $L^2(\omega \times \delta I)$ with Dirichlet boundary conditions (see Subsection 2.4.2).

As it is usual in the dimension reduction framework, the first step is to perform a rescaling and a change of variables in order to transform problem (3.1.2) into an equivalent one defined in the fixed domain $\omega \times I$. To each point $x^\delta = (\bar{x}^\delta, x_3^\delta) \in \omega \times \delta I$ we associate the point $x = (\bar{x}, x_3) = (\bar{x}^\delta, \delta^{-1} x_3^\delta) \in \omega \times I$, and we define $v \in H_0^1(\omega \times I)$ by $v(x) := \tilde{v}(x^\delta)$ whenever $\tilde{v} \in H_0^1(\omega \times \delta I)$. Accordingly, we rescale the energy in (3.1.3) by dividing it by δ so that the new energy becomes $E_\varepsilon^\delta : L^2(\omega \times I) \to [0, +\infty]$,

$$E_\varepsilon^\delta(v) := \begin{cases} \displaystyle\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x}) \bar{\nabla} v(x) \bar{\nabla} v(x) + \frac{a_{33}^\varepsilon(\bar{x})}{\delta^2} |\nabla_3 v(x)|^2 \, \mathrm{d}x & \text{if } v \in H_0^1(\omega \times I), \\ +\infty & \text{otherwise}. \end{cases} \tag{3.1.4}$$

The rescaled spectral problem reads

$$\begin{cases} -\mathrm{div}_{\bar{x}}(\bar{A}_\varepsilon \bar{\nabla} v_\varepsilon^\delta) - \dfrac{a_{33}^\varepsilon}{\delta^2} \Delta_3 v_\varepsilon^\delta = \lambda_\varepsilon^\delta \, v_\varepsilon^\delta & \text{a.e. in } \omega \times I, \\ v_\varepsilon^\delta \in H_0^1(\omega \times I). \end{cases} \tag{3.1.5}$$

We stress that problems (3.1.2) and (3.1.5) are equivalent.

Before stating our main results, we will introduce some notation. Since we are interested in the cases $\varepsilon \approx \delta$, $\varepsilon \ll \delta$ and $\varepsilon \gg \delta$, we consider $\delta = \varepsilon^{\tau}$ for each $\tau \in (0, +\infty)$, and we introduce the $L^2(Y)$-normalized first eigenpair $(\mu^{\tau}_{\varepsilon,0}, \phi^{\tau}_{\varepsilon,0})$ for the bidimensional periodic spectral problem

$$\begin{cases} -\varepsilon^{2(\tau-1)} \operatorname{div}(\bar{A}\bar{\nabla}\phi^{\tau}_{\varepsilon}) + a_{33}\pi^2\phi^{\tau}_{\varepsilon} = \mu^{\tau}_{\varepsilon}\phi^{\tau}_{\varepsilon} \quad \text{a.e. in } Y, \\ \phi^{\tau}_{\varepsilon} \in H^1_{\#}(Y). \end{cases} \tag{3.1.6}$$

We recall that the eigenvalue $\mu^{\tau}_{\varepsilon,0}$ is real, positive and simple, and the associated $L^2(Y)$-normalized eigenfunction $\phi^{\tau}_{\varepsilon,0}$ belongs to $H^1_{\#}(Y) \cap C^{0,s}_{\#}(Y)$, for some $0 < s < 1$, and may be chosen to be a strictly positive function (see Remark 2.4.32).

We will distinguish three cases: $\tau = 1$, $\tau < 1$ and $\tau > 1$. Notice that if $\tau = 1$ then problem (3.1.6) does not depend on ε, and for that reason we simply write (μ_0, ϕ_0) to denote its $L^2(Y)$-normalized first eigenpair.

Let us also introduce the following unidimensional spectral problem in the interval I:

$$\begin{cases} -\theta'' = \varsigma\theta \quad \text{a.e. in } I, \\ \theta \in H^1_0(I), \end{cases} \tag{3.1.7}$$

whose n^{th} $L^2(I)$-normalized eigenpair is represented by $\big(\varsigma_n, \theta_n\big)$ for each $n \in \mathbb{N}$, with $(\varsigma_1, \theta_1) := (\pi^2, \sqrt{2}\cos(\pi x_3))$, $x_3 \in I$. The following statement characterizes the behavior of $\sigma^{\delta}_{\varepsilon}$ in the case $\delta \approx \varepsilon$.

Theorem 3.1.1. *($\varepsilon \approx \delta$) Under the above hypotheses, let $\big(\lambda_{\varepsilon,k}, v_{\varepsilon,k}\big)$ be a k^{th} eigenpair associated with problem (3.1.5) for $\delta = \varepsilon$, and let (ν_k, φ_k) be a k^{th} eigenpair associated with the bidimensional homogenized spectral problem in the cross section ω*

$$\begin{cases} -\operatorname{div}(\bar{B}^h\bar{\nabla}\varphi) = \nu\varphi \quad \text{a.e. in } \omega, \\ \varphi \in H^1_0(\omega), \end{cases}$$

where the 2×2 constant matrix $\bar{B}^h$ is the homogenized limit of the family of εY-periodic matrices $\{\bar{B}_{\varepsilon}\}_{\varepsilon>0}$, $\bar{B}_{\varepsilon} := (b^{\varepsilon}_{\alpha\beta})$ with

$$b^{\varepsilon}_{\alpha\beta}(\bar{x}) := \left|\phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)\right|^2 a_{\alpha\beta}\Big(\frac{\bar{x}}{\varepsilon}\Big).$$

Then, there exists a self-adjoint operator $\mathcal{A}_{\varepsilon} : H_{\varepsilon} \to H_{\varepsilon}$, where H_{ε} coincides algebraically with $L^2(\omega \times I)$ endowed with the scalar product $(\,\cdot\,|\,\cdot\,)_{\varepsilon}$ defined by

$$(u|v)_{\varepsilon} := \int_{\omega\times I} \left|\phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)\right|^2 u(x)v(x)\,\mathrm{d}x, \;\; u, v \in L^2(\omega \times I),$$

such that $D(\mathcal{A}_{\varepsilon})$ is dense in $H^1_0(\omega \times I)$ and

$$\lambda_{\varepsilon,k} = \frac{\mu_0}{\varepsilon^2} + \nu_{\varepsilon,k}, \qquad v_{\varepsilon,k}(\bar{x}, x_3) = \phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)u_{\varepsilon,k}(\bar{x}, x_3) \quad \text{a.e. } (\bar{x}, x_3) \in \omega \times I, \tag{3.1.8}$$

where $(\nu_{\varepsilon,k}, u_{\varepsilon,k})$ is a k^{th} eigenpair of $\mathcal{A}_{\varepsilon}$, that is,

$$u_{\varepsilon,k} \in D(\mathcal{A}_{\varepsilon}), \quad \mathcal{A}_{\varepsilon}u_{\varepsilon,k} = \nu_{\varepsilon,k}u_{\varepsilon,k}, \quad \nu_{\varepsilon,1} \leqslant \nu_{\varepsilon,2} \leqslant \cdots \leqslant \nu_{\varepsilon,k} \leqslant \cdots, \quad (u_{\varepsilon,k}|u_{\varepsilon,l})_{\varepsilon} = \delta_{kl}.$$

Furthermore, $\nu_{\varepsilon,k} \to \nu_k$ as $\varepsilon \to 0^+$ and, up to a subsequence that we do not relabel, $u_{\varepsilon,k} \rightharpoonup u_k$ weakly in $H^1_0(\omega \times I)$ as $\varepsilon \to 0^+$, where u_k is the product of an eigenfunction associated with ν_k and θ_1.

Conversely, any product of eigenfunctions $u_k = \varphi_k \theta_1$ is the weak limit of a particular sequence of eigenfunctions associated with $\nu_{\varepsilon,k}$.

We next provide the characterization of $\sigma_\varepsilon^\delta$ when $\varepsilon \ll \delta$. For each $j \in \mathbb{N}_0$ define

$$\varrho_j := \pi^2 \int_Y a_{33}(\bar{y})\psi_j(\bar{y})\,\mathrm{d}\bar{y}, \tag{3.1.9}$$

where $\psi_0 \equiv 1$ in Y and, for $j \geqslant 1$, ψ_j are the solutions of the recurrence problems in $H^1_\#(Y)$

$$-\operatorname{div}(\bar{A}(\bar{y})\bar{\nabla}\psi_j) = -a_{33}(\bar{y})\pi^2\psi_{j-1} + \sum_{\ell=0}^{j-1} \varrho_\ell \psi_{j-1-\ell}, \quad \int_Y \psi_j(\bar{y})\,\mathrm{d}\bar{y} = 0. \tag{3.1.10}$$

Theorem 3.1.2. ($\varepsilon \ll \delta$) *Suppose that the above hypotheses are fulfilled and that in addition $a_{\alpha\beta}$ are uniformly Lipschitz continuous in Y. Let $(\lambda_{\varepsilon,k}, v_{\varepsilon,k})$ be a k^{th} eigenpair associated with the problem (3.1.5) with $\delta = \varepsilon^\tau$ for some $\tau \in (0,1)$, and let $(\mu^\tau_{\varepsilon,0}, \phi^\tau_{\varepsilon,0})$ be the $L^2(\omega)$-normalized first eigenpair of (3.1.6). Let $i \in \mathbb{N}$ be such that $\frac{i-1}{i} < \tau \leqslant \frac{i}{i+1}$, and let (ν_k, φ_k) be a k^{th} eigenpair associated with the bidimensional homogenized spectral problem in the cross section ω*

$$\begin{cases} -\operatorname{div}(\bar{A}^h \bar{\nabla}\varphi) = \nu\varphi & \text{a.e. in } \omega, \\ \varphi \in H^1_0(\omega), \end{cases} \tag{3.1.11}$$

where the 2×2 constant matrix $\bar{A}^h$ is the homogenized limit of the sequence $\{\bar{A}_\varepsilon\}_{\varepsilon>0}$. Then, $\mu^\tau_{\varepsilon,0} \to \pi^2 \int_Y a_{33}(\bar{y})\,\mathrm{d}\bar{y} = \varrho_0$ as $\varepsilon \to 0^+$, $\phi^\tau_{\varepsilon,0}(\bar{x}/\varepsilon) \to 1 = \psi_0$ uniformly in ω as $\varepsilon \to 0^+$, and there exists a self-adjoint operator $\mathcal{A}_\varepsilon : H_\varepsilon \to H_\varepsilon$, where H_ε coincides algebraically with $L^2(\omega \times I)$ endowed with the scalar product $(\,\cdot\,|\,\cdot\,)_\varepsilon$ defined by

$$(u|v)_\varepsilon := \int_{\omega\times I} \left|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\right|^2 u(x)v(x)\,\mathrm{d}x, \;\; u, v \in L^2(\omega \times I),$$

such that $D(\mathcal{A}_\varepsilon)$ is dense in $H^1_0(\omega \times I)$ and

$$\lambda_{\varepsilon,k} = \sum_{j=0}^{i} \frac{\varrho_j}{\varepsilon^{\tau(2j+2)-2j}} + \rho^\tau_\varepsilon + \nu_{\varepsilon,k}, \quad v_{\varepsilon,k}(\bar{x}, x_3) = \phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big) u_{\varepsilon,k}(\bar{x}, x_3) \;\; \text{a.e. } (\bar{x}, x_3) \in \omega \times I, \tag{3.1.12}$$

where $(\nu_{\varepsilon,k}, u_{\varepsilon,k})$ is a k^{th} eigenpair of $\mathcal{A}_\varepsilon$, that is,

$$u_{\varepsilon,k} \in D(\mathcal{A}_\varepsilon), \quad \mathcal{A}_\varepsilon u_{\varepsilon,k} = \nu_{\varepsilon,k} u_{\varepsilon,k}, \quad \nu_{\varepsilon,1} \leqslant \nu_{\varepsilon,2} \leqslant \cdots \leqslant \nu_{\varepsilon,k} \leqslant \cdots, \quad (u_{\varepsilon,k}|u_{\varepsilon,l})_\varepsilon = \delta_{kl}.$$

Furthermore, $\rho^\tau_\varepsilon \to 0$ as $\varepsilon \to 0^+$, $\nu_{\varepsilon,k} \to \nu_k$ as $\varepsilon \to 0^+$, and, up to a subsequence that we will not relabel, $u_{\varepsilon,k} \rightharpoonup u_k$ weakly in $H^1_0(\omega \times I)$ as $\varepsilon \to 0^+$, where u_k is the product between an eigenfunction associated with ν_k and θ_1. Conversely, any product of eigenfunctions $u_k = \varphi_k\theta_1$ is the weak limit of a particular sequence of eigenfunctions associated with $\nu_{\varepsilon,k}$.

Remark 3.1.3. *If the series $\sum_{j \geqslant 0} \|\psi_j\|_{L^2(Y)}$ converges, the same happens with $\sum_{j \geqslant 0} |\varrho_j|$ since from (3.1.9) we obtain that $|\rho_j| \leqslant C\|\psi_j\|_{L^2(Y)}$, where C is a constant independent of j. Moreover,*

$$\sum_{j \geqslant 0} \varrho_j = \mu_0 \text{ and } \sum_{j \geqslant 0} \psi_j = \frac{\phi_0}{\int_Y \phi_0\,\mathrm{d}\bar{y}},$$

where (μ_0, ϕ_0) is the $L^2(Y)$-normalized first eigenpair of (3.1.6) for $\tau = 1$. In fact, the sum on the right hand side of (3.1.10) is the general term of the Cauchy convolution of the series $\psi := \sum_{j \geqslant 0} \psi_j$ and $\rho := \sum_{j \geqslant 0} \rho_j$. Summing (3.1.10) in $j \in \mathbb{N}_0$ and passing to the limit, we get

$$\begin{cases} -\operatorname{div}(\bar{A}\bar{\nabla}\psi) + a_{33}\,\pi^2\,\psi = \rho\,\psi \quad \text{a.e. in } Y, \\ \psi \in H^1_\#(Y), \ \int_Y \psi\, d\bar{y} = 1, \end{cases}$$

which implies $\psi = \phi_0 / \int_Y \phi_0 \,\mathrm{d}\bar{y}$ and $\rho = \mu_0$.

We further observe that since $\frac{i-1}{i} < \tau \leqslant \frac{i}{i+1}$, the convergence of $\sum_{j \geqslant 0} |\varrho_j|$ implies that for fixed $\varepsilon > 0$ and as $\tau \to 1^-$, we have

$$\sum_{j=0}^{i} \frac{\varrho_j}{\varepsilon^{\tau(2j+2)-2j}} \to \frac{\mu_0}{\varepsilon^2}. \tag{3.1.13}$$

This shows that as τ tends to 1, the development (3.1.12) tends to the development (3.1.8). To prove (3.1.13) we fix $\varepsilon > 0$ and we consider, for an arbitrary $\delta > 0$, $j_0 \in \mathbb{N}$ such that $\sum_{j > j_0} |\rho_j| < \delta$. As τ is of order $\frac{i}{i+1}$, to obtain the desired convergence it is enough to prove that

$$\lim_{i \to +\infty} \sum_{j=0}^{i} \varepsilon^{\frac{2(j+1)}{i+1}} \rho_j = \rho = \mu_0. \tag{3.1.14}$$

Since

$$\sum_{j=0}^{i} \varepsilon^{\frac{2(j+1)}{i+1}} \rho_j = \sum_{j=0}^{j_0} \varepsilon^{\frac{2(j+1)}{i+1}} \rho_j + \sum_{j=j_0+1}^{i} \varepsilon^{\frac{2(j+1)}{i+1}} \rho_j, \quad \left| \sum_{j=j_0+1}^{i} \varepsilon^{\frac{2(j+1)}{i+1}} \rho_j \right| \leqslant \sum_{j > j_0} |\rho_j| \leqslant \delta,$$

and

$$\lim_{i \to +\infty} \sum_{j=0}^{j_0} \varepsilon^{\frac{2(j+1)}{i+1}} \rho_j = \sum_{j=0}^{j_0} \rho_j,$$

the arbitrariness of δ yields (3.1.14).

The case $\varepsilon \gg \delta$, say $\delta = \varepsilon^\tau$ with $\tau \in (1, +\infty)$, seems a lot more difficult to handle due to the degeneracy of the corresponding problem (3.1.6). Indeed, in the case $\tau > 1$ the asymptotic behavior of $\mu^\tau_{\varepsilon,0}$ depends strongly on the behavior of the potential a_{33} (see, for instance, Kozlov and Piatnitski [56], [57]). An interesting case is when the potential a_{33} oscillates between two different values, as it is the case of a two media mixture. In that direction we introduce new hypotheses on a_{33}. In Theorem 3.1.4 we identify the asymptotic expansion of the first eigenvalue. In Theorem 3.1.7 we provide a characterization of the limit spectrum in the sense of Kuratowsky.

Theorem 3.1.4. ($\varepsilon \gg \delta$) *Under the general hypotheses stated above, assume in addition that $a_{\alpha\beta}$ are smooth functions and that there exists an open, connected and smooth subset Q of Y, $Q \subset\subset Y$, such that a_{33} coincides with its minimum, $a_{\min}$, in Q and is a smooth function strictly greater than $a_{\min}$ in $Y \backslash Q$. Let (ν_0, q_0) be the $L^2(Q)$-normalized first eigenpair of the bidimensional spectral problem in Q*

$$\begin{cases} -\operatorname{div}(\bar{A}\bar{\nabla} q) = \nu q \quad \text{a.e. in } Q, \\ q \in H^1_0(Q). \end{cases} \tag{3.1.15}$$

Let $\sigma_\varepsilon := \{\lambda_{\varepsilon,k} \in \mathbb{R}^+ : \ k \in \mathbb{N}\}$ be the spectrum of problem (3.1.5) with $\delta = \varepsilon^\tau$ for some $\tau \in (1, +\infty)$. Let $k \in \mathbb{N}$ be such that $k \geqslant \frac{2}{\tau - 1}$, and let $(\mu^\tau_{\varepsilon,0}, \phi^\tau_{\varepsilon,0})$ be the $L^2(Y)$-normalized first eigenpair of (3.1.6).

Then $\mu^\tau_{\varepsilon,0} \to a_{\min}\pi^2$, $\phi^\tau_{\varepsilon,0} \rightharpoonup q_0$ weakly in $H^1_\#(Y)$ as $\varepsilon \to 0^+$, where we identify q_0 with its extension by zero to the whole Y, and

$$\lambda_{\varepsilon,1} = \frac{a_{\min}\pi^2}{\varepsilon^{2\tau}} + \frac{\nu_0}{\varepsilon^2} + \varepsilon^{\tau-3}\mu_1 + \cdots + \varepsilon^{k(\tau-1)-2}\mu_k + \rho^\tau_\varepsilon + \nu^\tau_{\varepsilon,1},$$

where μ_i, $i \in \{1,\cdots,k\}$, are well determined constants, $|\rho^\tau_\varepsilon| \leqslant C\varepsilon^{(k+\frac{1}{2})\tau-(k+\frac{5}{2})} \to 0$ as $\varepsilon \to 0^+$, for some constant C independent of ε, and

$$\nu^\tau_{\varepsilon,1} := \inf_{\substack{\psi \in H^1_0(\omega) \\ \left\|\phi^\tau_{\varepsilon,0}\left(\frac{\cdot}{\varepsilon}\right)\psi\right\|_{L^2(\omega)}=1}} \left\{ \int_\omega \left|\phi^\tau_{\varepsilon,0}\left(\frac{\bar{x}}{\varepsilon}\right)\right|^2 \bar{A}_\varepsilon \bar{\nabla}\psi \bar{\nabla}\psi \,\mathrm{d}\bar{x} \right\}$$

vanishes as $\varepsilon \to 0^+$.

Remark 3.1.5. *Theorem 3.1.4 is valid under weaker regularity hypotheses on the coefficients. In fact, as it will become clear within the proof, instead of smoothness it suffices to assume that $a_{\alpha\beta}$ are C^{k+2} functions and that in $Y\backslash Q$ a_{33} is also a C^{k+2} function, where k is the smallest natural number satisfying $k \geqslant \frac{2}{\tau-1}$. In particular, the smaller the positive number $\tau - 1$ is, the more regularity on the coefficients is required.*

Remark 3.1.6. *Hypotheses of Theorem 3.1.4 cover the important case where a_{33} oscillates between two different values, but rule out the case in which a_{33} is constant. Nevertheless, it is easy to see that under the general hypotheses stated at the beginning of the present chapter, if a_{33} is constant, then for any $\tau \in (0,+\infty)$, $\mu^\tau_{\varepsilon,0} \equiv a_{33}\pi^2$ and $\phi^\tau_{\varepsilon,0} \equiv 1$. Moreover, as it will become clear from our arguments, if $(\lambda_{\varepsilon,k}, v_{\varepsilon,k})$ is a k^{th} eigenpair associated with problem (3.1.5) with $\delta = \varepsilon^\tau$, then*

$$\lambda_{\varepsilon,k} = \frac{a_{33}\pi^2}{\varepsilon^{2\tau}} + \nu_{\varepsilon,k},$$

where $\nu_{\varepsilon,k} \to \nu_k$ as $\varepsilon \to 0^+$ and, up to a subsequence that we do not relabel, $v_{\varepsilon,k} \rightharpoonup v_k = \varphi_k\theta_1$ weakly in $H^1_0(\omega \times I)$ as $\varepsilon \to 0^+$, being (ν_k, φ_k) a k^{th} eigenpair associated with (3.1.11).

Finally, under quite more general hypotheses than those of Theorem 3.1.4, the next theorem characterizes the limit spectrum in the sense of Kuratowsky.

Theorem 3.1.7. *($\varepsilon \gg \delta$) Assume the general hypotheses stated at the beginning of the present chapter and, in addition, assume that ω is connected and that a_{33} attains a minimum value, $a_{\min}$, at some $\bar{y}_0 \in \mathbb{R}^2$ such that $a_{\alpha\beta}$ and a_{33} are continuous in some neighborhood of $\bar{y}_0$. Then,*

$$\lim_{\varepsilon \to 0^+} \left(\varepsilon^{2\tau}\sigma_\varepsilon\right) = \left[a_{\min}\pi^2, +\infty\right], \tag{3.1.16}$$

where the limit in (3.1.16) is to be understood in the sense of Kuratowsky, that is, $\left[a_{\min}\pi^2, +\infty\right]$ is the set of all cluster points of sequences $\{\lambda_\varepsilon\}_{\varepsilon>0}$, $\lambda_\varepsilon \in \varepsilon^{2\tau}\sigma_\varepsilon$.

The remaining part of this chapter is organized as follows. In Section 3.2 we prove some auxiliary results. Section 3.3 is devoted to the proof of Theorem 3.1.1, while Section 3.4 to the proof of Theorem 3.1.2. Finally, in Section 3.5 we prove Theorems 3.1.4 and 3.1.7.

In this section we prove two preliminary results that play an important role in the subsequent sections. The first result concerns a classical change of unknowns (see Vanninathan [75]; see also Allaire and Malige [4]). In the cases $\varepsilon \approx \delta$ and $\varepsilon \ll \delta$ it will allow us to transform the energies (3.1.4) into functionals for which Proposition 2.5.28 applies.

Proposition 3.2.1. *For fixed $\tau, \varepsilon > 0$, consider the functions u and v related by*

$$v(x) = \phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)u(x) \quad \text{for a.e. } x = (\bar{x}, x_3) \in \omega \times I. \tag{3.2.1}$$

Then $v \in H^1_0(\omega \times I)$ if, and only if, $u \in H^1_0(\omega \times I)$. Moreover, if $v \in H^1_0(\omega \times I)$, then

$$\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla} v(x)\bar{\nabla} v(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|v(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v(x)|^2 \,\mathrm{d}x = \int_{\omega\times I} \Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \bar{A}_\varepsilon(\bar{x})\bar{\nabla} u(x)\bar{\nabla} u(x)\,\mathrm{d}x. \tag{3.2.2}$$

Proof. We will proceed in two steps.

Step 1. We begin by proving that equality (3.2.2) holds for every $u \in H^1_0(\omega \times I) \cap L^\infty(\omega \times I)$.

Fix $u \in H^1_0(\omega \times I) \cap L^\infty(\omega \times I)$. Since $\phi^\tau_{\varepsilon,0} \in H^1_\#(Y) \cap C^{0,s}_\#(Y)$ for some $0 < s < 1$, the function v defined by (3.2.1) belongs to $H^1_0(\omega \times I) \cap L^\infty(\omega \times I)$. Furthermore, we have

$$\begin{aligned}
&\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla} v(x)\bar{\nabla} v(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|v(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v(x)|^2 \,\mathrm{d}x \\
&\quad= \int_{\omega\times I} \Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \bar{A}_\varepsilon(\bar{x})\bar{\nabla} u(x)\bar{\nabla} u(x) + \frac{1}{\varepsilon^2}\bar{A}_\varepsilon(\bar{x})\,\bar{\nabla}\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\,\bar{\nabla}\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)|u(x)|^2\,\mathrm{d}x \\
&\qquad+ \int_{\omega\times I} \frac{2}{\varepsilon}\bar{A}_\varepsilon(\bar{x})\,\bar{\nabla}\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\bar{\nabla} u(x)\,\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)u(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2|u(x)|^2\,\mathrm{d}x \\
&\qquad- \int_{\omega\times I} \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2|u(x)|^2\,\mathrm{d}x.
\end{aligned}$$

We claim that for a.e. $x_3 \in I$,

$$\begin{aligned}
&\int_{\omega} \frac{1}{\varepsilon^2}\bar{A}_\varepsilon(\bar{x})\,\bar{\nabla}\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\,\bar{\nabla}\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)|u(x)|^2 + \frac{2}{\varepsilon}\bar{A}_\varepsilon(\bar{x})\,\bar{\nabla}\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\,\bar{\nabla} u(x)\,\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)u(x)\,\mathrm{d}\bar{x} \\
&\quad+ \int_{\omega} \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2|u(x)|^2\,\mathrm{d}\bar{x} = \int_{\omega} \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2|u(x)|^2\,\mathrm{d}\bar{x},
\end{aligned} \tag{3.2.3}$$

from which Step 1 will follow.

To prove claim (3.2.3) we start by observing that since ω is an open and bounded subset of $\mathbb{R}^2$ there exist m_ε pairwise disjoint translated sets of Y, denoted by Y_i, $i = 1, \cdots, m_\varepsilon$, such that

$$\omega \subset \widetilde{\omega}, \quad \text{where } \widetilde{\omega} := \operatorname{int}\Big(\bigcup_{i=1}^{m_\varepsilon} \varepsilon\,\overline{Y}_i\Big).$$

Representing by $\tilde{u}$ the extension by zero of the function u to the set $\widetilde{\omega} \times I$, we have $\tilde{u} \in H^1_0(\widetilde{\omega} \times I) \cap L^\infty(\widetilde{\omega} \times I)$. Using the change of variables $\bar{y} := \frac{\bar{x}}{\varepsilon}$, defining $w(\bar{y}, x_3) := \tilde{u}(\varepsilon\bar{y}, x_3)$, and recalling that

$a^\varepsilon_{ij}(\bar{x}) = a_{ij}\big(\frac{\bar{x}}{\varepsilon}\big)$, we obtain for a.e. $x_3 \in I$:

$$\begin{aligned}
&\int_\omega \frac{1}{\varepsilon^2}\bar{A}_\varepsilon(\bar{x})\,\bar\nabla\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\,\bar\nabla\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\,|u(x)|^2 + \frac{2}{\varepsilon}\bar{A}_\varepsilon(\bar{x})\,\bar\nabla\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\,\bar\nabla u(x)\,\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)u(x)\,\mathrm{d}\bar{x}\\
&\qquad + \int_\omega \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2|u(x)|^2\,\mathrm{d}\bar{x}\\
&\quad = \varepsilon^2\sum_{i=1}^{m_\varepsilon}\bigg[\int_{Y_i}\frac{1}{\varepsilon^2}\bar{A}(\bar{y})\bar\nabla\phi^\tau_{\varepsilon,0}(\bar{y})\bar\nabla\phi^\tau_{\varepsilon,0}(\bar{y})|\tilde{u}(\varepsilon\bar{y},x_3)|^2\,\mathrm{d}\bar{y}\\
&\qquad + \int_{Y_i}\frac{2}{\varepsilon}\bar{A}(\bar{y})\bar\nabla\phi^\tau_{\varepsilon,0}(\bar{y})\bar\nabla\tilde{u}(\varepsilon\bar{y},x_3)\phi^\tau_{\varepsilon,0}(\bar{y})\,\tilde{u}(\varepsilon\bar{y},x_3)\,\mathrm{d}\bar{y} + \int_{Y_i}\frac{a_{33}(\bar{y})}{\varepsilon^{2\tau}}\pi^2\big|\phi^\tau_{\varepsilon,0}(\bar{y})\big|^2|\tilde{u}(\varepsilon\bar{y},x_3)|^2\,\mathrm{d}\bar{y}\bigg]\\
&\quad = \varepsilon^2\sum_{i=1}^{m_\varepsilon}\bigg[\int_{Y_i}\frac{1}{\varepsilon^2}\bar{A}(\bar{y})\bar\nabla\phi^\tau_{\varepsilon,0}(\bar{y})\bar\nabla\big(\phi^\tau_{\varepsilon,0}(\bar{y})|w(\bar{y},x_3)|^2\big) + \frac{a_{33}(\bar{y})}{\varepsilon^{2\tau}}\pi^2\phi^\tau_{\varepsilon,0}(\bar{y})\big(\phi^\tau_{\varepsilon,0}(\bar{y})|w(\bar{y},x_3)|^2\big)\,\mathrm{d}\bar{y}\bigg].
\end{aligned} \tag{3.2.4}$$

For a.e. $x_3 \in I$, let $w_{x_3} \in H^1_\#(Y)\cap L^\infty(Y)$ be the function defined by

$$w_{x_3}(\,\cdot\,) := \sum_{i=1}^{m_\varepsilon} w^2(\,\cdot\, + z_i, x_3),$$

where $z_i \in \mathbb{Z}^2$ are such that $Y_i = Y + z_i$, $i = 1,\cdots,m_\varepsilon$. Notice that $\phi^\tau_{\varepsilon,0}w_{x_3} \in H^1_\#(Y)$. Using the Y-periodicity of a_{ij} and $\phi^\tau_{\varepsilon,0}$, and using $\phi^\tau_{\varepsilon,0}\,w_{x_3}$ as a test function in the variational formulation of $\phi^\tau_{\varepsilon,0}$, we get for a.e. $x_3 \in I$:

$$\begin{aligned}
&\varepsilon^2\sum_{i=1}^{m_\varepsilon}\bigg[\int_{Y_i}\frac{1}{\varepsilon^2}\bar{A}(\bar{y})\bar\nabla\phi^\tau_{\varepsilon,0}(\bar{y})\bar\nabla\big(\phi^\tau_{\varepsilon,0}(\bar{y})|w(\bar{y},x_3)|^2\big) + \frac{a_{33}(\bar{y})}{\varepsilon^{2\tau}}\pi^2\phi^\tau_{\varepsilon,0}(\bar{y})\big(\phi^\tau_{\varepsilon,0}(\bar{y})|w(\bar{y},x_3)|^2\big)\,\mathrm{d}\bar{y}\bigg]\\
&\quad = \varepsilon^2\bigg[\int_Y\frac{1}{\varepsilon^2}\bar{A}(\bar{y})\bar\nabla\phi^\tau_{\varepsilon,0}(\bar{y})\bar\nabla\big(\phi^\tau_{\varepsilon,0}(\bar{y})\,w_{x_3}(\bar{y})\big) + \frac{a_{33}(\bar{y})}{\varepsilon^{2\tau}}\pi^2\phi^\tau_{\varepsilon,0}(\bar{y})\big(\phi^\tau_{\varepsilon,0}(\bar{y})w_{x_3}(\bar{y})\big)\,\mathrm{d}\bar{y}\bigg]\\
&\quad = \varepsilon^2\int_Y\frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}\phi^\tau_{\varepsilon,0}(\bar{y})\big(\phi^\tau_{\varepsilon,0}(\bar{y})w_{x_3}(\bar{y})\big)\,\mathrm{d}\bar{y} = \int_\omega\frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2|u(x)|^2\,\mathrm{d}\bar{x},
\end{aligned} \tag{3.2.5}$$

where in the last equality we used the definition of w_{x_3}, w and $\tilde{u}$, the Y-periodicity of $\phi^\tau_{\varepsilon,0}$ and the change of variables $\bar{x} := \varepsilon\bar{y}$.

From (3.2.4)–(3.2.5) we derive claim (3.2.3).

Step 2. We establish (3.2.2).

We start by proving that if $v \in H^1_0(\omega\times I)$, then the function u given by (3.2.1) belongs to $H^1_0(\omega\times I)$ and

$$\int_{\omega\times I}\bar{A}_\varepsilon(\bar{x})\bar\nabla v(x)\bar\nabla v(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|v(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v(x)|^2\,\mathrm{d}x \geqslant \int_{\omega\times I}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2\bar{A}_\varepsilon(\bar{x})\bar\nabla u(x)\bar\nabla u(x)\,\mathrm{d}x. \tag{3.2.6}$$

Fix $v \in H^1_0(\omega\times I)$. Since $\phi^\tau_{\varepsilon,0} \in H^1_\#(Y)\cap C^{0,s}_\#(Y)$ is strictly positive, the function

$$u(x) := \frac{v(x)}{\phi^\tau_{\varepsilon,0}\big(\frac{\bar{x}}{\varepsilon}\big)},\quad \text{a.e. } x = (\bar{x},x_3)\in\omega\times I,$$

is well defined and belongs to $L^2(\omega\times I)$. Moreover, $\nabla_3 u \in L^2(\omega\times I)$.

Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\omega\times I)$ such that $v_n \to v$ in $H_0^1(\omega\times I)$ as $n\to+\infty$. Setting $u_n := v_n/\phi^\tau_{\varepsilon,0}$, we have $u_n \to u$ and $\nabla_3 u_n \to \nabla_3 u$ in $L^2(\omega\times I)$ as $n\to+\infty$. Furthermore, for all $n\in\mathbb{N}$, $u_n \in H_0^1(\omega\times I)\cap L^\infty(\omega\times I)$, and so, by Step 1,

$$\begin{aligned}\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla} v_n(x)\bar{\nabla} v_n(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|v_n(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v_n(x)|^2\,\mathrm{d}x\\ = \int_{\omega\times I}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2\bar{A}_\varepsilon(\bar{x})\bar{\nabla} u_n(x)\bar{\nabla} u_n(x)\,\mathrm{d}x.\end{aligned}\tag{3.2.7}$$

The convergence $v_n\to v$ in $H_0^1(\omega\times I)$ as $n\to+\infty$ yields

$$\begin{aligned}\lim_{n\to+\infty}\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla} v_n(x)\bar{\nabla} v_n(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|v_n(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v_n(x)|^2\,\mathrm{d}x\\ = \int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla} v(x)\bar{\nabla} v(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|v(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v(x)|^2\,\mathrm{d}x,\end{aligned}\tag{3.2.8}$$

which, together with (3.2.7), implies that

$$\sup_{n\in\mathbb{N}}\bigg\{\int_{\omega\times I}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2\bar{A}_\varepsilon(\bar{x})\bar{\nabla} u_n(x)\bar{\nabla} u_n(x)\,\mathrm{d}x\bigg\} < +\infty.$$

Consequently, since there is a constant $c_\varepsilon > 0$ such that $\phi^\tau_{\varepsilon,0}(\cdot/\varepsilon) > c_\varepsilon$, from (3.1.1) we get $\sup_n \|\bar{\nabla} u_n\|_{L^2(\omega\times I;\mathbb{R}^2)} < +\infty$. Therefore, $u\in H_0^1(\omega\times I)$ and $u_n \rightharpoonup u$ weakly in $H_0^1(\omega\times I)$ as $n\to+\infty$.

Using the sequential lower semicontinuity with respect to the weak topology of $L^2(\omega\times I;\mathbb{R}^2)$ of the convex functional $F : L^2(\omega\times I;\mathbb{R}^2)\to\mathbb{R}$ defined by

$$F(w) := \int_{\omega\times I}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2\bar{A}_\varepsilon(\bar{x})w(x)w(x)\,\mathrm{d}x,\quad w\in L^2(\omega\times I;\mathbb{R}^2),$$

we conclude that

$$\liminf_{n\to+\infty}\int_{\omega\times I}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2\bar{A}_\varepsilon(\bar{x})\bar{\nabla} u_n(x)\bar{\nabla} u_n(x)\,\mathrm{d}x \geqslant \int_{\omega\times I}\Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2\bar{A}_\varepsilon(\bar{x})\bar{\nabla} u(x)\bar{\nabla} u(x)\,\mathrm{d}x.\tag{3.2.9}$$

From (3.2.7)–(3.2.9) we deduce (3.2.6).

Changing the roles between u and v we conclude that if $u\in H_0^1(\omega\times I)$ then v also belongs to $H_0^1(\omega\times I)$, and the converse of (3.2.6) holds true. □

Unfortunately, the lack of a positive uniform lower bound for $\{\phi^\tau_{\varepsilon,0}\}_{\varepsilon>0}$ when $\tau > 1$ will prevent us from using Proposition 2.5.26, and consequently Proposition 2.5.28, in the case $\varepsilon \gg \delta$. To treat this last case we will make use of an alternative result that shows that the spectrum $\sigma^\delta_\varepsilon$ associated with the tridimensional problem (3.1.5) equals a countable union of spectra associated with certain bidimensional problems.

Proposition 3.2.2. *Let $B\in M_S(\zeta,\eta,\omega)$ and let $b\in L^\infty(\omega)$ be such that $\zeta\leqslant b(\cdot)\leqslant\eta$ a.e. in ω. For each $n\in\mathbb{N}$, let $\big\{\lambda^{(n)}_k\big\}_{k\in\mathbb{N}}$ be the nondecreasing sequence formed by the eigenvalues associated with the bidimensional spectral problem*

$$\begin{cases} -\operatorname{div}(B(\bar{x})\bar{\nabla}\varphi_n) + b(\bar{x})\varsigma_n\varphi_n = \lambda_n\varphi_n & \text{a.e. } \bar{x}\in\omega,\\ \varphi_n\in H_0^1(\omega),\end{cases}\tag{3.2.10}$$

where each eigenvalue is repeated according to its multiplicity, and, we recall, (ς_n, θ_n) represents the n^{th} eigenpair of problem (3.1.7). Then $\{\lambda_k^{(n)}\}_{k,n\in\mathbb{N}}$ can be written as a nondecreasing sequence $\{\tilde{\lambda}_m\}_{m\in\mathbb{N}}$, where eigenvalues are repeated according to their multiplicity, which coincides with the spectral sequence of the tridimensional spectral problem

$$\begin{cases} -\operatorname{div}_{\bar{x}}(B(\bar{x})\bar{\nabla}v) - b(\bar{x})\Delta_3 v = \lambda v & \text{a.e. } (\bar{x}, x_3) \in \omega \times I, \\ v \in H_0^1(\omega \times I). \end{cases} \tag{3.2.11}$$

In particular, $\lambda_1 = \tilde{\lambda}_1 = \lambda_1^{(1)}$.

Proof. Denote by $(\lambda_k^{(n)}, \varphi_k^{(n)})$ a $L^2(\omega)$-normalized k^{th} eigenpair of problem (3.2.10). It can be checked that

(1) the family of functions $\{v_k^{(n)} := \varphi_k^{(n)}(\bar{x})\theta_n(x_3) \colon n \in \mathbb{N}, k \in \mathbb{N}\}$ forms an orthonormal basis of $L^2(\omega \times I)$;

2) for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $(\lambda_k^{(n)}, v_k^{(n)})$ is an eigenpair of (3.2.11).

Furthermore, in view of Theorem 2.4.30 applied to $V := H_0^1(\omega \times I)$, $H := L^2(\omega \times I)$, and $a(\cdot,\cdot) : H_0^1(\omega \times I) \times H_0^1(\omega \times I) \to \mathbb{R}$ defined by

$$a(u, v) := \int_{\omega\times I} B(\bar{x})\bar{\nabla}u(x)\bar{\nabla}v(x) + b(\bar{x})\nabla_3 u(x)\nabla_3 v(x)\,\mathrm{d}x = \int_{\omega\times I} C(x)\nabla u(x)\nabla u(x)\,\mathrm{d}x$$

for $u, v \in H_0^1(\omega \times I)$, where $C = (c_{ij})_{1\leqslant i,j\leqslant 3} \in M_S(\zeta, \eta, \omega \times I)$ is the matrix given by $(c_{\alpha\beta}) := B$, $c_{\alpha 3} := 0$, and $c_{33} := b$, we conclude that all eigenvalues of (3.2.11) belong to $\{\lambda_k^{(n)}\}_{k,n\in\mathbb{N}}$. This completes the proof. □

3.3. Proof of Theorem 3.1.1 ($\varepsilon \approx \delta$).

In this section we prove Theorem 3.1.1. Let us recall that (μ_0, ϕ_0) is the first $L^2(Y)$-normalized eigenpair for problem (3.1.6) with $\tau = 1$, while $(\varsigma_1, \theta_1) = (\pi^2, \sqrt{2}\cos(\pi x_3))$ is the first $L^2(I)$-normalized eigenpair for problem (3.1.7). Since we are expecting the asymptotic behavior mentioned in (3.1.8) for the shifted spectrum $\sigma_\varepsilon - \frac{\mu_0}{\varepsilon^2}$, instead of the energy defined in (3.1.4) for $\delta = \varepsilon$, we consider the functional $I_\varepsilon : L^2(\omega \times I) \to [0, +\infty]$, defined by

$$I_\varepsilon(v) := \begin{cases} \displaystyle\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla}v(x)\bar{\nabla}v(x) + \frac{a_{33}^\varepsilon(\bar{x})}{\varepsilon^2}|\nabla_3 v(x)|^2 - \frac{\mu_0}{\varepsilon^2}|v(x)|^2\,\mathrm{d}x & \text{if } v \in H_0^1(\omega \times I), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Using Proposition 3.2.1 with $\tau = 1$, we conclude that $I_\varepsilon(v) = G_\varepsilon(u)$, where $G_\varepsilon : L^2(\omega \times I) \to [0, +\infty]$ is the functional given by

$$G_\varepsilon(u) := \begin{cases} \displaystyle\int_{\omega\times I} \bar{B}_\varepsilon(\bar{x})\bar{\nabla}u(x)\bar{\nabla}u(x) + \frac{b_{33}^\varepsilon(\bar{x})}{\varepsilon^2}\Big(|\nabla_3 u(x)|^2 - \pi^2|u(x)|^2\Big)\,\mathrm{d}x & \text{if } u \in H_0^1(\omega \times I), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3.2}$$

where, a.e. $\bar{x} \in \omega$,

$$\bar{B}_\varepsilon(\bar{x}) := \big(b_{\alpha\beta}^\varepsilon(\bar{x})\big) \in \mathbb{M}^{2\times 2}, \quad b_{\alpha\beta}^\varepsilon(\bar{x}) := \Big|\phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 a_{\alpha\beta}\Big(\frac{\bar{x}}{\varepsilon}\Big), \quad b_{33}^\varepsilon(\bar{x}) := \Big|\phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 a_{33}\Big(\frac{\bar{x}}{\varepsilon}\Big).$$

Remark 3.3.1. *Notice that since ϕ_0 belongs to $H_\#^1(Y) \cap C_\#^{0,s}(Y)$ for some $0 < s < 1$, and is strictly positive, we have that $B_\varepsilon := |\phi_0(\frac{\cdot}{\varepsilon})|^2 A_\varepsilon$ is an εY-periodic matrix belonging to $M_S(\bar{\zeta}, \bar{\eta}, \omega \times I)$ for* some $\bar{\zeta}, \bar{\eta} \in \mathbb{R}$ *independent of* ε *and such that* $0 < \bar{\zeta} < \bar{\eta}$. *In particular, for all* $\xi \in \mathbb{R}^3$ *and for a.e.* $x \in \omega$, *one has*

$$\inf_{\varepsilon>0} (B_\varepsilon(\bar{x})\xi|\xi) \geqslant \bar{\zeta}\|\xi\|^2. \tag{3.3.3}$$

In order to prove Theorem 3.1.1, we start by showing that that the sequence $\{G_\varepsilon\}_{\varepsilon>0}$ satisfies the hypotheses of Proposition 2.5.28.

Proposition 3.3.2. *Let* G_ε *be the functional in (3.3.2). Then the sequence* $\{G_\varepsilon\}_{\varepsilon>0}$ Γ*-converges in* $L^2(\omega \times I)$ *as* $\varepsilon \to 0^+$ *to the functional* $G : L^2(\omega \times I) \to [0, +\infty]$ *defined by*

$$G(u) := \begin{cases} \displaystyle\int_\omega \bar{B}^h \bar{\nabla}\varphi(\bar{x})\bar{\nabla}\varphi(\bar{x})\,\mathrm{d}\bar{x} & \text{if } u(\bar{x}, x_3) = \varphi(\bar{x})\,\theta_1(x_3),\ \varphi \in H_0^1(\omega), \\ +\infty & \text{otherwise} \end{cases}$$

for $u \in L^2(\omega \times I)$, *where the constant matrix* $\bar{B}^h$ *is the homogenized limit of the sequence* $\{\bar{B}_\varepsilon\}_{\varepsilon>0}$. *Moreover,* G_ε *also satisfies conditions (i) and (ii) in Proposition 2.5.28.*

Proof. We will proceed in two steps (see Remark 2.5.2).

Step 1. We prove that if $u_\varepsilon, u \in L^2(\omega \times I)$ are such that $u_\varepsilon \to u$ in $L^2(\omega \times I)$ as $\varepsilon \to 0^+$, then $G(u) \leqslant \liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon)$. Furthermore, conditions (i) and (ii) in Proposition 2.5.28 are satisfied.

We start by observing that if $w \in H_0^1(\omega\times I)$, then for a.e. $\bar{x} \in \omega$, $w(\bar{x}, \cdot\,) \in H_0^1(I)$. Thus, since $\varsigma_1 = \pi^2$ is the first eigenvalue associated with problem (3.1.7), we have, a.e. $\bar{x} \in \omega$,

$$\int_I \left(|\nabla_3 w|^2 - \pi^2|w|^2\right) \mathrm{d}x_3 \geqslant 0. \tag{3.3.4}$$

This and (3.3.3) ensure that $G_\varepsilon \geqslant 0$ in $L^2(\omega\times I)$. Hence, condition (i) in Proposition 2.5.28 is satisfied.

Let $u_\varepsilon, u \in L^2(\omega \times I)$ be as in the statement of Step 1. Up to a subsequence (which we will not relabel), we may assume without loss of generality that

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon) < +\infty.$$

Then $\{u_\varepsilon\}_{\varepsilon>0} \subset H_0^1(\omega \times I)$ and $\sup_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty$. Using (3.3.4), (3.3.3) and the uniform bound of $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^2(\omega \times I)$, we get

$$\int_{\omega\times I} |\bar{\nabla} u_\varepsilon|^2\,\mathrm{d}x \leqslant C, \quad \int_{\omega\times I} |\nabla_3 u_\varepsilon|^2\,\mathrm{d}x \leqslant C\,\varepsilon^2 + \pi^2 \int_{\omega\times I} |u_\varepsilon|^2\,\mathrm{d}x \leqslant \overline{C}, \tag{3.3.5}$$

where C and $\overline{C}$ are constants independent of ε. Consequently, $\sup_\varepsilon \|u_\varepsilon\|_{H_0^1(\omega\times I)} < +\infty$ and $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\omega \times I)$ as $\varepsilon \to 0^+$. The sequential lower semicontinuity of the L^2-norm with respect to the weak topology and (3.3.5) yield

$$\int_{\omega\times I} \left(|\nabla_3 u|^2 - \pi^2|u|^2\right) \mathrm{d}x \leqslant 0.$$

Hence, taking into account (3.3.4), $\int_I (|\nabla_3 u|^2 - \pi^2|u|^2)\,\mathrm{d}x_3 = 0$ for a.e. $\bar{x} \in \omega$, from which we deduce that there is a function $\varphi \in H_0^1(\omega)$ such that $u(\bar{x}, x_3) = \varphi(\bar{x})\,\theta_1(x_3)$ for a.e. $(\bar{x}, x_3) \in \omega \times I$.

Using Fubini's Theorem, Fatou's Lemma, Proposition 2.5.26 (see also Remark 3.3.1) and the condition $\|\theta_1\|_{L^2(I)} = 1$, we obtain

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon) \geqslant \liminf_{\varepsilon\to 0^+} \int_{\omega\times I} \bar{B}_\varepsilon(\bar{x})\bar{\nabla}u_\varepsilon(x)\bar{\nabla}u_\varepsilon(x)\,\mathrm{d}x \geqslant \int_I \left[\int_\omega \bar{B}^h \bar{\nabla}u(x)\bar{\nabla}u(x)\,\mathrm{d}\bar{x}\right] \mathrm{d}x_3 = G(u).$$

Finally, to conclude Step 1, we observe that if $\sup_\varepsilon G_\varepsilon(u_\varepsilon) < +\infty$ and $\sup_\varepsilon \|u_\varepsilon\|_{L^2(\omega\times I)} < +\infty$, then (3.3.5) holds. Consequently, condition (ii) in Proposition 2.5.28 is also satisfied.

Step 2. We prove that for any $u \in L^2(\omega\times I)$ there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(\omega\times I)$ satisfying $u_\varepsilon \to u$ in $L^2(\omega\times I)$ as $\varepsilon\to 0^+$ and $G(u) = \lim\limits_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon)$.

Given $u \in L^2(\omega\times I)$, the only nontrivial case is when $u(\bar{x},x_3) = \varphi(\bar{x})\,\theta_1(x_3)$ with $\varphi\in H^1_0(\omega)$, otherwise, considering Step 1, it is enough to take $u_\varepsilon \equiv u$.

By Proposition 2.5.26, there exists a sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}\subset H^1_0(\omega)$ converging to φ in $L^2(\omega)$ and such that

$$\lim_{\varepsilon\to 0^+}\int_\omega \bar{B}_\varepsilon(\bar{x})\bar{\nabla}\varphi_\varepsilon(\bar{x})\bar{\nabla}\varphi_\varepsilon(\bar{x})\,\mathrm{d}\bar{x} = \int_\omega \bar{B}^h\bar{\nabla}\varphi(\bar{x})\bar{\nabla}\varphi(\bar{x})\,\mathrm{d}\bar{x}.$$

Recalling that $\int_I \left(|\theta_1'|^2 - \pi^2|\theta_1|^2\right)\mathrm{d}x_3 = 0$, in order to obtain the intended equality it suffices to define $u_\varepsilon(\bar{x},x_3) := \varphi_\varepsilon(\bar{x})\,\theta_1(x_3)$. This concludes Step 2 as well as the proof of Proposition 3.3.2. □

We now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let H_ε be the Hilbert space $H := L^2(\omega\times I)$ endowed with the scalar product $(\,\cdot\,|\,\cdot\,)_\varepsilon$, where

$$(u|v)_\varepsilon := \int_{\omega\times I}\left|\phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)\right|^2 u(x)v(x)\,\mathrm{d}x, \quad u,v\in L^2(\omega\times I).$$

Since $\phi_0 \in H^1_\#(Y)\cap C^{0,s}_\#(Y)$ is a strictly positive function, there exist positive constants c_1 and c_2 with $0<c_1<c_2$ and such that for all $\bar{y}\in Y$ we have $c_1 < \phi_0(\bar{y}) < c_2$. Moreover, by Riemann-Lebesgue's Lemma,

$$\left|\phi_0\Big(\frac{\cdot}{\varepsilon}\Big)\right|^2 \overset{\star}{\rightharpoonup} \int_Y \left|\phi_0(\bar{y})\right|^2\mathrm{d}\bar{y} = 1 \;\text{ weakly-}\star \text{ in } L^\infty(\mathbb{R}^2) \text{ as } \varepsilon\to 0^+.$$

Hence conditions (2.5.3) and (2.5.4) hold. On the other hand, for each $\varepsilon>0$, G_ε defined in (3.3.2) is a nonnegative lower semicontinuous quadratic form in $L^2(\omega\times I)$ (see Proposition 2.4.24). Consequently, in view of Theorem 2.4.22 and using Riesz Representation Theorem, the associated unbounded linear operator in H_ε, $\mathcal{A}_\varepsilon : D(\mathcal{A}_\varepsilon)\subset H_\varepsilon\to H_\varepsilon$, is a densely defined self-adjoint operator in H_ε. We further observe that $D(\mathcal{A}_\varepsilon)$ is also dense in $H^1_0(\omega\times I)$. Let $\{(\nu_{\varepsilon,k},u_{\varepsilon,k})\}_{k\in\mathbb{N}}$ and $\{(\nu_k,\varphi_k)\}_{k\in\mathbb{N}}$ be such that

$$u_{\varepsilon,k}\in H^1_0(\omega\times I),\qquad \mathcal{A}_\varepsilon u_{\varepsilon,k} = \nu_{\varepsilon,k}u_{\varepsilon,k},\qquad \nu_{\varepsilon,1}\leqslant\nu_{\varepsilon,2}\leqslant\cdots\leqslant\nu_{\varepsilon,k}\leqslant\cdots,\quad (u_{\varepsilon,k}|u_{\varepsilon,l})_\varepsilon = \delta_{kl},$$

$$\varphi_k\in H^1_0(\omega),\qquad -\operatorname{div}_{\bar{x}}(\bar{B}^h\bar{\nabla}\varphi_k) = \nu_k\varphi_k,\qquad \nu_1\leqslant\nu_2\leqslant\cdots\leqslant\nu_k\leqslant\cdots,\qquad (\varphi_k|\varphi_l) = \delta_{kl},$$

where $(\cdot|\cdot)$ represents the standard scalar product in $L^2(\omega)$.

By Propositions 2.5.28 and 3.3.2 (see also Propositions 2.4.26 and 2.5.9), $\nu_{\varepsilon,k}\to\nu_k$ as $\varepsilon\to 0^+$. Moreover, up to a subsequence that we do not relabel, $u_{\varepsilon,k}\rightharpoonup u_k$ weakly in $H^1_0(\omega\times I)$ as $\varepsilon\to 0^+$, where u_k is the product between an eigenfunction associated with ν_k and θ_1. Conversely, any eigenfunction $u_k = \varphi_k\theta_1$ is the weak limit of a particular sequence of eigenfunctions associated with $\nu_{\varepsilon,k}$.

To finish the proof of Theorem 3.1.1 we are left to show that (3.1.8) holds. Considering for each $k\in\mathbb{N}$, $\mu_k\in\mathbb{R}$ and functions w_k and $\tilde{w}_k$ such that

$$w_k(x) = \phi_0\Big(\frac{\bar{x}}{\varepsilon}\Big)\tilde{w}_k(x),\;\text{ a.e. } x=(\bar{x},x_3)\in\omega\times I,$$

Proposition 3.2.1 implies that w_k belongs to $H^1_0(\omega\times I)$ if, and only if, $\tilde{w}_k$ belongs to $H^1_0(\omega\times I)$, and also that the equalities

$$G_\varepsilon(\tilde{w}_k) = (\mathcal{A}_\varepsilon\tilde{w}_k|\tilde{w}_k)_\varepsilon = \mu_k(\tilde{w}_k|\tilde{w}_k)_\varepsilon,\quad (\tilde{w}_k|\tilde{w}_l)_\varepsilon = \delta_{kl}$$

hold true if, and only if, the equalities

$$I_\varepsilon(w_k) = \Big(-\operatorname{div}_{\bar{x}}(\bar{A}_\varepsilon \bar{\nabla} w_k) - \frac{a_{33}^\varepsilon}{\varepsilon^2}\Delta_3 w_k - \frac{\mu_0}{\varepsilon^2} w_k \Big| w_k \Big) = \mu_k (w_k|w_k), \quad (w_k|w_l) = \delta_{kl}$$

are satisfied, where $(\cdot|\cdot)$ represents the standard scalar product in $L^2(\omega \times I)$ and I_ε is the functional in (3.3.1). Replacing μ_k by $\nu_{\varepsilon,k}$, w_k by $v_{\varepsilon,k}$ and $\tilde{w}_k$ by $u_{\varepsilon,k}$, we conclude the proof of (3.1.8). □

3.4. Proof of Theorem 3.1.2 ($\varepsilon \ll \delta$)

This section is devoted to the proof of Theorem 3.1.2. The arguments are similar to those of Theorem 3.1.1, however, in this case problem (3.1.6) does depend on ε; this compels us to study the asymptotic behavior of its first $L^2(Y)$-normalized eigenpair $(\mu^\tau_{\varepsilon,0}, \phi^\tau_{\varepsilon,0})$ as $\varepsilon \to 0^+$. Throughout this section we assume that $\tau \in (0,1)$ is fixed, and that $\delta = \varepsilon^\tau$.

