Topics in rank-based stochastic differential equations

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Abstract

In this thesis, we tackle two problems. In the first problem, we study fluctuations of a system of diffusions interacting through the ranks when the number of diffusions goes to infinity. It is known that the empirical cumulative distribution function of such diffusions converges to a non-random limiting cumulative distribution function which satisfies the porous medium PDE. We show that the fluctuations of the empirical cumulative distribution function around its limit are governed by a suitable SPDE.

In the second problem, we introduce common noise that has a rank preserving structure into systems of diffusions interacting through the ranks and study the behaviour of such diffusion processes as the number of diffusions goes to infinity. We show that the limiting distribution function is no longer deterministic and furthermore, it satisfies a suitable SPDE.

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To my Teachers

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1
Introduction

Consider the following systems of interacting diffusion processes ("particles") on the real line whose dynamics are given by the SDEs

$$\begin{aligned}\mathrm{d}X_i^{(n,\gamma)}(t) = b\big(F_{\rho^{(n,\gamma)}(t)}\big(X_i^{(n,\gamma)}(t)\big)\big)\,\mathrm{d}t + \sigma\big(F_{\rho^{(n,\gamma)}(t)}\big(X_i^{(n,\gamma)}(t)\big)\big)\,\mathrm{d}B_i^{(n)}(t) \\ + \gamma\big(t, \rho^{(n,\gamma)}(t)\big)\mathrm{d}W(t), \quad i = 1,\, 2,\, \ldots,\, n.\end{aligned} \tag{1.0.1}$$

Here b, σ are functions from $[0,1]$ to $\mathbb{R}$, $(0,\infty)$, respectively, γ is a function from $[0,\infty) \times M_1(\mathbb{R})$ to $\mathbb{R}$, where $M_1(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$ equipped with the Lévy metric (inducing the topology of weak convergence), $\rho^{(n,\gamma)}(t) := \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i^{(n,\gamma)}(t)}$ is the empirical measure of the particle system at time t, $F_{\rho^{(n,\gamma)}(t)}$ is the cumulative distribution function of $\rho^{(n,\gamma)}(t)$, and W, $B_1^{(n)}$, $B_2^{(n)}$, $\ldots$, $B_n^{(n)}$ are independent standard Brownian motions. In the absence of the common noise W, the SDEs in (1.0.1) reduce to the following system of SDEs

$$\mathrm{d}X_i^{(n)}(t) = b\big(F_{\rho^{(n)}(t)}\big(X_i^{(n)}(t)\big)\big)\,\mathrm{d}t + \sigma\big(F_{\rho^{(n)}(t)}\big(X_i^{(n)}(t)\big)\big)\,\mathrm{d}B_i^{(n)}(t), \qquad i = 1,\,2,\,\ldots,\,n. \tag{1.0.2}$$

Here $\rho^{(n)}(t) := \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i^{(n)}(t)}$ is the empirical measure of the particle system at time t, $F_{\rho^{(n)}(t)}$ is the cumulative distribution function of $\rho^{(n)}(t)$. We observe that the drift and diffusion coefficients of a process $X_i^{(n)}$ take the values $b\big(\frac{k}{n}\big)$ and $\sigma\big(\frac{k}{n}\big)$ whenever the *rank* (from the left) of $X_i^{(n)}(t)$ within $\big(X_1^{(n)}(t), X_2^{(n)}(t), \ldots, X_n^{(n)}(t)\big)$ is k. Since the drift and diffusion coefficients of each particle in (1.0.2) depends on its rank, these SDEs are also called as rank-based SDEs. Moreover, these SDEs can be identified with rank-based models in stochastic portfolio theory introduced by Fernholz and Karatzas (see [FK, Section 13]).

The weak existence for rank-based SDEs in (1.0.2) follows from a general result in [SV, Exercise 12.4.3]) and weak uniqueness was established in [BP]. We remark that the questions about weak existence and weak uniqueness for the particle system in (1.0.1) will be addressed in chapter 5.

We are interested in the behaviour of the SDEs in (1.0.1) and the rank-based SDEs in (1.0.2) as the the number of particles goes to infinity. The limiting behaviour of the rank-based SDEs was first studied in [S], wherein the hydrodynamic limit of the particle system was derived under some restrictive assumptions (see [S, Theorem 1.2]). These restrictive assumptions were lifted and the result was generalised in [JR, Proposition 2.1] under suitable regularity conditions. To be precise, let $C([0,\infty), M_1(\mathbb{R}))$ be the space of continuous functions from $[0,\infty)$ to $M_1(\mathbb{R})$ endowed with the topology of locally uniform convergence. Given that the initial positions $X_1^{(n)}(0), X_2^{(n)}(0), \ldots, X_n^{(n)}(0)$ are i.i.d. according to a probability measure λ with a finite first moment and that b and σ in (1.0.2) are continuous, the functions $t \mapsto \rho^{(n)}(t)$, $n \in \mathbb{N}$ converge in probability in $C([0,\infty), M_1(\mathbb{R}))$ to a deterministic limit $t \mapsto \rho(t)$ and the associated cumulative distribution functions $R(t,\cdot) := F_{\rho(t)}(\cdot)$, $t \geq 0$ satisfy the following porous medium equation in a weak sense (see 2.2.1 for the definition)

$$R_t = -B(R)_x + \Sigma(R)_{xx}, \quad R(0,\cdot) = F_\lambda(\cdot), \tag{1.0.3}$$

where $B(r) := \int_0^r b(a)\,\mathrm{d}a$ and $\Sigma(r) := \int_0^r \frac{1}{2}\,\sigma(a)^2\,\mathrm{d}a$.
We will show in chapter 2 that R is the law of the SDE (1.1.1), consequently, it is well defined for all $t \geq 0$ and $x \in \mathbb{R}$. Furthermore, proposition 2.3.2 in chapter 2 reveals that the porous medium PDE in (1.0.3) has a classical solution on compact time intervals.

Finally, a large deviations result was obtained for the rank-based SDEs in [DSVZ, Theorem 1.4] under appropriate assumptions.

1.1 Problems and Intuition

In this thesis, we are concerned with the rate of convergence of $F_{\rho^{(n)}(t)}(\cdot)$ to $R(t,\cdot)$ and the *fluctuations* of the particle system (1.0.2). Next, we are interested in the hydrodynamic limit of the particle system in (1.0.1). To be more specific, we would like to characterize the limit of the the empirical cumulative distribution functions $F_{\rho^{(n,\gamma)}(t)}(\cdot)$.

Before we explain the approaches taken to tackle these problems, we will state the assumptions that we will be making for the remainder of the thesis.

Assumption 1.1.1. *(a) There exist $\eta > 0$ and $\lambda \in M_1(\mathbb{R})$ such that λ has a bounded density and finite moments up to order $(2+\eta)$ and the initial positions $X_1^{(n)}(0), X_2^{(n)}(0), \ldots, X_n^{(n)}(0)$, $X_1^{(n,\gamma)}(0), X_2^{(n,\gamma)}(0), \ldots, X_n^{(n,\gamma)}(0)$ are i.i.d. according to λ for all $n \in \mathbb{N}$.*

(b) The functions b and σ in (1.0.1) and in (1.0.2) are differentiable with locally Hölder continuous derivatives of the order β. Furthermore, $\min_{a\in[0,1]} \sigma^2(a) > 0$.

(c) The function γ is bounded and satisfies $|\gamma(t,\mu) - \gamma(t,\nu)| \leq C\,W_1(\mu,\nu)$ $\forall\ t$ in $[0,\infty)$, where W_1 is the 1 Wasserstein distance (see 2.1.8 for its definition), μ and ν are probability measures on the real line with finite first moment and C is a contant.

The key to answering the question about the rate of convergence of $F_{\rho^{(n)}(t)}(\cdot)$ to $R(t,\cdot)$ is to analyze the following independent diffusion processes ("particles")

$$\begin{aligned} \mathrm{d}\bar{X}_i^{(n)}(t) = b\big(R(t,\bar{X}_i^{(n)}(t))\big)\,\mathrm{d}t + \sigma\big(R(t,\bar{X}_i^{(n)}(t))\big)\,\mathrm{d}B_i^{(n)}(t), \\ \bar{X}_i^{(n)}(0) = X_i^{(n)}(0),\ i = 1,\, 2,\, \dots,\, n, \end{aligned} \tag{1.1.1}$$

where $B_1^{(n)}$, $B_2^{(n)}$, $\dots$, $B_n^{(n)}$ are the standard Brownian motions from (1.0.2). We refer to the discussion following Proposition 2.3.1 in chapter 2 for the existence of a unique strong solution of (1.1.1) and to show that the law of $\bar{X}_i^{(n)}(t)$ is $R(t,\cdot)$. We denote $\bar{\rho}^{(n)}(t) := \frac{1}{n}\sum_{i=1}^n \delta_{\bar{X}_i^{(n)}(t)}$, $t \geq 0$ for the path of empirical measures associated with the i.i.d. particles $\bar{X}_1^{(n)}$, $\bar{X}_2^{(n)}$, $\dots$, $\bar{X}_n^{(n)}$. As the number of particles becomes large, we would expect $X_i^{(n)}(t)$ to be close to $\bar{X}_i^{(n)}(t)$ $\forall\ i = 1,\, 2,\, \dots,\, n$. Consequently, we would also expect the measures $\rho^{(n)}(t)$ and $\bar{\rho}^{(n)}(t)$ to be close in some sense. The main idea behind the proof of estimating the rate of convergence is to estimate the distance between the probability measures $\rho^{(n)}(t)$ and $\bar{\rho}^{(n)}(t)$. We call this distance estimate, the propagation of chaos estimate. Next, we appeal to known results to obtain the rate of convergence of $F_{\bar{\rho}^{(n)}(t)}(\cdot)$ to $R(t,\cdot)$. Finally, we combine both these estimates to obtain the rate of convergence of $F_{\rho^{(n)}(t)}(\cdot)$ to $R(t,\cdot)$.

To study the fluctuations of the particle system in (1.0.2), we introduce the space $M_{\mathrm{fin}}(\mathbb{R})$ of finite signed measures on $\mathbb{R}$ and the space $C_0(\mathbb{R})$ of continuous functions vanishing at infinity. We note that the space $M_{\mathrm{fin}}(\mathbb{R})$ can be viewed as the dual of $C_0(\mathbb{R})$ and we endow it with the associated weak-$*$ topology. Similarly, we define the spaces $M_{\mathrm{fin}}([0,t]\times\mathbb{R})$ for $t > 0$ and equip each of them with the respective weak-$*$ topology. The fluctuations of the particle system (1.0.2) are studied via the $M_{\mathrm{fin}}(\mathbb{R})$-valued processes

$$t \mapsto G_n(t)(\mathrm{d}x) := \sqrt{n}\,(F_{\rho^{(n)}(t)}(x) - R(t,x))\,\mathrm{d}x, \quad n \in \mathbb{N} \tag{1.1.2}$$

indexed by $t \in [0,\infty)$, as well as the processes

$$t \mapsto H_n(t)(\mathrm{d}s,\mathrm{d}x) := \sqrt{n}\,(F_{\rho^{(n)}(s)}(x) - R(s,x))\,\mathrm{d}x\,\mathrm{d}s, \quad n \in \mathbb{N} \tag{1.1.3}$$

taking values in $M_{\mathrm{fin}}([0,t] \times \mathbb{R})$, $t > 0$, respectively. We show that the $M_{\mathrm{fin}}(\mathbb{R})$-valued processes G_n, and the $M_{\mathrm{fin}}([0,t] \times \mathbb{R})$-valued processes H_n, $n \in \mathbb{N}$ converge in the finite-dimensional distribution sense to $t \mapsto G(t,x)\,\mathrm{d}x$ and $t \mapsto G(s,x)\,\mathbf{1}_{[0,t]\times\mathbb{R}}(s,x)\,\mathrm{d}s\,\mathrm{d}x$, respectively and also jointly. Furthermore, the process $G(t,x)$ satisfies a suitable SPDE. The main idea behind the proof is to apply Ito's formula to the entity $\int_{\mathbb{R}} \gamma(t,x)\,G_n(t)(\mathrm{d}x)$ for some smooth function $\gamma(t,x)$, and then proceed to the limit.

Next, to characterize the limit of the the empirical cumulative distribution functions $F_{\rho^{(n,\gamma)}(t)}(\cdot)$, we make a simple yet clever observation that reduces the particles in (1.0.1) to the rank-based SDEs in (1.0.2). We then characterise the limiting distribution in terms of the solution of the porous medium equation $R(t,\cdot)$.

1.2 Outline of the Thesis

The rest of the thesis is structured as follows. In chapter 2 we will introduce the notion of generalised solutions and prove that the deterministic limit of $F_{\rho^{(n)}(t)}$ satisfies the porous medium equation in a weak sense. Furthermore, we will address the issue of uniqueness and regularity of solutions of the porous medium equation. Then, we will elucidate the connection between the porous medium equation and the diffusion process in (1.1.1) and prove that the transition density of the diffusion process in (1.1.1) admits lower and upper gaussian bounds. Most of the results in chapter 2 are taken from [JR] and [Gi] and have been included in the thesis for completeness. The main contributions of this thesis are contained in chapters 3, 4 and 5. In chapter 3 we will prove the propagation of chaos estimate 3.2.1 by reducing it to the estimate of [BL, Theorem 4.8] on the expected Wasserstein distance between the empirical measure of an i.i.d. sample from the uniform distribution and the uniform distribution itself. Theorem 3.2.1 is then used along with Dvoretzky-Kiefer-Wolfowitz inequality in the form of [Mas, Corollary 1] to prove the other propagation of chaos estimate 3.3.1. In chapter 4 we will use Theorem 3.2.1 to establish the tightness of the finite-dimensional distributions of the processes G_n, $n \in \mathbb{N}$ and H_n, $n \in \mathbb{N}$ and then proceed to the proof of the central limit theorem 4.0.1 by identifying the limit points of the finite-dimensional distributions of G_n, $n \in \mathbb{N}$ and H_n, $n \in \mathbb{N}$. Most of the results in chapters 3 and 4 are taken from the paper [KoS]. In chapter 5, we will derive the hydrodynamic limit of the particle system in (1.0.1) by reducing it to the rank-based SDEs in (1.0.2). The results in this chapter are based on the paper [Ko].

2
Porous Medium Equation

We recall that $\rho^{(n)}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}(t)}$ is the empirical measure of the particle system (1.0.2) and $F_{\rho^{(n)}(t)}$ is the cumulative distribution function of $\rho^{(n)}(t)$. We also note that $M_1(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$. This chapter is mainly devoted to prove that the limit of the empirical cumulative distribution function $F_{\rho^{(n)}(t)}$ satisfies the porous medium equation 1.0.3. We will also address the properties of the solution of the porous medium equation. We first prove a tightness result that establishes existence of subsequential limits for the distributions of the random mappings $t \mapsto \rho^{(n)}(t)$ in $C([0,\infty), M_1(\mathbb{R}))$ and then show that the limit of $F_{\rho^{(n)}(t)}$ satisfies the porous medium equation in a weak sense. Next, we address uniqueness and regularity of the solutions to the porous medium equation. Lastly, we reveal the connection between the porous medium equation and the SDE (1.1.1) and show that the transition density of the SDE in (1.1.1) admits gaussian lower and upper bounds.

2.1 Tightness of the Empirical Distributions

In this section, we will prove the existence of limit points for the sequence of the empirical distribution functions $F_{\rho^{(n)}(t)}$. We will show this by proving an appropriate tightness result, but first, we will prove an elementary that shows that if the particles at time 0 have a finite p^{th} moment, then the particles have a finite p^{th} moment at all times. We remark that the estimate also proves that the family of random variables $\left(\left|X_1^{(n)}(s)\right|^2\right)_{n\geq 1}$ in (1.0.2) is uniformly integrable. This moment estimate will be used in subsequent lemmata.

Lemma 2.1.1. *Suppose the Assumption 1.1.1 holds, then we have the following moment estimate for the rank-based particles in* (1.0.2)

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}\left|X_i^{(n)}(s)\right|^{(2+\eta)}\Big] \leq C_0 + C_1\, t^{\frac{2+\eta}{2}} + C_2\, t^{2+\eta}\,, \quad i = 1, 2, \ldots, n, \tag{2.1.1}$$

where C_0 is a constant that depends on $\int_{\mathbb{R}} |x|^{2+\eta}\,\lambda(\mathrm{d}x)$, C_1 and C_2 are constants that depend on the L^∞ norm of σ and b respectively.

Proof. Integrating equation (1.0.2), we get

$$\begin{aligned} X_i^{(n)}(s) = X_i^{(n)}(0) &+ \int_0^s b\big(F_{\rho^{(n)}(u)}\big(X_i^{(n)}(u)\big)\big)\,\mathrm{d}u \\ &+ \int_0^s \sigma\big(F_{\rho^{(n)}(u)}\big(X_i^{(n)}(u)\big)\big)\,\mathrm{d}W_i^{(n)}(u), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2.1.2}$$

Taking absolute value, applying triangular inequality and raising both sides to the power of $(2+\eta)$, we obtain

$$\begin{aligned} \left|X_i^{(n)}(s)\right|^{2+\eta} \leq C\bigg(&\left|X_i^{(n)}(0)\right|^{2+\eta} + \Big[\int_0^s \left|b\big(F_{\rho^{(n)}(u)}\big(X_i^{(n)}(u)\big)\big)\right|\,\mathrm{d}u\Big]^{2+\eta} \\ &+ \left|\int_0^s \sigma\big(F_{\rho^{(n)}(u)}\big(X_i^{(n)}(u)\big)\big)\,\mathrm{d}W_i^{(n)}(u)\right|^{2+\eta}\bigg), \end{aligned} \tag{2.1.3}$$

where C is a constant. Next, we take supremum in time on the right hand side followed up taking supremum on the left hand side and then take expectation and apply Burkholder-Davis-Gundy inequality (see e.g. [KS, Chapter

3, Theorem 3.28]) to the stochastic integral to obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le s\le t}\big|X_i^{(n)}(s)\big|^{2+\eta}\Big] \le C\,\bigg\{&\int_{\mathbb{R}}|x|^{2+\eta}\lambda(\mathrm{d}x)\\ &+\mathbb{E}\Big[\int_0^t\big|b\big(F_{\rho^{(n)}(u)}\big(X_i^{(n)}(u)\big)\big)\big|\,\mathrm{d}u\Big]^{2+\eta}\\ &+\mathbb{E}\Big[\int_0^t\sigma^2\big(F_{\rho^{(n)}(u)}\big(X_i^{(n)}(u)\big)\big)\,\mathrm{d}u\Big]^{\frac{2+\eta}{2}}\bigg\}.\end{aligned} \tag{2.1.4}$$

Since b and σ are continuous function defined on the compact set $[0,1]$, it follows that they are bounded. The result then follows trivially. $\square$

We are now ready to prove the main tightness result and to this end let $\rho^{(n)} = \frac{1}{n}\sum_{i=1}^n \delta_{X_i^{(n)}}$. We combine the arguments in [JR, Proposition 2.1] and [JR, Lemma 2.3] to obtain the following tightness result.

