Mean-Field Games

Theory, numerics and applications

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Abstract

In dynamical systems with a large number of agents, competitive, and cooperative phenomena occur in a broad range of designed and natural settings. Such as communications, environmental, biological, transportation, trading, and energy systems, and they underlie much economic and financial behavior. Analysis of such systems is intractable using the classical finite N-players game theoretic methods is often intractable. The mean-field games (MFG) framework was developed to study these large systems, modeling them as a continuum of rational agents that interact in a non-cooperative way.

In this thesis, we address some theoretical aspects and propose a definition of relaxed solution for MFG that allows establishing uniqueness under minimal regularity hypothesis. We also propose a price impact model, that is a modification of the Merton's portfolio problem where we consider that assets' transactions influence their prices.

We also study numerical methods for continuous time finite-state MFG that satisfy a monotonicity condition, and for time-dependent first-order nonlocal MFG. MFG is determined by a system of differential equations with initial and terminal boundary conditions. These non-standard conditions make the numerical approximation of MFG difficult. Using the monotonicity condition, we build a flow that is a contraction and whose fixed points solve both for stationary and time-dependent MFG.

We also develop Fourier approximation methods for the solutions of first-order nonlocal mean-field games (MFG) systems. Using Fourier expansion techniques, we approximate a given MFG system by a simpler one that is equivalent to a convex optimization problem over a finite-dimensional subspace of continuous curves. We solve this problem using a variant of a primal-dual hybrid gradient method.

Finally, we introduce a price-formation model where a large number of small players can store and trade electricity. Our model is a constrained MFG where the price is a Lagrange multiplier for the supply versus demand balance condition. We establish the existence of a unique solution using a fixed-point argument. Then, we study linear-quadratic models that hold specific solutions, and we find that the dynamic price depends linearly on the instant aggregated consumption.

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Chapter 1

Introduction

In this work we study different aspects of the mean-field games (MFG) framework, namely, the theory behind it, in Part I, numerical methods, in Part II, and applications, in Part III.

The mean-field games setting models a large number of identical rational agents that interact in a non-cooperative way. These agents seek to minimize their cost (or maximize a reward function) using statistical information on the distribution of the whole population. This setting can be formulated as a differential game between a large number of players and the aim is to find a Nash equilibrium where a generic player cannot unilaterally improve his position.

These models were introduced in the engineering community by [117, 114] and in the mathematical community by [131–134]. These games were modeled as a coupled system of partial differential equations (PDE) that comprises a Hamilton–Jacobi and a Fokker–Planck equation.

In a differential game with a small number of players, the contribution of each player determines the state of the system. Therefore, instabilities and intractable mathematical problems arise when we are looking for feedback-form Nash equilibria. Trying to extend this methods for a large number of players either using particle or agent-based models frequently lead to intractable problems from the analytical point of view. Moreover, they often do not provide insight on the qualitative properties of the models.

For a large number of players, the contribution of each individual player to the overall state of the system is small and no single player can change the evolution of the system by itself. Therefore the macroscopic state of the system is determined by the average behavior of the population. The MFG framework corresponds to a major paradigm change in the analysis of N-players differential games, see [24]. In contrast with the previous existing methods, the system of PDE in MFG is amenable to analytical tools and suitable to address a wide range of applications and to provide quantitative and qualitative insight. Using these models we want to find the optimal strategy of each player as well as to determine the value function and characterize the evolution of the density of the population. In these models the information about the population state is encoded in a probability measure, m. Given the density of players m, each agent solves an optimal control problem. This problem determines a value function, V, and the optimal action to be performed. This gives rise to a coupled system of PDE (Hamilton–Jacobi and a Fokker–Planck equation) for which the couple m and u is a solution.

This thesis is organized in three different parts. We now detail some background, motivation and a brief discussion of the contribution in each part.

Part I: In the first part, of this work, we address the theoretical aspects of the mean-field games framework. This part is based on the paper [99].

The literature on mean field games and its applications is grew fast in the recent years, for a survey see [137] and reference therein, as well as the excellent lecture notes [53], and the books [32] and [85].

The first results on MFG using PDE methods were established by [131, 132] and later described in [140]. Later Gomes and his collaborators have also considered the discrete time, finite state problem [88], and the continuous-time finite-state problem [89], [80]. Such problems have also been addressed in [108] and [110]. Various applications and additional models have been worked out in detail in [107], [22], [97], [107], [97], [128], [142], [152], [159], [165], [166], [75], [111]. Problems motivated by applications with mixed populations or with a major player were studied in [112, 113]. Mean field games have also been analyzed using backwards-forwards stochastic differential equations, see, for instance [149], and [64, 63, 66, 65]. Linear quadratic problems have been considered from distinct points of view, for instance, in [114], [19], [115], [149], [23], [34], and [138].

The rigorous derivation of mean-field models was considered in some models in the original papers by Lions and Lasry. Further developments, using the theory of nonlinear Markov processes were obtained in [125], [123], and [122] (see also the monograph [124]), and using PDE methods in [22]. For finite state problems, the N player problem was studied in [89] where a convergence result was established. For earlier works in the context of statistical physics and interacting particle systems see [162].

In Part I we address reduced mean-field models and mean-field games in master form, which are discussed, respectively, in Chapter 2 and Chapter 3.

In Chapter 2, we start by discussing the derivation of reduced MFG. The reduced form of MFG was first studied by and Huang Malhamé and Caines, [117, 114], in the engineering community and Lions and Lasry, [131–134], in the mathematical community. Later, it was made clear that this formulation is actually a particular case of MFG where agents are also coupled through common Brownian noise. In the general setting we have a infinite dimensional PDE, the *Master equation*, that we will address in Chapter 3. The reduced mean field games can be seen as the equations of characteristics of the Master equation.

Reduced mean-field models can be formulated as systems of a Hamilton-Jacobi-Bellman equation coupled with a Fokker–Planck or transport equation. We start by discussing the derivation of those models. Then we discuss various existence results both for first and second order equations as well as for problems with local dependence on the measure. Then we address uniqueness questions and propose a definition of relaxed solution that allows to establish uniqueness under minimal regularity hypothesis. A special class of mean-field games can be regarded as the Euler-Lagrange equation of suitable functionals. This section ends with a brief overview of the random variables point of view and some applications to extended mean-field games models. These extended models arise in problems where the costs incurred by the agents depend not only on the distribution of the other agents but also on their actions.

Then in Chapter 3 we continue the discussion on mean-field games by considering mean-field games in master form. These were introduced by Lions in [140]. Such master form is particularly useful for the study of problems where agents share a common noise. We present various of these models as well as an application to price formation problems.

The main contributions of this first part is the theory of relaxed solutions and its applications to uniqueness and stability of mean-field games, some of the extended models and the price formation model discussed at the end of the paper.

Part II: In the second part, we address some numerical aspects of mean-field games. This part is based on the papers [105] and [155].

The MFG system consists of two coupled equations, one is backward in time (Hamilton–Jacobi equation) and the other is forward in time (Fokker–Planck), and satisfy terminal-initial boundary conditions. This combined with a fully coupling of the system is the main source of difficulties in the numerical simulation of time-dependent MFG.

Numerical methods for MFG problems were first studied by Achdou and Capuzzo-Dolceta in [6]. Several improvements and developments followed in [1] and [10].

There is also a growing interest in numerical methods for these problems [130], [6], [3], [52], [10]. For a survey of numerical methods see [1]. In [4], the authors proposed a discretization of the Hamilton–Jacobi equation via a monotone scheme and the adjoint of the linearization of that equation is used to discretize the Fokker–Planck equation. These equations were solved using Newton's method. In alternative to Newton's method we can use monotonicity properties. This was done in [12] for the stationary case. And we extend this approach for the time-dependent case for finite-state MFGs in the Chapter 4, where we use projection operators to preserve the boundary conditions.

In Chapter 5 we address first-order nonlocal MFG problems. In this setting the Hamilton–Jacobi and the Fokker-Planck equations are coupled through a nonlocal coupling in the Hamilton-Jacobi equation, of the kernel convolution type. The nonlocal MFG models problems where each individual agent takes into account not only the density of players in the same sate but also the density of players in neighboring states. We used Fourier expansion methods from [153] to approximate the nonlocal operator, and developed a primal-dual hybrid gradient based numerical method.

The main contributions are the numerical method for time-dependent finite-state MFG that satisfy a monotonicity condition. And another numerical method for nonlocal MFG.

Part III: On the third part, we address some applications of MFG models to price formation, namely in electricity markets. This part is based on the paper [100].

Applications of mean-field games are several. From economic models, algorithmic trading in finance, socio-economic models (opinion dynamics), biology and engineering problems.

Applications of MFG on economic models are several, a good reference is [104]. Different models in MFG have been used to study exploration of non-renewable resources, for example, oil ([137]) and minerals ([9]). To model inequality [84], economic equilibrium [130], and growth theory [136].

In mathematical finance, MFG models were used in high-frequency [129] and algorithmic [67] trading. And, also price formation [48, 47, 145, 42, 41, 87].

Regarding applications in engineering, MFG models were used to study energy markets and for the management of the power-grid, see [18, 118–120, 146]. Other applications include risk-sensitive or robust control, [76, 77, 165, 166], adaptive control ([121, 151]), massive communications in 5G networks [36], among others ([115, 163, 164]).

MFG models for crowd and population dynamics were investigated by [74, 43–45], for traffic by [46, 69], and and to related problems on networks and graphs in [28, 50, 51, 111].

Many authors studied socio-economic problems such as opinion dynamics ([27, 160, 161, 35]) and consensus ([38, 150, 152, 158, 152]), paradigm shifts ([37, 102]), social choice and dynamics ([116, 26]), and corruption ([126, 127]).

The main contributions of this part is the price formation model and the proof of existence of solutions to the first and second order MFG model with constraints, and uniqueness in the second order case. Also we solved explicitly the linear-quadratic models of the price-formation MFG formulation, and our results suggest that a price determined by a supply versus demand condition may help stabilize the oscillations of the price.

Part I

Mean Field Games Theory

Chapter 2

Reduced mean-field models

In this chapter we consider reduced mean-field models. The reasons we start discussing the reduced form of MFG are twofold. The first is the fact that it is a simpler and better understood problem and appeared first in the literature. The second reason is that the main results presented in this work regard the reduced formulation.

The models originally studied by Lasry and Lions [131–134] which consist in a system of a Hamilton-Jacobi type equation and an associated transport or Fokker-Planck equation. We present the derivation of such models and discuss various methods to prove existence and uniqueness of solutions. Stationary models are then briefly discussed. These are quite interesting in their own right but also, under appropriate conditions, encode the long-time asymptotic for mean-field games, as shown in [59, 60] (see also [88] and [89] for discrete models). Following [92], we consider also stationary extended models in which the cost for a reference player depends not only in the other players distribution but also on their actions. Then we look at certain variational structures that some of these problems enjoy, and the connections between mean-field models and other now classical problems such as optimal transport and Aubry-Mather theory. We then describe the random variables point of view. This formulation is very close to the one in [117] (although many of the problems considered in this survey are deterministic), but the presentation here reflects also the ideas and methods from the lectures of P. L. Lions in Collège de France [140]. We will show that mean-field games can be set up as a system of Hamilton-Jacobi equation coupled with an ODE in a space of random variables. In this part we discuss only deterministic control problems. This allows us to avoid using backwards stochastic differential equations (see for instance [149], and [64, 63]) and therefore keeping the presentation elementary. Mean-field models with correlations and the master equation will be considered in section 3. The random variable point of view makes it easy to consider models where the costs incurred by players depend not only on the distribution of other players but also on their actions. Such models were first studied in [103] and are also briefly considered here.

2.1 Derivation of reduced models

Let $U \subset \mathbb{R}^m$ be a convex closed set. As it is usual in stochastic optimal control problems (see [83], for instance), we consider a vector field $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ and a diffusion matrix $\sigma : \mathbb{R}^d \times U \to \mathcal{M}_{\mathbb{R}}^{d\times m}$, where $\mathcal{M}_{\mathbb{R}}^{d\times m}$ is the set of $d \times m$ real matrices. We suppose that both f and σ are globally Lipschitz in the first coordinate, that is, for all $v \in U$

$$|f(x, v) - f(y, v)|, |\sigma(x, v) - \sigma(y, v)| \leq C|x - y|,$$

where the constant C is independent of the control variable v and $x, y \in \mathbb{R}^d$. We also assume the following growth condition

$$|f(x, v)|, |\sigma(x, v)| \leq C(1 + |x| + |v|).$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a set, $\mathcal{F}$ a σ-algebra on Ω and P a probability measure. Let W_t be a Brownian motion on Ω and $\mathcal{F}_t$ the associated filtration. Fix an initial time $t_0 \in [0, T]$. Let $\mathcal{B}_r$ be the Borel σ-algebra on $[t_0, r]$. A control process $\mathbf{v} : [t_0, T] \times \Omega \to U$ is called $\{\mathcal{F}_r\}$-*progressively measurable* if the map $(s, \omega) \to \mathbf{v}(s, \omega)$ from $[t_0, r] \times \Omega$ into U is $\mathcal{B}_r \times \{\mathcal{F}_r\}$-measurable. We denote by $\mathcal{U}$ the set of all progressively measurable control processes.

We consider a population of agents where each agent is allowed to choose a progressively measurable control $\mathbf{v} \in \mathcal{U}$. This control determines the agent's dynamics through the stochastic differential equation (SDE)

$$d\mathbf{x} = f(\mathbf{x}, \mathbf{v})dt + \sigma(\mathbf{x}, \mathbf{v})dW_t. \tag{2.1.1}$$

We will assume that each agent dynamics' is driven by an independent Brownian motion in (2.1.1).

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of Borel probability measures in $\mathbb{R}^d$. Let $\theta : [t_0, T] \to \mathcal{P}(\mathbb{R}^d)$ be, for each time t, a probability distribution of agents in $\mathbb{R}^d$. Assume for the moment that the trajectory of each agent is determined by an independent copy of (2.1.1) where the control $\mathbf{v}$ is given as a *(non-time homogeneous) feedback Markovian control*, that is

$$\mathbf{v}(t) = v(\mathbf{x}, t),$$

for some function $v : \mathbb{R}^d \times [t_0, T] \to U$. Thus each agent of this population will follow the diffusion

$$d\mathbf{x} = f(\mathbf{x}, v(\mathbf{x}, t))dt + \sigma(\mathbf{x}, v(\mathbf{x}, t))dW_t. \tag{2.1.2}$$

Because the Brownian motion driving each agent is independent from the remaining ones, the population distribution will evolve according to the following *Fokker-Planck equation*

$$\theta_t + \operatorname{div}(b(x,t)\theta) = \partial^2_{ij}(a_{ij}(x,t)\theta),$$

where

$$b(x,t) = f(x, v(x,t)), \qquad a_{ij} = \frac{1}{2}\sum_{k=1}^{m} \sigma_{ik}(x, v(x,t))\sigma_{jk}(x, v(x,t)),$$

and the initial condition $\theta(x,t)$ is given.