Proposition 3.4.1. *Assume that, in addition to the hypotheses made at the beginning of Chapter 3, $a_{\alpha\beta}$ are uniformly Lipschitz continuous in Y. Let $\{(\varrho_j, \psi_j)\}_{j\in\mathbb{N}_0}$ be given by (3.1.9)–(3.1.10), and let $i \in \mathbb{N}$ be such that $\frac{i-1}{i} < \tau \leqslant \frac{i}{i+1}$. Then $\phi^\tau_{\varepsilon,0}(\cdot/\varepsilon) \to 1 \equiv \psi_0$ uniformly in ω as $\varepsilon \to 0^+$, and $\mu^\tau_{\varepsilon,0}$ behaves as follows:*

$$\mu^\tau_{\varepsilon,0} = \varrho_0 + \varepsilon^{2(1-\tau)}\varrho_1 + \cdots + \varepsilon^{2i(1-\tau)}\varrho_i + o\big(\varepsilon^{2i(1-\tau)}\big). \tag{3.4.1}$$

Proof. Let us start by proving that $\mu^\tau_{\varepsilon,0} \to \varrho_0 = \pi^2 \int_Y a_{33}(\bar{y})\,\mathrm{d}\bar{y} > 0$ as $\varepsilon \to 0^+$, and that all the other eigenvalues of problem (3.1.6) tend to $+\infty$ as $\varepsilon \to 0^+$. By Rayleigh's formula for $\mu^\tau_{\varepsilon,0}$ (see Theorem 2.4.30-(v)),

$$\mu^\tau_{\varepsilon,0} = \min_{\substack{\phi\in H^1_\#(Y) \\ \|\phi\|_{L^2(Y)}=1}} \left\{ \int_Y \frac{1}{\varepsilon^{2(1-\tau)}} \bar{A}(\bar{y})\bar{\nabla}\phi(\bar{y})\bar{\nabla}\phi(\bar{y}) + a_{33}(\bar{y})\pi^2|\phi(\bar{y})|^2\,\mathrm{d}\bar{y} \right\}. \tag{3.4.2}$$

Using (3.1.1) and $\phi \equiv 1$ as a test function in (3.4.2), we conclude that $\zeta\pi^2 \leqslant \mu^\tau_{\varepsilon,0} \leqslant \varrho_0$. In particular,

$$\limsup_{\varepsilon\to 0^+} \mu^\tau_{\varepsilon,0} \leqslant \varrho_0.$$

Since $\phi^\tau_{\varepsilon,0}$ is a minimizer for $\mu^\tau_{\varepsilon,0}$, using again (3.1.1) we deduce that $\|\bar{\nabla}\phi^\tau_{\varepsilon,0}\|_{L^2(Y)} \to 0$ as $\varepsilon \to 0^+$. Consequently, $\phi^\tau_{\varepsilon,0} \to 1$ in $H^1_\#(Y)$ as $\varepsilon \to 0^+$. In turn, this implies

$$\liminf_{\varepsilon\to 0^+} \mu^\tau_{\varepsilon,0} \geqslant \liminf_{\varepsilon\to 0^+} \int_Y a_{33}(\bar{y})\pi^2|\phi^\tau_{\varepsilon,0}(\bar{y})|^2\,\mathrm{d}\bar{y} = \varrho_0.$$

Therefore, $\mu^\tau_{\varepsilon,0} \to \varrho_0$ as $\varepsilon \to 0^+$.

Similarly, using Rayleigh's formula for $\mu_{\varepsilon,1}$ and admitting that the latter is bounded, we are led to a contradiction, since we would conclude that any minimizing sequence of eigenfunctions must converge on the one hand to the constant function $\psi_0 \equiv 1$ and on the other hand to a function having zero mean (by the orthogonality condition). So, except for the first, all the eigenvalues of problem (3.1.6) tend to $+\infty$ as $\varepsilon \to 0^+$.

We now prove the statement on the asymptotic behavior of $\phi^\tau_{\varepsilon,0}$. If, in addition, we suppose that $a_{\alpha\beta}$ are uniformly Lipschitz continuous in Y, then (see Gilbarg and Trudinger [52, Thm. 8.8]) $\{\phi^\tau_{\varepsilon,0}\}_{\varepsilon>0}$ is uniformly bounded in $H^2(Y)$. Due to the compact injection of $H^2(Y)$ in $C(\overline{Y})$, we conclude that

$\phi^\tau_{\varepsilon,0}(\bar{y}) \to 1$ uniformly in $\overline{Y}$ as $\varepsilon \to 0^+$. Finally, the Y-periodicity of $\phi^\tau_{\varepsilon,0}$ ensures that $\phi^\tau_{\varepsilon,0}(\cdot/\varepsilon) \to 1$ uniformly in ω as $\varepsilon \to 0^+$.

We are left to establish (3.4.1), for which we will base ourselves on the Vishik-Lyusternik Lemma (i.e., Lemma 2.4.17). For the sake of simplicity we will present the proof only for $i = 1$, the argument being easily generalized for $i > 1$.

We start by setting $\epsilon := \varepsilon^{2(1-\tau)}$, $\mu_\epsilon := \mu^\tau_{\varepsilon,0}$, $b(\bar{y}) := \pi^2 a_{33}(\bar{y})$, and we define the unbounded linear operator $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \subset L^2_\#(Y) \to L^2_\#(Y)$ in $L^2_\#(Y)$ by

$$\mathcal{A}_\epsilon \varphi := -\frac{1}{\epsilon}\,\mathrm{div}(\bar{A}(\bar{y})\bar{\nabla}\varphi) + b(\bar{y})\varphi$$

for

$$\varphi \in D(\mathcal{A}_\epsilon) := \big\{\varphi \in H^1_\#(Y)\colon \mathrm{div}(\bar{A}\bar{\nabla}\varphi) \in L^2_\#(Y)\big\}.$$

Then (see Subsection 2.4.2), $\mathcal{A}_\epsilon^{-1}$ belongs to $\mathfrak{L}(L^2_\#(Y))$ and is a compact self-adjoint operator in $L^2_\#(Y)$. We then apply Lemma 2.4.17 to $H := L^2_\#(Y)$, to the operator $\mathcal{A}_\epsilon^{-1}$, to the real number $\gamma_\epsilon := (\varrho_0 + \epsilon\varrho_1)^{-1}$, and to the function $f_\epsilon := \tilde{f}_\epsilon / \|\tilde{f}_\epsilon\|_H$, where $\tilde{f}_\epsilon := \mathcal{A}_\epsilon \psi_\epsilon$ with $\psi_\epsilon := \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2$. Observe that $\|\tilde{f}_\epsilon\|_H \to \varrho_0$ as $\epsilon \to 0^+$ and recall that $\varrho_0 > 0$.

Since $\mathcal{A}_\epsilon^{-1} f_\epsilon - \gamma_\epsilon f_\epsilon = \psi_\epsilon - \gamma_\epsilon \mathcal{A}_\epsilon \psi_\epsilon =: w_\epsilon$, using (3.1.9)–(3.1.10) we conclude that

$$w_\epsilon = \Big(\epsilon^2\big((b - \varrho_0)\psi_2 - \varrho_1\psi_1\big) - \epsilon^3\varrho_1\psi_2\Big)(\varrho_0 + \epsilon\varrho_1)^{-1}.$$

Using the condition $\varrho_0 > 0$, we deduce that for all $\epsilon > 0$ small enough and for a constant c independent of ϵ, $\|w_\epsilon\|_H \leqslant \epsilon^2 c$. Consequently,

$$\|\mathcal{A}_\epsilon^{-1}\tilde{f}_\epsilon - \gamma_\epsilon \tilde{f}_\epsilon\|_H = \frac{\|w_\epsilon\|_H}{\|\tilde{f}_\epsilon\|_H} \leqslant \epsilon^2 c$$

for all $\epsilon > 0$ small enough and for some other constant c independent of ϵ. For any such ϵ, Lemma 2.4.17 yields the existence of an eigenvalue $\bar{\lambda}_\epsilon$ of $\mathcal{A}_\epsilon^{-1}$ satisfying $|\bar{\lambda}_\epsilon - (\varrho_0 + \epsilon\varrho_1)^{-1}| \leqslant \epsilon^2 c$. Since all the eigenvalues of $\mathcal{A}_\epsilon^{-1}$ tend to zero, except for the first, which converges to $\varrho_0^{-1} > 0$, we conclude that for all ϵ small enough, $\bar{\lambda}_\epsilon = \mu_\epsilon^{-1}$. Hence,

$$|\mu_\epsilon - (\varrho_0 + \epsilon\varrho_1)| \leqslant \epsilon^2 c,$$

for some other constant c independent of ϵ. This concludes the proof for $i = 1$. □

As it was already mentioned, the main ideas of the proof of Theorem 3.1.2 are those of Theorem 3.1.1. We are expecting the asymptotic behavior referred in (3.1.12) for the shifted spectrum $\sigma_\varepsilon - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}$ (see also (3.4.1)), and so instead of the energy defined in (3.1.4) for $\delta = \varepsilon^\tau$, we consider the functional $I^\tau_\varepsilon : L^2(\omega \times I) \to [0, +\infty]$ defined by

$$I^\tau_\varepsilon(v) := \begin{cases} \displaystyle\int_{\omega\times I} \bar{A}_\varepsilon(\bar{x})\bar{\nabla}v(x)\bar{\nabla}v(x) + \frac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}|\nabla_3 v(x)|^2 - \frac{\mu^\tau_{\varepsilon,0}}{\varepsilon^{2\tau}}|v(x)|^2\,\mathrm{d}x & \text{if } v \in H^1_0(\omega \times I),\\ +\infty & \text{otherwise}\end{cases}$$

for $v \in L^2(\omega \times I)$. By Proposition 3.2.1, we have that $I^\tau_\varepsilon(v) = G^\tau_\varepsilon(u)$, where $G^\tau_\varepsilon : L^2(\omega \times I) \to [0, +\infty]$ is the functional given by

$$G^\tau_\varepsilon(u) := \begin{cases} \displaystyle\int_{\omega\times I} \bar{B}^\tau_\varepsilon(\bar{x})\bar{\nabla}u(x)\bar{\nabla}u(x) + \frac{b^{\tau,\varepsilon}_{33}(\bar{x})}{\varepsilon^2}\Big(|\nabla_3 u(x)|^2 - \pi^2|u(x)|^2\Big)\,\mathrm{d}x & \text{if } u \in H^1_0(\omega \times I),\\ +\infty & \text{otherwise}\end{cases} \tag{3.4.3}$$

for $u \in L^2(\omega \times I)$, where for a.e. $\bar{x} \in \omega$,

$$\bar{B}^\tau_\varepsilon(\bar{x}) := \big(b^{\tau,\varepsilon}_{\alpha\beta}(\bar{x})\big) \in \mathbb{M}^{2\times 2}, \quad b^{\tau,\varepsilon}_{\alpha\beta}(\bar{x}) := \Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 a_{\alpha\beta}\Big(\frac{\bar{x}}{\varepsilon}\Big), \quad b^{\tau,\varepsilon}_{33}(\bar{x}) := \Big|\phi^\tau_{\varepsilon,0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 a_{33}\Big(\frac{\bar{x}}{\varepsilon}\Big).$$

The analogue to Proposition 3.3.2 reads as follows.

Proposition 3.4.2. *Let G^τ_ε be the functional in (3.4.3). Then the sequence $\{G^\tau_\varepsilon\}_{\varepsilon>0}$ Γ-converges in $L^2(\omega \times I)$ as $\varepsilon \to 0^+$ to the functional $G^\tau : L^2(\omega\times I) \to [0,+\infty]$ defined by*

$$G^\tau(u) := \begin{cases} \displaystyle\int_\omega \bar{A}^h \bar{\nabla}\varphi(\bar{x})\bar{\nabla}\varphi(\bar{x})\,\mathrm{d}\bar{x} & \text{if } u(\bar{x},x_3) = \varphi(\bar{x})\,\theta_1(x_3),\ \varphi \in H^1_0(\omega),\\ +\infty & \text{otherwise}\end{cases}$$

for $u \in L^2(\omega\times I)$, where $\bar{A}^h$ is the homogenized limit of the sequence $\{\bar{A}_\varepsilon\}_{\varepsilon>0}$. Moreover, G^τ_ε also satisfies conditions (i) and (ii) in Proposition 2.5.28.

Proof. The proof is very similar to that of Proposition 3.3.2, and so we only outline the main differences.

Step 1. We prove that if $u_\varepsilon, u \in L^2(\omega\times I)$ are such that $u_\varepsilon \to u$ in $L^2(\omega \times I)$ as $\varepsilon \to 0^+$, then $G^\tau(u) \leqslant \liminf_{\varepsilon\to 0^+} G^\tau_\varepsilon(u_\varepsilon)$. Furthermore, conditions (i) and (ii) in Proposition 2.5.28 are satisfied.

Without loss of generality we may assume that $\liminf_{\varepsilon\to0^+} G^\tau_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to0^+} G^\tau_\varepsilon(u_\varepsilon) < +\infty$. Then, using (3.1.1) and the uniform convergence $\phi^\tau_{\varepsilon,0}(\cdot/\varepsilon) \to 1$ in ω as $\varepsilon \to 0^+$ (see Proposition 3.4.1), we conclude that (3.3.5) holds. Consequently, $u_\varepsilon \rightharpoonup u$ weakly in $H^1_0(\omega\times I)$ as $\varepsilon \to 0^+$, where $u(\bar{x},x_3) = \varphi(\bar{x})\theta_1(x_3)$ for some $\varphi \in H^1_0(\omega)$, a.e. $(\bar{x},x_3) \in \omega \times I$.

Fix $0 < \gamma < 1$. Then for all ε sufficiently small, $\big|\phi^\tau_{\varepsilon,0}(\cdot/\varepsilon)\big|^2 \geqslant 1-\gamma$. Therefore, Fubini's Theorem, (3.1.1), Fatou's Lemma, Proposition 2.5.26 and the condition $\|\theta_1\|_{L^2(I)} = 1$ ensure that

$$\begin{aligned}\liminf_{\varepsilon\to0^+} G^\tau_\varepsilon(u_\varepsilon) &\geqslant \liminf_{\varepsilon\to0^+}\int_{\omega\times I} \bar{B}^\tau_\varepsilon(\bar{x})\bar{\nabla}u_\varepsilon(x)\bar{\nabla}u_\varepsilon(x)\,\mathrm{d}x\\ &\geqslant (1-\gamma)\int_I\Big[\liminf_{\varepsilon\to0^+}\int_\omega \bar{A}_\varepsilon(\bar{x})\bar{\nabla}u_\varepsilon(x)\bar{\nabla}u_\varepsilon(x)\,\mathrm{d}\bar{x}\Big]\mathrm{d}x_3 \geqslant (1-\gamma)\,G^\tau(u),\end{aligned}$$

from which we conclude that $G^\tau(u) \leqslant \liminf_{\varepsilon\to0^+} G^\tau_\varepsilon(u_\varepsilon)$ by letting $\gamma \to 0^+$.

To prove that G^τ_ε satisfies conditions (i) and (ii) in Proposition 2.5.28 it suffices to repeat the corresponding arguments in Step 1 of Proposition 3.3.2. This concludes Step 1.

Step 2. We prove that for any $u \in L^2(\omega\times I)$ there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(\omega\times I)$ satisfying $u_\varepsilon \to u$ in $L^2(\omega\times I)$ as $\varepsilon\to0^+$, and $G^\tau(u) = \lim_{\varepsilon\to0^+} G^\tau_\varepsilon(u_\varepsilon)$.

Given $u \in L^2(\omega \times I)$, the only nontrivial case is when $u(\bar{x},x_3) = \varphi(\bar{x})\,\theta_1(x_3)$ for some $\varphi \in H^1_0(\omega)$, otherwise, considering Step 1, it's enough to take $u_\varepsilon \equiv u$.

By Proposition 2.5.26, there exists a sequence $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset H^1_0(\omega)$ converging in $L^2(\omega)$ to φ and such that

$$\lim_{\varepsilon\to0^+}\int_\omega \bar{A}_\varepsilon(\bar{x})\bar{\nabla}\varphi_\varepsilon(\bar{x})\bar{\nabla}\varphi_\varepsilon(\bar{x})\,\mathrm{d}\bar{x} = \int_\omega \bar{A}^h\bar{\nabla}\varphi(\bar{x})\bar{\nabla}\varphi(\bar{x})\,\mathrm{d}\bar{x}. \tag{3.4.4}$$

Fix $\gamma > 0$. Let $\varepsilon_0 > 0$ be such that for all $0 < \varepsilon \leqslant \varepsilon_0$, $\big|\phi^\tau_{\varepsilon,0}(\cdot/\varepsilon)\big|^2 \leqslant 1+\gamma$. Define $u_\varepsilon(\bar{x},x_3) := \varphi_\varepsilon(\bar{x})\,\theta_1(x_3)$. Recalling that $\int_I \big(|\theta_1'|^2 - \pi^2|\theta_1|^2\big)\,\mathrm{d}x_3 = 0$, from (3.4.4) and (3.1.1) we conclude that

$$\limsup_{\varepsilon\to0^+} G^\tau_\varepsilon(u_\varepsilon) = \limsup_{\varepsilon\to0^+}\int_\omega \bar{B}^\tau_\varepsilon(\bar{x})\bar{\nabla}\varphi_\varepsilon(x)\bar{\nabla}\varphi_\varepsilon(x)\,\mathrm{d}x \leqslant (1+\gamma)\int_\omega \bar{A}^h\bar{\nabla}\varphi(\bar{x})\bar{\nabla}\varphi(\bar{x})\,\mathrm{d}\bar{x} = (1+\gamma)\,G^\tau(u).$$

Letting $\gamma \to 0^+$ and using Step 1, we conclude Step 2 as well as the proof of Proposition 3.4.2. $\square$

Proof of Theorem 3.1.2. Replacing G_ε by G_ε^τ, (μ_0, ϕ_0) by $(\mu_{\varepsilon,0}^\tau, \psi_{\varepsilon,0}^\tau)$, and recalling Propositions 3.4.1 and 3.4.2, the proof of Theorem 3.1.2 is analogous to that of Theorem 3.1.1. $\square$

3.5. Proof of Theorems 3.1.4 and 3.1.7 ($\varepsilon \gg \delta$)

Throughout this section we assume that $\tau \in (1, +\infty)$ is fixed, and that $\delta = \varepsilon^\tau$. As we mentioned before, the lack of a positive uniform lower bound for $\{\phi_{\varepsilon,0}^\tau\}_{\varepsilon>0}$ will prevent us from using Proposition 2.5.28. So, in order to prove Theorems 3.1.4 and 3.1.7, we will take advantage essentially of Propositions 3.2.1 and 3.2.2, and of the asymptotic behavior of the eigenpair $(\mu_{\varepsilon,0}^\tau, \phi_{\varepsilon,0}^\tau)$ introduced in (3.1.6), which is the aim of the following lemmas.

To simplify the statements and the proof of the lemmas, we introduce some notations:

$$b := (a_{33} - a_{\min})\pi^2, \quad \epsilon := \varepsilon^{\tau-1}, \quad \mu_\epsilon := \frac{\mu_{\varepsilon,0}^\tau - a_{\min}\pi^2}{\varepsilon^{2(\tau-1)}}, \quad \phi_\epsilon := \phi_{\varepsilon,0}^\tau.$$

Problem (3.1.6) then reads

$$\begin{cases} -\operatorname{div}(\bar{A}\bar{\nabla}\phi_\epsilon) + \dfrac{b}{\epsilon^2}\phi_\epsilon = \mu_\epsilon\phi_\epsilon & \text{a.e. in } Y, \\ \phi_\epsilon \in H^1_\#(Y), \quad \|\phi_\epsilon\|_{L^2(Y)} = 1. \end{cases} \tag{3.5.1}$$

The asymptotic behavior of $(\mu_\epsilon, \phi_\epsilon)$ depends strongly on the behavior of the potential b. As we referred at the beginning of this chapter, an interesting case is when b oscillates between two different values and this justifies the present hypotheses on the coefficients.

Lemma 3.5.1. *Under the hypotheses of Theorem 3.1.4 and using the above notations, let (ν_0, q_0) represent the $L^2(Q)$-normalized first eigenpair of problem (3.1.15), and consider q_0 extended by zero to the whole Y. Let also $\mu_{\epsilon,1}$ represent the second eigenvalue of problem (3.5.1) and ν_1 the second eigenvalue of problem (3.1.15). Then $\{\phi_\epsilon\}_{\epsilon>0}$ converges in norm to q_0 in $L^2(Y)$ and weakly in $H^1(Y)$. Moreover,*

$$\mu_\epsilon \to \nu_0 \quad \textit{and} \quad \liminf_{\varepsilon\to 0^+} \mu_{\epsilon,1} \geqslant \nu_1. \tag{3.5.2}$$

In particular, there exist a positive constant C and $\epsilon_0 > 0$ such that for all $0 < \epsilon \leqslant \epsilon_0$ we have $\mu_{\epsilon,1} - \mu_\epsilon \geqslant C$.

Proof. We will proceed in several steps.

Step 1: We prove that $\mu_\epsilon \leqslant \nu_0$.

Noticing that b vanishes in Q, the eigenvalue μ_ϵ is given by the Rayleigh's formula

$$\mu_\epsilon = \inf_{\substack{\phi\in H^1_\#(Y)\\ \|\phi\|_{L^2(Y)}=1}} \left\{ \int_Y \bar{A}\bar{\nabla}\phi\bar{\nabla}\phi \, \mathrm{d}\bar{y} + \frac{1}{\epsilon^2}\int_{Y\setminus Q} b|\phi|^2 \, \mathrm{d}\bar{y} \right\}. \tag{3.5.3}$$

Using in (3.5.3) test functions $q \in H_0^1(Q)$, with $\|q\|_{L^2(Q)} = 1$, extended by zero to the whole Y, we obtain

$$\mu_\epsilon \leqslant \inf_{\substack{q\in H^1_0(Q)\\ \|q\|_{L^2(Q)}=1}} \left\{ \int_Q \bar{A}\bar{\nabla}q\bar{\nabla}q \, \mathrm{d}\bar{y} \right\} = \nu_0, \tag{3.5.4}$$

which concludes Step 1.

Step 2. We establish the convergence of $\{\phi_\epsilon\}_{\epsilon>0}$.

In the previous step we proved that

$$\mu_\epsilon = \int_Y \bar{A}\bar{\nabla}\phi_\epsilon\bar{\nabla}\phi_\epsilon \,\mathrm{d}\bar{y} + \frac{1}{\epsilon^2}\int_{Y\backslash Q} b|\phi_\epsilon|^2 \,\mathrm{d}\bar{y} \leqslant \nu_0 = \int_Q \bar{A}\bar{\nabla}q_0\bar{\nabla}q_0 \,\mathrm{d}\bar{y}. \tag{3.5.5}$$

Consequently,

$$\int_Y \bar{A}\bar{\nabla}\phi_\epsilon\bar{\nabla}\phi_\epsilon \,\mathrm{d}\bar{y} \leqslant \int_Q \bar{A}\bar{\nabla}q_0\bar{\nabla}q_0 \,\mathrm{d}\bar{y}, \qquad \int_{Y\backslash Q} b|\phi_\epsilon|^2 \,\mathrm{d}\bar{y} \leqslant \epsilon^2\nu_0. \tag{3.5.6}$$

Using (3.1.1), from the first estimate in (3.5.6) we conclude that $\|\nabla\phi_\epsilon\|_{L^2(Y)}$ is bounded independently of ϵ. Hence, up to a subsequence, $\{\phi_\epsilon\}_{\epsilon>0}$ converges to some $\phi_0 \in H^1_\#(Y)$ weakly in $H^1(Y)$ and strongly in $L^2(Y)$. A lower semicontinuity argument then yields

$$\int_Q \bar{A}\bar{\nabla}\phi_0\bar{\nabla}\phi_0 \,\mathrm{d}\bar{y} \leqslant \int_Y \bar{A}\bar{\nabla}\phi_0\bar{\nabla}\phi_0 \,\mathrm{d}\bar{y} \leqslant \liminf_{\epsilon\to0+}\int_Y \bar{A}\bar{\nabla}\phi_\epsilon\bar{\nabla}\phi_\epsilon \,\mathrm{d}\bar{y} \leqslant \int_Q \bar{A}\bar{\nabla}q_0\bar{\nabla}q_0 \,\mathrm{d}\bar{y}. \tag{3.5.7}$$

Fix $c>0$ such that $b(\cdot)\geqslant c$ in $Y\backslash Q$. Then, in view of the second estimate in (3.5.6),

$$\|\phi_\epsilon\|^2_{L^2(Y\backslash\overline{Q})} = \int_{Y\backslash\overline{Q}} |\phi_\epsilon|^2 \,\mathrm{d}\bar{y} \leqslant \frac{\epsilon^2}{c}\nu_0 \underset{\epsilon\to0^+}{\longrightarrow} 0.$$

Thus $\phi_0 = 0$ a.e. in $Y\backslash\overline{Q}$. Consequently, $\phi_0 \in H^1_0(Q)$ and $\|\phi_0\|_{L^2(Q)} = 1$. Finally, from (3.5.7) and since ϕ_0 is admissible in the variational definition of ν_0, we obtain $\phi_0 \equiv q_0$, as well as the convergence of the whole sequence $\{\phi_\epsilon\}_{\epsilon>0}$.

Step 3. We prove that $\mu_\epsilon \to \nu_0$ as $\epsilon\to0^+$.

By (3.5.5), we have

$$\mu_\epsilon = \int_Y \bar{A}\bar{\nabla}\phi_\epsilon\bar{\nabla}\phi_\epsilon \,\mathrm{d}\bar{y} + \frac{1}{\epsilon^2}\int_{Y\backslash Q} b|\phi_\epsilon|^2 \,\mathrm{d}\bar{y} \geqslant \int_Y \bar{A}\bar{\nabla}\phi_\epsilon\bar{\nabla}\phi_\epsilon \,\mathrm{d}\bar{y},$$

and so, in view of (3.5.7) and since $\phi_0 \equiv q_0$,

$$\liminf_{\epsilon\to0^+}\mu_\epsilon \geqslant \int_Q \bar{A}\bar{\nabla}q_0\bar{\nabla}q_0 \,\mathrm{d}\bar{y} = \nu_0,$$

which, together with (3.5.4), concludes Step 3.

Step 4. We prove that $\liminf\limits_{\varepsilon\to0^+}\mu_{\epsilon,1} \geqslant \nu_1$.

Let $\phi_{\epsilon,1}$ be a $L^2(Y)$-normalized eigenfunction associated with $\mu_{\epsilon,1}$. Then $\phi_{\epsilon,1} \in H^1_\#(Y)$, $\|\phi_{\epsilon,1}\|_{L^2(Y)} = 1$, $\int_Y \phi_\epsilon(\bar{y})\,\phi_{\epsilon,1}(\bar{y})\,\mathrm{d}\bar{y} = 0$, and

$$\begin{aligned}\mu_{\epsilon,1} &= \inf_{\substack{\phi\in H^1_\#(Y),\int_Y \phi\phi_\epsilon \mathrm{d}\bar{y}=0\\ \|\phi\|_{L^2(Y)}=1}} \left\{\int_Y \bar{A}\bar{\nabla}\phi\bar{\nabla}\phi \,\mathrm{d}\bar{y} + \frac{1}{\epsilon^2}\int_{Y\backslash Q} b|\phi|^2 \,\mathrm{d}\bar{y}\right\}\\ &= \int_Y \bar{A}\bar{\nabla}\phi_{\epsilon,1}\bar{\nabla}\phi_{\epsilon,1} \,\mathrm{d}\bar{y} + \frac{1}{\epsilon^2}\int_{Y\backslash Q} b|\phi_{\epsilon,1}|^2 \,\mathrm{d}\bar{y}.\end{aligned} \tag{3.5.8}$$

If $\liminf\limits_{\varepsilon\to 0^+} \mu_{\epsilon,1} = +\infty$ there is nothing to prove, so that, extracting a subsequence if necessary, we may assume without loss of generality that

$$\liminf_{\varepsilon\to 0^+} \mu_{\epsilon,1} = \lim_{\varepsilon\to 0^+} \mu_{\epsilon,1} < +\infty.$$

Then, in view of (3.5.8) and (3.1.1), and arguing as in Step 2, we conclude that up to a subsequence that we do not relabel,

$$\phi_{\varepsilon,1} \rightharpoonup \tilde{\phi} \quad \text{weakly in } H^1(Y) \text{ as } \varepsilon \to 0^+,$$

for some $\tilde{\phi} \in H^1_\#(Y)$, and $\lim_{\varepsilon\to 0^+} \|\phi_{\varepsilon,1}\|_{L^2(Y\setminus\overline{Q})} = 0$. In particular, $\tilde{\phi} \in H^1_0(Q)$, $\|\tilde{\phi}\|_{L^2(Q)} = 1$ and

$$0 = \int_Y \phi_\epsilon(\bar{y})\, \phi_{\epsilon,1}(\bar{y})\, \mathrm{d}\bar{y} \underset{\epsilon\to 0^+}{\longrightarrow} \int_Y \phi_0(\bar{y})\, \tilde{\phi}(\bar{y})\, \mathrm{d}\bar{y} = \int_Q q_0(\bar{y})\, \tilde{\phi}(\bar{y})\, \mathrm{d}\bar{y}.$$

Hence,

$$\nu_1 = \inf_{\substack{q\in H^1_0(Q),\, \int_Q q q_0 \,\mathrm{d}\bar{y}=0 \\ \|q\|_{L^2(Q)}=1}} \left\{ \int_Q \bar{A}\bar{\nabla} q \bar{\nabla} q \,\mathrm{d}\bar{y} \right\} \leqslant \int_Q \bar{A}\bar{\nabla}\tilde{\phi}\bar{\nabla}\tilde{\phi}\,\mathrm{d}\bar{y}. \tag{3.5.9}$$

Finally, from (3.5.8), (3.5.9), and using a lower semicontinuity argument, we get

$$\liminf_{\varepsilon\to 0^+} \mu_{\epsilon,1} \geqslant \liminf_{\varepsilon\to 0^+} \int_Y \bar{A}\bar{\nabla}\phi_{\epsilon,1}\bar{\nabla}\phi_{\epsilon,1}\,\mathrm{d}\bar{y} \geqslant \int_Y \bar{A}\tilde{\phi}(\bar{y})\,\tilde{\phi}(\bar{y})\,\mathrm{d}\bar{y} = \int_Q \bar{A}\tilde{\phi}(\bar{y})\,\tilde{\phi}(\bar{y})\,\mathrm{d}\bar{y} \geqslant \nu_1.$$

This yields Step 4.

To conclude the proof of Lemma 3.5.1 it suffices to observe that from Steps 3 and 4 we obtain

$$\liminf_{\varepsilon\to 0^+}(\mu_{\varepsilon,1} - \mu_\varepsilon) \geqslant \nu_1 - \nu_0 > 0,$$

where we also used the fact that ν_0 is simple. □

Lemma 3.5.2. *Under the hypotheses of Theorem 3.1.4 and using the previous notations, the $L^2(Y)$-normalized first eigenpair $(\mu_\epsilon, \phi_\epsilon)$ of problem (3.5.1) has the following asymptotic behavior for any integer $n \in \mathbb{N}$:*

$$\mu_\epsilon = \nu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots + \epsilon^n\mu_n + \rho_{n,\epsilon},$$

where μ_i, $i \in \{1, \cdots, n\}$, are well determined constants and $|\rho_{n,\epsilon}| \leqslant c_n\epsilon^{n+\frac{1}{2}}$, for some positive constant c_n independent of ϵ, and

$$\phi_\epsilon = q_0 + \epsilon\phi_{1,\epsilon} + \epsilon^2\phi_{2,\epsilon} + \cdots + \epsilon^n\phi_{n,\epsilon} + r_{n,\epsilon},$$

where $\phi_{i,\epsilon}$, $i \in \{1, \cdots, n\}$, are well-defined functions in $L^2(Y)$ and $\|r_{n,\epsilon}\|_{L^2(Y)} \leqslant \bar{c}_n\epsilon^{n+\frac{1}{2}}$ for a certain positive constant $\bar{c}_n$ independent of ϵ.

Proof. The proof is based on the asymptotic expansion technique. We will detail the proof for $n = 1$, being clear how to extend it for the higher orders.

For $\gamma > 0$ we define $Q_\gamma := \{\bar{y} \in Y : \mathrm{dist}(\bar{y}, Q) < \gamma\}$. Let $\gamma_0 > 0$ be such that the outward normals to ∂Q of length $2\gamma_0$ do not intersect. Consider a system of local coordinates (s, θ) on $Q_{2\gamma_0}\setminus Q$, where θ represents the local coordinate on ∂Q and $s \in [0, 2\gamma_0)$ stands for the distance to ∂Q in the outward normal direction. In these local coordinates, equation (3.5.1) in $Q_{2\gamma_0}\setminus\overline{Q}$ reads

$$-\operatorname{div}(\bar{A}^\star\bar{\nabla}\varphi_\epsilon) + b^\star\cdot\bar{\nabla}\varphi_\epsilon + \frac{b}{\epsilon^2}\varphi_\epsilon = \mu_\epsilon\varphi_\epsilon, \tag{3.5.10}$$

for a certain uniformly elliptic matrix $\bar{A}^\star = (a^\star_{\alpha\beta})$ with smooth coefficients as functions of s and θ, and for a certain vector $b^\star = (b^\star_1, b^\star_2)$, where $b^\star_1, b^\star_2$ are also smooth functions of s and θ.

In the sequel we will deal with different coordinates on each side of ∂Q. For the sake of simplicity we will abusively identify $f(\bar{y})$ with $f(\bar{y}(s,\theta))$ or, conversely, $g(s,\theta)$ with $g(s(\bar{y}),\theta(\bar{y}))$.

For small $\epsilon > 0$ we search for ϕ_ϵ and μ_ϵ with the following development

$$\mu_\epsilon = \nu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots, \tag{3.5.11}$$

$$\phi_\epsilon = q_0 + \epsilon\phi_{1,\epsilon} + \epsilon^2\phi_{2,\epsilon} + \cdots, \tag{3.5.12}$$

where, we recall, (ν_0, q_0) is the $L^2(Q)$-normalized first eigenpair of problem (3.1.15), and for each $i \in \mathbb{N}$, $\phi_{i,\epsilon}$ have the form

$$\phi_{i,\epsilon}(\bar{y}) := \begin{cases} \phi_i^-(\bar{y}) & \text{in } Q, \\ \phi_i^+\left(\dfrac{s}{\epsilon},\theta\right) & \text{in } Q_{2\gamma_0}\backslash Q, \\ 0 & \text{in } Y\backslash Q_{2\gamma_0}. \end{cases} \tag{3.5.13}$$

In view of the regularity assumptions on the coefficients $a_{\alpha\beta}$ and b and in $Y\backslash Q$, the following Taylor expansions for θ fixed hold true

$$a^\star_{\alpha\beta}(s,\theta) = a^\star_{\alpha\beta}(0,\theta) + \frac{\partial a^\star_{\alpha\beta}}{\partial s}(0,\theta)s + \frac{\partial^2 a^\star_{\alpha\beta}}{\partial s^2}(0,\theta)\frac{s^2}{2} + \frac{\partial^3 a^\star_{\alpha\beta}}{\partial s^3}(\iota(s),\theta)\frac{s^3}{3!}, \tag{3.5.14}$$

$$b(s,\theta) = b(0,\theta) + \frac{\partial b}{\partial s}(0,\theta)s + \frac{\partial^2 b}{\partial s^2}(0,\theta)\frac{s^2}{2} + \frac{\partial^3 b}{\partial s^3}(\kappa(s),\theta)\frac{s^3}{3!}. \tag{3.5.15}$$

Setting $\tau = s/\epsilon$, $\tau \in [0, 2\gamma_0/\epsilon)$, substituting (3.5.11), (3.5.12) and (3.5.13) in (3.5.10), using expressions (3.5.14)–(3.5.15) and collecting like powers of ϵ, we obtain in $Q_{2\gamma_0}\backslash\overline{Q}$, for the power ϵ^{-1}, that ϕ_1^+ must satisfy

$$-a^\star_{11}(0,\theta)\frac{\partial^2\phi_1^+}{\partial\tau^2} + b(0,\theta)\phi_1^+ = 0,$$

where θ is a parameter.

Denote by ψ_1^+ the solution, for fixed θ, of

$$\begin{cases} -a^\star_{11}(0,\theta)\dfrac{\partial^2\psi_1^+}{\partial\tau^2} + b(0,\theta)\psi_1^+ = 0, \\ \lim\limits_{\tau\to+\infty}\psi_1^+(\tau,\theta) = 0, \quad \dfrac{\partial\psi_1^+}{\partial\tau}(0,\theta) = -\dfrac{1}{a^\star_{11}(0,\theta)}. \end{cases}$$

Then

$$\psi_1^+(\tau,\theta) = \frac{1}{\sqrt{a^\star_{11}(0,\theta)b(0,\theta)}}e^{-\sqrt{\frac{b(0,\theta)}{a^\star_{11}(0,\theta)}}\,\tau},$$

and we define

$$\phi_1^+(\tau,\theta) := -\big[(\bar{A}\bar{\nabla}q_0 n_Q)(\bar{y}(0,\theta))\big]\psi_1^+(\tau,\theta),$$

where n_Q represents the outward normal to ∂Q at $\bar{y}(0,\theta)$, so that we may have

$$\left(a^\star_{11}\frac{\partial\phi_1^+}{\partial\tau}\right)(0,\theta) = \big(\bar{A}\bar{\nabla}q_0 n_Q\big)(\bar{y}(0,\theta)).$$

Also, ϕ_1^- must satisfy

$$\begin{cases} -\operatorname{div}(\bar{A}\bar{\nabla}\phi_1^-) = \nu_0\phi_1^- + \mu_1 q_0 & \text{a.e. in } Q, \\ \phi^-_{1\,|\partial Q}(\bar{y}(0,\theta)) = \phi_1^+(0,\theta), \end{cases} \tag{3.5.16}$$

and, from the compatibility condition (see Theorem 2.4.15)

$$\int_{\partial Q} (\bar{A}\bar{\nabla} q_0 n_Q)\phi_1^+ \,\mathrm{d}\sigma = \mu_1 \int_Q |q_0|^2 \,\mathrm{d}\bar{y},$$

we obtain

$$\mu_1 = \int_{\partial Q} (\bar{A}\bar{\nabla} q_0 n_Q)\phi_1^+ \,\mathrm{d}\sigma = -\int_{\partial Q} \left|\bar{A}\bar{\nabla} q_0 n_Q\right|^2 \psi_1^+ \,\mathrm{d}\sigma < 0. \tag{3.5.17}$$

So, ϕ_1^- is uniquely defined as the solution of (3.5.16) with μ_1 given by (3.5.17), and satisfying

$$\int_Q q_0 \phi_1^- \,\mathrm{d}\bar{y} = 0. \tag{3.5.18}$$

Collecting the terms of order ϵ^0 we conclude that in $Q_{2\gamma_0}\backslash\overline{Q}$, ϕ_2^+ must satisfy

$$-a_{11}^\star(0,\theta)\frac{\partial^2 \phi_2^+}{\partial \tau^2} + b(0,\theta)\phi_2^+ = R(\tau,\theta),$$

where $R(\tau,\theta)$ is a finite sum of functions of the type $f(\theta)\tau e^{-g(\theta)\tau}$ with f and g bounded as functions of θ and g is strictly positive. Therefore, problem

$$\begin{cases} -a_{11}^\star(0,\theta)\dfrac{\partial^2 \phi_2^+}{\partial \tau^2} + b(0,\theta)\varphi = R(\tau,\theta), \\ \lim\limits_{\tau\to+\infty} \phi_2^+(\tau,\theta) = 0, \quad a_{11}^\star(0,\theta)\dfrac{\partial \phi_2^+}{\partial \tau}(0,\theta) = (\bar{A}\bar{\nabla}\phi_1^- n_Q)(\bar{y}(0,\theta)), \end{cases}$$

has a unique solution ϕ_2^+, which is smooth in (τ,θ) and decays exponentially to zero as $\tau \to +\infty$.

We now define ϕ_2^- in Q as the solution in Q of

$$\begin{cases} -\operatorname{div}(\bar{A}\bar{\nabla}\phi_2^-) = \nu_0\phi_2^- + \mu_1\phi_1^- + \mu_2 q_0 \quad \text{a.e. in } Q, \\ \phi_2^-{}_{|\partial Q}(\bar{y}(0,\theta)) = \phi_2^+(0,\theta), \end{cases}$$

with

$$\mu_2 := \int_{\partial Q} (\bar{A}\bar{\nabla} q_0 n_Q)\phi_2^+ \,\mathrm{d}\sigma$$

so that the compatibility condition is satisfied.

Now, in order to make the function ϕ_ϵ and its derivatives continuous on ∂Q, we introduce a smooth function ψ_2^-, defined in Q, such that $\psi_2^-{}_{|\partial Q} = 0$ and $\bar{A}\bar{\nabla}\psi_2^- n_Q = -\bar{A}\bar{\nabla}\phi_2^- n_Q$. Consider also a cut-off function $\phi_{\gamma_0} \in C^\infty(\mathbb{R};[0,1])$ such that $\phi_{\gamma_0}(s) = 1$ if $s \leqslant \gamma_0$, and $\phi_{\gamma_0}(s) = 0$ if $s \geqslant 2\gamma_0$.

Finally, we set

$$w_\epsilon(\bar{y}) := \begin{cases} q_0(\bar{y}) + \epsilon\phi_1^-(\bar{y}) + \epsilon^2\phi_2^-(\bar{y}) + \epsilon^2\ \psi_2^-(\bar{y}) & \text{if } \bar{y} \in Q, \\ \Big(\epsilon\phi_1^+\Big(\dfrac{s(\bar{y})}{\epsilon},\theta(\bar{y})\Big) + \epsilon^2\phi_2^+\Big(\dfrac{s(\bar{y})}{\epsilon},\theta(\bar{y})\Big)\Big)\phi_{2\gamma_0}(s(\bar{y})) & \text{if } \bar{y} \in Q_{2\gamma_0}\backslash Q, \\ 0 & \text{if } \bar{y} \in Y\backslash Q_{2\gamma_0}, \end{cases} \tag{3.5.19}$$

and

$$\Lambda_\epsilon := \nu_0 + \epsilon\mu_1 + \epsilon^2\mu_2. \tag{3.5.20}$$

Then, it can be checked that for suitable constants c_0 and c_1 independent of ϵ, the following estimates hold true

$$\|w_\epsilon\|_{L^2(Y)} \leqslant 1 + c_0\epsilon^2, \tag{3.5.21}$$

$$\Big\| -\operatorname{div}(\bar{A}\bar{\nabla} w_\epsilon) + \frac{1}{\epsilon^2} b w_\epsilon - \Lambda_\epsilon w_\epsilon \Big\|_{L^2(Y)} \leqslant c_1\epsilon^{3/2}. \tag{3.5.22}$$

Indeed, from (3.5.18) and the fact that q_0 vanishes outside Q it follows that $\phi_{1,\epsilon}$ is orthogonal to q_0. Thus, considering the normalization $\|q_0\|_{L^2(Y)} = 1$, we obtain

$$\|q_0 + \epsilon\phi_{1,\epsilon}\|^2_{L^2(Y)} = 1 + \epsilon^2\|\phi_{1,\epsilon}\|^2_{L^2(Y)}. \tag{3.5.23}$$

This implies in particular that $\|q_0 + \epsilon\phi_{1,\epsilon}\|_{L^2(Y)} \geqslant 1$ and $\|q_0 + \epsilon\phi_{1,\epsilon}\|_{L^2(Y)} \leqslant \|q_0 + \epsilon\phi_{1,\epsilon}\|^2_{L^2(Y)}$. Therefore, (3.5.21) is a consequence of (3.5.23).

To justify (3.5.22), we use (3.5.19) and (3.5.20) and the definitions of all functions q_0, $\phi_1^\pm$, $\phi_2^\pm$, ψ_2^- and ϕ_{γ_0}. After straightforward rearrangements we obtain

$$-\operatorname{div}(\bar{A}\bar{\nabla}w_\epsilon) + \frac{1}{\epsilon^2}bw_\epsilon - \Lambda_\epsilon w_\epsilon = \begin{cases} \epsilon^2 r_\epsilon^-(\bar{y}) & \text{if } \bar{y} \in Q, \\ \epsilon r_\epsilon^+\Big(s(\bar{y}), \theta(\bar{y}), \dfrac{s(\bar{y})}{\epsilon}\Big) & \text{if } \bar{y} \in Q_{2\gamma_0}\backslash Q, \\ 0 & \text{if } \bar{y} \in Y\backslash Q_{2\gamma_0}, \end{cases} \tag{3.5.24}$$

where

$$\|r_\epsilon^-\|_{L^2(Q)} \leqslant c_2 \quad \text{and} \quad |r_\epsilon^+(s,\theta,\tau)| \leqslant c_3\tau^j e^{-c_4\tau} \tag{3.5.25}$$

with positive constants c_2, c_3, c_4, and $j \in \mathbb{N}$, independent of ϵ. It follows from the second upper bound in (3.5.25) that for some positive constant c_5,

$$\|r_\epsilon^+\|^2_{L^2(Y\backslash Q)} \leqslant c_5\epsilon. \tag{3.5.26}$$

Then, in view of the first upper bound in (3.5.25) and thanks to (3.5.26) and (3.5.24), we obtain estimate (3.5.22).

In order to obtain the announced estimates we notice that by Lemma 3.5.1 we can find $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ the ground state μ_ϵ and the second eigenvalue $\mu_{\epsilon,1}$ of problem (3.5.1) satisfy the inequality $\mu_{\epsilon,1} - \mu_\epsilon \geqslant \bar{c} > 0$. So, using Vishik-Lyusternik Lemma (see Lemma 2.4.17), from (3.5.21) and (3.5.22) we get

$$|\Lambda_\epsilon - \mu_\epsilon| \leqslant c_6\epsilon^{3/2}, \quad \|w_\epsilon - \phi_\epsilon\|_{L^2(Y)} \leqslant c_7\epsilon^{3/2}, \tag{3.5.27}$$

for some positive constants c_6, c_7 independent of ϵ. Considering the definitions of w_ϵ and Λ_ϵ we conclude, from (3.5.27), that

$$|\mu_\epsilon - (\nu_0 + \epsilon\mu_1)| \leqslant c_8\epsilon^{3/2}, \quad \|\phi_\epsilon - (q_0 + \epsilon\phi_{1,\epsilon})\|_{L^2(Y)} \leqslant c_9\epsilon^{3/2},$$

for some constants $c_8 > 0$ and $c_9 > 0$. This completes the proof for $n = 1$. □

We now prove Theorems 3.1.4 and 3.1.7.

Proof of Theorem 3.1.4. By Proposition 3.2.2, the first eigenvalue $\lambda_{\varepsilon,1}$ of the tridimensional problem (3.1.5) coincides with the first eigenvalue $\lambda^{(1)}_{\varepsilon,1}$ of the following bidimensional problem

$$\begin{cases} -\operatorname{div}\big(\bar{A}_\varepsilon(\bar{x})\bar{\nabla}\varphi^{(1)}_\varepsilon\big) + \dfrac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2\varphi^{(1)}_\varepsilon = \lambda^{(1)}_\varepsilon\varphi^{(1)}_\varepsilon & \text{a.e. } \bar{x} \in \omega, \\ \varphi^{(1)}_\varepsilon \in H^1_0(\omega). \end{cases}$$

Also, the corresponding L^2-normalized eigenfunctions $v_{\varepsilon,1}$ and $\varphi^{(1)}_{\varepsilon,1}$ satisfy the following relation

$$v_{\varepsilon,1}(x) = \varphi^{(1)}_{\varepsilon,1}(\bar{x})\theta_1(x_3), \quad \text{a.e. } x = (\bar{x}, x_3) \in \omega \times I,$$

where θ_1 is the first $L^2(I)$-normalized eigenfunction of problem (3.1.7).

On the other hand, recalling the proof of Proposition 3.2.1, relation (3.2.2) holds true if we restrict (3.2.1) to v and u only depending on $\bar{x}$. Using (3.2.2) for $\varphi, \psi \in H_0^1(\omega)$ satisfying

$$\varphi(\bar{x}) = \phi_{\varepsilon,0}^{\tau}\Big(\frac{\bar{x}}{\varepsilon}\Big)\psi(\bar{x}), \quad \text{a.e. } \bar{x} \in \omega,$$

we obtain

$$\begin{aligned} \lambda_{\varepsilon,1} = \lambda_{\varepsilon,1}^{(1)} &= \inf_{\substack{\varphi \in H_0^1(\omega) \\ \varphi \not\equiv 0}} \frac{\displaystyle\int_\omega \bar{A}_\varepsilon(\bar{x})\bar{\nabla}\varphi(\bar{x})\bar{\nabla}\varphi(\bar{x}) + \frac{a_{33}^{\varepsilon}(\bar{x})}{\varepsilon^{2\tau}}\pi^2|\varphi(\bar{x})|^2 \,\mathrm{d}\bar{x}}{\displaystyle\int_\omega |\varphi(\bar{x})|^2 \,\mathrm{d}\bar{x}} \\ &= \frac{\mu_{\varepsilon,0}^{\tau}}{\varepsilon^{2\tau}} + \inf_{\substack{\psi \in H_0^1(\omega) \\ \psi \not\equiv 0}} \frac{\displaystyle\int_\omega \Big|\phi_{\varepsilon,0}^{\tau}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \bar{A}_\varepsilon(\bar{x})\bar{\nabla}\psi(\bar{x})\bar{\nabla}\psi(\bar{x})\,\mathrm{d}\bar{x}}{\displaystyle\int_\omega \Big|\phi_{\varepsilon,0}^{\tau}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 |\psi(\bar{x})|^2 \,\mathrm{d}\bar{x}} = \frac{\mu_{\varepsilon,0}^{\tau}}{\varepsilon^{2\tau}} + \nu_{\varepsilon,1}^{\tau}. \end{aligned} \tag{3.5.28}$$

Using Lemma 3.5.1 and recalling the notations introduced at the beginning of this section, we get $\mu_{\varepsilon,0}^{\tau} \to a_{\min}\pi^2$ and $\phi_{\varepsilon,0}^{\tau} \rightharpoonup q_0$ weakly in $H_\#^1(Y)$ as $\varepsilon \to 0^+$, and

$$\lambda_{\varepsilon,1} = \frac{a_{\min}\pi^2}{\varepsilon^{2\tau}} + \frac{\nu_0}{\varepsilon^2} + \varepsilon^{\tau-3}\mu_1 + \cdots + \varepsilon^{k(\tau-1)-2}\mu_k + \rho_\varepsilon^{\tau} + \nu_{\varepsilon,1}^{\tau},$$

where $|\rho_\varepsilon^{\tau}| \leqslant C\varepsilon^{(k+\frac{1}{2})\tau-(k+\frac{5}{2})} \to 0$ as $\varepsilon \to 0^+$.

To conclude the proof of Theorem 3.1.4 we are left to prove that $\nu_{\varepsilon,1}^{\tau} \to 0$ as $\varepsilon \to 0^+$. Construct a sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ in $H_0^1(\omega)$ as follows: for each $\varepsilon > 0$ let $K_\varepsilon := \{\kappa \in \mathbb{Z}^2 \colon \varepsilon(\kappa + Y) \subset \omega\}$, and define

$$T_\varepsilon := \mathrm{int}\Big(\bigcup_{\kappa \in K_\varepsilon} \varepsilon(\kappa_i + \overline{Y})\Big).$$

Consider the cut-off function ϕ_{γ_0} introduced in Lemma 3.5.2 in the definition of w_ϵ (see (3.5.19)). Extend ϕ_{γ_0} to the whole $\mathbb{R}^2$ by Y-periodicity, and define ψ_ε by $\psi_\varepsilon(\bar{x}) := \phi_{\gamma_0}\big(\frac{\bar{x}}{\varepsilon}\big)$ if $\bar{x} \in T_\varepsilon$, and $\psi_\varepsilon(\bar{x}) := 0$ if $\bar{x} \in \omega \backslash T_\varepsilon$.

Using the definition of $\nu_{\varepsilon,1}^{\tau}$, taking ψ_ε as test function, using the uniform bounds in (3.1.1), the usual change of scales $\bar{y} = \varepsilon^{-1}\bar{x}$, together with the Y-periodicity of ϕ_{γ_0} and $\phi_{\varepsilon,0}^{\tau}$, and since we have $\|\bar{\nabla}\phi_{\gamma_0}\|_{L^\infty(Y)} \leqslant c/\gamma_0$, we obtain

$$0 \leqslant \nu_{\varepsilon,1}^{\tau} \leqslant \frac{\eta}{\varepsilon^2} \frac{\displaystyle\int_{T_\varepsilon} \Big|\phi_{\varepsilon,0}^{\tau}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \Big|\bar{\nabla}\phi_{\gamma_0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \,\mathrm{d}\bar{x}}{\displaystyle\int_{T_\varepsilon} \Big|\phi_{\varepsilon,0}^{\tau}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \Big|\phi_{\gamma_0}\Big(\frac{\bar{x}}{\varepsilon}\Big)\Big|^2 \,\mathrm{d}\bar{x}} \leqslant \frac{\eta c^2}{\varepsilon^2\gamma_0^2} \frac{\displaystyle\int_{Q_{2\gamma_0}\backslash Q_{\gamma_0}} \big|\phi_{\varepsilon,0}^{\tau}(\bar{y})\big|^2 \,\mathrm{d}\bar{y}}{\displaystyle\int_Q \big|\phi_{\varepsilon,0}^{\tau}(\bar{y})\big|^2 \,\mathrm{d}\bar{y}}. \tag{3.5.29}$$

Using Lemma 3.5.2 with $k = n$ and recalling the definitions and the estimates therein, we obtain for $\bar{y} = \bar{y}(s,\theta) \in Q_{2\gamma_0}\backslash Q$,

$$\phi_{\varepsilon,0}^{\tau}(\bar{y}(s,\theta)) = \varepsilon^{\tau-1}P(s,\theta) + r_{k,\varepsilon} \tag{3.5.30}$$

where, since $\tau \geqslant (k+2)/k$,

$$\|r_{k,\varepsilon}\|_{L^2(Y\backslash Q)} \leqslant \bar{c}_k\varepsilon^{(\tau-1)(k+\frac{1}{2})} \leqslant \bar{c}_k\varepsilon^{2+\frac{1}{k}}, \tag{3.5.31}$$

and P satisfies the following pointwise estimate

$$|P(s,\theta)|^2 \leqslant \sum_{m=1}^{k} a_m \left(\frac{s}{\varepsilon}\right)^{j_m} e^{-b_m \frac{s}{\varepsilon}} \tag{3.5.32}$$

for some positive constants a_m, b_m and for some $j_m \in \mathbb{N}$, independent of ε.

Consequently, putting together (3.5.30), (3.5.31) and (3.5.32), and in view of (3.5.29), we conclude that

$$\begin{aligned} 0 \leqslant \nu^\tau_{\varepsilon,1} \leqslant \frac{\bar{c}}{\varepsilon^2}\left(\int_Q \left|\phi^\tau_{\varepsilon,0}(\bar{y})\right|^2 \mathrm{d}\bar{y}\right)^{-1} \sum_{m=1}^{k} a_m \varepsilon^{\tau-1-j_m} e^{-b_m \frac{\gamma_0}{\varepsilon}} \int_{Q_{2\gamma_0}\setminus Q_{\gamma_0}} s^{j_m}\,\mathrm{d}s \\ + \frac{\bar{c}}{\varepsilon^2}\left(\int_Q \left|\phi^\tau_{\varepsilon,0}(\bar{y})\right|^2 \mathrm{d}\bar{y}\right)^{-1} \varepsilon^{2(2+\frac{1}{k})}, \end{aligned} \tag{3.5.33}$$

for some constant $\bar{c}$ independent of ε. Having in mind that from Lemma 3.5.1, $\int_Q \left|\phi^\tau_{\varepsilon,0}(\bar{y})\right|^2 \mathrm{d}\bar{y} \to 1$ as $\varepsilon \to 0^+$, we may pass in (3.5.33) to the limit as $\varepsilon \to 0^+$ to conclude the proof of Theorem 3.1.4. □

Proof of Theorem 3.1.7. We start by observing that we may assume without loss of generality that $0 \in \omega$.

In view of the definition of $\lambda_{\varepsilon,1}$ (see (3.5.28)), we deduce that

$$\lambda_{\varepsilon,1} \geqslant \frac{a_{\min}}{\varepsilon^{2\tau}}\pi^2,$$

and so

$$\lim_{\varepsilon\to 0^+} \left(\varepsilon^{2\tau}\sigma_\varepsilon\right) \subset [a_{\min}\pi^2, +\infty]. \tag{3.5.34}$$

To prove the opposite inclusion we fix $\varepsilon > 0$ and we recall the notations of Proposition 3.2.2 with B, b and $\lambda^{(n)}_k$ replaced by $\bar{A}_\varepsilon$, $\frac{a^\varepsilon_{33}}{\varepsilon^{2\tau}}$ and $\lambda^{(n)}_{\varepsilon,k}$, respectively. Let also $\sigma^{(n)}_\varepsilon := \{\lambda^{(n)}_{\varepsilon,k} \colon k \in \mathbb{N}\}$.

For fixed $\varepsilon > 0$ we have that $\varepsilon^{2\tau}\lambda^{(1)}_{\varepsilon,k} \to +\infty$ as $k \to +\infty$. Using a diagonal argument we can find a sequence $\{\lambda_\varepsilon\}_{\varepsilon>0} \subset \sigma_\varepsilon$ such that $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$. Thus,

$$+\infty \in \lim_{\varepsilon\to 0^+} \left(\varepsilon^{2\tau}\sigma_\varepsilon\right). \tag{3.5.35}$$

Moreover, by Proposition 3.2.2 one has

$$\sigma_\varepsilon = \bigcup_{n\in\mathbb{N}} \sigma^{(n)}_\varepsilon.$$

We claim that

$$\lim_{\varepsilon\to 0^+} \left(\varepsilon^{2\tau}\sigma^{(1)}_\varepsilon\right) \supset [a_{\min}\pi^2, +\infty). \tag{3.5.36}$$

Assume that (3.5.36) holds. Then the inclusion $\sigma_\varepsilon \supset \sigma^{(1)}_\varepsilon$ yields

$$\lim_{\varepsilon\to 0^+} \left(\varepsilon^{2\tau}\sigma_\varepsilon\right) \supset \lim_{\varepsilon\to 0^+} \left(\varepsilon^{2\tau}\sigma^{(1)}_\varepsilon\right) \supset [a_{\min}\pi^2, +\infty),$$

which, together with (3.5.34) and (3.5.35), establishes (3.1.16).