Lemma 2.1.2. *Under Assumption 1.1.1, the sequence $(\pi^n)_{n\ge1}$ of the distributions of the random mappings $t\mapsto\rho^{(n)}(t)$ in $C([0,\infty),M_1(\mathbb{R}))$ is tight.*

Proof. Noting that $X_i^{(n)}$ is an element of $C([0,\infty),\mathbb{R})$ $\forall\ i=1,2,\dots,n$, we infer that $\rho^{(n)}$ is an element of $M_1\big(C([0,\infty),\mathbb{R})\big)$, where $M_1\big(C([0,\infty),\mathbb{R})\big)$ is the space of probability measures on $C([0,\infty),\mathbb{R})$. Let $\tilde{\pi}^n$ denote the distribution of $\rho^{(n)}$. Since the mapping $M_1\big(C([0,\infty),\mathbb{R})\big)\to C([0,\infty),M_1(\mathbb{R}))$ is continuous, proving a stronger result that the sequence $\tilde{\pi}^n$ is tight implies that the sequence π^n is tight. We begin by noting that the distribution of $\big(X_1^{(n)},X_2^{(n)},\cdots,X_n^{(n)}\big)$ in $C\big([0,\infty),\mathbb{R}^n\big)$ is symmetric. In view of [Sz2, Proposition 2.2, Pg 177], $\tilde{\pi}^n$ is tight if and only if the sequence of distributions of the variables $X_1^{(n)}\in C\big([0,\infty),\mathbb{R}\big)$ is tight. A computation similar to the one in lemma 2.1.1 reveals that $\mathbb{E}\big[X_1^{(n)}(r)-X_1^{(n)}(s)\big]^4 \le C(t)|r-s|^2\ \forall\ 0\le s,r\le t$, where $C(t)$ is a non random function of t. Noting that $E|X_1^{(n)}(0)| = \int_{\mathbb{R}}|x|\lambda\,(\mathrm{d}x) < \infty$, we appeal to [KS, Problem 4.11, Pg 64] and conclude that the sequence of distributions of the variables $X_1^{(n)}\in C\big([0,\infty),\mathbb{R}\big)$ is tight. This finishes the proof of the lemma. $\square$

The tightness result establishes the existence of converging subsequences for π^n and $\tilde{\pi}^n$. We remark that we will use the same index n for the converging subsequences. Let ρ be a variable in $M_1\big(C([0,\infty),\mathbb{R})\big)$ whose distribution is the limit point $\tilde{\pi}^\infty$ of the convergent subsequence $\tilde{\pi}^n$. In view of the Skorokhod representation theorem in the form of [Du, Theorem 3.5.1], we can assume that the sequence of random variables $\rho^{(n)}$ and ρ are defined on the same probability space with $\rho^{(n)} \xrightarrow{a.s.} \rho$ in $M_1\big(C([0,\infty),\mathbb{R})\big)$. Since the mapping $M_1\big(C([0,\infty),\mathbb{R})\big) \to C([0,\infty), M_1(\mathbb{R}))$ is continuous, we also have the almost sure convergence $\rho^{(n)} \to \rho$ in $C([0,\infty), M_1(\mathbb{R}))$.

In the next lemma, we show that the limit points of the empirical measures $\rho^{(n)}$ have 2 moments. We will be needing this lemma to strengthen the almost sure convergence of $\rho^{(n)}$ to ρ in the sense of Wasserstein distance. This is taken from [JR, Lemma 2.4].

Lemma 2.1.3. *Under Assumption 1.1.1,* $\mathbb{E}\Big[\sup\limits_{0\leq s\leq t} \int_{\mathbb{R}} |x|^2\, \rho(s)(\mathrm{d}x)\Big] < \infty.$

Proof. For all $M \geq 0$, the function $f_M : \mu \to \sup\limits_{0\leq s\leq t} \int_{\mathbb{R}} \big(|x|^2 \wedge M\big)\, \mu_s(\mathrm{d}x)$ is continuous and bounded on $C([0,\infty), M_1(\mathbb{R}))$. For a fixed n, we have

$$\mathbb{E}\big(f_M(\rho^{(n)})\big) \leq \frac{\sum\limits_{i=1}^{n} \mathbb{E}\big[\sup\limits_{0\leq s\leq t} |X_i^{(n)}(s)|^2\big]}{n} \leq K(t), \tag{2.1.5}$$

where the last inequality is a consequence of lemma 2.1.1. Note that $K(t)$ is a deterministic function of t independent of M and n. As a consequence of the discussion following lemma 2.1.2, we have $\mathbb{E}\big(f_M(\rho)\big) = \lim\limits_{n\to\infty} \mathbb{E}\big(f_M(\rho^{(n)})\big) \leq K(t)$. We then apply Fatou's lemma to obtain

$$\mathbb{E}\big(\liminf_{M\to\infty} f_M(\rho)\big) \leq \liminf_{M\to\infty} \mathbb{E}\big(f_M(\rho)\big) \leq K(t). \tag{2.1.6}$$

An elementary inequality coupled with montone convergence theorem yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq s\leq t} \liminf_{M\to\infty} \int_{\mathbb{R}} \big(|x|^2 \wedge M\big)\, \rho(s)(\mathrm{d}x)\Big) \leq \mathbb{E}\big(\liminf_{M\to\infty} f_M(\rho)\big) \leq K(t),\\
&\mathbb{E}\Big(\sup_{0\leq s\leq t} \int_{\mathbb{R}} |x|^2 \rho(s)(\mathrm{d}x)\Big) \leq K(t).
\end{aligned} \tag{2.1.7}$$

and this completes the proof of the lemma. $\square$

Before we state the next lemma, we will give a brief description of Wasserstein distance.

2.1.1 Wasserstein distance

Wasserstein distance is one of the many metrics that is used to measure the distance between probability measures. It's defined as follows

$$W_p(\mu,\nu) = \inf_{(Y_1,Y_2)} \mathbb{E}\big[|Y_1 - Y_2|^p\big]^{1/p}, \tag{2.1.8}$$

where the infimum is taken over all random vectors (Y_1, Y_2) such that Y_1 is distributed according to μ and Y_2 according to ν. We are interested in the probability measures on $\mathbb{R}$ and in this case there are equivalent representations of the formula 2.1.8 which we will summarize below (see e.g. [BL, Section 2.3]). For $p \geq 1$ consider two probability measures μ, ν on $\mathbb{R}$ having finite moments up to order p. Let F_μ, F_ν be their cumulative distribution functions and q_μ, q_ν be their quantile functions. Then, we have

Proposition 2.1.4.

$$W_1(\mu,\nu) = \int_{\mathbb{R}} \big|F_\mu(x) - F_\nu(x)\big| \,\mathrm{d}x, \tag{2.1.9}$$

$$W_p(\mu,\nu) = \left(\int_0^1 \big|q_\mu(a) - q_\nu(a)\big|^p \,\mathrm{d}a \right)^{1/p}, \quad p \geq 1. \tag{2.1.10}$$

The next lemma strengthens the convergence of the measures $\rho^{(n)}(t)$ in the sense of Wasserstein distance and this lemma is taken from [JR, Corollary 2.16].

Lemma 2.1.5. *Let Assumption 1.1.1 be satisfied, then we have the following convergences* $\lim\limits_{n\to\infty} \mathbb{E}\Big[W_2^2\big(\rho^{(n)}(s), \rho(s)\big)\Big] = 0$ *and* $\lim\limits_{n\to\infty} \int\limits_0^t \mathbb{E}\Big[W_2^2\big(\rho^{(n)}(s), \rho(s)\big)\Big]\,\mathrm{d}s = 0$ $\forall$ s *and* t *in* $[0,\infty)$ *respectively.*

Proof. We will first prove the first convergence and to this end, let $M \geq 0$ and $X_{(1)}^{(n)}(s) \leq X_{(2)}^{(n)}(s) \leq \cdots \leq X_{(n)}^{(n)}(s)$ be the order statistics of the vector $\big(X_1^{(n)}(s), X_2^{(n)}(s), \ldots, X_n^{(n)}(s)\big)$. Using the representation in proposition

2.1.4 we have

$$
\begin{aligned}
W_2^2\big(\rho^{(n)}(s),\rho(s)\big) &= \sum_{i=1}^{n} \int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(\big|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\big|^2 - M\Big)^+ \,\mathrm{d}u \\
&\quad + \sum_{i=1}^{n} \int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} \Big(\big|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\big|^2 \wedge M\Big) \,\mathrm{d}u \\
&\le \sum_{i=1}^{n} \int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} \big|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\big|^2 \mathbb{I}\Big(\big|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\big|^2 \ge M\Big) \,\mathrm{d}u \\
&\quad + \int\limits_{0}^{1} \Big(\big|F_{\rho^{(n)}(s)}^{-1}(u) - F_{\rho(s)}^{-1}(u)\big|^2 \wedge M\Big) \,\mathrm{d}u,
\end{aligned}
\tag{2.1.11}
$$

where $\mathbb{I}$ is the indicator function. The function $\mu \in M_1(\mathbb{R}) \to \int\limits_0^1 \big|F_\mu^{-1}(u) - F_{\rho(s)}^{-1}(u)\big|^2 \wedge M \,\mathrm{d}u$ is continuous and bounded. As a consequence of the lemma 2.1.2 and the discussion following it, we get

$$
\lim_{n\to\infty} \mathbb{E}\Bigg[\int\limits_0^1 \Big(\big|F_{\rho^{(n)}(s)}^{-1}(u) - F_{\rho(s)}^{-1}(u)\big|^2 \wedge M\Big)\,\mathrm{d}u\Bigg] = 0. \tag{2.1.12}
$$

Next, for all x, y in $\mathbb{R}$, we have the following inequality

$$
\begin{aligned}
|x-y|^p \mathbb{I}\big(|x-y|^p \ge M\big) &\le |x-y|^p \mathbb{I}\big(|x| \ge |y| \vee M^{1/p}/2\big) \\
&\quad + |x-y|^p \mathbb{I}\big(|y| \ge |x| \vee M^{1/p}/2\big) \\
&\le 2^p |x|^p \mathbb{I}\big(|x|^p \ge M/2^p\big) + 2^p |y|^p \mathbb{I}\big(|y|^p \ge M/2^p\big).
\end{aligned}
\tag{2.1.13}
$$

Applying this inequality to the term

$\sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\right|^2 \mathbb{I}\left(\left|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\right|^2 \geq M\right) \mathrm{d}u$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\right|^2 \mathbb{I}\left(\left|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\right|^2 \geq M\right) \mathrm{d}u \\
&\qquad \leq \frac{2^2}{n} \sum_{i=1}^{n} \left|X_{(i)}^{(n)}(s)\right|^2 \mathbb{I}\left(\left|X_{(i)}^{(n)}(s)\right|^2 \geq M/2^2\right) \\
&\qquad +2^2 \int_0^1 \left|F_{\rho(s)}^{-1}(u)\right|^2 \mathbb{I}\left(\left|F_{\rho(s)}^{-1}(u)\right|^2 \geq M/2^2\right) \mathrm{d}u \\
&\qquad = \frac{2^2}{n} \sum_{i=1}^{n} \left|X_{i}^{(n)}(s)\right|^2 \mathbb{I}\left(\left|X_{i}^{(n)}(s)\right|^2 \geq M/2^2\right) \\
&\qquad +2^2 \int_0^1 \left|F_{\rho(s)}^{-1}(u)\right|^2 \mathbb{I}\left(\left|F_{\rho(s)}^{-1}(u)\right|^2 \geq M/2^2\right) \mathrm{d}u.
\end{aligned} \tag{2.1.14}$$

Next, we note that the random variables $X_1^{(n)}(s)$, $X_2^{(n)}(s)$, $\ldots$, $X_n^{(n)}(s)$ are exchangeable. Furthermore, the moment estimate in lemma 2.1.1 reveals that the family of random variables $\left(\left|X_1^{(n)}(s)\right|^2\right)_{n\geq 1}$ is uniformly integrable and lemma 2.1.3 implies that $\mathbb{E}\left[\int_0^1 \left|F_{\rho(s)}^{-1}(u)\right|^2 \mathrm{d}u\right] < \infty$. Combining all these observations gives

$$\sup_{n\geq 1} \mathbb{E}\left[\sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\right|^2 \mathbb{I}\left(\left|X_{(i)}^{(n)}(s) - F_{\rho(s)}^{-1}(u)\right|^2 \geq M\right) \mathrm{d}u\right] \overset{M\to\infty}{\longrightarrow} 0 \tag{2.1.15}$$

which yields $\lim_{n\to\infty} \mathbb{E}\left[W_2^2\big(\rho^{(n)}(s), \rho(s)\big)\right] = 0$.

We now turn our attention to the second convergence $\lim_{n\to\infty} \int_0^t \mathbb{E}\left[W_2^2\big(\rho^{(n)}(s), \rho(s)\big)\right] \mathrm{d}s = 0$. In view of the first convergence that we just proved and the dominated convergence theorem, it suffices to show that $\sup_{0\leq s\leq t} \mathbb{E}\left[W_2^2\big(\rho^{(n)}(s), \rho(s)\big)\right] < C(t)$, where $C(t)$ is a constant depending on t.

To this end, we begin by making the observation

$$W_2^2\big(\rho^{(n)}(s), \rho(s)\big) \leq 2 \sum_{i=1}^{n} \frac{\big|X_i^{(n)}(s)\big|^2}{n} + 2 \int_{\mathbb{R}} |x|^2 \rho(s)\,(\mathrm{d}x). \tag{2.1.16}$$

Taking supremum on the right-hand side followed by taking expectation and using the moment estimates in lemmas 2.1.1 and 2.1.3 and then taking the supremum on the left-hand side gives the desired result. □

2.2 Derivation of the Porous Medium Equation

In this section, we will show that the cumulative distribution function $F_{\rho(t)}$ satisfies the porous medium equation 1.0.3 in the sense defined below.

Definition 2.2.1. *A bounded continuous nonnegative function R with $R(0,\cdot) = F_\lambda(\cdot)$ is called a generalized solution of the problem* (1.0.3) *if the following holds*

$$\begin{aligned}&\int_{-\infty}^{\infty} h(t,x)\, R(t,x)\, \mathrm{d}x \;-\; \int_{-\infty}^{\infty} h(0,x)\, R(0,x)\, \mathrm{d}x = \\ &\int_0^t \int_{-\infty}^{\infty} \Big(h_s(s,x)\, R(s,x) + h_x(s,x)\, B\big(R(s,x)\big) + h_{xx}(s,x)\, \Sigma\big(R(s,x)\big) \Big)\, \mathrm{d}x\, \mathrm{d}s\end{aligned} \tag{2.2.1}$$

for all functions $h \in C_c^{1,2}\big([0,\infty) \times \mathbb{R}\big)$, the space of functions on $[0,\infty) \times \mathbb{R}$ which are continuously differentiable in s, twice continuously differentiable in x and compactly supported.

The main proposition in this section is as follows

Proposition 2.2.2. *Suppose that the Assumption 1.1.1 holds. Then $F_{\rho(t)}$ satisfies the PDE* (1.0.3) *in the sense of definition 2.2.1.*

Proof. The proof relies on a suitable prelimit version of its statement. For every fixed $n \in \mathbb{N}$, let B_n and Σ_n be functions that are defined at finitely many points on $[0,1]$ with $k = 0,\, 1,\, \ldots,\, n$ and

$$B_n(k/n) = \frac{1}{n} \sum_{j=1}^{k} b(j/n), \quad \Sigma_n(k/n) = \frac{1}{2n} \sum_{j=1}^{k} \sigma(j/n)^2\ . \tag{2.2.2}$$

Let $H(s,x)=\int_{-\infty}^{x} h(s,y)\,\mathrm{d}y$, where the function $h\in C_c^{1,2}\big([0,\infty)\times\mathbb{R}\big)$. We apply Ito's formula to the function H to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} H(t,x)\,\rho^{(n)}(t)(\mathrm{d}x)=\int_{\mathbb{R}} H(0,x)\,\rho^{(n)}(0)(\mathrm{d}x)\\
&+(1/n)\sum_{i=1}^{n}\int_0^t h\big(s,X_i^{(n)}(s)\big)\,\sigma\big(F_{\rho^{(n)}(s)}(X_i^{(n)}(s))\big)\mathrm{d}B_i^{(n)}(s)\\
&+\int_0^t\int_{\mathbb{R}}\Big(H_s(s,x)+h(s,x)\,b\big(F_{\rho^{(n)}(s)}(x)\big)\\
&\qquad\qquad+h_x(s,x)\frac{\sigma^2\big(F_{\rho^{(n)}(s)}(x)\big)}{2}\Big)\,\rho^{(n)}(s)(\mathrm{d}x)\,\mathrm{d}s.
\end{aligned}
\tag{2.2.3}
$$

Next, we use integration by parts to obtain each of the following equalities

$$
\begin{aligned}
&\int_{\mathbb{R}} H(t,x)\,\rho^{(n)}(t)(\mathrm{d}x)-\int_{\mathbb{R}} H(0,x)\,\rho^{(n)}(0)(\mathrm{d}x)\\
&-\int_0^t\int_{\mathbb{R}} H_s(s,x)\,\rho^{(n)}(s)(\mathrm{d}x)\,\mathrm{d}s=-\int_{\mathbb{R}} h(t,x)F_{\rho^{(n)}(t)}(x)\,\mathrm{d}x\\
&+\int_{\mathbb{R}} h(0,x)F_{\rho^{(n)}(0)}(x)\,\mathrm{d}x+\int_0^t\int_{\mathbb{R}} h_s(s,x)F_{\rho^{(n)}(s)}(x)\;\mathrm{d}x\,\mathrm{d}s,\\
&\int_{\mathbb{R}} h(s,x)\,b\big(F_{\rho^{(n)}(s)}(x)\big)\,\rho^{(n)}(s)(\mathrm{d}x)=-\int_{\mathbb{R}} h_x(s,x)\,B_n\big(F_{\rho^{(n)}(s)}(x)\big)\,\mathrm{d}x,\\
&\int_{\mathbb{R}} h_x(s,x)\,\frac{\sigma^2\big(F_{\rho^{(n)}(s)}(x)\big)}{2}\,\rho^{(n)}(s)(\mathrm{d}x)=-\int_{\mathbb{R}} h_{xx}(s,x)\,\Sigma_n\big(F_{\rho^{(n)}(s)}(x)\big)\,\mathrm{d}x.
\end{aligned}
\tag{2.2.4}
$$

In view of the above equalities, we express (2.2.3) equivalently as follows

$$
\begin{aligned}
&\int_{\mathbb{R}} h(t,x) F_{\rho^{(n)}(t)}(x)\,\mathrm{d}x - \int_{\mathbb{R}} h(0,x) F_{\rho^{(n)}(0)}(x)\,\mathrm{d}x \\
&= \int_0^t \int_{\mathbb{R}} \Big(h_s(s,x)\, F_{\rho^{(n)}(s)}(x) + h_x(s,x)\, B\big(F_{\rho^{(n)}(s)}(x)\big) \\
&\qquad + h_{xx}(s,x) \Sigma\big(F_{\rho^{(n)}(s)}(x)\big)\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{R}} h_x(s,x) \Big(B\big(F_{\rho^{(n)}(s)}(x)\big) - B_n\big(F_{\rho^{(n)}(s)}(x)\big)\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{R}} h_{xx}(s,x) \Big(\Sigma\big(F_{\rho^{(n)}(s)}(x)\big) - \Sigma_n\big(F_{\rho^{(n)}(s)}(x)\big)\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&- (1/n) \sum_{i=1}^n \int_0^t h\big(s, X_i^{(n)}(s)\big)\, \sigma\big(F_{\rho^{(n)}(s)}(X_i^{(n)}(s))\big) \mathrm{d}B_i^{(n)}(s).
\end{aligned}
\tag{2.2.5}
$$

We note that $\sup_{s\in[0,t],\, x\in\mathbb{R}} \Big| B\big(F_{\rho^{(n)}(s)}(x)\big) - B_n\big(F_{\rho^{(n)}(s)}(x)\big)\Big|$ is $O(\frac{1}{n})$ and consequently $\int_0^t \int_{\mathbb{R}} h_x(s,x) \Big(B\big(F_{\rho^{(n)}(s)}(x)\big) - B_n\big(F_{\rho^{(n)}(s)}(x)\big)\Big)\,\mathrm{d}x\,\mathrm{d}s \xrightarrow{a.s.} 0$ as $n \to \infty$. A similar argument reveals that $\int_0^t \int_{\mathbb{R}} h_{xx}(s,x) \Big(\Sigma\big(F_{\rho^{(n)}(s)}(x)\big) - \Sigma_n\big(F_{\rho^{(n)}(s)}(x)\big)\Big)\,\mathrm{d}x\,\mathrm{d}s \xrightarrow{a.s.} 0$ as $n \to \infty$. Next, the quadratic variation of the martingale term is $(1/n) \int_0^t \int_{\mathbb{R}} h^2(x) \sigma^2\big(F_{\rho^{(n)}(s)}(x)\big)\, \rho^{(n)}(s)(\mathrm{d}x)\,\mathrm{d}s \xrightarrow{a.s.} 0$ as $n \to \infty$.