We consider a Lagrangian $L : \mathbb{R}^d \times U \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, and a terminal cost $\Psi : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. Suppose L, and Ψ are continuous, bounded by below, and satisfy the following quadratic growth condition

$$|L(x,v,\theta)| \le C(1 + |x|^2 + |v|^2), \quad |\Psi(x,\theta)| \le C(1 + |x|^2),$$

for positive constants C independent of (x, v, θ). Assume further, if U is unbounded, that

$$\lim_{|v|\to\infty} \frac{L(x,v,\theta)}{|v|} \to \infty.$$

Fix an agent, which knows the strategy v used by the other players and whose objective is to find a progressively measurable control $\mathbf{v}$ which minimizes the following *cost functional*

$$J(x,t;\mathbf{v}) = E\int_t^T L(\mathbf{x}, \mathbf{v}, \theta)ds + \psi(\mathbf{x}(T), \theta(T)).$$

From the point of view of this agent, its *value function* is

$$V(x,t) = \inf_{\mathbf{v}\in\mathcal{U}} J(x,t;\mathbf{v}).$$

It is well known, see Appendix A, that V is then a viscosity solution to the *Hamilton-Jacobi equation*

$$-V_t + H(x, D_xV, D^2_{xx}V, \theta) = 0, \tag{2.1.3}$$

where $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}^{d\times d}_{\mathbb{R}} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$H(x,p,M,\theta) = \sup_{v\in U}\left[-f(x,v)\cdot p - \frac{1}{2}\sigma(x,v)\sigma^T(x,v) : M - L(x,v,\theta)\right],$$

and : is the *trace* that is defined as

$$A : B = \sum_{i,j=1}^{d} A_{ij}B_{ij},$$

for square matrices $A, B \in \mathcal{M}_{\mathbb{R}}^{d\times d}$. Furthermore, V satisfies the terminal condition

$$V(x,T) = \psi(x, \theta(T)).$$

Suppose that V is a smooth enough solution to (2.1.3), and that H is differentiable. Assume further that there exists a function $\bar{\upsilon} : \mathbb{R}^d \times [t,T] \to U$ such that

$$H(x, D_x V, D_{xx}^2 V, \theta) = -f(x,\bar{\upsilon}) \cdot D_x V - \frac{1}{2}\sigma(x,\bar{\upsilon})\sigma^T(x,\bar{\upsilon}) : D_{xx}^2 V - L(x,\bar{\upsilon},\theta).$$

If, for all $x \in \mathbb{R}^d$ and $m \in \mathcal{P}(\mathbb{R}^d)$, $H(x,\cdot,\cdot,m)$ is smooth enough, then a simple argument shows that

$$f(x,\bar{\upsilon}) = -D_p H(x, D_x V, D_{xx}^2 V, \theta), \qquad \frac{1}{2}\sigma(x,\bar{\upsilon})\sigma^T(x,\bar{\upsilon}) = -D_M H(x, D_x V, D_{xx}^2 V, \theta),$$

and that the control $\bar{\upsilon}$ is optimal.

We assume now that all players have access to the same information and therefore will use the same strategy $\bar{\upsilon}$ in (2.1.2). This gives rise to the *second order mean-field games system*

$$\begin{cases} -V_t + H(x, D_x V, D^2 V, \theta) = 0 \\ \theta_t - \operatorname{div}(D_p H\theta) - \partial_{ij}(D_{M_{ij}} H\theta) = 0, \end{cases} \tag{2.1.4}$$

coupled with the initial-terminal conditions

$$\begin{cases} V(x,T) = \psi(x,\theta(T)) \\ \theta(x,0) = \theta_0. \end{cases} \tag{2.1.5}$$

The boundary conditions in this problem are non-standard in the sense that part of the unknowns are subject to initial conditions and the rest of them are subject to terminal conditions. Therefore existence of solutions is not obvious and requires some justification. This will be discussed in section 2.2 for three model problems.

In addition to the initial-terminal conditions it is also interesting from the point of view of applications to consider the planning problem, see [3], and [156]. In this problem we are given two probability measures θ_0 and θ_1 and one looks for a pair (V,θ) solving (2.1.4) under the boundary conditions

$$\theta(x,0) = \theta_0, \qquad \theta(x,T) = \theta_1. \tag{2.1.6}$$

In [156] the existence of weak solutions for the planning problem for the second order case was established.

At this stage the key points to address are existence and uniqueness for solutions to (2.1.4). This will be done in the following sections.

2.2 Existence of solutions

We now discuss the existence of solutions for the initial-terminal value problem for mean-field games. Rather than considering the most general problem, we consider three model cases. The first two concern first and second order Hamilton-Jacobi equations with smooth dependence on the measure. The third case concerns local dependence on the measure.

We will follow closely in the first two parts of this section the lecture notes by P. Cardaliaguet [54]. As such, we will not detail the more technical arguments that can be found in that reference. The case of local potentials will be addressed by establishing various a-priori estimates, using the techniques in [131, 132, 98, 96, 95].

2.2.1 First order case

Here we look into the case where the dynamics of the players are deterministic. Therefore, given by an ordinary differential equation (ODE)

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{v}).$$

In this case we obtain the *first order mean-field games* system:

$$\begin{cases} -V_t + H(x, D_x V, \theta) = 0 & \text{in } \mathbb{R}^d \times [0,T) \\ \theta_t - \operatorname{div}(D_p H \theta) = 0 & \text{in } \mathbb{R}^d \times (0,T], \end{cases} \tag{2.2.1}$$

with initial-terminal condition

$$\begin{cases} V(x,T) = \psi(x, \theta(\cdot, T)) \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{2.2.2}$$

We will study the particular case where the Hamiltonian is given by

$$H(x, D_x V, \theta) = \frac{1}{2}|D_x V(x,t)|^2 - F(x, \theta(t)), \tag{2.2.3}$$

where F is a (nonlocal) operator on probability measures. A solution to (2.2.1) is a pair (V, θ), where where V is a bounded locally Lipschitz continuous solution to the Hamilton-Jacobi equation and θ is a weak solution to the transport equation. We denote by $\mathcal{P}_1(\mathbb{R}^d)$ the set of Borel probability measures in $\mathbb{R}^d$ with finite first moments endowed with the $1-$Wasserstein distance. We recall (see [167]) that the $1-$ *Wasserstein distance* between two probability measures θ_1 and θ_2 is defined as

$$d_1(\theta_1, \theta_2) = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y),$$

where the infimum is taken over the set $\Pi(\theta_1, \theta_2)$ of all probability measures π in $\mathbb{R}^d \times \mathbb{R}^d$ whose first marginal is θ_1 and the second marginal is θ_2. We define the norm $\|\cdot\|_{C^2}$ as

$$\|g\|_{C^2} = \sup_{x\in\mathbb{R}^d} \left(|g(x)| + |D_x g(x)| + |D^2_{xx} g(x)| \right),$$

for any $g \in C^2(\mathbb{R}^d)$.

Theorem 1. Suppose that F, and ψ in (2.2.2) are continuous on $\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$, θ_0 is absolutely continuous with respect to the Lebesgue measure, and that there exists a constant $C > 0$ such that

$$\|F(\cdot,\theta)\|_{C^2}, \|\psi(\cdot,\theta)\|_{C^2} \leq C,$$

uniformly for $\theta \in \mathcal{P}_1$. Then the system (2.2.1), for the Hamiltonian (2.2.3), and under initial-terminal conditions (2.2.2) admits a solution.

Proof. We outline in what follows the proof by a fixed point argument from [140], as detailed in [54].

Semiconcavity estimates We recall that a function $\psi : \mathbb{R}^d \to \mathbb{R}$ is semiconcave if there exists a constant C such that $\psi - C|x|^2$ is a concave function. The first step on the fixed point argument consists in proving that the solution to the equation

$$-V_t + H(x, D_x V, \theta) = 0, \tag{2.2.4}$$

for a fixed $\theta : [0,T] \to \mathcal{P}_1$ is semiconcave in x, with semiconcavity modulus uniform in θ. This follows from standard viscosity solution techniques, see for instance [20].

Optimal trajectory synthesis Though viscosity solutions may fail to be differentiable, by semiconcavity they are differentiable almost everywhere. Furthermore, if $x \in \mathbb{R}^d$ is a point of differentiability of $V(x,0)$ then the trajectory

$$\begin{cases} \dot{\mathbf{x}} = -D_p H(\mathbf{x}, \varpi, \theta) \\ \dot{\mathbf{p}} = D_x H(\mathbf{x}, \varpi, \theta), \end{cases}$$

with

$$\mathbf{x}(0) = x, \qquad \varpi(0) = D_x V(x, 0),$$

is an optimal trajectory for the optimal control associated with (2.2.4), and V is differentiable at $(\mathbf{x}(t), t)$, with $\varpi(t) = D_x V(\mathbf{x}(t), t)$, for $0 < t < T$.

Transport equation As in [54], we can define, using a measurable selection argument a flow $\Phi(x,t,s)$ satisfying

$$\Phi_s(x,t,s) = -D_pH(\Phi(x,t,s), D_xV(\Phi(x,t,s),\theta), \qquad \Phi(x,t,t) = x. \tag{2.2.5}$$

Furthermore Φ satisfies the following properties

1. $$|\Phi(x,t,s') - \Phi(x,t,s)| \leq C|s-s'|$$

2. $$|x-y| \leq C|\Phi(x,t,s) - \Phi(y,t,s)|.$$

We then define $\zeta(t) = \Phi(\cdot,0,t)\sharp\theta_0$. It is not hard to check that $\zeta : [0,T] \to \mathcal{P}_1$ is continuous and it is a weak solution to

$$\partial_t\zeta - \operatorname{div}(D_pH(x,D_xV,\theta)\zeta) = 0.$$

Additionally, since θ_0 is absolutely continuous, so is ζ due to the properties of the flow. The key issue is uniqueness. If the vector field $b(x,t) = -D_pH(x,D_xV,\theta)$ were Lipschitz in the x variable, the uniqueness of solution of the conservative transport equation would follow by standard methods. Unfortunately the above vector field may be discontinuous. Consequently, to establish uniqueness one needs to use a approach due to Ambrosio [14], [15], as explained in detail in [54].

Stability and fixed point argument The last step of the proof consists in a fixed point argument which depends on the following stability result: for $m \in C([0,T],\mathcal{P}_1)$ denote by $U[m]$ the solution to

$$-V_t + H(x,D_xV,m) = 0$$

with $V(x,T) = \Psi(x,m(T))$. Denote by $\Phi[m]$ the flow induced by $U[m]$ through (2.2.5), and $\Theta[m] = \Phi[m]\sharp\theta_0$. Then by stability of viscosity solutions if $m_n \to m$ then $U[m_n] \to U[m]$. By the semiconcavity estimates, we have almost everywhere convergence of $D_xU[m_n]$ to $D_xU[m]$. In addition, $\Theta[m_n]$ is (uniformly) absolutely continuous and so any sublimit will be absolutely continuous. But then $\lim_{n\to\infty}\Theta[m_n]$ is a solution to the transport equation for $D_xU[m]$ and by uniqueness $\Theta[m] = \lim_{n\to\infty}\Theta[m_n]$. This then shows that the map $m \mapsto \Theta[m]$ is continuous. It is also easy to see that it is compact since the properties of the flow imply Lipschitz continuity of $\Theta[m]$ as a map from $[0,T] \to \mathcal{P}_1$. Therefore this map admits a fixed point to which corresponds a solution to (2.2.1), as claimed. □

2.2.2 Second order case

Now we consider the second order reduced mean field model (2.1.4) with initial-terminal conditions (2.1.5). In order to simplify the presentation, and to focus in the main arguments, we assume that the Hamiltonian has the following structure

$$H(x, D_xV, D_x^2V, \theta) = -\Delta V + \frac{1}{2}|D_xV|^2 - F(x, \theta).$$

We assume further

A) F and ψ are uniformly bounded over $\mathbb{R}^d \times \mathcal{P}_1$, and also Lipschitz continuous,

B) θ_0 is absolutely continuous with a continuous density function with finite second moment:
$$\int_{\mathbb{R}^d} |x|^2\theta_0(x)dx < +\infty.$$

Once more we follow the argument proposed in [140] and detailed in [54] to prove the existence of solutions for the mean-field equations.

Theorem 2. Assume that conditions A) and B) hold. Then the reduced mean field game

$$\begin{cases} -V_t - \Delta V + \frac{1}{2}|D_xV|^2 = F(x, \theta(t)) & \mathbb{R}^d \times [0, T) \\ \theta_t - \Delta\theta - \operatorname{div}(D_xV\theta) = 0 & \mathbb{R}^d \times (0, T] \end{cases}$$

with initial-terminal conditions

$$\begin{cases} V(x, T) = \Psi(x, \theta(T)) \\ \theta(x, 0) = \theta_0(x), \end{cases}$$

has a solution (V, θ).

Proof. The proof, as in the previous section is based upon a fixed point argument, of which we outline the main steps.

Fokker-Planck equation The first step consists in studying weak solutions, $\theta \in L^1([0, T], \mathcal{P}_1)$, of the Fokker-Planck equation

$$\begin{cases} \theta_t - \Delta\theta - \operatorname{div}(B\theta) = 0 \\ \theta(x, 0) = \theta_0(x), \end{cases} \tag{2.2.6}$$

where $B : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ is a vector field assumed to be continuous, bounded, and uniformly Lipschitz continuous in x. To do so, consider the stochastic differential equation

$$d\mathbf{x} = B(\mathbf{x}, t)dt + \sqrt{2}dW_t,$$

where W_t is a d-dimensional Brownian motion. Suppose further that $\mathcal{L}(\mathbf{x}_0) = \theta_0$. Then it is well known that $\theta(t) = \mathcal{L}(\mathbf{x})$ is a weak solution to the Fokker-Planck equation (2.2.6).