In order to show (3.5.36) we first perform a change of variables that will transform problem

$$\begin{cases} -\operatorname{div}\left(\bar{A}_\varepsilon(\bar{x})\bar{\nabla}\varphi^{(1)}_\varepsilon\right) + \dfrac{a^\varepsilon_{33}(\bar{x})}{\varepsilon^{2\tau}}\pi^2\varphi^{(1)}_\varepsilon = \lambda^{(1)}_\varepsilon\varphi^{(1)}_\varepsilon \quad \text{a.e. } \bar{x} \in \omega, \\ \varphi_\varepsilon \in H^1_0(\omega), \end{cases} \tag{3.5.37}$$

into an equivalent one allowing us to pass to the limit as $\varepsilon \to 0^+$. Recall that problem (3.5.37) corresponds to (3.2.10) for n=1, with B replaced by $\bar{A}_\varepsilon$ and b replaced by $\frac{a_{33}^\varepsilon}{\varepsilon^{2\tau}}$.

Let $\omega_\varepsilon := \frac{\omega}{\varepsilon^\tau} - \frac{\bar{y}_0}{\varepsilon^{\tau-1}}$, where $\bar{y}_0$ is a point of minimum of a_{33}. Notice that if $\mathcal{Z} \subset \mathbb{R}^2$ is a bounded set, then for all $\varepsilon > 0$ small enough, $\mathcal{Z} \subset \omega_\varepsilon$, since ω is connected, $0 \in \omega$, and $\tau > 1$. Associating to each function $\varphi \in H_0^1(\omega)$ the function $\psi \in H_0^1(\omega_\varepsilon)$ defined by $\psi(\bar{z}) := \varphi(\varepsilon^\tau \bar{z} + \varepsilon \bar{y}_0)$ and using the change of variables $\bar{z} := \varepsilon^{-\tau}\bar{x} - \varepsilon^{1-\tau}\bar{y}_0$, (3.5.37) becomes

$$\begin{cases} -\operatorname{div}(\bar{D}_\varepsilon(\bar{z})\bar{\nabla}\psi_\varepsilon^{(1)}) + d_\varepsilon(\bar{z})\psi_\varepsilon^{(1)} = \rho_\varepsilon^{(1)}\psi_\varepsilon^{(1)} \quad \text{a.e. } \bar{z} \in \omega_\varepsilon, \\ \psi_\varepsilon^{(1)} \in H_0^1(\omega_\varepsilon), \end{cases} \tag{3.5.38}$$

where $\rho_\varepsilon^{(1)} := \varepsilon^{2\tau}\lambda_\varepsilon^{(1)}$, while $\bar{D}_\varepsilon$ and d_ε are defined by

$$\bar{D}_\varepsilon(\bar{z}) := \bar{A}(\varepsilon^{\tau-1}\bar{z} + \bar{y}_0), \quad d_\varepsilon(\bar{z}) := a_{33}(\varepsilon^{\tau-1}\bar{z} + \bar{y}_0)\pi^2, \quad \bar{z} \in \mathbb{R}^2, \tag{3.5.39}$$

respectively. In view of (3.1.1), for all $\xi \in \mathbb{R}^2$ and for a.e. $\bar{z} \in \mathbb{R}^2$ one has

$$\zeta\|\xi\|^2 \leqslant (\bar{D}_\varepsilon(\bar{z})\xi|\xi) \leqslant \eta\|\xi\|^2, \quad \zeta \leqslant d_\varepsilon(\bar{z}) \leqslant \eta. \tag{3.5.40}$$

Notice also that $\bar{D}_\varepsilon \in M_S(\zeta, \eta, \mathbb{R}^2)$. Hence, up to a subsequence that we do not relabel, the sequence $\{\bar{D}_\varepsilon\}_{\varepsilon>0}$ G-converges to some matrix $\bar{D}_0$ in any open and bounded subset of $\mathbb{R}^2$ (see Theorem 2.5.23) and the sequence $\{d_\varepsilon\}_{\varepsilon>0}$ weakly-$\star$ converges in $L^\infty(\mathbb{R}^2)$ to some $d_0 \in L^\infty(\mathbb{R}^2)$. On the other hand, since $a_{\alpha\beta}$ and a_{33} are continuous in a neighborhood of $\bar{y}_0$, $\bar{D}_\varepsilon \to \bar{A}(\bar{y}_0)$ and $d_\varepsilon \to a_{33}(\bar{y}_0)\pi^2$ uniformly on each compact subset of $\mathbb{R}^2$ as $\varepsilon \to 0^+$. Thus, by definition of G-limit, we conclude that $\bar{D}_0 \equiv \bar{A}(\bar{y}_0)$ and $d_0 \equiv a_{33}(\bar{y}_0)\pi^2 = a_{\min}\pi^2$. In particular, the whole sequences $\{\bar{D}_\varepsilon\}_{\varepsilon>0}$ and $\{d_\varepsilon\}_{\varepsilon>0}$ converge.

Let $\mathcal{S}_\varepsilon$ represent the self-adjoint operator $\big(-\operatorname{div}(\bar{D}_\varepsilon\bar{\nabla}) + d_\varepsilon\big)$ in $L^2(\omega_\varepsilon)$ with Dirichlet boundary conditions. Then its spectrum is $\sigma(\mathcal{S}_\varepsilon) = \varepsilon^{2\tau}\sigma_\varepsilon^{(1)}$. Therefore, proving (3.5.36) is equivalent to proving

$$\lim_{\varepsilon\to0^+} \sigma(\mathcal{S}_\varepsilon) \supset [d_0, +\infty). \tag{3.5.41}$$

Consider now the inverse operator, $\mathcal{S}_\varepsilon^{-1}$, of $\mathcal{S}_\varepsilon$, i.e., the compact self-adjoint operator in $L^2(\omega_\varepsilon)$ that to each $f_\varepsilon \in L^2(\omega_\varepsilon)$ associates the function $\mathcal{S}_\varepsilon^{-1}f_\varepsilon := \psi_\varepsilon$, where $\psi_\varepsilon \in H_0^1(\omega_\varepsilon)$ is the solution of

$$\begin{cases} -\operatorname{div}(\bar{D}_\varepsilon\bar{\nabla}\psi_\varepsilon) + d_\varepsilon\psi_\varepsilon = f_\varepsilon \quad \text{a.e. in } \omega_\varepsilon, \\ \psi_\varepsilon \in H_0^1(\omega_\varepsilon). \end{cases} \tag{3.5.42}$$

For the sake of simplicity we will not distinguish a function in $H_0^1(\omega_\varepsilon)$ from its zero extension to the whole $\mathbb{R}^2$.

Let us also introduce the self-adjoint operators in $L^2(\mathbb{R}^2)$, $\mathcal{S} := \big(-\operatorname{div}(\bar{D}_0\bar{\nabla}) + d_0\big)$ and its inverse operator $\mathcal{S}^{-1}$ that to each $f \in L^2(\mathbb{R}^2)$ associates the function $\mathcal{S}^{-1}f := \psi$, where $\psi \in H^1(\mathbb{R}^2)$ is the solution of

$$\begin{cases} -\operatorname{div}(\bar{D}_0\bar{\nabla}\psi) + d_0\psi = f \quad \text{a.e. in } \mathbb{R}^2, \\ \psi \in H^1(\mathbb{R}^2). \end{cases} \tag{3.5.43}$$

Since $\bar{D}_0$ is a positive definite constant matrix and $d_0 > 0$, $\sigma(S) = [d_0, +\infty)$. Hence, if we prove that

$$\lim_{\varepsilon\to0^+} \sigma\big(\mathcal{S}_\varepsilon^{-1}\big) \supset \sigma\big(\mathcal{S}^{-1}\big), \tag{3.5.44}$$

it follows that $\lim_{\varepsilon\to 0^+}\sigma(\mathcal{S}_\varepsilon)\supset\sigma(\mathcal{S})=[d_0,+\infty)$, which is precisely (3.5.41). In order to show (3.5.44), we start by proving that $\mathcal{S}_\varepsilon^{-1}$ converges strongly to $\mathcal{S}^{-1}$ as $\varepsilon\to 0^+$; more precisely, if $f\in L^2(\mathbb{R}^2)$, then $\mathcal{S}_\varepsilon^{-1}f\chi_{\omega_\varepsilon}\to\mathcal{S}^{-1}f$ in $L^2(\mathbb{R}^2)$ as $\varepsilon\to 0^+$.

Let $f\in L^2(\mathbb{R}^2)$, and define $f_\varepsilon:=f\chi_{\omega_\varepsilon}\in L^2(\omega_\varepsilon)$. Let $\psi_\varepsilon:=\mathcal{S}_\varepsilon^{-1}f_\varepsilon$ (extended by zero outside ω_ε) and $\psi:=\mathcal{S}^{-1}f$. Thanks to (3.5.40), we have, up to a subsequence that we do not relabel, $\psi_\varepsilon\rightharpoonup\varphi$ weakly in $H^1(\mathbb{R}^2)$ as $\varepsilon\to 0^+$, for some $\varphi\in H^1(\mathbb{R}^2)$. Moreover, since ψ_ε is the solution of (3.5.42), if $\vartheta\in C_c^\infty(\mathbb{R}^2)$ then we have, for all $\varepsilon>0$ small enough, $\operatorname{supp}\vartheta\subset\omega_\varepsilon$ and

$$\int_{\mathbb{R}^2}\bar{D}_\varepsilon\bar{\nabla}\psi_\varepsilon\bar{\nabla}\vartheta+d_\varepsilon\psi_\varepsilon\vartheta\,\mathrm{d}\bar{z}=\int_{\mathbb{R}^2}f\vartheta\,\mathrm{d}\bar{z}.\tag{3.5.45}$$

Letting $\varepsilon\to 0^+$ we obtain

$$\int_{\mathbb{R}^2}\bar{D}_0\bar{\nabla}\varphi\bar{\nabla}\vartheta+d_0\varphi\vartheta\,\mathrm{d}\bar{z}=\int_{\mathbb{R}^2}f\vartheta\,\mathrm{d}\bar{z}.\tag{3.5.46}$$

Since $\vartheta\in C_c^\infty(\mathbb{R}^2)$ was arbitrary, we deduce that $\varphi=\psi$ a.e. in $\mathbb{R}^2$. Thus, $\psi_\varepsilon\rightharpoonup\psi$ weakly in $H^1(\mathbb{R}^2)$ as $\varepsilon\to 0^+$ and so, to establish the strong convergence in $L^2(\mathbb{R}^2)$ it suffices to prove that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2}|\psi_\varepsilon|^2\,\mathrm{d}\bar{z}=\int_{\mathbb{R}^2}|\psi|^2\,\mathrm{d}\bar{z}.$$

Let $L:=\liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}^2}|\psi_\varepsilon|^2\,\mathrm{d}\bar{z}$. Without loss of generality we may assume that the inferior limit defining L is actually a limit, otherwise we would extract a subsequence. By the sequential lower semicontinuity of the norm with respect to the weak topology of $L^2(\mathbb{R}^2)$, $L\geqslant\int_{\mathbb{R}^2}|\psi|^2\,\mathrm{d}\bar{z}$.

To prove the converse inequality, we start by proving that if $\{\upsilon_\varepsilon\}_{\varepsilon>0}\subset H^1(\mathbb{R}^2)$ and $\upsilon\in H^1(\mathbb{R}^2)$ are such that $\upsilon_\varepsilon\rightharpoonup\upsilon$ weakly in $H^1(\mathbb{R}^2)$ as $\varepsilon\to 0^+$, then

$$F_0(\upsilon)\leqslant\liminf_{\varepsilon\to 0^+}F_\varepsilon(\upsilon_\varepsilon),\tag{3.5.47}$$

where for each $\varepsilon>0$, $F_\varepsilon:H^1(\mathbb{R}^2)\to[0,+\infty]$ is defined by

$$F_\varepsilon(\upsilon):=\begin{cases}\displaystyle\int_{\omega_\varepsilon}\bar{D}_\varepsilon(\bar{z})\bar{\nabla}\upsilon(\bar{z})\bar{\nabla}\upsilon(\bar{z})+(d_\varepsilon(\bar{z})-c)|\upsilon(\bar{z})|^2\,\mathrm{d}\bar{z} & \text{if } \upsilon\in H_0^1(\omega_\varepsilon),\\ +\infty & \text{otherwise,}\end{cases}$$

for some $c\in\mathbb{R}$ such that $\inf_\varepsilon d_\varepsilon>c>0$, and $F_0:H^1(\mathbb{R}^2)\to[0,+\infty]$ is given by

$$F_0(\upsilon):=\int_{\mathbb{R}^2}\bar{D}_0\bar{\nabla}\upsilon(\bar{z})\bar{\nabla}\upsilon(\bar{z})+(d_0-c)|\upsilon(\bar{z})|^2\,\mathrm{d}\bar{z}.$$ [3.1]

In order to show (3.5.47) we may assume without loss of generality that the limit inferior on the right hand-side of (3.5.47) is actually a limit and that this limit is finite. Then, by definition of F_ε, $\upsilon_\varepsilon\in H_0^1(\omega_\varepsilon)$ for all $\varepsilon>0$.

Defining $g:=-\operatorname{div}(\bar{D}_0\bar{\nabla}\upsilon)+(d_0-c)\upsilon$, we have that $g\in H^{-1}(\mathbb{R}^2)$. Moreover, if $\tilde{\upsilon}_\varepsilon\in H_0^1(\omega_\varepsilon)$ is the solution of $-\operatorname{div}(\bar{D}_\varepsilon\bar{\nabla}\tilde{\upsilon}_\varepsilon)+(d_\varepsilon-c)\tilde{\upsilon}_\varepsilon$ a.e. in ω_ε, then given an arbitrary $\vartheta\in C_c^\infty(\mathbb{R}^2)$ the equality

$$\int_{\omega_\varepsilon}\bar{D}_\varepsilon\bar{\nabla}\tilde{\upsilon}_\varepsilon\bar{\nabla}\vartheta+(d_\varepsilon-c)\tilde{\upsilon}_\varepsilon\vartheta\,\mathrm{d}\bar{z}=\langle g,\vartheta\rangle_{H^{-1}(\omega_\varepsilon),H_0^1(\omega_\varepsilon)}\tag{3.5.48}$$

[3.1] It can be shown that actually the sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges with respect to the weak topology of $H^1(\mathbb{R}^2)$ as $\varepsilon\to 0^+$ to the functional F_0 (see also Dal Maso [31, Thm. 13.5]).

holds for all $\varepsilon > 0$ small enough. In view of (3.5.40), we have, up to a subsequence that we do not relabel, $\tilde{\upsilon}_\varepsilon \rightharpoonup \tilde{\upsilon}$ weakly in $H^1(\mathbb{R}^2)$ as $\varepsilon \to 0^+$, for some $\tilde{\upsilon} \in H^1(\mathbb{R}^2)$. Letting $\varepsilon \to 0^+$ in (3.5.48), we obtain for all $\vartheta \in C_c^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \bar{D}_0 \bar{\nabla} \tilde{\upsilon} \bar{\nabla} \vartheta + (d_0 - c)\tilde{\upsilon}\vartheta \,\mathrm{d}\bar{z} = \langle g, \vartheta \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)}. \tag{3.5.49}$$

Consequently, $\tilde{\upsilon}$ is the solution of

$$\begin{cases} -\operatorname{div}(\bar{D}_0 \bar{\nabla} \tilde{\upsilon}) + (d_0 - c)\tilde{\upsilon} = g \quad \text{a.e. in } \mathbb{R}^2, \\ \tilde{\upsilon} \in H^1(\mathbb{R}^2). \end{cases}$$

Since $g = -\operatorname{div}(\bar{D}_0 \bar{\nabla} \upsilon) + (d_0 - c)\upsilon$, we conclude that $\tilde{\upsilon} = \upsilon$ a.e. in $\mathbb{R}^2$. On the other hand, using the fact that (3.5.48) holds for all $\vartheta \in H_0^1(\omega_\varepsilon)$ and (3.5.49) holds for all $\vartheta \in H^1(\mathbb{R}^2)$, and using the weak convergence $\tilde{\upsilon}_\varepsilon \rightharpoonup \tilde{\upsilon} = \upsilon$ in $H^1(\mathbb{R}^2)$, we get

$$\begin{aligned} \int_{\omega_\varepsilon} \bar{D}_\varepsilon \bar{\nabla} \tilde{\upsilon}_\varepsilon \bar{\nabla} \tilde{\upsilon}_\varepsilon + (d_\varepsilon - c)|\tilde{\upsilon}_\varepsilon|^2 \,\mathrm{d}\bar{z} &= \langle g, \tilde{\upsilon}_\varepsilon \rangle_{H^{-1}(\omega_\varepsilon), H_0^1(\omega_\varepsilon)} \\ \underset{\varepsilon \to 0^+}{\longrightarrow} \langle g, \upsilon \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \bar{D}_0 \bar{\nabla} \upsilon \bar{\nabla} \upsilon + (d_0 - c)|\upsilon|^2 \,\mathrm{d}\bar{z}. \end{aligned} \tag{3.5.50}$$

Moreover, since $\tilde{\upsilon}_\varepsilon$ minimizes $F_\varepsilon(w) - 2\langle g, w \rangle_{H^{-1}(\omega_\varepsilon), H_0^1(\omega_\varepsilon)}$, $w \in H_0^1(\omega_\varepsilon)$, we have that $F_\varepsilon(\upsilon_\varepsilon) - 2\langle g, \upsilon_\varepsilon \rangle_{H^{-1}(\omega_\varepsilon), H_0^1(\omega_\varepsilon)} \geqslant F_\varepsilon(\tilde{\upsilon}_\varepsilon) - 2\langle g, \tilde{\upsilon}_\varepsilon \rangle_{H^{-1}(\omega_\varepsilon), H_0^1(\omega_\varepsilon)}$. This, together with (3.5.50) and the convergences

$$\langle g, \upsilon_\varepsilon \rangle_{H^{-1}(\omega_\varepsilon), H_0^1(\omega_\varepsilon)} \underset{\varepsilon \to 0^+}{\longrightarrow} \langle g, \upsilon \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)}, \quad \langle g, \tilde{\upsilon}_\varepsilon \rangle_{H^{-1}(\omega_\varepsilon), H_0^1(\omega_\varepsilon)} \underset{\varepsilon \to 0^+}{\longrightarrow} \langle g, \upsilon \rangle_{H^{-1}(\mathbb{R}^2), H^1(\mathbb{R}^2)},$$

yield

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(\upsilon_\varepsilon) \geqslant \liminf_{\varepsilon \to 0^+} F_\varepsilon(\tilde{\upsilon}_\varepsilon) = F_0(\upsilon),$$

which establishes (3.5.47).

In view of (3.5.47), and since $\psi_\varepsilon \rightharpoonup \psi$ in $H^1(\mathbb{R}^2)$ as $\varepsilon \to 0^+$, we conclude that

$$\liminf_{\varepsilon \to 0^+} \int_{\omega_\varepsilon} \bar{D}_\varepsilon \bar{\nabla} \psi_\varepsilon \bar{\nabla} \psi_\varepsilon + (d_\varepsilon - \bar{\zeta})|\psi_\varepsilon|^2 \,\mathrm{d}\bar{z} \geqslant \int_{\mathbb{R}^2} \bar{D}_0 \bar{\nabla} \psi \bar{\nabla} \psi + (d_0 - \bar{\zeta})|\psi|^2 \,\mathrm{d}\bar{z} \tag{3.5.51}$$

for any $0 < \bar{\zeta} < \zeta$. Furthermore, using in addition the strong convergence $f_\varepsilon \to f$ in $L^2(\mathbb{R}^2)$ as $\varepsilon \to 0^+$, and the fact that (3.5.45) holds for all $\vartheta \in H_0^1(\omega_\varepsilon)$ and (3.5.46) holds for all $\vartheta \in H^1(\mathbb{R}^2)$, we deduce that

$$\int_{\omega_\varepsilon} \bar{D}_\varepsilon \bar{\nabla} \psi_\varepsilon \bar{\nabla} \psi_\varepsilon + d_\varepsilon |\psi_\varepsilon|^2 \,\mathrm{d}\bar{z} = \int_{\omega_\varepsilon} f \,\psi_\varepsilon \,\mathrm{d}\bar{z} \underset{\varepsilon \to 0^+}{\longrightarrow} \int_{\mathbb{R}^2} f \,\psi \,\mathrm{d}\bar{z} = \int_{\mathbb{R}^2} \bar{D}_0 \bar{\nabla} \psi \bar{\nabla} \psi + d_0 |\psi|^2 \,\mathrm{d}\bar{z}.$$

Consequently,

$$\begin{aligned} \liminf_{\varepsilon \to 0^+} \int_{\omega_\varepsilon} \bar{D}_\varepsilon \bar{\nabla} \psi_\varepsilon \bar{\nabla} \psi_\varepsilon + (d_\varepsilon - \bar{\zeta})|\psi_\varepsilon|^2 \,\mathrm{d}\bar{z} &= \lim_{\varepsilon \to 0^+} \left(\int_{\omega_\varepsilon} \bar{D}_\varepsilon \bar{\nabla} \psi_\varepsilon \bar{\nabla} \psi_\varepsilon + d_\varepsilon |\psi_\varepsilon|^2 \,\mathrm{d}\bar{z} \right) - \bar{\zeta} L \\ &= \int_{\mathbb{R}^2} \bar{D}_0 \bar{\nabla} \psi \bar{\nabla} \psi + d_0 |\psi|^2 \,\mathrm{d}\bar{z} - \bar{\zeta} L, \end{aligned} \tag{3.5.52}$$

where we also used the definition of L. From (3.5.51) and (3.5.52) we deduce that $L \leqslant \int_{\mathbb{R}^2} |\psi|^2 \,\mathrm{d}\bar{z}$. Hence, $L = \int_{\mathbb{R}^2} |\psi|^2 \,\mathrm{d}\bar{z}$ and $\mathcal{S}_\varepsilon^{-1} f_\varepsilon = \psi_\varepsilon \to \psi = \mathcal{S}^{-1} f$ in $L^2(\mathbb{R}^2)$ as $\varepsilon \to 0^+$.

Finally, we prove (3.5.44). Assume by contradiction that there is $\gamma \in \sigma(\mathcal{S}^{-1})$ which is not a cluster point of $\sigma(\mathcal{S}_\varepsilon^{-1})$. Then there exist $c > 0$ and $\varepsilon_0 > 0$ such that for all $\gamma_\varepsilon \in \sigma(\mathcal{S}_\varepsilon^{-1})$ with $0 < \varepsilon < \varepsilon_0$ one has

$$|\gamma_\varepsilon - \gamma| > c.$$

Let $f \in L^2(\mathbb{R}^2)$, and set $f_\varepsilon := f\chi_{\omega_\varepsilon} \in L^2(\omega_\varepsilon)$. If $\gamma_\varepsilon \in \sigma(\mathcal{S}_\varepsilon^{-1})$ with $0 < \varepsilon < \varepsilon_0$, then

$$\|\mathcal{S}_\varepsilon^{-1} f_\varepsilon - \gamma f_\varepsilon\|_{L^2(\mathbb{R}^2)} = \|\mathcal{S}_\varepsilon^{-1} f_\varepsilon - \gamma f_\varepsilon\|_{L^2(\omega_\varepsilon)} \geqslant |\gamma_\varepsilon - \gamma| \|f_\varepsilon\|_{L^2(\omega_\varepsilon)} \geqslant c\|f_\varepsilon\|_{L^2(\mathbb{R}^2)}. \tag{3.5.53}$$

Using the strong convergence of $\mathcal{S}_\varepsilon^{-1}$ established above together with the strong convergence $f_\varepsilon \to f$ in $L^2(\mathbb{R}^2)$ as $\varepsilon \to 0^+$, and letting $\varepsilon \to 0^+$ in (3.5.53), we get

$$\|\mathcal{S}^{-1} f - \gamma f\|_{L^2(\mathbb{R}^2)} \geqslant c\|f\|_{L^2(\mathbb{R}^2)},$$

which contradicts the fact that $\gamma \in \sigma(\mathcal{S}^{-1})$ since $f \in L^2(\mathbb{R}^2)$ was taken arbitrarily. Thus (3.5.44) holds, and this finishes the proof of Theorem 3.1.7. □

Chapter 4

Multiscale Convergence of Sequences of Radon Measures

Under the motivation mentioned in the Introduction (see Subsection 1.2), in this chapter we are concerned with the characterization of $(n+1)$-scale limit pairs (u, U) of sequences $\{(u_\varepsilon \mathcal{L}^N\lfloor_\Omega, Du_\varepsilon\lfloor_\Omega)\}_{\varepsilon>0} \subset \mathcal{M}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV(\Omega; \mathbb{R}^d)$. This characterization, established in Ferreira and Fonseca [43], is useful in the study of the asymptotic behavior of periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation and described by $n \in \mathbb{N}$ microscales, undertaken in Chapter 5.

The notion of two-scale convergence was first introduced by Nguetseng [64] and further developed by Allaire [1]. It was used to provide a mathematical rigorous justification of the formal asymptotic expansions that used to be commonly adopted in the study of homogenization problems (see, for example, Bensoussan, Lions and Papanicolaou [13], Jikov, Kozlov and Oleĭnik [53] and Sanchez-Palencia [69]).

In Allaire and Briane [2] the authors extended that notion to the case of multiple separated scales of periodic oscillations. Precisely,

Definition 4.0.1. *Let $n, N \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, and let $Q := [0,1]^N$. Let $\varrho_1, ..., \varrho_n : (0,\infty) \to (0,\infty)$ satisfy for all $i \in \{1, \cdots, n\}$ and for all $j \in \{2, \cdots, n\}$,*

$$\lim_{\varepsilon\to 0^+} \varrho_i(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0^+} \frac{\varrho_j(\varepsilon)}{\varrho_{j-1}(\varepsilon)} = 0. \tag{4.0.1}$$

A sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega)$ is said to $(n+1)$-scale converge to a function $u_0 \in L^2(\Omega \times Q_1 \times \cdots \times Q_n)$, where each Q_i is a copy of Q, if for every $\varphi \in L^2(\Omega; C_\#(Q_1 \times \cdots \times Q_n))$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_\Omega u_\varepsilon(x)\varphi\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\Big)\,\mathrm{d}x \\ = \int_{\Omega\times Q_1\times\cdots\times Q_n} u_0(x, y_1, \cdots, y_n)\varphi(x, y_1, \cdots, y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,\end{aligned}$$

in which case we write $u_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} u_0$.

Remark 4.0.2. *In the context of multiscale composites, the functions $\varrho_1, ..., \varrho_n$ stand for the length scales or scales of oscillation. The second condition in (4.0.1) is known as a separation of scales hypothesis.*

Also, Allaire and Briane [2] established a compactness result concerning this notion and provided the relationship between the $(n+1)$-scale limit and the usual weak limit in $L^2(\Omega)$ (see [2, Thms. 2.4 and 2.5]). Precisely,

Theorem 4.0.3. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L^2(\Omega)$. Then, there exist a (not relabeled) subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ and a function $u_0 \in L^2(\Omega \times Q_1 \times \cdots \times Q_n)$ such that $u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} u_0$. Furthermore, $u_\varepsilon \rightharpoonup \bar{u}_0$ weakly in $L^2(\Omega)$ as $\varepsilon \to 0^+$, where $\bar{u}_0(x) := \int_{Q_1\times\cdots\times Q_n} u_0(x, y_1, \cdots, y_n)\,\mathrm{d}y_1 \cdots \mathrm{d}y_n$, and $\lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_{L^2(\Omega)} \geqslant \|u_0\|_{L^2(\Omega\times Q_1\times\cdots\times Q_n)} \geqslant \|\bar{u}_0\|_{L^2(\Omega)}$.*

In general the $(n+1)$-scale limit differs from the weak limit in $L^2(\Omega)$, with the $(n+1)$-scale limit capturing more information on the oscillatory behavior of a bounded sequence in $L^2(\Omega)$ than its weak limit in $L^2(\Omega)$. The proof of Theorem 4.0.3 follows the arguments introduced in the case $n = 1$ treated in Allaire [1] (see also Nguetseng [64]).

Moreover, in order to study the asymptotic behavior of the solutions of certain partial differential equations with periodically oscillating coefficients in the space $H^1(\Omega)$, the $(n+1)$-scale limit of gradients was fully characterized in Allaire and Briane [2, Thm. 1.2]. Precisely,

Theorem 4.0.4. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $H^1(\Omega)$. Then there exist $u \in H^1(\Omega)$ and n functions $u_i \in L^2(\Omega \times Q_1 \times \cdots \times Q_{i-1}; H^1_\#(Q_i))$, for $i \in \{1, \cdots, n\}$, such that*

$$u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} u, \tag{4.0.2}$$

and, up to a not relabeled subsequence,

$$\nabla u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \nabla u + \sum_{i=1}^{n} \nabla_{y_i} u_i. \tag{4.0.3}$$

Furthermore, given any $u \in H^1(\Omega)$ and $u_i \in L^2(\Omega \times Q_1 \times \cdots \times Q_{i-1}; H^1_\#(Q_i))$, $i \in \{1, \cdots, n\}$, there exists a bounded sequence $\{u_\varepsilon\}_{\varepsilon>0}$ for which (4.0.2) and (4.0.3) hold.

Remark 4.0.5. *In the theorem above, the function u is the weak limit in $H^1(\Omega)$ of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$. The terms $\nabla_{y_i} u_i$ in (4.0.3) may be interpreted as the gradient limits at each scale.*

Remark 4.0.6. *Definition 4.0.1 and Theorem 4.0.4 admit simple generalizations to the cases $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, for any $p \in (1, \infty)$.*

Theorem 4.0.4 extends Prop. 1.14 (i) in Allaire [1] to the case in which $n \geqslant 2$, but its proof requires significant changes and is rather more difficult. By means of this result, Allaire and Briane [2] completely characterize the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the family of boundary value problems

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{a.e. in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$, $A_\varepsilon(x) := A\big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\big)$, and A is a $N \times N$ matrix satisfying appropriate coercivity and boundedness hypotheses, and such that $A(x, \cdot)$ is $Q_1 \times \cdots \times Q_n$-periodic (see Allaire and Briane [2, Thm. 1.3]).

A similar analysis was undertaken in Allaire [1] in the case $n = 1$. Also in Allaire [1] (see [1, Thms. 3.1 and 3.3]), the author provides a simple and elegant proof for the homogenized functional of a sequence $\{I_\varepsilon\}_{\varepsilon>0}$ of functionals of the form

$$u \in W_0^{1,p}(\Omega; \mathbb{R}^d) \mapsto I_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u(x)\Big)\,\mathrm{d}x.$$

Following this last approach, in Amar [5] the author extended the notion of two-scale convergence to the case of bounded sequences of Radon measures with finite total variation, and characterized the two-scale limit associated with a bounded sequence in $BV(\Omega)$ (see Amar [5, Thm. 3.6]). Using this characterization, the asymptotic behavior as $\varepsilon \to 0^+$ of sequences of positively 1-homogeneous and periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation, of the form

$$u \in BV(\Omega) \mapsto I_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}Du}{\mathrm{d}\|Du\|}(x)\Big)\,\mathrm{d}\|Du\|(x)$$

is given in Amar [5, Thm 4.1].

The purpose of the present chapter is to present an extension of the notion of two-scale convergence for sequences of Radon measures with finite total variation introduced in Amar [5] to the case of multiple periodic length scales of oscillations, and the characterization of the $(n+1)$-limit associated with a bounded sequence in $BV(\Omega;\mathbb{R}^d)$. The remaining part of this chapter is organized as follows. In Section 4.1 we state our main results, whose proofs are presented in Section 4.2. This study was elaborated in the joint work with Fonseca [43].

4.1. Main Results.

The notations introduced in Subsection 2.1.6 and the analysis undertaken in Subsection 2.3.2 will play an important role in the sequel.

Definition 4.1.1. *Let $m, n, N \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^N$ be an open set and define $Y := (0,1)^N$. Let $\varrho_1, ..., \varrho_n$ be positive functions in $(0,\infty)$ satisfying (4.0.1). We say that a sequence $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ of Radon measures $(n+1)$-scale converges to a Radon measure $\mu_0 \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$, where each Y_i is a copy of Y, if for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$ we have*

$$\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\Big)\cdot \mathrm{d}\mu_\varepsilon(x) = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n)\cdot \mathrm{d}\mu_0(x, y_1, \cdots, y_n), \quad (4.1.1)$$

in which case we write $\mu_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \mu_0$.

Remark 4.1.2. *The $(n+1)$-scale limit μ_0 may depend on the sequence $\{\varepsilon\}$. Indeed, let $n = 1$, $\varrho_1(\varepsilon) = \varepsilon$ for all $\varepsilon > 0$, let $\Omega \subset \mathbb{R}^N$ be open and bounded, and let $\vartheta \in C_\#(Y)$. Define $\mu_\varepsilon := \vartheta\big(\frac{\cdot}{\varepsilon}\big)\mathcal{L}^N\lfloor\Omega$. If $\varphi \in C_0(\Omega; C_\#(Y))$, then by the Riemann–Lebesgue's Lemma*

$$\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}\mu_\varepsilon(x) = \lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\vartheta\Big(\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{\Omega\times Y}\varphi(x,y)\vartheta(y)\,\mathrm{d}x\mathrm{d}y =: \langle \mathcal{L}^N\lfloor\Omega \otimes \vartheta\mathcal{L}^N_y, \varphi\rangle$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}\mu_{\varepsilon^2}(x) &= \lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\vartheta\Big(\frac{x}{\varepsilon^2}\Big)\,\mathrm{d}x = \int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1)\vartheta(y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &= \int_{\Omega\times Y}\varphi(x,y)\Big(\int_{Y_2}\vartheta(y_2)\,\mathrm{d}y_2\Big)\mathrm{d}x\mathrm{d}y =: \langle \bar\vartheta\mathcal{L}^N\lfloor\Omega \otimes \mathcal{L}^N_y, \varphi\rangle,\end{aligned}$$

where $\bar\vartheta := \int_Y \vartheta(y)\,\mathrm{d}y$. Hence $\mu_\varepsilon \xrightarrow[\varepsilon]{2\text{-}sc} \mathcal{L}^N\lfloor\Omega \otimes \vartheta\mathcal{L}^N_y$, while $\mu_{\varepsilon^2} \xrightarrow[\varepsilon]{2\text{-}sc} \bar\vartheta\mathcal{L}^N\lfloor\Omega \otimes \mathcal{L}^N_y$. This example shows that it may be the case that $\mu_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \mu_0$ and $\mu_{\varepsilon'} \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \lambda_0$, with $\varepsilon' \prec \varepsilon$, but $\mu_0 \neq \lambda_0$. What *we can guarantee is that* $\mu_{\varepsilon'} \xrightarrow[\varepsilon']{(n+1)\text{-}sc} \mu_0$. *This is due to the dependence of the test functions on the length scales.*

The notion of $(n+1)$-scale convergence is justified in view of the following compactness result asserting that every bounded sequence $\{\mu_\varepsilon\}_{\varepsilon>0}$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$ admits a $(n+1)$-scale convergent subsequence.

Theorem 4.1.3. *Let* $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ *be a bounded sequence. Then there exist a subsequence* $\{\mu_{\varepsilon'}\}_{\varepsilon'>0}$ *of* $\{\mu_\varepsilon\}_{\varepsilon>0}$ *and a measure* $\mu_0 \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ *such that* $\mu_{\varepsilon'} \xrightarrow[\varepsilon']{(n+1)\text{-}sc} \mu_0$.

As in the cases studied in Allaire [1], Allaire and Briane [2], and Amar [5], the $(n+1)$-scale limit contains more information on the oscillations of a bounded sequence in $\mathcal{M}(\Omega;\mathbb{R}^m)$ than its weak-$\star$ limit, in that the latter is the canonical projection of the $(n+1)$-scale limit onto Ω.

Proposition 4.1.4. *Let* $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ *and* $\mu_0 \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ *be such that* $\mu_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \mu_0$. *Then* $\mu_\varepsilon \overset{\star}{\rightharpoonup} \bar{\mu}_0$ *weakly-*$\star$ *in* $\mathcal{M}(\Omega;\mathbb{R}^m)$ *as* $\varepsilon \to 0^+$, *where* $\bar{\mu}_0 \in \mathcal{M}(\Omega;\mathbb{R}^m)$ *is the measure defined for all* $B \in \mathcal{B}(\Omega)$ *by*

$$\bar{\mu}_0(B) := \mu_0(B \times Y_1 \times \cdots \times Y_n).$$

Moreover, $\|\bar{\mu}_0\|(\Omega) \leqslant \|\mu_0\|(\Omega \times Y_1 \times \cdots \times Y_n) \leqslant \liminf_{\varepsilon\to 0^+} \|\mu_\varepsilon\|(\Omega)$.

Remark 4.1.5. *In view of Proposition 4.1.4, since every weakly-*$\star$ *convergent sequence in* $\mathcal{M}(\Omega;\mathbb{R}^m)$ *is bounded, the same holds for any* $(n+1)$*-scale convergent sequence in* $\mathcal{M}(\Omega;\mathbb{R}^m)$.

Assume that $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega;\mathbb{R}^d)$ is a bounded sequence. By Theorem 4.1.3, there exist subsequences of $\left\{u_\varepsilon \mathcal{L}^N{\lfloor_\Omega}\right\}_{\varepsilon>0}$ and $\{Du_\varepsilon\}_{\varepsilon>0}$ that $(n+1)$-scale converge. The next theorem provides a characterization of these $(n+1)$-scale limits as well as the relationship between them. We will assume a stronger separation of scales hypothesis than the one in (4.0.1), precisely (cf. Allaire and Briane [2]),

Definition 4.1.6. *The scales* $\varrho_1, ..., \varrho_n$ *are said to be well-separated if there exists* $m \in \mathbb{N}$ *such that for all* $i \in \{2, \cdots, n\}$,

$$\lim_{\varepsilon\to 0^+} \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)} = 0. \tag{4.1.2}$$

The case in which $\varrho_i(\varepsilon) := \varepsilon^i$ is a simple example of well-separated scales. Indeed, it suffices to take $m = n+1$.

Theorem 4.1.7. *Let* $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega;\mathbb{R}^d)$ *be a sequence such that* $u_\varepsilon \overset{\star}{\rightharpoonup} u$ *weakly-*$\star$ *in* $BV(\Omega;\mathbb{R}^d)$ *as* $\varepsilon \to 0^+$, *for some* $u \in BV(\Omega;\mathbb{R}^d)$. *Assume that the length scales* $\varrho_1, ..., \varrho_n$ *satisfy (4.0.1) and (4.1.2). Then*

a) $u_\varepsilon \mathcal{L}^N{\lfloor_\Omega} \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \tau_u$, *where* $\tau_u \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^d)$ *is the measure defined by*

$$\tau_u := u\,\mathcal{L}^N{\lfloor_\Omega} \otimes \mathcal{L}^{nN}_{y_1,\cdots,y_n},$$

i.e., if $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^d))$ *then*

$$\langle \tau_u, \varphi\rangle = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot u(x)\,\mathrm{d}x\mathrm{d}y_1 \cdots \mathrm{d}y_n.$$

b) there exist a subsequence $\{Du_{\varepsilon'}\}_{\varepsilon'>0}$ of $\{Du_\varepsilon\}_{\varepsilon>0}$ and n measures $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$, $i \in \{1, \cdots, n\}$, such that

$$Du_{\varepsilon'} \xrightarrow[\varepsilon']{(n+1)\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n},$$

where $\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n} \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^{d\times N})$ is the measure

$$\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n} := Du\lfloor_\Omega \otimes \mathcal{L}^{nN}_{y_1,\cdots,y_n} + \sum_{i=1}^{n-1} \lambda_i \otimes \mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n} + \lambda_n, \tag{4.1.3}$$

i.e., if $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^{d\times N}))$ then

$$\begin{aligned}\langle \lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}, \varphi\rangle = &\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}Du(x)\mathrm{d}y_1 \cdots \mathrm{d}y_n \\ &+ \sum_{i=1}^{n-1} \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_i(x, y_1, \cdots, y_i)\mathrm{d}y_{i+1} \cdots \mathrm{d}y_n \\ &+ \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_n(x, y_1, \cdots, y_n),\end{aligned}$$

and each $\lambda_i \in \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i; \mathbb{R}^{d\times N})$ is the measure associated with $D_{y_i}\boldsymbol{\mu}_i$, $i \in \{1, \cdots, n\}$.

The proof of Theorem 4.1.7 is not a simple generalization of the analogous result in the case $n = 1$ treated in Amar [5]. When $n \geqslant 2$, and similarly to Allaire and Briane [2], some new arguments are needed. We also show that Theorem 4.1.7 fully characterizes the $(n + 1)$-scale limit of bounded sequences in $BV(\Omega; \mathbb{R}^d)$, in that:

Proposition 4.1.8. *Let $u \in BV(\Omega; \mathbb{R}^d)$ and let $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i; \mathbb{R}^d))$, $i \in \{1, \cdots, n\}$. Then there exists a bounded sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega; \mathbb{R}^d)$ for which a) and b) of Theorem 4.1.7 hold (with ε' replaced by ε).*

Remark 4.1.9. *Proposition 4.1.8 together with Theorem 4.1.7 represent the BV version of Theorem 4.0.4.*

4.2. Multiscale Convergence in BV.

We start this section by proving Theorem 4.1.3 and Proposition 4.1.4, which are simple generalizations of Amar [5, Thm 3.5] (see also Allaire [1]) and Amar [5, Lemmas 3.3 and 3.4], respectively. Here, the letter $\mathcal{C}$ represents a generic positive constant, whose value may change from expression to expression.

Proof of Theorem 4.1.3. Let $c := \sup_{\varepsilon>0} \|\mu_\varepsilon\|(\Omega) < +\infty$. For all $\varepsilon > 0$ and $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$ we have that

$$\left|\int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) \cdot \mathrm{d}\mu_\varepsilon(x)\right| \leqslant c\|\varphi\|_\infty. \tag{4.2.1}$$

Hence $F_\varepsilon : \varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)) \mapsto F_\varepsilon(\varphi) := \int_\Omega \varphi\big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\big) \cdot \mathrm{d}\mu_\varepsilon(x)$ is a linear and continuous functional in $C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$. By Riesz Representation Theorem (see Theorem 2.1.46), there exists a measure $\lambda_\varepsilon \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ such that for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$,

$$\begin{aligned}F_\varepsilon(\varphi) &= \int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) \cdot \mathrm{d}\mu_\varepsilon(x) = \langle \lambda_\varepsilon, \varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^m), C_0(\Omega; C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m))}\\ &= \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}\lambda_\varepsilon(x, y_1, \cdots, y_n).\end{aligned}$$

Using (4.2.1) we have $\sup_{\varepsilon>0} \|\lambda_\varepsilon\|(\Omega \times Y_1 \times \cdots \times Y_n) \leqslant c$, and so there exist a subsequence $\{\lambda_{\varepsilon'}\}_{\varepsilon'>0}$ of $\{\lambda_\varepsilon\}_{\varepsilon>0}$ and a measure $\lambda_0 \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ such that $\lambda_{\varepsilon'} \overset{\star}{\rightharpoonup} \lambda_0$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ as $\varepsilon' \to 0^+$, that is,

$$\lim_{\varepsilon'\to 0^+} \langle \lambda_{\varepsilon'}, \varphi\rangle_{\mathcal{M},C_0} = \langle \lambda_0, \varphi\rangle_{\mathcal{M},C_0},$$

or, equivalently,

$$\lim_{\varepsilon'\to 0^+} \int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon')}, \cdots, \frac{x}{\varrho_n(\varepsilon')}\right) \cdot \mathrm{d}\mu_{\varepsilon'}(x) = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}\lambda_0(x, y_1, \cdots, y_n)$$

for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$. This proves that $\mu_{\varepsilon'} \xrightarrow[\varepsilon']{(n+1)\text{-}sc} \lambda_0$. □

Proof of Proposition 4.1.4. Let $\phi \in C_0(\Omega;\mathbb{R}^m)$ be given. By (4.1.1) we have

$$\lim_{\varepsilon\to 0^+} \int_\Omega \phi(x) \cdot \mathrm{d}\mu_\varepsilon(x) = \int_{\Omega\times Y_1\times\cdots\times Y_n} \phi(x) \cdot \mathrm{d}\mu_0(x, y_1 \cdots, y_n) = \int_\Omega \phi(x) \cdot \mathrm{d}\bar{\mu}_0(x).$$

Thus $\mu_\varepsilon \overset{\star}{\rightharpoonup} \bar{\mu}_0$ weakly-$\star$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$ as $\varepsilon \to 0^+$.

Furthermore,

$$\begin{aligned}&\|\mu_0\|(\Omega \times Y_1 \times \cdots \times Y_n)\\ &\quad= \sup\left\{ \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}\mu_0(x, y_1 \cdots, y_n) \colon \right.\\ &\qquad\qquad \left. \varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)),\ \|\varphi\|_\infty \leqslant 1 \right\}\\ &\quad= \sup\left\{ \lim_{\varepsilon\to 0^+} \int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) \cdot \mathrm{d}\mu_\varepsilon(x) \colon \right.\\ &\qquad\qquad \left. \varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)),\ \|\varphi\|_\infty \leqslant 1 \right\}\\ &\quad\leqslant \liminf_{\varepsilon\to 0^+} \left[\sup\left\{ \int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) \cdot \mathrm{d}\mu_\varepsilon(x) \colon \right.\right.\\ &\qquad\qquad \left.\left. \varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m)),\ \|\varphi\|_\infty \leqslant 1 \right\}\right]\\ &\quad\leqslant \liminf_{\varepsilon\to 0^+} \left[\sup\left\{ \int_\Omega \phi(x) \cdot \mathrm{d}\mu_\varepsilon(x) \colon\ \phi \in C_0(\Omega;\mathbb{R}^m),\ \|\phi\|_\infty \leqslant 1 \right\}\right]\\ &\quad= \liminf_{\varepsilon\to 0^+} \|\mu_\varepsilon\|(\Omega),\end{aligned}$$

where we have used the fact that if $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$, then for each $\varepsilon > 0$, the function $\phi_\varepsilon(x) := \varphi\big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\big)$, $x \in \Omega$, belongs to $C_0(\Omega;\mathbb{R}^m)$ and

$$\|\phi_\varepsilon\|_\infty = \sup_{x\in\Omega} \left| \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right)\right| \leqslant \sup_{\substack{x\in\Omega\\ y_i\in\mathbb{R}^N, i\in\{1,\cdots,n\}}} |\varphi(x, y_1, \cdots, y_n)| = \|\varphi\|_\infty.$$

On the other hand,

$$
\begin{aligned}
\|\bar{\mu}_0\|(\Omega) &= \sup\left\{ \int_\Omega \phi(x)\cdot \mathrm{d}\bar{\mu}_0(x) \colon\ \phi \in C_0(\Omega;\mathbb{R}^m),\ \|\phi\|_\infty \leqslant 1\right\}\\
&= \sup\left\{ \int_{\Omega\times Y_1\times\cdots\times Y_n} \phi(x)\cdot \mathrm{d}\mu_0(x,y_1,\cdots,y_n) \colon\ \phi \in C_0(\Omega;\mathbb{R}^m),\ \|\phi\|_\infty \leqslant 1\right\}\\
&\leqslant \sup\left\{ \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n)\cdot \mathrm{d}\mu_0(x,y_1\cdots,y_n) \colon \right.\\
&\qquad\qquad\qquad \left. \varphi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m)),\ \|\varphi\|_\infty\leqslant 1\right\}\\
&= \|\mu_0\|(\Omega\times Y_1\times\cdots\times Y_n),
\end{aligned}
$$

which concludes the proof of Proposition 4.1.4. □

In order to prove Theorem 4.1.7 we we need an auxiliary lemma, which is an extension of Amar [5, Thm. 2.5] (see also Allaire and Briane [2, Lemma 3.7]). We first introduce some notation.

Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be the function defined by

$$
\rho(x) := \begin{cases} c\, e^{\frac{1}{|x|^2-1}}, & |x|<1,\\ 0, & |x|\geqslant 1,\end{cases}
$$

where $c>0$ is such that $\int_{\mathbb{R}^N}\rho(x)\,\mathrm{d}x = 1$. For each $0<\varepsilon<1$ let

$$
\rho_\varepsilon(x) := \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right). \tag{4.2.2}
$$

Then $\rho_\varepsilon\in C_c^\infty(\mathbb{R}^N)$ and

$$
\int_{\mathbb{R}^N}\rho_\varepsilon(x)\,\mathrm{d}x = 1,\quad \operatorname{supp}\rho_\varepsilon\subset\overline{B(0,\varepsilon)},\quad \rho_\varepsilon\geqslant 0,\quad \rho_\varepsilon(-x)=\rho_\varepsilon(x), \tag{4.2.3}
$$

for all $x\in\mathbb{R}^N$.

For $0<\varepsilon<1/2$, let η_ε denote the extension to $\mathbb{R}^N$ by $(-\frac{1}{2},\frac{1}{2})^N$-periodicity of the function $\rho_{\varepsilon|(-\frac{1}{2},\frac{1}{2})^N}$. Then $\eta_\varepsilon\in C_\#^\infty(Y)$ is such that

$$
\int_Q \eta_\varepsilon(y)\,\mathrm{d}y = 1,\quad \eta_\varepsilon\geqslant 0,\quad \eta_\varepsilon(-x)=\eta_\varepsilon(x), \tag{4.2.4}
$$

for any unit cube $Q\subset\mathbb{R}^N$ and $x\in\mathbb{R}^N$.

Lemma 4.2.1. *Let $\lambda\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^N)$ be given. The following conditions are equivalent:*

i) for all $i\in\{1,\cdots,n\}$ there exists a measure $\boldsymbol{\mu}_i\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i))$ such that

$$
\lambda := \begin{cases} \lambda_1 & \text{if } n=1,\\ \displaystyle\sum_{i=1}^{n-1}\lambda_i\otimes\mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n}+\lambda_n & \text{if } n\geqslant 2,\end{cases}
$$

where each $\lambda_i\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^N)$ is the measure associated with $D_{y_i}\boldsymbol{\mu}_i$;

ii) for all $\varphi \in C_c^\infty(\Omega; C_\#^\infty(Y_1 \times \cdots \times Y_n; \mathbb{R}^N))$ such that $\operatorname{div}_{y_n} \varphi = 0$ and, if $n \geqslant 2$, for all $k \in \{1, \cdots, n-1\}$, $x \in \Omega$, $y_i \in Y_i$, $i \in \{1, \cdots, n\}$,

$$\int_{Y_{k+1} \times \cdots \times Y_n} \operatorname{div}_{y_k} \varphi(x, y_1, \cdots, y_n) \, dy_{k+1} \cdots dy_n = 0,$$

we have

$$\int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot d\lambda(x, y_1, \cdots, y_n) = 0.$$

PROOF. We will give the proof only for $n = 2$, the argument being easily adapted for any $n \in \mathbb{N}$.

Step 1. Assume first that *i)* holds, and let $\varphi \in C_c^\infty(\Omega; C_\#^\infty(Y_1 \times Y_2; \mathbb{R}^N))$ be such that $\operatorname{div}_{y_2} \varphi = 0$ and

$$\int_{Y_2} \operatorname{div}_{y_1} \varphi(x, y_1, y_2) \, dy_2 = 0.$$

Using the decomposition of λ as in *i)*, we have

$$\int_{\Omega \times Y_1 \times Y_2} \varphi \cdot d\lambda(x, y_1, y_2) = \int_{\Omega \times Y_1 \times Y_2} \varphi \cdot d\lambda_1(x, y_1) dy_2 + \int_{\Omega \times Y_1 \times Y_2} \varphi \cdot d\lambda_2(x, y_1, y_2). \tag{4.2.5}$$

We will show that both integrals on the right-hand side of (4.2.5) are equal to zero. Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a sequence of the form $\varphi_j(x, y_1, y_2) = \sum_{k=1}^{m_j} \phi_k^{(j)}(x) \psi_k^{(j)}(y_1) \theta_k^{(j)}(y_2)$, where $m_j \in \mathbb{N}$ and for all $k \in \{1, \cdots, m_j\}$, $\phi_k^{(j)} \in C_c^\infty(\Omega)$, $\psi_k^{(j)} \in C_\#^\infty(Y_1)$, $\theta_k^{(j)} \in C_\#^\infty(Y_2; \mathbb{R}^N)$, converging to φ in $C_0^\infty(\Omega; C_\#^\infty(Y_1 \times Y_2; \mathbb{R}^N))$. Then,

$$\int_{Y_2} \operatorname{div}_{y_1} \varphi_j \, dy_2 = \sum_{k=1}^{m_j} \left(\phi_k^{(j)} \nabla \psi_k^{(j)} \cdot \int_{Y_2} \theta_k^{(j)} \, dy_2 \right) \to \int_{Y_2} \operatorname{div}_{y_1} \varphi \, dy_2 = 0 \text{ in } C_0(\Omega; C_\#(Y_1)), \tag{4.2.6}$$

$$\operatorname{div}_{y_2} \varphi_j = \sum_{k=1}^{m_j} \phi_k^{(j)} \psi_k^{(j)} \operatorname{div} \theta_k^{(j)} \to \operatorname{div}_{y_2} \varphi = 0 \text{ in } C_0(\Omega; C_\#(Y_1 \times Y_2)). \tag{4.2.7}$$

as $j \to +\infty$. The convergence $\varphi_j \to \varphi$ in $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^N))$ as $j \to +\infty$ and Lemma 2.3.20 (see also Remark 2.3.21) yield

$$\begin{aligned}
&\int_{\Omega \times Y_1 \times Y_2} \varphi(x, y_1, y_2) \cdot d\lambda_1(x, y_1) dy_2 = \lim_{j \to +\infty} \int_{\Omega \times Y_1 \times Y_2} \varphi_j(x, y_1, y_2) \cdot d\lambda_1(x, y_1) dy_2 \\
&\quad = \lim_{j \to +\infty} \left\{ \sum_{k=1}^{m_j} \int_{\Omega \times Y_1} \phi_k^{(j)}(x) \psi_k^{(j)}(y_1) \, d\lambda_1(x, y_1) \cdot \int_{Y_2} \theta_k^{(j)}(y_2) \, dy_2 \right\} \\
&\quad = \lim_{j \to +\infty} \left\{ - \sum_{k=1}^{m_j} \int_{Y_1} \left(\int_\Omega \phi_k^{(j)}(x) \, d\boldsymbol{\mu}_1(x) \right)(y_1) \, \nabla \psi_k^{(j)}(y_1) \, dy_1 \cdot \int_{Y_2} \theta_k^{(j)}(y_2) \, dy_2 \right\} \\
&\quad = \lim_{j \to +\infty} \left\{ - \sum_{k=1}^{m_j} \int_{Y_1} \left(\int_\Omega \phi_k^{(j)}(x) \, d\boldsymbol{\mu}_1(x) \right)(y_1) \, \tilde{\psi}_k^{(j)}(y_1) \, dy_1 \right\},
\end{aligned} \tag{4.2.8}$$

where $\tilde{\psi}_k^{(j)} := \nabla \psi_k^{(j)} \cdot \int_{Y_2} \theta_k^{(j)} \, dy_2$. By (4.2.6), $\sum_{k=1}^{m_j} \phi_k^{(j)} \tilde{\psi}_k^{(j)} \to 0$ in $C_0(\Omega; C_\#(Y_1))$ as $j \to +\infty$, and so, using (4.2.8) and Definition 2.3.18, we obtain

$$\int_{\Omega \times Y_1 \times Y_2} \varphi(x, y_1, y_2) \cdot d\lambda_1(x, y_1) dy_2 = \int_{\Omega \times Y_1} 0 \, d\boldsymbol{\mu}_1(x) dy_1 = 0. \tag{4.2.9}$$

Similarly, in view of (2.3.19), (2.3.22) and (4.2.7), we get

$$\begin{aligned}
\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2) &= \lim_{j\to+\infty}\int_{\Omega\times Y_1\times Y_2} \varphi_j(x,y_1,y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2)\\
&= \lim_{j\to+\infty}\left\{\sum_{k=1}^{m_j}\int_{\Omega\times Y_1\times Y_2}\phi_k^{(j)}(x)\psi_k^{(j)}(y_1)\theta_k^{(j)}(y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2)\right\}\\
&= \lim_{j\to+\infty}\left\{-\sum_{k=1}^{m_j}\int_{Y_2}\left(\int_{\Omega\times Y_1}\phi_k^{(j)}(x)\psi_k^{(j)}(y_1)\,\mathrm{d}\boldsymbol{\mu}_2(x,y_1)\right)(y_2)\operatorname{div}\theta_k^{(j)}(y_2)\,\mathrm{d}y_2\right\}\\
&= \int_{\Omega\times Y_1\times Y_2} 0\,\mathrm{d}\boldsymbol{\mu}_2(x,y_1)\mathrm{d}y_2 = 0.
\end{aligned}\tag{4.2.10}$$

From (4.2.5), (4.2.9) and (4.2.10), we conclude that

$$\int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda(x,y_1,y_2)=0,$$

which proves *ii)*.

Step 2. Conversely, assume by contradiction that *ii)* holds but $\lambda\notin\mathcal{E}$, where $\mathcal{E}$ is the space of all measures $\tau\in\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$ for which there exist two measures $\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ and $\boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$ such that

$$\tau=\lambda_1\otimes\mathcal{L}^N_{y_2}+\lambda_2,$$

where $\lambda_1\in\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ and $\lambda_2\in\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$ are the measures associated with $D_{y_1}\boldsymbol{\mu}_1$ and $D_{y_2}\boldsymbol{\mu}_2$, respectively.

Note that $\mathcal{E}$ is a vectorial subspace of $\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$. We claim that it is weakly-$\star$ closed.