Next, we define the bounded continuous function $G : M_1\big(C([0,\infty),\mathbb{R})\big) \to \mathbb{R}$ as follows

$$
\begin{aligned}
G(\mu) &= \int_{\mathbb{R}} h(t,x) F_{\mu_t}(x)\,\mathrm{d}x - \int_{\mathbb{R}} h(0,x) F_{\mu_0}(x)\,\mathrm{d}x \\
&- \int_0^t \int_{-\infty}^{\infty} \Big(h_s(s,x) F_{\mu_s}(x) + h_x(s,x) B\big(F_{\mu_s}(x)\big) + h_{xx}(s,x) \Sigma\big(F_{\mu_s}(x)\big)\Big) \mathrm{d}x\,\mathrm{d}s,
\end{aligned}
\tag{2.2.6}
$$

where F_{μ_s} is the cumulative distribution function of the probability measure μ_s on $\mathbb{R}$ and h is the smooth test function that we defined earlier. The discussion following lemma 2.1.2 implies that $\mathbb{E}\big(G^2(\rho^{(n)})\big) \xrightarrow{n\to\infty} \mathbb{E}\big(G^2(\rho)\big)$ and our aforementioned arguments using the prelimit version reveal that $\lim_{n\to\infty} \mathbb{E}\big(G^2(\rho^{(n)})\big) = 0$. Combining these two arguments finishes the proof of the proposition. $\square$

2.3 Uniqueness and Regularity of Solutions

The aim of this section is to address uniqueness and the regularity properties of the solutions of the cauchy problem (1.0.3) in the sense of definition 2.2.1. We remark that these results and the results in the next section will be used extensively in chapters 3, 4 and 5.

In view of Assumption 1.1.1 we can combine [Gi, Theorems 4 and 7] to obtain the following proposition.

Proposition 2.3.1. *Let Assumption 1.1.1 be satisfied. Then, the Cauchy problem* (1.0.3) *admits a unique generalized solution R. Moreover, its distributional derivative R_x can be represented by a bounded function on any strip of the form* $[0,t]\times\mathbb{R}$.

The next proposition taken from [JR, Lemma 2.7] gives us the conditions to obtain classical regularity of the generalized solution and to this end let $C_b^{1,2}\big([0,t],\mathbb{R}\big)$ denote the set of $C^{1,2}\big([0,t],\mathbb{R}\big)$ functions that are bounded together with their derivatives and let $H^l\big(\mathbb{R}\big)$ denote the Hölder space, defined as in [LSU, Pg. 7].

Proposition 2.3.2. *Let Assumption 1.1.1 be satisfied and let $F_\lambda(\cdot)$ be in the Hölder space $H^l\big(\mathbb{R}\big)$, with $l=3+\beta$, where β is the Hölder exponent defined in the Assumption 1.1.1. Then for all finite $t>0$, $R\in C_b^{1,2}\big([0,t],\mathbb{R}\big)$. In particular, it is a classical solution.*

2.4 Porous Medium Equation and the associated diffusion process

In this section, we will discuss the connection between the SDE (1.1.1) and the porous medium equation (1.0.3). Consider the following SDE

$$\mathrm{d}\bar{X}(t)=b\big(R(t,\bar{X}(t))\big)\,\mathrm{d}t+\sigma\big(R(t,\bar{X}(t))\big)\,\mathrm{d}B(t) \tag{2.4.1}$$

satisfied by each of the processes $\bar{X}_i^{(n)}$. Assumption 1.1.1 and Proposition 2.3.1 guarantee that the functions $x\mapsto b\big(R(t,x)\big)$ and $x\mapsto\sigma\big(R(t,x)\big)$ are

Lipschitz with uniformly bounded Lipschitz constants on every compact interval of t's. Consequently, there exists a unique strong solution of (2.4.1) for the initial condition λ of Assumption 1.1.1 or any deterministic initial condition (see e.g. [KS, Chapter 5, Theorems 2.5 and 2.9]). In addition, $\bar{X}$ is the unique solution of the martingale problem associated with the operators $b(R(t,\cdot))\,\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\sigma(R(t,\cdot))^2}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $t\geq 0$ and therefore a strong Markov process (see [SV, Theorems 7.2.1 and 6.2.2]). For the initial condition λ, Assumption 1.1.1 allows us to apply [JR, Corollary 2.13] to identify the one-dimensional distributions of the solution to the nonlinear martingale problem therein with $\rho(t)$, $t\geq 0$, so that the solution itself is given by the law $\mathcal{L}(\bar{X})$ of $\bar{X}$ and therefore

$$\mathcal{L}(\bar{X}(t))=\rho(t),\quad t\geq 0. \tag{2.4.2}$$

We now aim to apply the results of [Ar] to conclude that under Assumption 1.1.1 the transition density of $\bar{X}$ exists and satisfies Gaussian lower and upper bounds. To identify the transition density of $\bar{X}$ with the weak fundamental solution of a parabolic PDE as in [Ar, Theorem 5] we fix a $T>0$ and consider the Cauchy problem

$$u_t+b(R)\,u_x+\frac{\sigma(R)^2}{2}\,u_{xx}=f,\quad u(T,\cdot)=0, \tag{2.4.3}$$

where $f\in L^2([0,T]\times\mathbb{R})\cap L^\infty([0,T]\times\mathbb{R})$. We note that $b(R)$ and $\frac{\sigma(R)^2}{2}$ are bounded and that $x\mapsto\frac{\sigma(R(t,x))^2}{2}$ are Lipschitz with uniformly bounded Lipschitz constants for $t\in[0,T]$. Hence, according to [Kr2, Theorem 2.1 and Remark 2.2] there exists a unique solution u of (2.4.3) with $u,u_t,u_x,u_{xx}\in L^2([0,T]\times\mathbb{R})$ and it is given by

$$u(t,x)=-\mathbb{E}\bigg[\int_t^T f(r,\bar{X}(r))\,\mathrm{d}r\,\bigg|\,\bar{X}(t)=x\bigg],\quad (t,x)\in[0,T]\times\mathbb{R}. \tag{2.4.4}$$

In particular, $u\in L^\infty([0,T]\times\mathbb{R})$, so that with $g(t,x):=-f(T-t,x)$, $S(t,x):=R(T-t,x)$, $(t,x)\in[0,T]\times\mathbb{R}$ the function $v(t,x):=u(T-t,x)$,

$(t,x) \in [0,T] \times \mathbb{R}$ is a weak solution of

$$v_t - \big(b(S) - \sigma(S)\,\sigma'(S)\,S_x\big)\,v_x - \left(\frac{\sigma(S)^2}{2}\,v_x\right)_x = g, \quad v(0,\cdot) = 0 \qquad (2.4.5)$$

in the sense of [Ar, Theorem 5(ii)]. The latter theorem is applicable, since $b(S) - \sigma(S)\,\sigma'(S)\,S_x$ and $\frac{\sigma(S)^2}{2}$ are bounded on $[0,T] \times \mathbb{R}$ and $\frac{\sigma(S)^2}{2}$ is bounded away from 0 on $[0,T] \times \mathbb{R}$ by Assumption 1.1.1 and Proposition 2.3.1. Comparing the conclusion of [Ar, Theorem 5(ii)] with (2.4.4) we obtain the existence of the transition density $p(t,x;r,z)$ of $\bar{X}$ and recognize $p(T-r,x;T-t,z)$ as the weak fundamental solution corresponding to the PDE in (2.4.5). Thus, [Ar, Theorem 10(ii)] yields the following result.

Proposition 2.4.1. *Let Assumption 1.1.1 be satisfied. Then, the process $\bar{X}$ has a transition density p such that*

$$C^{-1}(r-t)^{-1/2}\,e^{-C(z-x)^2/(r-t)} \le p(t,x;r,z) \le C(r-t)^{-1/2}\,e^{-C^{-1}(z-x)^2/(r-t)}$$
$$\forall\, T > 0 \;\; , 0 \le t < r \le T, \;\; x, z \in \mathbb{R}. \qquad (2.4.6)$$

with $C \in (1,\infty)$ possibly depending on T. In particular, if $\bar{X}(0)$ is distributed according to λ, then

$$\forall\, T > 0: \quad \sup_{0 \le t \le T} \mathbb{E}\big[|\bar{X}(t)|^{2+\eta}\big] < \infty. \qquad (2.4.7)$$

3

Propagation of Chaos

In chapter 1, we relied on intuition and claimed that the particle $X_i^{(n)}(t)$ in (1.0.2) should be close to $\bar{X}_i^{(n)}(t)$ in (1.1.1) $\forall\ i = 1, 2, \ldots, n$. Furthermore, we also asserted that the measures $\rho^{(n)}(t)$ and $\bar{\rho}^{(n)}(t)$ should also be close, where $\bar{\rho}^{(n)}(t)$ is the empirical measure of the particle system (1.1.1). In this chapter we will prove propagation of chaos estimates that tell us precisely the sense in which the particles and the measures are close.

3.1 Preliminary Estimates

We will be needing an estimate of the expected Wasserstein distance between the empirical measure of an i.i.d. sample from the uniform distribution and the uniform distribution itself. These are taken from [BL, Theorem 4.8].

Proposition 3.1.1. *Let $U_1, U_2, \ldots$ be i.i.d. according to the uniform dis-*

tribution υ on $[0,1]$. Then, there exists a constant $C < \infty$ such that

$$\mathbb{E}\left[W_p\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{U_i},\upsilon\right)^p\right]^{1/p} \leq C\,p^{1/2}\,n^{-1/2}, \quad p \geq 1, \quad n \in \mathbb{N}. \tag{3.1.1}$$

Next, we recall the Functional Central Limit Theorem for empirical cumulative distribution functions from [dGM, Theorem 2.1] (see also [BL, Corollary 3.9 and the discussion of the functional J_1 on p. 25]). This result gives rise to the initial condition in (4.0.1).

Proposition 3.1.2. *Let Assumption 1.1.1(a) be satisfied. Then, the sequence $G_n(0,\cdot)$ in 1.1.2, $n \in \mathbb{N}$ converges in law weakly in $L^1(\mathbb{R})$ (and therefore in $M_{\text{fin}}(\mathbb{R})$) to $\beta(F_\lambda(\cdot))$, where β is a standard Brownian bridge.*

We remark that we denote $\sqrt{n}\,(F_{\rho^{(n)}(t)}(\cdot) - R(t,\cdot))$ with $G_n(t,\cdot)$ and we would like to warn the reader to note the difference between the function $G_n(t,\cdot)$ and the measure $G_n(t)(dx)$ defined in 1.1.2.

3.2 Propagation of chaos estimate 1

We are now ready to prove the main theorem in this chapter and this theorem provides an estimate of the distance between the measures $\rho^{(n)}(t)$ and $\bar{\rho}^{(n)}(t)$.

Theorem 3.2.1. *Suppose that Assumption 1.1.1 holds. Then, for all $p > 0$ and $T > 0$ there exists a constant $C = C(p,T) < \infty$ such that*

$$\forall\, n \in \mathbb{N},\ 1 \leq i \leq n: \quad \mathbb{E}\Big[\sup_{0\leq t\leq T}\big|X_i^{(n)}(t) - \bar{X}_i^{(n)}(t)\big|^p\Big] \leq C\,n^{-p/2}. \tag{3.2.1}$$

In particular, when $p \geq 1$ one has

$$\forall\, n \in \mathbb{N}: \quad \mathbb{E}\Big[\sup_{0\leq t\leq T} W_p\big(\rho^{(n)}(t),\bar{\rho}^{(n)}(t)\big)^p\Big] \leq C\,n^{-p/2}. \tag{3.2.2}$$

Proof of Theorem 3.2.1. Step 1. Fix any $p \geq 2$ and $T > 0$. We aim to employ Proposition 3.1.1 and to do so we are going to estimate the left-hand side of (3.2.1) by a quantity involving the left-hand side of (3.1.1). To this end, we first observe that the pairs $(X_i^{(n)},\bar{X}_i^{(n)})$, $i = 1, 2, \ldots, n$

have the same distribution (due to the weak uniqueness for (1.0.2) and the strong uniqueness for (1.1.1)) and therefore the left-hand side of (3.2.1) can be rewritten in the symmetrized form

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{0\le t\le T}\big|X_i^{(n)}(t)-\bar{X}_i^{(n)}(t)\big|^p\Big]. \tag{3.2.3}$$

Next, we use the SDEs (1.0.2) and (1.1.1) satisfied by $X_i^{(n)}$ and $\bar{X}_i^{(n)}$, the elementary inequality

$$(r_1+r_2)^p\le 2^{p-1}(r_1^p+r_2^p),\quad r_1,r_2\ge 0, \tag{3.2.4}$$

the Burkholder-Davis-Gundy inequality (see e.g. [KS, Chapter 3, Theorem 3.28]) and the Lipschitz property of b and σ to find for all $t\in[0,T]$ and $i=1,\,2,\,\dots,\,n$:

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le s\le t}\big|X_i^{(n)}(s)-\bar{X}_i^{(n)}(s)|^p\Big]\le\\ &C\,\mathbb{E}\bigg[\bigg(\int_0^t\big|F_{\rho^{(n)}(s)}(X_i^{(n)}(s))-R(s,\bar{X}_i^{(n)}(s))\big|\,\mathrm{d}s\bigg)^p\bigg]\\ &+C\,\mathbb{E}\bigg[\bigg(\int_0^t\big|F_{\rho^{(n)}(s)}(X_i^{(n)}(s))-R(s,\bar{X}_i^{(n)}(s))\big|^2\,\mathrm{d}s\bigg)^{p/2}\bigg],\end{aligned} \tag{3.2.5}$$

where $C<\infty$ depends only on p and the Lipschitz constants of b and σ. Applying Jensen's inequality to each of the summands on the right-hand side of (3.2.5) we obtain the further upper bound

$$C\,\mathbb{E}\bigg[\int_0^t\big|F_{\rho^{(n)}(s)}(X_i^{(n)}(s))-R(s,\bar{X}_i^{(n)}(s))\big|^p\,\mathrm{d}s\bigg], \tag{3.2.6}$$

where $C<\infty$ can be chosen in terms of T, p and the Lipschitz constants of b and σ.

Another application of (3.2.4) gives

$$
\begin{aligned}
& C\,\mathbb{E}\bigg[\int_0^t \big|F_{\rho^{(n)}(s)}(X_i^{(n)}(s)) - R(s,\bar{X}_i^{(n)}(s))\big|^p\,\mathrm{d}s\bigg] \\
& \leq C\,\mathbb{E}\bigg[\int_0^t \big|F_{\rho^{(n)}(s)}(X_i^{(n)}(s)) - R(s,X_i^{(n)}(s))\big|^p\,\mathrm{d}s\bigg] \\
& + C\,\mathbb{E}\bigg[\int_0^t \big|R(s,X_i^{(n)}(s)) - R(s,\bar{X}_i^{(n)}(s))\big|^p\,\mathrm{d}s\bigg],
\end{aligned}
\tag{3.2.7}
$$

where $C < \infty$ is still a function of T, p and the Lipschitz constants of b and σ only. Now, we take the average of the first summands on the right-hand side of (3.2.7) over $i = 1, 2, \ldots, n$ and get

$$
\begin{aligned}
& \frac{C}{n}\sum_{i=1}^n \mathbb{E}\bigg[\int_0^t \big|F_{\rho^{(n)}(s)}(X_i^{(n)}(s)) - R(s,X_i^{(n)}(s))\big|^p\,\mathrm{d}s\bigg] \\
& = C\int_0^t \mathbb{E}\bigg[\frac{1}{n}\sum_{k=1}^n \big|F_{\rho^{(n)}(s)}(X_{(k)}^{(n)}(s)) - R(s,X_{(k)}^{(n)}(s))\big|^p\bigg]\,\mathrm{d}s,
\end{aligned}
\tag{3.2.8}
$$

where $X_{(1)}^{(n)}(s) \leq X_{(2)}^{(n)}(s) \leq \cdots \leq X_{(n)}^{(n)}(s)$ are the order statistics of the vector $\big(X_1^{(n)}(s), X_2^{(n)}(s), \ldots, X_n^{(n)}(s)\big)$.

At this point, [Kr1, Theorem on p. 439] for the function $y \mapsto \sum_{1\leq i<j\leq n} \mathbf{1}_{\{y_i=y_j\}}$ on $\mathbb{R}^n$ reveals that with probability one it holds $F_{\rho^{(n)}(s)}(X_{(k)}^{(n)}(s)) = \frac{k}{n}$, $k = 1, 2, \ldots, n$ for Lebesgue a.e. $s \in [0,T]$. This and (3.2.4) allow to estimate the end result of (3.2.8) from above by

$$
\begin{aligned}
& C\int_0^t \mathbb{E}\bigg[\frac{1}{n}\sum_{k=1}^n \bigg|\frac{k}{n} - R(s,\bar{X}_{(k)}^{(n)}(s))\bigg|^p\bigg] \\
& + \mathbb{E}\bigg[\frac{1}{n}\sum_{k=1}^n \big|R(s,\bar{X}_{(k)}^{(n)}(s)) - R(s,X_{(k)}^{(n)}(s))\big|^p\bigg]\,\mathrm{d}s,
\end{aligned}
\tag{3.2.9}
$$

where $\bar{X}_{(1)}^{(n)}(s) \leq \bar{X}_{(2)}^{(n)}(s) \leq \cdots \leq \bar{X}_{(n)}^{(n)}(s)$ are the order statistics of the vector $\big(\bar{X}_1^{(n)}(s), \bar{X}_2^{(n)}(s), \ldots, \bar{X}_n^{(n)}(s)\big)$ and $C < \infty$ depends on T, p and the Lipschitz constants of b and σ only.

Step 2. Relying on the representation (2.1.10) we readily identify the quantity $\mathbb{E}\big[\frac{1}{n}\sum_{k=1}^n \big|\frac{k}{n} - R(s, \bar{X}_{(k)}^{(n)}(s))\big|^p\big]$ in (3.2.9) as

$$\mathbb{E}\bigg[W_p\bigg(\frac{1}{n}\sum_{k=1}^n \delta_{k/n},\, \frac{1}{n}\sum_{k=1}^n \delta_{R(s,\bar{X}_{(k)}^{(n)}(s))}\bigg)^p\bigg]. \tag{3.2.10}$$

The observation (2.4.2) reveals $R(s, \bar{X}_{(1)}^{(n)}(s)) \le R(s, \bar{X}_{(2)}^{(n)}(s)) \le \cdots \le R(s, \bar{X}_{(n)}^{(n)}(s))$ as the order statistics of an i.i.d. sample from the uniform distribution on $[0, 1]$. This, the triangle inequality for W_p and (3.2.4) imply that the expectation in (3.2.10) is bounded above by

$$2^{p-1}\, W_p\bigg(\frac{1}{n}\sum_{k=1}^n \delta_{k/n},\, \upsilon\bigg)^p + 2^{p-1}\,\mathbb{E}\bigg[W_p\bigg(\upsilon,\, \frac{1}{n}\sum_{i=1}^n \delta_{U_i}\bigg)^p\bigg] \tag{3.2.11}$$

in the notation of Proposition 3.1.1. Using the representation (2.1.10) for the first expectation in (3.2.11) and Proposition 3.1.1 for the second expectation in (3.2.11) we end up with the upper bound

$$2^{p-1}\, n^{-p} + 2^{p-1}\, C^p\, p^{p/2}\, n^{-p/2}, \tag{3.2.12}$$

where C is the constant in Proposition 3.1.1.