Furthermore, there exists a constant depending only on the terminal time T, $C_0 = C_0(T)$ such that

$$d_1(\theta(t), \theta(s)) \le C_0 (\|f\|_\infty + \|\sigma\|_\infty) |t-s|^{\frac{1}{2}}. \tag{2.2.7}$$

Indeed, let $s < t$, and consider the random variables $\mathbf{x}_t$, $\mathbf{x}_s$ with law $\mathcal{L}(\mathbf{x}_t) = \theta(t)$, and $\mathcal{L}(\mathbf{x}_s) = \theta(s)$. Using the definition of the *Kantorovitch-Rubinstein distance*, and observing that the joint law $\gamma \in \Pi(\theta(t), \theta(s))$ we have

$$d_1(\theta(t), \theta(s)) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| d\gamma(x,y) = E[|\mathbf{x}_t - \mathbf{x}_s|].$$

Since both $\mathbf{x}_t, \mathbf{x}_s$ satisfy (2.1.1), with the same initial condition, we get

$$E|\mathbf{x}_t - \mathbf{x}_s| \le E\left[\int_s^t |f(\mathbf{x}_r, r)| dr + \left|\int_s^t \sigma(\mathbf{x}_r, r) dW_t\right|\right] \le K(|f\|_\infty + \|\sigma\|_\infty)\sqrt{t-s}.$$

Additionally, elementary computations show that there exists a constant $C_0 = C_0(T)$ such that

$$\int_{\mathbb{R}^d} |x|^2 d\theta(t,x) \le K\left(\int_{\mathbb{R}^d} |x|^2 d\theta_0(x) + \|f\|_\infty^2 + \|\sigma\|_\infty^2\right). \tag{2.2.8}$$

The idea in [54] is to consider the set $\mathcal{K}$ given by

$$\mathcal{K} = \left\{ m \in C^0([0,T], \mathcal{P}) : \sup_{s \neq t} \frac{d_1(m(s), m(t))}{|t-s|^{\frac{1}{2}}} \le C_0, \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |x|^2 dm(t,x) \le C_0 \right\},$$

for C_0 large enough. This set is a convex compact subset of $C^0([0,T], \mathcal{P})$, which is essential to apply a fixed point argument.

Second-order Hamilton-Jacobi equation The second step consists in looking at the Hamilton-Jacobi equation

$$\begin{cases} -V_t + -\Delta V + \frac{1}{2}|D_x V|^2 = F(x, m(t)) & \mathbb{R}^d \times [0,T) \\ V(x,T) = \Psi(x, m(T)) & \mathbb{R}^d, \end{cases}$$

where m is a given density measure in $\mathcal{K}$. Since F and Ψ satisfy A. and B., and $m \in \mathbb{C}^0([0,T], \mathcal{K})$, using the Cole-Hopf transform is is possible to show that the Hamilton Jacobi equation has unique solution V with $D_x V$ Lipschitz.

Fixed point argument For the last step of the argument, define the map

$$\begin{aligned}\Upsilon : \mathcal{K} &\to \mathcal{K}\\ m &\mapsto \theta = \Upsilon(m),\end{aligned}$$

where $\mathcal{K}$ is the convex, compact set defined before. Note that θ solves the Fokker-Planck equation and is continuous. Since $V \in \mathbb{C}^{2+\frac{1}{2}}$ we have uniqueness of solutions for the Fokker-Planck equation and also $\theta \in \mathbb{C}^{2+\frac{1}{2}}$. From (2.2.7) and (2.2.8) we conclude that $\theta \in \mathcal{K}$.

We now prove that Υ is continuous. Take a sequence (θ_n) in $\mathcal{K}$ convergent to θ. Then using the local uniformly convergence of Ψ, and F we obtain that (V_n) is also uniformly convergent, say, to V. Using interior regularity estimates we prove that (D_xV_n) is locally uniformly Holder continuous so it converges local uniformly to D_xV. Further we have that (θ_n) converges to θ. So now we have all the results we need to state and prove the existence result.

Use the above defined map Υ, between the convex set $\mathcal{K}$ and apply the Schauder fixed point theorem. The solution to the reduced mean field game is precisely the fixed point (V,θ). □

2.2.3 A-priori estimates methods

We will address a different dependence of the mean-field game on the measure θ. Rather than a smoothing one, we consider a local dependence on the measure. Again, to simplify the presentation we consider the following equation

$$\begin{cases} -V_t + \frac{|D_xV|^2}{2} = \Delta V + g(\theta) \\ \theta_t - \operatorname{div}(D_xV\theta) = \Delta\theta, \end{cases} \tag{2.2.9}$$

under initial-terminal data

$$V(x,T) = \psi(x) \qquad \theta(x,0) = \theta_0(x), \tag{2.2.10}$$

periodic boundary conditions in the spatial variable, that is $x \in \mathbb{T}^d$, and $g(\theta) = G'(\theta)$, where G is a convex increasing function. We should note that for quadratic Hamiltonians such as (2.2.9) smooth solutions are known to exist, see [59]. However the proof depends on the Hopf-Cole transformation and does not generalize in any obvious way for Hamiltonians which satisfy, for instance, quadratic-type growth conditions. The techniques present here can be generalized easily, and so more general Hamiltonians can be studied with similar techniques, as well as somewhat more general dependence on the measure of the form $g(x,\theta)$.

Lasry-Lions estimates One can easily obtain the following estimate, see [132, 133], by multiplying the first equation in (2.2.9) by θ and the second equation by V, subtracting these two and integrating by parts:

$$\int_0^T \int_{\mathbb{T}^d} \frac{|D_x V|^2}{2}\theta + G(\theta) \leq C. \tag{2.2.11}$$

This estimate is related with the optimality of certain mean-field games as described in section 2.5.

In the case $g \geq 0$ one can obtain the additional estimate

$$\int_0^T \int_{\mathbb{T}^d} |D_x V|^2 dxdt \leq C. \tag{2.2.12}$$

By combining the estimates (2.2.11) and (2.2.12), Lions and Lasry in [132, 133] obtained existence of weak solutions for various mean-field games, see also the papers [157] and [58].

In [93, 94] the following estimate for mean-field games, also in the case $g \geq 0$ was obtained

$$\int_0^T \int_{\mathbb{T}^d} g'(\theta)|D\theta|^2 + |D^2 V|\theta \leq C, \tag{2.2.13}$$

which extends a similar estimate for the stationary case in [78], as well as in [98], for second order problems (a similar estimate was also obtained by P. L. Lions). This result can be established by applying the Laplacian operator to the first equation of (2.2.9) and integrating with respect to θ.

Fokker-Planck equation The previous estimates were obtained by looking at the first equation, the Hamilton-Jacobi equation, in (2.2.9). However, the second equation also has regularizing properties. By combining the previous results with iterative methods for parabolic equations Gomes and his co-authors obtained in [93, 94] the following result, which only depends on $g \geq 0$:

Theorem 3. $\theta \in L^\infty((0,T), L^r(\mathbb{T}^d))$, for all $0 < r < \frac{2^*}{2}$, where $2^* = \frac{2d}{d-2}$ is the Sobolev conjugated exponent to 2.

The proof of this theorem uses an iterative procedure. First we know a-priori that $\theta \in L^\infty((0,T), L^1)$, since the Fokker-Planck equation conserves mass. The idea is to construct a sequence β_n such that at each step one has $\theta \in L^\infty((0,T), L^{1+\beta_n})$. One first obtains the identity

$$\begin{aligned}\int_{\mathbb{T}^d} \theta^{\beta+1}(x,\tau)dx + \frac{4\beta}{\beta+1}\int_0^\tau \int_{\mathbb{T}^d} |D\theta^{\frac{\beta+1}{2}}|^2 dx\, dt \\ = \int_{\mathbb{T}^d} \theta^{\beta+1}(x,0)dx + \beta\int_0^\tau \int_{\mathbb{T}^d} \mathrm{div}(DV)\theta^{\beta+1} dxdt.\end{aligned} \tag{2.2.14}$$

The last term can be controlled by

$$\int_0^\tau \int_{\mathbb{T}^d} \Delta V \theta^{\beta+1} \leq C \int_0^\tau \int_{\mathbb{T}^d} (\Delta V)^2 \theta + \delta \int_0^\tau \int_{\mathbb{T}^d} \theta^{2\beta+1} dx dt$$

The first term in the right hand side can be estimated by the inequality (2.2.13) and the second one is handled by using a combination of Sobolev inequalities and Hölder inequality. This allows to establish an estimate for the $L^\infty((0,T), L^{1+\beta_{n+1}})$ norm of θ in terms of the $L^\infty((0,T), L^{1+\beta_n})$ norm of θ. For the details we refer the reader to [93, 94].

Regularity for Hamilton-Jacobi equation From the integrability properties for θ, we can now look back at the Hamilton-Jacobi equation. We consider the reference case $g(\theta) = \theta^\alpha$. For quadratic Hamiltonians the existence of smooth solutions was established in [59]. The proof in that paper relies on the Hopf-Cole transformation and depends strongly on the specific quadratic form of the Hamiltonian and does not extend (except perhaps in very specific perturbation regimes) to general Hamiltonians. For Hamiltonians with sub-quadratic growth P.L. Lions, established in [141], the following result:

Theorem 4. Consider the Hamiltonian

$$H(p,x) = (1+|p|^2)^{\frac{\gamma}{2}} + V(x)$$

with $1 \leq \gamma < 2$. If $\alpha < \frac{2}{d-2}$ or $1 \leq \gamma < 1 + \frac{1}{d+1}$ and $\alpha > 0$, then $D_t V, D^2_{xx} V \in L^p([0,T] \times \mathbb{T}^d)$ for any p, and $\theta \in L^\infty([0,T], L^p)$.

Once this regularity is obtained then further regularity results can also be established by bootstrapping and standard methods. In [93] this result was improved in the subquadratic case and, through a completely different proof, in [94] the authors were also able to study also the superquadratic case. In particular the following result was proved in those papers:

Theorem 5. Consider the Hamiltonian

$$H(p,x) = (1+|p|^2)^{\frac{\gamma}{2}} + V(x).$$

Then if $1 + \frac{1}{d+1} < \gamma < 2$ there exists $\alpha_{\gamma,d} > \frac{2}{d-2}$ and for $2 \leq \gamma < 3$ for $\alpha_{\gamma,d} = \frac{2}{d\gamma-2}$, the solutions of the corresponding mean-field game

$$-V_t + H(x, DV) = \theta^\alpha + \Delta V, \qquad \theta_t - \operatorname{div}(D_p H \theta) = \Delta\theta,$$

for $\alpha < \alpha_{\gamma,d}$, with smooth initial-terminal data and $\theta(x,0)$ bounded away from zero satisfy $D_t V, D^2_{xx} V \in L^p([0,T] \times \mathbb{T}^d)$ for any p, and $\theta \in L^\infty([0,T], L^p)$.

As remarked previously, from this regularity it follows the existence of smooth solutions.

2.3 Uniqueness by the Lions-Lasry monotonicity method

We now address the uniqueness of classical solutions for the initial-terminal value problem for mean-field games. We start by reviewing the Lions-Lasry monotonicity method. Then we present a definition of weak solution which allows for an improved uniqueness result.

2.3.1 Monotonicity method

We discuss here uniqueness for classical solutions in the second order case (the first order case is just a special case of the second order case) using the technique by Lions and Lasry. This proof yields uniqueness for classical solutions. In the next section we show how to modify the proof so that one can prove uniqueness for viscosity solutions without any regularity hypothesis on the solutions. We consider mean-field games in $\mathbb{R}^d$, but the argument extends to the periodic case without any difficulty.

Theorem 6. Consider a smooth Hamiltonian of the form $H(x,p,M,\theta) = H_0(x,p,M) - F(x,\theta)$, where $F : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$. Assume further that H_0 is jointly convex in p and M, that F is strictly monotone in θ, that is, if $\theta_1, \theta_2 \in \mathcal{P}_1$, $\theta_1 \neq \theta_2$ then

$$\int_0^T \int_{\mathbb{R}^d} (F(x,\theta_1) - F(x,\theta_2))(\theta_1 - \theta_2)(x) > 0, \tag{2.3.1}$$

and

$$\int_{\mathbb{R}^d} (\psi(x,\theta_1) - \psi(x,\theta_2))\,(\theta_1 - \theta_2) \geq 0, \quad \forall \theta_1, \theta_2 \in \mathcal{P}_1. \tag{2.3.2}$$

Then the initial-terminal value problem for the mean-field game given by

$$\begin{cases} -V_t + H_0(x, D_x V, D_x^2 V) = F(x,\theta) \\ \theta_t - \operatorname{div}(D_p H_0 \theta) - \partial^2_{ij}(D_{M_{ij}} H_0 \theta) = 0, \end{cases} \tag{2.3.3}$$

together with conditions (2.1.5), has at most a classical solution (V,θ).

Proof. We now follow the Lions-Lasry's strategy to prove uniqueness. Suppose, by contradiction that there exist two solutions, (V_1,θ_1), and (V_2,θ_2) of the above mean-field game. We have

$$\int_0^T \int_{\mathbb{R}^d} \frac{d}{dt}(V_1 - V_2)(\theta_1 - \theta_2) = 0.$$

In fact, for the uniqueness proof it suffice to have ≥ 0 in the previous expression. The expression for the left hand side can be obtained by considering the equations for $\bar{V} = V_1 - V_2$ and $\bar{\theta} = \theta_1 - \theta_2$ and multiply them by $\bar{\theta}$ and $\bar{V}$, respectively. After subtracting the later from the former, rearranging various terms we obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}^d} \bar{V}(x,0)\bar{\theta}_0(x) - (\psi(x,\theta_1(T)) - \psi(x,\theta_2(T)))\,\bar{\theta}(x,T) \qquad (2.3.4)\\ &- \int_0^T \int_{\mathbb{R}^d} \Big(D_pH_0(x,D_xV_1,D^2_{xx}V_1)\theta_1 - D_pH_0(x,D_xV_2,D^2_{xx}V_2)\theta_2\Big)\nabla\bar{V}\\ &+ \int_0^T \int_{\mathbb{R}^d} \Big(-D_{M_{ij}}H_0(x,D_xV_1,D^2_{xx}V_1)\theta_1 + D_{M_{ij}}H_0(x,D_xV_2,D^2_{xx}V_2)\theta_2\Big)\,\partial^2_{ij}\bar{V}\\ &+ \int_0^T \int_{\mathbb{R}^d} \Big(H_0(x,D_xV_1,D^2_{xx}V_1) - H_0(x,D_xV_2,D^2_{xx}V_2)\Big)\,\bar{\theta} + \int_0^T \int_{\mathbb{R}^d} (F(x,\theta_2) - F(x,\theta_1))\,\bar{\theta}\end{aligned}$$

where we have assumed enough regularity and decay to integrate by parts. Using the condition (2.3.2), and the convexity of H_0, which implies

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} \Big(H_0(x,D_xV_2,D^2_{xx}V_2) - H_0(x,D_xV_1,D^2_{xx}V_1) - D_pH_0(x,D_xV_1,D^2_{xx}V_1)(\nabla V_2 - \nabla V_1)\\ &\qquad -D_{M_{ij}}H_0(x,D_xV_1,D^2_{xx}V_1)(\partial^2_{ij}V_2 - \partial^2_{ij}V_1)\Big)\theta_1\\ &+ \int_0^T \int_{\mathbb{R}^d} \Big(H_0(x,D_xV_1,D^2_{xx}V_1) - H_0(x,D_xV_2,D^2_{xx}V_2) - D_pH_0(x,D_xV_2,D^2_{xx}V_2)(\nabla V_1 - \nabla V_2)\\ &\qquad -D_{M_{ij}}H_0(x,D_xV_2,D^2_{xx}V_2)(\partial^2_{ij}V_1 - \partial^2_{ij}V_2)\Big)\theta_2 \geq 0,\end{aligned}$$

we conclude that

$$\int_0^T \int_{\mathbb{R}^d} (F(x,\theta_1) - F(x,\theta_2))(\theta_1 - \theta_2)(x) \leq 0,$$

which contradicts (2.3.1). This yields $\theta_1 = \theta_2$. Then uniqueness for viscosity solutions implies $V_1 = V_2$, therefore the solution is unique. □

In the local case one can also use a similar argument to obtain uniqueness. As discussed in P.L. Lions course [140], as described in [109], take $H(x,p,z) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+_0 \to \mathbb{R}$. Then uniqueness for the mean-field game

$$\begin{cases} -V_t - \Delta V + H(x,D_xV,\theta) = 0\\ \theta_t - \Delta\theta - \operatorname{div}(D_pH\theta) = 0\end{cases}$$

holds if the H satisfies

$$\begin{bmatrix} zD^2_{pp}H & \frac{1}{2}zD^2_{pz}H\\ \frac{1}{2}zD^2_{zp}H & -D_zH\end{bmatrix} > 0$$

for any $(x,p,z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$.