Substep 2a. Assume that the claim holds. Recalling that in a Banach space, a convex set is weakly closed if, and only if, it is closed, then by a corollary to the Hahn–Banach Theorem (see, for example, [22, Cor. I.8]), there exists a function $\varphi\in C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))$ such that for all $\tau\in\mathcal{E}$,

$$\begin{aligned}
\langle\tau,\varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))} &= \int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1,y_2)\cdot\mathrm{d}\tau(x,y_1,y_2)=0,\\
\langle\lambda,\varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))} &= \int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1,y_2)\cdot\mathrm{d}\lambda(x,y_1,y_2)\neq 0.
\end{aligned}\tag{4.2.11}$$

Let $f\in C_c^\infty(\Omega)$, $g\in C_\#^\infty(Y_1)$ and $h\in C_\#^\infty(Y_2)$ be arbitrary. Define $\boldsymbol{\mu}_1:\mathcal{B}(\Omega)\to BV_\#(Y_1)$, $\boldsymbol{\mu}_2:\mathcal{B}(\Omega\times Y_1)\to BV_\#(Y_2)$ by

$$\boldsymbol{\mu}_1(B):=\left(\int_B f(x)\,\mathrm{d}x\right)g,\ \ B\in\mathcal{B}(\Omega),\quad \boldsymbol{\mu}_2(E):=\left(\int_E f(x)g(y_1)\,\mathrm{d}x\mathrm{d}y_1\right)h,\ \ E\in\mathcal{B}(\Omega\times Y_1).$$

Clearly, $\boldsymbol{\mu}_1\in\mathcal{M}(\Omega;BV_\#(Y_1))$ and $\boldsymbol{\mu}_2\in\mathcal{M}(\Omega\times Y_1;BV_\#(Y_2))$. Moreover, for all $B\in\mathcal{B}(\Omega)$, $E\in\mathcal{B}(\Omega\times Y_1)$,

$$D_{y_1}(\boldsymbol{\mu}_1(B))=\left(\int_B f(x)\,\mathrm{d}x\right)\nabla g\mathcal{L}^N{\lfloor_{Y_1}},\quad D_{y_2}(\boldsymbol{\mu}_2(E))=\left(\int_E f(x)g(y_1)\,\mathrm{d}x\mathrm{d}y_1\right)\nabla h\mathcal{L}^N{\lfloor_{Y_2}}.$$

Hence $\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ and $\boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$, with

$$\lambda_1=f\mathcal{L}^N_{\lfloor\Omega}\otimes\nabla g\mathcal{L}^N{\lfloor_{Y_1}}\quad\text{and}\quad\lambda_2=\left(fg\mathcal{L}^N_{\lfloor\Omega}\otimes\mathcal{L}^N{\lfloor_{Y_1}}\right)\otimes\nabla h\mathcal{L}^N{\lfloor_{Y_2}},$$

respectively. Thus $\lambda_1 \otimes \mathcal{L}^N_{y_2}$, $\lambda_2 \in \mathcal{E}$, and so by the first condition in (4.2.11), and denoting by $\langle\cdot\,,\cdot\rangle$ the duality pairing in the sense of distributions, we conclude that

$$
\begin{aligned}
0 &= \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_1(x,y_1)\mathrm{d}y_2 = \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot (f(x)\nabla g(y_1))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&= \int_{\Omega\times Y_1}\left(\int_{Y_2}\varphi(x,y_1,y_2)\,\mathrm{d}y_2\right)\cdot (f(x)\nabla g(y_1))\,\mathrm{d}x\mathrm{d}y_1 = -\left\langle \int_{Y_2}\operatorname{div}_{y_1}\varphi\,\mathrm{d}y_2, fg\right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2) = \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot (f(x)g(y_1)\nabla h(y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&= -\left\langle \operatorname{div}_{y_2}\varphi, fgh\right\rangle.
\end{aligned}
$$

The arbitrariness of $f \in C_c^\infty(\Omega)$, $g \in C^\infty_\#(Y_1)$ and $h \in C^\infty_\#(Y_2)$ yields

$$
\int_{Y_2}\operatorname{div}_{y_1}\varphi\,\mathrm{d}y_2 = 0 \quad\text{and}\quad \operatorname{div}_{y_2}\varphi = 0, \tag{4.2.12}
$$

in the sense of distributions.

Substep 2b. We show that (4.2.12) and *ii)* contradict the second condition in (4.2.11). We will derive such contradiction by proving that there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega; C^\infty_\#(Y_1\times Y_2;\mathbb{R}^N))$ such that $\operatorname{div}_{y_2}\varphi_j = 0$, $\int_{Y_2}\operatorname{div}_{y_1}\varphi_j\,\mathrm{d}y_2 = 0$ and $\varphi_j \to \varphi$ in $C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))$ as $j\to+\infty$.

Let $0<\varepsilon<1/2$, and let $\rho_\varepsilon \in C_c(\mathbb{R}^N)$ and $\eta_\varepsilon \in C_\#(Y)$ be the functions introduced above (see (4.2.2), (4.2.3) and (4.2.4)). For $x\in\Omega$, $y_1,y_2\in\mathbb{R}^N$, define

$$
\varphi_\varepsilon(x,y_1,y_2) := \int_{Y_1\times Y_2}\varphi(x,y_1',y_2')\eta_\varepsilon(y_1-y_1')\eta_\varepsilon(y_2-y_2')\,\mathrm{d}y_1'\mathrm{d}y_2'.
$$

Then $\varphi_\varepsilon \in C_0(\Omega; C^\infty_\#(Y_1\times Y_2;\mathbb{R}^N))$ and $\varphi_\varepsilon\to\varphi$ in $C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))$ as $\varepsilon\to0^+$. Moreover, by (4.2.12) $\operatorname{div}_{y_2}\varphi_\varepsilon = 0$ in $\Omega\times\mathbb{R}^N\times\mathbb{R}^N$ and $\int_{Y_2}\operatorname{div}_{y_1}\varphi_\varepsilon\,\mathrm{d}y_2 = 0$ in $\Omega\times\mathbb{R}^N$. In fact, let $f\in C_c^\infty(\Omega)$. We have that

$$
\begin{aligned}
\int_\Omega \operatorname{div}_{y_2}\varphi_\varepsilon(x,y_1,y_2)f(x)\,\mathrm{d}x &= \int_\Omega\left(\int_{Y_1\times Y_2}[\varphi(x,y_1',y_2')\eta_\varepsilon(y_1-y_1')]\cdot\nabla_{y_2}\eta_\varepsilon(y_2-y_2')\,\mathrm{d}y_1'\mathrm{d}y_2'\right)f(x)\,\mathrm{d}x\\
&= -\int_{\Omega\times Y_1\times Y_2}[\varphi(x,y_1',y_2')\eta_\varepsilon(y_1-y_1')]\cdot\nabla_{y_2'}\eta_\varepsilon(y_2-y_2')f(x)\,\mathrm{d}x\mathrm{d}y_1'\mathrm{d}y_2'\\
&= \left\langle \operatorname{div}_{y_2'}\varphi, \eta_\varepsilon(y_1-\cdot)\eta_\varepsilon(y_2-\cdot)f\right\rangle = 0,
\end{aligned}
$$

where we used (4.2.12). The continuity of $\operatorname{div}_{y_2}\varphi_\varepsilon$ and the arbitrariness of $f\in C_c^\infty(\Omega)$ yield $\operatorname{div}_{y_2}\varphi_\varepsilon = 0$ in $\Omega\times\mathbb{R}^N\times\mathbb{R}^N$, for all $0<\varepsilon<1/2$.

Similarly, using (4.2.4), (4.2.12) and Fubini's Theorem, we deduce that

$$
\begin{aligned}
&\int_\Omega\left(\int_{Y_2}\operatorname{div}_{y_1}\varphi_\varepsilon(x,y_1,y_2)\,\mathrm{d}y_2\right)f(x)\,\mathrm{d}x\\
&\quad= -\int_{\Omega\times Y_2}\left(\int_{Y_1\times Y_2}\varphi(x,y_1',y_2')\cdot\nabla_{y_1'}\eta_\varepsilon(y_1-y_1')\eta_\varepsilon(y_2-y_2')\mathrm{d}y_1'\mathrm{d}y_2'\right)f(x)\,\mathrm{d}x\mathrm{d}y_2\\
&\quad= -\int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1',y_2')\cdot\nabla_{y_1'}\eta_\varepsilon(y_1-y_1')\left(\int_{Y_2}\eta_\varepsilon(y_2-y_2')\mathrm{d}y_2\right)f(x)\,\mathrm{d}x\mathrm{d}y_1'\mathrm{d}y_2'\\
&\quad= -\int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1',y_2')\cdot\nabla_{y_1'}\eta_\varepsilon(y_1-y_1')f(x)\,\mathrm{d}x\mathrm{d}y_1'\mathrm{d}y_2' = \left\langle\int_{Y_2}\operatorname{div}_{y_1'}\varphi\,\mathrm{d}y_2', \eta_\varepsilon(y_1-\cdot)f\right\rangle = 0,
\end{aligned}
$$

from which we conclude that for all $0 < \varepsilon < 1/2$, $\int_{Y_2} \operatorname{div}_{y_1} \varphi_\varepsilon \, \mathrm{d}y_2 = 0$ in $\Omega \times \mathbb{R}^N$.

Extend φ_ε to $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ by zero outside $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$, and for each $j \in \mathbb{N}$ let

$$K_j := \Big\{x \in \Omega \colon \ |x| \leqslant j, \ \operatorname{dist}(x, \mathbb{R}^N \backslash \Omega) \geqslant \frac{2}{j}\Big\}, \quad \varphi_j^{(\varepsilon)}(x, y_1, y_2) := \varphi_\varepsilon(x, y_1, y_2)\chi_{K_j}(x),$$
$$\tilde{\varphi}_j^{(\varepsilon)}(x, y_1, y_2) := \int_{\mathbb{R}^N} \varphi_j^{(\varepsilon)}(x', y_1, y_2)\rho_{\frac{1}{j}}(x - x') \, \mathrm{d}x',$$

for all $(x, y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, where $\rho_{\frac{1}{j}}$ is the function given by (4.2.2) with ε replaced by $1/j$. Notice that $K_j \subset K_{j+1}$, and $\cup_{j\in\mathbb{N}} K_j = \Omega$. Moreover, since $\operatorname{supp} \rho_{\frac{1}{j}} \subset \overline{B(0, 1/j)}$ we have

$$\begin{aligned} \operatorname{supp} \tilde{\varphi}_j^{(\varepsilon)} &\subset \Big\{(x, y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \colon \operatorname{dist}(x, K_j) \leqslant \frac{1}{j}\Big\} \\ &\subset \Big\{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) \geqslant \frac{1}{j}\Big\} \times \mathbb{R}^N \times \mathbb{R}^N. \end{aligned}$$

Hence,

$$\tilde{\varphi}_j^{(\varepsilon)} \in C_c^\infty(\Omega; C_\#^\infty(Y_1 \times Y_2; \mathbb{R}^N)), \quad \operatorname{div}_{y_2} \tilde{\varphi}_j^{(\varepsilon)} = 0, \quad \int_{Y_2} \operatorname{div}_{y_1} \tilde{\varphi}_j^{(\varepsilon)} \, \mathrm{d}y_2 = 0.$$

Furthermore, arguing as in [48, Thm 2.78], we have that $\tilde{\varphi}_j^{(\varepsilon)} \to \varphi_\varepsilon$ in $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^N))$ as $j \to +\infty$. Finally, using a diagonalization argument we can find a subsequence $j_\varepsilon \prec j$ such that $\tilde{\varphi}_\varepsilon := \tilde{\varphi}_{j_\varepsilon}^{(\varepsilon)} \in C_c^\infty(\Omega; C_\#^\infty(Y_1 \times Y_2; \mathbb{R}^N))$, $\operatorname{div}_{y_2} \tilde{\varphi}_\varepsilon = 0$, $\int_{Y_2} \operatorname{div}_{y_1} \tilde{\varphi}_\varepsilon \, \mathrm{d}y_2 = 0$ and $\tilde{\varphi}_\varepsilon \to \varphi$ in $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^N))$ as $\varepsilon \to 0^+$. Using *ii)*,

$$0 = \int_{\Omega\times Y_1\times Y_2} \tilde{\varphi}_\varepsilon(x, y_1, y_2) \, \mathrm{d}\lambda(x, y_1, y_2) \to \int_{\Omega\times Y_1\times Y_2} \varphi(x, y_1, y_2) \, \mathrm{d}\lambda(x, y_1, y_2) \text{ as } \varepsilon \to 0^+,$$

which contradicts the second condition in (4.2.11).

It remains to prove the claim, i.e., $\mathcal{E}$ is weakly-$\star$ closed.

Substep 2c. We start by proving that the set $\mathcal{E}_1$ of all measures $\tau \in \mathcal{M}_{y\#}(\Omega \times Y_1; \mathbb{R}^N)$ for which there exists a measure $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ such that τ is the measure associated with $D_{y_1}\boldsymbol{\mu}_1$ (i.e., for all $B \in \mathcal{B}(\Omega)$, $E \in \mathcal{B}(Y_1)$, $\tau(B \times E) = D_{y_1}(\boldsymbol{\mu}_1(B))(E))$ is weakly-$\star$ closed.

Since the weak-$\star$ topology is metrizable on every closed ball of $\mathcal{M}_{y\#}(\Omega\times Y_1; \mathbb{R}^N)$, by the Krein–Smulian Theorem to prove that $\mathcal{E}_1$ is weakly-$\star$ closed it suffices to show that $\mathcal{E}_1$ is sequentially weakly-$\star$ closed. Let $\{\tau_j\}_{j\in\mathbb{N}} \subset \mathcal{E}_1$ and $\tau \in \mathcal{M}_{y\#}(\Omega \times Y_1; \mathbb{R}^N)$ be such that $\tau_j \overset{\star}{\rightharpoonup} \tau$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega \times Y_1; \mathbb{R}^N)$ as $j \to +\infty$, that is, for all $\varphi \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^N))$ we have

$$\lim_{j\to+\infty} \int_{\Omega\times Y_1} \varphi(x, y_1) \, \mathrm{d}\tau_j(x, y_1) = \int_{\Omega\times Y_1} \varphi(x, y_1) \, \mathrm{d}\tau(x, y_1).$$

We want to prove that $\tau \in \mathcal{E}_1$. Let $\{\boldsymbol{\mu}_j^{(1)}\}_{j\in\mathbb{N}} \subset \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ be such that τ_j is the measure associated with $D_{y_1}\boldsymbol{\mu}_j^{(1)}$ for each $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$, and let $\tilde{\boldsymbol{\mu}}_j^{(1)} : \mathcal{B}(\Omega) \to BV_\#(Y_1)$ be defined by

$$\tilde{\boldsymbol{\mu}}_j^{(1)}(B) := \boldsymbol{\mu}_j^{(1)}(B) - \int_{Y_1} \boldsymbol{\mu}_j^{(1)}(B) \, \mathrm{d}y_1, \ \ B \in \mathcal{B}(\Omega).$$

It can be seen that each $\tilde{\boldsymbol{\mu}}_j^{(1)}$ satisfies conditions *i)* and *ii)* of Definition 2.3.13. In fact, since $\boldsymbol{\mu}_j^{(1)} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$, we deduce that

$$\tilde{\boldsymbol{\mu}}_j^1(\emptyset) = 0$$

and, proceeding as in the proof of (2.3.2), given any sequence $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega)$ of mutually disjoint Borel sets,

$$\begin{aligned}
\tilde{\boldsymbol{\mu}}_j^{(1)}\Big(\bigcup_{i=1}^\infty B_i\Big) &= \boldsymbol{\mu}_j^{(1)}\Big(\bigcup_{i=1}^\infty B_i\Big) - \int_{Y_1} \boldsymbol{\mu}_j^{(1)}\Big(\bigcup_{i=1}^\infty B_i\Big)\,\mathrm{d}y_1 = \sum_{i=1}^\infty \boldsymbol{\mu}_j^{(1)}(B_i) - \int_{Y_1}\sum_{i=1}^\infty \boldsymbol{\mu}_j^{(1)}(B_i)\,\mathrm{d}y_1 \\
&= \sum_{i=1}^\infty \Big(\boldsymbol{\mu}_j^{(1)}(B_i) - \int_{Y_1}\boldsymbol{\mu}_j^{(1)}(B_i)\,\mathrm{d}y_1\Big) = \sum_{i=1}^\infty \tilde{\boldsymbol{\mu}}_j^{(1)}(B_i).
\end{aligned}$$

Moreover, for all $B \in \mathcal{B}(\Omega)$, $D_{y_1}\big(\tilde{\boldsymbol{\mu}}_j^{(1)}(B)\big) = D_{y_1}\big(\boldsymbol{\mu}_j^{(1)}(B)\big)$ and

$$\begin{aligned}
\|\tilde{\boldsymbol{\mu}}_j^{(1)}\|(\Omega) &= \sup\Big\{\sum_{i=1}^\infty \|\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i)\|_{BV_\#(Y_1)} \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\Big\} \\
&\leqslant 2\sup\Big\{\sum_{i=1}^\infty \|\boldsymbol{\mu}_j^{(1)}(B_i)\|_{BV_\#(Y_1)} \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\Big\} \\
&= 2\|\boldsymbol{\mu}_j^{(1)}\|(\Omega) < \infty.
\end{aligned}$$

Thus $\tilde{\boldsymbol{\mu}}_j^{(1)} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$, being τ_j the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}_j^{(1)}$. Furthermore,

$$\begin{aligned}
\|\tilde{\boldsymbol{\mu}}_j^{(1)}\|_{\mathcal{M}(\Omega; L^{1^\star}_\#(Y_1))} &= \sup\Big\{\sum_{i=1}^\infty \|\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i)\|_{L^{1^\star}_\#(Y_1)} \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\Big\} \\
&\leqslant \mathcal{C}\sup\Big\{\sum_{i=1}^\infty \|D_{y_1}(\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i))\|(Y_1) \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\Big\} \\
&= \mathcal{C} \sup_{\substack{\{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \\ \text{partition of } \Omega}} \sum_{i=1}^\infty \sup_{\substack{\{E_k\}_{k\in\mathbb{N}}\subset\mathcal{B}(Y_1) \\ \text{partition of } Y_1}} \sum_{k=1}^\infty \big|D_{y_1}(\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i))(E_k)\big| \\
&= \mathcal{C} \sup_{\substack{\{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \\ \text{partition of } \Omega}} \sum_{i=1}^\infty \sup_{\substack{\{E_k\}_{k\in\mathbb{N}}\subset\mathcal{B}(Y_1) \\ \text{partition of } Y_1}} \sum_{k=1}^\infty |\tau_j(B_i \times E_k)| \\
&\leqslant \mathcal{C} \sup_{\substack{\{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \\ \text{partition of } \Omega}} \sum_{i=1}^\infty \sup_{\substack{\{E_k\}_{k\in\mathbb{N}}\subset\mathcal{B}(Y_1) \\ \text{partition of } Y_1}} \sum_{k=1}^\infty \|\tau_j\|(B_i \times E_k) \leqslant \mathcal{C}\|\tau_j\|(\Omega \times Y_1),
\end{aligned} \tag{4.2.13}$$

where $1^\star$ is the Sobolev conjugate of N, and where we have used a Poincaré inequality in BV (see Theorem 2.3.10) taking into account that for each $B \in \mathcal{B}(\Omega)$, $\tilde{\boldsymbol{\mu}}_j^{(1)}$ is a function in $BV_\#(Y_1)$ with zero mean value.

Since $\sup_{j\in\mathbb{N}} \|\tau_j\|(\Omega \times Y_1) < \infty$, and as $\mathcal{M}(\Omega; L^{1^\star}_\#(Y_1)) \simeq \big(C_0(\Omega; L^N_\#(Y_1))\big)'$ (see, for example, [38, p.182]), from (4.2.13) we deduce the existence of a (not relabeled) subsequence of $\{\tilde{\boldsymbol{\mu}}_j^{(1)}\}_{j\in\mathbb{N}}$ and of a measure $\tilde{\boldsymbol{\mu}} \in \mathcal{M}(\Omega; L^{1^\star}_\#(Y_1))$ such that

$$\tilde{\boldsymbol{\mu}}_j^{(1)} \overset{\star}{\rightharpoonup} \tilde{\boldsymbol{\mu}} \text{ weakly-}\star \text{ in } \mathcal{M}(\Omega; L^{1^\star}_\#(Y_1)) \text{ as } j \to +\infty.$$

In particular, for all $\varphi \in C_0(\Omega; C_\#(Y_1))$ we have

$$\lim_{j\to+\infty} \int_{\Omega\times Y_1} \varphi(x, y_1)\,\mathrm{d}\tilde{\boldsymbol{\mu}}_j^{(1)}(x)\mathrm{d}y_1 = \int_{\Omega\times Y_1} \varphi(x, y_1)\,\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\mathrm{d}y_1, \tag{4.2.14}$$

where the integrals are to be understood in the sense of Subsection 2.3.2.

We want to prove that $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ and that τ is the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}$, thus proving that $\tau \in \mathcal{E}$. We start by showing that $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$. Let $\phi \in C_0(\Omega)$ and $\psi \in C^1_\#(Y_1;\mathbb{R}^N)$ be given. Taking into account that τ_j is the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}_j^{(1)}$, Lemma 2.3.20 and the weak-$\star$ convergence $\tau_j \overset{\star}{\rightharpoonup} \tau$ in $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ as $j \to +\infty$, we have

$$\begin{aligned} \lim_{j\to+\infty} \int_{\Omega\times Y_1} \phi(x) \operatorname{div}\psi(y_1)\, \mathrm{d}\tilde{\boldsymbol{\mu}}_j^{(1)}(x)\mathrm{d}y_1 &= \lim_{j\to+\infty} \int_{Y_1} \Big(\int_\Omega \phi(x)\, \mathrm{d}\tilde{\boldsymbol{\mu}}_j^{(1)}(x)\Big)(y_1) \operatorname{div}\psi(y_1)\,\mathrm{d}y_1 \\ = -\lim_{j\to+\infty} \int_{\Omega\times Y_1} \phi(x)\,\psi(y_1)\cdot \mathrm{d}\tau_j(x,y_1) &= -\int_{\Omega\times Y_1} \phi(x)\,\psi(y_1)\cdot\mathrm{d}\tau(x,y_1). \end{aligned} \tag{4.2.15}$$

From (4.2.14) and (4.2.15), we get

$$\int_{Y_1} \Big(\int_\Omega \phi(x)\,\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\Big)(y_1) \operatorname{div}\psi(y_1)\,\mathrm{d}y_1 = -\int_{\Omega\times Y_1} \phi(x)\,\psi(y_1)\cdot\mathrm{d}\tau(x,y_1), \tag{4.2.16}$$

for all $\phi \in C_0(\Omega)$ and $\psi \in C^1_\#(Y_1;\mathbb{R}^N)$.

We claim that for all $B \in \mathcal{B}(\Omega)$ and $\psi \in C^1_\#(Y_1;\mathbb{R}^N)$, we have

$$\int_{Y_1} \tilde{\boldsymbol{\mu}}(B)(y_1) \operatorname{div}\psi(y_1)\,\mathrm{d}y_1 = -\int_{Y_1} \psi(y_1)\cdot\mathrm{d}\tau_B(y_1), \tag{4.2.17}$$

where $\tau_B(\cdot) := \tau(B\times\cdot)$, thus showing that $\tilde{\boldsymbol{\mu}}(B) \in BV_\#(Y_1)$ with $D_{y_1}(\tilde{\boldsymbol{\mu}}(B)) = \tau_B$.

Indeed, proceeding as in Lemma 2.3.17, it can be proved that for all bounded, Borel functions $\phi : \Omega \to \mathbb{R}$, we have

$$\int_Y \left|\int_\Omega \phi(x)\,\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\right| \mathrm{d}y \leqslant \int_\Omega |\phi(x)|\,\mathrm{d}\|\tilde{\boldsymbol{\mu}}\|(x). \tag{4.2.18}$$

Fix $\delta > 0$. Since $\|\tilde{\boldsymbol{\mu}}\| \in \mathcal{M}(\Omega;\mathbb{R})$ and $\|\tau\| \in \mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R})$ are positive, finite Radon measures, we may find an open set $A_\delta \supset B$ and a closed set $C_\delta \subset B$ such that

$$\|\tilde{\boldsymbol{\mu}}\|(A_\delta\backslash C_\delta) < \delta, \quad \|\tau\|((A_\delta\backslash C_\delta)\times Y_1) < \delta. \tag{4.2.19}$$

By Urysohn's Lemma, we may also find a function $\phi_\delta \in C_0(\Omega;[0,1])$ such that $\phi_\delta = 0$ in $\Omega\backslash A_\delta$ and $\phi_\delta = 1$ in C_δ. Then, in view of (4.2.18),

$$\begin{aligned} &\left|\int_{Y_1} \Big(\int_\Omega \phi_\delta(x)\,\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\Big)(y_1)\operatorname{div}\psi(y_1)\,\mathrm{d}y_1 - \int_{Y_1} \tilde{\boldsymbol{\mu}}(B)(y_1)\operatorname{div}\psi(y_1)\,\mathrm{d}y_1\right| \\ &\qquad \leqslant \mathcal{C}\|\nabla\psi\|_\infty \int_{Y_1}\left|\int_\Omega \big(\phi_\delta(x) - \chi_B(x)\big)\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\right|\mathrm{d}y_1 \leqslant 2\mathcal{C}\|\nabla\psi\|_\infty \|\tilde{\boldsymbol{\mu}}\|(A_\delta\backslash C_\delta). \end{aligned} \tag{4.2.20}$$

From (4.2.19) and (4.2.20), we get

$$\lim_{\delta\to0^+} \int_{Y_1} \Big(\int_\Omega \phi_\delta(x)\,\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\Big)(y_1)\operatorname{div}\psi(y_1)\,\mathrm{d}y_1 = \int_{Y_1} \tilde{\boldsymbol{\mu}}(B)(y_1)\operatorname{div}\psi(y_1)\,\mathrm{d}y_1. \tag{4.2.21}$$

Similarly,

$$\lim_{\delta\to0^+} \int_{\Omega\times Y_1} \phi_\delta(x)\,\psi(y_1)\cdot\mathrm{d}\tau(x,y_1) = \int_{Y_1} \psi(y_1)\cdot\mathrm{d}\tau_B(y_1). \tag{4.2.22}$$

Considering (4.2.16) with ϕ replaced by ϕ_δ, passing to the limit as $\delta \to 0^+$ taking into account (4.2.21) and (4.2.22), we deduce (4.2.17). In particular, for all $B \in \mathcal{B}(\Omega)$, $E \in \mathcal{B}(Y_1)$,

$$D_{y_1}(\tilde{\boldsymbol{\mu}}(B))(E) = \tau_B(E) = \tau(B \times E). \tag{4.2.23}$$

To conclude that $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ it remains to prove that $\tilde{\boldsymbol{\mu}}$ has finite total variation. As in (4.2.13), by (4.2.23) we get

$$\sup\left\{\sum_{i=1}^{\infty} \|D_{y_1}(\tilde{\boldsymbol{\mu}}(B_i))\|(Y_1) \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\} \leqslant \|\tau\|(\Omega \times Y_1).$$

Consequently,

$$\begin{aligned}
\|\tilde{\boldsymbol{\mu}}\|(\Omega) &= \sup\left\{\sum_{i=1}^{\infty} \|\tilde{\boldsymbol{\mu}}(B_i)\|_{BV_\#(Y_1)} \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\} \\
&\leqslant \mathcal{C} \sup\left\{\sum_{i=1}^{\infty} \big(\|\tilde{\boldsymbol{\mu}}(B_i)\|_{L^{1\star}_\#(Y_1)} + \|D_{y_1}(\tilde{\boldsymbol{\mu}}(B_i))\|(Y_1)\big) \colon \{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\} \\
&\leqslant \mathcal{C}\left(\sup_{j\in\mathbb{N}} \|\tau_j\|(\Omega \times Y_1) + \|\tau\|(\Omega \times Y_1)\right) < \infty,
\end{aligned}$$

where we have also used (4.2.13). Thus, $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ and τ is the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}$, which shows that $\tau \in \mathcal{E}_1$, and this concludes the proof that $\mathcal{E}_1$ is a weakly-$\star$ closed subspace of $\mathcal{M}_{y\#}(\Omega \times Y_1; \mathbb{R}^N)$.

Substep 2d. Similarly to Substep 2c, one can show that the space $\mathcal{E}_2$ of all measures $\tau \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2; \mathbb{R}^N)$ for which there exists a measure $\boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$ such that τ is the measure associated with $D_{y_2}\boldsymbol{\mu}_2$ (i.e., for all $B \in \mathcal{B}(\Omega \times Y_1)$, $E \in \mathcal{B}(Y_2)$, $\tau(B \times E) = D_{y_2}(\boldsymbol{\mu}_2(B))(E)$) is weakly-$\star$ closed.

Substep 2e. We are now in position to prove that $\mathcal{E}$ is a weakly-$\star$ closed vectorial subspace of $\mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2; \mathbb{R}^N)$. As before, it suffices to show that $\mathcal{E}$ is sequentially weakly-$\star$ closed. Let $\{\tau_j\}_{j\in\mathbb{N}} \subset \mathcal{E}$ be a sequence such that $\tau_j \overset{\star}{\rightharpoonup} \tau$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2; \mathbb{R}^N)$ as $j \to +\infty$. We want to prove that $\tau \in \mathcal{E}$.

For each $j \in \mathbb{N}$ write $\tau_j = \tau_j^{(1)} \otimes \mathcal{L}^N_{y_2} + \tau_j^{(2)}$, where $\tau_j^{(1)} \in \mathcal{M}_{y\#}(\Omega \times Y_1; \mathbb{R}^N)$ and $\tau_j^{(2)} \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2; \mathbb{R}^N)$ are the measures associated with $D_{y_1}\boldsymbol{\mu}_j^{(1)}$ and $D_{y_2}\boldsymbol{\mu}_j^{(2)}$ for some $\boldsymbol{\mu}_j^{(1)} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ and $\boldsymbol{\mu}_j^{(2)} \in \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$, respectively.

Let $\vartheta \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^N))$ be such that $\|\vartheta\|_\infty \leqslant 1$. Then ϑ can be seen as an element of $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^N))$, still with norm less than or equal to 1. Moreover,

$$\begin{aligned}
\langle \tau_j, \vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N), C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))} &= \int_{\Omega\times Y_1\times Y_2} \vartheta(x, y_1)\, \mathrm{d}\tau_j(x, y_1, y_2) \\
&= \int_{\Omega\times Y_1\times Y_2} \vartheta(x, y_1)\, \mathrm{d}\tau_j^{(1)}(x, y_1)\mathrm{d}y_2 + \int_{\Omega\times Y_1\times Y_2} \vartheta(x, y_1)\, \mathrm{d}\tau_j^{(2)}(x, y_1, y_2) \\
&= \int_{\Omega\times Y_1} \vartheta(x, y_1)\, \mathrm{d}\tau_j^{(1)}(x, y_1) = \langle \tau_j^{(1)}, \vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N), C_0(\Omega; C_\#(Y_1;\mathbb{R}^N))},
\end{aligned}$$

since $\int_{\Omega\times Y_1\times Y_2}\vartheta(x,y_1)\,\mathrm{d}\tau_j^{(2)}(x,y_1,y_2)=0$ by (2.3.21) (with $i=2$ and $\psi\equiv 1$). This implies that

$$\begin{aligned}
&\|\tau_j\|(\Omega\times Y_1\times Y_2)\\
&\quad=\sup\left\{\langle\tau_j,\varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))}\colon \varphi\in C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N)),\ \|\varphi\|_\infty\leqslant 1\right\}\\
&\quad\geqslant\ \sup\left\{\langle\tau_j,\vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))}\colon \vartheta\in C_0(\Omega;C_\#(Y_1;\mathbb{R}^N)),\ \|\vartheta\|_\infty\leqslant 1\right\}\\
&\quad=\sup\left\{\langle\tau_j^{(1)},\vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1;\mathbb{R}^N))}\colon \vartheta\in C_0(\Omega;C_\#(Y_1;\mathbb{R}^N)),\ \|\vartheta\|_\infty\leqslant 1\right\}\\
&\quad=\|\tau_j^{(1)}\|(\Omega\times Y_1).
\end{aligned}$$

Hence $\{\tau_j^{(1)}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$, and so there exist a subsequence $\{\tau_{j_k}^{(1)}\}_{k\in\mathbb{N}}$ of $\{\tau_j^{(1)}\}_{j\in\mathbb{N}}$ and a measure $\tau_1\in\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ such that $\tau_{j_k}^{(1)}\overset{\star}{\rightharpoonup}\tau_1$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ as $k\to+\infty$. Since $\tau_{j_k}^{(1)}\in\mathcal{E}_1$ for all $k\in\mathbb{N}$, and $\mathcal{E}_1$ is a weakly-$\star$ closed subspace of $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ (see Substep 2c), we conclude that $\tau_1\in\mathcal{E}_1$. Let $\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ be such that τ_1 is the measure associated with $D_{y_1}\boldsymbol{\mu}_1$.

Next, write $\tau_{j_k}^{(2)}=\tau_{j_k}-\tau_{j_k}^{(1)}\otimes\mathcal{L}_{y_2}^N$, so that $\tau_{j_k}^{(2)}\overset{\star}{\rightharpoonup}\tau-\tau_1\otimes\mathcal{L}_{y_2}^N=:\tau_2$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$ as $k\to+\infty$. Since $\tau_{j_k}^{(2)}\in\mathcal{E}_2$ for all $k\in\mathbb{N}$, by Substep 2c we conclude that $\tau_2\in\mathcal{E}_2$. Thus we can find $\boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$ such that τ_2 is the measure associated with $D_{y_2}\boldsymbol{\mu}_2$. Finally,

$$\tau=\tau_1\otimes\mathcal{L}_{y_2}^N+\tau_2\in\mathcal{E},$$

and this concludes the proof of the claim. $\square$

Proof of Theorem 4.1.7. *a)* We claim that for all $\varphi\in C_0\big(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d)\big)$ we have

$$\lim_{\varepsilon\to0^+}\int_\Omega\varphi\bigg(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\bigg)\cdot u_\varepsilon(x)\,\mathrm{d}x=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\cdot u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{4.2.24}$$

If $\varphi\in C(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d))$, then by Riemann–Lebesgue's Lemma

$$\varphi\bigg(\cdot,\frac{\cdot}{\varrho_1(\varepsilon)},\cdots,\frac{\cdot}{\varrho_n(\varepsilon)}\bigg)\overset{\star}{\rightharpoonup}\int_{Y_1\times\cdots\times Y_n}\varphi(\cdot,y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n \tag{4.2.25}$$

weakly-$\star$ in $L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ as $\varepsilon\to0^+$, from which (4.2.24) follows since by hypothesis $u_\varepsilon\to u$ (strongly) in $L^1(\Omega;\mathbb{R}^d)$ as $\varepsilon\to0^+$, and since if $\varphi\in C_0\big(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d)\big)$ then (4.2.25) holds weakly-$\star$ in $L^\infty(\Omega;\mathbb{R}^d)$.

b) By reasoning component by component, we may assume without loss of generality that $d=1$. Since $\{Du_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $\mathcal{M}(\Omega;\mathbb{R}^N)$, by Theorem 4.1.3, and up to a subsequence (not relabeled),

$$Du_\varepsilon\xrightharpoonup[\varepsilon]{(n+1)\text{-}sc}\mu_0, \tag{4.2.26}$$

for some $\mu_0\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^N)$.

We claim that if $\varphi\in C_c^\infty(\Omega;C_\#^\infty(Y_1\times\cdots\times Y_n;\mathbb{R}^N))$ is such that $\operatorname{div}_{y_n}\varphi=0$ and, if $n\geqslant2$, for all $k\in\{1,\cdots,n-1\}$, $x\in\Omega$, $y_i\in Y_i$, $i\in\{1,\cdots,n\}$,

$$\int_{Y_{k+1}\times\cdots\times Y_n}\operatorname{div}_{y_k}\varphi(x,y_1,\cdots,y_n)\,\mathrm{d}y_{k+1}\cdots\mathrm{d}y_n=0, \tag{4.2.27}$$

then we have

$$\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n)\cdot \mathrm{d}\mu_0(x,y_1,\cdots,y_n) = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n)\cdot \mathrm{d}Du(x)\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{4.2.28}$$

If the claim holds, then by Lemma 4.2.1 there exist n measures $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#(Y_i))$, $i\in\{1,\cdots,n\}$, such that

$$\mu_0 - Du_{\lfloor\Omega}\otimes\mathcal{L}^{nN}_{y_1,\cdots,y_n} = \sum_{i=1}^{n-1}\lambda_i\otimes\mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n} + \lambda_n,$$

where each $\lambda_i\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^N)$ is the measure associated with $D_{y_i}\boldsymbol{\mu}_i$. This will establish statement *b)*.

Let us prove (4.2.28). Let $\varphi\in C_c^\infty(\Omega;C_\#^\infty(Y_1\times\cdots\times Y_n;\mathbb{R}^N))$ be such that $\operatorname{div}_{y_n}\varphi=0$. Using the fact that $u_\varepsilon\in BV(\Omega)$ we obtain

$$\begin{aligned}
&\int_\Omega \varphi\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)\cdot\mathrm{d}Du_\varepsilon(x)\\
&\quad= -\int_\Omega(\operatorname{div}_x\varphi)\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)u_\varepsilon(x)\,\mathrm{d}x\\
&\qquad-\sum_{k=1}^{n-1}\frac{1}{\varrho_k(\varepsilon)}\int_\Omega(\operatorname{div}_{y_k}\varphi)\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)u_\varepsilon(x)\,\mathrm{d}x.
\end{aligned}\tag{4.2.29}$$

By *a)* and Fubini's Theorem, we deduce that

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_\Omega(\operatorname{div}_x\varphi)\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)u_\varepsilon(x)\,\mathrm{d}x\\
&\quad=\int_{\Omega\times Y_1\times\cdots\times Y_n}(\operatorname{div}_x\varphi)(x,y_1,\cdots,y_n)u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\quad=-\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\cdot\mathrm{d}Du(x)\mathrm{d}y_1\cdots\mathrm{d}y_n.
\end{aligned}\tag{4.2.30}$$

We claim that, if in addition φ is such that for $n\geqslant 2$ and for all $k\in\{1,\cdots,n-1\}$,

$$\int_{Y_{k+1}\times\cdots\times Y_n}\operatorname{div}_{y_k}\varphi(x,y_1,\cdots,y_n)\,\mathrm{d}y_{k+1}\cdots\mathrm{d}y_n=0,$$

then for all $k\in\{1,\cdots,n-1\}$,

$$\lim_{\varepsilon\to0^+}\frac{1}{\varrho_k(\varepsilon)}\int_\Omega(\operatorname{div}_{y_k}\varphi)\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)u_\varepsilon(x)\,\mathrm{d}x=0. \tag{4.2.31}$$

Assume that (4.2.31) holds. Then passing (4.2.29) to the limit as $\varepsilon\to0^+$, from (4.2.26), (4.2.30) and (4.2.31) we get (4.2.28), which concludes the proof of Theorem 4.1.7.

It remains to establish (4.2.31). The main ideas to prove (4.2.31) are those of Allaire and Briane [2, Thm. 3.3, Cor. 3.4], which we will include here for the sake of completeness. Let $n\geqslant 2$, fix $k\in\{1,\cdots,n-1\}$ and define $\vartheta_k:=\operatorname{div}_{y_k}\varphi$. By (4.2.27), we can write

$$\vartheta_k(x,y_1,\cdots,y_n)=\sum_{i=k+1}^{n}\vartheta_i^{(k)}(x,y_1,\cdots,y_i),$$

where the functions $\vartheta_i^{(k)}$ are given by the inductive formulae

$$\begin{cases} \vartheta_n^{(k)} := \vartheta_k - \displaystyle\int_{Y_n} \vartheta_k \, \mathrm{d}y_n, \\ \vartheta_i^{(k)} := \displaystyle\int_{Y_{i+1}\times\cdots\times Y_n} \vartheta_k \, \mathrm{d}y_{i+1}\cdots\mathrm{d}y_n - \int_{Y_i\times\cdots\times Y_n} \vartheta_k \, \mathrm{d}y_i\cdots\mathrm{d}y_n \quad \text{if } i\in\{k+1,\cdots,n-1\}. \end{cases}$$

By construction, for each $i \in \{k+1,\cdots,n\}$ one has

$$\vartheta_i^{(k)} \in \mathcal{O}_i := \Big\{ \vartheta \in C_c^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i)) : \int_{Y_i} \vartheta(x, y_1, \cdots, y_i)\,\mathrm{d}y_i = 0 \Big\}.$$

Moreover, for $n \geqslant 2$ and $k \in \{1,\cdots,n-1\}$,

$$\begin{aligned} \frac{1}{\varrho_k(\varepsilon)} \int_\Omega (\mathrm{div}_{y_k}\,\varphi)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x &= \frac{1}{\varrho_k(\varepsilon)} \int_\Omega \vartheta_k\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x \\ = \sum_{i=k+1}^n \frac{\varrho_i(\varepsilon)}{\varrho_k(\varepsilon)} \frac{1}{\varrho_i(\varepsilon)} \int_\Omega \vartheta_i^{(k)}\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x. \end{aligned}$$

Hence, using the boundedness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $BV(\Omega)$ and (4.0.1), to prove (4.2.31) it suffices to show that for each $i\in\{k+1,\cdots,n\}$ there exists a constant $\mathcal{C}_i = \mathcal{C}\big(\vartheta_i^{(k)}\big)$, independent of ε, such that

$$\left| \frac{1}{\varrho_i(\varepsilon)} \int_\Omega \vartheta_i^{(k)}\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x \right| \leqslant \mathcal{C}_i \|u_\varepsilon\|_{BV(\Omega)}. \tag{4.2.32}$$

Fix $i\in\{k+1,\cdots,n\}$. To simplify the notation, in the remaining part of the proof we will drop the dependence on i and k of the function $\vartheta_i^{(k)}$, so that $\vartheta_i^{(k)} = \vartheta \in \mathcal{O}_i$.

As shown in Allaire and Briane [2, Lemma 3.6], there exists a linear operator $S : \vartheta \in \mathcal{O}_i \mapsto S\vartheta \in \mathcal{O}_i^N$ such that $\mathrm{div}_{y_i}(S\vartheta) = \vartheta$ and $\|S\vartheta\|_\infty \leqslant \mathcal{C}\|\vartheta\|_\infty$, for some constant $\mathcal{C}$. Then we can write

$$\begin{aligned} &\frac{1}{\varrho_i(\varepsilon)} \vartheta\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) \\ &\quad = \mathrm{div}\left((S\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right)\right) - \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right) \frac{1}{\varrho_i(\varepsilon)} (T_\varepsilon\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right), \end{aligned} \tag{4.2.33}$$

where T_ε is the linear operator given by

$$T_\varepsilon\vartheta := \varrho_{i-1}(\varepsilon)\,\mathrm{div}_x(S\vartheta) + \sum_{j=1}^{i-1} \frac{\varrho_{i-1}(\varepsilon)}{\varrho_j(\varepsilon)} \mathrm{div}_{y_j}(S\vartheta).$$

Note that $T_\varepsilon\vartheta \in \mathcal{O}_i$. Indeed, $T_\varepsilon\vartheta \in \mathcal{O}_i$ inherits the same regularity of $S\vartheta$, and

$$\int_{Y_i} \mathrm{div}_x(S\vartheta)\,\mathrm{d}y_i = \mathrm{div}_x \int_{Y_i} S\vartheta\,\mathrm{d}y_i = 0, \quad \int_{Y_i} \mathrm{div}_{y_j}(S\vartheta)\,\mathrm{d}y_i = \mathrm{div}_{y_j} \int_{Y_i} S\vartheta\,\mathrm{d}y_i = 0,$$

for all $j \in \{1,\cdots,i-1\}$, and so $\int_{Y_i} T_\varepsilon\vartheta\,\mathrm{d}y_i = 0$.

Let us now analyze the right-hand side of (4.2.33). On the one hand we have that

$$\begin{aligned} \left| \int_\Omega \mathrm{div}\left((S\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right)\right) u_\varepsilon(x)\,\mathrm{d}x \right| &= \left| - \int_\Omega (S\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) \cdot \mathrm{d}Du_\varepsilon(x) \right| \\ &\leqslant \|S\vartheta\|_\infty \|Du_\varepsilon\|(\Omega) \leqslant \mathcal{C}\|\vartheta\|_\infty \|Du_\varepsilon\|(\Omega). \end{aligned}$$

On the other hand, the function $\frac{1}{\varrho_i(\varepsilon)}(T_\varepsilon\vartheta)\big(\cdot, \frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_i(\varepsilon)}\big)$ is of the same type as the function $\frac{1}{\varrho_i(\varepsilon)}\vartheta\big(\cdot, \frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_i(\varepsilon)}\big)$.

Applying (4.2.33) to $T_\varepsilon\vartheta$ instead of ϑ, and reiterating this process m times, with m as in (4.1.2), we get

$$\begin{aligned} &\frac{1}{\varrho_i(\varepsilon)}\vartheta\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\Big) \\ &\quad= \sum_{j=0}^{m-1}(-1)^j\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^j \operatorname{div}\Big((S(T_\varepsilon)^j\vartheta)\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\Big)\Big) \\ &\qquad+(-1)^m\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)}((T_\varepsilon)^m\vartheta)\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\Big). \end{aligned} \tag{4.2.34}$$

Reasoning as above,

$$\begin{aligned} &\left|\int_\Omega (-1)^j\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^j \operatorname{div}\Big((S(T_\varepsilon)^j\vartheta)\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\Big)\Big)\, u_\varepsilon(x)\,\mathrm{d}x\right| \\ &\qquad\leqslant \mathcal{C}\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^j \|(T_\varepsilon)^j\vartheta\|_\infty \|Du_\varepsilon\|(\Omega) \leqslant \mathcal{C}\|(T_\varepsilon)^j\vartheta\|_\infty\|Du_\varepsilon\|(\Omega) \end{aligned} \tag{4.2.35}$$

for all $j \in \{0, \cdots, m-1\}$, while

$$\begin{aligned} &\left|\int_\Omega (-1)^m\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)}((T_\varepsilon)^m\vartheta)\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\Big)\, u_\varepsilon(x)\,\mathrm{d}x\right| \\ &\qquad\leqslant \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)}\|(T_\varepsilon)^m\vartheta\|_\infty\|u_\varepsilon\|_{L^1(\Omega)} \leqslant \mathcal{C}\|(T_\varepsilon)^m\vartheta\|_\infty\|u_\varepsilon\|_{L^1(\Omega)}, \end{aligned} \tag{4.2.36}$$

where we used (4.0.1) and (4.1.2).

Finally using the definition of the operator T_ε, we deduce that for all $j \in \{0, \cdots, m\}$,

$$\sup_{\varepsilon>0}\|(T_\varepsilon)^j\vartheta\|_\infty \leqslant \mathcal{C}\big(\|\mathcal{S}\vartheta\|_{C^j(\Omega;C^j_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^N))} + \|\vartheta\|_{C^j(\Omega;C^j_\#(Y_1\times\cdots\times Y_i))}\big), \tag{4.2.37}$$

so that (4.2.32) follows from (4.2.34)–(4.2.37). □

The proof of the converse of Theorem 4.1.7, that is, of Proposition 4.1.8, is hinged on a version for $BV_\#(Y;\mathbb{R}^d)$-valued measures of the classical Meyers–Serrin's (density) Theorem. We will need some auxiliary results.

For $0 < \varepsilon < 1/2$, let $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ and $\eta_\varepsilon \in C_\#^\infty(Y)$ be functions satisfying (4.2.3) and (4.2.4), respectively. Fix $i \in \{1, \cdots, n\}$, let $\boldsymbol{\mu} \in \mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#(Y_i,\mathbb{R}^d))$ and denote by λ the measure associated with $D_{y_i}\boldsymbol{\mu}$. We define

$$\begin{aligned} &\psi^\varepsilon_{\boldsymbol{\mu}}(x, y_1, \cdots, y_i) \\ &\quad:= \int_{Y_i}\Big(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \rho_\varepsilon(x-x')\prod_{\kappa=1}^{i-1}\eta_\varepsilon(y_\kappa - y'_\kappa)\,\mathrm{d}\boldsymbol{\mu}(x', y'_1, \cdots, y'_{i-1})\Big)(y'_i)\,\eta_\varepsilon(y_i - y'_i)\,\mathrm{d}y'_i, \end{aligned} \tag{4.2.38}$$

for $x \in \Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and $y_1, ..., y_i \in \mathbb{R}^N$.

Lemma 4.2.2. *The function $\psi^\varepsilon_{\boldsymbol{\mu}}$ defined in (4.2.38) belongs to $C^\infty(\Omega_\varepsilon; C^\infty_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$.*

Proof. The proof is similar to the usual mollification case (see, for example, Ambrosio, Fusco and Pallara [7]). It is done by induction on the order of the derivative, and the key ingredients are the difference quotients and the Lebesgue Dominated Convergence Theorem, taking into account the regularity of ρ_ε and η_ε. $\square$

Lemma 4.2.3. *Let $\Omega' \subset\subset \Omega$ be an open, bounded set, and let $\psi^\varepsilon_{\boldsymbol{\mu}}$ be the function defined in (4.2.38). Then $\psi^\varepsilon_{\boldsymbol{\mu}}\mathcal{L}^{(i+1)N}\lfloor_{\Omega'\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_i}$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega'\times Y_1\times\cdots\times Y_i;\mathbb{R}^d)$ as $\varepsilon\to 0^+$, that is, for all $\varphi\in C_0(\Omega';C_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ we have*

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{\Omega'\times Y_1\times\cdots\times Y_i} &\varphi(x,y_1,\cdots,y_i)\cdot\psi^\varepsilon_{\boldsymbol{\mu}}(x,y_1,\cdots,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i\\ =\int_{\Omega'\times Y_1\times\cdots\times Y_i} &\varphi(x,y_1,\cdots,y_i)\cdot\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i,\end{aligned}\tag{4.2.39}$$

where the last integral is to be understood in the sense of Subsection 2.3.2.

Proof. To simplify the notation, set $\widetilde{Y}:=Y_1\times\cdots\times Y_{i-1}$, $Y:=Y_i$, $\tilde{y}:=(y_1,\cdots,y_{i-1})$ and $y:=y_i$ with the obvious conventions if $i=1$. Set also $\bar{\eta}_\varepsilon(\tilde{y}):=\prod_{\kappa=1}^{i-1}\eta_\varepsilon(y_\kappa)$. Notice that due to (4.2.4), for all $\tilde{y}'\in\mathbb{R}^{(i-1)N}$, $y'\in\mathbb{R}^N$, we have

$$\int_{\widetilde{Y}}\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\,\mathrm{d}\tilde{y}=1,\qquad \int_Y\eta_\varepsilon(y-y')\,\mathrm{d}y=1.\tag{4.2.40}$$

Fix $0<\varepsilon\leqslant\varepsilon_0$ where $0<\varepsilon_0<1/2$ is such that $\Omega'\subset\Omega_{\varepsilon_0}$. By (4.2.3), for all such ε and for all $x'\in\Omega'$, one has

$$\int_\Omega\rho_\varepsilon(x-x')\,\mathrm{d}x=1.\tag{4.2.41}$$

We will proceed in two steps.

Step 1. We start by proving that for every function $f\in C(\Omega;C_\#(\widetilde{Y}\times Y;C(Z;\mathbb{R}^d)))\cap L^\infty(\Omega\times\widetilde{Y}\times Y\times Z;\mathbb{R}^d)$, where $Z\subset\mathbb{R}^m$ is an open and bounded set, we have

$$\int_{\Omega\times\widetilde{Y}\times Y\times Z} f(x',\tilde{y}',y',z)\cdot\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y'\mathrm{d}z=\int_{\Omega\times\tilde{Y}\times Y}\left(\int_Z f(x',\tilde{y}',y',z)\,\mathrm{d}z\right)\cdot\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y',\tag{4.2.42}$$

where the integrals are to be understood in the sense of Subsection 2.3.2.

In fact, let $f_j(x',\tilde{y}',y',z):=\sum_{k=1}^{m_j}\vartheta_k^{(j)}(x',\tilde{y}')\psi_k^{(j)}(y')\phi_k^{(j)}(z)$, where $m_j\in\mathbb{N}$, and for all $k\in\{1,\cdots,m_j\}$, $\vartheta_k^{(j)}\in C(\Omega;C_\#(\widetilde{Y}))\cap L^\infty(\Omega\times\widetilde{Y})$, $\psi_k^{(j)}\in C_\#(Y)$, $\phi_k^{(j)}\in C(Z;\mathbb{R}^d)\cap L^\infty(Z;\mathbb{R}^d)$, be such that $\{f_j\}_{j\in\mathbb{N}}$ converges to f with respect to the supremum norm $\|\cdot\|_\infty$ (such a sequence exists as a consequence of the Stone-Weierstrass Theorem).

Then, by definition,

$$\begin{aligned}&\int_{\Omega\times\widetilde{Y}\times Y\times Z} f(x',\tilde{y}',y',z)\cdot\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y'\mathrm{d}z\\ &\quad=\lim_{j\to+\infty}\sum_{k=1}^{m_j}\int_{Y\times Z}\left[\left(\int_{\Omega\times\widetilde{Y}}\vartheta_k^{(j)}(x',\tilde{y}')\,\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\right)(y')\cdot\left(\psi_k^{(j)}(y')\,\phi_k^{(j)}(z)\right)\right]\mathrm{d}y'\mathrm{d}z.\end{aligned}\tag{4.2.43}$$

On the other hand, since $\int_Z\phi_k^{(j)}(z)\,\mathrm{d}z\in\mathbb{R}^d$, by Fubini's Theorem we have that for all $j\in\mathbb{N}$ and $k\in\{1,\cdots,m_j\}$,

$$\begin{aligned}&\int_{Y\times Z}\left[\left(\int_{\Omega\times\widetilde{Y}}\vartheta_k^{(j)}(x',\tilde{y}')\,\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\right)(y')\cdot\left(\psi_k^{(j)}(y')\,\phi_k^{(j)}(z)\right)\right]\mathrm{d}y'\mathrm{d}z\\ &\quad=\int_Y\left[\left(\int_{\Omega\times\widetilde{Y}}\left(\int_Z\phi_k^{(j)}(z)\,\mathrm{d}z\,\vartheta_k^{(j)}(x',\tilde{y}')\right)\cdot\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\right)(y')\,\psi_k^{(j)}(y')\right]\mathrm{d}y'.\end{aligned}\tag{4.2.44}$$

Finally, we observe that since $\int_Z f_j \,\mathrm{d}z = \sum_{k=1}^{m_j} \big(\int_Z \phi_i^{(j)} \mathrm{d}z\, \vartheta_i^{(j)} \psi_i^{(j)} \big)$ converges to $\int_Z f \,\mathrm{d}z$ with respect to the supremum norm in $\Omega \times \tilde{Y} \times Y$, then, in view of Remark 2.3.21,

$$
\begin{aligned}
\lim_{j\to+\infty} \sum_{k=1}^{m_j} \int_Y \bigg[\bigg(\int_{\Omega\times\tilde{Y}} \bigg(\int_Z \phi_i^{(j)}(z)\,\mathrm{d}z\, \vartheta_i^{(j)}(x',\tilde{y}') \bigg) \cdot \mathrm{d}\boldsymbol{\mu}(x',\tilde{y}') \bigg)(y')\, \psi_i^{(j)}(y') \bigg] \mathrm{d}y' \\
= \int_{\Omega\times\tilde{Y}\times Y} \bigg(\int_Z f(x',\tilde{y}',y',z)\,\mathrm{d}z \bigg) \cdot \mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y'.
\end{aligned}
\tag{4.2.45}
$$

From (4.2.43)–(4.2.45) we obtain (4.2.42).