Step 3. Putting the estimates (3.2.5), (3.2.7), (3.2.9) and (3.2.12) together we arrive at the inequality

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[\sup_{0\le s\le t}\big|X_i^{(n)}(s) - \bar{X}_i^{(n)}(s)\big|^p\Big] \\
&\le C\int_0^t \bigg(n^{-p} + n^{-p/2} + \mathbb{E}\bigg[\frac{1}{n}\sum_{k=1}^n \big|R(s, X_{(k)}^{(n)}(s)) - R(s, \bar{X}_{(k)}^{(n)}(s))\big|^p\bigg] \\
&\qquad\qquad + \mathbb{E}\bigg[\frac{1}{n}\sum_{i=1}^n \big|R(s, X_i^{(n)}(s)) - R(s, \bar{X}_i^{(n)}(s))\big|^p\bigg]\bigg)\,\mathrm{d}s
\end{aligned} \tag{3.2.13}$$

for all $t \in [0, T]$, where $C < \infty$ is a function of T, p and the Lipschitz constants of b and σ. Moreover, the functions $x \mapsto R(s, x)$ are Lipschitz with uniformly bounded Lipschitz constants as s varies in $[0, T]$ by Proposition

2.3.1 and

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\big|X^{(n)}_{(k)}(s)-\bar X^{(n)}_{(k)}(s)\big|^p &= W_p\big(\rho^{(n)}(s),\bar\rho^{(n)}(s)\big)^p\\ &\le \frac{1}{n}\sum_{i=1}^{n}\big|X^{(n)}_{i}(s)-\bar X^{(n)}_{i}(s)\big|^p\end{aligned} \tag{3.2.14}$$

by the representation (2.1.10) and the definition of W_p in (2.1.8), so that for all $t\in[0,T]$:

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{0\le s\le t}\big|X^{(n)}_{i}(s)-\bar X^{(n)}_{i}(s)\big|^p\Big]\le\; C\big(n^{-p}+n^{-p/2}\big)t\\ &+C\int_0^t\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{0\le r\le s}\big|X^{(n)}_{i}(r)-\bar X^{(n)}_{i}(r)\big|^p\Big]\,\mathrm{d}s,\end{aligned} \tag{3.2.15}$$

where $C<\infty$ depends on T, p, the Lipschitz constants of b and σ and the supremum of R_x on $[0,T]\times\mathbb{R}$ only. The desired estimate (3.2.1) is a consequence of (3.2.15) due to the representation (3.2.3) and Gronwall's lemma.

Step 4. For $p\in(0,2)$, we choose a $p'\in[2,\infty)$ and deduce (3.2.1) for p from (3.2.1) for p' by means of the inequality

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{0\le t\le T}\big|X^{(n)}_{i}(t)-\bar X^{(n)}_{i}(t)\big|^p\Big]\\ &\le\bigg(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{0\le t\le T}\big|X^{(n)}_{i}(t)-\bar X^{(n)}_{i}(t)\big|^{p'}\Big]\bigg)^{p/p'}.\end{aligned} \tag{3.2.16}$$

Finally, we obtain (3.2.2) from (3.2.1) via the chain of estimates

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le T}W_p\big(\rho^{(n)}(t),\bar\rho^{(n)}(t)\big)\Big]^p\le\mathbb{E}\Big[\sup_{0\le t\le T}W_p\big(\rho^{(n)}(t),\bar\rho^{(n)}(t)\big)^p\Big]\\ &\le\mathbb{E}\bigg[\sup_{0\le t\le T}\frac{1}{n}\sum_{i=1}^{n}\big|X^{(n)}_{i}(t)-\bar X^{(n)}_{i}(t)\big|^p\bigg]\\ &\le\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\Big[\sup_{0\le t\le T}\big|X^{(n)}_{i}(t)-\bar X^{(n)}_{i}(t)\big|^p\Big]\end{aligned} \tag{3.2.17}$$

valid for all $p \geq 1$. $\square$

The following corollary settles the question of rate of convergence of $F_{\rho^{(n)}(t)}(\cdot)$ to $R(t,\cdot)$.

Corollary 3.2.2. *Suppose the Assumption 1.1.1 holds. There exists a constant $C(t)$ depending on t such that, $\mathbb{E}\Big[W_1\big(\rho^{(n)}(t),\rho(t)\big)\Big] \leq \frac{C(t)}{\sqrt{n}}$ holds for all $t \geq 0$.*

Proof. Thanks to the triangular inequality, we have

$$\mathbb{E}\Big[W_1\big(\rho^{(n)}(t),\rho(t)\big)\Big] < \mathbb{E}\Big[W_1\big(\rho^{(n)}(t),\bar{\rho}^{(n)}(t)\big)\Big] + \mathbb{E}\Big[W_1\big(\bar{\rho}^{(n)}(t),\rho(t)\big)\Big] \tag{3.2.18}$$

Let C be a constant depending on t. (3.2.2) immediately gives $\mathbb{E}\Big[W_1\big(\rho^{(n)}(t),\bar{\rho}^{(n)}(t)\big)\Big] \leq \frac{C}{\sqrt{n}}$. In view of [BL, Theorem 3.2 and the discussion of the functional J_1 on p. 25], (2.4.7) and (2.4.2), we obtain $\mathbb{E}\Big[W_1\big(\bar{\rho}^{(n)}(t),\rho(t)\big)\Big] \leq \frac{C}{\sqrt{n}}$. Combining these two estimates finishes the proof. $\square$

3.3 Propagation of chaos estimate 2

The first propogation of chaos estimate tells us that $\big|X_i^{(n)}(t) - \bar{X}_i^{(n)}(t)\big|$ is of the order $O\big(\frac{1}{\sqrt{n}}\big)$. We set out to show that $X_i^{(n)}(t) = \bar{X}_i^{(n)}(t) + Z_i^{(n)}(t) + o\big(\frac{1}{\sqrt{n}}\big)$, where the process $Z_i^{(n)}(t)$ depends on the processes $\bar{X}_i^{(n)}(t)$, $\forall\ i = 1, 2, \ldots, n$. This approach was used in [Sz1, Theorem 2.3] and would allow us to characterize the fluctuations in terms of the processes $\bar{X}_i^{(n)}(t)$ and $Z_i^{(n)}(t)$. This endeavour was unsuccessful because of the non smooth interaction between the particles in (1.0.2). However, we were able to prove the following estimate in this direction.

Theorem 3.3.1. *Under the Assumption 1.1.1 and the stronger assumption in proposition 2.3.2, the following holds for all positive p, T*

$$\begin{aligned} n^{p/2-1}\sum_{i=1}^{n}\mathbb{E}\Big[\Big|&F_{\bar{\rho}_n(t)}\big(X_i^{(n)}(t)\big) - F_{\bar{\rho}_n(t)}\big(\bar{X}_i^{(n)}(t)\big) \\ &- R_x\big(t,\bar{X}_i^{(n)}(t)\big)\big(X_i^{(n)}(t) - \bar{X}_i^{(n)}(t)\big)\Big|^p\Big] \xrightarrow{n\to\infty} 0 \end{aligned} \tag{3.3.1}$$

uniformly in $t \in [0,T]$.

Proof of Theorem 3.3.1. Step 1. We first fix an $i \in \{1,2,\ldots,n\}$ and consider the expectation in (3.3.1) for that i. Clearly, for any $K \in (0,\infty)$, it can decomposed into the expectations over the events

$$\begin{aligned} &\Big\{\bar{X}_i^{(n)}(t) \le X_i^{(n)}(t) \le \bar{X}_i^{(n)}(t) + \frac{K}{n^{1/2}}\Big\} \\ &\Big\{\bar{X}_i^{(n)}(t) - \frac{K}{n^{1/2}} \le X_i^{(n)}(t) < \bar{X}_i^{(n)}(t)\Big\} \\ \text{and} \quad &\Big\{\big|X_i^{(n)}(t) - \bar{X}_i^{(n)}(t)\big| > \frac{K}{n^{1/2}}\Big\}. \end{aligned} \tag{3.3.2}$$

The expectation over the first event can be bounded above further by

$$\begin{aligned} \mathbb{E}\Big[\sup_{\bar{X}_i^{(n)}(t) \le x \le \bar{X}_i^{(n)}(t) + \frac{K}{n^{1/2}}} \Big| F_{\bar{\rho}_n(t)}(x) - F_{\bar{\rho}_n(t)}\big(\bar{X}_i^{(n)}(t)\big) \\ - R_x\big(t, \bar{X}_i^{(n)}(t)\big)\big(x - \bar{X}_i^{(n)}(t)\big)\Big|^p\Big]. \end{aligned} \tag{3.3.3}$$

Moreover, in view of (3.2.4), we can estimate the random variable inside the expectation in (3.3.3) conditional on

$$\bar{X}_i^{(n)}(t) = \bar{x} \quad \text{and} \quad \Big|\Big\{1 \le j \le n :\, \bar{x} < \bar{X}_j^{(n)}(t) \le \bar{x} + \frac{K}{n^{1/2}}\Big\}\Big| = m \tag{3.3.4}$$

for some $\bar{x} \in \mathbb{R}$ and $m \in \mathbb{N}$ by

$$\begin{aligned} &\frac{Cm^p}{n^p} \sup_{\bar{x} \le x \le \bar{x} + \frac{K}{n^{1/2}}} \Big| \frac{n}{m}\big(F_{\bar{\rho}_n(t)}(x) - F_{\bar{\rho}_n(t)}(\bar{x})\big) - \frac{R(t,x) - R(t,\bar{x})}{R(t,\bar{x} + K/n^{1/2}) - R(t,\bar{x})}\Big|^p \\ &+ C \sup_{\bar{x} \le x \le \bar{x} + \frac{K}{n^{1/2}}} \Big| \frac{m}{n}\, \frac{R(t,x) - R(t,\bar{x})}{R(t,\bar{x} + K/n^{1/2}) - R(t,\bar{x})} - R_x(t,\bar{x})(x - \bar{x})\Big|^p, \end{aligned} \tag{3.3.5}$$

where $C = 2^{p-1}$ is the constant in (3.2.4).

Thanks to the independence of $\bar{X}_1^{(n)}(t)$, $\bar{X}_2^{(n)}(t)$, $\ldots$, $\bar{X}_n^{(n)}(t)$ we can now apply the Dvoretzky-Kiefer-Wolfowitz inequality in the form of [Mas, Corollary 1] to conclude that the conditional expectation of the first summand in

(3.3.5) given the events in (3.3.4) is at most $\frac{Cm^{p/2}}{n^p}$, where $C<\infty$ is a constant depending only on p. In addition, we use (3.2.4) to bound the second term in (3.3.5) from above by

$$\begin{aligned}
&C \sup_{\bar{x}\le x\le \bar{x}+\frac{K}{n^{1/2}}} \left|\frac{m}{n}\frac{R(t,x)-R(t,\bar{x})}{\big(R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})\big)} - \frac{n-1}{n}\big(R(t,x)-R(t,\bar{x})\big)\right|^p \\
&+C \sup_{\bar{x}\le x\le \bar{x}+\frac{K}{n^{1/2}}} \left|\frac{n-1}{n}\big(R(t,x)-R(t,\bar{x})\big) - R_x(t,\bar{x})(x-\bar{x})\right|^p \\
&= \frac{C}{n^p}\left|m-(n-1)\big(R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})\big)\right|^p \\
&\quad +C \sup_{\bar{x}\le x\le \bar{x}+\frac{K}{n^{1/2}}} \left|\frac{n-1}{n}\big(R(t,x)-R(t,\bar{x})\big) - R_x(t,\bar{x})(x-\bar{x})\right|^p,
\end{aligned} \tag{3.3.6}$$

where the constant $C<\infty$ depends only on p.

At this point, we consider the conditional expectation of the random variable inside the expectation in (3.3.3) given $\bar{X}_i^{(n)}(t)=\bar{x}$. Writing Y for a binomial random variable with parameters $(n-1)$ and $R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})$ we conclude from the above that the contribution of the first summand in (3.3.6) is at most $\frac{C}{n^p}\,\mathbb{E}[Y^{p/2}]$. Moreover, we claim

$$\frac{C}{n^p}\,\mathbb{E}[Y^{p/2}] = o(n^{-p/2}), \tag{3.3.7}$$

where the right-hand side can be chosen to depend only on n, p, the supremum norm $\|R_x(t,\cdot)\|_\infty$, and K. Indeed, for $p\ge 2$, we can view Y as a sum of $(n-1)$ i.i.d. Bernoulli random variables with parameter $R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})$ and then use Jensen's inequality to bound $\mathbb{E}[Y^{p/2}]$ by the $(n-1)^{p/2}$ multiple of the $p/2$-th moment of such a Bernoulli random variable. The latter can be further dominated by the $p/2$-th moment of a Bernoulli random variable with parameter $\|R_x(t,\cdot)\|_\infty K/n^{1/2}$, which is $o(1)$, so that (3.3.7) for $p\ge 2$ follows. To obtain (3.3.7) for $p\in(0,2)$ it suffices to combine (3.3.7) for some $p'\ge 2$ with Jensen's inequality.

In addition, the contribution of the first summand on the right-hand side of (3.3.6) to the conditional expectation given $\bar{X}_i^{(n)}(t)=\bar{x}$ can be bounded above by

$$\frac{C}{n^p}\,\mathbb{E}\Big[\Big|Y-(n-1)\big(R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})\big)\Big|^p\Big], \tag{3.3.8}$$

where Y is a binomial random variable as before. We claim that the expression in (3.3.8) can be estimated by a function of n, p, $\|R_x(t,\cdot)\|_\infty$, and K of order $o(n^{-p/2})$. It suffices to consider even integer p's, since for other p's we can combine the estimate for an even integer $p'>p$ with Jensen's inequality to obtain the estimate for p. For an even integer p, we write

$$Y-(n-1)\big(R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})\big)$$

as a sum of i.i.d. centered (by their mean) Bernoulli random variables with parameter $R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})$. Raising this sum to power p, taking the expectation, dropping the terms equal to zero, and collecting equal terms we end up with a degree $p/2$ polynomial of $(n-1)$ with leading coefficient

$$\frac{1}{(p/2)!}\,\frac{p!}{2^{p/2}}\,\Big(\big(R(t,\bar{x}+K/n^{1/2})-R(t,\bar{x})\big)\big(1-R(t,\bar{x}+K/n^{1/2})+R(t,\bar{x})\big)\Big)^{p/2}. \tag{3.3.9}$$

In view of the inequality

$$R(t,\bar{x}+K/n^{1/2})-R(t,x)\le\|R_x(t,\cdot)\|_\infty K/n^{1/2}, \tag{3.3.10}$$

the latter polynomial is, in fact, of order $o(n^{p/2})$, which gives the desired estimate on the expression in (3.3.8).

Next, we apply the Mean Value Theorem to the term $R(t,x)-R(t,\bar{x})-R_x(t,\bar{x})(x-\bar{x})$ to find that the second summand on the right-hand side of (3.3.6) is of the order

$$\frac{C}{n^p}, \tag{3.3.11}$$

where the constant $C<\infty$ depends only on p and $\|R_{xx}(t,\cdot)\|_\infty$.

Putting together all the obtained estimates (noting that they do not depend on $\bar{x}$) and using exactly the same arguments for the expectation over the second event in (3.3.2) we find that the limit superior as $n\to\infty$ of the expression in (3.3.1) equals to

$$
\begin{aligned}
&\limsup_{n\to\infty} n^{p/2-1}\sum_{i=1}^{n}\mathbb{E}\Big[\Big|F_{\bar\rho_n(t)}(X_i^{(n)}(t))-F_{\bar\rho_n(t)}(\bar X_i^{(n)}(t))\\
&-R_x(t,\bar X_i^{(n)}(t))(X_i^{(n)}(t)-\bar X_i^{(n)}(t))\Big|^p\mathbf{1}_{\left\{|X_i^{(n)}(t)-\bar X_i^{(n)}(t)|>\frac{K}{n^{1/2}}\right\}}\Big].
\end{aligned}
\tag{3.3.12}
$$

Step 2. To control the limit superior in (3.3.12) we use (3.2.4) to bound the expectation in (3.3.12) for a fixed $i\in\{1,2,\dots,n\}$ from above by

$$
\begin{aligned}
&C\,\mathbb{E}\Big[\Big|F_{\bar\rho_n(t)}(X_i^{(n)}(t))-R(t,X_i^{(n)}(t))-F_{\bar\rho_n(t)}(\bar X_i^{(n)}(t))+R(t,\bar X_i^{(n)}(t))\Big|^p\mathbf{1}_{A_{i,K}^{(n)}}\Big]\\
&+C\,\mathbb{E}\Big[\Big|R(t,X_i^{(n)}(t))-R(t,\bar X_i^{(n)}(t))\\
&-R_x(t,\bar X_i^{(n)}(t))(X_i^{(n)}(t)-\bar X_i^{(n)}(t))\Big|^p\,\mathbf{1}_{A_{i,K}^{(n)}}\Big],
\end{aligned}
\tag{3.3.13}
$$

where $A_{i,K}^{(n)}:=\{|X_i^{(n)}(t)-\bar X_i^{(n)}(t)|>K/n^{1/2}\}$ and $C=2^{p-1}$ is the constant in (3.2.4). With the help of the Cauchy-Schwarz inequality the first summand in (3.3.13) can be estimated further by

$$
C\,\mathbb{E}\Big[\sup_{x\in\mathbb{R}}\big|F_{\bar\rho_n(t)}(x)-R(t,x)\big|^{2p}\Big]^{1/2}\,\mathbb{P}\big(A_{i,K}^{(n)}\big)^{1/2},
\tag{3.3.14}
$$

where $C=2^{2p-1}$. In view of the Dvoretzky-Kiefer-Wolfowitz inequality in the form of [Mas, Corollary 1], the latter estimate is at most $Cn^{-p/2}\,\mathbb{P}(A_{i,K}^{(n)})^{1/2}$, where the constant $C<\infty$ depends only on p. In addition, the second summand in (3.3.13) can be bounded above by $C\,\mathbb{E}\big[|X_i^{(n)}(t)-\bar X_i^{(n)}(t)|^p\,\mathbf{1}_{A_{i,K}^{(n)}}\big]$, where $C<\infty$ depends only on p and $\|R_x(t,\cdot)\|_\infty$. The latter bound is at most $C\,\mathbb{E}\big[|X_i^{(n)}(t)-\bar X_i^{(n)}(t)|^{2p}\big]^{1/2}\,\mathbb{P}(A_{i,K}^{(n)})^{1/2}$ by the Cauchy-Schwarz inequality.