2.3.2 Relaxed solutions and uniqueness

We now introduce a notion of relaxed solutions for mean-field games that allows to prove uniqueness under minimal regularity assumptions. To simplify the discussion we consider first order mean-field games. Let $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ be a continuous function. Let $L(x, \cdot, \theta)$ be the Legendre transform of $H(x, \cdot, \theta)$, for all $x \in \mathbb{R}^d$ and $\theta \in \mathcal{P}_1(\mathbb{R}^d)$, which to simplify we assume bounded by below.

A relaxed solution for the mean-field game

$$\begin{cases} -u_t + H(x, Du, \theta) = 0 \\ \theta_t - \operatorname{div}(D_p H\theta) = 0. \end{cases} \tag{2.3.5}$$

is a triplet (u, θ, J) where $u : \mathbb{R}^d \times [0, T] \to \mathbb{R}$, $\theta \in C([0, T], \mathcal{P}(\mathbb{R}^d))$ and J is a vector valued measure in $[0, T] \times \mathbb{R}^d$ absolutely continuous with respect to θ, satisfying the following properties:

1. $u \in C(\mathbb{R}^d)$ is a viscosity solution of $-u_t + H(x, Du, \theta) = 0$;
2. for $0 \leq t \leq T$, $u(\cdot, t) \in L^1(d\theta(\cdot, t))$;
3. as a distribution
$$\theta_t + \operatorname{div}(J) = 0;$$
4. since J is absolutely continuous with respect to θ denote by $v(x, t)$ its Radon-Nykodym derivative. Then we require
$$\int u(x, 0) d\theta(x, 0) \geq \int_0^T \int_{\mathbb{R}^d} L(x, v(x, t), \theta) d\theta + \int_{\mathbb{R}^d} u(x, T) d\theta(x, T).$$

Any classical solution to the mean-field game is in fact a relaxed solution, for $J = -D_p H\theta$. Also we observe that (under very mild standard assumptions) from the optimal control representation for viscosity solutions of Hamilton-Jacobi equations, for any pair $(\tilde{\theta}, \tilde{J})$ with $\tilde{\theta} \in C([0, T], \mathcal{P}(\mathbb{R}^d))$ $\tilde{J}$ absolutely continuous with respect to $\tilde{\theta}$ such that $\tilde{J} = \tilde{v}\theta$ we have

$$\int u(x, 0) d\tilde{\theta}(x, 0) \leq \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x, t), \theta) d\tilde{\theta} + \int_{\mathbb{R}^d} u(x, T) d\tilde{\theta}(x, T). \tag{2.3.6}$$

Indeed, under mild standard regularity hypothesis it is possible to build a sequence of C^1 functions, such that $u^n \to u$ uniformly, and

$$-u_t^n + H(x, D_x u^n, \theta) \leq o(1),$$

as $n \to \infty$. Then

$$-u_t^n - \tilde{v} D_x u^n \leq o(1) - H(x, D_x u^n, \theta) - -\tilde{v} D_x u^n \leq o(1) + L(x, \tilde{v}, \theta).$$

Integrating and passing to the limit we obtain (2.3.6).

One advantage of this notion of relaxed solution is that uniqueness can be proved without any regularity or differentiability assumptions on u.

Theorem 7. Suppose L satisfies

$$\int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \theta)d\tilde{\theta} + L(x, v(x,t), \tilde{\theta})d\theta - L(x, v(x,t), \theta)d\theta - L(x, \tilde{v}(x,t), \tilde{\theta})d\tilde{\theta} < 0, \tag{2.3.7}$$

whenever $(\theta, J) \neq (\tilde{\theta}, \tilde{J})$, where $J = v\theta$ and $\tilde{J} = \tilde{v}\tilde{\theta}$.

Then the initial-terminal value problem for (2.3.5) has at most one relaxed solution.

Proof. Let (u, θ, J) and $(\tilde{u}, \tilde{\theta}, \tilde{J})$ be relaxed solutions to (2.3.5). Then

$$\int u(x,0)d\theta(x,0) = \int_0^T \int_{\mathbb{R}^d} L(x, v(x,t), \theta)d\theta + \int_{\mathbb{R}^d} u(x,T)d\theta(x,T),$$

$$\int \tilde{u}(x,0)d\tilde{\theta}(x,0) = \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \tilde{\theta})d\tilde{\theta} + \int_{\mathbb{R}^d} \tilde{u}(x,T)d\tilde{\theta}(x,T),$$

$$\int \tilde{u}(x,0)d\tilde{\theta}(x,0) \leq \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \theta)d\tilde{\theta} + \int_{\mathbb{R}^d} \tilde{u}(x,T)d\tilde{\theta}(x,T),$$

and

$$\int u(x,0)d\theta(x,0) \leq \int_0^T \int_{\mathbb{R}^d} L(x, v(x,t), \tilde{\theta})d\theta + \int_{\mathbb{R}^d} u(x,T)d\theta(x,T).$$

Adding the last two inequalities and subtracting the first two equalities yields:

$$0 \leq \int_0^T \int_{\mathbb{R}^d} L(x, \tilde{v}(x,t), \theta)\tilde{\theta} + L(x, v(x,t), \tilde{\theta})\theta - L(x, v(x,t), \theta)\theta - L(x, \tilde{v}(x,t), \tilde{\theta})\tilde{\theta},$$

which contradicts (2.3.7) unless $(\theta, J) = (\tilde{\theta}, \tilde{J})$. Then $u = \tilde{u}$ by uniqueness for viscosity solutions. □

An example where the previous theorem applies is the separated case

$$L = \frac{|v|^2}{2} + g(\theta),$$

where g is a monotone function (not necessarily local) in the sense that

$$\int_{\mathbb{R}^d} (g(\theta) - g(\tilde{\theta}))(\theta - \tilde{\theta}) > 0,$$

if $\theta \neq \tilde{\theta}$.

2.4 Stationary problems

In addition to the terminal-initial value or planning problems discussed in the previous section, stationary problems play an important role in many applications. The stationary problem corresponding to (2.1.4) consists in finding a triplet $(u, \theta, \bar{H})$, where $u : \mathbb{R}^d \to \mathbb{R}$, $\theta \in \mathcal{P}(\mathbb{R}^d)$ and $\overline{H} \in \mathbb{R}$ which solves

$$\begin{cases} H(D_x u, x, \theta) = \Delta u + \overline{H} \\ -\operatorname{div}(D_p H \theta) - \Delta\theta = 0. \end{cases} \tag{2.4.1}$$

The constant $\overline{H}$ is called the effective Hamiltonian as it arises in related problems in homogenization theory, see [33], as well as in Aubry-Mather theory.

There are three natural questions that arise immediately when considering (2.4.1). First, of course, is existence (and regularity) of solutions, secondly uniqueness of the constant $\overline{H}$ and of solutions, and finally to what extend one can expect time-dependent mean-field games to converge to stationary solutions.

Concerning existence, one can use similar proofs to the ones in the time dependent case. In particular it is possible to prove suitable a-priori bounds for (2.4.1), see [131], [98, 96], for instance. In certain cases uniqueness can be established by monotonicity methods using a procedure similar to the one in Section 2.3. An important class of stationary mean-field games admits a variational formulation. In those cases (2.4.1) is the Euler-Lagrange of a (possibly non-coercive) convex functional. For these variational cases, once existence is established, uniqueness follows by standard convexity arguments in the calculus of variations see [98] for instance. We will discuss some variational structures for mean-field games in Section 2.5. The last question, the trend to equilibrium, will be addressed in what follows. We end this section by presenting some results on extended mean-field games.

2.4.1 Trend to equilibrium

A natural question in the initial-terminal value problem the following: suppose one is given an initial probability measure $m^T(x, 0) = m_0(x)$ and a terminal cost $u^T(x, T) = u_0(x)$ and then lets $T \to \infty$ - is it true that $m^T(x, t)$ and $u^T(x, t)$ converge to a stationary solution? In some sense this would mean that by taking an initial probability distribution far in the past and a terminal cost far in the future, the present behaves like an equilibrium. The first positive answer to this problem in discrete state and time was given in [88] and then further extended to the continuous time setting in [89]. The key idea in these papers is to adapt the uniqueness proof by monotonicity to extract the convergence result. A similar idea was also independently used in [59] and in [60] where the authors studied the continuous time problem both for local coupling and non-local coupling. More recently, the first order case was addressed in [55]. Further results on the finite state problem were established in [80] using Γ convergence through a different set of ideas. For Hamilton-Jacobi equations, a new class of ideas using the adjoint method is discussed in [49].

The long time convergence results in [59] hold for fairly general Hamiltonians as long as smooth enough solutions are known to exist. In particular the results apply to the following mean-field game:

$$\begin{cases} -u_t - \Delta u + \frac{1}{2}|Du|^2 = F(x, \theta) \\ \theta_t - \Delta\theta - \operatorname{div}(\theta Du) = 0, \end{cases} \tag{2.4.2}$$

coupled with initial-terminal conditions

$$\begin{cases} \theta(x, 0) = \theta_0(x) \\ u(x, T) = u_0(x), \end{cases}$$

as well as periodic boundary conditions in the spatial variable, that is $x \in \mathbb{T}^d$. Let (u^T, θ^T) be a solution of (2.4.2) satisfying, the above, initial-terminal conditions. The existence and uniqueness of solution (u^T, θ^T) of (2.4.2) was already discussed in sections 2.2.2 and 2.3. In order to state the convergence results in [59] we need to consider the following stationary problem (in fact in [59] more general initial-terminal conditions are considered):

$$\begin{cases} \overline{H} - \Delta\bar{u} + \frac{1}{2}|D\bar{u}|^2 = F(x, \bar{\theta}) \\ -\Delta\bar{\theta} - \operatorname{div}(\bar{\theta} D\bar{u}) = 0 \end{cases} \tag{2.4.3}$$

where $\bar{u}$ and $\bar{m}$ satisfy the normalization conditions:

$$\begin{cases} \int_{\mathbb{T}^d} \bar{u} dx = 0 \\ \int_{\mathbb{T}^d} \bar{\theta} dx = 1. \end{cases}$$

We denote by $(\overline{H}\bar{u}, \bar{\theta})$ the solution to the above problem.

A convergence result In the above mentioned paper, the convergence is proved assuming the following conditions:

A1. $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, is C^1, and $\mathbb{Z}^d-$*periodic* in the space variable x and increasing in m,

A2. $u_0 : \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic and of class C^2,

Also, for convenience, one considers a rescaled version of (2.4.2), $v^T(x, t) = v(x, Tt$, and $\theta^T(x, t) = \theta(x, Tt)$, which satisfies:

$$\begin{cases} -\frac{1}{T} v_t^T - \Delta v^T + \frac{1}{2}|Dv^T|^2 = F(x, \theta^T) \\ \frac{1}{T}\theta_t^T - \Delta\theta^T - \operatorname{div}(\theta^T Dv^T) = 0, \end{cases} \tag{2.4.4}$$

with $\theta^T(x,0)=\theta_0(x)$, $v^T(x,1)=u_0(x)$.

Provided the above conditions hold, the convergence results obtained in [59] are the following:

Theorem 8. Let (v^T,θ^T) be a solution to (2.4.4). Then

B1. $v^T(t,\cdot)/T$ converges uniformly in $L^1(\mathbb{T}^d)$ to $(1-t)\overline{H}$, for $t\in[0,1]$,

B2. v^T converges uniformly to $\tilde{u}$ in $L^2((0,1)\times\mathbb{T}^d)$, where $\tilde{u}=(1-t)\overline{H}$,

B3. $v^T-\int_{\mathbb{T}^d} v^T(t,y)dy$ converges to $\bar{u}$ in $L^2((0,1)\times\mathbb{T}^d)$,

B4. $\theta^T\to\bar{\theta}$ in $L^p((0,1)\times\mathbb{T}^d)$ for $p<\frac{d+2}{d}$ provided $d>2$, and for $p<2$ if $d=2$,

B5. $F(\cdot,\theta^T)\to F(\cdot,\bar{\theta})$ and $F(\cdot,\theta^T)\theta^T\to F(\cdot,\bar{\theta})\bar{\theta}$ both in $L^1((0,1)\times\mathbb{T}^d)$.

The proof of this theorem relies on the following: first by applying the same technique as in the uniqueness proof one obtains the following identity:

$$\int_0^1\int_{\mathbb{T}^d}\frac{\theta^T+\bar{\theta}}{2}|Dv^T-D\bar{u}|^2+(F(x,\theta^T)-F(x,\bar{\theta}))(\theta^T-\bar{\theta})dxdt=-\frac{1}{T}\left[\int_{\mathbb{T}^d}(v^T-\bar{u})(\theta^T-\bar{\theta})\right]_0^1.$$

The second key step consists in obtaining estimates to show that the right hand side is in fact controlled and therefore implies convergence. The proof presented in that paper uses the specific form of the Hamiltonian through the use of the Hopf-Cole transform to obtain various estimates.

Convergence rate The convergence rate's result proved in [59] assumes, in addition to conditions A1.-A.2, the following one:

A3. The growth rate of F is bounded by below. Let $s\geq t$, so there is a $\gamma>0$ such that

$$F(x,s)-F(x,t)\geq\gamma(s-t),\quad\forall x\in\mathbb{T}^d.$$

Let $C>0$ be a constant, and define $\tilde{u}^T(x,t)=u^T(x,t)-\int_{\mathbb{T}^d}u^T(x,t)$, where u^T is a solution to (2.4.2).

Theorem 9. Under the assumptions A1.-A.3 then for $\forall t\in(0,T)$, the following holds:

$$\|\tilde{u}^T(t)-\bar{u}\|_{L^1(\mathbb{T}^d)}\leq\frac{C}{T-t}\left(e^{-C(T-t)}+e^{-Ct}\right),$$
$$\|\theta^T(t)-\bar{\theta}\|_{L^1(\mathbb{T}^d)}\leq\frac{C}{t}\left(e^{-C(T-t)}+e^{-Ct}\right),$$

and for all $t\in(0,T-1)$

$$\left\|\frac{u^T(t)}{T}-\overline{H}\left(1-\frac{t}{T}\right)\right\|_{L^1(\mathbb{T}^d)}\leq\frac{C}{T}.$$

where $(\overline{H}, \bar{u}, \bar{\theta})$ is a solution to the associated ergodic problem (2.4.3).