Step 2. We establish (4.2.39). Fix $\varphi \in C_0(\Omega'; C_\#(Y_1 \times \cdots \times Y_i; \mathbb{R}^d))$. Using Fubini's Theorem and Step 1, with $Z := \Omega' \times \tilde{Y} \times Y$ and $f(x',\tilde{y}',y',x,\tilde{y},y) := \rho_\varepsilon(x-x')\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\eta_\varepsilon(y-y')\varphi(x,\tilde{y},y)$, and considering $\varphi(\cdot,\tilde{y},y)$ extended by zero outside Ω', we get

$$
\begin{aligned}
&\int_{\Omega'\times Y_1\times\cdots\times Y_i} \varphi(x,y_1,\cdots,y_i)\psi^\varepsilon_{\boldsymbol{\mu}}(x,y_1,\cdots,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i \\
&\quad = \int_{\Omega\times\widetilde{Y}\times Y} \varphi(x,\tilde{y},y) \bigg[\int_Y \bigg(\int_{\Omega\times\widetilde{Y}} \rho_\varepsilon(x-x')\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\,\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}') \bigg)(y')\,\eta_\varepsilon(y-y')\,\mathrm{d}y' \bigg] \mathrm{d}x\mathrm{d}\tilde{y}\mathrm{d}y \\
&\quad = \int_{Y\times\Omega\times\widetilde{Y}\times Y} \bigg(\int_{\Omega\times\widetilde{Y}} \rho_\varepsilon(x-x')\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\,\mathrm{d}\boldsymbol{\mu}(x',\tilde{y}') \bigg)(y')\,\eta_\varepsilon(y-y')\varphi(x,\tilde{y},y)\,\mathrm{d}y'\mathrm{d}x\mathrm{d}\tilde{y}\mathrm{d}y \\
&\quad = \int_{\Omega\times\widetilde{Y}\times Y} \bigg(\int_{\Omega\times\widetilde{Y}\times Y} \rho_\varepsilon(x-x')\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\eta_\varepsilon(y-y')\varphi(x,\tilde{y},y)\,\mathrm{d}x\mathrm{d}\tilde{y}\mathrm{d}y \bigg)\, \mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y',
\end{aligned}
$$

and, using in addition (4.2.40) and (4.2.41),

$$
\begin{aligned}
&\int_{\Omega'\times Y_1\times\cdots\times Y_i} \varphi(x',y_1',\cdots,y_i')\,\mathrm{d}\boldsymbol{\mu}(x',y_1',\cdots,y_{i-1}')\mathrm{d}y_i' \\
&\quad = \int_{\Omega'\times\widetilde{Y}\times Y} \bigg[\bigg(\int_\Omega \rho_\varepsilon(x-x')\,\mathrm{d}x \bigg) \bigg(\int_{\widetilde{Y}} \bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\,\mathrm{d}\tilde{y} \bigg) \\
&\qquad\qquad \bigg(\int_Y \eta_\varepsilon(y-y')\,\mathrm{d}y \bigg) \varphi(x',\tilde{y}',y') \bigg]\, \mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y' \\
&\quad = \int_{\Omega\times\widetilde{Y}\times Y} \bigg(\int_{\Omega\times\widetilde{Y}\times Y} \rho_\varepsilon(x-x')\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\eta_\varepsilon(y-y')\varphi(x',\tilde{y}',y')\,\mathrm{d}x\mathrm{d}\tilde{y}\mathrm{d}y \bigg)\, \mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y'.
\end{aligned}
$$

Thus, by Remark 2.3.21,

$$
\begin{aligned}
&\bigg| \int_{\Omega'\times Y_1\times\cdots\times Y_i} \varphi(x,y_1,\cdots,y_i)\psi^\varepsilon_{\boldsymbol{\mu}}(x,y_1,\cdots,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i \\
&\qquad\qquad - \int_{\Omega'\times Y_1\times\cdots\times Y_i} \varphi(x,y_1,\cdots,y_i)\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i \bigg| \\
&\quad = \bigg| \int_{\Omega\times\widetilde{Y}\times Y} \bigg(\int_{\Omega\times\widetilde{Y}\times Y} \big(\varphi(x,\tilde{y},y) - \varphi(x',\tilde{y}',y')\big) \\
&\qquad\qquad \rho_\varepsilon(x-x')\bar{\eta}_\varepsilon(\tilde{y}-\tilde{y}')\eta_\varepsilon(y-y')\,\mathrm{d}x\mathrm{d}\tilde{y}\mathrm{d}y \bigg)\, \mathrm{d}\boldsymbol{\mu}(x',\tilde{y}')\mathrm{d}y' \bigg| \\
&\quad \leqslant \max_{A_\varepsilon} \big|\varphi(x,\tilde{y},y) - \varphi(x',\tilde{y}',y')\big| \|\boldsymbol{\mu}\|(\Omega\times\widetilde{Y}),
\end{aligned}
\tag{4.2.46}
$$

where $A_\varepsilon := \big\{(x,\tilde{y},y),(x',\tilde{y}',y') \in \Omega\times\widetilde{Y}\times Y :\ |x-x'| \leqslant \varepsilon,\ |\tilde{y}-\tilde{y}'| \leqslant \varepsilon,\ |y-y'| \leqslant \varepsilon\big\}$, and where we have also used the inclusions $\operatorname{supp}\rho_\varepsilon$, $\operatorname{supp}\eta_\varepsilon \cap Y \subset \overline{B(0,\varepsilon)}$, as well as (4.2.40) and (4.2.41).

The uniform continuity of φ entails

$$\lim_{\varepsilon\to 0^+}\max_{A_\varepsilon}\big|\varphi(x,\tilde y,y)-\varphi(x',\tilde y',y')\big|=0,$$

and so, to conclude Step 2 it suffices to pass (4.2.46) to the limit as $\varepsilon\to 0^+$. □

Lemma 4.2.4. *Let $\Omega'\subset\subset\Omega$ be an open, bounded set, and let $\psi^\varepsilon_{\boldsymbol\mu}$ be the function defined in (4.2.38). Then $\nabla_{y_i}\psi^\varepsilon_{\boldsymbol\mu}\mathcal{L}^{(i+1)N}\lfloor_{\Omega'\times Y_1\times\cdots\times Y_i}\overset{\star}{\rightharpoonup}\lambda$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega'\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})$ as $\varepsilon\to 0^+$, and*

$$\lim_{\varepsilon\to 0^+}\int_{\Omega'\times Y_1\times\cdots\times Y_i}|\nabla_{y_i}\psi^\varepsilon_{\boldsymbol\mu}(x,y_1,\cdots,y_i)|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i=\|\lambda\|(\Omega'\times Y_1\times\cdots\times Y_i).$$

Proof. Fix $x\in\Omega_\varepsilon$ and $y_1,\cdots,y_i\in\mathbb{R}^N$. Set $\widetilde Y:=Y_1\times\cdots\times Y_{i-1}$, $Y:=Y_i$, $\tilde y:=(y_1,\cdots,y_{i-1})$, $y:=y_i$, and $\bar\eta_\varepsilon(\tilde y):=\prod_{\kappa=1}^{i-1}\eta_\varepsilon(y_\kappa)$. As in the previous proof, (4.2.40) holds. Using (2.3.21) and (4.2.40), we get

$$\begin{aligned}\nabla_y\psi^\varepsilon_{\boldsymbol\mu}(x,\tilde y,y)&=\int_Y\bigg(\int_{\Omega\times\widetilde Y}\rho_\varepsilon(x-x')\bar\eta_\varepsilon(\tilde y-\tilde y')\,\mathrm{d}\boldsymbol\mu(x',\tilde y')\bigg)(y')\otimes\nabla_y\eta_\varepsilon(y-y')\,\mathrm{d}y'\\&=-\int_Y\bigg(\int_{\Omega\times\widetilde Y}\rho_\varepsilon(x-x')\bar\eta_\varepsilon(\tilde y-\tilde y')\,\mathrm{d}\boldsymbol\mu(x',\tilde y')\bigg)(y')\otimes\nabla_{y'}\eta_\varepsilon(y-y')\,\mathrm{d}y'\\&=\int_{\Omega\times\widetilde Y\times Y}\rho_\varepsilon(x-x')\bar\eta_\varepsilon(\tilde y-\tilde y')\eta_\varepsilon(y-y')\,\mathrm{d}\lambda(x',\tilde y',y').\end{aligned}$$

Hence $\nabla_{y_i}\psi^\varepsilon_{\boldsymbol\mu}=\varphi_\varepsilon*\lambda$ in $\Omega_\varepsilon\times\mathbb{R}^{iN}$, where $\varphi_\varepsilon(x,y_1,\cdots,y_i):=\rho_\varepsilon(x)\prod_{\kappa=1}^{i}\eta_\varepsilon(y_i)$, and well known results on mollification of measures yield the desired convergences (see, for example, Ambrosio, Fusco and Pallara [7, Thm. 2.2]). □

Remark 4.2.5. *Let $\phi\in C_c(\Omega)$ and $\boldsymbol\mu\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$ be given, and define $\boldsymbol\nu(B):=\int_B\phi(x)\,\mathrm{d}\boldsymbol\mu(x,y_1,\cdots,y_{i-1})$ for all $B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$. By Remark 2.3.21, $\boldsymbol\nu\in\mathcal{M}(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$. Note that $\operatorname{supp}\boldsymbol\nu\subset\operatorname{supp}\phi\times\mathbb{R}^{(i-1)N}$.*

Considering first functions $\tilde\varphi$, φ of the form $\tilde\varphi(x,y_1,\cdots,y_i)=\tilde\vartheta(x,y_1,\cdots,y_{i-1})\tilde\psi(y_i)$ and $\varphi(x,y_1,\cdots,y_i)=\vartheta(x,y_1,\cdots,y_{i-1})\psi(y_i)$ with $\tilde\vartheta,\vartheta\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_{i-1}))$, $\tilde\psi\in C_\#(Y_i)$ and $\psi\in C^1_\#(Y_i)$, using (2.3.21), arguing component by component, and finally considering a density argument, we conclude that $\boldsymbol\nu\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$, with $\tau:=\phi\,\mathrm{d}\lambda$ being the measure associated with $D_{y_i}\boldsymbol\nu$, so that

$$\begin{aligned}&\int_{\Omega\times Y_1\times\cdots\times Y_i}\tilde\varphi(x,y_1,\cdots,y_i)\cdot\mathrm{d}\boldsymbol\nu(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i\\&\qquad=\int_{\Omega\times Y_1\times\cdots\times Y_i}\big(\tilde\varphi(x,y_1,\cdots,y_i)\phi(x)\big)\cdot\mathrm{d}\boldsymbol\mu(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i,\end{aligned}\tag{4.2.47}$$

$$\begin{aligned}&\int_{\Omega\times Y_1\times\cdots\times Y_i}\varphi(x,y_1,\cdots,y_i):\mathrm{d}\tau(x,y_1,\cdots,y_i)\\&\qquad=\int_{\Omega\times Y_1\times\cdots\times Y_i}\big(\varphi(x,y_1,\cdots,y_i)\phi(x)\big):\mathrm{d}\lambda(x,y_1,\cdots,y_i),\end{aligned}\tag{4.2.48}$$

for all $\tilde\varphi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ and $\varphi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}))$.

Notice that the domain of the function $\psi^\varepsilon_{\boldsymbol\nu}$ given by (4.2.38), is $\Omega_\varepsilon\times\mathbb{R}^{iN}$. In order to have it defined on the whole $\Omega\times\mathbb{R}^{iN}$, we extend $\boldsymbol\nu$ by zero. Precisely, for $B\in\mathcal{B}(\mathbb{R}^N\times Y_1\times\cdots\times Y_{i-1})$, *let* $\bar{\boldsymbol{\nu}}(B) := \boldsymbol{\nu}(B \cap \Omega \times Y_1 \times \cdots \times Y_{i-1})$. *Then* $\bar{\boldsymbol{\nu}} \in \mathcal{M}_\star(\mathbb{R}^N \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$, *and* $\operatorname{supp}\bar{\boldsymbol{\nu}} = \operatorname{supp}\boldsymbol{\nu}$.

In this setting, the function $\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}$ *defined in (4.2.38) (with* $\boldsymbol{\mu}$ *and* Ω *replaced by* $\bar{\boldsymbol{\nu}}$ *and* $\mathbb{R}^N$*, respectively) belongs to* $C^\infty_c\big(\mathbb{R}^N; C^\infty_\#\big(Y_1 \times \cdots \times Y_i;\mathbb{R}^d\big)\big)$. *Furthermore,*

$$\operatorname{supp}\psi^\varepsilon_{\bar{\boldsymbol{\nu}}} \subset \Omega \times \mathbb{R}^{iN} \quad \textit{for all } \varepsilon > 0 \textit{ small enough,} \tag{4.2.49}$$

since for all $y_1, ..., y_i \in \mathbb{R}^N$, $\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}(\cdot, y_1, \cdots, y_i) = 0$ *in* $\{x \in \mathbb{R}^N : \operatorname{dist}(x, \operatorname{supp}\phi) > \varepsilon\}$. *Arguing as in Lemmas 4.2.3 and 4.2.4, we conclude that*

$$\begin{aligned}
&\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}\mathcal{L}^{(i+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \boldsymbol{\nu}\mathcal{L}^N\lfloor_{Y_i} \ \textit{weakly-}\star \textit{ in } \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_i;\mathbb{R}^d) \textit{ as } \varepsilon \to 0^+,\\
&\nabla_{y_i}\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}\mathcal{L}^{(i+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \tau \ \textit{weakly-}\star \textit{ in } \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_i;\mathbb{R}^{d\times N}) \textit{ as } \varepsilon \to 0^+,\\
&\lim_{\varepsilon\to0^+}\int_{\Omega\times Y_1\times\cdots\times Y_i} |\nabla_{y_i}\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}(x, y_1, \cdots, y_i)|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i = \|\tau\|(\Omega \times Y_1 \times \cdots \times Y_i).
\end{aligned} \tag{4.2.50}$$

Proposition 4.2.6. *Fix* $i \in \{1, \cdots, n\}$*, and let* $\boldsymbol{\mu} \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$. *Denote by* λ *the measure associated with* $D_{y_i}\boldsymbol{\mu}$. *Then there exists a sequence* $\{\psi_j\}_{j\in\mathbb{N}} \subset C^\infty(\Omega; C^\infty_\#(Y_1 \times \cdots \times Y_i;\mathbb{R}^d)) \cap L^1(\Omega \times Y_1 \times \cdots \times Y_{i-1}; W^{1,1}(Y_i;\mathbb{R}^d))$ *satisfying*

$$\begin{aligned}
&\psi_j\mathcal{L}^{(i+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_i} \ \textit{weakly-}\star \textit{ in } \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_i;\mathbb{R}^d) \textit{ as } j \to +\infty,\\
&\nabla_{y_i}\psi_j\mathcal{L}^{(i+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \lambda \ \textit{weakly-}\star \textit{ in } \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_i;\mathbb{R}^{d\times N}) \textit{ as } j \to +\infty,\\
&\lim_{j\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_i} |\nabla_{y_i}\psi_j(x, y_1, \cdots, y_i)|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i = \|\lambda\|(\Omega \times Y_1 \times \cdots \times Y_i).
\end{aligned} \tag{4.2.51}$$

Proof. For simplicity we will assume that $i = 1$. The case $i \geqslant 2$ may be treated similarly.

Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_k \subset\subset \Omega_{k+1}$ and

$$\Omega = \bigcup_{k=1}^\infty \Omega_k,$$

and consider a smooth partition of unity subordinated to the open cover $\big\{\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big\}_{k\in\mathbb{N}}$ of Ω, where $\Omega_0 := \emptyset$, that is, a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ such that

$$\phi_k \in C^\infty_c\big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}; [0, 1]\big), \qquad \sum_{k=1}^\infty \phi_k(x) = 1 \text{ for all } x \in \Omega. \tag{4.2.52}$$

For each $k \in \mathbb{N}$, define $\boldsymbol{\nu}_k := \phi_k\,\mathrm{d}\boldsymbol{\mu}$ in the sense of Remark 4.2.5. In particular, $\operatorname{supp}\boldsymbol{\nu}_k \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big)$. Let $\{\tilde{\varphi}_j\}_{j\in\mathbb{N}}$ and $\{\varphi_j\}_{j\in\mathbb{N}}$ be dense in $C_0(\Omega; C_\#(Y_1;\mathbb{R}^d))$ and $C_0(\Omega; C_\#(Y_1;\mathbb{R}^{d\times N}))$, respectively.

By induction and by (4.2.49) and (4.2.50) (with $\boldsymbol{\nu}$ replaced by $\boldsymbol{\nu}_k$), given $j \in \mathbb{N}$ we can find a sequence $\{\varepsilon^{(j)}_k\}_{k\in\mathbb{N}}$ of positive numbers converging to zero, with $\varepsilon^{(j)}_k < \varepsilon^{(j-1)}_k$ (and $\varepsilon^{(0)}_k := 1/2$), such that for all $k \in \mathbb{N}$ and $l \in \{1, \cdots, j\}$ we have

$$\operatorname{supp}\psi^{\varepsilon^{(j)}_k}_{\bar{\boldsymbol{\nu}}_k} \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\,\big) \times \mathbb{R}^N, \tag{4.2.53}$$

$$\left|\int_{\Omega\times Y_1} \tilde{\varphi}_l(x, y_1) \cdot \psi^{\varepsilon^{(j)}_k}_{\bar{\boldsymbol{\nu}}_k}(x, y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1} \tilde{\varphi}_l(x, y_1) \cdot \mathrm{d}\boldsymbol{\nu}_k(x)\mathrm{d}y_1\right| \leqslant \frac{1}{j\,2^k}, \tag{4.2.54}$$

$$\left|\int_{\Omega\times Y_1} \varphi_l(x, y_1) : \nabla_{y_1}\psi^{\varepsilon^{(j)}_k}_{\bar{\boldsymbol{\nu}}_k}(x, y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1} \varphi_l(x, y_1) : \mathrm{d}\tau_k(x, y_1)\right| \leqslant \frac{1}{j\,2^k}, \tag{4.2.55}$$

$$\left|\int_{\Omega\times Y_1} \left|\nabla_{y_1}\psi^{\varepsilon^{(j)}_k}_{\bar{\boldsymbol{\nu}}_k}(x, y_1)\right|\mathrm{d}x\mathrm{d}y_1 - \|\tau_k\|(\Omega \times Y_1)\right| \leqslant \frac{1}{2^k}, \tag{4.2.56}$$

where τ_k is the measure associated with $D_{y_1}\boldsymbol{\nu}_k$. For every open, bounded $\Omega' \subset\subset \Omega$ only finitely many $\Omega_{k+1}\backslash\overline{\Omega_{k-1}}$ cover Ω', and so, in view of (4.2.53), for each $j \in \mathbb{N}$ the function ψ_j defined by

$$\psi_j(x,y_1) := \sum_{k=1}^{\infty} \psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1) \tag{4.2.57}$$

belongs to $C^\infty(\Omega; C^\infty_\#(Y_1;\mathbb{R}^d))$, with $\nabla_{y_1}\psi_j = \sum_{k=1}^{\infty} \nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}$. Moreover, $\psi_j \in L^1(\Omega; W^{1,1}(Y_1;\mathbb{R}^d))$ and

$$\sup_{j\in\mathbb{N}} \|\psi_j\|_{L^1(\Omega\times Y_1;\mathbb{R}^d)} =: M < \infty, \quad \sup_{j\in\mathbb{N}} \|\nabla_{y_1}\psi_j\|_{L^1(\Omega\times Y_1;\mathbb{R}^{d\times N})} =: \tilde{M} < \infty. \tag{4.2.58}$$

Indeed, thanks to (4.2.53), and defining $\psi^{\varepsilon_0^{(j)}}_{\bar{\boldsymbol{\nu}}_0} := 0$, we obtain

$$\begin{aligned}\int_{\Omega\times Y_1} |\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 &\leqslant \sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} |\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \\ &= \sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left|\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1) + \psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1) + \psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\right| \mathrm{d}x\mathrm{d}y_1 \\ &\leqslant \sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left[\left|\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1)\right| + \left|\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\right| + \left|\psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\right|\right] \mathrm{d}x\mathrm{d}y_1,\end{aligned} \tag{4.2.59}$$

and

$$\begin{aligned}\int_{\Omega\times Y_1} |\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 &\leqslant \sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} |\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \\ &= \sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left|\nabla_{y_1}\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1) + \nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1) + \nabla_{y_1}\psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\right| \mathrm{d}x\mathrm{d}y_1 \\ &\leqslant \sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left[\left|\nabla_{y_1}\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1)\right| + \left|\nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\right| + \left|\nabla_{y_1}\psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\right|\right] \mathrm{d}x\mathrm{d}y_1.\end{aligned} \tag{4.2.60}$$

We have that

$$\begin{aligned}&\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left|\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\right| \mathrm{d}x\mathrm{d}y_1 \\ &\quad= \int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left|\int_{Y_1}\left(\int_{\mathbb{R}^N} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\bar{\nu}_k(x')\right)(y_1')\,\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}y_1'\right| \mathrm{d}x\mathrm{d}y_1 \\ &\quad\leqslant \int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})}\int_{Y_1}\left[\int_{Y_1}\left|\left(\int_{\Omega} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\bar{\nu}_k(x')\right)(y_1')\right| \eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}y_1\right]\mathrm{d}y_1'\mathrm{d}x \\ &\quad= \int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})}\left[\int_{Y_1}\left|\left(\int_{\Omega} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\bar{\nu}_k(x')\right)(y_1')\right|\mathrm{d}y_1'\right]\mathrm{d}x \\ &\quad\leqslant \int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})}\int_{\Omega} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\|\bar{\nu}_k\|(x')\,\mathrm{d}x \leqslant \|\bar{\nu}_k\|(\Omega_{k+1}\backslash\overline{\Omega}_{k-1}) \leqslant \|\boldsymbol{\mu}\|(\Omega_{k+1}\backslash\overline{\Omega}_{k-1}),\end{aligned}$$

where we used Fubini's Theorem, (4.2.4), Lemma 2.3.17 (see also Remark 2.3.21), (4.2.53) and (4.2.3) in this order. Thus,

$$\sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\backslash\overline{\Omega}_{k-1})\times Y_1} \left|\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\right| \mathrm{d}x\mathrm{d}y_1 \leqslant 2\|\boldsymbol{\mu}\|(\Omega). \tag{4.2.61}$$

Similarly,

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_{(\Omega_k\setminus\overline{\Omega}_{k-1})\times Y_1}\left|\psi^{\varepsilon^{(j)}_{k-1}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1)\right|\mathrm{d}x\mathrm{d}y_1 \leqslant 2\|\boldsymbol{\mu}\|(\Omega),\\
&\sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k})\times Y_1}\left|\psi^{\varepsilon^{(j)}_{k+1}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\right|\mathrm{d}x\mathrm{d}y_1 \leqslant 2\|\boldsymbol{\mu}\|(\Omega).
\end{aligned}\tag{4.2.62}$$

From (4.2.59), (4.2.61) and (4.2.62), we deduce the first condition in (4.2.58). To prove the second condition in (4.2.58), we observe that from (4.2.53), (4.2.56), (4.2.52) and equality $\tau_k = \phi_k\,\mathrm{d}\lambda$ (see Remark 4.2.5), we have that

$$\begin{aligned}
\sum_{k=1}^{\infty}\int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1}\left|\nabla_{y_1}\psi^{\varepsilon^{(j)}_k}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\right|\mathrm{d}x\mathrm{d}y_1 &\leqslant \sum_{k=1}^{\infty}\left(\|\tau_k\|(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})+\frac{1}{2^k}\right)\\
&\leqslant \sum_{k=1}^{\infty}\|\lambda\|(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})+1 \leqslant 2\|\lambda\|(\Omega)+1.
\end{aligned}$$

Arguing as above, and taking into account (4.2.60),

$$\int_{\Omega\times Y_1}|\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant 6\|\lambda\|(\Omega\times Y_1)+3,$$

which concludes the proof of (4.2.58).

Now we prove the first convergence in (4.2.51). Let $\tilde{\varphi}\in C_0(\Omega;C_\#(Y_1;\mathbb{R}^d))$ be given, and fix $\eta>0$. There exists $m\in\mathbb{N}$ such that

$$\|\tilde{\varphi}-\tilde{\varphi}_m\|_{C_0(\Omega;C_\#(Y_1;\mathbb{R}^d))}\leqslant\eta.$$

Using (4.2.58), (4.2.57), (4.2.52), (4.2.53), (2.3.15) (see also Remark 2.3.21), (4.2.47) and (4.2.54), we obtain for any $j>m$

$$\begin{aligned}
&\left|\int_{\Omega\times Y_1}\tilde{\varphi}(x,y_1)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1-\int_{\Omega\times Y_1}\tilde{\varphi}(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\quad\leqslant\left|\int_{\Omega\times Y_1}\big(\tilde{\varphi}(x,y_1)-\tilde{\varphi}_m(x,y_1)\big)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1\right|\\
&\qquad+\left|\int_{\Omega\times Y_1}\tilde{\varphi}_m(x,y_1)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1-\int_{\Omega\times Y_1}\tilde{\varphi}_m(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\qquad+\left|\int_{\Omega\times Y_1}\big(\tilde{\varphi}_m(x,y_1)-\tilde{\varphi}(x,y_1)\big)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\quad\leqslant\eta M+\sum_{k=1}^{\infty}\left|\int_{\Omega\times Y_1}\tilde{\varphi}_m(x,y_1)\cdot\psi^{\varepsilon^{(j)}_k}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1-\int_{\Omega\times Y_1}\big(\tilde{\varphi}_m(x,y_1)\phi_k(x)\big)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\qquad+\eta\|\boldsymbol{\mu}\|(\Omega)\\
&\quad\leqslant\mathcal{C}\eta+\frac{1}{j}\cdot
\end{aligned}$$

Letting first $j\to+\infty$ and then $\eta\to0^+$, we conclude that

$$\lim_{j\to+\infty}\int_{\Omega\times Y_1}\tilde{\varphi}(x,y_1)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1=\int_{\Omega\times Y_1}\tilde{\varphi}(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1.$$

Since $\tilde{\varphi}\in C_0(\Omega;C_\#(Y_1;\mathbb{R}^d))$ was taken arbitrarily, this proves that

$$\psi_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1}\overset{\star}{\rightharpoonup}\boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_1}\ \text{ weakly-}\star\text{ in }\ \mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^d)\text{ as }j\to+\infty.$$

We will now prove the second convergence in (4.2.51). Let $\varphi \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$ be given, and fix $\eta > 0$. There exists $m \in \mathbb{N}$ such that

$$\|\varphi - \varphi_m\|_\infty \leqslant \eta.$$

Using (4.2.58), (4.2.57), (4.2.52), (4.2.53), (4.2.48) and (4.2.55), we get for every $j > m$

$$\begin{aligned}
&\left| \int_{\Omega\times Y_1} \varphi(x,y_1) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1} \varphi(x,y_1) : \mathrm{d}\lambda(x,y_1)\right| \\
&\quad\leqslant \left| \int_{\Omega\times Y_1} \big(\varphi(x,y_1) - \varphi_m(x,y_1)\big) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1\right| \\
&\qquad + \left| \int_{\Omega\times Y_1} \varphi_m(x,y_1) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1} \varphi_m(x,y_1) : \mathrm{d}\lambda(x,y_1)\right| \\
&\qquad + \left| \int_{\Omega\times Y_1} \big(\varphi_m(x,y_1) - \varphi(x,y_1)\big) : \mathrm{d}\lambda(x,y_1)\right| \\
&\quad\leqslant \eta\tilde{M} + \sum_{k=1}^{\infty} \left| \int_{\Omega\times Y_1} \varphi_m(x,y_1) : \nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1} (\varphi_m(x,y_1)\phi_k(x)) : \mathrm{d}\lambda(x,y_1)\right| \\
&\qquad + \eta\|\lambda\|(\Omega\times Y_1) \\
&\quad\leqslant \mathcal{C}\eta + \frac{1}{j}\cdot
\end{aligned}$$

Letting first $j \to +\infty$ and then $\eta \to 0^+$, we conclude that

$$\lim_{j\to+\infty} \int_{\Omega\times Y_1} \varphi(x,y_1) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 = \int_{\Omega\times Y_1} \varphi(x,y_1) : \mathrm{d}\lambda(x,y_1).$$

Since $\varphi \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$ was taken arbitrarily, we have just proved that

$$\nabla_{y_1}\psi_j \mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup} \lambda \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1; \mathbb{R}^{d\times N}) \text{ as } j \to +\infty. \tag{4.2.63}$$

Using the lower semicontinuity of the total variation, convergence (4.2.63) yields

$$\liminf_{j\to+\infty} \int_{\Omega\times Y_1} |\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \geqslant \|\lambda\|(\Omega\times Y_1). \tag{4.2.64}$$

To prove the converse inequality, let $\varphi \in C_c(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$ be such that $\|\varphi\|_\infty \leqslant 1$. Using similar arguments to those in the proof of Lemma 4.2.4, Fubini's Theorem, the symmetry of $\rho_{\varepsilon_k^{(j)}}$ and $\eta_{\varepsilon_k^{(j)}}$ with respect to the origin, (4.2.48) and the inclusion $\operatorname{supp}\varphi \subset \Omega_l \times \mathbb{R}^N$ for some $l \in \mathbb{N}$, we deduce that

$$\begin{aligned}
&\int_{\Omega\times Y_1} \varphi(x,y_1) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 = \sum_{k=1}^{l} \int_{\Omega\times Y_1} \varphi(x,y_1) : \nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 \\
&\quad= \sum_{k=1}^{l} \int_{\Omega\times Y_1} \varphi(x,y_1) : \left[\int_{\mathbb{R}^N\times Y_1} \rho_{\varepsilon_k^{(j)}}(x-x')\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}\bar{\tau}_k(x',y_1')\right]\mathrm{d}x\mathrm{d}y_1 \\
&\quad= \sum_{k=1}^{l} \int_{\mathbb{R}^N\times Y_1} \left[\int_{\Omega\times Y_1} \varphi(x,y_1)\rho_{\varepsilon_k^{(j)}}(x-x')\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}x\mathrm{d}y_1\right] : \mathrm{d}\bar{\tau}_k(x',y_1') \\
&\quad= \sum_{i=1}^{l} \int_{\mathbb{R}^N\times Y_1} \big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}}) * \varphi\big)(x',y_1') : \mathrm{d}\bar{\tau}_k(x',y_1') \\
&\quad= \sum_{k=1}^{l} \int_{\Omega\times Y_1} \big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}}) * \varphi\big)(x,y_1) : \mathrm{d}\tau_k(x,y_1) \\
&\quad= \int_{\Omega\times Y_1} \sum_{k=1}^{l} \Big[\big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}}) * \varphi\big)(x,y_1)\,\phi_k(x)\Big] : \mathrm{d}\lambda(x,y_1) = \int_{\Omega\times Y_1} \bar{\varphi}_j(x,y_1) : \mathrm{d}\lambda(x,y_1),
\end{aligned} \tag{4.2.65}$$

where $\bar{\varphi}_j(x, y_1) := \sum_{k=1}^{l} \Big[\big((\rho_{\varepsilon_k^{(j)}} \eta_{\varepsilon_k^{(j)}}) * \varphi\big)(x, y_1)\, \phi_k(x)\Big]$. Notice that $\|\bar{\varphi}_j\|_\infty \leqslant 1$. Indeed, for all $x \in \Omega$, $y_1 \in Y_1$, we have

$$\begin{aligned}|\bar{\varphi}_j(x, y_1)| &= \left|\sum_{k=1}^{l} \left(\int_{\Omega\times Y_1} \rho_{\varepsilon_k^{(j)}}(x - x')\eta_{\varepsilon_k^{(j)}}(y_1 - y_1')\varphi(x', y_1')\, \mathrm{d}x'\mathrm{d}y_1'\, \phi_k(x)\right)\right| \\ &\leqslant \|\varphi\|_\infty \sum_{k=1}^{l} \left(\int_{\Omega\times Y_1} \rho_{\varepsilon_k^{(j)}}(x - x')\eta_{\varepsilon_k^{(j)}}(y_1 - y_1')\, \mathrm{d}x'\mathrm{d}y_1'\, \phi_k(x)\right) \leqslant \|\varphi\|_\infty \sum_{k=1}^{l} \phi_k(x) \leqslant 1,\end{aligned}$$

where we used (4.2.3), (4.2.4), (4.2.52) and the condition $\|\varphi\|_\infty \leqslant 1$. Taking the supremum over $x \in \Omega$ and $y_1 \in Y_1$, we get $\|\bar{\varphi}_j\|_\infty \leqslant 1$. Moreover, $\bar{\varphi}_j \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$ and so, from (4.2.65), we deduce that

$$\int_{\Omega\times Y_1} \varphi(x, y_1) : \nabla_{y_1}\psi_j(x, y_1)\, \mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega \times Y_1). \tag{4.2.66}$$

By density, taking into account (4.2.58) and using Lebesgue Dominated Convergence Theorem, we conclude that (4.2.66) holds for all $\varphi \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$ with $\|\varphi\|_\infty \leqslant 1$. Hence

$$\int_{\Omega\times Y_1} |\nabla_{y_1}\psi_j(x, y_1)|\, \mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega \times Y_1),$$

which together with (4.2.64) yield

$$\lim_{j\to+\infty} \int_{\Omega\times Y_1} |\nabla_{y_1}\psi_j(x, y_1)|\, \mathrm{d}x\mathrm{d}y_1 = \|\lambda\|(\Omega \times Y_1). \qquad \square$$

Corollary 4.2.7. *Fix $i \in \{1, \cdots, n\}$, and let $\boldsymbol{\mu} \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i; \mathbb{R}^d))$. Denote by λ the measure associated with $D_{y_i}\boldsymbol{\mu}$. Then there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega; C_\#^\infty(Y_1 \times \cdots \times Y_i; \mathbb{R}^d))$ satisfying (4.2.51).*

Proof. As in the previous proof, we may assume without loss of generality that $i = 1$. Let $\{\psi_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega; C_\#^\infty(Y_1; \mathbb{R}^d))$ be the sequence given by Proposition 4.2.6. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_j \subset\subset \Omega_{j+1}$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$, and let $\{\phi_j\}_{j\in\mathbb{N}}$ be a sequence of cut-off functions $\phi_j \in C_c^\infty(\Omega; [0, 1])$ satisfying $\phi_j = 1$ in Ω_j and $\phi_j = 0$ in $\Omega\backslash\Omega_{j+1}$, for all $j \in \mathbb{N}$. Define

$$\tilde{\psi}_{j,k}(x, y_1) := \phi_j(x)\psi_k(x, y_1).$$

We have that $\tilde{\psi}_{j,k} \in C_c^\infty(\Omega; C_\#^\infty(Y_1; \mathbb{R}^d))$. Let $\tilde{\varphi} \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^d))$ and $\varphi \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$ be given. Then for all $j \in \mathbb{N}$, $\tilde{\varphi}\phi_j \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^d))$ and $\varphi\phi_j \in C_0(\Omega; C_\#(Y_1; \mathbb{R}^{d\times N}))$. Using the first two convergences in (4.2.51), Remark 2.3.19 (iii) (see also Remark 2.3.21), the convergence $\lim_{j\to+\infty} \|\boldsymbol{\mu}\|(\Omega\backslash\Omega_j) = 0$, the pointwise convergence $\phi_j \to 1$ in Ω as $j \to +\infty$, and Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned}\lim_{j\to+\infty}\lim_{k\to+\infty} \int_{\Omega\times Y_1} \tilde{\varphi}(x, y_1) \cdot \tilde{\psi}_{j,k}(x, y_1)\, \mathrm{d}x\mathrm{d}y_1 &= \lim_{j\to+\infty}\lim_{k\to+\infty} \int_{\Omega\times Y_1} \big(\tilde{\varphi}(x, y_1)\phi_j(x)\big) \cdot \psi_k(x, y_1)\, \mathrm{d}x\mathrm{d}y_1 \\ = \lim_{j\to+\infty} \int_{\Omega\times Y_1} \big(\tilde{\varphi}(x, y_1)\phi_j(x)\big) \cdot \mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1 &= \int_{\Omega\times Y_1} \tilde{\varphi}(x, y_1) \cdot \mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1,\end{aligned}$$

and

$$\begin{aligned}&\lim_{j\to+\infty}\lim_{k\to+\infty} \int_{\Omega\times Y_1} \varphi(x, y_1) : \nabla_{y_1}\tilde{\psi}_{j,k}(x, y_1)\, \mathrm{d}x\mathrm{d}y_1 \\ &\quad= \lim_{j\to+\infty}\lim_{k\to+\infty} \int_{\Omega\times Y_1} \varphi(x, y_1) : (\phi_j(x)\nabla_{y_1}\psi_k(x, y_1))\, \mathrm{d}x\mathrm{d}y_1 \\ &\quad= \lim_{j\to+\infty} \int_{\Omega\times Y_1} (\varphi(x, y_1)\phi_j(x)) : \mathrm{d}\lambda(x, y_1) = \int_{\Omega\times Y_1} \varphi(x, y_1) : \mathrm{d}\lambda(x, y_1).\end{aligned}$$

On the other hand,

$$\int_{\Omega\times Y_1} |\nabla_{y_1}\tilde{\psi}_{j,k}(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 = \int_{\Omega\times Y_1} |\phi_j(x)\nabla_{y_1}\psi_k(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \int_{\Omega\times Y_1} |\nabla_{y_1}\psi_k(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1,$$

and so

$$\limsup_{j\to+\infty}\limsup_{k\to+\infty}\int_{\Omega\times Y_1} |\nabla_{y_1}\tilde{\psi}_{j,k}(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega\times Y),$$

where we have used the third convergence in (4.2.51). Using a diagonal argument together with the separability of the spaces $C_0(\Omega;C_\#(Y_1;\mathbb{R}^d))$ and $C_0(\Omega;C_\#(Y_1;\mathbb{R}^{d\times N}))$, we can find a subsequence $k_j \prec k$ such that $\tilde{\psi}_j := \tilde{\psi}_{j,k_j} \in C_c^\infty(\Omega;C_\#^\infty(Y_1;\mathbb{R}^d))$ and

$$\begin{aligned}
&\psi_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup} \boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_1} \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^d) \text{ as } j\to+\infty,\\
&\nabla_{y_1}\tilde{\psi}_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup} \lambda \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^{d\times N}) \text{ as } j\to+\infty,\\
&\limsup_{j\to+\infty}\int_{\Omega\times Y_1} |\nabla_{y_1}\tilde{\psi}_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega\times Y_1).
\end{aligned}$$

Finally, the convergence $\nabla_{y_1}\tilde{\psi}_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^{d\times N})$ as $j\to+\infty$ implies

$$\liminf_{j\to+\infty}\int_{\Omega\times Y_1} |\nabla_{y_1}\tilde{\psi}_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \geqslant \|\lambda\|(\Omega\times Y_1),$$

which concludes the proof. □

Corollary 4.2.8. *Assume that $\partial\Omega$ is Lipschitz. Let $u\in BV(\Omega;\mathbb{R}^d)$ and for each $i\in\{1,\cdots,n\}$, let $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$. Then there exist sequences $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}\subset C_c^\infty(\Omega;C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ satisfying*

$$\begin{aligned}
&u_j \overset{\star}{\rightharpoonup} u \ \textit{weakly-}\star \textit{ in } BV(\Omega;\mathbb{R}^d) \textit{ as } j\to+\infty, \quad \lim_{j\to+\infty}\int_\Omega |\nabla u_j(x)|\,\mathrm{d}x = \|Du\|(\Omega),\\
&\Big(\nabla u_j + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}\Big)\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n}\\
&\qquad \overset{\star}{\rightharpoonup} \lambda_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n} \ \textit{weakly-}\star \textit{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}) \textit{ as } j\to+\infty,\\
&\lim_{j\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\qquad = \|\lambda_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n),
\end{aligned} \tag{4.2.67}$$

where $\lambda_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}$ is the measure defined in (4.1.3).

Proof. We will proceed in two steps.

Step 1. We first prove that there are sequences $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}\subset C^\infty(\Omega;C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ satisfying (4.2.67).

Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_k \subset\subset \Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^\infty \Omega_k$, and consider a smooth partition of unity $\{\phi_k\}_{k\in\mathbb{N}}$ subordinated to the open cover $\big\{\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big\}_{k\in\mathbb{N}}$ of Ω, where $\Omega_0 := \emptyset$, as in (4.2.52).

For each $k\in\mathbb{N}$ and $i\in\{1,\cdots,n\}$, define $\boldsymbol{\nu}_i^k := \phi_k\,\mathrm{d}\boldsymbol{\mu}_i$ in the sense of Remark 4.2.5, and let $\{\varphi_j^{(i)}\}_{j\in\mathbb{N}}$ be dense in $C_0(\Omega;C_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}))$. Arguing as in the proof of Proposition 4.2.6 and as in

Ambrosio, Fusco and Pallara [7, Thm 3.9] (see Theorem 2.3.6), for each $j \in \mathbb{N}$ we can find a sequence $\{\varepsilon_k^{(j)}\}_{k\in\mathbb{N}}$ of positive numbers converging to zero, with $\varepsilon_k^{(j)} < \varepsilon_k^{(j-1)}$ (and $\varepsilon_k^{(0)} := 1/2$), such that for all $k \in \mathbb{N}$, $l \in \{1, \cdots, j\}$ and $i \in \{1, \cdots, n\}$ one has

$$\begin{aligned}
&\operatorname{supp}\big(\rho_{\varepsilon_k^{(j)}} * (u\phi_k)\big) \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\,\big),\\
&\int_\Omega \big[|\rho_{\varepsilon_k^{(j)}} * (u\phi_k) - u\phi_k| + |\rho_{\varepsilon_k^{(j)}} * (u\otimes\nabla\phi_k) - u\otimes\nabla\phi_k|\big]\,\mathrm{d}x \leqslant \frac{1}{j\,2^k}, \qquad (4.2.68)\\
&\operatorname{supp}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}} \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\,\big)\times\mathbb{R}^{iN},\\
&\bigg|\int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi_l^{(i)}(x,y_1,...,y_i) : \nabla_{y_i}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}(x,y_1,...,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i\\
&\qquad\qquad - \int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi_l^{(i)}(x,y_1,...,y_i) : \mathrm{d}\tau_i^k(x,y_1,...,y_i)\bigg| \leqslant \frac{1}{j\,2^k},\\
&\bigg|\int_{\Omega\times Y_1\times\cdots\times Y_i} \Big|\nabla_{y_i}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}(x,y_1,...,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i - \|\tau_i^k\|(\Omega\times Y_1\times\cdots\times Y_i)\bigg| \leqslant \frac{1}{2^k},
\end{aligned}$$

where $\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}$ were introduced in (4.2.38) and τ_i^k is the measure associated with $D_{y_i}\boldsymbol{\nu}_i^k$.

Similarly to the proof of Proposition 4.2.6 and as in Ambrosio, Fusco and Pallara [7, Thm 3.9], for each $j \in \mathbb{N}$ and $i \in \{1, \cdots, n\}$ the functions u_j and $\psi_j^{(i)}$ defined by

$$u_j(x) := \sum_{k=1}^{\infty}((\rho_{\varepsilon_k^{(j)}} * (u\phi_k))(x), \quad \psi_j^{(i)}(x,y_1,\cdots,y_i) := \sum_{k=1}^{\infty}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}(x,y_1,\cdots,y_i), \tag{4.2.69}$$

belong to $C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$ and $C^\infty(\Omega;C^\infty_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$, respectively, and are such that

$$\begin{aligned}
&u_j \to u \text{ in } L^1(\Omega;\mathbb{R}^d) \text{ as } j\to+\infty, \quad \lim_{j\to+\infty}\int_\Omega|\nabla u_j(x)|\,\mathrm{d}x = \|Du\|(\Omega),\\
&\sup_{j\in\mathbb{N}}\|\nabla_{y_i}\psi_j^{(i)}\|_{L^1(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})} < \infty, \qquad (4.2.70)\\
&\nabla_{y_i}\psi_j^{(i)}\mathcal{L}^{(i+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \lambda_i \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}) \text{ as } j\to+\infty. \qquad (4.2.71)
\end{aligned}$$

In particular, $u_j \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^d)$ as $j\to+\infty$. In turn, this implies that

$$\nabla u_j\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup} Du\lfloor_\Omega\otimes\mathcal{L}^{nN}_{y_1,\cdots,y_n}$$

weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N})$ as $j\to+\infty$. Also, convergences (4.2.71) imply that $\nabla_{y_i}\psi_j^{(i)}\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup} \lambda_i\otimes\mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n}$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N})$ as $j\to+\infty$. Hence,

$$\Big(\nabla u_j + \sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}\Big)\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup} \lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n} \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N})$$

as $j\to+\infty$. Using the lower semicontinuity of the total variation,

$$\begin{aligned}
\liminf_{j\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}&\Big|\nabla u_j(x) + \sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\geqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).
\end{aligned} \tag{4.2.72}$$

Finally, let $\varphi \in C_c(\Omega; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^{d\times N}))$ with $\|\varphi\|_\infty \leqslant 1$ be given. Let $m \in \mathbb{N}$ be such that $\operatorname{supp}\varphi \subset \Omega_m \times \mathbb{R}^{iN}$. Taking into account (4.2.4), similar arguments to those of Proposition 4.2.6 (see (4.2.65)) show that

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_n} &\varphi(x,y_1,\cdots,y_n) : \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &= \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar{\varphi}_j(x,y_1,\cdots,y_n) : \mathrm{d}\lambda_i(x,y_1,\cdots,y_i)\mathrm{d}y_{i+1}\cdots\mathrm{d}y_n,\end{aligned} \tag{4.2.73}$$

where $\bar{\varphi}_j(x,y_1,\cdots,y_n) := \sum_{k=1}^m \Big[\big((\rho_{\varepsilon_k^{(j)}} \prod_{i=1}^n \eta_{\varepsilon_k^{(j)}}) * \varphi\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big]$ is such that

$$\bar{\varphi}_j \in C_0(\Omega; C_\#(Y_1\times\cdots\times Y_n; \mathbb{R}^{d\times N})), \quad \|\bar{\varphi}_j\|_\infty \leqslant 1. \tag{4.2.74}$$

On the other hand, using the identity

$$\nabla u_j = \sum_{k=1}^\infty \rho_{\varepsilon_k^{(j)}} * (\phi_k\,\mathrm{d}Du) + \sum_{k=1}^\infty \big[\rho_{\varepsilon_k^{(j)}} * (u\otimes\nabla\phi_k) - u\otimes\nabla\phi_k\big],$$

the estimate (4.2.68) and the condition $\|\varphi\|_\infty \leqslant 1$, we deduce that

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_n} &\varphi(x,y_1,\cdots,y_n) : \nabla u_j(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &\leqslant \sum_{k=1}^m \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \big(\rho_{\varepsilon_k^{(j)}} * (\phi_k\,\mathrm{d}Du)\big)(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n + \frac{1}{j}\cdot\end{aligned} \tag{4.2.75}$$

In turn, using (4.2.3), (4.2.4) and Fubini's Theorem,

$$\begin{aligned}\sum_{k=1}^m &\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \big(\rho_{\varepsilon_k^{(j)}} * (\phi_k\,\mathrm{d}Du)\big)(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &= \sum_{k=1}^m \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \left(\int_{\mathbb{R}^N} \rho_{\varepsilon_k^{(j)}}(x-x')\phi_k(x')\,\mathrm{d}Du(x')\right)\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &= \sum_{k=1}^m \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \left(\int_{\mathbb{R}^N\times Y_1\times\cdots\times Y_n} \phi_k(x')\rho_{\varepsilon_k^{(j)}}(x-x')\right. \\ &\qquad\qquad \left.\prod_{i=1}^n \eta_{\varepsilon_k^{(j)}}(y_i-y_i')\,\mathrm{d}Du(x')dy_1'\cdots\mathrm{d}y_n'\right)\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &= \sum_{k=1}^m \int_{\mathbb{R}^N\times Y_1\times\cdots\times Y_n} \left[\left(\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n)\rho_{\varepsilon_k^{(j)}}(x'-x)\right.\right. \\ &\qquad\qquad \left.\left.\prod_{i=1}^n \eta_{\varepsilon_k^{(j)}}(y_i'-y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\right)\phi_k(x')\right] : \mathrm{d}Du(x')\mathrm{d}y_1'\cdots\mathrm{d}y_n' \\ &= \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar{\varphi}_j(x',y_1',\cdots,y_n') : \mathrm{d}Du(x')\mathrm{d}y_1'\cdots\mathrm{d}y_n'.\end{aligned} \tag{4.2.76}$$

Thus, from (4.2.73), (4.2.75) and (4.2.76) we conclude that

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_n} &\varphi(x,y_1,\cdots,y_n) : \Big(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &\leqslant \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar{\varphi}_j(x,y_1,\cdots,y_n) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n) + \frac{1}{j} \\ &\leqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n) + \frac{1}{j},\end{aligned} \tag{4.2.77}$$

where in the last inequality we have used (4.2.74). Lebesgue Dominated Convergence Theorem, (4.2.70) and an approximation argument ensure that for all $\varphi \in C_0(\Omega; C_\#(Y; \mathbb{R}^{d\times N}))$ with $\|\varphi\|_\infty \leqslant 1$ one has

$$\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \Big(\nabla u_j(x) + \sum_{i=1}^{n} \nabla_{y_i}\psi_j^{(i)}(x, y_1, \cdots, y_i)\Big)\, \mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n$$
$$\leqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n) + \frac{1}{j}\cdot$$

Hence,

$$\limsup_{j\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n} \Big|\nabla u_j(x)+\sum_{i=1}^{n} \nabla_{y_i}\psi_j^{(i)}(x, y_1, \cdots, y_i)\Big|\, \mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \leqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n),$$

which, together with (4.2.72), concludes Step 1.

Step 2. We prove that the sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}$ may be taken in $C^\infty(\overline{\Omega}; \mathbb{R}^d)$ and $C_c^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i; \mathbb{R}^d))$, respectively.

The argument is similar to that of Corollary 4.2.7. Let $\{u_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ be the sequences constructed in Step 1. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_k \subset\subset \Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty}\Omega_k$, and let $\{\theta_k\}_{k\in\mathbb{N}}$ be a sequence of cut-off functions $\theta_k \in C_c^\infty(\Omega; [0,1])$ satisfying for all $k\in\mathbb{N}$, $\theta_k = 1$ in Ω_k. Define

$$\psi_{j,k}^{(i)}(x, y_1, \cdots, y_i) := \theta_k(x)\psi_j^{(i)}(x, y_1, \cdots, y_i).$$

We have that $\psi_{j,k}^{(i)} \in C_c^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i; \mathbb{R}^d))$, with $\nabla_{y_i}\psi_{j,k}^{(i)} = \theta_k\nabla_{y_i}\psi_j^{(i)}$. For each $j\in\mathbb{N}$, let $\{u_k^{(j)}\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega}; \mathbb{R}^d)$ be a sequence such that

$$u_k^{(j)} \to u_j \text{ in } W^{1,1}(\Omega; \mathbb{R}^d) \text{ as } k\to+\infty. \tag{4.2.78}$$

We observe that here, and only here, we use the hypothesis that $\partial\Omega$ is Lipschitz. We have that

$$\lim_{j\to+\infty}\lim_{k\to+\infty}\int_\Omega \Big|u_k^{(j)}(x) - u(x)\Big|\, \mathrm{d}x = 0, \quad \lim_{j\to+\infty}\lim_{k\to+\infty}\int_\Omega \Big|\nabla u_k^{(j)}(x)\Big|\, \mathrm{d}x = \|Du\|(\Omega). \tag{4.2.79}$$

Let $\varphi \in C_0(\Omega; C_\#(Y_1\times\cdots\times Y_n; \mathbb{R}^{d\times N}))$ be given. Using on the one hand convergence (4.2.78), and on the other hand the pointwise convergence $\theta_k \to 1$ in Ω as $k\to+\infty$ together with Lebesgue Dominated Convergence Theorem and taking into account estimate (4.2.70), we obtain

$$\begin{aligned}
&\lim_{j\to+\infty}\lim_{k\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \Big(\nabla u_k^{(j)}(x) + \sum_{i=1}^{n}\nabla_{y_i}\psi_{j,k}^{(i)}(x, y_1, \cdots, y_i)\Big)\, \mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\
&\quad= \lim_{j\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \Big(\nabla u_j(x) + \sum_{i=1}^{n}\nabla_{y_i}\psi_j^{(i)}(x, y_1, \cdots, y_i)\Big)\, \mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\
&\quad= \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x, y_1, \cdots, y_n),
\end{aligned} \tag{4.2.80}$$

where in the last equality we have used Step 1. By similar arguments, and since we can write

$$\nabla u_k^{(j)} + \sum_{i=1}^{n}\nabla_{y_i}\psi_{j,k}^{(i)} = \nabla u_k^{(j)} - \nabla u_j + \theta_k\nabla u_j + \theta_k\sum_{i=1}^{n}\nabla_{y_i}\psi_j^{(i)} + (1-\theta_k)\nabla u_j,$$

we have

$$\begin{aligned}
&\limsup_{j\to+\infty}\limsup_{k\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_k^{(j)}(x)+\sum_{i=1}^n\nabla_{y_i}\psi_{j,k}^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\quad\leqslant\lim_{j\to+\infty}\lim_{k\to+\infty}\Big\{\int_\Omega\Big|\nabla u_k^{(j)}(x)-\nabla u_j(x)\Big|\,\mathrm{d}x\\
&\qquad+\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\qquad+\int_\Omega(1-\theta_k(x))|\nabla u_j(x)|\,\mathrm{d}x\Big\}\\
&\quad=\lim_{j\to+\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\quad=\|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).
\end{aligned}\tag{4.2.81}$$

From (4.2.79), (4.2.80) and (4.2.81), using the separability of $C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}))$ and a diagonal argument, and finally the lower semicontinuity of the total variation, we can find sequences as in the statement of Corollary 4.2.8. $\square$

Remark 4.2.9. *As it was observed within the previous proof, if $\partial\Omega$ fails to be Lipschitz, then Corollary 4.2.8 holds replacing the condition "$\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)$" by "$\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$".*

We are now in place to prove Proposition 4.1.8.

Proof of Proposition 4.1.8. Let $u\in BV(\Omega;\mathbb{R}^d)$ and for $i\in\{1,\cdots,n\}$, let $\boldsymbol{\mu}_i\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#(Y_i;\mathbb{R}^d))$. Let $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}\subset C_c^\infty(\Omega;C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ be sequences satisfying (4.2.67).

For each $\varepsilon>0$ and $j\in\mathbb{N}$, define

$$u_{\varepsilon,j}(x):=u_j(x)+\sum_{i=1}^n\varrho_i(\varepsilon)\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big),\ \ x\in\Omega.$$

Then $u_{\varepsilon,j}\in W^{1,1}(\Omega;\mathbb{R}^d)$, and

$$\begin{aligned}
\nabla u_{\varepsilon,j}(x)=\nabla u_j(x)&+\sum_{i=1}^n\varrho_i(\varepsilon)\,\nabla_x\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)\\
&+\sum_{i=2}^n\sum_{k=1}^{i-1}\frac{\varrho_i(\varepsilon)}{\varrho_k(\varepsilon)}\nabla_{y_k}\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big).
\end{aligned}$$

Let $\tilde\varphi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d))$ and $\varphi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}))$ be given. Since for fixed $j\in\mathbb{N}$ and $i\in\{1,\cdots,n\}$, and for all $(y_1,\cdots,y_i)\in\mathbb{R}^{iN}$, $x\mapsto\psi_j^{(i)}(x,y_1,\cdots,y_i)$ has compact support in $\mathbb{R}^N$, from (4.0.1) and (4.2.25) we deduce that

$$\begin{aligned}
\lim_{\varepsilon\to0^+}\int_\Omega\tilde\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_{\varepsilon,j}(x)\,\mathrm{d}x&=\lim_{\varepsilon\to0^+}\int_\Omega\tilde\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_j(x)\,\mathrm{d}x\\
&=\int_{\Omega\times Y_1\times\cdots\times Y_n}\tilde\varphi(x,y_1,\cdots,y_n)\cdot u_j(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,
\end{aligned}$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big):\nabla u_{\varepsilon,j}(x)\,\mathrm{d}x\\
&\quad=\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big):\Big(\nabla u_j(x)+\sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)\Big)\,\mathrm{d}x\\
&\quad=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\Big(\nabla u_j(x)+\sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n.
\end{aligned}
$$

Thus, in view of (4.2.67),

$$
\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \tilde\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_{\varepsilon,j}(x)\,\mathrm{d}x=\int_{\Omega\times Y_1\times\cdots\times Y_n}\tilde\varphi(x,y_1,\cdots,y_n)\cdot u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n, \tag{4.2.82}
$$

and

$$
\begin{aligned}
&\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big):\nabla u_{\varepsilon,j}(x)\,\mathrm{d}x\\
&\qquad=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n).
\end{aligned} \tag{4.2.83}
$$

We claim that we may find a sequence $\{j_\varepsilon\}_{\varepsilon>0}$ such that $j_\varepsilon\to+\infty$ as $\varepsilon\to 0^+$, and if we define $v_\varepsilon:=u_{\varepsilon,j_\varepsilon}$, then $\{v_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $W^{1,1}(\Omega;\mathbb{R}^d)$ satisfying *a)* and *b)* of Theorem 4.1.7.

In fact, let $\{\tilde\varphi_m\}_{m\in\mathbb{N}}$ and $\{\varphi_m\}_{m\in\mathbb{N}}$ be dense in $C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d))$ and $C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}))$, respectively. For each $\varepsilon>0$, $j,m\in\mathbb{N}$, define

$$
\begin{aligned}
\tilde\Psi_{\varepsilon,j,m}&:=\int_\Omega\tilde\varphi_m\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_{\varepsilon,j}(x)\,\mathrm{d}x,\\
\tilde L_m&:=\int_{\Omega\times Y_1\times\cdots\times Y_n}\tilde\varphi_m(x,y_1,\cdots,y_n)\cdot u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,\\
\Psi_{\varepsilon,j,m}&:=\int_\Omega\varphi_m\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big):\nabla u_{\varepsilon,j}(x)\,\mathrm{d}x,\\
L_m&:=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi_m(x,y_1,\cdots,y_n):\mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n).
\end{aligned}
$$

By (4.2.82) and (4.2.83), for all $m\in\mathbb{N}$, we have

$$
\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\tilde\Psi_{\varepsilon,j,m}=\tilde L_m,\quad \lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\Psi_{\varepsilon,j,m}=L_m. \tag{4.2.84}
$$

For each $\varepsilon>0$, $j\in\mathbb{N}$, set

$$
\Theta_{\varepsilon,j}:=\sum_{m=1}^\infty\bigg[\frac{1}{2^m}\bigg(\frac{|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}{1+|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}+\frac{|\Psi_{\varepsilon,j,m}-L_m|}{1+|\Psi_{\varepsilon,j,m}-L_m|}\bigg)\bigg].
$$

Fix $\delta>0$, and let $m_\delta\in\mathbb{N}$ be such that $\displaystyle\sum_{m=m_\delta+1}^\infty\frac{1}{2^m}\leqslant\delta/2$. Then,

$$
0\leqslant\Theta_{\varepsilon,j}\leqslant\sum_{m=1}^{m_\delta}\bigg[\frac{1}{2^m}\bigg(\frac{|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}{1+|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}+\frac{|\Psi_{\varepsilon,j,m}-L_m|}{1+|\Psi_{\varepsilon,j,m}-L_m|}\bigg)\bigg]+\delta
$$

and so, using (4.2.84),

$$0 \leqslant \limsup_{j\to+\infty} \limsup_{\varepsilon\to 0^+} \Theta_{\varepsilon,j} \leqslant \delta, \quad 0 \leqslant \limsup_{j\to+\infty} \liminf_{\varepsilon\to 0^+} \Theta_{\varepsilon,j} \leqslant \delta.$$

Letting $\delta \to 0^+$, we obtain

$$\lim_{j\to+\infty} \limsup_{\varepsilon\to 0^+} \Theta_{\varepsilon,j} = \lim_{j\to+\infty} \liminf_{\varepsilon\to 0^+} \Theta_{\varepsilon,j} = 0.$$

By a diagonalization argument, we may find a sequence $\{j_\varepsilon\}_{\varepsilon>0}$ such that $j_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, and

$$\lim_{\varepsilon\to 0^+} \Theta_{\varepsilon,j_\varepsilon} = 0. \tag{4.2.85}$$

This way, given $m \in \mathbb{N}$, by definition of $\Theta_{\varepsilon,j_\varepsilon}$ and by (4.2.85), we have

$$0 \leqslant \frac{1}{2^m}\left(\frac{|\tilde{\Psi}_{\varepsilon,j_\varepsilon,m} - \tilde{L}_m|}{1+|\tilde{\Psi}_{\varepsilon,j_\varepsilon,m} - \tilde{L}_m|} + \frac{|\Psi_{\varepsilon,j_\varepsilon,m} - L_m|}{1+|\Psi_{\varepsilon,j_\varepsilon,m} - L_m|}\right) \leqslant \Theta_{\varepsilon,j_\varepsilon} \to 0 \text{ as } \varepsilon \to 0^+,$$

which implies

$$\lim_{\varepsilon\to 0^+} \tilde{\Psi}_{\varepsilon,j_\varepsilon,m} = \tilde{L}_m, \quad \lim_{\varepsilon\to 0^+} \Psi_{\varepsilon,j_\varepsilon,m} = L_m. \tag{4.2.86}$$

Finally, the existence of a sequence $\{v_\varepsilon\}_{\varepsilon>0}$ as claimed above follows from (4.2.86), taking into account the boundedness of $\{u_{\varepsilon,j_\varepsilon}\}_{\varepsilon>0}$ in $W^{1,1}(\Omega;\mathbb{R}^d)$. $\square$

We finish this section by proving an extension of Corollary 4.2.8 to the case in which Ω is bounded, and that will play an important role in our application to homogenization in Chapter 5.