Summing over i we conclude that the limit superior in (3.3.12) is less or equal to

$$
\begin{aligned}
&\limsup_{n\to\infty} Cn^{-1}\sum_{i=1}^{n}\mathbb{P}(A_{i,K}^{(n)})^{1/2}\\
&+\limsup_{n\to\infty} Cn^{p/2-1}\sum_{i=1}^{n}\mathbb{E}\big[|X_i^{(n)}(t)-\bar{X}_i^{(n)}(t)|^{2p}\big]^{1/2}\mathbb{P}(A_{i,K}^{(n)})^{1/2}.
\end{aligned}
\tag{3.3.15}
$$

Next, we apply Jensen's inequality to the first sum and the Cauchy-Schwarz inequality to the second sum and then estimate $\mathbb{P}(A_{i,K}^{(n)})$ using Markov's inequality to obtain the further upper bound

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{C}{K}\bigg(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big[|X_i^{(n)}(t)-\bar{X}_i^{(n)}(t)|^{2}\big]n\bigg)^{1/2}\\
&+\limsup_{n\to\infty}\frac{Cn^{p/2}}{K}\bigg\{\bigg(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big[|X_i^{(n)}(t)-\bar{X}_i^{(n)}(t)|^{2p}\big]\bigg)^{1/2}\\
&\qquad\bigg(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big[|X_i^{(n)}(t)-\bar{X}_i^{(n)}(t)|^{2}\big]n\bigg)^{1/2}\bigg\}.
\end{aligned}
\tag{3.3.16}
$$

At this point, the estimate (3.2.1) of Theorem 3.2.1 reveals that the latter upper bound is not larger than $\frac{C}{K}$, with a constant $C<\infty$ depending only on p, $\|R_x(t,\cdot)\|_\infty$, and T. Noting that R has classical regularity owing to proposition 2.3.2, we pass to the limit $K\uparrow\infty$ keeping in mind that all the constants appearing in the present proof can be chosen to not depend on $t\in[0,T]$. $\square$

4

Central Limit Theorem

In this chapter, we will study the fluctuations of the particle system (1.0.2). We begin by proving a tightness result that guarantees the existence of subsequential limits. We will then derive the limit and prove its uniqueness. The main theorem in this chapter, which is also the main result of this thesis is as follows :

Theorem 4.0.1. *Suppose that Assumption 1.1.1 holds and consider the mild solution G of the SPDE*

$$G_t = -\big(b(R)\,G\big)_x + \left(\frac{\sigma(R)^2}{2}\,G\right)_{xx} + \sigma(R)\,R_x^{1/2}\,\dot{W}, \quad G(0,\cdot) = \beta(F_\lambda(\cdot)), \tag{4.0.1}$$

where R is the unique generalized solution to the Cauchy problem (1.0.3)*, $\dot{W}$ is a space-time white noise and β is a standard Brownian bridge independent* *of $\dot{W}$. More specifically, let G be the random field defined by*

$$G(t,x) = \int_{\mathbb{R}} \beta(F_\lambda(y))\, p(0,y;t,x)\,\mathrm{d}y$$
$$+ \int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2}\, p(s,y;t,x)\,\mathrm{d}W(s,y), (t,x) \in [0,\infty) \times \mathbb{R}, \tag{4.0.2}$$

where p denotes the transition density of the solution to the martingale problem associated with the operators $b(R(t,\cdot))\,\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma(R(t,\cdot))^2}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $t \geq 0$ and the double integral should be understood in the Itô sense.

Then, one has the following convergences:

(a) The $M_{\mathrm{fin}}(\mathbb{R})$-valued processes G_n, $n \in \mathbb{N}$ 1.1.2 tend in the finite-dimensional distribution sense to $t \mapsto G(t,x)\,\mathrm{d}x$.

(b) The processes H_n, $n \in \mathbb{N}$ 1.1.3 taking values in $M_{\mathrm{fin}}([0,t] \times \mathbb{R})$, $t > 0$ converge in the finite-dimensional distribution sense to $t \mapsto G(s,x)\,\mathbf{1}_{[0,t]\times\mathbb{R}}(s,x)\,\mathrm{d}s\,\mathrm{d}x$, also jointly with the processes in (a).

4.1 Existence of subsequential limits

The main result of this section is the next proposition establishing the existence of subsequential limits for the finite-dimensional distributions of the fluctuation processes G_n, $n \in \mathbb{N}$ and H_n, $n \in \mathbb{N}$. It serves as a key ingredient in the proof of Theorem 4.0.1.

Proposition 4.1.1. *Suppose that Assumption 1.1.1 is satisfied. Then, for all $m \in \mathbb{N}$ and $0 < t_1 < \cdots < t_m$ every subsequence of*

$$\big(G_n(0), G_n(t_1), \ldots, G_n(t_m), H_n(t_1), H_n(t_2), \ldots, H_n(t_m)\big), \quad n \in \mathbb{N} \tag{4.1.1}$$

has a further subsequence which converges in law in

$$M_{\mathrm{fin}}(\mathbb{R})^{m+1} \times M_{\mathrm{fin}}([0,t_1] \times \mathbb{R}) \times M_{\mathrm{fin}}([0,t_2] \times \mathbb{R}) \times \cdots \times M_{\mathrm{fin}}([0,t_m] \times \mathbb{R}).$$

Proof. By Prokhorov's Theorem in the form of [FGH, Corollary on p. 119] it suffices to show that the laws of the random vectors in (4.1.1) form a uniformly tight sequence. Moreover, since products of compact sets are compact, we only need to prove that for all $s \geq 0$ and $t > 0$ the laws associated with the sequences $G_n(s)$, $n \in \mathbb{N}$ and $H_n(t)$, $n \in \mathbb{N}$ are uniformly tight. In view of the Banach-Alaoglu Theorem (see e.g. [La, Chapter 12, Theorem 3]), this is the case for any fixed $s \geq 0$ and $t > 0$ if for all $\epsilon > 0$ there exists a $C_\epsilon < \infty$ such that

$$\forall\, n \in \mathbb{N}: \quad \mathbb{P}\big(\|G_n(s)\|_{TV} > C_\epsilon\big) < \epsilon \quad \text{and} \quad \mathbb{P}\big(\|H_n(t)\|_{TV} > C_\epsilon\big) < \epsilon, \tag{4.1.2}$$

where $\|\cdot\|_{TV}$ stands for the total variation norm.

By the definitions of $G_n(s)$, $n \in \mathbb{N}$ and $H_n(t)$, $n \in \mathbb{N}$ in (1.1.2) and (1.1.3) the two inequalities of (4.1.2) can be rewritten as

$$\mathbb{P}\bigg(\sqrt{n}\,\int_{\mathbb{R}} \big|F_{\rho^{(n)}(s)}(x) - R(s,x)\big|\,\mathrm{d}x > C_\epsilon\bigg) < \epsilon, \tag{4.1.3}$$

$$\mathbb{P}\bigg(\sqrt{n}\,\int_0^t \int_{\mathbb{R}} \big|F_{\rho^{(n)}(r)}(x) - R(r,x)\big|\,\mathrm{d}x\,\mathrm{d}r > C_\epsilon\bigg) < \epsilon. \tag{4.1.4}$$

The representation (2.1.9) allows to rewrite these further as

$$\begin{aligned} &\mathbb{P}\Big(\sqrt{n}\,W_1(\rho^{(n)}(s),\rho(s)) > C_\epsilon\Big) < \epsilon \\ &\mathbb{P}\bigg(\sqrt{n}\,\int_0^t W_1(\rho^{(n)}(r),\rho(r))\,\mathrm{d}r > C_\epsilon\bigg) < \epsilon. \end{aligned} \tag{4.1.5}$$

Applying Markov's inequality, the triangle inequality for W_1 and Fubini's Theorem we bound the two probabilities in (4.1.5) from above by

$$\frac{\sqrt{n}}{C_\epsilon}\,\mathbb{E}\big[W_1(\rho^{(n)}(s),\bar{\rho}^{(n)}(s))\big] + \frac{\sqrt{n}}{C_\epsilon}\,\mathbb{E}\big[W_1(\bar{\rho}^{(n)}(s),\rho(s))\big], \tag{4.1.6}$$

$$\frac{\sqrt{n}}{C_\epsilon}\int_0^t \Big(\mathbb{E}\big[W_1(\rho^{(n)}(r),\bar{\rho}^{(n)}(r))\big] + \mathbb{E}\big[W_1(\bar{\rho}^{(n)}(r),\rho(r))\big]\Big)\,\mathrm{d}r \tag{4.1.7}$$

respectively. In view of (3.2.2), [BL, Theorem 3.2 and the discussion of the functional J_1 on p. 25], (2.4.7) and (2.4.2), we can make the estimates (4.1.6), (4.1.7) smaller than ϵ for all $n \in \mathbb{N}$ by choosing a large enough $C_\epsilon < \infty$. □

4.2 Identification of subsequential limits

In this section we identify the subsequential limits of Proposition 4.1.1 and complete the proof of Theorem 4.0.1. The next proposition is the first step towards such an identification.

Proposition 4.2.1. *Suppose that Assumption 1.1.1 holds and let*

$$\big(G_\infty(0), G_\infty(t_1), \ldots, G_\infty(t_m), H_\infty(t_1), H_\infty(t_2), \ldots, H_\infty(t_m)\big) \tag{4.2.1}$$

be a limit point in law of the sequence in (4.1.1)*. Then, with the notation of Theorem 4.0.1 and*

$$\begin{aligned}(\mathcal{A}_s\gamma)(s,x) :=& \gamma_s(s,x) + \gamma_x(s,x)\, b(R(s,x)) \\ &+ \gamma_{xx}(s,x)\, \frac{\sigma(R(s,x))^2}{2}, \quad (s,x) \in [0,t] \times \mathbb{R},\end{aligned} \tag{4.2.2}$$

the joint distribution of

$$\begin{aligned}&\int_{\mathbb{R}} \gamma(t_\ell,x)\, G_\infty(t_\ell)(\mathrm{d}x) - \int_{\mathbb{R}} \gamma(0,x)\, G_\infty(0)(\mathrm{d}x) \\ &- \int_0^{t_\ell} \int_{\mathbb{R}} (\mathcal{A}_s\gamma)(s,x)\, H_\infty(t_\ell)(\mathrm{d}s, \mathrm{d}x), \int_{\mathbb{R}} \gamma(0,x)\, G_\infty(0)(\mathrm{d}x),\end{aligned} \tag{4.2.3}$$

as ℓ and γ vary over $\{1,2,\ldots,m\}$ and the space of functions on $[0,t_\ell] \times \mathbb{R}$ which are continuously differentiable in s, twice continuously differentiable in x and compactly supported, coincides with that of

$$\int_0^{t_\ell} \int_{\mathbb{R}} \gamma(s,x)\, \sigma(R(s,x))\, R_x(s,x)^{1/2}\, \mathrm{d}W(s,x), \qquad \int_{\mathbb{R}} \gamma(0,x)\, \beta(F_\lambda(x))\, \mathrm{d}x. \tag{4.2.4}$$

The proof of Proposition 4.2.1 relies on a suitable prelimit version of its statement. For every fixed $n \in \mathbb{N}$ let B_n, Σ_n be the piecewise constant functions on $[0,1]$ with jumps at $\frac{1}{n}$, $\frac{2}{n}$, $\ldots$, 1 and

$$B_n(k/n) = \frac{1}{n} \sum_{j=1}^{k} b(j/n), \;\; \Sigma_n(k/n) = \frac{1}{n} \sum_{j=1}^{k} \frac{\sigma(j/n)^2}{2}, \quad k = 0,\, 1,\, \ldots,\, n. \tag{4.2.5}$$

Lemma 4.2.2. *Suppose that Assumption 1.1.1 is satisfied. Then, for any $n \in \mathbb{N}$, $t > 0$ and function γ on $[0,t] \times \mathbb{R}$ which is continuously differentiable in s, twice continuously differentiable in x and compactly supported it holds*

$$
\begin{aligned}
&\int_{\mathbb{R}} \gamma(t,x)\, G_n(t)(\mathrm{d}x) - \int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x) \\
&- \int_0^t \int_{\mathbb{R}} \int_0^1 \bigg(\gamma_s(s,x) + \gamma_x(s,x)\, b(aF_{\rho^{(n)}(s)}(x) + (1-a)R(s,x)) \\
&\qquad\qquad + \gamma_{xx}(s,x)\, \frac{\sigma(aF_{\rho^{(n)}(s)}(x) + (1-a)R(s,x))^2}{2} \bigg)\, \mathrm{d}a\ H_n(t)(\mathrm{d}s, \mathrm{d}x) \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^t \gamma(s, X_i^{(n)}(s))\, \sigma(F_{\rho^{(n)}(s)}(X_i^{(n)}(s)))\, \mathrm{d}B_i^{(n)}(s) \\
&\quad + \sqrt{n} \int_0^t \int_{\mathbb{R}} \big(\gamma_x(s,x)\, (B_n - B)(F_{\rho^{(n)}(s)}(x)) \\
&\qquad\qquad + \gamma_{xx}(s,x)\, (\Sigma_n - \Sigma)(F_{\rho^{(n)}(s)}(x)) \big)\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned}
\tag{4.2.6}
$$

Proof of Lemma 4.2.2. Fixing n, t and γ as described we observe that Definition 2.2.1 of a generalized solution to the Cauchy problem (1.0.3) implies

$$
\begin{aligned}
&\int_{\mathbb{R}} \gamma(t,x)\, R(t,x)\, \mathrm{d}x - \int_{\mathbb{R}} \gamma(0,x)\, R(0,x)\, \mathrm{d}x \\
&= \int_0^t \int_{\mathbb{R}} \gamma_s(s,x)\, R(s,x) + \gamma_x(s,x)\, B(R(s,x)) + \gamma_{xx}(s,x)\, \Sigma(R(s,x))\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned}
\tag{4.2.7}
$$

To find a version of the identity (4.2.7) with $F_{\rho^{(n)}(\cdot)}(\cdot)$ in place of $R(\cdot,\cdot)$ we apply Itô's formula for $\Gamma(s,x) := -\int_x^\infty \gamma(s,y)\,\mathrm{d}y$ and obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \Gamma(t,x)\,\rho^{(n)}(t)(\mathrm{d}x) - \int_{\mathbb{R}} \Gamma(0,x)\,\rho^{(n)}(0)(\mathrm{d}x) \\
&= \frac{1}{n}\sum_{i=1}^{n} \int_0^t \gamma(s, X_i^{(n)}(s))\,\sigma(F_{\rho^{(n)}(s)}(X_i^{(n)}(s)))\,\mathrm{d}B_i^{(n)}(s) \\
&\quad + \int_0^t\int_{\mathbb{R}} \Big(\Gamma_s(s,x) + \Gamma_x(s,x) b(F_{\rho^{(n)}(s)}(x)) \\
&\qquad\qquad + \Gamma_{xx}(s,x)\frac{\sigma(F_{\rho^{(n)}(s)}(x))^2}{2}\Big)\rho^{(n)}(s)(\mathrm{d}x)\,\mathrm{d}s.
\end{aligned} \tag{4.2.8}$$

Next, we use summation by parts (note that $\lim_{x\to\infty}\Gamma(s,x) = 0$ and $\lim_{x\to\infty}\Gamma_s(s,x) = 0$ for all $s \in [0,t]$ by the compact support assumption on γ) to compute

$$\int_{\mathbb{R}} \Gamma(s,x)\rho^{(n)}(s)(\mathrm{d}x) = -\int_{\mathbb{R}} \gamma(s,x) F_{\rho^{(n)}(s)}(x)\,\mathrm{d}x, \tag{4.2.9}$$

$$\int_{\mathbb{R}} \Gamma_s(s,x)\rho^{(n)}(s)(\mathrm{d}x) = -\int_{\mathbb{R}} \gamma_s(s,x) F_{\rho^{(n)}(s)}(x)\,\mathrm{d}x, \tag{4.2.10}$$

$$\begin{aligned}
&\int_{\mathbb{R}} \Gamma_x(s,x) b(F_{\rho^{(n)}(s)}(x))\rho^{(n)}(s)(\mathrm{d}x) \\
&\qquad = -\int_{\mathbb{R}} \gamma_x(s,x) B_n(F_{\rho^{(n)}(s)}(x))\,\mathrm{d}x,
\end{aligned} \tag{4.2.11}$$

$$\begin{aligned}
&\int_{\mathbb{R}} \Gamma_{xx}(s,x)\frac{\sigma(F_{\rho^{(n)}(s)}(x))^2}{2}\rho^{(n)}(s)(\mathrm{d}x) \\
&\qquad = -\int_{\mathbb{R}} \gamma_{xx}(s,x) \Sigma_n(F_{\rho^{(n)}(s)}(x))\,\mathrm{d}x,
\end{aligned} \tag{4.2.12}$$

where B_n, Σ_n are defined according to (4.2.5). Inserting the identities

(4.2.9)-(4.2.12) into (4.2.8) we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}} \gamma(t,x)\, F_{\rho^{(n)}(t)}(x)\,\mathrm{d}x - \int_{\mathbb{R}} \gamma(0,x)\, F_{\rho^{(n)}(0)}(x)\,\mathrm{d}x \\
&= -\frac{1}{n}\sum_{i=1}^{n}\int_0^t \gamma(s,X_i^{(n)}(s))\,\sigma(F_{\rho^{(n)}(s)}(X_i^{(n)}(s)))\,\mathrm{d}B_i^{(n)}(s) \\
&\quad + \int_0^t\int_{\mathbb{R}} \big(\gamma_s(s,x)F_{\rho^{(n)}(s)}(x) + \gamma_x(s,x)B_n(F_{\rho^{(n)}(s)}(x)) \\
&\qquad\qquad\qquad + \gamma_{xx}(s,x)\Sigma_n(F_{\rho^{(n)}(s)}(x))\big)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{4.2.13}$$

At this point, we take the difference between the equations (4.2.13) and (4.2.7), multiply the resulting equation by $\sqrt{n}$, use the Fundamental Theorem of Calculus in the forms

$$\begin{aligned}
&B_n(F_{\rho^{(n)}(s)}(x)) - B(R(s,x)) = B_n(F_{\rho^{(n)}(s)}(x)) - B(F_{\rho^{(n)}(s)}(x)) \\
&+ \int_0^1 b(aF_{\rho^{(n)}(s)}(x) + (1-a)R(s,x))\,(F_{\rho^{(n)}(s)}(x) - R(s,x))\,\mathrm{d}a,
\end{aligned} \tag{4.2.14}$$

$$\begin{aligned}
&\Sigma_n(F_{\rho^{(n)}(s)}(x)) - \Sigma(R(s,x)) = \Sigma_n(F_{\rho^{(n)}(s)}(x)) - \Sigma(F_{\rho^{(n)}(s)}(x)) \\
&+ \int_0^1 \frac{\sigma(aF_{\rho^{(n)}(s)}(x) + (1-a)R(s,x))^2}{2}\,(F_{\rho^{(n)}(s)}(x) - R(s,x))\,\mathrm{d}a
\end{aligned} \tag{4.2.15}$$

and rearrange terms to end up with (4.2.6). $\square$

We are now ready to give the proof of Proposition 4.2.1.

Proof of Proposition 4.2.1. Step 1. By definition the random variables in (4.2.3) are the limits in law of

$$\begin{aligned}
&\int_{\mathbb{R}} \gamma(t_\ell,x)\, G_n(t_\ell)(\mathrm{d}x) - \int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x) \\
&- \int_0^{t_\ell}\int_{\mathbb{R}} (\mathcal{A}_s\gamma)(s,x)\, H_n(t_\ell)(\mathrm{d}s,\mathrm{d}x),\ \int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x)
\end{aligned} \tag{4.2.16}$$

along a suitable sequence of $n \in \mathbb{N}$.