2.4.2 Extended stationary models

In many applications it is important to consider mean-field games where the running cost or the dynamics of the players depend not only on the distribution of players but also on their actions. This leads to the class of extended mean-field games considered in [103] and [92]. In this section we describe briefly the stationary models and refer the reader to [92] for details and additional results.

We denote by $\chi(\mathbb{T}^d)$ the set of continuous vector fields on $\mathbb{T}^d$, and by $\mathcal{P}^{ac}$ the set of absolutely continuous probability measures in $\mathbb{T}^d$. Let

$$H\colon \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}^{ac}(\mathbb{T}^d) \times \chi(\mathbb{T}^d) \to \mathbb{R}.$$

Then one can consider the system:

$$\begin{cases} -\Delta u(x) + H(x, Du(x), \theta, V) = \overline{H} \\ -\Delta\theta(x) + \operatorname{div}(V(x)\theta(x)) = 0 \\ V(x) = -D_p H(x, Du(x), \theta, V). \end{cases} \tag{2.4.5}$$

The unknowns for this problems are $u\colon \mathbb{T}^d \to \mathbb{R}$, identified with a $\mathbb{Z}^d$-periodic function on $\mathbb{R}^d$ whenever convenient, a probability measure $m \in \mathcal{P}(\mathbb{T}^d)$, the effective Hamiltonian $\overline{H} \in \mathbb{R}$ and the effective velocity field $V \in \chi(\mathbb{T}^d)$. Among the problems considered in [92] the following example was investigated:

$$H(x, p, \theta, V) = h(x, p) + \delta p \cdot \int_{\mathbb{T}^d} V d\theta - g(\theta) \tag{2.4.6}$$

where h is a coercive and satisfies quadratic growth-type conditions and $g : \mathbb{R}^+ \to \mathbb{R}$ is either $g(z) = z^\alpha$ or $g(z) = \ln z$ (other possible functions can also be handled with similar methods but these two are representative of the main techniques and difficulties). In the above reference, the authors proved:

Theorem 10. The system (2.4.5) where H is given by (2.4.6) admits a unique classical solution, for δ small enough, for $g = \ln m$ in any dimension, and for $g(z) = z^\alpha$, if $d \leq 4$ for any exponent α and if $d \geq 5$ for $\alpha < \frac{1}{d-4}$.

The proof of this theorem, which rather is lengthy depends upon establishing careful a-priori estimates for the solutions and applying a continuation argument.

2.5 Potential mean-field models

Certain mean-field games admit variational formulations which allows the use of duality and calculus of variations techniques in their study. Some of these structures were already discussed in the papers [131–134], and used to study the planning problem in [3] or the long-time behavior in [55]. Also, existence of weak solutions for first order problems was addressed by variational methods in [58]. These will be addressed in section 2.5.1 and consist in optimization problems in the space of measures whose optimality conditions are equivalent to mean-field games. Another related class of variational structures, which are written in terms of integral variational problems, was discovered in the study of the stochastic Evans-Aronsson problem in [98] (see also [86]). These will be discussed in section 2.5.2 together with some applications and extensions. Then, in section 2.5.3, we investigate, through duality, the connection between these problems defined through multiple integrals and optimization problems in the space of measures.

Rather than developing here a complete theory we present several examples and applications. More general problems can be handled by adapting the ideas outlined in the present paper. Throughout this section we will work on the periodic setting, that is the state space is $\mathbb{T}^d$, the d-dimensional torus, identified with $[0,1]^d$. The main reason is to avoid problems that could arise by computing integral functionals on non-compact domains. By similar methods, one can consider boundary value problems of various types.

2.5.1 Optimal control in the space of measures

We discuss in this section a class of planning problems for mean-field games which can be seen as optimal control problems in the space of measures.

Let $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ be a convex function. Suppose that F is differentiable with gradient with respect to the L^2 inner product $\nabla F(\rho)$. We will work in the setting of section 2.1 under the following simplifying assumptions $U = \mathbb{R}^d$, $f(x,v) = v$, and $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$. Consider the problem of minimizing over all (smooth enough) vector fields $b : \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ and measures ρ in $\mathbb{T}^d \times [0,T]$ the functional

$$\int_0^T \int_{\mathbb{T}^d} F(\rho) + L(x, b(x,t))\rho \tag{2.5.1}$$

under the constraint

$$\rho_t + \operatorname{div}(b\rho) = \Delta\rho, \tag{2.5.2}$$

with $\rho(x,0) = \theta_0$ and $\rho(x,T) = \theta_1$.

In order to study this problem and obtain optimality conditions we will introduce the dual problem through the minimax principle. Our discussion will be mostly informal as our main objective in this section is to obtain optimality conditions. However, a rigorous discussion of duality in this setting can be found in [3].

Let $\phi : \mathbb{T}^d \times [0,T] \to \mathbb{R}$ be a smooth function that will act as the Lagrange multiplier for (2.5.2). The problem of minimizing (2.5.1) under the constraint (2.5.2) is equivalent to

$$\min_{b,\rho} \max_{\phi} \int_0^T \int_{\mathbb{T}^d} F(\rho) + L(x, b(x,t))\rho - \phi(\rho_t + \operatorname{div}(b\rho) - \Delta\rho). \tag{2.5.3}$$

By definition, the dual problem is the variational problem obtained by switching the minimum with the maximum. In general, the value for the dual problem is a lower bound for the original problem. In many cases, it is possible to show that their values coincide using the Legendre-Fenchel-Rockafellar theorem, see [3]. Note that the dual problem is simply

$$\max_{\phi} \min_{b,\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) + L(x, b(x,t))\rho - \phi(\rho_t + \operatorname{div}(b\rho) - \Delta\rho).$$

By integrating by parts and performing the minimization over the vector fields b we obtain that the dual problem is simply

$$\max_{\phi} \min_{\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) + (\Delta\phi - H(x, D_x\phi) + \phi_t)\,\rho + \int_{\mathbb{T}^d} \phi(x,0)\theta_0 - \int_{\mathbb{T}^d} \phi(x,T)\theta_1, \tag{2.5.4}$$

where

$$H(x,p) = \sup_{v \in \mathbb{R}^d} \left[-v \cdot p - L(x,v)\right].$$

Proposition 1. Let (V, θ) be a solution to

$$\begin{cases} -V_t - \Delta V + H(x, D_x V) = \nabla F(\theta) \\ \theta_t - \Delta\theta - \operatorname{div}(D_p H(x, D_x V)\theta) = 0 \end{cases}$$

satisfying $\theta(x,0) = \theta_0$, $\theta(x,T) = \theta_T$. Then V is optimal for (2.5.4), $(\rho, b) = (\theta, -D_p H(x, D_x V))$ is optimal for (2.5.1). Furthermore there is no duality gap, that is, the value of the primal agrees with the one of the dual.

Proof. Denote by P the value in (2.5.3), and Q the value of (2.5.4). We always have $P \geq Q$.

Clearly, by choosing $\phi = V$ in (2.5.4) we obtain the following lower bound:

$$\begin{aligned} Q &\geq \min_{\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) + (\Delta V - H(x, D_x u) + V_t)\,\rho + \int_{\mathbb{T}^d} V(x,0)\theta_0 - \int_{\mathbb{T}^d} V(x,T)\theta_1 \\ &= \min_{\rho} \int_0^T \int_{\mathbb{T}^d} F(\rho) - \nabla F(\theta)\rho + \int_{\mathbb{T}^d} V(x,0)\theta_0 - \int_{\mathbb{T}^d} V(x,T)\theta_1. \end{aligned}$$

By convexity we have therefore

$$Q+\int_{\mathbb{T}^d}V(x,T)\theta_1-\int_{\mathbb{T}^d}V(x,0)\theta_0\geq\int_0^T\int_{\mathbb{T}^d}F(\rho)-\nabla F(\theta)\rho\geq\int_0^T\int_{\mathbb{T}^d}F(\theta)-\nabla F(\theta)\theta.$$

Choosing in (2.5.1) $\rho=\theta$ and $b=-D_pH(x,D_xV)$, which by definition satisfy (2.5.2) we have the following upper bound:

$$P\leq\int_0^T\int_{\mathbb{T}^d}F(\theta)+L(x,-D_pH(x,D_xV))\theta.$$

Using the identity

$$L(x,-D_pH(x,D_xV))-D_xV\cdot D_pH(x,D_xV)=-H(x,D_xV),$$

we have that

$$P\leq\int_0^T\int_{\mathbb{T}^d}F(\theta)+(V_t-H(x,D_xV)+\Delta V)\,\theta+(-V_t-\Delta V+D_pH(x,D_xV)D_xV)\,\theta.$$

From this we get

$$P+\int_{\mathbb{T}^d}V(x,T)\theta_1-\int_{\mathbb{T}^d}V(x,0)\theta_0\leq\int_0^T\int_{\mathbb{T}^d}F(\theta)-\nabla F(\theta)\theta,$$

that is $P\leq Q$. This shows that there is no duality gap and the optimality of $(\theta,-D_pH(x,D_xu))$ and u. □

It is important to observe, however, that not every mean-field game will have such a variational structure. Additionally, it is also not true, in general, that variational problems which involve general costs $L(x,b,\rho)$ rather than a sum of a linear functional in ρ and a nonlinear function of ρ have optimality conditions equivalent to mean-field games.

2.5.2 Calculus of variations with convex non-linear integrands

We consider now a class of variational problems than gives rise to mean-field games through the minimization of functionals defined by multiple integrals. The discussion here is based upon the ideas first developed in [98], [96], and [86]. The connection by duality theory between these problems and the ones considered in the previous section will be discussed in section 2.5.3.

In this setting it is more convenient to start in a somewhat more general setting which includes various important examples, such as initial-terminal, planning and stationary problems. Let W be a compact set, in most examples either $W=\mathbb{T}^d\times[0,T]$ or $W=\mathbb{T}^d$. Consider a nonlinear operator denoted by $\mathcal{N}:C^\infty(W)\to C^\infty(W)$. Important examples of such operators are

$$\mathcal{N}(V) = -V_t + H(x, D_x V) - \Delta V, \tag{2.5.5}$$

with $W = \mathbb{T}^d \times [0, T]$ and

$$\mathcal{N}(V) = H(x, D_x V) - \Delta V, \tag{2.5.6}$$

for $W = \mathbb{T}^d$. To simplify we will assume H smooth. Note that many other variations, including first order (simply by omitting the Laplacian), general fully non-linear elliptic operators or even non-local operators can also be considered by the same methods. We will assume $\mathcal{N}$ to be differentiable with respect to u, that is, for any $v \in C_c(W)$ the following directional derivative exists and defines a linear operator $\mathcal{L}_u : C^\infty(W) \to C^\infty(W)$

$$\frac{d}{d\epsilon}\mathcal{N}(u + \epsilon v)\bigg|_{\epsilon=0} = \mathcal{L}_u v.$$

The linear operators corresponding to examples (2.5.5), and (2.5.6) are

$$\mathcal{L}_V v = -v_t + D_p H(x, D_x V) D_x v - \Delta v,$$

and

$$\mathcal{L}_V v = D_p H(x, D_x V) D_x v - \Delta v.$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a convex increasing function. Consider the integral functional

$$\int_W G(\mathcal{N}(V))dx. \tag{2.5.7}$$

Let V be a minimizer of (2.5.7). Then, assuming enough regularity, an elementary computation shows that the Euler-Lagrange equation in weak form is

$$\int_W G'(\mathcal{N}(V))\mathcal{L}_V v dx = 0, \text{ for all } \ v \in C_c^\infty(W).$$

If we define

$$\theta = G'(\mathcal{N}(V)),$$

the above Euler-Lagrange equation can be written in strong form as

$$\begin{cases} \mathcal{N}(V) = (G')^{-1}(\theta) \\ \mathcal{L}_V^* \theta = 0, \end{cases} \tag{2.5.8}$$

where $\mathcal{L}_V^*$ is the adjoint of $\mathcal{L}_V$ with respect to the L^2 inner product.

For illustration purposes, we consider now some possible choices of G and $\mathcal{N}$. First set $G(z) = z^\alpha$, and $W = [0,T] \times \mathbb{T}^d$ in (2.5.7). Then, for $\alpha \neq 1$, (2.5.8) is simply

$$\begin{cases} -V_t + H(x, D_x V) - \Delta V = \left(\frac{\theta}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ \theta_t - \operatorname{div}(D_p H(x, D_x V)\theta) - \Delta\theta = 0. \end{cases}$$

Consider also the case $G(z) = e^z$, then (2.5.8) becomes

$$\begin{cases} -V_t + H(x, D_x V) - \Delta V = \ln\theta \\ \theta_t - \operatorname{div}(D_p H(x, D_x V)\theta) - \Delta\theta = 0. \end{cases}$$

This variational interpretation of mean-field games is quite remarkable as it shows that various problems which have been researched intensely in the last few years are closely related to mean-field games. Take for instance

$$\mathcal{N}(V) = |DV|, \qquad G(z) = z^p.$$

The Euler-Lagrange equation for this functional is simply the p-Laplace equation.

Also certain mean-field games have surprising regularizing properties. Take for instance

$$\mathcal{N}(V) = \frac{|DV|^2}{2} + W(x), \qquad G(z) = e^z.$$

This corresponds to the mean-field game

$$\begin{cases} \frac{|DV|^2}{2} + W(x) = \ln\theta \\ -\operatorname{div}(DV\theta) = 0. \end{cases}$$

Though in general first order equations have only Lipschitz or semiconcave solutions, this mean-field game in fact has smooth classical solutions. This was proved in the periodic setting in [78]. In [98] the second order case, which is associated with the non-coercive, convex functional

$$\int_{\mathbb{T}^d} e^{-\Delta V + \frac{|DV|^2}{2} + W(x)},$$

was also studied and shown to admit smooth solutions which are minimizers of the above functional. For further results, see also [92].

In certain applications, it may be necessary to modify the structure of the mean-field game equations. For instance, there may be a source f of agents, or they may die at a rate γ.

In this case it is natural to consider equations of the form

$$\begin{cases}\gamma V + H(x, DV) = \Delta u + \ln\theta \\ \gamma\theta - \operatorname{div}(D_pH(x, DV)\theta) = \Delta\theta + f(x).\end{cases}$$

The previous equation is also a Euler-Lagrange equation of the functional

$$\int e^{\gamma V + H(x,DV) - \Delta V} - fV.$$

Various other modifications can also be considered to study optimal switching and obstacle type problems, see [91]. Mean-field games with a non-linear Fokker-Planck equation were considered in [97].