Proposition 4.2.10. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set such that $\partial\Omega$ is Lipschitz. Let $u \in BV(\Omega;\mathbb{R}^d)$ and for each $i \in \{1,\cdots,n\}$, let $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$. Then there exist sequences $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ satisfying (4.2.67), and such that*

$$\begin{aligned} &\tilde{\lambda}_j \overset{\star}{\rightharpoonup} \tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n} \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}) \text{ as } j\to+\infty,\\ &\lim_{j\to+\infty} \|\tilde{\lambda}_j\|(\Omega\times Y_1\times\cdots\times Y_n) = \|\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n), \end{aligned} \tag{4.2.87}$$

where, for any $B \in \mathcal{B}(\Omega\times Y_1\times\cdots\times Y_n)$,

$$\begin{aligned} \tilde{\lambda}_j(B) &:= \left(\int_B \Big(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n, \mathcal{L}^{(n+1)N}(B)\right),\\ \tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}(B) &:= \Big(\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}(B), \mathcal{L}^{(n+1)N}(B)\Big). \end{aligned}$$

Proof. The proof is very similar to that of Corollary 4.2.8. We will just point out the main differences. In Step 1 of the proof of Corollary 4.2.8, for each $j \in \mathbb{N}$ we require the sequence $\{\varepsilon_k^{(j)}\}_{k\in\mathbb{N}}$ to satisfy the additional conditions

$$\operatorname{supp}\big(\rho_{\varepsilon_k^{(j)}} * \phi_k\big) \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big), \quad \sup_{x\in\Omega}\big|\phi_k(x) - \rho_{\varepsilon_k^{(j)}} * \phi_k(x)\big| \leqslant \frac{1}{j\,2^k}\cdot \tag{4.2.88}$$

This is possible since if $\phi \in C(\Omega)$, then $\rho_\varepsilon * \phi$ converges uniformly to ϕ as $\varepsilon \to 0^+$ on every compact subset of Ω, and $\operatorname{supp}\phi_k \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big)$.

Defining $u_j \in C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d))$ and $\psi_j^{(i)} \in C^\infty(\Omega; C^\infty_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ as in (4.2.69), then (4.2.67) holds. Moreover, we clearly have $\tilde\lambda_j \overset{\star}{\rightharpoonup} \tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R})$ as $j\to+\infty$, which in turn implies that

$$\liminf_{j\to+\infty}\|\tilde\lambda_j\|(\Omega\times Y_1\times\cdots\times Y_n)\geqslant \|\tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).$$

Furthermore, given $\psi=(\varphi,\theta)\in C_c(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}))\times C_c(\Omega;C_\#(Y_1\times\cdots\times Y_n))$ with $\|\psi\|_\infty\leqslant 1$, then by (4.2.77)

$$\begin{aligned}
&\int_{\Omega\times Y_1\times\cdots\times Y_n}\psi(x,y_1,\cdots,y_n)\cdot \mathrm{d}\tilde\lambda_j(x,y_1,\cdots,y_n)\\
&\quad=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\Big(\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\qquad+\int_{\Omega\times Y_1\times\cdots\times Y_n}\theta(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\quad\leqslant\int_{\Omega\times Y_1\times\cdots\times Y_n}\bar\varphi_j(x,y_1,\cdots,y_n):\mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n)+\frac{1}{j}\\
&\qquad+\int_{\Omega\times Y_1\times\cdots\times Y_n}\theta(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,
\end{aligned}$$

where $\bar\varphi_j(x,y_1,\cdots,y_n):=\sum_{k=1}^m\Big[\big((\rho_{\varepsilon_k^{(j)}}\prod_{i=1}^n\eta_{\varepsilon_k^{(j)}})*\varphi\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big]$. Similarly, setting

$$\bar\theta_j(x,y_1,\cdots,y_n):=\sum_{k=1}^m\Big[\big((\rho_{\varepsilon_k^{(j)}}\prod_{i=1}^n\eta_{\varepsilon_k^{(j)}})*\theta\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big],$$

then, using (4.2.88) and Fubini's Theorem, we deduce that

$$\left|\int_{\Omega\times Y_1\times\cdots\times Y_n}\theta(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n-\int_{\Omega\times Y_1\times\cdots\times Y_n}\bar\theta_j(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\right|\leqslant\frac{\mathcal{L}^N(\Omega)}{j}.$$

Hence, defining $\bar\psi_j(x,y_1,\cdots,y_n):=\sum_{k=1}^m\Big[\big((\rho_{\varepsilon_k^{(j)}}\prod_{i=1}^n\eta_{\varepsilon_k^{(j)}})*\psi\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big]$, we conclude that

$$\begin{aligned}
&\int_{\Omega\times Y_1\times\cdots\times Y_n}\psi(x,y_1,\cdots,y_n)\cdot\mathrm{d}\tilde\lambda_j(x,y_1,\cdots,y_n)\\
&\quad\leqslant\int_{\Omega\times Y_1\times\cdots\times Y_n}\bar\psi_j(x,y_1,\cdots,y_n)\cdot\mathrm{d}\tilde\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n)+\frac{1+\mathcal{L}^N(\Omega)}{j} \qquad (4.2.89)\\
&\quad\leqslant\|\tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n)+\frac{1+\mathcal{L}^N(\Omega)}{j},
\end{aligned}$$

where in the last inequality we have used the fact that $\bar\psi_j\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}))$ and $\|\bar\psi_j\|_\infty\leqslant 1$. Using a density argument, together with Lebesgue Dominated Convergence Theorem, we deduce that (4.2.89) holds for every $\psi\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}))$ with $\|\psi\|_\infty\leqslant 1$. Consequently,

$$\limsup_{j\to+\infty}\|\tilde\lambda_j\|(\Omega\times Y_1\times\cdots\times Y_n)\leqslant\|\tilde\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).$$

Thus (4.2.87) holds. We proceed as in Step 2 of Corollary 4.2.8 to prove that the sequence $\{u_j\}_{j\in\mathbb{N}}$ may be taken in $C^\infty(\overline{\Omega};\mathbb{R}^d)$ and that the sequences $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}$ may be taken in $C_c^\infty(\Omega;C^\infty_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$.
□

Chapter 5

Reiterated Homogenization in BV via Multiscale Convergence

Under the motivation mentioned in the Introduction (see Subsection 1.2), in this chapter we treat multiple-scale homogenization problems in the space BV of functions of bounded variation, using the notion of multiple-scale convergence developed in Chapter 4. In the case of one microscale we recover Amar's result [5] under more general conditions; for two or more microscales we obtain new results. This study was elaborated in the joint work with Fonseca [44].

As we referred in the previous chapter, in Amar [5] the author extended the notion of two-scale convergence to the case of bounded sequences of Radon measures with finite total variation. This was used to study the asymptotic behavior of sequences of positively 1-homogeneous and periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation. Precisely, the following result is given in Amar [5].

Theorem A (cf. Amar [5, Thm. 4.1]). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with $\partial\Omega$ Lipschitz, let $Q := [0,1]^N$, and let $f : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ be a function such that*

(A1) for all $\xi \in \mathbb{R}^N$, $f(\cdot,\xi)$ is continuous and Q-periodic;

(A2) for all $y \in Q$, $f(y,\cdot)$ is convex, positively 1-homogeneous, and of class $C^1(\mathbb{R}^N\backslash\{0\})$;

(A3) there exists a constant $C > 0$ such that for all $y \in Q$, $\xi \in \mathbb{R}^N$, $\frac{1}{C}|\xi| \leqslant f(y,\xi) \leqslant C|\xi|$.

For each $\varepsilon > 0$, let $I_\varepsilon : BV(\Omega) \to \mathbb{R}$ be the functional defined by

$$I_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}Du}{\mathrm{d}\|Du\|}(x)\Big)\,\mathrm{d}\|Du\|(x) + \int_\Omega |v(x) - u(x)|^p\,\mathrm{d}x,$$

where $v \in L^{N/(N-1)}(\Omega)$, $p \in (1, N/(N-1)]$ if $N > 1$, and $p \in (1,\infty)$ if $N = 1$. Then for each $\varepsilon > 0$, there exists a unique $u_\varepsilon \in BV(\Omega)$ such that

$$I_\varepsilon(u_\varepsilon) = \min_{w\in BV(\Omega)} I_\varepsilon(w) = \inf_{w\in W^{1,1}(\Omega)} \Big\{ \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla w(x)\Big)\,\mathrm{d}x + \int_\Omega |v(x) - w(x)|^p\,\mathrm{d}x \Big\}.$$

Moreover, there exist $u \in BV(\Omega)$ and $\boldsymbol{\mu} \in \mathcal{M}_\star(\Omega; BV_\#(Q))$[5.1], such that $\{u_\varepsilon\}_{\varepsilon>0}$ weakly-$\star$ converges to u in $BV(\Omega)$ as $\varepsilon \to 0^+$ and, up to a subsequence, $\{Du_\varepsilon\}_{\varepsilon>0}$ two-scale converges to the measure

[5.1] In Amar [5] no considerations on the application $D_y\boldsymbol{\mu}$ were made; in particular, the subspace $\mathcal{M}_\star(\Omega; BV_\#(Q))$ was not introduced. In view of Theorem 4.1.7 we believe this is the correct setting, and so we use here the same notations as in Subsection 2.3.2 and Chapter 4.

$\lambda_{u,\boldsymbol{\mu}} \in \mathcal{M}_{y\#}(\Omega \times Q; \mathbb{R}^N)$ *given by (4.1.3) for* $n = 1$. *Furthermore,*

$$\lim_{\varepsilon \to 0^+} I_\varepsilon(u_\varepsilon) = \inf_{\substack{w \in BV(\Omega) \\ \boldsymbol{\nu} \in \mathcal{M}_\star(\Omega; BV_\#(Q))}} I^{\mathrm{sc}}(w, \boldsymbol{\nu}) = I^{\mathrm{sc}}(u, \boldsymbol{\mu}),$$

where I^{sc} *is the two-scaled homogenized functional defined for* $w \in BV(\Omega)$ *and* $\boldsymbol{\nu}_\star \in \mathcal{M}(\Omega; BV_\#(Q))$ *by*

$$I^{\mathrm{sc}}(w, \boldsymbol{\nu}) := \int_{\Omega \times Q} f\Big(y, \frac{\mathrm{d}\lambda_{w,\boldsymbol{\nu}}}{\mathrm{d}\|\lambda_{w,\boldsymbol{\nu}}\|}(x,y)\Big)\, \mathrm{d}\|\lambda_{w,\boldsymbol{\nu}}\|(x,y) + \int_\Omega |v(x) - w(x)|^p \, \mathrm{d}x.$$

Finally, in the minimizing pair $(u, \boldsymbol{\mu})$ *the function* $u \in BV(\Omega)$ *is uniquely determined.*

The proof of Theorem A is based on the so-called two-scale convergence method, which has the virtue of taking full advantage of the periodic microscopic properties of the media, enabling the explicit characterization of the local behavior of the system: The asymptotic behavior as $\varepsilon \to 0^+$ of the energies F_ε and of the respective minimizers u_ε is given with regard to both macroscopic and microscopic levels, through the two space variables x (the macroscopic one) and y (the microscopic one), and through the two unknowns u and $\boldsymbol{\mu}$. The next step of the two-scale convergence method is to obtain the effective or homogenized problem, that is, the limit problem only involving the macroscopic space variable x, and which has as solution the function $\bar{u}(x) := \int_Q u(x,y)\, \mathrm{d}y$. This is usually done via an average process with respect to the "fast variable" y of the two-scale homogenized problem. It should be noticed that in some cases this averaging process leads to very complicated expressions for the homogenized problem, and consequently, the *nice* form of the two-scale homogenized problem is lost (see Allaire [1] for several references exemplifying such a phenomenon). Therefore, in particular in these cases, the two-scale homogenized limit problem seems to be preferable.

For the class of functions f considered by Amar [5], Theorem A provides an alternative characterization of the homogenized problem previously obtained by Bouchitté [16], [17], and summarizes as follows:

Theorem B (cf. Bouchitté [16, Thm. 2.1])**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open and bounded set, let* $Y := (0,1)^N$, *and let* $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *be a function such that*

- *(B1) for all* $\xi \in \mathbb{R}^N$, $f(\cdot, \xi)$ *is measurable and* Y*-periodic;*
- *(B2) for all* $y \in Y$, $f(y, \cdot)$ *is convex;*
- *(B3) there exists a constant* $C > 0$ *such that for all* $y \in Y$, $\xi \in \mathbb{R}^N$, $\frac{1}{C}|\xi| - C \leqslant f(y,\xi) \leqslant C(1 + |\xi|)$.

For each $\varepsilon > 0$, *let* $F_\varepsilon : L^1(\Omega) \to (-\infty, \infty]$ *be the functional defined by*

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u(x)\Big)\, \mathrm{d}x & \text{if } u \in W^{1,1}(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Then, the sequence of functionals $\{F_\varepsilon\}_{\varepsilon > 0}$ Γ*-converges in* $L^1(\Omega)$ *as* $\varepsilon \to 0^+$ [5.2], *to the functional* $F_0 : L^1(\Omega) \to (-\infty, \infty]$ *given by*

$$F_0(u) := \begin{cases} F^{\mathrm{h}}(u) & \text{if } u \in BV(\Omega), \\ \infty & \text{otherwise,} \end{cases}$$

where, for $u \in BV(\Omega)$,

$$F^{\mathrm{h}}(u) := \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\, \mathrm{d}x + \int_\Omega (f_{\mathrm{hom}})^\infty \Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\, \mathrm{d}\|D^s u\|(x),$$

5.2 See Definition 2.5.1 (see also Remark 2.5.2).

with

$$f_{\mathrm{hom}}(\xi) := \inf\left\{ \int_Y f(y, \xi + \nabla\psi(y))\,\mathrm{d}y\colon\ \psi \in W^{1,1}_{\#}(Y)\right\}, \quad (f_{\mathrm{hom}})^\infty(\xi) := \lim_{t\to+\infty} \frac{f_{\mathrm{hom}}(t\xi)}{t}\cdot$$

In view of Theorem 2.5.16, under the coercivity condition in (B3), if we consider the analogous functional I_ε of Amar [5], i.e., the functional $I_\varepsilon(u) := F_\varepsilon(u) + \int_\Omega |v-u|^p\,\mathrm{d}x$ for $u \in L^1(\Omega)$, where F_ε is as in Theorem B, and v and p are as in Theorem A, then, assuming $\partial\Omega$ Lipschitz and using the continuous injection of $BV(\Omega)$ in $L^p(\Omega)$ (see Theorem 2.3.9),

$$\lim_{\varepsilon\to0^+} \inf_{w\in L^1(\Omega)} I_\varepsilon(w) = \lim_{\varepsilon\to0^+} \inf_{w\in W^{1,1}(\Omega)} I_\varepsilon(w) = \min_{w\in L^1(\Omega)} I_0(w) = \min_{w\in BV(\Omega)} I^{\mathrm{h}}(w),$$

where $I_0(w) := F_0(w) + \int_\Omega |v-w|^p\,\mathrm{d}x$, $I^{\mathrm{h}}(w) := F^{\mathrm{h}}(w) + \int_\Omega |v-w|^p\,\mathrm{d}x$, and F_0 and F^{h} were introduced in Theorem B. In particular, if f satisfies conditions (A1), (A2) and (A3), then $I^{\mathrm{h}}(u) = I^{\mathrm{sc}}(u, \boldsymbol{\mu})$, where I^{sc} and $(u,\boldsymbol{\mu}) \in BV(\Omega) \times \mathcal{M}_\star(\Omega; BV_{\#}(Q))$ are as in the statement of Theorem A.

The proof of Theorem B relies on integral functionals of measures and their formulation by duality, while, as we mentioned before, the proof of Theorem A is based on the two-scale convergence method and is very similar to that of Allaire [1, Thm. 3.3] in which the subdifferentiability of f and the regularity and boundedness of $\nabla_\xi f$ play a crucial role. In particular, the arguments used in Amar [5] do not apply neither under weaker regularity hypotheses than those in (A2) nor under more general linear estimates from above and from below than those in (A3).

Some questions then naturally arise: Is it possible to derive the two-scale homogenized functional under weaker hypotheses than those considered in Amar [5]? May we establish the relation between the two-scale homogenized functional I^{sc} and the homogenized functional I^{hom} in a systematic and direct way? How to generalize this analysis to the case of multiple microscales? And to the vectorial case? The goal of this chapter is precisely to give answers to these questions.

In particular, using Theorem 4.1.7 (and having in mind Proposition 4.1.4) we seek to characterize and relate the functionals

$$F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) := \inf\left\{ \liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in BV(\Omega;\mathbb{R}^d),\ Du_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\right\} \tag{5.0.1}$$

and

$$F^{\mathrm{hom}}(u) := \inf\left\{ \liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon)\colon\ u_\varepsilon \in BV(\Omega;\mathbb{R}^d),\ u_\varepsilon \overset{\star}{\rightharpoonup}_\varepsilon u \text{ weakly-}\star \text{ in } BV(\Omega;\mathbb{R}^d)\right\} \tag{5.0.2}$$

for $u \in BV(\Omega;\mathbb{R}^d)$ and $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_{\#}(Y_i;\mathbb{R}^d))$, $i\in\{1,\cdots,n\}$, where F_ε is of the form

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)},\nabla u(x)\Big)\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)},\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\mathrm{d}\|D^s u\|(x) \tag{5.0.3}$$

for $u \in BV(\Omega;\mathbb{R}^d)$, where

$$f^\infty(y_1,\cdots,y_n,\xi) := \limsup_{t\to+\infty} \frac{f(y_1,\cdots,y_n,t\xi)}{t}$$

is the recession function of a real valued function $f : \mathbb{R}^{nN}\times\mathbb{R}^{d\times N} \to \mathbb{R}$, separately periodic in the first n variables.

Before we state our main result, we introduce some notation. Fix $k \in \mathbb{N}$ and let $g : \mathbb{R}^{kN} \times \mathbb{R}^{d\times N} \to \overline{\mathbb{R}}$ be a Borel function. We recall that the effective domain of g, $\mathrm{dom}_e g$, is the set

$$\mathrm{dom}_e g := \big\{(y_1, \cdots, y_k, \xi) \in \mathbb{R}^{kN} \times \mathbb{R}^{d\times N} :\ g(y_1, \cdots, y_k, \xi) < \infty\big\},$$

while the conjugate function of g is the function $g^* : \mathbb{R}^{kN} \times \mathbb{R}^{d\times N} \to \overline{\mathbb{R}}$ defined by

$$g^*(y_1, \cdots, y_k, \xi^*) := \sup_{\xi\in\mathbb{R}^{d\times N}} \big\{\xi : \xi^* - g(y_1, \cdots, y_k, \xi)\big\}, \quad y_1, ..., y_k \in \mathbb{R}^N,\ \xi^* \in \mathbb{R}^{d\times N}, \tag{5.1.1}$$

and the biconjugate function of g is the function $g^{**} : \mathbb{R}^{kN} \times \mathbb{R}^{d\times N} \to \overline{\mathbb{R}}$ defined by

$$g^{**}(y_1, \cdots, y_k, \xi) := \sup_{\xi^*\in\mathbb{R}^{d\times N}} \big\{\xi^* : \xi - g^*(y_1, \cdots, y_k, \xi^*)\big\}, \quad y_1, ..., y_k \in \mathbb{R}^N,\ \xi^* \in \mathbb{R}^{d\times N}. \tag{5.1.2}$$

We define a function $g_{\mathrm{hom}_k} : \mathbb{R}^{(k-1)N} \times \mathbb{R}^{d\times N} \to \overline{\mathbb{R}}$ by setting

$$g_{\mathrm{hom}_k}(y_1, \cdots, y_{k-1}, \xi) := \inf\left\{ \int_{Y_k} g(y_1, \cdots, y_{k-1}, y_k, \xi + \nabla\psi_k(y_k))\,\mathrm{d}y_k :\ \psi_k \in W^{1,1}_{\#}(Y_k; \mathbb{R}^d)\right\} \tag{5.1.3}$$

for $y_1, ..., y_{k-1} \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$.

Let $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function. If $n = 1$, we set $f_{\mathrm{hom}} := f_{\mathrm{hom}_1}$, where f_{hom_1} is given by (5.1.3) for $k = 1$ and with g replaced by f, that is,

$$f_{\mathrm{hom}}(\xi) := \inf\left\{ \int_{Y_1} f(y_1, \xi + \nabla\psi_1(y_1))\,\mathrm{d}y_1 :\ \psi_1 \in W^{1,1}_{\#}(Y_1; \mathbb{R}^d)\right\}.$$

If $n = 2$, we define $f_{\mathrm{hom}} := (f_{\mathrm{hom}_2})_{\mathrm{hom}_1}$, which is the function given by (5.1.3) for $k = 1$ and with g replaced by f_{hom_2}, where the latter is the function given by (5.1.3) for $k = 2$ and with g replaced by f. Precisely,

$$f_{\mathrm{hom}}(\xi) := \inf\left\{ \int_{Y_1} f_{\mathrm{hom}_2}(y_1, \xi + \nabla\psi_1(y_1))\,\mathrm{d}y_1 :\ \psi_1 \in W^{1,1}_{\#}(Y_1; \mathbb{R}^d)\right\},$$

where

$$f_{\mathrm{hom}_2}(y_1, \xi) := \inf\left\{ \int_{Y_2} f(y_1, y_2, \xi + \nabla\psi_2(y_2))\,\mathrm{d}y_2 :\ \psi_2 \in W^{1,1}_{\#}(Y_2; \mathbb{R}^d)\right\}.$$

Similarly, if $n = 3$ we define $f_{\mathrm{hom}} := \big((f_{\mathrm{hom}_3})_{\mathrm{hom}_2}\big)_{\mathrm{hom}_1}$, i.e.

$$f_{\mathrm{hom}}(\xi) := \inf\left\{ \int_{Y_1} (f_{\mathrm{hom}_3})_{\mathrm{hom}_2}(y_1, \xi + \nabla\psi_1(y_1))\,\mathrm{d}y_1 :\ \psi_1 \in W^{1,1}_{\#}(Y_1; \mathbb{R}^d)\right\},$$

where

$$(f_{\mathrm{hom}_3})_{\mathrm{hom}_2}(y_1, \xi) := \inf\left\{ \int_{Y_2} f_{\mathrm{hom}_3}(y_1, y_2, \xi + \nabla\psi_2(y_2))\,\mathrm{d}y_2 :\ \psi_2 \in W^{1,1}_{\#}(Y_2; \mathbb{R}^d)\right\},$$

with

$$f_{\mathrm{hom}_3}(y_1, y_2, \xi) := \inf\left\{ \int_{Y_3} f(y_1, y_2, y_3, \xi + \nabla\psi_3(y_3))\,\mathrm{d}y_3 :\ \psi_3 \in W^{1,1}_{\#}(Y_3; \mathbb{R}^d)\right\}.$$

Recursively, for $n \in \mathbb{N}$ we set

$$f_{\text{hom}} := \Big((f_{\text{hom}_n})_{\text{hom}_{n-1}} \cdots \Big)_{\text{hom}_1}. \tag{5.1.4}$$

Consider the following conditions:

($\mathcal{F}1$) for all $\xi \in \mathbb{R}^{d\times N}$, $f(\cdot,\xi)$ is $Y_1 \times \cdots \times Y_n$-periodic;

($\mathcal{F}2$) for all $y_1, ..., y_n \in \mathbb{R}^N$, $f(y_1,\cdots,y_n,\cdot)$ is convex;

($\mathcal{F}3$) there exists $C>0$ such that for all $y_1,...,y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1,\cdots,y_n,\xi) \leqslant C(1+|\xi|);$$

($\mathcal{F}4$) for all $\delta > 0$ there exist $c_\delta \in \mathbb{R}^N$, $b_\delta \in \mathbb{R}$, such that $|c_\delta| \to 0$ as $\delta \to 0^+$, and for all $y_1,...,y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1,\cdots,y_n,\xi) + c_\delta \cdot \xi + b_\delta \geqslant 0;$$

($\mathcal{F}4$)' there exists $C>0$ such that for all $y_1,...,y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1,\cdots,y_n,\xi) \geqslant \frac{1}{C}|\xi| - C;$$

($\mathcal{F}5$) for every $y_1',...,y_n' \in \mathbb{R}^N$, $\delta>0$, there exists $\tau = \tau(y_1',\cdots,y_n',\delta)$ such that for all $y_1,...,y_n \in \mathbb{R}^N$ with $|(y_1',\cdots,y_n') - (y_1,\cdots,y_n)| \leqslant \tau$, and for all $\xi \in \mathbb{R}^{d\times N}$,

$$|f(y_1',\cdots,y_n',\xi) - f(y_1,\cdots,y_n,\xi)| \leqslant \delta(1+|\xi|);$$

($\mathcal{F}6$) for all $\delta > 0$ there exists $\tilde{a}_\delta \in L^1_\#(Y_1 \times \cdots \times Y_n)$ such that $\delta \|\tilde{a}_\delta\|_{L^1_\#(Y_1\times\cdots\times Y_n)} \to 0$ as $\delta \to 0^+$, and there exits $\tau_\delta > 0$ such that for all $y_1,...,y_{n-1}, y_1',...,y_{n-1}' \in \mathbb{R}^N$ with $|(y_1,\cdots,y_{n-1}) - (y_1',\cdots,y_{n-1}')| \leqslant \tau_\delta$, and for all y_n, $\xi \in \mathbb{R}^{d\times N}$,

$$f(y_1,\cdots,y_{n-1},y_n,\xi) \geqslant \delta\, \tilde{a}_\delta(y_1',\cdots,y_{n-1}',y_n) + (1+o(1)) f(y_1',\cdots,y_{n-1}',y_n,\xi)$$

(as $\delta \to 0^+$). If $n \geqslant 3$, then we assume in addition that for a.e. $y_{n-1}, y_n \in \mathbb{R}^N$ we have $\tilde{a}_\delta(\cdot,y_{n-1},y_n) \in C_\#(Y_1\times\cdots\times Y_{n-2})$ with $\|\tilde{a}_\delta(\cdot,y_{n-1},y_n)\|_{C_\#(Y_1\times\cdots\times Y_{n-2})} \in L^1(Y_{n-1}\times Y_n)$;

($\mathcal{F}7$) there exist $\alpha \in (0,1)$ and $L, C > 0$, such that for all $y_1,...,y_n \in \mathbb{R}^N$, for all $\xi \in \mathbb{R}^{d\times N}$ with $|\xi| = 1$, and for all $t \geqslant L$,

$$\left| f^\infty(y_1,\cdots,y_n,\xi) - \frac{f(y_1,\cdots,y_n,t\xi)}{t} \right| \leqslant \frac{C}{t^\alpha};$$

($\mathcal{F}8$) the conjugate function f^* of f is a bounded function on its effective domain, $\text{dom}_e f^*$.

The next proposition will be used to establish integral representations for the multiple-scale functional F^{sc} in (5.0.1) and for the homogenized functional F^{hom} in (5.0.2).

Proposition 5.1.1. *Let $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying hypotheses ($\mathcal{F}1$), ($\mathcal{F}3$) and ($\mathcal{F}4$). For $\eta > 0$, let f_η be the function defined by $f_\eta(y_1,\cdots,y_n,\xi) := f(y_1,\cdots,y_n,\xi) + \eta|\xi|$. Then,* *(i) For all $y_1, ..., y_n \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$, the limit*

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})^\infty(y_1, ..., y_n, \xi) =: ((f_{0^+})^{**})^\infty(y_1, ..., y_n, \xi) \tag{5.1.5}$$

exists, $((f_{0^+})^{**})^\infty : \mathbb{R}^{nN} \times \mathbb{R}^N \to \mathbb{R}$ is positively 1-homogeneous and convex in the last variable, and $(f^{**})^\infty \leqslant ((f_{0^+})^{**})^\infty \leqslant (f^\infty)^{**}$.

Furthermore, if in addition

a) f also satisfies ($\mathcal{F}$2), then $((f_{0^+})^{**})^\infty \equiv f^\infty$;

b) $d = 1$ and f also satisfies ($\mathcal{F}$7), then $((f_{0^+})^{**})^\infty \equiv (f^\infty)^{**}$.

(ii) For all $\xi \in \mathbb{R}^N$, the limit

$$\lim_{\eta\to 0^+} \big(((f_\eta)^{**})_{\mathrm{hom}}\big)^\infty(\xi) =: \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty(\xi) \tag{5.1.6}$$

exists, with $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty : \mathbb{R}^N \to \mathbb{R}$ positively 1-homogeneous, convex, and such that

$$\big((f^{**})_{\mathrm{hom}}\big)^\infty \leqslant \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \leqslant \big(((f_{0^+})^{**})^\infty\big)_{\mathrm{hom}} \leqslant \big((f^\infty)^{**}\big)_{\mathrm{hom}}.$$

Furthermore, if in addition

a) f also satisfies ($\mathcal{F}$2) and ($\mathcal{F}$8), then $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \equiv (f_{\mathrm{hom}})^\infty = (f^\infty)_{\mathrm{hom}}$;

b) f also satisfies ($\mathcal{F}$2) and ($\mathcal{F}$7), then $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \equiv (f^\infty)_{\mathrm{hom}}$;

c) $d = 1$ and f also satisfies ($\mathcal{F}$7), then $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty \equiv \big((f^\infty)^{**}\big)_{\mathrm{hom}}$.

Remark 5.1.2. *Hypothesis ($\mathcal{F}$7) is common within variational problems with linear growth conditions (see, for example, Bouchitté, Fonseca and Mascarenhas [18, Sect. 4], Babadjian, Zappale and Zorgati [11]). We will prove (see Lemma 5.2.11 below) that under hypotheses ($\mathcal{F}$1), ($\mathcal{F}$3), ($\mathcal{F}$4)' and ($\mathcal{F}$7), we have $(f_{\mathrm{hom}})^\infty = (f^\infty)_{\mathrm{hom}}$; in the scalar case, these conditions also ensure the equality $(f^{**})^\infty = (f^\infty)^{**}$. Other sufficient conditions to guarantee that $(f_{\mathrm{hom}})^\infty = (f^\infty)_{\mathrm{hom}}$ are ($\mathcal{F}$1)–($\mathcal{F}$4) and ($\mathcal{F}$8) (see Lemma 5.2.10 below), which is an hypothesis on f^* that is often considered when dealing with duality problems (see, for example, Témam [74, Ch. II.4]).*

Unless stated otherwise, we will always assume that the length scales $\varrho_1, ..., \varrho_n$ satisfy (4.0.1) and (4.1.2). Our main result is the following.

Theorem 5.1.3. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with $\partial\Omega$ Lipschitz, let $Y_i := (0,1)^N$, $i \in \{1, \cdots, n\}$, and let $f : \mathbb{R}^{nN} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying ($\mathcal{F}$1)–($\mathcal{F}$4), ($\mathcal{F}$5) and ($\mathcal{F}$6). Then, for all $(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \in BV(\Omega;\mathbb{R}^d) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1;\mathbb{R}^d)) \times \cdots \times \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{n-1}; BV_\#(Y_n;\mathbb{R}^d))$,*

$$\begin{aligned} F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) = &\int_{\Omega\times Y_1\times\cdots\times Y_n} f\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x, y_1, \cdots, y_n)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &+ \int_{\Omega\times Y_1\times\cdots\times Y_n} f^\infty\Big(y_1, \cdots, y_n, \frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|}(x, y_1, \cdots, y_n)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(x, y_1, \cdots, y_n). \end{aligned} \tag{5.1.7}$$

Moreover, for all $u \in BV(\Omega;\mathbb{R}^d)$,

$$\begin{aligned} F^{\mathrm{hom}}(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d)),\ldots,\\ \boldsymbol{\mu}_n\in\mathcal{M}_\star(\Omega\times Y_1\times\cdots\times Y_{n-1};BV_\#(Y_n;\mathbb{R}^d))}} F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \\ &= \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{0^+,\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x), \end{aligned} \tag{5.1.8}$$

where $(f_{0^+,\mathrm{hom}})^\infty := \big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty$ *is the function defined by (5.1.6) (note that in view of (*$\mathcal{F}2$*),* $(f_\eta)^{**} \equiv f_\eta$*).*

Furthermore, if in addition

(i) f *satisfies one of the two conditions (*$\mathcal{F}4$*)' or (*$\mathcal{F}8$*), then* $(f_{0^+,\mathrm{hom}})^\infty \equiv (f_{\mathrm{hom}})^\infty$*;*

(ii) f *satisfies (*$\mathcal{F}7$*), then* $(f_{0^+,\mathrm{hom}})^\infty \equiv (f^\infty)_{\mathrm{hom}}$*.*

We remark that in Theorem 5.1.3 we do not assume coercivity nor boundedness from below of f. The main ingredients of the proof are the unfolding operator (see Cioranescu, Damlamian and De Arcangelis [25], Cioranescu, Damlamian and Griso [27]; see also Fonseca and Krömer [47]) and Reshetnyak's continuity- and lower semicontinuity-type results. The approach via the unfolding operator, in connection with the notion of two-scale convergence and in the framework of homogenization problems, sometimes referred as periodic unfolding method, has already been adopted by other authors in the Sobolev setting (see, for example, Cioranescu, Damlamian and De Arcangelis [25], Cioranescu, Damlamian and De Arcangelis [26], Fonseca and Krömer [47]).

We use the convexity hypothesis ($\mathcal{F}2$) when establishing the lower bound for the infimum defining F^{sc}, which is based on a sequential lower semicontinuity argument. We start by proving that the $(n+1)$-scale convergence of a sequence of measures absolutely continuous with respect to the Lebesgue measure is equivalent to the weak-$\star$ convergence in the product space $\Omega \times Y_1 \times \cdots \times Y_n$ in the sense of measures of the unfolded sequence, i.e., the image through the unfolding operator of the original sequence (see Lemma 5.2.4). Then we prove that the energy F_ε does not increase by means of the unfolding operator (see Lemma 5.2.2). In order to conclude we need sequential lower semicontinuity of the functional

$$\begin{aligned} F(\lambda) := &\int_{\Omega\times Y_1\times\cdots\times Y_n} f\Big(y_1,\cdots,y_n,\frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x,y_1,...,y_n)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &+\int_{\Omega\times Y_1\times\cdots\times Y_n} f^\infty\Big(y_1,\cdots,y_n,\frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(x,y_1,...,y_n)\Big)\,\mathrm{d}\|\lambda^s\|(x,y_1,...,y_n)\end{aligned}$$

for $\lambda \in \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N})$, with respect to weak-$\star$ convergence in the sense of measures, which requires convexity of f in the last variable (see, for example, Ambrosio and Buttazzo [6]). In the scalar case $d=1$ we can overcome this difficulty by a relaxation argument with respect to the weak topology of $W^{1,1}(\Omega)$, which cannot be applied in the vectorial case since quasiconvexity is a weaker condition than convexity (see, for example, Dacorogna [29]). As a corollary of Theorem 5.1.3, we obtain the following result concerning the scalar case $d=1$.

Corollary 5.1.4. *Let* $\Omega \subset \mathbb{R}^N$ *be an open and bounded set with* $\partial\Omega$ *Lipschitz, let* $Y_i := (0,1)^N$*,* $i \in \{1,\cdots,n\}$*, and let* $f : \mathbb{R}^{nN}\times\mathbb{R}^N \to \mathbb{R}$ *be a Borel function satisfying conditions (*$\mathcal{F}1$*), (*$\mathcal{F}3$*), (*$\mathcal{F}4$*), (*$\mathcal{F}5$*) and (*$\mathcal{F}6$*) with* $d=1$ *and with* $o(1)$ *replaced by* $-|o(1)|$ *in (*$\mathcal{F}6$*). Then, for all* $(u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n) \in BV(\Omega)\times\mathcal{M}_\star(\Omega;BV_\#(Y_1))\times\cdots\times\mathcal{M}_\star(\Omega\times Y_1\times\cdots Y_{n-1};BV_\#(Y_n))$*,*

$$\begin{aligned} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n) = &\int_{\Omega\times Y_1\times\cdots\times Y_n} f^{**}\Big(y_1,\cdots,y_n,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}}{\mathrm{d}\mathcal{L}^{(n+1)N}}(x,y_1,\cdots,y_n)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &+\int_{\Omega\times Y_1\times\cdots\times Y_2}((f_{0^+})^{**})^\infty\Big(y_1,\cdots,y_n,\frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}\|}(x,y_1,\cdots,y_n)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}\|(x,y_1,\cdots,y_n), \end{aligned} \tag{5.1.9}$$

where $((f_{0^+})^{**})^\infty$ *is the function defined by (5.1.5). Moreover, for all* $u \in BV(\Omega)$,

$$
\begin{aligned}
F^{\text{hom}}(u) &= \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1)), \dots, \\ \boldsymbol{\mu}_n \in \mathcal{M}_\star(\Omega \times Y_1 \times \dots \times Y_{n-1}; BV_\#(Y_n))}} F^{\text{sc}}(u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n) \\
&= \int_\Omega (f^{**})_{\text{hom}}(\nabla u(x))\, \mathrm{d}x + \int_\Omega \big(((f_{0^+})^{**})_{\text{hom}}\big)^\infty \Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x) \Big)\, \mathrm{d}\|D^s u\|(x),
\end{aligned} \tag{5.1.10}
$$

where $\big(((f_{0^+})^{**})_{\text{hom}}\big)^\infty$ *is the function defined by (5.1.6).*

Furthermore, if in addition

(i) f *satisfies the coercivity condition* ($\mathcal{F}4$)', *then* $((f_{0^+})^{**})^\infty \equiv (f^{**})^\infty$ *and* $\big(((f_{0^+})^{**})_{\text{hom}}\big)^\infty \equiv \big((f^{**})_{\text{hom}}\big)^\infty$;

(ii) f *satisfies* ($\mathcal{F}7$), *then* $((f_{0^+})^{**})^\infty \equiv (f^\infty)^{**}$ *and* $\big(((f_{0^+})^{**})_{\text{hom}}\big)^\infty \equiv \big((f^\infty)^{**}\big)_{\text{hom}}$.

Remark 5.1.5. (Comments on the hypotheses) *(i) If* f *is bounded from below, then* ($\mathcal{F}4$) *is satisfied: it suffices to take* $c_\delta \equiv 0$ *and* $b_\delta \equiv -b$, *where* $b := \inf f \in \mathbb{R}$. *Hypothesis* ($\mathcal{F}4$) *may be regarded as a stronger version of the condition*

($\mathcal{F}4)^\star$ *for all* $\delta > 0$ *there exists* $b_\delta \in \mathbb{R}$ *such that for all* $y_1, ..., y_n$, $\xi \in \mathbb{R}^N$,

$$
f(y_1, \cdots, y_n, \xi) + \delta|\xi| + b_\delta \geqslant 0,
$$

so f *cannot decrease as* $-|\xi|$ *but it can decrease as* $-|\xi|^\alpha$ *with* $\alpha \in (0,1)$: *If* $\tilde{f} : \mathbb{R}^{nN} \times \mathbb{R}^{d \times N} \to [0, \infty)$ *is a nonnegative function, and* $b \in \mathbb{R}$, $c > 0$, *then for all* $\alpha \in (0,1)$,

$$
f(y_1, \cdots, y_n, \xi) := \tilde{f}(y_1, \cdots, y_n, \xi) - c|\xi|^\alpha + b
$$

is a function satisfying ($\mathcal{F}4)^\star$. *We do not assume* ($\mathcal{F}4)^\star$ *in place of* ($\mathcal{F}4$) *in Theorem 5.1.3 and Corollary 5.1.4 because in general the former is not inherited neither by* f_{hom} *nor by* f^{**} *from* f, *whereas the latter is.*

We observe that if f *is lower semicontinuous and independent of* $(y_1, \cdots, y_n)$, *then* f *satisfies* ($\mathcal{F}4)^\star$ *if, and only if, it satisfies*

$$
\liminf_{|\xi| \to +\infty} \frac{f(\xi)}{|\xi|} \geqslant 0. \tag{5.1.11}
$$

Moreover, if f *is in addition convex, then (5.1.11) is a necessary and sufficient condition for the sequentially lower semicontinuity with respect to weak-$\star$ convergence in the sense of measures of the functional*

$$
u \in L^1(\Omega; \mathbb{R}^{d \times N}) \mapsto \int_\Omega f(u(x))\, \mathrm{d}x.
$$

Furthermore, (5.1.11) yields

$$
\liminf_{\varepsilon \to 0^+} \int_\Omega f(u_\varepsilon(x))\, \mathrm{d}x \geqslant \int_\Omega f\Big(\frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^N}(x)\Big)\, \mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(x)\Big)\, \mathrm{d}\|\lambda^s\|(x)
$$

whenever $u_\varepsilon \mathcal{L}^N{\lfloor_\Omega} \overset{\star}{\rightharpoonup} \lambda$ *weakly-$\star$ in* $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ *(see Fonseca and Leoni [48, Thm. 5.21]). This fact will be used when establishing (5.1.8) and (5.1.10).*

(ii) If f *satisfies a growth condition of the form* $|f(y_1, \cdots, y_n, \xi)| \leqslant C(1 + |\xi|)$ *and is convex in the last variable, then (see Boni [15])* ($\mathcal{F}5$) *holds if, and only if, the function* $\bar{f} : \mathbb{R}^{nN} \times \mathbb{R}^{d \times N} \times [0, \infty) \to \mathbb{R}$ *defined by*

$$
\bar{f}(y_1, \cdots, y_n, \xi, t) := \begin{cases} t f(y_1, \cdots, y_n, \frac{\xi}{t}) & \text{if } t > 0, \\ f^\infty(y_1, \cdots, y_n, \xi) & \text{if } t = 0, \end{cases}
$$

is continuous. In particular, if f is continuous, positively 1-homogeneous in the last variable, and satisfies ($\mathcal{F}2$), ($\mathcal{F}3$), and ($\mathcal{F}4$)*, then it also satisfies ($\mathcal{F}5$) since in this setting $\bar{f}$ is continuous.

The continuity of $\bar{f}$ will be crucial in our analysis in order to apply Reshetnyak's continuity- and lower semicontinuity-type results (see Lemmas 5.2.5 and 5.2.6 below).

(iii) Hypothesis ($\mathcal{F}6$) is a weaker version of the hypothesis

($\mathcal{F}6$)' *there exist a continuous, positive function ω satisfying $\omega(0) = 0$, and a function $a \in L^1_{\#}(Y_n)$ such that for all $y_1, ..., y_{n-1}$, $y'_1, ..., y'_{n-1}$, y_n, $\xi \in \mathbb{R}^{d\times N}$, we have*

$$\begin{aligned}&|f(y_1, \cdots, y_{n-1}, y_n, \xi) - f(y'_1, \cdots, y'_{n-1}, y_n, \xi)|\\ &\qquad \leqslant \omega(|(y_1, \cdots, y_{n-1}) - (y'_1 \cdots, y'_{n-1})|)\big(a(y_n) + f(y_1, \cdots, y_n, \xi)\big),\end{aligned}$$

which often appears in the literature (see, for example, Braides and Defranceschi [21], Serrin [70]).

If f is of the form $f(y_1, \cdots, y_n, \xi) := g(y_1, \cdots, y_{n-1})h(y_n, \xi)$, where g is a continuous and $Y_1\times\cdots\times Y_{n-1}$-periodic function, and h is a function satisfying ($\mathcal{F}1$)–($\mathcal{F}5$), then f satisfies ($\mathcal{F}1$)–($\mathcal{F}6$); in particular, we may consider $g \equiv 1$, which corresponds to the case of one microscale (i.e., $n = 1$) and so, in this situation, ($\mathcal{F}6$) is trivially satisfied. Other simple examples of functions satisfying ($\mathcal{F}1$)–($\mathcal{F}6$) are functions of the form $f(y_1, \cdots, y_n, \xi) := g(y_1, \cdots, y_n)h(\xi)$, where g is continuous and $Y_1 \times \cdots \times Y_n$-periodic, and h satisfies ($\mathcal{F}2$)–($\mathcal{F}4$).

Remark 5.1.6. *(i) Equalities (5.1.7) and the first one in (5.1.8) are valid under the more general growth condition from below ($\mathcal{F}4$)* (introduced in Remark 5.1.5 (i)). The reason why this condition is not enough in order to conclude the second equality in (5.1.8) is that in general it is not inherited by f_{hom}, while ($\mathcal{F}4$) is and this ensures that f_{hom} satisfies (5.1.11), which, as we will see, will play a crucial role in the proof.*

(ii) In Theorem 5.1.3 and Corollary 5.1.4, we need the length scales to satisfy condition (4.1.2) only to establish the equalities (5.1.8) and (5.1.10) involving F^{hom}.

In the case in which $n = 1$ and $d = 1$, we recover Amar's integral representation [5] of the two-scale homogenized functional F^{sc} under more general conditions (see Remark 5.1.5 (ii) and (iii)). Furthermore, if we assume *a priori* compactness of a diagonal infimizing sequence for the sequence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$, we recover Amar's result [5] under more general conditions. We observe that even if *a priori* compactness of a diagonal infimizing sequence is assumed in Theorem A, the coercivity condition is still needed to validate the arguments in Amar [5]. We also recover Bouchitté's integral representation [16] of the effective energy F^{hom} without assuming coercivity of f and without assuming convexity of f in the second variable, but assuming continuity in the first one in order to apply Reshetnyak Continuity Theorem, while in Bouchitté [16] f is assumed to be convex in the second variable and coercive, but only measurable and Y-periodic in the first variable.

If $n = 1$ and $d > 1$ in Theorem 5.1.3, then we recover De Arcangelis and Gargiulo's integral representation [34] of the effective energy F^{hom} without assuming f to be bounded from below, but assuming f to be continuous in the first variable and convex in the second one, while in De Arcangelis and Gargiulo [34] f is only required to be nonnegative, measurable and Y-periodic in the first variable and continuous in the second one. As we mentioned before, our hypotheses are related to the periodic unfolding method and Reshetnyak Continuity Theorem's hypotheses.

In the case in which $n \geqslant 2$, Theorem 5.1.3 and Corollary 5.1.4 provide new results in the literature in that, to the best of our knowledge, the homogenization of nonlinear periodically oscillating functionals with linear growth and characterized by $n \geqslant 2$ microscales has not yet been carried out.

Finally, in the framework of homogenization by Γ-convergence in the BV setting and for $n = 1$ we also mention the works by Braides and Chiatò Piat [20] and Carbone, Cioranescu, De Arcangelis and Gaudiello [23] concerning the convex case; and Bouchitté, Fonseca and Mascarenhas [18, Sect. 4.3], Attouch, Buttazzo and Michaille [9, Sect. 12.3] and Babadjian and Millot [10] regarding the nonconvex case.

The remaining part of the present chapter is organized as follows. In Section 5.2 we prove Proposition 5.1.1 and Theorem 5.1.3, and in Section 5.3 we prove Corollary 5.1.4.

5.2. Proof of Theorem 5.1.3.

Throughout this section we will assume that $n = 2$. The cases in which $n = 1$ or $n \geqslant 3$ do not bring any additional technical difficulties. Given $x \in \mathbb{R}^N$, we write $[x]$ and $\langle x \rangle$ to denote the integer and the fractional part of x componentwise, respectively, so that $x = [x] + \langle x \rangle$ and $[x] \in \mathbb{Z}^N$, $\langle x \rangle \in Y$. We denote the Lipschitz constant of a function g on a set D by $\mathrm{Lip}(g; D)$; if D coincides with the domain of g we omit its dependence. The letter $\mathcal{C}$ represents a generic positive constant, whose value may change from expression to expression.

For $n = 2$ the energies F_ε in (5.0.3) take the form

$$F_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, \nabla u(x)\Big)\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{x}{\varrho_1(\varepsilon)}, \frac{x}{\varrho_2(\varepsilon)}, \frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\mathrm{d}\|D^s u\|(x) \qquad (5.2.1)$$

for $u \in BV(\Omega; \mathbb{R}^d)$, where, we recall, $\varrho_1, \varrho_2 : (0, \infty) \to (0, \infty)$ are functions satisfying (4.0.1) (with $n = 2$) and f^∞ is the recession function associated with f. Due to the convexity hypothesis $(\mathcal{F}2)$, the limit superior defining f^∞ is actually a limit (see, for example, Fonseca and Leoni [48]), so that $f^\infty : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is given by

$$f^\infty(y_1, y_2, \xi) := \lim_{t\to+\infty} \frac{f(y_1, y_2, t\xi)}{t}.$$

Moreover, under hypotheses $(\mathcal{F}1)$–$(\mathcal{F}3)$ and $(\mathcal{F}4)^\star$ on f, we have that f^∞ is a Borel function satisfying $(\mathcal{F}1)$, $(\mathcal{F}2)$, and the growth condition

$$0 \leqslant f^\infty(y_1, y_2, \xi) \leqslant C|\xi|. \qquad (5.2.2)$$

Notice that in view of $(\mathcal{F}3)$, $(\mathcal{F}4)^\star$ and (5.2.2), the functional F_ε is well defined (in $\mathbb{R}$) for every $u \in BV(\Omega; \mathbb{R}^d)$.

In Theorem 5.2.1 below we will establish (5.1.7). We will use the unfolding operator (see Cioranescu, Damlamian and De Arcangelis [25], Cioranescu, Damlamian and Griso [27]; see also Fonseca and Krömer [47]): For $\varrho > 0$, $\mathcal{T}_\varrho : L^1(\Omega; \mathbb{R}^m) \to L^1(\mathbb{R}^N; L^1_\#(Y_2; \mathbb{R}^m))$ is defined by

$$\mathcal{T}_\varrho(g)(x, y_2) := \tilde{g}\Big(\varrho\Big[\frac{x}{\varrho}\Big] + \varrho(y_2 - [y_2])\Big) \quad \text{for } x, y_2 \in \mathbb{R}^N,\ g \in L^1(\Omega; \mathbb{R}^m),$$

where $\tilde{g}$ is the extension by zero of g to $\mathbb{R}^N$. Clearly $\mathcal{T}_\varrho$ is linear, and for every $g \in L^1(\Omega; \mathbb{R}^m)$

$$\|\mathcal{T}_\varrho(g)\|_{L^1(\Omega\times Y_2;\mathbb{R}^m)} \leqslant \|\mathcal{T}_\varrho(g)\|_{L^1(\mathbb{R}^N\times Y_2;\mathbb{R}^m)} = \|\tilde{g}\|_{L^1(\mathbb{R}^N;\mathbb{R}^m)} = \|g\|_{L^1(\Omega;\mathbb{R}^m)}, \qquad (5.2.3)$$

and

$$\lim_{\varrho\to 0^+}\int_{\mathbb{R}^N\times Y_2}|\tilde{g}(x)-\mathcal{T}_\varrho(g)(x,y_2)|\,\mathrm{d}x\mathrm{d}y_2=0 \tag{5.2.4}$$

(see Fonseca and Krömer [47, Prop. A.1]).

Similarly, we define the operator $\mathcal{A}_\varrho : L^1(\Omega\times Y_2;\mathbb{R}^m)\to L^1(\mathbb{R}^N;L^1_{\#}(Y_1;L^1(Y_2;\mathbb{R}^m)))$ by

$$\mathcal{A}_\varrho(h)(x,y_1,y_2):=\tilde{h}\Big(\varrho\Big[\frac{x}{\varrho}\Big]+\varrho(y_1-[y_1]),y_2\Big)=\mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)$$

for $x,y_1\in\mathbb{R}^N$, $y_2\in Y_2$, $h\in L^1(\Omega\times Y_2;\mathbb{R}^m)$, where $\tilde{h}$ is the extension by zero of h to $\mathbb{R}^N\times Y_2$. $\mathcal{A}_\varrho$ is linear, and for all $h\in L^1(\Omega\times Y_2;\mathbb{R}^m)$,

$$\|\mathcal{A}_\varrho(h)\|_{L^1(\Omega\times Y_1\times Y_2;\mathbb{R}^m)}\leqslant\|\mathcal{A}_\varrho(h)\|_{L^1(\mathbb{R}^N\times Y_1\times Y_2;\mathbb{R}^m)}=\|\tilde{h}\|_{L^1(\mathbb{R}^N\times Y_2;\mathbb{R}^m)}=\|h\|_{L^1(\Omega\times Y_2;\mathbb{R}^m)} \tag{5.2.5}$$

by (5.2.3) and Fubini's Theorem. Moreover, we notice that for a.e. $y_2\in Y_2$, we have

$$\lim_{\varrho\to 0^+}\int_{\mathbb{R}^N\times Y_1}\big|\tilde{h}(x,y_2)-\mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)\big|\,\mathrm{d}x\mathrm{d}y_1=0$$

by (5.2.4), and

$$\int_{\mathbb{R}^N\times Y_1}\big|\tilde{h}(x,y_2)-\mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)\big|\,\mathrm{d}x\mathrm{d}y_1\leqslant 2\int_{\mathbb{R}^N}\big|\tilde{h}(x,y_2)\big|\,\mathrm{d}x\in L^1(Y_2),$$

where we used (5.2.3) to obtain

$$\int_{\mathbb{R}^N\times Y_1}\big|\mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)\big|\,\mathrm{d}x\mathrm{d}y_1=\int_{\mathbb{R}^N}\big|\tilde{h}(x,y_2)\big|\,\mathrm{d}x.$$

Thus, Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned}&\lim_{\varrho\to 0^+}\int_{\mathbb{R}^N\times Y_1\times Y_2}\big|\tilde{h}(x,y_2)-\mathcal{A}_\varrho(h)(x,y_1,y_2)\big|\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\&=\lim_{\varrho\to 0^+}\int_{\mathbb{R}^N\times Y_1\times Y_2}\big|\tilde{h}(x,y_2)-\mathcal{T}_\varrho(h(\cdot,y_2))(x,y_1)\big|\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2=0.\end{aligned}$$

Theorem 5.2.1. *Let $\Omega\subset\mathbb{R}^N$ be an open, bounded set with $\partial\Omega$ Lipschitz, let $Y_1=Y_2:=(0,1)^N$, and let $f:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ be a Borel function satisfying conditions ($\mathcal{F}$1)–($\mathcal{F}$3), ($\mathcal{F}$4)*, ($\mathcal{F}$5), ($\mathcal{F}$6) for $n=2$. Then (5.1.7) holds (with $n=2$).*

The proof of Theorem 5.2.1 is hinged on some lemmas. The first lemma "unfolds" the rapidly oscillating sequence.

Lemma 5.2.2. *Under the same hypotheses of Theorem 5.2.1, if $\{v_\varepsilon\}_{\varepsilon>0}\subset L^1(\Omega;\mathbb{R}^{d\times N})$ is a bounded sequence then, for all $\eta>0$,*

$$\begin{aligned}&\liminf_{\varepsilon\to 0^+}\int_\Omega f_\eta\Big(\frac{x}{\varrho_1(\varepsilon)},\frac{x}{\varrho_2(\varepsilon)},v_\varepsilon(x)\Big)\,\mathrm{d}x\\&\quad\geqslant\liminf_{\varepsilon\to 0^+}\int_{\Omega\times Y_1\times Y_2}f_\eta\big(y_1,y_2,\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,\end{aligned} \tag{5.2.6}$$

where $f_\eta(y_1,y_2,\xi):=f(y_1,y_2,\xi)+\eta|\xi|$.