To proceed we note that the convergence $\rho^{(n)} \to \rho$ in probability in $C([0,\infty), M_1(\mathbb{R}))$ and the regularity result of Proposition 2.3.1 imply the convergences in probability

$$\sup_{(s,x)\in[0,t_\ell]\times\mathbb{R}} \big|F_{\rho^{(n)}(s)}(x) - R(s,x)\big| \to 0, \quad \ell = 1,\, 2,\, \ldots,\, m. \tag{4.2.17}$$

To this end, we apply the Skorokhod Representation Theorem in the form of [Du, Theorem 3.5.1] to the sequence $\rho^{(n)}$, $n \in \mathbb{N}$, and use Prokhorov's Theorem for the compact set $\{\rho(s),\, s \in [0,t_\ell]\}$ to find for every fixed $\epsilon > 0$ some $-\infty < \underline{x} < \overline{x} < \infty$ such that $\max\big(R(s,\underline{x}), 1 - R(s,\overline{x})\big) \le \epsilon$, $s \in [0,t_\ell]$. Moreover, by the regularity result of Proposition 2.3.1 the ϵ-modulus of continuity $\theta = \theta(\epsilon) > 0$ of the function R on $[0,t_\ell] \times [\underline{x},\overline{x}]$ is well-defined. In addition, for every $n \in \mathbb{N}$ large enough the Lévy distance between $\rho^{(n)}(s)$ and $\rho(s)$ is less than θ for all $s \in [0,t_\ell]$. Finally, choosing points $\underline{x} = x_1 < x_2 < \cdots < x_J = \overline{x}$ at most θ apart, with a suitable $J \in \mathbb{N}$, we have with the conventions $x_0 := -\infty$, $x_{J+1} := \infty$,

$$\begin{aligned}
&\sup_{(s,x)\in[0,t_\ell]\times\mathbb{R}} \big|F_{\rho^{(n)}(s)}(x) - R(s,x)\big| \\
&\le \sup_{s\in[0,t_\ell]} \max_{1\le j\le J+1} \max\Big(\big|F_{\rho^{(n)}(s)}(x_j) - R(s,x_{j-1})\big|, \\
&\qquad\qquad\qquad\qquad \big|F_{\rho^{(n)}(s)}(x_{j-1}) - R(s,x_j)\big|\Big) \\
&\le \theta + \sup_{s\in[0,t_\ell]} \max_{1\le j\le J+1} \max\Big(\big|R(s,x_j-\theta) - R(s,x_{j-1})\big|, \\
&\qquad \big|R(s,x_j+\theta) - R(s,x_{j-1})\big|, \big|R(s,x_{j-1}-\theta) - R(s,x_j)\big|, \\
&\qquad \big|R(s,x_{j-1}+\theta) - R(s,x_j)\big|\Big).
\end{aligned}$$

The latter bound is at most $\theta + 2\epsilon$, and (4.2.17) follows from the arbitrariness of $\epsilon > 0$.

The convergences of (4.2.17) in conjunction with the Lipschitz property of b, $\frac{\sigma^2}{2}$ (cf. Assumption 1.1.1(b)) show that the limit in law of the random variables in (4.2.16) along a sequence of $n \in \mathbb{N}$ is the same as the limit in law of

$$
\begin{aligned}
&\int_{\mathbb{R}} \gamma(t_\ell, x)\, G_n(t_\ell)(\mathrm{d}x) - \int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x) \\
&- \int_0^{t_\ell} \int_{\mathbb{R}} \int_0^1 \bigg(\gamma_s(s,x) + \gamma_x(s,x)\, b(aF_{\rho^{(n)}(s)}(x) + (1-a)R(s,x)) \\
&\qquad\qquad + \gamma_{xx}(s,x)\, \frac{\sigma(aF_{\rho^{(n)}(s)}(x) + (1-a)R(s,x))^2}{2} \bigg) \mathrm{d}a\, H_n(t_\ell)(\mathrm{d}s, \mathrm{d}x), \\
&\int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x)
\end{aligned}
$$

along the same sequence of $n \in \mathbb{N}$.

Next, we apply Lemma 4.2.2 and find that the latter limit in law must be equal to the limit in law of

$$
\begin{aligned}
&- \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^{t_\ell} \gamma(s, X_i^{(n)}(s))\, \sigma(F_{\rho^{(n)}(s)}(X_i^{(n)}(s)))\, \mathrm{d}B_i^{(n)}(s) \\
&+ \sqrt{n} \int_0^{t_\ell} \int_{\mathbb{R}} \big(\gamma_x(s,x)(B_n - B)(F_{\rho^{(n)}(s)}(x)) \\
&\qquad\qquad + \gamma_{xx}(s,x)(\Sigma_n - \Sigma)(F_{\rho^{(n)}(s)}(x)) \big)\, \mathrm{d}x\, \mathrm{d}s, \\
&\quad \int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x)
\end{aligned}
$$

along the same sequence of $n \in \mathbb{N}$. Moreover, since the functions b and $\frac{\sigma^2}{2}$ are Lipschitz by Assumption 1.1.1(b), the suprema $\sup_{[0,1]} |B_n - B|$ and $\sup_{[0,1]} |\Sigma_n - \Sigma|$ can be bounded above by Cn^{-1} with a constant $C < \infty$ depending only on the Lipschitz constants of b and $\frac{\sigma^2}{2}$. Consequently, it suffices to study the limit in law of

$$
\begin{aligned}
&- \frac{1}{\sqrt{n}} \sum_{i=1}^n \int_0^{t_\ell} \gamma(s, X_i^{(n)}(s))\, \sigma(F_{\rho^{(n)}(s)}(X_i^{(n)}(s)))\, \mathrm{d}B_i^{(n)}(s), \\
&\quad \int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x)
\end{aligned}
\tag{4.2.18}
$$

along the same sequence of $n \in \mathbb{N}$ as before.

Step 2. Consider the sequences of continuous martingales

$$\int_{\mathbb{R}} \gamma(0,x)\, G_n(0)(\mathrm{d}x) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^t \gamma(s, X_i^{(n)}(s))\, \sigma(F_{\rho^{(n)}(s)}(X_i^{(n)}(s)))\, \mathrm{d}B_i^{(n)}(s),\ t \in [0, t_\ell] \tag{4.2.19}$$

indexed by $n \in \mathbb{N}$, where ℓ and γ vary over $\{1, 2, \dots, m\}$ and a countable dense subset $\mathcal{C}_\ell$ of the space of functions on $[0, t_\ell] \times \mathbb{R}$ which are continuously differentiable in s, twice continuously differentiable in x and compactly supported. One easily verifies the tightness of each such sequence via the tightness criterion of [Bi, Theorem 7.3] by recalling Proposition 3.1.2, writing each of the martingales as a time-changed standard Brownian motion with the same initial value (cf. [KS, Chapter 3, Problem 4.7]) and using the assumed boundedness of γ and σ. In particular, every sequence of $n \in \mathbb{N}$ admits a subsequence along which the continuous martingales of (4.2.19) converge to the respective limiting processes M^γ for all $\gamma \in \mathcal{C}_\ell$, $\ell \in \{1, 2, \dots, m\}$.

Now, letting $\Gamma(s,x) := -\int_x^\infty \gamma(s,y)\,\mathrm{d}y$ as before, integrating by parts, recalling Assumption 1.1.1(a), applying the inequality (3.2.4) with $p = 2$ and using the Itô isometry we arrive at the estimate

$$\begin{aligned} &2\left(\mathbb{E}\big[\Gamma(0, X_1^{(n)}(0))^2\big] - \mathbb{E}\big[\Gamma(0, X_1^{(n)}(0))\big]^2\right) \\ &+ 2\,\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma(s,x)^2\, \sigma(F_{\rho^{(n)}(s)}(x))^2\, \rho^{(n)}(s)(\mathrm{d}x)\,\mathrm{d}s\bigg] \end{aligned} \tag{4.2.20}$$

on the second moment of the random variable in (4.2.19) with the same value of t. The latter quantities tend to

$$\begin{aligned} &2\left(\mathbb{E}\big[\Gamma(0, X_1^{(1)}(0))^2\big] - \mathbb{E}\big[\Gamma(0, X_1^{(1)}(0))\big]^2\right) \\ &+ 2\,\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma(s,x)^2\, \sigma(R(s,x))^2\, \rho(s)(\mathrm{d}x)\,\mathrm{d}s\bigg] \end{aligned} \tag{4.2.21}$$

in the limit $n \to \infty$, as can be seen by applying the Skorokhod Representation Theorem in the form of [Du, Theorem 3.5.1] to the sequence $\rho^{(n)}$, $n \in \mathbb{N}$, using the almost sure weak convergences

$$\begin{aligned}&\sigma(F_{\rho^{(n)}(s)}(x))^2\,\rho^{(n)}(s)(\mathrm{d}x)=2\,\mathrm{d}\Sigma_n(F_{\rho^{(n)}(s)}(\cdot))\rightarrow 2\,\mathrm{d}\Sigma(R(s,\cdot))\\&2\,\mathrm{d}\Sigma(R(s,\cdot))=\sigma(R(s,x))^2\,\rho(s)(\mathrm{d}x),\, s\in[0,t]\end{aligned} \tag{4.2.22}$$

and appealing to the Dominated Convergence Theorem (recall that γ and σ are bounded by assumption). In particular, the one-dimensional distributions of the continuous martingales in (4.2.19) are uniformly integrable as n varies, so that the limiting processes M^γ must be themselves continuous martingales for all $\gamma\in\mathcal{C}_\ell$, $\ell\in\{1,2,\ldots,m\}$.

Finally, for any $\gamma\in\mathcal{C}_\ell$, $\tilde{\gamma}\in\mathcal{C}_{\tilde{\ell}}$ another application of the Skorokhod Representation Theorem to the sequence $\rho^{(n)}$, $n\in\mathbb{N}$, the convergences in (4.2.22) and the Dominated Convergence Theorem show that the quadratic covariation process on $[0,\min(t_\ell,t_{\tilde{\ell}})]$ between the continuous martingales of (4.2.19) associated with γ, $\tilde{\gamma}$ converges in law to

$$\int_0^t\int_{\mathbb{R}}\gamma(s,x)\,\tilde{\gamma}(s,x)\,\sigma(R(s,x))^2\,\rho(s)(\mathrm{d}x)\,\mathrm{d}s,\quad t\in[0,\min(t_\ell,t_{\tilde{\ell}})] \tag{4.2.23}$$

in the limit $n\to\infty$. Moreover, another uniform integrability argument relying on integration by parts, Assumption 1.1.1(a), the inequality (3.2.4) with $p=4$, the Burkholder-Davis-Gundy inequality (see e.g. [KS, Chapter 3, Theorem 3.28]) and the boundedness of γ and σ allows us to identify the process in (4.2.23) as the quadratic covariation process between M^γ and $M^{\tilde{\gamma}}$. This and Proposition 3.1.2 lead to the conclusion that the probability space supporting M^γ, $\gamma\in\mathcal{C}_\ell$, $\ell\in\{1,2,\ldots,m\}$ admits an orthogonal martingale measure $\mathrm{d}M(s,x)$ on $[0,t_m]\times\mathbb{R}$ in the sense of [Wa, definitions on pp. 287–288] with the quadratic variation measure

$$\mathrm{d}\langle M\rangle(s,x)=\sigma(R(s,x))^2\,\rho(s)(\mathrm{d}x)\,\mathrm{d}s\quad\text{on}\quad[0,t_m]\times\mathbb{R} \tag{4.2.24}$$

and a reparametrized Brownian bridge $\beta(F_\lambda(\cdot))$ independent of $\mathrm{d}M(s,x)$

satisfying

$$M^{\gamma}(t) = \int_{\mathbb{R}} \gamma(0,x)\,\beta(F_\lambda(x))\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \gamma(s,x)\,\mathrm{d}M(s,x), \quad t \in [0,t_\ell] \tag{4.2.25}$$

for all $\gamma \in \mathcal{C}_\ell$, $\ell \in \{1,2,\ldots,m\}$. It remains to use the positivity of σ throughout $[0,1]$ and the existence of a positive density $R_x(s,\cdot)$ of $\rho(s)$ for $s>0$ (cf. (2.4.2) and the lower bound of (2.4.6)) in order to define the white noise

$$\mathrm{d}W(s,x) := \sigma(R(s,x))^{-1}\,R_x(s,x)^{-1/2}\,\mathrm{d}M(s,x) \quad \text{on} \quad [0,t_m]\times\mathbb{R}, \tag{4.2.26}$$

ending up with the identification

$$\begin{aligned} M^{\gamma}(t) = & \int_{\mathbb{R}} \gamma(0,x)\,\beta(F_\lambda(x))\,\mathrm{d}x \\ & + \int_0^t \int_{\mathbb{R}} \gamma(s,x)\,\sigma(R(s,x))\,R_x(s,x)^{1/2}\,\mathrm{d}W(s,x),\ t \in [0,t_\ell] \end{aligned} \tag{4.2.27}$$

for all $\gamma \in \mathcal{C}_\ell$, $\ell \in \{1,2,\ldots,m\}$. The statement of the proposition for such ℓ and γ readily follows. To obtain the statement for arbitrary ℓ and γ it suffices to pick a sequence of functions from $\mathcal{C}_\ell$ converging to γ, use the statement for the latter and pass to the limit. $\square$

We proceed to an analogue of Proposition 4.2.1 for the mild solution G from (4.0.2).

Proposition 4.2.3. *Suppose that Assumption 1.1.1 holds. Then, for any $t>0$ the measures $G(t,x)\,\mathrm{d}x$ on $\mathbb{R}$ and $G(s,x)\,\mathbf{1}_{[0,t]\times\mathbb{R}}(s,x)\,\mathrm{d}s\,\mathrm{d}x$ on $[0,t]\times\mathbb{R}$, defined in terms of the mild solution G from* (4.0.2)*, are finite almost surely and for every function on $[0,t]\times\mathbb{R}$ which is continuously differentiable in s,* *twice continuously differentiable in x and compactly supported one has*

$$
\begin{aligned}
&\int_{\mathbb{R}} \gamma(t,x)\, G(t,x)\, \mathrm{d}x - \int_{\mathbb{R}} \gamma(0,x)\, G(0,x)\, \mathrm{d}x \\
&- \int_0^t \int_{\mathbb{R}} (\mathcal{A}_s \gamma)(s,x)\, G(s,x)\, \mathrm{d}x\, \mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} \gamma(s,x)\, \sigma(R(s,x))\, R_x(s,x)^{1/2}\, \mathrm{d}W(s,x).
\end{aligned} \tag{4.2.28}
$$

Proof. Step 1. We fix a $t > 0$ and aim to verify in this first step that

$$
\mathbb{E}\bigg[\int_{\mathbb{R}} |G(t,x)|\, \mathrm{d}x\bigg] < \infty \quad \text{and} \quad \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} |G(s,x)|\, \mathrm{d}x\, \mathrm{d}s\bigg] < \infty. \tag{4.2.29}
$$

To this end, we insert the right-hand side of (4.0.2) into the first expectation and bound the result using the triangle inequality, Fubini's Theorem and Jensen's inequality by

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_{\mathbb{R}} \big|\beta(F_\lambda(y))\big|\, \mathrm{d}y\bigg] \\
&+ \int_{\mathbb{R}} \bigg(\int_0^t \int_{\mathbb{R}} \sigma(R(s,y))^2\, R_x(s,y)\, p(s,y;t,x)^2\, \mathrm{d}y\, \mathrm{d}s\bigg)^{1/2} \mathrm{d}x.
\end{aligned} \tag{4.2.30}
$$

Fubini's Theorem and the scaling property of Gaussian distributions reveals further that the first summand in (4.2.30) is the product of the first absolute moment of the standard Gaussian distribution and $\int_{\mathbb{R}} \sqrt{F_\lambda(y)(1 - F_\lambda(y))}\, \mathrm{d}y$. The latter integral is finite due to Assumption 1.1.1(a) and [BL, discussion of the functional J_1 on p. 25].

To estimate the second summand in (4.2.30) we combine the boundedness of σ (cf. Assumption 1.1.1(b)), the inequality $p(s,y;t,x) \le C(t-s)^{-1/2}$ (cf. (2.4.6)) and the identity

$$
\int_{\mathbb{R}} R_x(s,y)\, p(s,y;t,x)\, \mathrm{d}y = R_x(t,x) \tag{4.2.31}
$$

(due to the Markov property of the diffusion $\bar{X}$, see the discussion following

(2.4.1)) to arrive at the upper bound

$$C \int_{\mathbb{R}} \bigg(\int_0^t \frac{1}{2}\,(t-s)^{-1/2}\,R_x(t,x)\,\mathrm{d}s\bigg)^{1/2}\,\mathrm{d}x = C\,t^{1/4}\,\int_{\mathbb{R}} R_x(t,x)^{1/2}\,\mathrm{d}x, \tag{4.2.32}$$

where $C<\infty$ depends only on $\sup_{[0,1]}\sigma$ and the constant in (2.4.6). At this point, Jensen's inequality with respect to the Cauchy distribution

$$\begin{aligned}\int_{\mathbb{R}} R_x(t,x)^{1/2}\,\mathrm{d}x &= \pi\int_{\mathbb{R}} R_x(t,x)^{1/2}(1+x^2)\frac{1}{\pi(1+x^2)}\mathrm{d}x \\ &\le \pi^{1/2}\bigg(\int_{\mathbb{R}} R_x(t,x)(1+x^2)\,\mathrm{d}x\bigg)^{1/2}\end{aligned} \tag{4.2.33}$$

and the estimate (2.4.7) imply that the first expectation in (4.2.29) is finite. Moreover, in view of Fubini's Theorem and since the just obtained estimate is uniformly bounded on every compact interval of t's, the second expectation in (4.2.29) is also finite.