2.5.3 Duality revisited

We now apply duality theory to the problems discussed in the previous section. We will work a specific example but it should be clear how to apply these ideas in different settings.

Consider the problem

$$\min_{\mathcal{C}} \int_{\mathbb{T}^d} G(-q(x) + H(p(x), x)),$$

where $\mathcal{C}$ is the set of smooth functions (ϕ, p, q), $\phi : \mathbb{T}^d \to \mathbb{R}$, $p : \mathbb{T}^d \to \mathbb{R}^d$, and $q : \mathbb{T}^d \to \mathbb{R}$, which satisfy the constraints

$$p = D_x\phi, \quad q = \Delta\phi.$$

We introduce two Lagrange multipliers $J : \mathbb{T}^d \to \mathbb{R}^d$ and $\theta : \mathbb{T}^d \to \mathbb{R}$. Proceeding as before, we look at the functional

$$\int_{\mathbb{T}^d} G(-q + H(p, x)) + J(p - D_x\phi) + \theta(q - \Delta\phi).$$

The Euler-Lagrange equation for the previous functional can be written as

$$\theta = G'(-q + H(p, x)), \qquad J = \theta D_pH(p, x), \tag{2.5.9}$$

and

$$\operatorname{div}(J) - \Delta\theta = 0. \tag{2.5.10}$$

Now note that if θ and J are given by (2.5.9) then

$$G(-q + H(p, x)) + Jp + \theta q = \inf_{p,q} \left[G(-q + H(p, x)) + Jp + \theta q\right] = Z(J, \theta).$$

That is objective functional of dual problem is then

$$\int_{\mathbb{T}^d} Z(J,\theta)dx,$$

together with the constraint (2.5.10).

2.6 Random variables point of view

We discuss in this section the random variables point of view for deterministic mean-field games. This allows us, in the first order case, to reformulate (2.1.4) as a system of a Hamilton-Jacobi equation and a ordinary differential equation in a space of random variables. This formulation is very close to the one originally considered in [117, 114]. The presentation here reflects also ideas discussed by P. L. Lions in [140]. The random variables point of view is also convenient to the study extended mean-field games, where the costs incurred by a player depend not only on the positions of the other players but also on their actions. A further application of this framework is the stochastic case where the players have a common noise. The latter problem will be briefly discussed in section 3. See also the recent papers [149, 64, 63, 66, 65] where these problems are addressed using backward-forward stochastic differential equations. This is approach is also natural to address the limit as the number of players N tends to infinity using the interaction particle framework. Such limit is a fundamental problem also in statistical physics, see for instance [162] and references therein.

2.6.1 Random variables

Let $(\Omega,\mathcal{F},P)$ be a probability space, where Ω is a non-empty set, $\mathcal{F}$ a σ-algebra on Ω and P a probability measure. As usual in probability theory we denote integration with respect to P by the *expected value*, that is, for any P-integrable real valued function φ we set

$$E\varphi \equiv \int_\Omega \varphi dP.$$

A $\mathbb{R}^d$ valued random variable X is a measurable map $X:\Omega\to\mathbb{R}^d$. For definiteness, we will consider random variables which are L^p integrable. The *law* of a $\mathbb{R}^d$ valued random variable is the probability measure $\mathcal{L}(X)$ in $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} \phi d\mathcal{L}(X) = E\phi(X),$$

for any bounded continuous function $\phi:\mathbb{R}^d\to\mathbb{R}$. We say that a function $\Psi: L^p(\Omega)\to\mathbb{R}$ depends only on the law of a random variable if for any pair of random variables $X,Y\in L^p(\Omega)$ such that $\mathcal{L}(X)=\mathcal{L}(Y)$ we have $\Psi(X)=\Psi(Y)$.

Let $\eta : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, $\eta(\theta)$. We define a function, $\tilde{\eta} : L^p(\Omega; \mathbb{R}^d) \to \mathbb{R}$ which depends only on the law of the argument, by

$$\tilde{\eta}(X) = \eta(\mathcal{L}(X)).$$

This allows us to identify functions in $\mathcal{P}(\Omega)$ with functions in $L^p(\Omega)$ which depend only on the law.

2.6.2 Dynamics

We regard the set Ω as the collection of all players. We will consider a time dependent family of random variables $\mathbf{X} : \Omega \times [t, T] \to \mathbb{R}^d$. If $\omega \in \Omega$, we interpret $\mathbf{X}_s(\omega)$ as the position of the player ω at time s. For the moment we suppose we are given a vector field determined by a function

$$B : \mathbb{R}^d \times L^p(\Omega) \times [t, T] \to \mathbb{R}^d,$$

depending only on the law on the second coordinate. We suppose the players in Ω follow the deterministic trajectory

$$\dot{\mathbf{X}}_s(\omega) = B(\mathbf{X}_s(\omega), \mathbf{X}_s, s).$$

We observe that for our purposes it is important to distinguish between the dependence of B on the position of a player ω at $\mathbf{X}_s(\omega)$ and the law of the random variable $\mathbf{X}_s$. We consider now a reference player, which has a dynamic

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{v}),$$

where f is as in Section 2.1, and $\mathbf{v} : [t, T] \to U$ is the control of this reference player. As before, we denote by $\mathcal{U}$ the set of bounded controls on $[t, T]$ with values in U.

In this new setting, the Lagrangian is a function $L : \mathbb{R}^d \times L^p(\Omega) \times U \to \mathbb{R}$, that we denote by $L(x, X, v)$, which in the second coordinate depends only on the law. An example of such a Lagrangian is

$$L(x, X, v) = \frac{|v|^2}{2} - EW(x, X),$$

where $W : \mathbb{R}^d \times \mathbb{R}^p(\Omega) \to \mathbb{R}$ is, for instance, a bounded Lipschitz function. The terminal cost is given by a function $\psi : \mathbb{R}^d \times L^p(\Omega) \to \mathbb{R}$ which depends only on the law of the second coordinate.

The objective of the reference player is to minimize

$$V(x, t) = \inf_{\mathbf{v} \in \mathcal{U}} \int_t^T L(\mathbf{x}, \mathbf{X}_s, \mathbf{v}) ds + \psi(\mathbf{x}(T), \mathbf{X}_T).$$

As before, for $(x, X, p) \in \mathbb{R}^d \times L^p(\Omega) \times \mathbb{R}^d$, the Hamiltonian is given by

$$H(x, X, p) = \sup_{v \in U} \left[-f(x, v) \cdot p - L(x, X, v) \right].$$

The Hamiltonian $H : \mathbb{R}^d \times L^p(\Omega) \times \mathbb{R}^d \to \mathbb{R}$, depends only on the law of the second coordinate.

Then, from standard viscosity solution methods, V is the unique viscosity solution of the Hamilton-Jacobi equation

$$-V_t(x, t) + H(x, \mathbf{X}_t, D_x V(x, t)) = 0$$

with the terminal condition $V(x, T) = \psi(x, \mathbf{X}_T)$. As before, the optimal feedback strategy for the reference player yields the dynamics

$$\dot{\mathbf{x}} = -D_p H(\mathbf{x}, \mathbf{X}_t, D_x V(\mathbf{x}, t)).$$

We assume at this stage that each of the players is faced with the same optimization problem. Thus they all have a similar strategy and consequently

$$B(x, X, t) = -D_p H(x, X, D_x V(x, t)).$$

Hence, for $\omega \in \Omega$

$$\dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, D_x V(\mathbf{X}_s(\omega), s)).$$

Therefore the mean-field equations can be written as

$$\begin{cases} -V_t + H(x, \mathbf{X}, D_x V) = 0 \\ \dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, D_x V(\mathbf{X}_s(\omega), s)), \end{cases}$$

with the initial-terminal condition

$$\begin{cases} V(x, T) = \psi(x, \mathbf{X}_T) \\ \mathbf{X}_0 = X_0. \end{cases}$$

2.7 Extended mean-field models

We now consider extended mean-field models. These differ from the ones discussed in the previous section because the cost function depends not only on the state of the other players, but also on their strategies. Here, as before, we will formulate the problem using the random variables point of view as in [103].

2.7.1 Model set up

We consider the same set up as in the previous subsection, except that the Lagrangian now depends also on the other players actions. More precisely we consider a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}$. We further assume that the $L(x, v, X, Z)$ depends only on the joint law of $(X, Z) \in L^q(\Omega) \times L^q(\Omega)$. The players positions are determined by a random variable X, and its velocities by the random variable Z.

We consider a reference player and assume as before that the dynamics of the remaining players is described by a differentiable trajectory $\mathbf{X} : [t, T] \to L^q(\Omega)$, $\dot{\mathbf{X}}_s(\omega) = B(\mathbf{X}_s(\omega), \mathbf{X}_s, s)$, with B fixed for now and known by the reference player. This player, which we assume to be at time t in the state $x \in \mathbb{R}^d$, faces the following optimal control problem:

$$V(x, t) = \inf_{\mathbf{v} \in \mathcal{U}} \int_t^T L(\mathbf{x}, \mathbf{v}, \mathbf{X}_s, B) ds + \Psi(\mathbf{x}(T), \mathbf{X}(T)).$$

The Hamiltonian $H : \mathbb{R}^d \times L^q(\Omega) \times \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R}$, is given by

$$H(x, X, p, Z) = \sup_{v \in U} \left[-f(x, v) \cdot p - L(x, X, v, Z) \right],$$

and depends only on the joint law of $(X, Z) \in L^q(\Omega) \times L^q(\Omega)$. Assuming enough regularity on the value function, V is the unique viscosity solution of the following Hamilton-Jacobi equation

$$-V_t(x, t) + H(x, \mathbf{X}_t, D_x V(x, t), \dot{\mathbf{X}}) = 0.$$

As before, all players act rationally, therefore follow optimal trajectories. Then the dynamics, for all players $\omega \in \Omega$, is

$$\dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \dot{\mathbf{X}}_s, D_x V(\mathbf{X}_s(\omega), s), \dot{\mathbf{X}}_s).$$

Henceforth the mean-field equations are

$$\begin{cases} -V_t(x, t) + H(x, \mathbf{X}, D_x V(x, t), \dot{\mathbf{X}}) = 0 \\ \dot{\mathbf{X}}_s(\omega) = -D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, D_x V(\mathbf{X}_s(\omega), s), \dot{\mathbf{X}}_s), \end{cases} \tag{2.7.1}$$

with the following initial-terminal condition

$$\begin{cases} V(x, T) = \psi(x, \mathbf{X}(T)) \\ \mathbf{X}(0) = X_0. \end{cases} \tag{2.7.2}$$

2.7.2 Existence

Following [103], we address now existence and uniqueness of solutions of the extended mean-field games, (2.7.1) with initial-terminal conditions (2.7.2). Consider that the following conditions, for $1 \leq q < \infty$ and a Lipschitz bounded function ψ, hold

1. For all $x \in \mathbb{R}^d$, $X, Z \in L^q(\Omega)$, the Lagrangian $L(x, v, X, Z)$ is strictly convex in v and satisfies the coercivity condition
$$\lim_{|v|\to\infty} \frac{L(x, v, X, Z)}{|v|} \to \infty,$$
uniformly in x.

2. $L(x, v, X, Z) > -c_0 E[|X|^q + |Z|^q + 1]$.

3. For all $X, Z \in L^q(\Omega)$ there exists a continuous function $v_0 : L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}^d$ such that $L(x, v_0(X, Z), X, Z) \leq c_1$.

4. $|D_v L|, |D^2_{vv}|L \leq (c_2 L + c_3) E[|X|^q + |Z|^q + 1]$, and $|D_x L|, |D^2_{x,v} L|, |D^2_{xx}| \leq c_2 L + c_3$.

5. $D_x H$ is Lipschitz in $\mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega)$, where $H = L^*$.

6. For any $X, Y, P \in L^q(\Omega)$ the equation $Z = -D_p H(X, P, Y, Z)$ can be solved with respect to Z as
$$Z = G(X, P, Y),$$
where $G : L^q(\Omega) \times L^q(\Omega) \times L^q(\Omega) \to L^q(\Omega)$ is a Lipschitz map.

7. The Hamiltonian H is continuous with respect to X, Z, and locally uniformly in x, p.

8. For any $R > 0$ there exists a constant $C(R)$ such that
$$|H(x, p, X, Z) - H(y, q, Y, W)| \leq C(R) \left(|x - y| + |p - q| + E[X - Y] + E[Z - W]\right),$$
for $|x|, |y|, |p|, |q|, \|X\|_q, \|Y\|_q, \|Z\|_q, \|W\|_q \leq R$.

Theorem 11. Let the above conditions on L and ψ hold. Suppose X_0 has an absolutely continuous law. Then there exists a solution $(V, X) \in \mathbb{R}^d \times C^{1,1}([0, T] \times L^q(\Omega))$ of the extended mean field game (2.7.1) with initial-terminal condition (2.7.2). Furthermore V is a semiconcave and Lipschitz continuous function.

Proof. The proof in [103] uses a fixed point argument that we sketch now, and is divided into the following main steps.

Expanded dynamics Let u be a Lipschitz function. Using the assumption 6. we consider the following system of ODEs in $L^q(\Omega)$,

$$\begin{cases} \dot{X}_s(\omega) = G(P_s(\omega), X_s(\omega), X_s) \\ \dot{P}_s(\omega) = D_x H(P_s(\omega), X_s(\omega), X_s, \dot{X}_s) \\ X_0 = X_0, \quad P_0 = D_x u(X_0). \end{cases} \tag{2.7.3}$$

By the assumptions on the Lagrangian L, and on the terminal cost Ψ, the value function u is Lipschitz continuous. Therefore by the Rademacher's theorem $D_x u$ exists almost everwhere. Furthermore since $\mathcal{L}(X_0)$ is supposed to be absolutely continuous P_0 is well defined. By standard arguments the Lipschitz condition on G and $D_x H$ implies uniqueness of solutions (X, P) for the above system (2.7.3).

Optimal control problem Given a solution to (2.7.3), we consider the following optimal control problem,

$$\tilde{V}(x,t) = \inf_{\mathbf{x}} \int_t^T L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{X}_s, \dot{\mathbf{X}}_s) ds + \psi(\mathbf{x}(T), \mathbf{X}_T) \tag{2.7.4}$$

where the take the infimum over all absolutely continuous trajectories $\dot{\mathbf{x}}(s)$ with $\dot{\mathbf{x}}(t) = x$.

Lemma 2.7.1. *The value function $\tilde{V}(x,t)$ is Lipschitz continuous and semi-concave. Therefore the following conditions hold*

1. $\tilde{V} \leq (T-t)c_1 + \|\psi\|_\infty \quad \forall x \in \mathbb{R}^d, 0 \leq t \leq T.$
2. $|\tilde{V}(x+y,t) - \tilde{V}(x,t)| \leq c_6|y| \quad \forall x, y, \in \mathbb{R}^d, 0 \leq t \leq T.$
3. $\tilde{V}(x+y,t) - \tilde{V}(x-y,t) - 2\tilde{u}(x,t) \leq c_7|y|^2 \quad \forall x, y \in \mathbb{R}^d, 0 \leq t \leq T.$

With constants c_6, c_7 depending only on L, ψ and T.