Proof. Fix $\eta > 0$ and $\delta > 0$. Let $b_\eta \in \mathbb{R}$ be given by $(\mathcal{F}4)^\star$ (see Remark 5.1.5), and let $\tilde{a}_\delta \in L^1_\#(Y_1 \times Y_2)$ and $\tau_\delta > 0$ be given by $(\mathcal{F}6)$. Then

$$f_\eta(\cdot,\cdot,\cdot) \geqslant -b_\eta, \tag{5.2.7}$$

and, for all $y_1, y_1', y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$ such that $|y_1 - y_1'| \leqslant \tau_\delta$,

$$f_\eta(y_1, y_2, \xi) \geqslant \delta\tilde{a}_\delta(y_1', y_2) + (1+o(1))f_\eta(y_1', y_2, \xi) - o(1)\eta|\xi| \ \text{ (as } \delta \to 0^+). \tag{5.2.8}$$

Set $c := \sup_\varepsilon \|v_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{d\times N})}$, $\varepsilon_1 := \varrho_1(\varepsilon)$ and $\varepsilon_2 := \varrho_2(\varepsilon)$. Define

$$Z_{\varepsilon_2} := \{\kappa \in \mathbb{Z}^N :\ \varepsilon_2(\kappa + Y_2) \cap \overline{\Omega} \neq \emptyset\}, \quad \Omega_{\varepsilon_2} := \mathrm{int}\Big(\bigcup_{\kappa \in Z_{\varepsilon_2}} \varepsilon_2(\kappa + \overline{Y}_2)\Big). \tag{5.2.9}$$

Notice that $\Omega \subset \Omega_{\varepsilon_2}$ and, by (5.2.3),

$$\sup_{\varepsilon>0} \|\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\|_{L^1(\mathbb{R}^N\times Y_2;\mathbb{R}^{d\times N})} \leqslant c. \tag{5.2.10}$$

Recalling that $\tilde{v}_\varepsilon$ stands for the extension by zero to the whole $\mathbb{R}^N$ of v_ε, using $(\mathcal{F}3)$, a change of variables and $(\mathcal{F}1)$, in this order, we obtain

$$\begin{aligned}
\int_\Omega f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, v_\varepsilon(x)\Big)\,\mathrm{d}x &= \int_{\Omega_{\varepsilon_2}} f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \tilde{v}_\varepsilon(x)\Big)\,\mathrm{d}x - \int_{\Omega_{\varepsilon_2}\setminus\Omega} f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, 0\Big)\,\mathrm{d}x \\
&\geqslant \sum_{\kappa\in Z_{\varepsilon_2}} \int_{\varepsilon_2(\kappa+Y_2)} f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \tilde{v}_\varepsilon(x)\Big)\,\mathrm{d}x - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\setminus\Omega\big) \\
&= \sum_{\kappa\in Z_{\varepsilon_2}} \int_{Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\kappa + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \tilde{v}_\varepsilon(\varepsilon_2\kappa + \varepsilon_2 y_2)\Big)\,\varepsilon_2^N\mathrm{d}y_2 - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\setminus\Omega\big).
\end{aligned} \tag{5.2.11}$$

Since $\big[\frac{x}{\varepsilon_2}\big] = \kappa$ whenever $x \in \varepsilon_2(\kappa + Y_2)$, $\mathcal{L}^N(\varepsilon_2(\kappa+Y_2)) = \varepsilon_2^N$ and $[y_2] = 0$ for all $y_2 \in Y_2$, in view of the definition of $\mathcal{T}_{\varepsilon_2}(v_\varepsilon)$, by Fubini's Theorem, and from (5.2.11) we get

$$\begin{aligned}
&\int_\Omega f_\eta\Big(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, v_\varepsilon(x)\Big)\,\mathrm{d}x \\
&\quad\geqslant \sum_{\kappa\in Z_{\varepsilon_2}} \int_{\varepsilon_2(\kappa+Y_2)} \bigg(\int_{Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \tilde{v}_\varepsilon\Big(\varepsilon_2\Big[\frac{x}{\varepsilon_2}\Big] + \varepsilon_2 y_2\Big)\Big)\,\mathrm{d}y_2\bigg)\,\mathrm{d}x - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\setminus\Omega\big) \\
&\quad= \int_{\Omega_{\varepsilon_2}\times Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 - C\mathcal{L}^N\big(\Omega_{\varepsilon_2}\setminus\Omega\big) \\
&\quad\geqslant \int_{\Omega\times Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2 - (b_\eta + C)\mathcal{L}^N\big(\Omega_{\varepsilon_2}\setminus\Omega\big),
\end{aligned} \tag{5.2.12}$$

where in the last inequality we used (5.2.7).

By (4.0.1) there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_\delta$ one has $0 < \varepsilon_2/\varepsilon_1 < \tau_\delta/2\sqrt{N}$. For any such ε,

$$\sup_{x\in\Omega, y_2\in Y_2} \Big|\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big] + \frac{\varepsilon_2}{\varepsilon_1}y_2 - \frac{x}{\varepsilon_1}\Big| = \sup_{x\in\Omega, y_2\in Y_2} \Big| -\frac{\varepsilon_2}{\varepsilon_1}\Big\langle\frac{x}{\varepsilon_2}\Big\rangle + \frac{\varepsilon_2}{\varepsilon_1}y_2\Big| < \tau_\delta,$$

thus (5.2.8) and (5.2.10) yield

$$\begin{aligned}
&\int_{\Omega\times Y_2} f_\eta\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}y_2, y_2, \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2\\
&\quad\geqslant \delta\int_{\Omega\times Y_2}\tilde{a}_\delta\Big(\frac{x}{\varepsilon_1},y_2\Big)\,\mathrm{d}x\mathrm{d}y_2+\big(1+o(1)\big)\int_{\Omega\times Y_2} f_\eta\Big(\frac{x}{\varepsilon_1},y_2,\mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2-|o(1)|\eta c.
\end{aligned}\tag{5.2.13}$$

Defining Z_{ε_1} and Ω_{ε_1} as in (5.2.9) (with ε_2 and Y_2 replaced by ε_1 and Y_1, respectively), and reasoning as in (5.2.11)–(5.2.12), we conclude that

$$\begin{aligned}
&\int_{\Omega\times Y_2} f_\eta\Big(\frac{x}{\varepsilon_1},y_2,\mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_2\\
&\quad\geqslant\int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1,y_2,\mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2-(b_\eta+C)\mathcal{L}^N\big(\Omega_{\varepsilon_1}\backslash\Omega\big).
\end{aligned}\tag{5.2.14}$$

By the Riemann–Lebesgue Lemma we have that for a.e. $y_2\in Y_2$, $\tilde{a}_\delta(\cdot/\varepsilon_1,y_2)\rightharpoonup\int_{Y_1}\tilde{a}_\delta(y_1,y_2)\,\mathrm{d}y_1$ weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Hence,

$$\liminf_{\varepsilon\to0^+}\int_{\Omega\times Y_2}\tilde{a}_\delta\Big(\frac{x}{\varepsilon_1},y_2\Big)\,\mathrm{d}x\mathrm{d}y_2\geqslant\mathcal{L}^N(\Omega)\int_{Y_1\times Y_2}\tilde{a}_\delta(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2,\tag{5.2.15}$$

where we have also used Fatou's Lemma and Fubini's Theorem.

In view of (5.2.12)–(5.2.15), we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to0^+}\int_\Omega f_\eta\Big(\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2},v_\varepsilon(x)\Big)\,\mathrm{d}x\\
&\quad\geqslant\big(1+o(1)\big)\liminf_{\varepsilon\to0^+}\int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1,y_2,\mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\qquad+\delta\mathcal{L}^N(\Omega)\int_{Y_1\times Y_2}\tilde{a}_\delta(y_1,y_2)\,\mathrm{d}y_1\mathrm{d}y_2-|o(1)|\eta c,
\end{aligned}\tag{5.2.16}$$

where we also used the convergences $\mathcal{L}^N\big(\Omega_{\varepsilon_1}\backslash\Omega\big),\mathcal{L}^N\big(\Omega_{\varepsilon_2}\backslash\Omega\big)\to0$ as $\varepsilon\to0^+$, since $\partial\Omega$ is Lipschitz and so $\mathcal{L}^N(\partial\Omega)=0$. Finally, recalling that $\delta\|\tilde{a}_\delta\|_{L^1_\#(Y_1\times Y_2)}\to0$ as $\delta\to0^+$, passing (5.2.16) to the limit as $\delta\to0^+$ we get (5.2.6). □

Remark 5.2.3. *The previous proof can be easily generalized to the case in which $n\geqslant3$ by using (2.2.2) in place of Riemann–Lebesgue Lemma (see (5.2.15)).*

We now show that, similarly to what happens in the L^p-case with $p\in(1,\infty)$ (see Cioranescu, Damlamian and Griso [27, Prop. 2.14]), 3-scale convergence of a sequence of measures absolutely continuous with respect to the Lebesgue measure is equivalent to a weak-$\star$ convergence in the sense of measures in a product space of the unfolded sequence.

Lemma 5.2.4. *Let $\Omega\subset\mathbb{R}^N$ be open and bounded, let $\{v_\varepsilon\}_{\varepsilon>0}\subset L^1(\Omega;\mathbb{R}^{d\times N})$ be a bounded sequence and let $\lambda\in\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^{d\times N})$. Then $v_\varepsilon\mathcal{L}^N\lfloor_\Omega\xrightharpoonup{3\text{-}sc}_{\varepsilon}\lambda$ if, and only if, $\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)\mathcal{L}^{3N}\lfloor_{\Omega\times Y_1\times Y_2}\overset{\star}{\rightharpoonup}\lambda$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^{d\times N})$ as $\varepsilon\to0^+$.*

Proof. For $\delta>0$, define the sets

$$W_\delta:=\big\{\kappa\in\mathbb{Z}^N:\ \delta(\kappa+Y)\subset\Omega\big\},\qquad\Omega_\delta:=\mathrm{int}\Big(\bigcup_{\kappa\in W_\delta}\delta(\kappa+\overline{Y})\Big).$$

Take $\phi \in C_c^1(\Omega)$, $\psi_1 \in C_\#^1(Y_1)$ and $\psi_2 \in C_\#^1(Y_2;\mathbb{R}^{d\times N})$, and let $\varphi := \phi\psi_1\psi_2$. Set $\varepsilon_1 := \varrho_1(\varepsilon)$ and $\varepsilon_2 := \varrho_2(\varepsilon)$. By (4.0.1) we can find $\bar\varepsilon > 0$ such that for all $0 < \varepsilon \leqslant \bar\varepsilon$ one has

$$\operatorname{dist}(\operatorname{supp}\phi, \Omega\backslash\Omega_{\varepsilon_1}) > 2\varepsilon_1\sqrt{N}, \quad \operatorname{dist}(\operatorname{supp}\phi, \Omega\backslash\Omega_{\varepsilon_2}) > 2\varepsilon_1\sqrt{N}. \tag{5.2.17}$$

Fix any such ε. Using (5.2.17), the definition of $\mathcal{A}_{\varepsilon_1}$, Fubini's Theorem, and the equalities $\left[\frac{x}{\varepsilon_1}\right] = \kappa$ if $x \in \varepsilon_1(\kappa + Y_1)$ and $[y_1] = 0$ if $y_1 \in Y_1$, in this order, we get

$$\begin{aligned}
&\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad= \int_{\Omega_{\varepsilon_1}\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1(y_1 - [y_1]), y_2\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad= \int_{Y_1\times Y_2}\bigg(\sum_{\kappa\in W_{\varepsilon_1}}\int_{\varepsilon_1(\kappa+Y_1)} \varphi(x,y_1,y_2) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(\varepsilon_1\kappa + \varepsilon_1 y_1, y_2)\,\mathrm{d}x\bigg)\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned} \tag{5.2.18}$$

Performing the change of variables $x = \varepsilon_1\kappa + \varepsilon_1\zeta$, by Fubini's Theorem the last integral in (5.2.18) becomes

$$\int_{Y_1\times Y_2}\bigg(\sum_{\kappa\in W_{\varepsilon_1}}\int_{Y_1} \varphi(\varepsilon_1\kappa + \varepsilon_1\zeta, y_1, y_2) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(\varepsilon_1\kappa + \varepsilon_1 y_1, y_2)\,\varepsilon_1^N\mathrm{d}y_1\bigg)\mathrm{d}\zeta\mathrm{d}y_2. \tag{5.2.19}$$

Considering now the change of variables $y_1 = \frac{x}{\varepsilon_1} - \kappa$, and using again Fubini's Theorem, (5.2.19) reduces to

$$\begin{aligned}
&\int_{Y_1\times Y_2}\bigg(\sum_{\kappa\in W_{\varepsilon_1}}\int_{\varepsilon_1(\kappa+Y_1)} \varphi\Big(\varepsilon_1\kappa + \varepsilon_1\zeta, \frac{x}{\varepsilon_1} - \kappa, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\bigg)\mathrm{d}\zeta\mathrm{d}y_2\\
&\quad= \int_{Y_1\times Y_2}\bigg(\int_{\Omega_{\varepsilon_1}} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1\zeta, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\bigg)\mathrm{d}\zeta\mathrm{d}y_2\\
&\quad= \int_{\Omega_{\varepsilon_1}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned} \tag{5.2.20}$$

where in the first equality we used the Y_1-periodicity of ψ_1.

We claim that if $x \in \Omega\backslash\Omega_{\varepsilon_1} \cup \Omega\backslash\Omega_{\varepsilon_2}$ then

$$\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 Y_1\Big) \cap \operatorname{supp}\phi = \emptyset. \tag{5.2.21}$$

In fact, if there was $z \in (\varepsilon_1[\frac{x}{\varepsilon_1}] + \varepsilon_1 Y_1) \cap \operatorname{supp}\phi$, then $z = \varepsilon_1[\frac{x}{\varepsilon_1}] + \varepsilon_1 y_1$ for some $y_1 \in Y_1$ and, by (5.2.17),

$$2\varepsilon_1\sqrt{N} < \operatorname{dist}(\operatorname{supp}\phi, x) \leqslant |z - x| = \Big|\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1 - x\Big| = \Big| - \varepsilon_1\Big\langle\frac{x}{\varepsilon_1}\Big\rangle + \varepsilon_1 y_1\Big| \leqslant 2\varepsilon_1\sqrt{N},$$

which is a contradiction. Hence, (5.2.21) holds. Consequently,

$$\begin{aligned}
&\int_{\Omega_{\varepsilon_1}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad= \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned} \tag{5.2.22}$$

Arguing as in (5.2.18)–(5.2.20), we have

$$\begin{aligned}
&\int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big]+\varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : \mathcal{T}_{\varepsilon_2}(v_\varepsilon)(x,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad = \int_{Y_1\times Y_2}\bigg(\sum_{\kappa\in W_{\varepsilon_2}}\int_{\varepsilon_2(\kappa+Y_2)} \varphi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big]+\varepsilon_1 y_1, \frac{x}{\varepsilon_1}, y_2\Big) : v_\varepsilon(\varepsilon_2\kappa+\varepsilon_2 y_2)\,\mathrm{d}x\bigg)\mathrm{d}y_1\mathrm{d}y_2\\
&\quad = \int_{Y_1\times Y_2}\bigg(\sum_{\kappa\in W_{\varepsilon_2}}\int_{Y_2} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\kappa+\frac{\varepsilon_2}{\varepsilon_1}\zeta\Big]+\varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\kappa+\frac{\varepsilon_2}{\varepsilon_1}\zeta, y_2\Big) : v_\varepsilon(\varepsilon_2\kappa+\varepsilon_2 y_2)\,\varepsilon_2^N\mathrm{d}y_2\bigg)\mathrm{d}y_1\mathrm{d}\zeta\\
&\quad = \int_{Y_1\times Y_2}\bigg(\sum_{\kappa\in W_{\varepsilon_2}}\int_{\varepsilon_2(\kappa+Y_2)} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\kappa+\frac{\varepsilon_2}{\varepsilon_1}\zeta\Big]+\varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\kappa+\frac{\varepsilon_2}{\varepsilon_1}\zeta, \frac{x}{\varepsilon_2}-\kappa\Big) : v_\varepsilon(x)\,\mathrm{d}x\bigg)\mathrm{d}y_1\mathrm{d}\zeta\\
&\quad = \int_{Y_1\times Y_2}\bigg(\int_{\Omega_{\varepsilon_2}} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}\zeta\Big]+\varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}\zeta, \frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\bigg)\mathrm{d}y_1\mathrm{d}\zeta\\
&\quad = \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \varphi\Big(\varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}y_2\Big]+\varepsilon_1 y_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}y_2, \frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned} \tag{5.2.23}$$

where in the fourth equality we used the Y_2-periodicity of ψ_2.

In view of (5.2.18)–(5.2.20) and (5.2.22)–(5.2.23), we conclude that

$$\begin{aligned}
&\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad = \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \phi(a_\varepsilon(x,y_1,y_2))\psi_1(b_\varepsilon(x,y_2))\psi_2\Big(\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned} \tag{5.2.24}$$

where

$$a_\varepsilon(x,y_1,y_2) := \varepsilon_1\Big[\frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}y_2\Big]+\varepsilon_1 y_1,\;\; b_\varepsilon(x,y_2) := \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{x}{\varepsilon_2}\Big]+\frac{\varepsilon_2}{\varepsilon_1}y_2,\;\; x,y_1,y_2\in\mathbb{R}^N.$$

Notice that for all $x\in\Omega$, $y_1\in Y_1$ and $y_2\in Y_2$,

$$|a_\varepsilon(x,y_1,y_2)-x| \leqslant 2\sqrt{N}(\varepsilon_1+\varepsilon_2), \quad \Big|b_\varepsilon(x,y_2)-\frac{x}{\varepsilon_1}\Big| \leqslant 2\sqrt{N}\,\frac{\varepsilon_2}{\varepsilon_1}. \tag{5.2.25}$$

Using (5.2.24) and (5.2.17), we obtain

$$\begin{aligned}
&\bigg|\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2) : \mathcal{A}_{\varepsilon_1}\big(\mathcal{T}_{\varepsilon_2}(v_\varepsilon)\big)(x,y_1,y_2)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 - \int_\Omega \varphi\Big(x,\frac{x}{\varepsilon_1},\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\bigg|\\
&\quad = \bigg|\int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \phi(a_\varepsilon(x,y_1,y_2))\psi_1(b_\varepsilon(x,y_2))\psi_2\Big(\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\qquad\qquad - \int_{\Omega_{\varepsilon_2}\times Y_1\times Y_2} \phi(x)\psi_1\Big(\frac{x}{\varepsilon_1}\Big)\psi_2\Big(\frac{x}{\varepsilon_2}\Big) : v_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\bigg|\\
&\quad \leqslant \|\psi_2\|_{L^\infty_\#(Y_2;\mathbb{R}^{d\times N})}\int_{\Omega\times Y_1\times Y_2}\Big|\phi(a_\varepsilon(x,y_1,y_2))\psi_1(b_\varepsilon(x,y_2)) - \phi(x)\psi_1\Big(\frac{x}{\varepsilon_1}\Big)\Big||v_\varepsilon(x)|\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad \leqslant \|\psi_2\|_{L^\infty_\#(Y_2;\mathbb{R}^{d\times N})}\int_{\Omega\times Y_1\times Y_2}\bigg(\|\phi\|_{L^\infty(\Omega)}\mathrm{Lip}(\psi)\Big|b_\varepsilon(x,y_2)-\frac{x}{\varepsilon_1}\Big|\\
&\qquad\qquad + \|\psi_1\|_{L^\infty_\#(Y_1)}\mathrm{Lip}(\phi)\big|a_\varepsilon(x,y_1,y_2)-x\big|\bigg)|v_\varepsilon(x)|\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad \leqslant \mathcal{C}\Big(\varepsilon_1+\varepsilon_2+\frac{\varepsilon_2}{\varepsilon_1}\Big),
\end{aligned} \tag{5.2.26}$$

where in the last inequality we used (5.2.25) and the fact that $\sup_\varepsilon \|v_\varepsilon\|_{L^1(\Omega;\mathbb{R}^{d\times N})} < \infty$.

Since functions of the form $\varphi = \phi\psi_1\psi_2$ are dense in $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^{d\times N}))$, and since $\big\{\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)\big\} \subset L^1(\Omega \times Y_1 \times Y_2; \mathbb{R}^{d\times N})$, $\{v_\varepsilon\} \subset L^1(\Omega; \mathbb{R}^{d\times N})$ are bounded sequences (see (5.2.3) and (5.2.5)), using a density argument, (4.0.1), and passing (5.2.26) to the limit as $\varepsilon \to 0^+$, we conclude that $v_\varepsilon \mathcal{L}^N\lfloor_\Omega \xrightharpoonup[\varepsilon]{3\text{-}sc} \lambda$ if, and only if, $\mathcal{A}_{\varrho_1(\varepsilon)}\big(\mathcal{T}_{\varrho_2(\varepsilon)}(v_\varepsilon)\big)\mathcal{L}^{3N}\lfloor_{\Omega\times Y_1\times Y_2} \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega \times Y_1 \times Y_2; \mathbb{R}^{d\times N})$ as $\varepsilon \to 0^+$. $\square$

The next lemma is a Reshetnyak continuity type result for functions not necessarily positively 1-homogeneous, and similar to Kristensen and Rindler [58, Thm. 5] (see Delladio [33] for related results).

Lemma 5.2.5. *Let $U \subset \mathbb{R}^l$ be an open set such that $\mathcal{L}^l(U) < \infty$. Let $g : U \times \mathbb{R}^m \to \mathbb{R}$ be a function such that $\bar{g} : U \times \mathbb{R}^m \times [0,\infty) \to \mathbb{R}$ defined by*

$$\bar{g}(z,\xi,t) := \begin{cases} tg\big(z, \frac{\xi}{t}\big) & \text{if } t > 0, \\ g^\infty(z,\xi) & \text{if } t = 0, \end{cases} \tag{5.2.27}$$

is continuous and bounded on $U \times \mathbb{S}^m$, where $g^\infty(z,\xi) := \limsup_{t\to+\infty} g(z,t\xi)/t$ is the recession function of g and $\mathbb{S}^m$ is the unit sphere in $R^m \times \mathbb{R}$. If $\lambda \in \mathcal{M}(U;\mathbb{R}^m)$, let $\tilde{\lambda} \in \mathcal{M}(U;\mathbb{R}^m \times \mathbb{R})$ denote the measure defined by $\tilde{\lambda}(\cdot) := \big(\lambda(\cdot), \mathcal{L}^l(\cdot)\big)$. Assume that $\lambda_j, \lambda \in \mathcal{M}(U;\mathbb{R}^m)$ are such that

$$\tilde{\lambda}_j \overset{\star}{\rightharpoonup}_j \tilde{\lambda} \quad \text{weakly-}\star \text{ in } \mathcal{M}(U;\mathbb{R}^m \times \mathbb{R}), \qquad \lim_{j\to+\infty} \|\tilde{\lambda}_j\|(U) = \|\tilde{\lambda}\|(U). \tag{5.2.28}$$

Then

$$\begin{aligned} \lim_{j\to+\infty} \bigg\{ \int_U g\Big(z, \frac{\mathrm{d}\lambda_j^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z, \frac{\mathrm{d}\lambda_j^s}{\mathrm{d}\|\lambda_j^s\|}(z)\Big)\,\mathrm{d}\|\lambda_j^s\|(z) \bigg\} \\ = \int_U g\Big(z, \frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z + \int_U g^\infty\Big(z, \frac{\mathrm{d}\lambda^s}{\mathrm{d}\|\lambda^s\|}(z)\Big)\,\mathrm{d}\|\lambda^s\|(z). \end{aligned} \tag{5.2.29}$$

Proof. Since $\bar{g}$ is a continuous and bounded function on $U \times \mathbb{S}^m$, in view of (5.2.28) Theorem 2.1.51 yields

$$\lim_{j\to+\infty} \int_U \bar{g}\Big(z, \frac{\mathrm{d}\tilde{\lambda}_j}{\mathrm{d}\|\tilde{\lambda}_j\|}(z)\Big)\mathrm{d}\|\tilde{\lambda}_j\|(z) = \int_U \bar{g}\Big(z, \frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\|\tilde{\lambda}\|}(z)\Big)\mathrm{d}\|\tilde{\lambda}\|(z). \tag{5.2.30}$$

We claim that (5.2.30) reduces to (5.2.29). In fact, writing the Lebesgue decomposition of an arbitrary $\mu \in \mathcal{M}(U;\mathbb{R}^m)$ with respect to $\mathcal{L}^l$ as

$$\mu = \frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}\mathcal{L}^l\lfloor_U + \mu^s,$$

then

$$\tilde{\mu} = \Big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}, 1\Big)\mathcal{L}^l\lfloor_U + (\mu^s, 0), \quad \|\tilde{\mu}\| = \Big|\Big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}, 1\Big)\Big|\mathcal{L}^l\lfloor_U + \|\mu^s\|, \tag{5.2.31}$$

are the Lebesgue decomposition of $\tilde{\mu}$ and $\|\tilde{\mu}\|$ with respect to $\mathcal{L}^l$, respectively.

In view of the Besicovitch Derivation Theorem, for $\mathcal{L}^l$-a.e. $z \in U$, we have

$$\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\|\tilde{\mu}\|}(z) = \frac{\big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z), 1\big)}{\big|\big(\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z), 1\big)\big|}, \tag{5.2.32}$$

and for $\|\mu^s\|$-a.e. $z \in U$, we have

$$\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\|\tilde{\mu}\|}(z) = \Big(\frac{\mathrm{d}\mu^s}{\mathrm{d}\|\mu^s\|}(z), 0\Big). \tag{5.2.33}$$

From (5.2.31)–(5.2.33), and taking into account the positive 1-homogeneity of $(\xi,t)\in\mathbb{R}^m\times[0,\infty)\mapsto\bar g(z,\xi,t)$, we deduce that

$$\begin{aligned}\int_U \bar g\Big(z,\frac{\mathrm{d}\tilde\mu}{\mathrm{d}\|\tilde\mu\|}(z)\Big)\mathrm{d}\|\tilde\mu\|(z)&=\int_U \bar g\Big(z,\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z),1\Big)\,\mathrm{d}z+\int_U \bar g\Big(z,\frac{\mathrm{d}\mu^{s}}{\mathrm{d}\|\mu^s\|}(z),0\Big)\,\mathrm{d}\|\mu^s\|(z)\\&=\int_U g\Big(z,\frac{\mathrm{d}\mu^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z+\int_U g^\infty\Big(z,\frac{\mathrm{d}\mu^{s}}{\mathrm{d}\|\mu^s\|}(z)\Big)\,\mathrm{d}\|\mu^s\|(z),\end{aligned}\tag{5.2.34}$$

where in the last equality we used the definition of $\bar g$. By (5.2.34) we conclude that (5.2.30) reduces to (5.2.29). □

Next we prove a Reshetnyak lower semicontinuity type result for functions not necessarily positively 1-homogeneous (see also Dal Maso [30], Giaquinta, Modica and Souček [51]).

Lemma 5.2.6. *Let $U\subset\mathbb{R}^l$ be an open set such that $\mathcal{L}^l(U)<\infty$. Let $g:U\times\mathbb{R}^m\to\mathbb{R}$ be a function satisfying $|g(z,\xi)|\leqslant C(1+|\xi|)$, for some $C>0$ and for every $(z,\xi)\in U\times\mathbb{R}^m$, and such that for all $z\in U$, $g(z,\cdot)$ is convex. Assume further that for all $\bar z\in U$ and $\delta>0$, there exists $\tau=\tau(\bar z,\delta)>0$ such that for all $z\in U$ with $|z-\bar z|<\tau$, and $\xi\in\mathbb{R}^m$, we have $|g(\bar z,\xi)-g(z,\xi)|\leqslant\delta(1+|\xi|)$. If $\lambda_j,\lambda\in\mathcal{M}(U;\mathbb{R}^m)$ are such that $\lambda_j\overset{\star}{\rightharpoonup}_j\lambda$ weakly-$\star$ in $\mathcal{M}(U;\mathbb{R}^m)$ as $j\to+\infty$, then*

$$\begin{aligned}&\liminf_{j\to+\infty}\Big\{\int_U g\Big(z,\frac{\mathrm{d}\lambda_j^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z+\int_U g^\infty\Big(z,\frac{\mathrm{d}\lambda_j^{s}}{\mathrm{d}\|\lambda_j^s\|}(z)\Big)\,\mathrm{d}\|\lambda_j^s\|(z)\Big\}\\&\qquad\geqslant\int_U g\Big(z,\frac{\mathrm{d}\lambda^{ac}}{\mathrm{d}\mathcal{L}^l}(z)\Big)\,\mathrm{d}z+\int_U g^\infty\Big(z,\frac{\mathrm{d}\lambda^{s}}{\mathrm{d}\|\lambda^s\|}(z)\Big)\,\mathrm{d}\|\lambda^s\|(z).\end{aligned}\tag{5.2.35}$$

Proof. Let $\lambda_j,\lambda\in\mathcal{M}(U;\mathbb{R}^m)$ be such that $\lambda_j\overset{\star}{\rightharpoonup}_j\lambda$ weakly-$\star$ in $\mathcal{M}(U;\mathbb{R}^m\times\mathbb{R})$. Defining $\tilde\lambda_j,\tilde\lambda\in\mathcal{M}(U;\mathbb{R}^m\times\mathbb{R})$ as in Lemma 5.2.5, we see that $\tilde\lambda_j\overset{\star}{\rightharpoonup}_j\tilde\lambda$ weakly-$\star$ in $\mathcal{M}(U;\mathbb{R}^m\times\mathbb{R})$.

Let $\bar g:U\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$ be the function introduced in (5.2.27). Then (see Remark 5.1.5 (ii)) $\bar g$ is a continuous function, and $|\bar g(z,\xi,t)|\leqslant 2C|(\xi,t)|$ for all $(z,\xi,t)\in U\times\mathbb{R}^m\times[0,\infty)$. Moreover, since for each $i\in\mathbb{N}$ there exist functions $a_i:U\to\mathbb{R}$ and $b_i:U\to\mathbb{R}^m$ such that

$$g(z,\xi)=\sup_{i\in\mathbb{N}}\big\{a_i(z)+b_i(z)\cdot\xi\big\},\quad g^\infty(z,\xi)=\sup_{i\in\mathbb{N}}\big\{b_i(z)\cdot\xi\big\},$$

(see Fonseca and Leoni [48, Prop. 2.77]), we have that for all $(z,\xi,t)\in U\times\mathbb{R}^m\times[0,\infty)$,

$$\bar g(z,\xi,t)=\sup_{i\in\mathbb{N}}\big\{a_i(z)t+b_i(z)\cdot\xi\big\}.$$

Thus for all $z\in U$, $(\xi,t)\in\mathbb{R}^m\times[0,\infty)\mapsto\bar g(z,\xi,t)$ is convex and positively 1-homogeneous. So, Theorem 2.1.53 yields

$$\liminf_{j\to+\infty}\int_U\bar g\Big(z,\frac{\mathrm{d}\tilde\lambda_j}{\mathrm{d}\|\tilde\lambda_j\|}(z)\Big)\mathrm{d}\|\tilde\lambda_j\|(z)\geqslant\int_U\bar g\Big(z,\frac{\mathrm{d}\tilde\lambda}{\mathrm{d}\|\tilde\lambda\|}(z)\Big)\mathrm{d}\|\tilde\lambda\|(z).\tag{5.2.36}$$

Finally, we observe that by (5.2.34), (5.2.36) reduces to (5.2.35). □

Proof of Theorem 5.2.1. Fix $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\in BV(\Omega;\mathbb{R}^d)\times\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\times\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))$, and set

$$\begin{aligned}G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2):=&\int_{\Omega\times Y_1\times Y_2}f\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\&+\int_{\Omega\times Y_1\times Y_2}f^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2).\end{aligned}$$

We will proceed in two steps.

Step 1. We start by proving that

$$F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \geqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2).$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero as $h \to +\infty$, and by Proposition 4.1.8 let $\{u_h\}_{h\in\mathbb{N}} \subset BV(\Omega; \mathbb{R}^d)$ be a bounded sequence such that $Du_h \xrightarrow[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$. We claim that

$$\liminf_{h\to+\infty} F_{\varepsilon_h}(u_h) \geqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{5.2.37}$$

Since $\{Du_h\}_{h\in\mathbb{N}}$ is bounded in $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ (see Remark 4.1.5), in view of $(\mathcal{F}3)$, $(\mathcal{F}4)^\star$ and (5.2.2), we have that $\{F_{\varepsilon_h}(u_h)\}_{h\in\mathbb{N}}$ is bounded. Therefore, we may assume without loss of generality that the limit inferior in (5.2.37) is actually a limit and that this limit is finite (which is true up to a subsequence).

By Proposition 4.2.10 (with $\boldsymbol{\mu}_i = 0$), for each $h \in \mathbb{N}$ we can find a sequence $\big\{u_j^{(h)}\big\}_{j\in\mathbb{N}} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ such that

$$\begin{aligned} &u_j^{(h)} \overset{\star}{\rightharpoonup}_j u_h \quad \text{weakly-}\star \text{ in } BV(\Omega; \mathbb{R}^d),\\ &\tilde\lambda_j^{(h)} \overset{\star}{\rightharpoonup}_j \tilde\lambda_h \quad \text{weakly-}\star \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d\times N} \times \mathbb{R}), \quad \lim_{j\to+\infty} \|\tilde\lambda_j^{(h)}\|(\Omega) = \|\tilde\lambda_h\|(\Omega), \end{aligned} \tag{5.2.38}$$

where, for $B \in \mathcal{B}(\Omega)$,

$$\tilde\lambda_j^{(h)}(B) := \bigg(\int_B \nabla u_j^{(h)}(x)\,\mathrm{d}x, \mathcal{L}^N(B)\bigg), \quad \tilde\lambda_h(B) := \big(Du_h(B), \mathcal{L}^N(B)\big).$$

Under hypotheses $(\mathcal{F}1)$–$(\mathcal{F}3)$, $(\mathcal{F}4)^\star$, $(\mathcal{F}5)$ (see also Remark 5.1.5 (ii)), it can be shown that for fixed $h \in \mathbb{N}$, Lemma 5.2.5 applies to $U := \Omega$ and $g(x,\xi) := f(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \xi)$, which ensures the continuity of the functional F_{ε_h} with respect to the convergence (5.2.38), that is, $\lim_{j\to+\infty} F_{\varepsilon_h}\Big(u_j^{(h)}\Big) = F_{\varepsilon_h}(u_h)$. Consequently,

$$\lim_{h\to+\infty} \lim_{j\to+\infty} F_{\varepsilon_h}\Big(u_j^{(h)}\Big) = \lim_{h\to+\infty} F_{\varepsilon_h}(u_h). \tag{5.2.39}$$

Moreover, given $\varphi \in C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^{d\times N}))$ we have

$$\begin{aligned} \lim_{h\to+\infty} \lim_{j\to+\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) : \nabla u_j^{(h)}(x)\,\mathrm{d}x &= \lim_{h\to+\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) : \mathrm{d}Du_h(x)\\ &= \int_{\Omega\times Y_1\times Y_2} \varphi(x, y_1, y_2) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(x, y_1, y_2), \end{aligned} \tag{5.2.40}$$

where we have used the weak-$\star$ convergence $\nabla u_j^{(h)} \mathcal{L}^N{\lfloor_\Omega} \overset{\star}{\rightharpoonup}_j Du_h$ in $\mathcal{M}(\Omega; \mathbb{R}^{d\times N})$, and the 3-scale convergence $Du_h \xrightarrow[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$. In addition, in view of (5.2.38),

$$\sup_{h\in\mathbb{N}} \sup_{j\in\mathbb{N}} \int_\Omega \big|\nabla u_j^{(h)}(x)\big|\,\mathrm{d}x < \infty. \tag{5.2.41}$$

Using the separability of $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^{d\times N}))$ and a diagonalization argument, from (5.2.39), (5.2.40) and (5.2.41), we can find a sequence $\{j_h\}$ such that $j_h \to +\infty$ as $h \to +\infty$, and such that $w_h := u_{j_h}^{(h)}$ satisfies

$$w_h \in W^{1,1}(\Omega; \mathbb{R}^d), \quad \nabla w_h \mathcal{L}^N{\lfloor_\Omega} \xrightarrow[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}, \quad \lim_{h\to+\infty} F_{\varepsilon_h}(w_h) = \lim_{h\to+\infty} F_{\varepsilon_h}(u_h). \tag{5.2.42}$$

Set $c := \sup_h \|\nabla w_h\|_{L^1(\Omega;\mathbb{R}^{d\times N})} < \infty$ and fix $\eta > 0$. Then by Lemmas 5.2.2 and 5.2.4, and by Lemma 5.2.6 applied to $U := \Omega \times Y_1 \times Y_2$ and $g(x, y_1, y_2, \xi) := f_\eta(y_1, y_2, \xi)$, where $f_\eta(y_1, y_2, \xi) := f(y_1, y_2, \xi) + \eta|\xi|$, we conclude that

$$\begin{aligned}\lim_{h\to+\infty} F_{\varepsilon_h}(u_h) + \eta c &= \lim_{h\to+\infty} F_{\varepsilon_h}(w_h) + \eta c \geqslant \liminf_{h\to+\infty} \int_\Omega f_\eta\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla w_h(x)\Big)\,\mathrm{d}x \\ &\geqslant \liminf_{h\to+\infty} \int_{\Omega\times Y_1\times Y_2} f_\eta\big(y_1, y_2, \mathcal{A}_{\varrho_1(\varepsilon_h)}\big(\mathcal{T}_{\varrho_2(\varepsilon_h)}(\nabla w_h)\big)(x, y_1, y_2)\big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \geqslant F^{\mathrm{sc}}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2),\end{aligned} \tag{5.2.43}$$

where

$$\begin{aligned}F^{\mathrm{sc}}_\eta(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) := &\int_{\Omega\times Y_1\times Y_2} f_\eta\Big(y_1, y_2, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x, y_1, y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\Omega\times Y_1\times Y_2} f^\infty_\eta\Big(y_1, y_2, \frac{\mathrm{d}\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x, y_1, y_2)\Big)\,\mathrm{d}\|\lambda^s_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x, y_1, y_2).\end{aligned} \tag{5.2.44}$$

Since $f^\infty_\eta(y_1, y_2, \xi) = f^\infty(y_1, y_2, \xi) + \eta|\xi|$, from (5.2.43) we deduce that

$$\lim_{h\to+\infty} F_{\varepsilon_h}(u_h) + \eta c \geqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) + \eta\|\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(\Omega\times Y_1\times Y_2).$$

Finally, letting $\eta \to 0^+$ we obtain (5.2.37).

Step 2. We prove that

$$F^{\mathrm{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \leqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{5.2.45}$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be a sequence of positive numbers converging to zero as $h \to +\infty$, and let $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega}; \mathbb{R}^d)$, $\{\psi^{(1)}_j\}_{j\in\mathbb{N}} \subset C^\infty_c(\Omega; C^\infty_\#(Y_1; \mathbb{R}^d))$ and $\{\psi^{(2)}_j\}_{j\in\mathbb{N}} \subset C^\infty_c(\Omega; C^\infty_\#(Y_1 \times Y_2; \mathbb{R}^d))$ be the sequences given by Proposition 4.2.10. For each $h, j \in \mathbb{N}$ define $u_{h,j} \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$ by

$$u_{h,j}(x) := u_j(x) + \varrho_1(\varepsilon_h)\psi^{(1)}_j\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}\Big) + \varrho_2(\varepsilon_h)\psi^{(2)}_j\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big). \tag{5.2.46}$$

Using (4.0.1), (2.2.1), and (4.2.67), in this order, we have that for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^{d\times N}))$

$$\lim_{j\to+\infty}\lim_{h\to+\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) : \nabla u_{h,j}(x)\,\mathrm{d}x = \int_{\Omega\times Y_1\times Y_2} \varphi(x, y_1, y_2) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(x, y_1, y_2). \tag{5.2.47}$$

Moreover,

$$\begin{aligned}F_{\varepsilon_h}(u_{h,j}) &= \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_{h,j}(x)\Big)\,\mathrm{d}x \\ &= \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_j(x) + \big(\nabla_{y_1}\psi^{(1)}_j\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}\Big)\,\mathrm{d}x \\ &\qquad + \int_\Omega \big(\nabla_{y_2}\psi^{(2)}_j\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) + \vartheta_{h,j}(x)\Big)\,\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}\vartheta_{h,j}(x) := &\ \varrho_1(\varepsilon_h)\big(\nabla_x\psi^{(1)}_j\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}\Big) + \varrho_2(\varepsilon_h)\big(\nabla_x\psi^{(2)}_j\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big) \\ &+ \frac{\varrho_2(\varepsilon_h)}{\varrho_1(\varepsilon_h)}\big(\nabla_{y_1}\psi^{(2)}_j\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big).\end{aligned}$$

We claim that if $K \subset \mathbb{R}^{d\times N}$ is a compact set then there exists a positive constant $C(K)$, depending only on K, such that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi, \xi' \in K$,

$$|f(y_1, y_2, \xi) - f(y_1, y_2, \xi')| \leqslant C(K)|\xi - \xi'|. \tag{5.2.48}$$

In fact, the continuity of f (see Remark 5.1.5 (ii)) and ($\mathcal{F}1$) ensure that there exists a positive constant $c(K)$ only depending on K such that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi \in K$,

$$|f(y_1, y_2, \xi)| \leqslant c(K). \tag{5.2.49}$$

On the other hand, by ($\mathcal{F}2$) (see, for example, Fonseca and Leoni [48, Thm. 4.36]) $f(y_1, y_2, \cdot)$ is locally Lipschitz with

$$\operatorname{Lip}(f(y_1, y_2, \cdot); B(0; r)) \leqslant \frac{\sqrt{d \times N}}{r} \sup \big\{ |f(y_1, y_2, \xi) - f(y_1, y_2, \xi')| : \xi, \xi' \in B(0, 2r) \big\}. \tag{5.2.50}$$

From (5.2.49) and (5.2.50), we deduce that (5.2.48) holds.

Taking into account (4.0.1), in view of (5.2.48) for each $j \in \mathbb{N}$ we can find a positive constant C_j independent of ε such that

$$\begin{aligned} &F_{\varepsilon_h}(u_{h,j}) \\ &\quad \leqslant \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_j(x) + \big(\nabla_{y_1} \psi_j^{(1)}\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}\Big) + \big(\nabla_{y_2} \psi_j^{(2)}\big)\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big)\Big)\, \mathrm{d}x \\ &\qquad + C_j \int_\Omega |\vartheta_{h,j}(x)|\, \mathrm{d}x, \end{aligned} \tag{5.2.51}$$

with, for all $j \in \mathbb{N}$,

$$\lim_{h \to +\infty} \int_\Omega |\vartheta_{h,j}(x)|\, \mathrm{d}x = 0. \tag{5.2.52}$$

Furthermore, the function

$$g_j(x, y_1, y_2) := f\big(y_1, y_2, \nabla u_j(x) + \big(\nabla_{y_1} \psi_j^{(1)}\big)(x, y_1) + \big(\nabla_{y_2} \psi_j^{(2)}\big)(x, y_1, y_2)\big)$$

belongs to $C(\overline{\Omega}; C_\#(Y_1 \times Y_2))$, hence by (2.2.1)

$$\lim_{h \to +\infty} \int_\Omega g_j\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big)\, \mathrm{d}x = \int_{\Omega \times Y_1 \times Y_2} g_j(x, y_1, y_2)\, \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2. \tag{5.2.53}$$

From (5.2.51)–(5.2.53) we conclude that

$$\begin{aligned} &\limsup_{j \to +\infty} \limsup_{h \to +\infty} F_{\varepsilon_h}(u_{h,j}) \\ &\quad \leqslant \limsup_{j \to +\infty} \int_{\Omega \times Y_1 \times Y_2} f\big(y_1, y_2, \nabla u_j(x) + \big(\nabla_{y_1} \psi_j^{(1)}\big)(x, y_1) + \big(\nabla_{y_2} \psi_j^{(2)}\big)(x, y_1, y_2)\big)\, \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad = G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2), \end{aligned} \tag{5.2.54}$$

where in the last equality we invoked Lemma 5.2.5 applied to $U := \Omega \times Y_1 \times Y_2$ and $g(x, y_1, y_2, \xi) := f(y_1, y_2, \xi)$, and also (4.2.87).

Using the separability of $C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^{d \times N}))$ and a diagonalization argument, from (5.2.47) and (5.2.54), and noticing that $\{u_{h,j}\}_{h,j \in \mathbb{N}}$ is a bounded sequence in $W^{1,1}(\Omega; \mathbb{R}^d)$, we can find subsequences $h_k \prec h$ and $j_k \prec j$ such that $u_{h_k, j_k} \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$ satisfies

$$\nabla u_{h_k, j_k} \mathcal{L}^N \lfloor \Omega \xrightharpoonup[\varepsilon_{h_k}]{\text{3-sc}} \lambda_{u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2}, \quad \limsup_{k \to +\infty} F_{\varepsilon_{h_k}}(u_{h_k, j_k}) \leqslant G(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{5.2.55}$$

Finally, consider the sequence $\{w_h\}_{h\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^d)$ defined by

$$w_h := \begin{cases} u_{h_k,j_k} & \text{if } h = h_k \text{ for some } k \in \mathbb{N}, \\ v_h & \text{if } h \neq h_k \text{ for all } k \in \mathbb{N}, \end{cases}$$

where $\{v_h\}_{h\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^d)$ is a sequence such that $Dv_h \xrightharpoonup[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ (which exists by Proposition 4.1.8). Then $Dw_h \xrightharpoonup[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$, and so by (5.2.55)

$$F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1\boldsymbol{\mu}_2) \leqslant \liminf_{h\to+\infty} F_{\varepsilon_h}(w_h) \leqslant \limsup_{k\to+\infty} F_{\varepsilon_{h_k}}(u_{h_k,j_k}) \leqslant G(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2).$$

This concludes the proof of Theorem 5.2.1. □

The next theorem concerns the first equality in (5.1.8) relating the three-scale homogenized functional, F^{sc}, and the effective energy, F^{hom}.

Theorem 5.2.7. *Under the hypotheses of Theorem 5.2.1, assume further that the length scales ϱ_1, ϱ_2 satisfy the condition (4.1.2). Then, for all $u \in BV(\Omega;\mathbb{R}^d)$,*

$$F^{\mathrm{hom}}(u) = \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2).$$

Proof. Let $u \in BV(\Omega;\mathbb{R}^d)$ be given. We will proceed in two steps.

Step 1. We prove that

$$F^{\mathrm{hom}}(u) \geqslant \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \tag{5.2.56}$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero as $h \to +\infty$, and let $\{u_h\}_{h\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^d)$ be a sequence weakly-$\star$ converging to u in $BV(\Omega;\mathbb{R}^d)$ as $h \to +\infty$. By $(\mathcal{F}3)$, $(\mathcal{F}4)^\star$ and (5.2.2), $\liminf_{h\to+\infty} F_{\varepsilon_h}(u_h) \in \mathbb{R}$. Using Theorem 4.1.7, we can find a subsequence $h_k \prec h$ and measures $\bar{\boldsymbol{\mu}}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))$, $\bar{\boldsymbol{\mu}}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))$, such that

$$\lim_{k\to+\infty} F_{\varepsilon_{h_k}}(u_{h_k}) = \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h), \quad u_{h_k} \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} u\mathcal{L}^N\lfloor_\Omega \otimes \mathcal{L}^{2N}_{y_1,y_2}, \quad Du_{h_k} \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} \lambda_{u,\bar{\boldsymbol{\mu}}_1,\bar{\boldsymbol{\mu}}_2}.$$

Hence, taking into account Theorem 5.2.1 (see (5.2.37)),

$$\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \leqslant F^{\mathrm{sc}}(u,\bar{\boldsymbol{\mu}}_1,\bar{\boldsymbol{\mu}}_2) \leqslant \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h).$$

Taking the infimum over all sequences $\{u_h\}_{h\in\mathbb{N}}$ as above, we deduce that (5.2.56) holds.

Step 2. We show that

$$F^{\mathrm{hom}}(u) \leqslant \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2). \tag{5.2.57}$$

Let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to zero as $h \to +\infty$, and take $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))$, $\boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))$. Reasoning as in the proof of (5.2.45), we can find a subsequence $h_k \prec h$ and a sequence $\{v_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ such that (see (5.2.46) and (5.2.55))

$$\lim_{k\to+\infty} \int_\Omega |v_k - u|\,\mathrm{d}x = 0, \quad \nabla v_k\mathcal{L}^N\lfloor_\Omega \xrightharpoonup[\varepsilon_{h_k}]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}, \quad \limsup_{k\to+\infty} F_{\varepsilon_{h_k}}(v_k) \leqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2).$$

Consequently, we also have that $Dv_k \xrightharpoonup{\star} Du$ weakly-$\star$ in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ as $k\to+\infty$. Finally, define

$$u_h := \begin{cases} v_k & \text{if } h = h_k \text{ for some } k\in\mathbb{N},\\ u & \text{otherwise.}\end{cases}$$

Then $u_h \xrightharpoonup{\star} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^d)$ as $h\to+\infty$, so that

$$F^{\text{hom}}(u) \leqslant \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h) \leqslant \limsup_{k\to+\infty} F_{\varepsilon_{h_k}}(v_k) \leqslant F^{\text{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2),$$

from which we get (5.2.57) by taking the infimum over all $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1;\mathbb{R}^d))$ and $\boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1; BV_\#(Y_2;\mathbb{R}^d))$. □

Remark 5.2.8. *We observe that Theorems 5.2.1 and 5.2.7 hold if $(\mathcal{F}4)^\star$ is replaced by $(\mathcal{F}4)$ (see also Remark 5.1.5 (i)).*

In order to establish the integral representation for the effective energy F^{hom} stated in Theorem 5.1.3 we will need some auxiliary results. The first one is a simple consequence of Kristensen and Rindler [58, Thm. 6] (see also Dal Maso [30] in the case in which $d = 1$ and g is coercive).

Lemma 5.2.9. *Assume that $\Omega\subset\mathbb{R}^N$ is an open and bounded set with $\partial\Omega$ Lipschitz, and let $g:\mathbb{R}^{d\times N}\to\mathbb{R}$ be a convex function such that for all $\xi\in\mathbb{R}^{d\times N}$ and for some constant $M>0$, $|g(\xi)|\leqslant M(1+|\xi|)$. Then, for all $\delta>0$ and for all $u\in BV(\Omega;\mathbb{R}^{d\times N})$, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}\subset W^{1,1}(\Omega;\mathbb{R}^{d\times N})$ such that $u_j \xrightharpoonup{\star} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^{d\times N})$ as $j\to+\infty$, and*

$$\int_\Omega g(\nabla u(x))\,\mathrm{d}x + \int_\Omega g^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x) + \delta \geqslant \lim_{j\to+\infty}\int_\Omega g(\nabla u_j(x))\,\mathrm{d}x.$$

The next two lemmas provide sufficient conditions under which equality $(g_{\text{hom}})^\infty = (g^\infty)_{\text{hom}}$ holds.

Lemma 5.2.10. *Let $g:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ be a Borel function satisfying conditions $(\mathcal{F}1)$–$(\mathcal{F}4)$ and $(\mathcal{F}8)$. Then,*

$$(g_{\text{hom}})^\infty = (g^\infty)_{\text{hom}}. \tag{5.2.58}$$

Proof. We start by observing that, arguing as in Attouch [8, Thm. 4], we can prove a similar result to Bouchitté [17, Lemme 3.5]: If $h:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ is a Borel function satisfying hypotheses $(\mathcal{F}1)$–$(\mathcal{F}4)$, then for all $y_1\in\mathbb{R}^N$, $\xi^*\in\mathbb{R}^{d\times N}$ (see (5.1.1) and (5.1.3)),

$$(h_{\text{hom}_2})^*(y_1,\xi^*) = \inf_{\Psi_2\in E_\#(Y_2;\mathbb{R}^{d\times N})}\int_{Y_2} h^*(y_1,y_2,\xi^*+\Psi_2(y_2))\,\mathrm{d}y_2 \tag{5.2.59}$$

where, for $k\in\mathbb{N}$,

$$E_\#(Y_k;\mathbb{R}^{d\times N}) := \Big\{\Psi = (\Psi_{ij}) \in L^\infty_\#(Y_k;\mathbb{R}^{d\times N}): \int_{Y_k}\Psi(y_k)\,\mathrm{d}y_k = 0,\ \operatorname{div}\Psi_{i\cdot} = 0 \text{ for all } i\in\{1,\cdots,d\}\Big\}.$$

Similarly, since $h_{\text{hom}_2}:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ is also a Borel function satisfying conditions $(\mathcal{F}1)$–$(\mathcal{F}4)$, we have that for all $\xi^*\in\mathbb{R}^{d\times N}$,

$$(h_{\text{hom}})^*(\xi^*) = \inf_{\Psi_1\in E_\#(Y_1;\mathbb{R}^{d\times N})}\int_{Y_1}(h_{\text{hom}_2})^*(y_1,\xi^*+\Psi_1(y_1))\,\mathrm{d}y_1. \tag{5.2.60}$$

Moreover, for all $y_1, y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$ (see, for example, Rockafellar [67, Thm. 13.3, Lemma 7.42]),

$$h^\infty(y_1,y_2,\xi) = \sup_{(y_1,y_2,\xi^*)\in \mathrm{dom}_e h^*} \xi:\xi^*, \quad (h_{\mathrm{hom}_2})^\infty(y_1,\xi) = \sup_{(y_1,\xi^*)\in \mathrm{dom}_e (h_{\mathrm{hom}_2})^*} \xi:\xi^*. \tag{5.2.61}$$

If, in addition, h^* is bounded from above in $\mathrm{dom}_e h^*$, then we claim that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi^* \in \mathbb{R}^{d\times N}$,

$$(h^\infty)^*(y_1,y_2,\xi^*) = \begin{cases} 0 & \text{if } (y_1,y_2,\xi^*)\in \mathrm{dom}_e h^*, \\ \infty & \text{otherwise.}\end{cases} \tag{5.2.62}$$

Indeed, under this additional hypothesis, we have that for each $y_1, y_2 \in \mathbb{R}^N$ the set $\{\xi^* \in \mathbb{R}^{d\times N} : (y_1,y_2,\xi^*) \in \mathrm{dom}_e h^*\}$ is convex and closed. Hence (see, for example, Ekeland and Témam [41], Rockafellar [67]), the indicator function $\chi_{\mathrm{dom}_e h^*}$, that is, the function defined by

$$\chi_{\mathrm{dom}_e h^*}(y_1,y_2,\xi^*) := \begin{cases} 0 & \text{if } (y_1,y_2,\xi^*)\in \mathrm{dom}_e h^*, \\ \infty & \text{otherwise,}\end{cases}$$

coincides with its biconjugate function $(\chi_{\mathrm{dom}_e h^*})^{**}$. On the other hand, defining for each $t>0$,

$$h_t(y_1,y_2,\xi) := \frac{h(y_1,y_2,t\xi)-h(y_1,y_2,0)}{t}, \ \ y_1,y_2\in\mathbb{R}^N, \xi\in\mathbb{R}^{d\times N},$$

due to the convexity hypothesis we have that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$, $t\in\mathbb{R}^+ \mapsto h_t(y_1,y_2,\xi)$ is nondecreasing and

$$\sup_{t>0} h_t(y_1,y_2,\xi) = \lim_{t\to+\infty} h_t(y_1,y_2,\xi) = h^\infty(y_1,y_2,\xi).$$

Furthermore, it can be shown that for all $y_1, y_2 \in \mathbb{R}^N$, $\xi, \xi^* \in \mathbb{R}^{d\times N}$,

$$\begin{aligned} &\inf_{t>0} h_t^*(y_1,y_2,\xi^*) = \lim_{t\to+\infty} h_t^*(y_1,y_2,\xi^*) = \chi_{\mathrm{dom}_e h^*}(y_1,y_2,\xi^*), \\ &\Big(\inf_{t>0} h_t^*\Big)^{**}(y_1,y_2,\xi) = \Big(\sup_{t>0} h_t\Big)^{*}(y_1,y_2,\xi) = (h^\infty)^*(y_1,y_2,\xi), \end{aligned}$$

so that (5.2.62) follows from the equality $(\chi_{\mathrm{dom}_e h^*})^{**} = \chi_{\mathrm{dom}_e h^*}$.

We now establish equality (5.2.58) in two steps. Notice that both g_{hom_2} and g_{hom}, as well as their respective recession functions, are real-valued Borel functions satisfying similar conditions to $(\mathcal{F}1)$–$(\mathcal{F}4)$.

Step 1. We prove that $(g_{\mathrm{hom}_2})^\infty = (g^\infty)_{\mathrm{hom}_2}$.

Inequality $(g_{\mathrm{hom}_2})^\infty \leqslant (g^\infty)_{\mathrm{hom}_2}$ follows from the definitions of both functions and using Lebesgue Dominated Convergence Theorem taking into account $(\mathcal{F}3)$ and $(\mathcal{F}4)$.

We claim that to prove that $(g_{\mathrm{hom}_2})^\infty \geqslant (g^\infty)_{\mathrm{hom}_2}$, it suffices to show that

$$\mathrm{dom}_e (g_{\mathrm{hom}_2})^* \supset \mathrm{dom}_e \big((g^\infty)_{\mathrm{hom}_2}\big)^*. \tag{5.2.63}$$

In fact, if (5.2.63) holds then by (5.2.61) we have that

$$(g_{\mathrm{hom}_2})^\infty \geqslant \big((g^\infty)_{\mathrm{hom}_2}\big)^\infty. \tag{5.2.64}$$

Since $(g^\infty)_{\mathrm{hom}_2}$ is positively 1-homogeneous in the last variable, we have that $\big((g^\infty)_{\mathrm{hom}_2}\big)^\infty = (g^\infty)_{\mathrm{hom}_2}$, which together with (5.2.64) yields $(g_{\mathrm{hom}_2})^\infty \geqslant (g^\infty)_{\mathrm{hom}_2}$.