Step 2. To derive the identity (4.2.28) we fix a function γ as described and deduce from the definition of G in (4.0.2) that

$$\begin{aligned}\int_{\mathbb{R}} \gamma(t,x)\,G(t,x)\,\mathrm{d}x &= \int_{\mathbb{R}} \gamma(t,x)\int_{\mathbb{R}} G(0,y)\,p(0,y;t,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{\mathbb{R}} \gamma(t,x)\int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\,R_x(s,y)^{1/2}\,p(s,y;t,x)\,\mathrm{d}W(s,y)\,\mathrm{d}x.\end{aligned} \tag{4.2.34}$$

Moreover, the boundedness of γ and σ and the estimates

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G(0,y)|\,p(0,y;t,x)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}} |G(0,y)|\,\mathrm{d}y < \infty, \tag{4.2.35}$$

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^t \int_{\mathbb{R}} R_x(s,y)\,p(s,y;t,x)^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \\ &\qquad \le C\int_{\mathbb{R}}\int_0^t (t-s)^{-1/2} R_x(t,x)\,\mathrm{d}s\,\mathrm{d}x < \infty\end{aligned} \tag{4.2.36}$$

(see Step 1 for more details) allow us to use the classical and the stochastic Fubini's Theorems (see [Wa, Theorem 2.6] and note that the dominating measure therein is $\delta_{\tilde{y}}(\mathrm{d}y)\,\mathrm{d}\tilde{y}\,\mathrm{d}s$ in our case) and to rewrite the right-hand side of (4.2.34) as

$$\begin{aligned}
&\int_{\mathbb{R}} G(0,y) \int_{\mathbb{R}} \gamma(t,x)\, p(0,y;t,x)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2} \int_{\mathbb{R}} \gamma(t,x)\, p(s,y;t,x)\,\mathrm{d}x\,\mathrm{d}W(s,y).
\end{aligned} \tag{4.2.37}$$

Next, we employ Itô's formula and Fubini's Theorem to find

$$\begin{aligned}
\int_{\mathbb{R}} \gamma(t,x)\, p(s,y;t,x)\,\mathrm{d}x &= \mathbb{E}\big[\gamma(t,\bar{X}(t))\big|\bar{X}(s)=y\big] \\
&= \gamma(s,y) + \mathbb{E}\bigg[\int_s^t (\mathcal{A}_r\gamma)(r,\bar{X}(r))\,\mathrm{d}r\bigg|\bar{X}(s)=y\bigg] \\
&= \gamma(s,y) + \int_s^t \int_{\mathbb{R}} (\mathcal{A}_r\gamma)(r,x)\, p(s,y;r,x)\,\mathrm{d}x\,\mathrm{d}r.
\end{aligned}$$

Applying this observation to the expression in (4.2.37) we get

$$\begin{aligned}
&\int_{\mathbb{R}} G(0,y)\,\gamma(0,y)\,\mathrm{d}y + \int_{\mathbb{R}} G(0,y) \int_0^t \int_{\mathbb{R}} (\mathcal{A}_r\gamma)(r,x)\, p(0,y;r,x)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}y \\
&+ \int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2}\,\gamma(s,y)\,\mathrm{d}W(s,y) \\
&+ \int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2} \int_s^t \int_{\mathbb{R}} (\mathcal{A}_r\gamma)(r,x)\, p(s,y;r,x)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}W(s,y).
\end{aligned} \tag{4.2.38}$$

At this point, thanks to the boundedness of $(\mathcal{A}_r\gamma)$ and σ (cf. Assumption 1.1.1(b)) and the estimates

$$\begin{aligned}
&\int_{\mathbb{R}} |G(0,y)| \int_0^t \int_{\mathbb{R}} p(0,y;r,x)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}y \\
&= \int_{\mathbb{R}} |G(0,y)|\, t\,\mathrm{d}y < \infty, \\
&\int_0^t \int_{\mathbb{R}} \int_0^r \int_{\mathbb{R}} R_x(s,y)\, p(s,y;r,x)^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}r \\
&\le C \int_0^t \int_{\mathbb{R}} \int_0^r (r-s)^{-1/2}\, R_x(r,x)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}r < \infty
\end{aligned} \tag{4.2.39}$$

the classical and the stochastic Fubini's Theorems are applicable to the second and fourth summands in (4.2.38), so that the overall expression in (4.2.38) equals to

$$\begin{aligned}
&\int_{\mathbb{R}} \gamma(0,y)\, G(0,y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} (\mathcal{A}_r\gamma)(r,x) \int_{\mathbb{R}} G(0,y)\, p(0,y;r,x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}r \\
&+ \int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2}\, \gamma(s,y)\,\mathrm{d}W(s,y) \\
&+ \int_0^t \int_{\mathbb{R}} (\mathcal{A}_r\gamma)(r,x) \int_0^r \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2}\, p(s,y;r,x)\,\mathrm{d}W(s,y)\,\mathrm{d}x\,\mathrm{d}r \\
&= \int_{\mathbb{R}} \gamma(0,y)\, G(0,y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} (\mathcal{A}_r\gamma)(r,x)\, G(r,x)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_0^t \int_{\mathbb{R}} \gamma(s,y)\, \sigma(R(s,y))\, R_x(s,y)^{1/2}\,\mathrm{d}W(s,y).
\end{aligned} \tag{4.2.40}$$

This finishes the proof of the proposition. □

We can now identify the subsequential limits of Proposition 4.1.1.

Proposition 4.2.4. *Suppose that Assumption 1.1.1 is satisfied. Then, any subsequential limit in law of the sequence in* (4.1.1) *has the same distribution as*

$$\begin{aligned}
&\big(G(0,x)\,\mathrm{d}x,\ G(t_1,x)\,\mathrm{d}x,\ \ldots,\ G(t_m,x)\,\mathrm{d}x, \\
&G(s,x)\mathbf{1}_{[0,t_1]\times\mathbb{R}}(s,x)\,\mathrm{d}s\,\mathrm{d}x,\ G(s,x)\mathbf{1}_{[0,t_2]\times\mathbb{R}}(s,x)\,\mathrm{d}s\,\mathrm{d}x,\ \ldots, \\
&G(s,x)\mathbf{1}_{[0,t_m]\times\mathbb{R}}(s,x)\,\mathrm{d}s\,\mathrm{d}x\big),
\end{aligned} \tag{4.2.41}$$

where G is the mild solution from (4.0.2)*.*

Proof. Step 1. We consider a probability space that supports a limit point in law

$$\big(G_\infty(0), G_\infty(t_1), \ldots, G_\infty(t_m), H_\infty(t_1), H_\infty(t_2), \ldots, H_\infty(t_m)\big) \tag{4.2.42}$$

of the sequence in (4.1.1) and aim to couple it with a mild solution of the SPDE (4.0.1).

To this end, for each $\ell \in \{1, 2, \ldots, m\}$ we pick a countable dense subset $\mathcal{C}_\ell$ of the space of functions on $[0,t_\ell]\times\mathbb{R}$ which are continuously differentiable in

s, twice continuously differentiable in x and compactly supported. We note that the random variables of (4.2.3) with ℓ and γ varying over $\{1, 2, \ldots, m\}$ and $\mathcal{C}_\ell$ are defined on the underlying probability space. Moreover, by Proposition 4.2.1 their joint distribution must be that of the random variables in (4.2.4). Hence, by [Ka, Theorem 5.3] we can define on an enlargement of the underlying probability space a countable collection of continuous processes whose conditional distribution given the random variables in (4.2.3) with ℓ and γ varying over $\{1, 2, \ldots, m\}$ and $\mathcal{C}_\ell$ is the same as the conditional distribution of the continuous processes

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \gamma(s,x)\, \sigma(R(s,x))\, R_x(s,x)^{1/2}\, \mathrm{d}W(s,x), \\ &\qquad t \in [0, t_\ell], \quad \gamma \in \mathcal{C}_\ell, \quad \ell = 1,\, 2,\, \ldots,\, m \end{aligned} \tag{4.2.43}$$

given

$$\begin{aligned} &\int_0^{t_\ell} \int_{\mathbb{R}} \gamma(s,x)\, \sigma(R(s,x))\, R_x(s,x)^{1/2}\, \mathrm{d}W(s,x), \quad \gamma \in \mathcal{C}_\ell, \quad \ell = 1,\, 2,\, \ldots,\, m, \\ &\int_{\mathbb{R}} \gamma(0,x)\, \beta(F_\lambda(x))\, \mathrm{d}x, \quad \gamma \in \mathcal{C}_\ell, \quad \ell = 1,\, 2,\, \ldots,\, m. \end{aligned} \tag{4.2.44}$$

It follows that the enlarged probability space supports an orthogonal martingale measure $\mathrm{d}M(s,x)$ on $[0, t_m] \times \mathbb{R}$ in the sense of [Wa, definitions on pp. 287–288] with the quadratic variation measure

$$\mathrm{d}\langle M\rangle(s,x) = \sigma(R(s,x))^2\, R_x(s,x)\, \mathrm{d}x\, \mathrm{d}s \quad \text{on} \quad [0, t_m] \times \mathbb{R} \tag{4.2.45}$$

and we can define a white noise $\mathrm{d}W(s,x)$ on $[0,t_m]\times\mathbb{R}$ as in (4.2.26). Finally, we let G be the mild solution of the SPDE (4.0.1) on $[0, t_m] \times \mathbb{R}$ given by

$$\begin{aligned} G(t,x) = &\int_{\mathbb{R}} p(0,y;t,x)\, G_\infty(0)(\mathrm{d}y) \\ &+ \int_0^t \int_{\mathbb{R}} \sigma(R(s,y))\, R_x(s,y)^{1/2}\, p(s,y;t,x)\, \mathrm{d}W(s,y), (t,x) \in [0, t_m] \times \mathbb{R}. \end{aligned} \tag{4.2.46}$$

In particular, Proposition 4.2.3 and our coupling construction ensure that

$$\begin{aligned}&\int_{\mathbb{R}} \gamma(t_\ell,x)\,G(t_\ell,x)\,\mathrm{d}x-\int_0^{t_\ell}\int_{\mathbb{R}} (\mathcal{A}_s\gamma)(s,x)\,G(s,x)\,\mathrm{d}x\,\mathrm{d}s\\&=\int_{\mathbb{R}} \gamma(t_\ell,x)\,G_\infty(t_\ell)(\mathrm{d}x)-\int_0^{t_\ell}\int_{\mathbb{R}} (\mathcal{A}_s\gamma)(s,x)\,H_\infty(t_\ell)(\mathrm{d}s,\mathrm{d}x),\\&\qquad \gamma\in\mathcal{C}_\ell,\ \ell=1,\,2,\,\ldots,\,m,\end{aligned} \tag{4.2.47}$$

with the notation of (4.2.2).

Step 2. We fix an $\ell\in\{1,2,\ldots,m\}$ and a continuous function $g:\,[0,t_\ell]\times\mathbb{R}\to\mathbb{R}$ with compact support and consider the backward Cauchy problem

$$\mathcal{A}_s u = g,\quad u(t_\ell,\cdot)=0 \tag{4.2.48}$$

on $[0,t_\ell]\times\mathbb{R}$. As explained in the paragraph following (2.4.3), the conditions of [Kr2, Theorem 2.1] apply to the equation (4.2.48) and guarantee the existence of a solution u with $u,u_t,u_x,u_{xx}\in L^2([0,t_\ell]\times\mathbb{R})$. We claim that (4.2.47) implies

$$\begin{aligned}&\int_{\mathbb{R}} u(t_\ell,x)\,G(t_\ell,x)\,\mathrm{d}x-\int_0^{t_\ell}\int_{\mathbb{R}} (\mathcal{A}_s u)(s,x)\,G(s,x)\,\mathrm{d}x\,\mathrm{d}s\\&=\int_{\mathbb{R}} u(t_\ell,x)\,G_\infty(t_\ell)(\mathrm{d}x)-\int_0^{t_\ell}\int_{\mathbb{R}} (\mathcal{A}_s u)(s,x)\,H_\infty(t_\ell)(\mathrm{d}s,\mathrm{d}x).\end{aligned} \tag{4.2.49}$$

Since the first integrals on both sides of (4.2.49) vanish due to the terminal condition in (4.2.48) and $g=\mathcal{A}_s u$ can be chosen arbitrarily from a countable dense subset of $C_0([0,t_\ell]\times\mathbb{R})$, it would follow from (4.2.49) that $G(s,x)\,\mathbf{1}_{[0,t_\ell]\times\mathbb{R}}\,\mathrm{d}s\,\mathrm{d}x = H_\infty(t_\ell)(\mathrm{d}s,\mathrm{d}x)$ for all $\ell\in\{1,2,\ldots,m\}$ and then from (4.2.47) that $G(t_\ell,x)\,\mathrm{d}x=G_\infty(t_\ell)(\mathrm{d}x)$ for all $\ell\in\{1,2,\ldots,m\}$, finishing the proof of the proposition.

To obtain (4.2.49) from (4.2.47) it suffices to show that one can pick functions $\gamma^{(\kappa)}$, $\kappa\in\mathbb{N}$ in $\mathcal{C}_\ell$ with

$$\gamma^{(\kappa)}(t_\ell,\cdot)\to u(t_\ell,\cdot)\ \text{ and }\ \mathcal{A}_s\gamma^{(\kappa)}\to\mathcal{A}_s u=g\ \text{ uniformly as }\ \kappa\to\infty. \tag{4.2.50}$$

To this end, we recall the solution $\bar{X}$ of the SDE (2.4.1) and observe that the time-homogeneous Markov process $(s,\bar{X}(s))$, $s\in[0,t_\ell]$ is the unique weak solution of the associated SDE, hence also of the local martingale problem for the operator $\mathcal{A}_s$ (see e.g. [Ka, Theorem 18.7]). The latter has bounded continuous coefficients, so that $(s,\bar{X}(s))$, $s\in[0,t_\ell]$ is a Feller process and its generator is the unique extension of $\mathcal{A}_s$ from the space of infinitely differentiable functions with compact support in $[0,t_\ell]\times\mathbb{R}$ to an appropriate domain within the space of continuous functions on $[0,t_\ell]\times\mathbb{R}$ vanishing at infinity (see e.g. [Ka, Theorem 18.11]).

Next, we employ the stochastic representation

$$u(s,x)=-\int_s^t \mathbb{E}\big[g(r,\bar{X}(r))\big|\bar{X}(s)=x\big]\,\mathrm{d}r,\quad (s,x)\in[0,t_\ell]\times\mathbb{R} \tag{4.2.51}$$

of the solution to (4.2.48) (cf. the explanation preceding (2.4.4)). Together with the Feller property of the process $(s,\bar{X}(s))$, $s\in[0,t]$ and the Dominated Convergence Theorem it shows that u is continuous. Moreover, since g has compact support and the diffusion $\bar{X}$ has bounded coefficients, u vanishes at infinity. Finally, the representation (4.2.51) reveals that the process

$$u(s,\bar{X}(s))-u(0,\bar{X}(0))-\int_0^s g(r,\bar{X}(r))\,\mathrm{d}r,\quad s\in[0,t] \tag{4.2.52}$$

is a martingale, so that by the converse of Dynkin's formula (see e.g. [RY, Chapter VII, Proposition 1.7]) u belongs to the domain of $\mathcal{A}_s$ with $\mathcal{A}_s u=g$. In particular, u admits an approximation as described in (4.2.50). □

We conclude the section with the proof of Theorem 4.0.1.

Proof of Theorem 4.0.1. By Proposition 4.1.1 every subsequence of the sequence in (4.1.1) has a further subsequence which converges in law. Moreover, by Proposition 4.2.4 the limit of the latter must have the distribution of the random vector in (4.2.41). Consequently, the whole sequence in (4.1.1) converges in law to the random vector in (4.2.41), which is precisely the content of Theorem 4.0.1. □

5

Limit of rank-based models with common noise

In this chapter, we study the hydrodynamic limit of the particle system in (1.0.1). We recall that $\rho^{(n,\gamma)}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n,\gamma)}(t)}$ is the empirical measure of the particle system (1.0.1) and $F_{\rho^{(n,\gamma)}(t)}$ is the cumulative distribution function of $\rho^{(n,\gamma)}(t)$. In the presence of the common noise W, the limit of the empirical cumulative distribution functions $F_{\rho^{(n,\gamma)}(t)}$ is no longer deterministic. We show that the limit of $F_{\rho^{(n,\gamma)}(t)}(\cdot)$ can be characterized using the solution of the porous medium equation (1.0.3) and furthermore, the limit is a strong solution of a suitable SPDE. We will also address the issues of existence and uniqueness for the SDEs in (1.0.1).

We are now ready to state the main result of this chapter and to this end let $\rho^{(n,\gamma)} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n,\gamma)}}$, $\tilde{\pi}^{n,\gamma}$ denote the distributions of $\rho^{(n,\gamma)}$ in $M_1\big(C([0,\infty),\mathbb{R})\big)$ and $T > 0$ be a constant.

Theorem 5.0.1. *Suppose that Assumption 1.1.1 holds, then the sequence $\tilde{\pi}^{n,\gamma}$ is tight and for any accumulation point $\tilde{\pi}^{\infty,\gamma}$ and random variable $\rho^{(\gamma)}$ distributed according $\tilde{\pi}^{\infty,\gamma}$, it holds that $F_{\rho^{(\gamma)}(t)}(\cdot)$ is the unique solution of the equation $F_{\rho^{(\gamma)}(t)}(\cdot) = R\Big(t, \cdot - \int\limits_0^t \gamma\big(s, \rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big)$, $t \geq 0$. Moreover under the stronger assumption in proposition 2.3.2, it is a strong solution of the following SPDE in $[0,T] \times \mathbb{R}$*

$$\mathrm{d}G = \big(- B(G)_x + \Sigma(G)_{xx} + \frac{1}{2} G_{xx}\gamma^2(t, G_x(\mathrm{d}x)\big)\mathrm{d}t \, - \gamma\big(t, G_x(\mathrm{d}x)\big)G_x \mathrm{d}W(t),$$
$$G(0,\cdot) = F_\lambda(\cdot). \tag{5.0.1}$$

5.1 Preliminaries

We will prove the theorem 5.0.1 as a sequence of lemmata. The following moment estimate will be used in the subsequent lemmas and is analogous to lemma 2.1.1.

Lemma 5.1.1. *Suppose the Assumption 1.1.1 holds, then we have the following moment estimate*

$$\mathbb{E}\big[\sup_{0\leq s \leq t} \big|X_i^{(n,\gamma)}(s)\big|^{(2+\eta)}\big] \leq C_0 + C_1 \, t^{\frac{2+\eta}{2}} + C_2 \, t^{2+\eta}\,, \quad i = 1,\, 2,\, \ldots,\, n, \tag{5.1.1}$$

where C_0 is a constant that depends on $\int_{\mathbb{R}} |x|^{2+\eta}\,\lambda(\mathrm{d}x)$, C_1 is a constant that depends on the L^∞ norms of σ and γ and C_2 is a constant that depends on the L^∞ norm of b.

Proof. Noting that the functions b, σ and γ are bounded, we apply Burkholder-Davis-Gundy inequality and repeat the arguments in lemma 2.1.1 to finish the proof. □

The next lemma establishes tightness of the distributions of $\rho^{(n,\gamma)}$.

Lemma 5.1.2. *Under Assumption 1.1.1, the sequence $\tilde{\pi}^{n,\gamma}$ of the distributions of $\rho^{(n,\gamma)}$ in $M_1\big(C([0,\infty),\mathbb{R})\big)$ is tight.*

Proof. Noting that the random variables $\big(X_1^{(n,\gamma)}, X_2^{(n,\gamma)}, \cdots, X_n^{(n,\gamma)}\big)$ are exchangeable, we apply [Sz2, Proposition 2.2, Pg 177] and repeat the arguments in lemma 2.1.2 to conclude that the sequence $\tilde{\pi}^{n,\gamma}$ is tight. $\square$

The tightness result establishes the existence of converging subsequences for $\pi^{n,\gamma}$. We remark that we will use the same index n for the converging subsequences. Let $\rho^{(\gamma)}$ be a variable in $M_1\big(C([0,\infty),\mathbb{R})\big)$ whose distribution is the limit point $\tilde{\pi}^{\infty,\gamma}$ of the convergent subsequence $\tilde{\pi}^{n,\gamma}$. In view of the Skorokhod representation theorem in the form of [Du, Theorem 3.5.1], we can assume that the sequence of random variables $\rho^{(n,\gamma)}$ and $\rho^{(\gamma)}$ are defined on the same probability space with $\rho^{(n,\gamma)} \xrightarrow{a.s.} \rho^{(\gamma)}$.

We state without proof the following lemmas that are analogous to lemma 2.1.3 and lemma 2.1.5, respectively.