Fixed point We consider the following map, for a Lipschitz function u we associate a trajectory $\mathbf{X}$ by solving (2.7.3). Then, we compute the solution $\tilde{V}$ to (2.7.4). Denote by Ψ the map $\Psi(u)(x) = \tilde{V}(x, 0)$.

Lemma 2.7.2. *Let $\mathcal{A}$ be the set of functions $u \in C^0(\mathbb{R}^d)$ with $|u| \leq c_8$, $Lip(u) \leq c_6$, and semi-concave with constant c_7. Then the mapping Ψ is a continuous and compact mapping from $\mathcal{A}$ into itself.*

Once this lemma is established the existence of fixed point of Ψ follows from Browder's fixed point theorem. Then the argument ends by proving that this fixed point satisfies the mean-field equations (2.7.1). □

2.7.3 Uniqueness

We now address the uniqueness of solutions to extended mean-field games (2.7.1). The key technique to prove uniqueness is based upon the Lions-Lasry monotonicity method. In the setting of random variables these monotonicity conditions can be formulated either in terms of the Hamiltonian, as in the original Lions-Lasry argument, or in terms of the Lagrangian (as considered before in section 2.3.2). We discuss both approaches here following [103].

Lasry-Lions monotonicity argument Recall that to prove uniqueness of the mean field game (2.1.4) the strategy is the following: suppose (V,θ) and $(\bar{V},\bar{\theta})$ are solutions satisfying (2.1.5). Then uniqueness, as explained in section 2.3, follows from considering the quantity

$$\frac{d}{dt}\int (V-\bar{V})(\theta-\bar{\theta}), \tag{2.7.5}$$

which together with appropriate monotonicity assumptions on H and the terminal condition yields a contradiction.

The analog idea in the random variable setting is the following: suppose that (V,X), and $(\bar{V},\bar{X})$ solve the extended mean-field equations (2.7.1) with initial-terminal condition (2.7.2). Then (2.7.5), in this setting, becomes

$$\frac{d}{dt}E\left[V(X,t)-\bar{V}(X,t)+\bar{V}(\bar{X},t)-V(\bar{X},t)\right].$$

In order to illustrate this technique we give a simple example which is also presented in [103]. Let the following conditions hold:

1. The Hamiltonian is given by

$$H(x,p,X,Z)=H_0(x,p+\beta E[Z])+F(x,X),$$

where H_0 is convex in p, and $\beta\geq 0$.

2. The following monotonicity condition holds true

$$E\left[F(X,X)-F(X,\bar{X})+F(\bar{X},\bar{X})-F(\bar{X},X)\right]<0,$$

for $X\neq\bar{X}$.

3. The terminal condition ψ satisfies

$$E\left[\psi(X,X)-\psi(X,\bar{X})+\psi(\bar{X},\bar{X})-\psi(\bar{X},X)\right]\geq 0.$$

Then we have the following result:

Theorem 12. Let the Hamiltonian in (2.7.1) be given by 1. above. Assume further that the conditions 2. and 3. hold. Then uniqueness of (classical) solutions to the extended mean-field game (2.7.1) with initial-terminal condition (2.7.2) holds.

Proof. Suppose that (V, X) and $(\bar{V}, \bar{X})$ are two distinct solutions of (2.7.1) satisfying the initial-terminal condition (2.7.2). By using the various assumptions we obtain

$$\frac{d}{dt} E\left[(V - \bar{V})(X, t) + (\bar{V} - V)(\bar{X}, t)\right] < 0. \tag{2.7.6}$$

However, by routine computations and using the monotonicity hypothesis we obtain the opposite inequality. Therefore (2.7.6) must be a contradiction, henceforth proving the uniqueness result. □

A Lagrangian approach We present now a uniqueness result in terms of monotonicity conditions for the Lagrangian, as presented in [103]. Suppose that the Lagrangian function satisfy the following monotonicity condition

$$E\left[L(X, Z, X, Z) - L(\tilde{X}, \tilde{Z}, X, Z) + L(\tilde{X}, \tilde{Z}, \tilde{X}, \tilde{Z}) - L(X, Z, \tilde{X}, \tilde{Z})\right] > 0, \tag{2.7.7}$$

provided that $X \neq \tilde{X}$ or $Z \neq \tilde{Z}$, where $X, \tilde{X}, Z, \tilde{Z} \in L^q(\Omega)$.

Remark 2.7.3. As shown in [103], provided that L is strictly convex the above monotonicity condition is equivalent to the following differential one

$$E\left[Z^T D^2_{vZ} LZ + X^T D^2_{xX} LX + Z^T D^2_{vX} LY + Y^T D^2_{xZ} LZ\right] > 0,$$

where the the Lagrangian is evaluated at an arbitrary point $(X_1, Z_1, X_2, Z_2) \in (L^q(\Omega))^4$.

We suppose further that the terminal cost function satisfies

$$E\left[\psi(X, X) - \psi(X, \tilde{X}) + \psi(\tilde{X}, \tilde{X}) - \psi(\tilde{X}, X)\right] \geq 0. \tag{2.7.8}$$

Theorem 13. Assume that (2.7.7) and (2.7.8) hold, then there exists a unique solution to (2.7.1).

Proof. Suppose that (X, V) and $(\tilde{X}, \tilde{V})$ are two solutions of (2.7.1). Henceforth, for a.e. $\omega \in \Omega$ we have that X, and $\tilde{X}$ are minimizers of optimal control problems for which the value functions are, respectively, given by

$$V(X(0), 0) = \int_0^T L(X(s), \dot{X}(s), X(s), \dot{X}(s))ds + \psi(X(T), X(T)), \tag{2.7.9}$$

and

$$\tilde{V}(\tilde{X}(0), 0) = \int_0^T L(\tilde{X}(s), \dot{\tilde{X}}(s), \tilde{X}(s), \dot{\tilde{X}}(s))ds + \psi(\tilde{X}(T), \tilde{X}(T)). \tag{2.7.10}$$

Furthermore we have

$$V(\tilde{X}(0),0) \leq \int_0^T L(\tilde{X}(s),\dot{\tilde{X}}(s),X(s),\dot{X}(s))ds + \psi(\tilde{X}(T),X(T)), \tag{2.7.11}$$

$$\tilde{V}(X(0),0) \leq \int_0^T L(X(s),\dot{X}(s),\tilde{X}(s),\dot{\tilde{X}}(s))ds + \psi(X(T),\tilde{X}(T)). \tag{2.7.12}$$

Combining the previous inequalities we easily obtain a contradiction to our assumptions (2.7.7) and (2.7.8). This implies that $X = \tilde{X}$. The identity $V = \tilde{V}$ then follows from the uniqueness of viscosity solutions. $\square$

Chapter 3

Mean-field models in master form

In this section we discuss a more general formulation for mean-field games, called the master equation. These ideas were introduced by Lions in [140]. Here we focus particularly in the random variables point of view and address both the deterministic and stochastic correlated cases, where the players are subject to a common Brownian motion.

3.1 Deterministic models

We now consider deterministic mean-field games and we derive the master form setting. To do so, we will use the notation and hypotheses from section 2.6.

We start by looking at the optimal control problem

$$V(x,X,t) = \inf_{\mathbf{v}} \left[\int_t^T L(\mathbf{v}(s), \mathbf{x}(s), \mathbf{X}_s) ds + \psi(\mathbf{x}(T), \mathbf{X}_T) \right], \tag{3.1.1}$$

where $\mathbf{x}$ is the trajectory of a player which starts at time t at point $\mathbf{x}(t) = x$, and is controlled by $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{v})$, and $\mathbf{X}_s(\omega)$ is the trajectory of the population of the players which move along a vector field $B : \mathbb{R}^d \times L^q(\Omega; \mathbb{R}^d) \times [0,T] \to L^q(\Omega; \mathbb{R}^d)$,

$$\dot{\mathbf{X}}_s = B(\mathbf{X}_s(\omega), \mathbf{X}_s, s), \qquad \mathbf{X}(t) = X,$$

as previously. The key difference here is that we are considering the value function as a function of both x and X. Then it, at least formally, that the value function V for (3.1.1) is a viscosity solution of the equation (see for instance [90, 154], where infinite dimensional optimal control in the space of random variables are considered and the viscosity solution property is established rigorously for related problems).

$$-V_t(x,X,t) - D_X V(x,X,t) \cdot B(X,X,t) + H(x,X,D_x V(x,X,t)) = 0, \tag{3.1.2}$$

where H is as before, and $D_X V$ denotes the Frechét derivative with respect to the random variable X.

Furthermore, if V is a smooth enough solution to (3.1.2) then any optimal control $\bar{v}$ for (3.1.1) satisfies $f(x, \bar{v}(x, X, t)) = -D_p H(x, X, D_x V(x, X, t))$. Since we assume all players act rationally they will all follow the optimal flow. This then yields $B(x, X, t) = -D_p H(x, X, D_x V(x, X, t))$. Thus we arrive to the equation

$$-V_t(x, X, t) + D_X V(x, X, t) \cdot D_p H(X, X, D_x V(X, X, t)) + H(x, X, D_x V(x, X, t)) = 0, \quad (3.1.3)$$

with terminal condition $V(x, X, T) = \psi(x, X)$. Equation (3.1.3) is called the master equation.

Though a general theory for this class of equations is still lacking, it is possible to establish some basic a-priori estimates for this equation. Namely, suppose we assume

1. ψ is bounded and is Lipschitz in x:
$$|\psi(x_1, X) - \psi(x_2, X)| \leq C|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^d.$$
2. There exist constants $c_0, c_1 > 0$ such that $L(v, x, X) \geq -c_0$ and $L(0, x, X) \leq c_1$ for all $x, v \in \mathbb{R}^d, X \in L^2(\Omega; \mathbb{R}^d)$.
3. L is twice differentiable in x, v and we have the following bounds
$$|D_x L(v, x, X)|, |D^2_{xx} L(v, x, X)|, |D^2_{xv} L(v, x, X)|, |D^2_{vv} L(v, x, X)| \leq C.$$
for all $x, v \in \mathbb{R}^d, X \in L^2(\Omega; \mathbb{R}^d)$.

We have then the following result from [103]:

Theorem 14. Assume that 1-3 hold. Then the function V defined in (3.1.1) for a fixed vector field B is finite, bounded, Lipschitz and semiconcave in x:

1. $$|V(x, Y, t)| \leq C, \quad \forall x, h \in \mathbb{R}^d, Y \in L^2(\Omega; \mathbb{R}^d).$$
2. $$|V(x + h, Y, t) - V(x, Y, t)| \leq C|h|, \quad \forall x, h \in \mathbb{R}^d, Y \in L^2(\Omega; \mathbb{R}^d).$$
3. $$V(x + h, Y, t) + V(x - h, Y, t) - 2V(x, Y, t) \leq C|h|^2, \quad \forall x, h \in \mathbb{R}^d, Y \in L^2(\Omega; \mathbb{R}^d).$$

3.2 Correlations

One important applications of master form framework concerns the case where agentsare subject to a common noise such as being driven by a common Brownian Motion. Let $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \times \mathbb{P}')$ be a probability space, where Ω is the events space, Ω' is the set of all players, $\mathcal{F} \times \mathcal{F}'$ the product σ-algebra on $\Omega \times \Omega'$, and $\mathbb{P}$, and $\mathbb{P}'$ probability measures. As in Subsection 2.6.2 we consider a $L^p(\Omega \times \Omega')$-integrable, time dependent family of random variables $\mathbf{X} : \Omega \times \Omega' \times [0, T] \to \mathbb{R}^d$. We interpret $\mathbf{X}_t(\omega, \omega')$ as the ω realization of the position of player ω' at time t. Let $W_t : [0, T] \times \Omega \to \mathbb{R}^d$ be a Brownian motion. In order to simplify the notation we may write, in the sequel, $\mathbf{X}$, or $\mathbf{X}(\omega')$ instead of $\mathbf{X}_t(\omega, \omega')$. Given a vector field $B : \mathbb{R}^d \times L^p(\Omega') \to \mathbb{R}^d$, and supposing the players follow the random trajectories

$$d\mathbf{X}(\omega') = B(\mathbf{X}(\omega'), \mathbf{X})dt + \sigma(\mathbf{X}(\omega'), \mathbf{v})dW_t, \quad \forall \omega' \in \Omega', \tag{3.2.1}$$

where W_t is a Brownian motion, and σ is as defined in the Subsection 2.1. As before, we consider a reference player with dynamics given by

$$d\mathbf{x} = f(\mathbf{x}, \mathbf{v})dt + \sigma(\mathbf{x}, \mathbf{v})dW_t. \tag{3.2.2}$$

We now consider the Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^m \times L^p(\Omega') \to \mathbb{R}$, along with a terminal cost $\psi : \mathbb{R}^d \times L^p(\Omega') \to \mathbb{R}$. Each player aims to minimize

$$V(x, X, t) = \inf_{\mathbf{v} \in \mathcal{U}} E\left[\int_t^T L(\mathbf{x}, \mathbf{v}, \mathbf{X})ds + \psi(\mathbf{x}_T, \mathbf{X}_T)\right],$$

where the expectation is taken with respect to the probability measure $\mathbb{P}$ in Ω. We define the Hamiltonian $H : \mathbb{R}^d \times L^p(\Omega') \times \mathbb{R}^d \to \mathbb{R}$ as

$$H(x, p, X) = \sup_{v \in U} \left[-f(x, v) \cdot p - L(x, v, X)\right].$$

We we now define certain directional derivatives of functions of random variables. These are the directional derivatives along constant directions on $\mathbb{R}^d$ and play an important role in problems with correlations. Let e_i be the i-th standard coordinate unit vector in $\mathbb{R}^d$. We define the directional first derivative operator as

$$\delta_i V(x, X, t) = \lim_{\varepsilon \to 0} \frac{V(x, X + \varepsilon e_i, t) - V(x, X, t)}{\varepsilon},$$

and the second derivative operator as

$$\delta_i^2 V(x, X, t) = \frac{d^2}{d\varepsilon^2} V(x, X + \varepsilon e_i, t)\Big|_{\varepsilon=0}.$$

Assuming the Dynamic Programming Principle, that $V \in C^{2,2,1}(\mathbb{R}^d \times \mathbb{R}^d \times [0,T])$, and that σ is a constant scalar, we find that V satisfies the following PDE,

$$-V_t(x,X,t) + H\left(x, D_x V(x,X,t), X\right) - B(X(\omega'),X) D_X V(x,X,t) - \frac{\sigma^2}{2}\sum_i^d \delta_i^2 V(x,X,t)$$
$$-\sigma^2 \sum_i^d \delta_i D_{x_i} V(x,X,t) - \frac{\sigma^2}{2}\Delta_x V(x,X,t) = 0,$$

with a terminal condition $V(\mathbf{x}_T, \mathbf{X}_T, T) = \psi(\mathbf{x}_T, \mathbf{X}_T)$. Now we assume that all players are rational, which means that when faced with the problem of minimizing their cost functions, all they will use the same optimal strategy. Therefore the feedback strategy followed by the players is

$$B(\mathbf{X}_t(\omega'), \mathbf{X}_t) = -D_p H\left(\mathbf{X}_t(\omega'), D_X V(\mathbf{X}_t(\omega'), \mathbf{X}_t, t), \mathbf{X}_t\right).$$

So the dynamics of the players will be given by

$$d\mathbf{X}_s(\omega') = -D_p H\left(\mathbf{X}_t(\omega'), D_X V(\mathbf{X}_t(\omega'), \mathbf{X}_t, t), \mathbf{X}_t\right) dt + \sigma dW_t.$$

Consequently, the mean-field equations are given by

$$\begin{cases} -V_t + H(x, D_x V, X) - B(X,X) D_X V - \frac{\sigma^2}{2}\sum_i^d \left(\delta_i^2 V + 2\delta_i D_{x_i} V + D^2_{x_i x_i} V\right) = 0 \\ B(x,X) = -D_p H\left(x, D_x V(x,X,t), X\right), \end{cases} \tag{3.2.3}$$

where the value function V is evaluated at (x,X,t). Furthermore we have the following terminal condition

$$V(x,X,T) = \psi(x,X).$$

The first equation in (3.2.3) is called *Master Equation*. As before, a solution of (3.2.3) is understood to be a viscosity solution of the first equation, with B fixed, coupled with the second equation, which determines B. Again this is a fixed point problem rather than a single PDE. As in the deterministic case one can prove various partial a-priori regularity results as in Theorem 14 (see [103]).