We now prove (5.2.63). Let $(y_1, \xi^*) \in \mathrm{dom}_e\big((g^\infty)_{\mathrm{hom}_2}\big)^*$. Then, by (5.2.59) (with g replaced by g^∞), there exists $\Psi_2 \in E_\#(Y_2; \mathbb{R}^{d\times N})$ such that

$$\int_{Y_2} (g^\infty)^*(y_1, y_2, \xi^* + \Psi_2(y_2))\,\mathrm{d}y_2 < \infty,$$

and so (5.2.62) ensures that for a.e. $y_2 \in Y_2$ we have $(y_1, y_2, \xi^* + \Psi_2(y_2)) \in \mathrm{dom}_e g^*$. From (5.2.59) and $(\mathcal{F}8)$ we conclude that

$$(g_{\mathrm{hom}_2})^*(y_1, \xi^*) \leqslant \int_{Y_2} g^*(y_1, y_2, \xi^* + \Psi_2(y_2))\,\mathrm{d}y_2 \leqslant \mathcal{C} < \infty.$$

Thus, $(y_1, \xi^*) \in \mathrm{dom}_e(g_{\mathrm{hom}_2})^*$, which proves (5.2.63). So, $(g_{\mathrm{hom}_2})^\infty = (g^\infty)_{\mathrm{hom}_2}$ and, consequently,

$$\Big((g_{\mathrm{hom}_2})^\infty\Big)_{\mathrm{hom}_1} = \Big((g^\infty)_{\mathrm{hom}_2}\Big)_{\mathrm{hom}_1} = (g^\infty)_{\mathrm{hom}}, \tag{5.2.65}$$

where in the last equality we used definition (5.1.4).

Step 2. We prove that $(g_{\mathrm{hom}})^\infty = (g^\infty)_{\mathrm{hom}}$.

It suffices to observe that $(\mathcal{F}3)$, $(\mathcal{F}8)$ and (5.2.59) imply that $(g_{\mathrm{hom}_2})^*$ is also bounded on its effective domain. Hence, reasoning as before and in view of (5.2.60),

$$\Big((g_{\mathrm{hom}_2})^\infty\Big)_{\mathrm{hom}_1} = \Big((g_{\mathrm{hom}_2})_{\mathrm{hom}_1}\Big)^\infty = (g_{\mathrm{hom}})^\infty. \tag{5.2.66}$$

Thus, from (5.2.65)–(5.2.66) we conclude that $(g_{\mathrm{hom}})^\infty = (g^\infty)_{\mathrm{hom}}$. □

Lemma 5.2.11. *Let $g : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying conditions $(\mathcal{F}1)$, $(\mathcal{F}3)$, $(\mathcal{F}4)$' and $(\mathcal{F}7)$. Then $(g_{\mathrm{hom}})^\infty = (g^\infty)_{\mathrm{hom}}$.*

Proof. Note that $(\mathcal{F}7)$ is equivalent to requiring that there exist constants $C, L > 0$ and $\alpha \in (0,1)$ such that given y_1, $y_2 \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{d\times N}$ arbitrarily, then for all $t \in \mathbb{R}$ such that $t|\xi| > L$,

$$\left| g^\infty(y_1, y_2, \xi) - \frac{g(y_1, y_2, t\xi)}{t} \right| \leqslant C\frac{|\xi|^{1-\alpha}}{t^\alpha}. \tag{5.2.67}$$

We now prove that

$$(g_{\mathrm{hom}_2})^\infty = (g^\infty)_{\mathrm{hom}_2}. \tag{5.2.68}$$

Inequality $(g_{\mathrm{hom}_2})^\infty \leqslant (g^\infty)_{\mathrm{hom}_2}$ follows from the definitions of both functions and Fatou's Lemma taking into account $(\mathcal{F}3)$ and $(\mathcal{F}4)$'.

Conversely, fix $y_1 \in \mathbb{R}$, $\xi \in \mathbb{R}^{d\times N}$. By definition of infimum, for each $t \geqslant 1$ we can find $\psi_t \in W^{1,1}_\#(Y_2; \mathbb{R}^d)$ such that

$$\int_{Y_2} \frac{g(y_1, y_2, t\xi + t\nabla\psi_t(y_2))}{t}\,\mathrm{d}y_2 \leqslant \frac{g_{\mathrm{hom}_2}(y_1, t\xi)}{t} + \frac{1}{t}. \tag{5.2.69}$$

In particular, (5.2.69), together with $(\mathcal{F}3)$ and $(\mathcal{F}4)$', yields

$$\int_{Y_2} |\xi + \nabla\psi_t(y_2)|\,\mathrm{d}y_2 \leqslant \bar{C}(1 + |\xi|), \tag{5.2.70}$$

for some positive constant $\bar{C}$ independent of t.

By definition of $(g^\infty)_{\hom_2}$,

$$\begin{aligned}(g^\infty)_{\hom_2}(y_1,\xi) &\leqslant \int_{Y_2} g^\infty(y_1,y_2,\xi+\nabla\psi_t(y_2))\,\mathrm{d}y_2\\ &\leqslant \frac{CL}{t}+\int_{Y_2\cap\{y_2\colon t|\xi+\nabla\psi_t(y_2)|>L\}} g^\infty(y_1,y_2,\xi+\nabla\psi_t(y_2))\,\mathrm{d}y_2,\end{aligned}$$

where we used the fact that in view of $(\mathcal{F}3)$, $g^\infty(y_1,y_2,\xi)\leqslant C|\xi|$. Invoking, in addition, (5.2.67), $(\mathcal{F}4)$' and (5.2.69), in this order, we have

$$\begin{aligned}(g^\infty)_{\hom_2}(y_1,\xi) &\leqslant \frac{CL}{t}+\int_{Y_2\cap\{y_2\colon t|\xi+\nabla\psi_t(y_2)|>L\}} \frac{g(y_1,y_2,t\xi+t\nabla\psi_t(y_2))}{t}+C\frac{|\xi+\nabla\psi_t(y_2)|^{1-\alpha}}{t^\alpha}\,\mathrm{d}y_2\\ &\leqslant \frac{C(L+1)}{t}+\int_{Y_2}\frac{g(y_1,y_2,t\xi+t\nabla\psi_t(y_2))}{t}\,\mathrm{d}y_2+\frac{C}{t^\alpha}\int_{Y_2}|\xi+\nabla\psi_t(y_2)|^{1-\alpha}\,\mathrm{d}y_2\\ &\leqslant \frac{C(L+1)+1}{t}+\frac{g_{\hom_2}(y_1,t\xi)}{t}+\frac{C}{t^\alpha}\big(\bar{C}(1+|\xi|)\big)^{1-\alpha},\end{aligned} \tag{5.2.71}$$

where in the last estimate we also used Hölder's Inequality together with (5.2.70). Letting $t\to+\infty$, we conclude that $(g^\infty)_{\hom_2}\leqslant(g_{\hom_2})^\infty$. Thus, (5.2.68) holds. Consequently,

$$\Big((g_{\hom_2})^\infty\Big)_{\hom_1}=\Big((g^\infty)_{\hom_2}\Big)_{\hom_1}=(g^\infty)_{\hom}.$$

Next we show that

$$\Big((g_{\hom_2})^\infty\Big)_{\hom_1}=\big((g_{\hom_2})_{\hom_1}\big)^\infty, \tag{5.2.72}$$

which will finish the proof since, by definition, $\big((g_{\hom_2})_{\hom_1}\big)^\infty=(g_{\hom})^\infty$.

In view of the hypotheses on g and using definition (5.1.3), it can be shown that $g_{\hom_2}:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ is a Borel function satisfying conditions $(\mathcal{F}1)$, $(\mathcal{F}3)$ and $(\mathcal{F}4)$'. If we prove that $g_{\hom_2}$ also satisfies $(\mathcal{F}7)$ then, reasoning as in the proof of (5.2.68), we deduce that (5.2.72) holds.

Let $C,L>0$ and $\alpha\in(0,1)$ be given by $(\mathcal{F}7)$ for g. Fix $y_1\in\mathbb{R}^N$ and $\xi\in\mathbb{R}^{d\times N}$ such that $|\xi|=1$. Let $t\geqslant\tilde{L}:=\max\{1,L\}$. Using (5.2.68) and (5.2.71), we have

$$\begin{aligned}(g_{\hom_2})^\infty(y_1,\xi)-\frac{g_{\hom_2}(y_1,t\xi)}{t} &= (g^\infty)_{\hom_2}(y_1,\xi)-\frac{g_{\hom_2}(y_1,t\xi)}{t}\\ &\leqslant \frac{C(L+1)+1}{t}+\frac{C}{t^\alpha}\big(2\bar{C}\big)^{1-\alpha}\leqslant\frac{C_1}{t^\alpha},\end{aligned} \tag{5.2.73}$$

where C_1 is a positive constant independent of t.

Conversely, for each $0<\delta<1$ we can find $\psi_\delta\in W^{1,1}_\#(Y_2,\mathbb{R}^d)$ such that

$$\int_{Y_2} g^\infty(y_1,y_2,\xi+\nabla\psi_\delta(y_2))\,\mathrm{d}y_2\leqslant(g^\infty)_{\hom_2}(y_1,\xi)+\delta, \tag{5.2.74}$$

so that, in view of $(\mathcal{F}3)$ and $(\mathcal{F}4)$',

$$\frac{1}{C}\int_{Y_2}|\xi+\nabla\psi_\delta(y_2)|\,\mathrm{d}y_2\leqslant C|\xi|+\delta<C+1. \tag{5.2.75}$$

From (5.2.68), (5.2.74) and (5.2.67), and taking into account that $g^\infty \geqslant 0$, we conclude that

$$\begin{aligned} &\frac{g_{\hom_2}(y_1, t\xi)}{t} - (g_{\hom_2})^\infty(y_1,\xi) \\ &\leqslant \int_{Y_2} \frac{g(y_1,y_2,t\xi + t\nabla\psi_\delta(y_2))}{t} - g^\infty(y_1,y_2,\xi+\nabla\psi_\delta(y_2))\,\mathrm{d}y_2 + \delta \\ &\leqslant C\int_{Y_2} \frac{|\xi+\nabla\psi_\delta(y_2)|^{1-\alpha}}{t^\alpha}\,\mathrm{d}y_2 + \int_{Y_2\cap\{y_2\colon t|\xi+\nabla\psi_\delta(y_2)|\leqslant L\}} \frac{g(y_1,y_2,t\xi+t\nabla\psi_\delta(y_2))}{t}\,\mathrm{d}y_2 + \delta \\ &\leqslant \frac{C(C^2+C)^{1-\alpha}}{t^\alpha} + \frac{C(1+L)}{t} + \delta, \end{aligned} \tag{5.2.76}$$

where in the last inequality we also used Hölder's Inequality together with (5.2.75), and ($\mathcal{F}3$). Letting $\delta \to 0^+$ in (5.2.76), using the fact that $t \geqslant t^\alpha$ whenever $t \geqslant 1$ together with (5.2.73), we deduce that $g_{\hom_2}$ satisfies ($\mathcal{F}7$). □

We now prove Proposition 5.1.1.

Proof of Proposition 5.1.1. Without loss of generality we may assume that the parameter $\eta > 0$ takes values on a sequence of positive numbers converging to zero.

(i) We start by observing that for fixed $y_1, y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$, the sequences $\{f_\eta(y_1,y_2,\xi)\}_{\eta>0}$, $\{(f_\eta)^{**}(y_1,y_2,\xi)\}_{\eta>0}$ and $\{((f_\eta)^{**})^\infty(y_1,y_2,\xi)\}_{\eta>0}$ are decreasing (as $\eta\to 0^+$), so that the respective limits as $\eta \to 0^+$ exist and are given by the infimum in $\eta > 0$.

Recalling definition (5.1.2) and in view of ($\mathcal{F}3$) and ($\mathcal{F}4$), we have that the biconjugate function f^{**} of f is such that for all $y_1, y_2 \in \mathbb{R}^N$, $f^{**}(y_1,y_2,\cdot)$ is a convex function which coincides with the convex envelope $\mathcal{C}f(y_1,y_2,\cdot)$ of $f(y_1,y_2,\cdot)$ (see, for example, Fonseca and Leoni [48, Thm. 4.92]). Precisely, for all $(y_1,y_2,\xi) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$,

$$f^{**}(y_1,y_2,\xi) = \mathcal{C}f(y_1,y_2,\xi) := \sup\big\{g(\xi)\colon\ g:\mathbb{R}^{d\times N}\to\mathbb{R} \text{ convex},\ g(\cdot)\leqslant f(y_1,y_2,\cdot)\big\}. \tag{5.2.77}$$

Note that the same holds true for $(f_\eta)^{**}$. Consequently, $((f_\eta)^{**})^\infty$ is a convex function, since the recession function of a convex function is a convex function. Moreover, for all $\eta > 0$, we have that

$$f^{**} \leqslant (f_\eta)^{**} \leqslant f_\eta, \tag{5.2.78}$$

and so, using the fact that the pointwise limit of a sequence of convex functions is a convex function, passing (5.2.78) to the limit as $\eta\to 0^+$ we get

$$\lim_{\eta\to 0^+} (f_\eta)^{**}(y_1,y_2,\xi) = f^{**}(y_1,y_2,\xi). \tag{5.2.79}$$

In view of (5.2.78), $(f^{**})^\infty \leqslant ((f_\eta)^{**})^\infty \leqslant (f_\eta)^\infty$; thus, letting $\eta \to 0^+$ and observing that $(f_\eta)^\infty(y_1,y_2,\xi) = f^\infty(y_1,y_2,\xi) + \eta|\xi|$, we have

$$(f^{**})^\infty(y_1,y_2,\xi) \leqslant ((f_{0^+})^{**})^\infty(y_1,y_2,\xi) \leqslant (f^\infty)^{**}(y_1,y_2,\xi), \tag{5.2.80}$$

where we also used the fact that both functions $(f^{**})^\infty$ and $((f_{0^+})^{**})^\infty$ are convex in the last variable, since the recession function of a convex function is also a convex function. We further observe that $((f_{0^+})^{**})^\infty$ is positively 1-homogeneous in the last variable because it is the pointwise limit of a sequence of positively 1-homogeneous functions in the last variable.

(i)–a) If, in addition, f also satisfies ($\mathcal{F}2$), then $(f^{**})^\infty = f^\infty = (f^\infty)^{**}$, which, together with (5.2.80), implies that $((f_{0^+})^{**})^\infty \equiv f^\infty$.

(i)–b) Assume that $d = 1$ and that, in addition, f also satisfies ($\mathcal{F}7$).

In the scalar case $d = 1$ the notions of convexity and quasiconvexity agree (see, for example, Dacorogna [29, Thms. 5.3, 6.9]), therefore f^{**} is alternatively given by

$$f^{**}(y_1, y_2, \xi) = \inf\left\{ \int_Y f(y_1, y_2, \xi + \nabla\varphi(y))\, dy\colon\ \varphi \in W_0^{1,\infty}(Y) \right\} \tag{5.2.81}$$

for $(y_1, y_2, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$.

Since f_η is a Borel function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)' and ($\mathcal{F}7$), using (5.2.81) and arguing as in the proof of Lemma 5.2.11, it can be shown that $(f_\eta)^{**}$ also satisfies ($\mathcal{F}7$) and that $((f_\eta)^{**})^\infty = ((f_\eta)^\infty)^{**}$. Consequently,

$$((f_{0^+})^{**})^\infty(y_1, y_2, \xi) = \lim_{\eta\to 0^+} ((f_\eta)^{**})^\infty(y_1, y_2, \xi) = \lim_{\eta\to 0^+} ((f_\eta)^\infty)^{**}(y_1, y_2, \xi) = (f^\infty)^{**}(y_1, y_2, \xi), \tag{5.2.82}$$

where the last equality may be proved in a similar way as (5.2.79) (with f replaced by f^∞).

(ii) Just as (i) above, it can be shown that the limit (5.1.6) exists and defines a positively 1-homogeneous convex function $\big(((f_{0^+})^{**})_{\text{hom}}\big)^\infty : \mathbb{R}^N \to \mathbb{R}$.

By (5.1.3), ($\mathcal{F}3$) and ($\mathcal{F}4$), there exists a constant $M > 0$ such that for all y_1, $y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$|f(y_1, y_2, \xi)| \leqslant M(1 + |\xi|), \quad |f_{\text{hom}_2}(y_1, \xi)| \leqslant M(1 + |\xi|), \quad |f_{\text{hom}}(\xi)| \leqslant M(1 + |\xi|). \tag{5.2.83}$$

Using in addition (5.2.79), Lebesgue Dominated Convergence Theorem yields

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})_{\text{hom}_2}(y_1, \xi) \leqslant (f^{**})_{\text{hom}_2},$$

which, together with inequality $((f_\eta)^{**})_{\text{hom}_2} \geqslant (f^{**})_{\text{hom}_2}$, implies that

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})_{\text{hom}_2}(y_1, \xi) = (f^{**})_{\text{hom}_2}(y_1, \xi).$$

Similar arguments ensure that

$$\lim_{\eta\to 0^+} ((f_\eta)^{**})_{\text{hom}}(\xi) = \lim_{\eta\to 0^+} (((f_\eta)^{**})_{\text{hom}_2})_{\text{hom}_1}(\xi) = ((f^{**})_{\text{hom}_2})_{\text{hom}_1}(\xi) = (f^{**})_{\text{hom}}(\xi), \tag{5.2.84}$$

and that

$$\lim_{\eta\to 0^+} \big(((f_\eta)^{**})^\infty\big)_{\text{hom}}(\xi) = \big(((f_{0^+})^{**})^\infty\big)_{\text{hom}}(\xi) \leqslant \big((f^\infty)^{**}\big)_{\text{hom}}(\xi), \tag{5.2.85}$$

with $((f_{0^+})^{**})^\infty$ the function defined by (5.1.5), where in the last inequality we used (5.2.80).

Using the fact that if g is a function satisfying ($\mathcal{F}3$) and ($\mathcal{F}4$) then $(g_{\text{hom}})^\infty \leqslant (g^\infty)_{\text{hom}}$, passing to the limit as $\eta \to 0^+$ the chain of inequalities

$$\big((f^{**})_{\text{hom}}\big)^\infty \leqslant \big(((f_\eta)^{**})_{\text{hom}}\big)^\infty \leqslant \big(((f_\eta)^{**})^\infty\big)_{\text{hom}},$$

from (5.2.85) we obtain

$$\big((f^{**})_{\text{hom}}\big)^\infty(\xi) \leqslant \big(((f_{0^+})^{**})_{\text{hom}}\big)^\infty(\xi) \leqslant \big(((f_{0^+})^{**})^\infty\big)_{\text{hom}}(\xi) \leqslant \big((f^\infty)^{**}\big)_{\text{hom}}(\xi). \tag{5.2.86}$$

(ii)–a) Assume that, in addition, f also satisfies ($\mathcal{F}2$) and ($\mathcal{F}8$).

In this case, from (5.2.86) we get

$$(f_{\text{hom}})^{\infty} \leqslant (f_{0^+,\text{hom}})^{\infty} \leqslant (f^{\infty})_{\text{hom}}, \tag{5.2.87}$$

where $(f_{0^+,\text{hom}})^{\infty} := \big(((f_{0^+})^{**})_{\text{hom}}\big)^{\infty} = \lim_{\eta\to 0^+} \big((f_\eta)_{\text{hom}}\big)^{\infty}(\xi)$, since $(f_\eta)^{**} = f_\eta$. To conclude that $(f_{0^+,\text{hom}})^{\infty} \equiv (f_{\text{hom}})^{\infty} = (f^{\infty})_{\text{hom}}$ it suffices to apply Lemma 5.2.10 to f, taking into account (5.2.87).

(ii)–b) Assume that, in addition, f also satisfies ($\mathcal{F}2$) and ($\mathcal{F}7$).

As before, using (5.1.3), equality $(f_\eta)^{\infty}(y_1, y_2, \xi) = f^{\infty}(y_1, y_2, \xi) + \eta|\xi|$, and Lebesgue Dominated Convergence Theorem together with (5.2.83), we obtain

$$\lim_{\eta\to 0^+} ((f_\eta)^{\infty})_{\text{hom}}(\xi) = (f^{\infty})_{\text{hom}}(\xi). \tag{5.2.88}$$

By Lemma 5.2.11 applied to f_η, we conclude that for all $\eta > 0$, $((f_\eta)_{\text{hom}})^{\infty} = ((f_\eta)^{\infty})_{\text{hom}}$, which, together with (5.2.88), yields $(f_{0^+,\text{hom}})^{\infty} \equiv (f^{\infty})_{\text{hom}}$.

(ii)–c) Assume that $d = 1$ and that, in addition, f also satisfies ($\mathcal{F}7$) (with $d = 1$).

As we observed in (i)–b), $(f_\eta)^{**}$ is a Borel function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)' and ($\mathcal{F}7$). Applying Lemma 5.2.11 to $(f_\eta)^{**}$, using the first equality in (5.2.85) and by (5.2.82),

$$\begin{aligned}\big(((f_{0^+})^{**})_{\text{hom}}\big)^{\infty}(\xi) &= \lim_{\eta\to 0^+} \big(((f_\eta)^{**})_{\text{hom}}\big)^{\infty}(\xi) = \lim_{\eta\to 0^+} \big(((f_\eta)^{**})^{\infty}\big)_{\text{hom}}(\xi)\\ &= \big(((f_{0^+})^{**})^{\infty}\big)_{\text{hom}}(\xi) = \big((f^{\infty})^{**}\big)_{\text{hom}}(\xi).\end{aligned}$$

This concludes the proof of Proposition 5.1.1. □

We finally prove Theorem 5.1.3.

Proof of Theorem 5.1.3. By Theorem 5.2.1 and Remark 5.2.8, we have that (5.1.7) holds.

We observe that in view of ($\mathcal{F}1$)–($\mathcal{F}4$), we have that both f_{hom_2} and f_{hom} are real-valued Borel functions, satisfying ($\mathcal{F}1$)–($\mathcal{F}4$), and we can find a constant $M > 0$ such that for all y_1, $y_2 \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$,

$$|f(y_1, y_2, \xi)| \leqslant M(1 + |\xi|), \quad |f_{\text{hom}_2}(y_1, \xi)| \leqslant M(1 + |\xi|), \quad |f_{\text{hom}}(\xi)| \leqslant M(1 + |\xi|). \tag{5.2.89}$$

Moreover, since ($\mathcal{F}4$) holds for f_{hom},

$$\liminf_{|\xi|\to+\infty} \frac{f_{\text{hom}}(\xi)}{|\xi|} \geqslant 0. \tag{5.2.90}$$

The first equality in (5.1.8) is given by Theorem 5.2.7 (see also Remark 5.2.8). To prove the second equality in (5.1.8) we will proceed in several steps.

Step 1. We show that for all $u \in BV(\Omega;\mathbb{R}^d)$,

$$\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_{\#}(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_{\#}(Y_2;\mathbb{R}^d))}} F^{\text{sc}}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \geqslant \int_\Omega f_{\text{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega (f_{\text{hom}})^{\infty}\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x). \tag{5.2.91}$$

Fix $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\in BV(\Omega;\mathbb{R}^d)\times\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\times\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))$, and let $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)$, $\{\psi_j^{(1)}\}_{j\in\mathbb{N}}\subset C_c^\infty(\Omega;C_\#^\infty(Y_1;\mathbb{R}^d))$ and $\{\psi_j^{(2)}\}_{j\in\mathbb{N}}\subset C_c^\infty(\Omega;C_\#^\infty(Y_1\times Y_2;\mathbb{R}^d))$ be sequences given by Proposition 4.2.10.

By (5.1.7), applying Lemma 5.2.5 to $U:=\Omega\times Y_1\times Y_2$ and $g(x,y_1,y_2,\xi):=f(y_1,y_2,\xi)$ (see also Remark 5.1.5 (ii)), and using the definitions of f_{hom_2} and f_{hom} together with Fubini's Theorem, we conclude that

$$\begin{aligned}
F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)&=\int_{\Omega\times Y_1\times Y_2} f\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad+\int_{\Omega\times Y_1\times Y_2} f^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2)\\
&=\lim_{j\to+\infty}\int_{\Omega\times Y_1\times Y_2} f\big(y_1,y_2,\nabla u_j(x)+\big(\nabla_{y_1}\psi_j^{(1)}\big)(x,y_1)+\big(\nabla_{y_2}\psi_j^{(2)}\big)(x,y_1,y_2)\big)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\geqslant \liminf_{j\to+\infty}\int_{\Omega\times Y_1} f_{\mathrm{hom}_2}\big(y_1,\nabla u_j(x)+\big(\nabla_{y_1}\psi_j^{(1)}\big)(x,y_1)\big)\,\mathrm{d}x\mathrm{d}y_1\\
&\geqslant \liminf_{j\to+\infty}\int_\Omega f_{\mathrm{hom}}(\nabla u_j(x))\,\mathrm{d}x\\
&\geqslant \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+\int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x),
\end{aligned}$$

where in the last inequality we have used [48, Thm. 5.21] (see also Remark 5.1.5 (i)) taking into account that $\nabla u_j\mathcal{L}^N\lfloor_\Omega \overset{\star}{\rightharpoonup} Du$ weakly-$\star$ in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ as $j\to+\infty$, and that f_{hom} is a real-valued convex function satisfying (5.2.90). Taking the infimum over all $\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))$ and $\boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))$, we obtain (5.2.91).

Step 2. We prove that for all $u\in W^{1,1}(\Omega;\mathbb{R}^d)$,

$$\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\leqslant\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x. \tag{5.2.92}$$

Fix $\eta>0$, and let $0<\tau<\eta$ be such that for all measurable sets $D\subset\Omega$ with $\mathcal{L}^N(D)\leqslant\tau$,

$$\int_D (1+|\nabla u(x)|)\,\mathrm{d}x\leqslant\eta. \tag{5.2.93}$$

In view of (5.2.83), without loss of generality we may assume that for all $x\in\Omega$,

$$f_{\mathrm{hom}}(\nabla u(x))\in\mathbb{R}. \tag{5.2.94}$$

Fix $0<\delta<\tau$, and consider the multifunction $\Gamma_1^\delta:\Omega\to 2^{W^{1,1}_\#(Y_1;\mathbb{R}^d)}$ defined, for each $x\in\Omega$, by

$$\Gamma_1^\delta(x):=\Big\{\psi_1\in W^{1,1}_\#(Y_1;\mathbb{R}^d)\colon\ \int_{Y_1} f_{\mathrm{hom}_2}(y_1,\nabla u(x)+\nabla\psi_1(y_1))\,\mathrm{d}y_1<f_{\mathrm{hom}}(\nabla u(x))+\delta\Big\}.$$

By (5.2.94), for all $x\in\Omega$ one has $\Gamma_1^\delta(x)\neq\emptyset$. Moreover, if $\{\psi_j\}_{j\in\mathbb{N}}\subset W^{1,1}_\#(Y_1;\mathbb{R}^d)\backslash\Gamma_1^\delta(x)$ is a sequence converging in $W^{1,1}_\#(Y_1;\mathbb{R}^d)$ to some ψ, then, taking into account (5.2.83) and the continuity of $f_{\mathrm{hom}_2}(y_1,\cdot)$, by Lebesgue Dominated Convergence Theorem we deduce that $\psi\in W^{1,1}_\#(Y_1;\mathbb{R}^d)\backslash\Gamma_1^\delta(x)$. Thus, $\Gamma_1^\delta(x)$ is an open subset of $W^{1,1}_\#(Y_1;\mathbb{R}^d)$. Furthermore, given $\psi_1\in W^{1,1}_\#(Y_1;\mathbb{R}^d)$, the measurability of the function

$$x\mapsto\int_{Y_1} f_{\mathrm{hom}_2}(y_1,\nabla u(x)+\nabla\psi_1(y_1))\,\mathrm{d}y_1-f_{\mathrm{hom}}(\nabla u(x))-\delta$$

ensures the measurability of the set $\{x \in \Omega : \ \psi_1 \in \Gamma_1^\delta(x)\}$. Thus, by Lemma 2.1.22 we can find a measurable selection $\bar\psi_1 : \Omega \to W^{1,1}_\#(Y_1;\mathbb{R}^d)$ of Γ_1^δ. Moreover, by Lusin's Theorem, $\bar\psi_1 \in L^1(\Omega_\delta; W^{1,1}_\#(Y_1;\mathbb{R}^d))$ for a suitable measurable set $\Omega_\delta \subset \Omega$ such that $\mathcal{L}^N(\Omega\backslash\Omega_\delta) \leqslant \delta$. Since for a.e. $x \in \Omega_\delta$ one has $\bar\psi_1(x) \in \Gamma_1^\delta(x)$, in view of (5.2.83) and (5.2.93) we obtain

$$\int_{\Omega_\delta\times Y_1} f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1))\,\mathrm{d}x\mathrm{d}y_1 \leqslant \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + M\eta + \eta\mathcal{L}^N(\Omega), \tag{5.2.95}$$

where we also used the fact that $0 < \delta < \tau < \eta$.

Similarly, let $0 < \bar\tau < \delta$ be such that for all measurable sets $E \subset \Omega_\delta \times Y$ with $\mathcal{L}^{2N}(E) \leqslant \bar\tau$,

$$\int_E (1 + |\nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1)|)\,\mathrm{d}x\mathrm{d}y_1 \leqslant \eta. \tag{5.2.96}$$

As before, we may assume without loss of generality that for all $(x,y_1) \in \Omega_\delta \times Y_1$ we have $f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1)) \in \mathbb{R}$. Moreover, fixed $0 < \gamma < \bar\tau$, the multifunction $\Gamma_2^\gamma : \Omega_\delta \times Y_1 \to 2^{W^{1,1}_\#(Y_2;\mathbb{R}^d)}$ defined, for each $(x,y_1) \in \Omega_\delta \times Y_1$, by

$$\begin{aligned}\Gamma_2^\gamma(x,y_1) := \Bigg\{\psi_2 \in W^{1,1}_\#(Y_2;\mathbb{R}^d)\colon \int_{Y_2} f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1) + \nabla\psi_2(y_2))\,\mathrm{d}y_2 \\ < f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1)) + \gamma\Bigg\},\end{aligned}$$

is such that for all $(x,y_1) \in \Omega_\delta \times Y_1$, $\Gamma_2^\gamma(x,y_1)$ is a nonempty and open subset of $W^{1,1}_\#(Y_2;\mathbb{R}^d)$, and for all $\psi_2 \in W^{1,1}_\#(Y_2;\mathbb{R}^d)$, the set $\{(x,y_1) \in \Omega_\delta \times Y_1 : \ \psi_2 \in \Gamma_2^\gamma(x,y_1)\}$ is measurable. Hence, by Lemma 2.1.22 we can find a measurable selection $\bar\psi_2 : \Omega_\delta \times Y_1 \to W^{1,1}_\#(Y_2;\mathbb{R}^d)$ of Γ_2^γ. Moreover, by Lusin's Theorem, $\bar\psi_2 \in L^1(E_\gamma; W^{1,1}_\#(Y_2;\mathbb{R}^d))$ for a suitable measurable set $E_\gamma \subset \Omega_\delta \times Y_1$ such that $\mathcal{L}^N(\Omega_\delta \times Y_1\backslash E_\gamma) \leqslant \gamma$. Since for a.e. $(x,y_1) \in E_\gamma$ one has $\bar\psi_2(x,y_1) \in \Gamma_2^\gamma(x,y_1)$, in view of (5.2.83) and (5.2.96) we get

$$\begin{aligned}&\int_{E_\gamma\times Y_2} f(y_1,y_2,\nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1) + \nabla_{y_2}\bar\psi_2(x,y_1y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\qquad\leqslant \int_{\Omega_\delta\times Y_1} f_{\mathrm{hom}_2}(y_1, \nabla u(x) + \nabla_{y_1}\bar\psi_1(x,y_1))\,\mathrm{d}x\mathrm{d}y_1 + M\eta + \eta\mathcal{L}^N(\Omega).\end{aligned} \tag{5.2.97}$$

Finally, define $\psi_1 \in L^1(\Omega; W^{1,1}_\#(Y_1;\mathbb{R}^d))$, $\psi_2 \in L^1(\Omega\times Y_1; W^{1,1}_\#(Y_2;\mathbb{R}^d))$ by setting $\psi_1(x) := \bar\psi_1(x)$ if $x \in \Omega_\delta$, $\psi_1(x) := 0$ if $x \in \Omega\backslash\Omega_\delta$, $\psi_2(x,y_1) := \bar\psi_2(x,y_1)$ if $(x,y_1) \in E_\gamma$, and $\psi_2(x,y_1) := 0$ if $(x,y_1) \in (\Omega\times Y_1)\backslash E_\gamma$. Using the usual identification of an integrable function with a measure, elements of $L^1(\Omega; W^{1,1}_\#(Y_1;\mathbb{R}^d))$ and $L^1(\Omega\times Y_1; W^{1,1}_\#(Y_2;\mathbb{R}^d))$ can be seen as elements of $\mathcal{M}_\star(\Omega; BV_\#(Y_1;\mathbb{R}^d))$ and $\mathcal{M}_\star(\Omega\times Y_1; BV_\#(Y_2;\mathbb{R}^d))$, respectively. Considering this identification (see also (4.1.3)), we have

$$\lambda_{u,\psi_1,\psi_2\lfloor\Omega\times Y_1\times Y_2} = \nabla u\mathcal{L}^{3N}_{\lfloor\Omega\times Y_1\times Y_2} + \nabla_{y_1}\psi_1\mathcal{L}^{3N}_{\lfloor\Omega\times Y_1\times Y_2} + \nabla_{y_2}\psi_2\mathcal{L}^{3N}_{\lfloor\Omega\times Y_1\times Y_2}. \tag{5.2.98}$$

From (5.1.7), (5.2.98), (5.2.83), (5.2.93), (5.2.96), (5.2.97) and (5.2.95), in this order, we deduce that

$$\begin{aligned}
&\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\\
&\quad= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} \bigg\{\int_{\Omega\times Y_1\times Y_2} f\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\qquad\qquad+\int_{\Omega\times Y_1\times Y_2} f^\infty\Big(y_1,y_2,\frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2)\bigg\}\\
&\quad\leqslant \int_{\Omega\times Y_1\times Y_2} f(y_1,y_2,\nabla u(x)+\nabla_{y_1}\psi_1(x,y_1)+\nabla_{y_2}\psi_2(x,y_1,y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad= \int_{(\Omega\backslash\Omega_\delta)\times Y_1\times Y_2} f(y_1,y_2,\nabla u(x))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\qquad+\int_{((\Omega_\delta\times Y_1)\backslash E_\gamma)\times Y_2} f(y_1,y_2,\nabla u(x)+\nabla_{y_1}\bar\psi_1(x,y_1))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\qquad+\int_{E_\gamma\times Y_2} f(y_1,y_2,\nabla u(x)+\nabla_{y_1}\bar\psi_1(x,y_1)+\nabla_{y_2}\bar\psi_2(x,y_1y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\
&\quad\leqslant 2M\eta+\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+2\big(M\eta+\eta\mathcal{L}^N(\Omega)\big).
\end{aligned}$$

Letting $\eta\to0^+$, we obtain (5.2.92).

Step 3. We prove that if ($\mathcal{F}$4)' is satisfied, then the converse of (5.2.91) holds for all $u\in BV(\Omega;\mathbb{R}^d)$.

Indeed, let $u\in BV(\Omega;\mathbb{R}^d)$. Since $f_{\mathrm{hom}}:\mathbb{R}^{d\times N}\to\mathbb{R}$ is a convex function satisfying (5.2.83), in view of Lemma 5.2.9 for all $\eta>0$ we can find a sequence $\{u_j\}_{j\in\mathbb{N}}\subset W^{1,1}(\Omega;\mathbb{R}^d)$ weakly-$\star$ converging to u in $BV(\Omega;\mathbb{R}^d)$ and such that

$$\lim_{j\to+\infty}\int_\Omega f_{\mathrm{hom}}(\nabla u_j(x))\,\mathrm{d}x\leqslant\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+\int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^su}{\mathrm{d}\|D^su\|}(x)\Big)\,\mathrm{d}\|D^su\|(x)+\eta.$$

Under the present hypotheses on f, it can be checked that F^{hom} is sequentially lower semicontinuous with respect to the weak-$\star$ convergence in $BV(\Omega;\mathbb{R}^d)$. Hence, using Theorem 5.2.7 and (5.2.92),

$$\begin{aligned}
&\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\leqslant\liminf_{j\to+\infty}\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}(u_j,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\\
&\quad\leqslant\lim_{j\to+\infty}\int_\Omega f_{\mathrm{hom}}(\nabla u_j(x))\,\mathrm{d}x\leqslant\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+\int_\Omega (f_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^su}{\mathrm{d}\|D^su\|}(x)\Big)\,\mathrm{d}\|D^su\|(x)+\eta,
\end{aligned}$$

from which we conclude Step 3 by letting $\eta\to0^+$.

Step 4. We establish the second equality in (5.1.8).

Let $u\in BV(\Omega;\mathbb{R}^d)$, and fix $\eta>0$ (which, without loss of generality, we assume will take values on a sequence of positive numbers converging to zero). Then f_η (we recall, $f_\eta(y_1,y_2,\xi):=f(y_1,y_2,\xi)+\eta|\xi|$) satisfies conditions ($\mathcal{F}$1)–($\mathcal{F}$3), ($\mathcal{F}$4)', ($\mathcal{F}$5); condition ($\mathcal{F}$6), which was only used in Lemma 5.2.2, reads slightly different for f_η than for f (see (5.2.8)), but it can be checked that this difference is innocuous. So, in view of Steps 1, 2 and 3 applied to f_η,

$$\inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\mathrm{sc}}_\eta(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)=\int_\Omega f_{\eta,\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x+\int_\Omega (f_{\eta,\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^su}{\mathrm{d}\|D^su\|}(x)\Big)\,\mathrm{d}\|D^su\|(x),\tag{5.2.99}$$

where $F_\eta^{\rm sc}$ is the functional given by (5.2.44), and where $f_{\eta,\rm hom} := (f_\eta)_{\rm hom}$.

In order to pass (5.2.99) to the limit as $\eta \to 0^+$, we start by observing that for fixed $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV(\Omega;\mathbb{R}^d) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1;\mathbb{R}^d)) \times \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2;\mathbb{R}^d))$, $\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ has finite total variation and $\{F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\}_{\eta>0}$ is a bounded decreasing sequence, and so

$$\begin{aligned}
\lim_{\eta\to 0^+} \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= \inf_{\eta>0} \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\\
= \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} \inf_{\eta>0} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= \inf_{\substack{\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2).
\end{aligned} \tag{5.2.100}$$

Furthermore, using Lebesgue Dominated Convergence Theorem together with (5.2.89), in view of (5.2.84) (observing that thanks to ($\mathcal{F}2$), $f^{**} = f$ and $(f_\eta)^{**} = f_\eta$) and of (5.1.6) we get

$$\lim_{\eta\to 0^+} \int_\Omega f_{\eta,\rm hom}(\nabla u(x))\,{\rm d}x = \int_\Omega f_{\rm hom}(\nabla u(x))\,{\rm d}x, \tag{5.2.101}$$

and

$$\lim_{\eta\to 0^+} \int_\Omega (f_{\eta,\rm hom})^\infty\Big(\frac{{\rm d}D^s u}{{\rm d}\|D^s u\|}(x)\Big)\,{\rm d}\|D^s u\|(x) = \int_\Omega (f_{0^+,\rm hom})^\infty\Big(\frac{{\rm d}D^s u}{{\rm d}\|D^s u\|}(x)\Big)\,{\rm d}\|D^s u\|(x). \tag{5.2.102}$$

From (5.2.99), (5.2.100), (5.2.101) and (5.2.102), we conclude Step 4.

Finally, we observe that

a) if, in addition, f satisfies ($\mathcal{F}4$)', then by Step 1–Step 4, we have that $(f_{0^+,\rm hom})^\infty \equiv (f_{\rm hom})^\infty$;

b) if, in addition, f satisfies ($\mathcal{F}8$), then by Proposition 5.1.1 (ii)–a), $(f_{0^+,\rm hom})^\infty \equiv (f_{\rm hom})^\infty = (f^\infty)_{\rm hom}$;

c) if, in addition, f satisfies ($\mathcal{F}7$), then Proposition 5.1.1 (ii)–b) yields $(f_{0^+,\rm hom})^\infty \equiv (f^\infty)_{\rm hom}$. □

5.3. Proof of Corollary 5.1.4.

As in the previous section, below we will assume, without loss of generality ,that $n = 2$, since the generalization to an arbitrary $n \in \mathbb{N}$ does not bring any additional technical difficulties.

The proof of Corollary 5.1.4 relies on Theorems 5.1.3 and on the next lemma concerning properties inherited by f^{**} from f.

Lemma 5.3.1. *Assume that $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)', ($\mathcal{F}5$) and ($\mathcal{F}6$) with $d = 1$. Then the biconjugate function f^{**} of f is a real-valued Borel function in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, and verifies conditions ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$)', ($\mathcal{F}5$) and ($\mathcal{F}6$) with $d = 1$.*

Proof. By (5.2.77) and since $f_1 \leqslant f_2$ implies that $\mathcal{C}f_1 \leqslant \mathcal{C}f_2$, the only nontrivial condition to verify is ($\mathcal{F}5$).

Fix $(y_1', y_2') \in \mathbb{R}^N \times \mathbb{R}^N$ and $\delta > 0$ arbitrarily. Set $\bar\delta := \delta/(1 + 2C^2)$, where C is given by ($\mathcal{F}3$) and ($\mathcal{F}4$)', and let $\bar\tau = \bar\tau(y_1', y_2', \bar\delta)$ be given by ($\mathcal{F}5$) for f and for such $\bar\delta$.

Fix $\xi \in \mathbb{R}^N$ and $(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$ such that $|(y_1', y_2') - (y_1, y_2)| \leqslant \bar\tau$. By (5.2.81), for each $\epsilon > 0$ we can find $\varphi_\epsilon \in W_0^{1,\infty}(Y)$ such that

$$\int_Y f(y_1, y_2, \xi + \nabla\varphi_\epsilon(y))\,{\rm d}y \leqslant f^{**}(y_1, y_2, \xi) + \epsilon, \tag{5.3.1}$$

and so,

$$
\begin{aligned}
f^{**}(y_1',y_2',\xi)-f^{**}(y_1,y_2,\xi) &\leqslant \int_Y \Big(f(y_1',y_2',\xi+\nabla\varphi_\epsilon(y))-f(y_1,y_2,\xi+\nabla\varphi_\epsilon(y))\Big)\,\mathrm{d}y+\epsilon \\
&\leqslant \int_Y \bar\delta(1+|\xi+\nabla\varphi_\epsilon(y)|)\,\mathrm{d}y+\epsilon,
\end{aligned}
\tag{5.3.2}
$$

where in the last inequality we used $(\mathcal{F}5)$ for f.

In view of (5.3.1), $(\mathcal{F}3)$ and $(\mathcal{F}4)$', we have that $\frac{1}{C}\|\xi+\nabla\varphi_\epsilon\|_{L^1(Y;\mathbb{R}^N)}-C\leqslant C(1+|\xi|)+\epsilon$. Thus, from (5.3.2) we deduce that

$$
f^{**}(y_1',y_2',\xi)-f^{**}(y_1,y_2,\xi)\leqslant \bar\delta(1+C^2(2+|\xi|))+(\bar\delta C+1)\epsilon\leqslant \delta(1+|\xi|)+(\delta C+1)\epsilon.
$$

Letting $\epsilon\to 0^+$, we conclude that

$$
f^{**}(y_1',y_2',\xi)-f^{**}(y_1,y_2,\xi)\leqslant \delta(1+|\xi|).
$$

Interchanging the roles between (y_1',y_2',ξ) and (y_1,y_2,ξ), we prove that $f^{**}(y_1,y_2,\xi)-f^{**}(y_1',y_2',\xi)$ $\leqslant \delta(1+|\xi|)$ also holds. Thus f^{**} satisfies $(\mathcal{F}5)$. ☐

Proof of Corollary 5.1.4. We proceed in two steps.

Step 1. We prove that if in addition f satisfies $(\mathcal{F}4)$', then (5.1.9) holds with $(f_{0^+}^{**})^\infty$ replaced by $(f^{**})^\infty$, and (5.1.10) holds with $\big(((f_{0^+})^{**})_{\mathrm{hom}}\big)^\infty$ replaced by $\big((f^{**})_{\mathrm{hom}}\big)^\infty$.

Substep 1.1. We show that the infima (5.0.1) and (5.0.2) remain unchanged if we substitute f by its biconjugate function f^{**}.

Fix $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\in BV(\Omega)\times\mathcal{M}_\star(\Omega;BV_\#(Y_1))\times\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$, and define

$$
F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2):=\inf\Big\{\liminf_{\varepsilon\to0^+}F_\varepsilon^{**}(u_\varepsilon)\,:\,u_\varepsilon\in BV(\Omega),\ Du_\varepsilon\xrightharpoonup[\varepsilon]{3\text{-}sc}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\Big\}
$$

and

$$
F^{**,\mathrm{hom}}(u):=\inf\Big\{\liminf_{\varepsilon\to0^+}F_\varepsilon^{**}(u_\varepsilon)\colon\ u_\varepsilon\in BV(\Omega),\ u_\varepsilon\overset{\star}{\rightharpoonup}_\varepsilon u \text{ weakly-}\star\text{ in } BV(\Omega)\Big\},
$$

where F_ε^{**} is the functional given by (5.2.1) for $d=1$ and with f replaced by f^{**}.

Notice that by Lemma 5.3.1 and Remark 5.1.5 (ii), f^{**} is a real-valued continuous function in $\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$ satisfying conditions $(\mathcal{F}1)$, $(\mathcal{F}3)$, $(\mathcal{F}4)$', $(\mathcal{F}5)$ and $(\mathcal{F}6)$ with $d=1$.

Since $f^{**}\leqslant f$, we have that $F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\leqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$ and $F^{**,\mathrm{hom}}(u)\leqslant F^{\mathrm{hom}}(u)$. To prove the opposite inequalities, we start by observing that in view of (5.2.38)–(5.2.42) the following equalities hold:

$$
\begin{aligned}
F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)&=\inf\Big\{\liminf_{\varepsilon\to0^+}F_\varepsilon^{**}(u_\varepsilon)\,:\,u_\varepsilon\in W^{1,1}(\Omega),\ \nabla u_\varepsilon\mathcal{L}^N\lfloor_\Omega\xrightharpoonup[\varepsilon]{3\text{-}sc}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\Big\}\\
&=\inf\Big\{\liminf_{\varepsilon\to0^+}\int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon)},\frac{x}{\varrho_2(\varepsilon)},\nabla u_\varepsilon(x)\Big)\mathrm{d}x\,:\,u_\varepsilon\in W^{1,1}(\Omega),\ \nabla u_\varepsilon\mathcal{L}^N\lfloor_\Omega\xrightharpoonup[\varepsilon]{3\text{-}sc}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\Big\}.
\end{aligned}
$$

Moreover, a similar argument to (5.2.38)–(5.2.42) ensures that also

$$
\begin{aligned}
F^{**,\mathrm{hom}}(u)&=\inf\Big\{\liminf_{\varepsilon\to0^+}F_\varepsilon^{**}(u_\varepsilon)\colon\ u_\varepsilon\in W^{1,1}(\Omega),\ u_\varepsilon\overset{\star}{\rightharpoonup}_\varepsilon u\text{ weakly-}\star\text{ in }BV(\Omega)\Big\}\\
&=\inf\Big\{\liminf_{\varepsilon\to0^+}\int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon)},\frac{x}{\varrho_2(\varepsilon)},\nabla u_\varepsilon(x)\Big)\mathrm{d}x\colon\ u_\varepsilon\in W^{1,1}(\Omega),\ u_\varepsilon\overset{\star}{\rightharpoonup}_\varepsilon u\text{ weakly-}\star\text{ in }BV(\Omega)\Big\}.
\end{aligned}
$$

Fix $\delta > 0$. We can find a sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$ of positive numbers converging to zero as $h \to +\infty$, and a sequence $\{u_h\}_{h\in\mathbb{N}} \subset W^{1,1}(\Omega)$ such that $\nabla u_h \mathcal{L}^N{\lfloor_\Omega} \xrightarrow[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ and

$$F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) + \delta \geqslant \lim_{h\to+\infty} \int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_h(x)\Big)\mathrm{d}x.$$

On the other hand (see, for example, Marcellini and Sbordone [61, Cor. 3.13]; see also Ekeland and Témam [41, Chapter X]), since f is a continuous function satisfying ($\mathcal{F}3$) and ($\mathcal{F}4$)', for each $h \in \mathbb{N}$ there exist a sequence $\big\{u_j^{(h)}\big\}_{j\in\mathbb{N}} \subset W^{1,1}(\Omega)$ weakly converging to u_h in $W^{1,1}(\Omega)$ and such that

$$\int_\Omega f^{**}\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_h(x)\Big)\mathrm{d}x = \lim_{j\to+\infty} \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_j^{(h)}(x)\Big)\mathrm{d}x.$$

Hence,

$$F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) + \delta \geqslant \lim_{h\to+\infty}\lim_{j\to+\infty} \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla u_j^{(h)}(x)\Big)\mathrm{d}x, \tag{5.3.3}$$

and for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times Y_2; \mathbb{R}^N))$,

$$\begin{aligned}\lim_{h\to+\infty}\lim_{j\to+\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big)\cdot \nabla u_j^{(h)}(x)\,\mathrm{d}x &= \lim_{h\to+\infty} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}\Big)\cdot \nabla u_h(x)\,\mathrm{d}x \\ &= \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(x,y_1,y_2).\end{aligned} \tag{5.3.4}$$

Using a diagonalization argument and the separability of $C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))$, from (5.3.3), (5.3.4) and ($\mathcal{F}4$)' we can find a sequence $\{j_h\}_{h\in\mathbb{N}}$ such that $j_h \to +\infty$ as $h \to +\infty$, $v_h := u_{j_h}^{(h)} \in W^{1,1}(\Omega)$, $\nabla v_h \mathcal{L}^N{\lfloor_\Omega} \xrightarrow[\varepsilon_h]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$ and

$$F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) + \delta \geqslant \lim_{h\to+\infty} \int_\Omega f\Big(\frac{x}{\varrho_1(\varepsilon_h)}, \frac{x}{\varrho_2(\varepsilon_h)}, \nabla v_h(x)\Big)\mathrm{d}x \geqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2),$$

where in the last inequality we used the definition of $F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$. Letting $\delta \to 0^+$, we conclude that $F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \geqslant F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$.

The proof of inequality $F^{**,\mathrm{hom}}(u) \geqslant F^{\mathrm{hom}}(u)$ is similar. Thus, we conclude that $F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) = F^{\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$ and $F^{**,\mathrm{hom}}(u) = F^{\mathrm{hom}}(u)$.

Substep 1.2. Finally, we observe that in view of Theorem 5.1.3 (i) and Lemma 5.3.1, we have that for all $(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \in BV(\Omega) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1)) \times \mathcal{M}_\star(\Omega\times Y_1; BV_\#(Y_2))$,

$$\begin{aligned}F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) := &\int_{\Omega\times Y_1\times Y_2} f^{**}\Big(y_1, y_2, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x,y_1,y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\Omega\times Y_1\times Y_2} (f^{**})^\infty\Big(y_1, y_2, \frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x,y_1,y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x,y_1,y_2)\end{aligned}$$

and

$$\begin{aligned}F^{**,\mathrm{hom}}(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1)) \\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))}} F^{**,\mathrm{sc}}(u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2) \\ &= \int_\Omega (f^{**})_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x + \int_\Omega ((f^{**})_{\mathrm{hom}})^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x),\end{aligned}$$

and this, together with Substep 1.1, completes the proof of Step 1.

Step 2. We establish Corollary 5.1.4.

Fix $\eta > 0$ (which, without loss of generality, we assume will take values on a sequence of positive numbers converging to zero), and let $F_\eta^{\rm sc}$ and $F_\eta^{\rm hom}$ be the functionals given by (5.0.1) and (5.0.2) for $d = 1$, respectively, with f replaced by f_η.

Assuming ($\mathcal{F}6$) with $o(1)$ replaced by $-|o(1)|$ in ($\mathcal{F}6$), it can be shown that we may use Step 1 for f_η. Thus, for every $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV(\Omega) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1)) \times \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$,

$$\begin{aligned} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = &\int_{\Omega\times Y_1\times Y_2} (f_\eta)^{**}\Big(y_1, y_2, \frac{\mathrm{d}\lambda^{ac}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\mathcal{L}^{3N}}(x, y_1, y_2)\Big)\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\Omega\times Y_1\times Y_2} ((f_\eta)^{**})^\infty\Big(y_1, y_2, \frac{\mathrm{d}\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}}{\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|}(x, y_1, y_2)\Big)\,\mathrm{d}\|\lambda^{s}_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}\|(x, y_1, y_2) \end{aligned} \tag{5.3.5}$$

and

$$\begin{aligned} F_\eta^{\rm hom}(u) &= \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))}} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \\ &= \int_\Omega ((f_\eta)^{**})_{\rm hom}(\nabla u(x))\,\mathrm{d}x + \int_\Omega \big(((f_\eta)^{**})_{\rm hom}\big)^\infty\Big(\frac{\mathrm{d}D^s u}{\mathrm{d}\|D^s u\|}(x)\Big)\,\mathrm{d}\|D^s u\|(x). \end{aligned} \tag{5.3.6}$$

In order to pass (5.3.5) and (5.3.6) to the limit as $\eta \to 0^+$, we start by observing that for fixed $(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in BV(\Omega) \times \mathcal{M}_\star(\Omega; BV_\#(Y_1)) \times \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$, the sequences $\{F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\}_{\eta>0}$ and $\{F_\eta^{\rm hom}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)\}_{\eta>0}$ are decreasing (as $\eta \to 0^+$), so that the respective limits as $\eta \to 0^+$ exist and are given by the infimum in $\eta > 0$.

Let $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega)$ be such that $Du_\varepsilon \xrightharpoonup[\varepsilon]{3\text{-}sc} \lambda_{u,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}$. Then $\{Du_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathcal{M}(\Omega; \mathbb{R}^N)$ (see Remark 4.1.5), and so since $(f_\eta)^\infty(y_1, y_2, \xi) = f^\infty(y_1, y_2, \xi) + \eta|\xi|$, we have

$$F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \leqslant \liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon) + \eta C,$$

where C is a constant independent of ε. Letting $\eta \to 0^+$ and then taking the infimum over all such sequences $\{u_\varepsilon\}_{\varepsilon>0}$, we conclude that $\lim_{\eta\to0^+} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \leqslant F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$. Conversely, since for all $\eta > 0$, $f_\eta \geqslant f$, we have that $F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \geqslant F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$. Hence,

$$\lim_{\eta\to0^+} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{5.3.7}$$

Similar arguments ensure that

$$\lim_{\eta\to0^+} F_\eta^{\rm hom}(u) = F^{\rm hom}(u). \tag{5.3.8}$$

Moreover, as in (5.2.100),

$$\lim_{\eta\to0^+} \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F_\eta^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = \inf_{\substack{\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1;\mathbb{R}^d))\\ \boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2;\mathbb{R}^d))}} F^{\rm sc}(u, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2). \tag{5.3.9}$$

So, letting $\eta \to 0^+$ in (5.3.5) and (5.3.6), thanks to (5.3.7), (5.3.8), (5.3.9), (5.2.79), (5.2.80), (5.2.84), (5.2.86) and Lebesgue Dominated Convergence Theorem together with ($\mathcal{F}3$) and ($\mathcal{F}4$), we obtain (5.1.9) and (5.1.10).

Finally, we observe that in view of Step 1, if f satisfies in addition ($\mathcal{F}4$)', then

$$((f_{0^+})^{**})^\infty \equiv (f^{**})^\infty \text{ and } \big(((f_{0^+})^{**})_{\rm hom}\big)^\infty \equiv \big((f^{**})_{\rm hom}\big)^\infty.$$

Moreover, if, in addition to ($\mathcal{F}1$), ($\mathcal{F}3$), ($\mathcal{F}4$), ($\mathcal{F}5$) and ($\mathcal{F}6$), with $o(1)$ replaced by $-|o(1)|$ in ($\mathcal{F}6$), f satisfies the condition ($\mathcal{F}7$), then by Proposition 5.1.1 (i)–b) and (ii)–c),

$$((f_{0^+})^{**})^\infty \equiv (f^\infty)^{**} \text{ and } \big(((f_{0^+})^{**})_{\rm hom}\big)^\infty \equiv \big((f^\infty)^{**}\big)_{\rm hom}. \qquad \square$$

Chapter 6
Future Research Projects

Wave propagation. In the sequence of the problem described in Chapter 3, we would like to address the case in which the coefficients $a_{\alpha 3}$, $\alpha \in \{1,2\}$, are not necessarily null and, in the case $\varepsilon \gg \delta$, when different hypotheses on a_{33} are assumed. Another interesting variant of the problem in Chapter 3 is when instead of $\Omega_\delta := \omega \times \delta I$ we consider $\Omega_\delta := \{(\bar{x}, x_3) \in \mathbb{R}^3 : \bar{x} \in \omega, |x_3| < \frac{\delta}{2} h_\delta(\bar{x})\}$, where h_δ determines the δ-dependent profile $x_3 = \pm h_\delta(\bar{x})$.

Effective energies for composite materials in the presence of fracture or cracks. Following the work described in Chapters 4 and 5, we would like to address a similar problem within the scope of second order derivatives theories. It amounts to characterize the multiscale limit of bounded sequences of the second-order distributional derivatives of functions of Bounded Hessian. The next steps are the characterization of multiscale homogenized functionals associated with homogenization problems with linear growth involving dependence on the Hessian and the study of the relation between the multiscale homogenized functional with the classical homogenized functional.

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