Lemma 5.1.3. *Under Assumption 1.1.1,* $\mathbb{E}\Big[\sup_{0\le s\le t} \int_{\mathbb{R}} |x|^2 \, \rho_s^{(\gamma)}(\mathrm{d}x)\Big] < \infty.$

Lemma 5.1.4. *Let Assumption 1.1.1 be satisfied, then we have the following convergences* $\lim_{n\to\infty} \mathbb{E}\Big[W_2^2\big(\rho^{(n,\gamma)}(s), \rho^{(\gamma)}(s)\big)\Big] = 0$ *and*
$\lim_{n\to\infty} \int_0^t \mathbb{E}\Big[W_2^2\big(\rho^{(n,\gamma)}(s), \rho^{(\gamma)}(s)\big)\Big] \,\mathrm{d}s = 0 \;\forall\; s$ *and* t *in* $[0,\infty)$ *respectively.*

5.2 Proof of Theorem 5.0.1

Step 1. Lemma 5.1.2 proves the first part of the claim. In this step, we will characterize the limit of $F_{\rho^{(n,\gamma)}(t)}$. The standard approach to derive the limit is to adapt the arguments in proposition 2.2.2; however, proposition 2.3.1 or proposition [JR, Proposition 2.2] cannot be extended to prove uniqueness of limits. We take a different and a much simpler route to derive the limit and to prove uniqueness of limits. The main idea is to reduce the particle system in (1.0.1) to the classical rank-based system in (1.0.2). We first make the substitution $Y_i^n(t) = X_i^{(n,\gamma)}(t) - \int_0^t \gamma\big(s, \rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)$ and let $\mu^{(n)}(t) := \frac{1}{n}\sum_{i=1}^n \delta_{Y_i^{(n)}(t)}$ be the empirical measure of the particles $\big(Y_1^{(n)}(t), Y_2^{(n)}(t), \ldots, Y_n^{(n)}(t)\big)$ at time t and $F_{\mu^{(n)}(t)}$ be the cumulative distribution function of $\mu^{(n)}(t)$. Next, we make the following simple observations:

$$\begin{aligned} F_{\rho^{(n,\gamma)}(t)}(x) &= \frac{\sum_{i=1}^{n} \mathbb{I}\big(X_i^{(n,\gamma)}(t) \le x\big)}{n} \\ &= \frac{\sum_{i=1}^{n} \mathbb{I}\big(Y_i^{(n)}(t) \le x - \int_0^t \gamma\big(s, \rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\big)}{n} \end{aligned} \tag{5.2.1}$$

$$F_{\rho^{(n,\gamma)}(t)}(x) = F_{\mu^{(n)}(t)}\Big(x - \int_0^t \gamma\big(s, \rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big). \tag{5.2.2}$$

$$\begin{aligned} F_{\rho^{(n,\gamma)}(t)}\big(X_i^{(n,\gamma)}(t)\big) &= F_{\mu^{(n)}(t)}\Big(X_i^{(n,\gamma)}(t) - \int_0^t \gamma\big(s, \rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big) \\ F_{\rho^{(n,\gamma)}(t)}\big(X_i^{(n,\gamma)}(t)\big) &= F_{\mu^{(n)}(t)}\big(Y_i^{(n)}(t)\big). \end{aligned} \tag{5.2.3}$$

Using the aforementioned observations, we reduce the particle system in (1.0.1) to the classical rank-based particle system

$$\begin{aligned} \mathrm{d}Y_i^{(n)}(t) = b\big(F_{\mu^{(n)}(t)}\big(Y_i^{(n)}(t)\big)\big)\,\mathrm{d}t + \sigma\big(F_{\mu^{(n)}(t)}\big(Y_i^{(n)}(t)\big)\big)\,\mathrm{d}B_i^{(n)}(t), \\ i = 1,\, 2,\, \ldots,\, n. \end{aligned} \tag{5.2.4}$$

Thanks to [JR, Proposition 2.1], we obtain the convergence $\mu^{(n)} \to \mu$ in probability in $C([0,\infty), M_1(\mathbb{R}))$, where $R(t,\cdot) = F_{\mu(t)}(\cdot)$. Furthermore, (4.2.17) implies that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \big|F_{\mu^{(n)}(t)}(x) - R(t,x)\big| \xrightarrow{\mathbb{P}} 0. \tag{5.2.5}$$

We claim that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \Big|F_{\rho^{(n,\gamma)}(t)}(x) - R\Big(t, x - \int_0^t \gamma\big(s, \rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big)\Big| \xrightarrow{\mathbb{P}} 0. \tag{5.2.6}$$

Thanks to (5.2.2), it suffices to show that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\Big|F_{\mu^{(n)}(t)}\Big(x-\int_0^t\gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big) \\ -R\Big(t,x-\int_0^t\gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big)\Big|\xrightarrow{\mathbb{P}}0. \tag{5.2.7}$$

We apply the triangle inequality to obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\Big|F_{\mu^{(n)}(t)}\Big(x-\int_0^t\gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big)- \\
&\qquad R\Big(t,x-\int_0^t\gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big)\Big| \\
&\le\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\Big|F_{\mu^{(n)}(t)}\Big(x-\int_0^t\gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big) \\
&\qquad -R\Big(t,x-\int_0^t\gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big)\Big| \\
&+\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\Big|R\Big(t,x-\int_0^t\gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)\Big) \\
&\qquad -R\Big(t,x-\int_0^t\gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big)\Big| \\
&\le\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\big|F_{\mu^{(n)}(t)}(x)-R(t,x)\big| \\
&+\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}R_x(t,x)\sup_{t\in[0,T]}\Big|\int_0^t\gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s) \\
&\qquad -\int_0^t\gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big|,
\end{aligned}$$

where the last expression in the above chain of inequalities is a consequence

of the mean value theorem.
Thanks to (5.2.5), $\sup\limits_{t\in[0,T]}\sup\limits_{x\in\mathbb{R}}\big|F_{\mu^{(n)}(t)}(x)-R(t,x)\big| \xrightarrow{\mathbb{P}} 0$ and $\sup\limits_{t\in[0,T]}\sup\limits_{x\in\mathbb{R}} R_x$ is bounded. In view of the regularity result in proposition 2.3.1, it remains to show that $\sup\limits_{t\in[0,T]}\Big|\int\limits_0^t \gamma\big(s,\rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)-\int\limits_0^t \gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big|$ goes to 0 in probability. Thanks to Burkholder-Davis-Gundy inequality (see e.g. [KS, Chapter 3, Theorem 3.28]) and assumption 1.1.1, it suffices to show that $\int\limits_0^T \mathbb{E}\Big[W_1^2(\rho^{(n,\gamma)}(s),\rho^{(\gamma)}(s))\Big]\,\mathrm{d}s \overset{n\to\infty}{\longrightarrow} 0$. Combining the basic inequality $W_1\le W_2$ with lemma 2.1.5 finishes the proof of the claim (5.2.6).

The tightness result in lemma 2.1.1 and the discussion following it imply for fixed t, almost surely, at all continuity points x of $F_{\rho^{(\gamma)}(t)}(x)$, $F_{\rho^{(n,\gamma)}(t)}(x) \to F_{\rho^{(\gamma)}(t)}(x)$. Since, $F_{\rho^{(\gamma)}(t)}(\cdot)$ is right-continuous with only finitely many discontinuities and R is continuous in the spatial variable x, the convergence in (5.2.6) implies that $\forall\ t\in[0,T],\ x\in\mathbb{R}$

$$F_{\rho^{(\gamma)}(t)}(x)=R\Big(t,x-\int\limits_0^t \gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)\Big). \tag{5.2.8}$$

Since $T>0$ is arbitrary, we can find a set of full probability measure on which (5.2.8) holds for all $t\ge 0$ and $x\in\mathbb{R}$. The stronger assumption in proposition 2.3.2 yields classical regularity for R, which allows us to apply Ito's formula in (5.2.8). This yields the SPDE (5.0.1).

Step 2.

We now turn our attention to proving uniqueness of limits. To this end, let $F_{\rho_1^{(\gamma)}(t)}(x)$ and $F_{\rho_2^{(\gamma)}(t)}(x)$ be two distribution functions that satisfy (5.2.8). Then, we have

$$\begin{aligned}\big|F_{\rho_1^{(\gamma)}(t)}(x)-F_{\rho_2^{(\gamma)}(t)}(x)\big| =&\Big|R\big(t,x-\int\limits_0^t \gamma\big(s,\rho_1^{(\gamma)}(s)\big)\,\mathrm{d}W(s)\big)\\ &-R\big(t,x-\int\limits_0^t \gamma\big(s,\rho_2^{(\gamma)}(s)\big)\mathrm{d}W(s)\big)\Big|.\end{aligned} \tag{5.2.9}$$

Integrating both sides and using the regularity result in proposition 2.3.1, we obtain

$$\int_{\mathbb{R}} \left|F_{\rho_1^{(\gamma)}(t)}(x) - F_{\rho_2^{(\gamma)}(t)}(x)\right| \mathrm{d}x \leq \left| \int_0^t \Big(\gamma\big(s, \rho_1^{(\gamma)}(s)\big) - \gamma\big(s, \rho_2^{(\gamma)}(s)\big)\Big) \,\mathrm{d}W(s)\right|. \tag{5.2.10}$$

Next, we use the representation in proposition 2.1.4, square both sides and then take expectation to obtain

$$\begin{aligned} \mathbb{E}\Big[W_1^2(\rho_1^{(\gamma)}(t), \rho_2^{(\gamma)}(t)\Big] &\leq \mathbb{E}\int_0^t \Big(\gamma\big(s, \rho_1^{(\gamma)}(s)\big) - \gamma\big(s, \rho_2^{(\gamma)}(s)\big)\Big)^2 \,\mathrm{d}s \\ &\leq C \int_0^t \mathbb{E}\Big[W_1^2(\rho_1^{(\gamma)}(s), \rho_2^{(\gamma)}(s)\Big] \,\mathrm{d}s. \end{aligned} \tag{5.2.11}$$

Gronwall's lemma implies $F_{\rho_1^{(\gamma)}(t)}(x) = F_{\rho_2^{(\gamma)}(t)}(x)$ a.s. completing the proof of the theorem. □

5.3 Existence and Uniqueness

In this section, we will establish weak existence and weak uniqueness for the SDEs in (1.0.1).

5.3.1 Sketch of the proof of weak existence

We go back to the substitution $Y_i^n(t) = X_i^{(n,\gamma)}(t) - \int_0^t \gamma\big(s, \rho^{(n,\gamma)}(s)\big)\mathrm{d}W(s)$ and recall that the particles $Y_i^n(t)$, $i = 1, \ldots, n$, satisfy (1.0.2). Since the SDEs governing the particles $Y_i^n(t)$, $i = 1, \ldots, n$, have a weak unique solution, we first construct a weak solution for the particles $Y_i^n(t)$, $i = 1, \ldots, n$. Then, we use an appropriate enlargement of the probability space and the filtration that support the common brownian motion W to construct a weak solution for the particles in (1.0.1). We will use the classical approximation technique to construct the weak solution and to this end, we define the approximation scheme ignoring the dependence on n for notational convenience

$$X_i^{(k,\gamma)}(t) = Y_i(t) + \int_0^t \gamma\big(s, \rho^{(k-1,\gamma)}(s)\big)\mathrm{d}W(s), \quad k = 1, 2, \ldots \tag{5.3.1}$$

where $X_i^{(0,\gamma)}(t) = Y_i(t)$ and $\rho^{(k,\gamma)}(t)$ is the empirical measure of the particles $X_i^{(k,\gamma)}(t)$, $i = 1, \ldots, n$. In view of (5.3.1), assumption 1.1.1, the inequality $W_1 \leq W_2$ and another elementary inequality on Wasserstein distances, we get the following chain of inequalities

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\mathbb{E}\Big[\Big(X_i^{(k+1,\gamma)}(t) - X_i^{(k,\gamma)}(t)\Big)^2\Big]}{n} \\
&\leq \int_0^t \mathbb{E}\Big[\Big(\gamma\big(s, \rho^{(k,\gamma)}(s)\big) - \gamma\big(s, \rho^{(k-1,\gamma)}(s)\big)\Big)^2\Big]\,\mathrm{d}s \\
&\leq C \int_0^t \mathbb{E}\Big[W_1^2\big(\rho^{(k,\gamma)}(s), \rho^{(k-1,\gamma)}(s)\big)\Big]\,\mathrm{d}s \\
&\leq C \int_0^t \mathbb{E}\Big[W_2^2\big(\rho^{(k,\gamma)}(s), \rho^{(k-1,\gamma)}(s)\big)\Big]\,\mathrm{d}s \\
&\leq C \int_0^t \sum_{i=1}^{n} \frac{\mathbb{E}\Big[\Big(X_i^{(k,\gamma)}(s) - X_i^{(k-1,\gamma)}(s)\Big)^2\Big]}{n}\,\mathrm{d}s.
\end{aligned} \tag{5.3.2}$$

Combining all the above inequalities, we obtain

$$\mathbb{E}\Big[\Big|\Big|X^{(k+1,\gamma)}(t) - X^{(k,\gamma)}(t)\Big|\Big|_2^2\Big] \leq C \int_0^t \mathbb{E}\Big[\Big|\Big|X^{(k,\gamma)}(s) - X^{(k-1,\gamma)}(s)\Big|\Big|_2^2\Big]\,\mathrm{d}s, \tag{5.3.3}$$

where $X^{(k,\gamma)} = \big(X_1^{(k,\gamma)}, X_2^{(k,\gamma)}, \cdots, X_n^{(k,\gamma)}\big)$. A simple induction argument reveals that $\mathbb{E}\Big[\Big|\Big|X^{(k+1,\gamma)}(t) - X^{(k,\gamma)}(t)\Big|\Big|_2^2\Big] \leq \frac{C\,t^{k+1}}{(k+1)!}$. Clearly $X^{(k,\gamma)}(t)$ is Cauchy. We let its L^2 limit be $X^{(\gamma)}(t)$, where $X^{(\gamma)} = \big(X_1^{(\gamma)}, X_2^{(\gamma)}, \cdots, X_n^{(\gamma)}\big)$.

It's easy to check that

$$\sup_{0\le s\le t} n\,\mathbb{E}\Big[W_2^2\big(\rho^{(k,\gamma)}(s),\rho^{(\gamma)}(s)\big)\Big] \le \sup_{0\le s\le t}\mathbb{E}\Big[\Big|\Big|X^{(k,\gamma)}(s)-X^{(\gamma)}(s)\Big|\Big|_2^2\Big]$$
$$\le C\,\frac{t^k}{k!}\overset{k\to\infty}{\longrightarrow} 0, \tag{5.3.4}$$

where $\rho^{(\gamma)}(t)$ is the empirical measure of the particles $X_i^{(\gamma)}(t)$, $i=1,\dots,n$. Taking the L^2 limit in (5.3.1), we obtain

$$X_i^{(\gamma)}(t) = Y_i(t) + \lim_{k\to\infty}\int_0^t \gamma\big(s,\rho^{(k,\gamma)}(s)\big)\mathrm{d}W(s), \quad i=1,2,\dots,n. \tag{5.3.5}$$

It remains to show that $\lim_{k\to\infty}\int_0^t \gamma\big(s,\rho^{(k,\gamma)}(s)\big)\mathrm{d}W(s) = \int_0^t \gamma\big(s,\rho^{(\gamma)}(s)\big)\mathrm{d}W(s)$ in L^2. Thanks to assumption 1.1.1, the basic inequality $W_1 \le W_2$ and (5.3.4), the following chain of inequalities gives the desired result.

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^t \mathbb{E}\Big[\Big(\gamma\big(s,\rho^{(k,\gamma)}(s)\big)-\gamma\big(s,\rho^{(\gamma)}(s)\big)\Big)^2\Big]\,\mathrm{d}s\\
&\le C\lim_{k\to\infty}\int_0^t \mathbb{E}\Big[W_1^2\big(\rho^{(k,\gamma)}(s),\rho^{(\gamma)}(s)\big)\Big]\,\mathrm{d}s\\
&\le \lim_{k\to\infty}\int_0^t \mathbb{E}\Big[W_2^2\big(\rho^{(k,\gamma)}(s),\rho^{(\gamma)}(s)\big)\Big]\,\mathrm{d}s = 0.
\end{aligned} \tag{5.3.6}$$

5.3.2 Sketch of the proof of weak uniqueness

In the previous section, we adapted the classical construction of strong solutions to establish weak existence for (1.0.1). Intuitively speaking, conditioned on the classical rank-based particles $Y_i^n(t)$, $i=1,\dots,n$ and given any common brownian motion, we were able to construct the solution for the SDEs in (1.0.1). We also note that the solution to the limiting SPDE that we obtained in (5.0.1) is adapted to the filtration generated by the common brownian motion. Consequently, we prove the following version of weak uniqueness. Let $\big(X^{(n,\gamma)}, W, B_1^{(n)}, B_2^{(n)}, \ldots, B_n^{(n)}\big)$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}$ and $\big(\widetilde{X}^{(n,\gamma)}, W, \widetilde{B}_1^{(n)}, \widetilde{B}_2^{(n)}, \ldots, \widetilde{B}_n^{(n)}\big)$, $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\{\widetilde{\mathcal{F}_t}\}$ be two weak solutions of (1.0.1). Then the two processes $X^{(n,\gamma)} = \big(X_1^{(n,\gamma)}, X_2^{(n,\gamma)}, \cdots, X_n^{(n,\gamma)}\big)$ and $\widetilde{X}^{(n,\gamma)} = \big(\widetilde{X}_1^{(n,\gamma)}, \widetilde{X}_2^{(n,\gamma)}, \cdots, \widetilde{X}_n^{(n,\gamma)}\big)$ have the same law.

Let the corresponding classical rank-based particles for the two weak solutions be $Y^n(t) = \big(Y_1^n(t), Y_2^n(t), \cdots, Y_n^n\big)$ and $\widetilde{Y}^n(t) = \big(\widetilde{Y}_1^n(t), \widetilde{Y}_2^n(t), \cdots, \widetilde{Y}_n^n\big)$. Since the classical rank-based particles have a weak solution which is unique in distribution, to show that the processes $X^{(n,\gamma)}(t)\,, \widetilde{X}^{(n,\gamma)}(t)$ have the same law it suffices to show that the distrubtion of $X^{(n,\gamma)}(t)$ conditioned on $Y^n(t)$ equals the distribution of $\widetilde{X}^{(n,\gamma)}(t)$ conditioned on $\widetilde{Y}^n(t)$. We will show this by proving a stronger statement : Given the classical rank-based particles $Y_i^n(t)$, $i = 1, \ldots, n$, then any process $X^{(n,\gamma)}(t)$ satisfying $Y^n(t) = X^{(n,\gamma)}(t) - \int_0^t \gamma\big(s, \rho^{(n,\gamma)}(s)\big) \mathrm{d}W(s)$ is unique in the strong sense.

We will ignore the dependence on n for notational simplicity and let $X^{(1,\gamma)}(t)$ and $X^{(2,\gamma)}(t)$ be two processes that satisfy

$$\begin{aligned} Y(t) &= X^{(1,\gamma)}(t) - \int_0^t \gamma\big(s, \rho^{(1,\gamma)}(s)\big) \mathrm{d}W(s) \\ Y(t) &= X^{(2,\gamma)}(t) - \int_0^t \gamma\big(s, \rho^{(2,\gamma)}(s)\big) \mathrm{d}W(s), \end{aligned} \tag{5.3.7}$$

where $X^{(1,\gamma)} = \big(X_1^{(1,\gamma)}, X_2^{(1,\gamma)}, \cdots, X_n^{(1,\gamma)}\big)$,
$X^{(2,\gamma)} = \big(X_1^{(2,\gamma)}, X_2^{(2,\gamma)}, \cdots, X_n^{(2,\gamma)}\big)$ and $\rho^{(1,\gamma)}$ and $\rho^{(2,\gamma)}$ are the corresponding empirical measures respectively. In view of (5.3.7), assumption 1.1.1 and

a few basic inequalities on Wasserstein distances, we have the following

$$
\begin{aligned}
&\sum_{i=1}^{n} \frac{\mathbb{E}\Big[\Big(X_i^{(1,\gamma)}(t) - X_i^{(2,\gamma)}(t)\Big)^2\Big]}{n} = \int_0^t \mathbb{E}\Big[\Big(\gamma\big(s, \rho^{(1,\gamma)}(s)\big) - \gamma\big(s, \rho^{(2,\gamma)}(s)\big)\Big)^2\Big] \\
&\leq C \int_0^t \mathbb{E}\Big[W_1^2\big(\rho^{(1,\gamma)}(s), \rho^{(2,\gamma)}(s)\big)\Big] \,\mathrm{d}s \\
&\leq C \int_0^t \mathbb{E}\Big[W_2^2\big(\rho^{(1,\gamma)}(s), \rho^{(2,\gamma)}(s)\big)\Big] \,\mathrm{d}s \\
&\leq C \int_0^t \sum_{i=1}^{n} \frac{\mathbb{E}\Big[\Big(X_i^{(1,\gamma)}(s) - X_i^{(2,\gamma)}(s)\Big)^2\Big]}{n} \,\mathrm{d}s.
\end{aligned}
\tag{5.3.8}
$$

Gronwall's lemma gives uniqueness in the strong sense and thereby finishing the proof of weak uniqueness.

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