3.3 Extended models

In this section we discuss an extended version of the mean-field games with correlations. Here we look at the case where the Lagrangian depends not only on the state of other players but also on the actions they take. We then present a price-formation model using this set up.

As before we suppose that the players follow the dynamics given by (3.2.1). And we consider a reference player which follows (3.2.2) and aims to minimizing a cost function. In this extended setting we suppose that the Lagrangian function, $L : \mathbb{R}^d \times \mathbb{R}^m \times L^p(\Omega') \times L^p(\Omega') \to \mathbb{R}$, depends also on the actions taken by the other players. Therefore the value function for the reference player is given by

$$V(x, X, t) = \inf_{\mathbf{v}} E\left[\int_t^T L(\mathbf{x}, \mathbf{v}, \mathbf{X}, B)ds + \psi(\mathbf{x}_T, \mathbf{X}_T)\right],$$

where, as before, $\psi : \mathbb{R}^d \times L^p(\Omega') \to \mathbb{R}$ is a terminal cost, and the expectation is taken with respect to the probability measure $\mathbb{P}$ in Ω. The Hamiltonian, $H : \mathbb{R}^d \times \mathbb{R}^m \times L^p(\Omega) \times L^p(\Omega) \to \mathbb{R}$ is now given by

$$H(x, p, X, Z) = \sup_{v \in U} \left[-f(x, v) \cdot p - L(x, v, X, Z)\right].$$

Note that both the Lagrangian and the Hamiltonian are functions depending only on the joint law of $X, Z \in L^p(\Omega)$.

By standard arguments, the value function V is a viscosity solution of the following PDE

$$\begin{aligned} -V_t(x, X, t) + H\left(x, D_x V(x, X, t), X, B(X, X, t)\right) - D_X V(x, X, t) \cdot B(X, X, t) \\ -\frac{\sigma^2}{2}\sum_i \delta_i^2 V(x, X, t) - \sigma^2 \sum_i \delta_i D_{x_i} V(x, X, t) - \frac{\sigma^2}{2}\Delta_x V(x, X, t) = 0. \end{aligned} \tag{3.3.1}$$

Provided V is smooth enough, and for a fixed B, then the optimal control v^* satisfies

$$f(x, v^*) = -D_p H(x, D_x V(x, X, t), X, B).$$

By our assumptions of indistinguishability and rationality of players, every player $\omega' \in \Omega'$ will follow the optimal flow given by

$$B(X(\omega'), X, t) = -D_p H\left(X(\omega'), X, D_x V(X(\omega'), X, t), X, B(X(\omega'), X, t)\right).$$

So plugging the previous optimal flow in the equation (3.3.1), we obtain the *Master equation*

$$\begin{aligned} -V_t(x, X, t) + H\left(x, D_x V(x, X, t), X, B(x, X, t)\right) \\ - D_X V(x, X, t) \cdot D_p H\left(x, D_x V(x, X, t), X, B(x, X, t)\right) \\ - \frac{\sigma^2}{2}\sum_i \delta_i^2 V(x, X, t) - \sigma^2 \sum_i \delta_i D_{x_i} V(x, X, t) - \frac{\sigma^2}{2}\Delta_x V(x, X, t) = 0, \end{aligned}$$

where V satisfies the terminal condition $V(x, X, T) = \psi(x, X)$, where $D_X V$ is the Fréchet derivative of V and $\delta_i V$ and $\delta_i^2 V$ are defined as previously in section 3.2.

3.3.1 A price impact model

As an application of the extended formulation we present a modified Merton's portfolio problem where we consider that assets' transactions influence their prices. We formulate the problem for a large number of traders, each one aiming to maximize it's own reward function, while taking the point of view of a reference player. This formulation fits in the previously considered master form of mean-field games. We will continue using the random variables point of view.

Merton's portfolio problem We consider a financial market with two assets, a risk-free asset, *bond* B_t, and a risky asset, *stock* S_t. The dynamics of these variables is given by

$$\begin{cases} dB_t = rB_t dt \\ dS_t = \mu S_t dt + \sigma S_t dW_t, \end{cases}$$

where W_t is the Brownian motion, $r \in (0, \infty)$ is the interest rate, $\mu \in \mathbb{R}$ the drift, and $\sigma \in \mathbb{R}$ the volatility of the stock.

The Merton's problem is to decide what portion of the wealth should be allocated to bonds or stocks, in order to maximize a given reward function. We now consider the Merton's problem for a large number of players. We formulate the problem using the mean-field models in master form discussed in the previous sections. The various players states and actions are encoded by random variables $\mathbf{X}, \mathbf{Y}, \mathbf{C}, \mathbf{L} : \Omega \times \Omega' \times [0, T] \to \mathbb{R}$. At time t the players have an amount of money $\mathbf{X}(t)$ invested in bonds, and an amount $\mathbf{Y}(t)$ in stocks. The players are allowed to consume their wealth, which amounts to withdraw a $\mathbf{C}(t)$ amount from the money invested in bonds. They also can re-allocate their investments by selling a money amount $\mathbf{L}(t)$ of stocks in order to buy bonds. As before we also consider a reference player which allocates an amount $\mathbf{x}(t)$ of his wealth in bonds, and $\mathbf{y}(t)$ in stocks, at a given moment t. This player is allowed to consume an amount $\mathbf{c}(t) \geq 0$ and to change the amount investing in stocks by selling an amount $\mathbf{l}(t)$ (either positive or negative) of stocks in order to buying bonds.

Dynamics of the reference player The reference player has the following dynamics

$$\begin{cases} d\mathbf{x} &= r\mathbf{x}dt + \mathbf{l}dt - \alpha \mathbf{l} E'[\mathbf{L}]dt - \mathbf{c}dt \\ d\mathbf{y} &= \mu \mathbf{y} dt + \sigma \mathbf{y} dW_t - \mathbf{l}dt. \end{cases}$$

where $r, \mu \in \mathbb{R}$ are, respectively, the interest rate and the drift values as before, and $\alpha \geq 0$ is an impact factor of the selling/buying process, and the expectation E' is taken with respect to the probability measure $\mathbb{P}'$ in Ω', that is

$$E'(Z) = \int_{\Omega'} Z(\omega, \omega') d\mathbb{P}'(\omega').$$

The term $-\alpha \mathbf{l} E'[\mathbf{L}]dt$ encodes the price impact cause by a non-equilibrium situation when the sellers are not matched by buyers. In the case, when $\mathbf{l}(t) > 0$, if the expected value of other players' actions is also positive $E[\mathbf{L}'] > 0$ this means that as a whole there are more shares being sold than bought. Therefore it this will adversely affect a player trying to sell. So a player sells what before was valued as a $\mathbf{l}(t)$ amount, and gets instead $\mathbf{l}(t) - \alpha \mathbf{l}(t)E'[\mathbf{L}]$.. In the case where a player $\omega' \in \Omega'$ acts in an opposite direction as the the population of players' average, the impact on the wealth is positive. So, for instance if the player buys $\mathbf{l}(t) < 0$ amount worth of stocks and while the players on average are selling the stock $E'[\mathbf{L}] > 0$, then there will be a positive impact price on the wealth, since $-\mathbf{l}(t)E'[\mathbf{L}(t)] > 0$. Note that, one should have in principle $E'[\mathbf{L}] = 0$. In this model this is not imposed as a constraint but it is natural to expect, as $\alpha \to \infty$, this constraint to be asymptotically satisfied.

Dynamics of the mean-field We assume all players have the same dynamics. Therefore the mean-field variables $\mathbf{X}$ and $\mathbf{Y}$ satisfy

$$
\begin{cases}
d\mathbf{X} &= r\mathbf{X}dt + \mathbf{L}dt - \alpha \mathbf{L}E'[\mathbf{L}]dt - \mathbf{C}dt \\
d\mathbf{Y} &= \mu \mathbf{Y}dt + \sigma \mathbf{Y}dW_t - \mathbf{L}dt,
\end{cases}
$$

where we assume for now that $\mathbf{L}$ and $\mathbf{C}$ are known and given in feedback form

$$
\mathbf{L} = \Theta(X, Y, X, Y), \qquad \mathbf{C} = \Pi(X, Y, X, Y).
$$

Optimization problem for the reference player In order to simplify the expressions that follow we set up some notation first: we write $\mathbf{z} = (\mathbf{x}, \mathbf{y})$, $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$, $\mathbf{m} = (\mathbf{l}, \mathbf{c})$, and $\mathbf{M} = (\mathbf{L}, \mathbf{C})$. Each player aims to maximize its reward function, which from the point of view of a reference player amounts to:

$$
V(z, Z) = \max_{\mathbf{l},\mathbf{c}} E\left[\int_0^\infty e^{-\beta t}\mathcal{L}(\mathbf{z}, \mathbf{m}, \mathbf{Z}, \mathbf{M})dt \Big| (\mathbf{z}, \mathbf{Z})(0) = (\mathbf{z}_0, \mathbf{Z}_0)\right], \tag{3.3.2}
$$

where the controls $(\mathbf{l}, \mathbf{c})$ are taken in $L^p(\Omega, \mathbb{R}) \times L^p(\Omega, \mathbb{R}_0^+)$, and $\mathcal{L} : \mathbb{R}^d \times \mathbb{R}^d \times L^p(\Omega') \times L^p(\Omega') \to \mathbb{R}$.

Master equation We define define the Hamiltonian function as

$$
H(z, p, q, Z, M) = \inf_{l,c}\left[-(rx + l - \alpha l E'[L] - c) \cdot p - (\mu y - l(z, Z)) \cdot q - \mathcal{L}(z, p, q, Z, M)\right],
$$

where $z = (x, y)$, and $Z = (X, Y)$. Now assuming enough regularity, on the value function (3.3.2), such that we can use the Itô's formula, and such that the dynamic programming principle applies we have that V satisfies the following PDE:

$$\begin{aligned}&\beta V(z,Z)+H\Big(z,D_xV(z,Z),D_yV(z,Z),Z,L(Z,Z),C(Z,Z)\Big)\\&\quad-(rX+L(Z,Z)-\alpha L(Z,Z)E'[L(Z,Z)]-C(Z,Z))D_XV(z,Z)\\&\quad-(\mu Y-L(Z,Z))D_YV(z,Z)\\&\quad-\frac{1}{2}\sigma^2(y^2D_{yy}V(z,Z)+Y^2\delta_i^2V(z,Z)+y\delta_iD_yV(z,Z))=0.\end{aligned}\tag{3.3.3}$$

So, provided V is a smooth enough solution to the previous equation, then an optimal control pair (c,l) satisfies

$$\begin{cases}rx+l(z,Z)-\alpha l(z,Z)E'[L(z,Z)]-c(z,Z)=-D_pH(z,D_xV,D_yV,Z,L,C)\\\mu y-l(z,Z)=-D_qH(z,D_xV,D_yV,Z,L,C).\end{cases}$$

Furthermore we assume that all players are indistinguishable and act rationally henceforth playing optimal strategies. Then this tells us that

$$\begin{cases}rx+\Theta(z,Z)(1-\alpha E'[\Theta(z,Z)])-\Pi(z,Z)\\\qquad=-D_pH(z,D_xV(z,Z,t),D_yV(z,Z,t),Z,\Theta(z,Z),\Pi(z,Z))\\\mu y-\Theta(z,Z)\\\qquad=-D_qH(z,D_xV(z,Z,t),D_yV(z,Z,t),Z,\Theta(z,Z),\Pi(z,Z)),\end{cases}\tag{3.3.4}$$

Plugging this controls into the above PDE gives rise to the master equation

$$\begin{aligned}&\beta V(z,Z)+H\Big(z,D_xV(z,Z),D_yV(z,Z),Z,L(z,Z),C(z,Z)\Big)\\&\quad+D_pH(Z,D_XV,D_YV,Z,\Theta,\Pi)D_XV(z,Z)+D_qH(Z,D_XV,D_YV,Z,\Theta,\Pi)D_YV(z,Z)\\&\quad-\frac{1}{2}\sigma^2(y^2D_{yy}V(z,Z)+Y^2\delta_i^2V(z,Z)+y\delta_iD_yV(z,Z))=0.\end{aligned}\tag{3.3.5}$$

Open questions This price formation model illustrates various of the open questions on this area of research. First, it is not clear at all the existence or regularity of solutions. The natural definition of solution is the following: for fixed controls for the mean-field, (in the price formation model Θ and Π) the function V is a viscosity solution of the Hamilton-Jacobi equation (in this case (3.3.3)), then the controls for the mean-field are determined by the optimality conditions (such as (3.3.4)). This is thus a fixed point problem. In order to study it new techniques to understand the regularity of viscosity solutions of Hamilton-Jacobi equations in infinite dimensions must be developed. Uniqueness, as far as we know is also open, though it may be possible to adapt some of monotonicity techniques developed by Lions in [140] to this setting. From the application point of view it would be extremely important to develop effective numerical methods. It is clear at this stage that any naive attempt to address this would suffer from the curse of dimensionality problem and therefore new ideas are necessary to address this class of problems. Finally, singular perturbation problems such as the one that arises by sending $\alpha \to \infty$ are important, natural, and should certainly be investigated in depth.

Part II

Numerical methods for MFG

Chapter 4

Monotone numerics

The mean-field game (MFG) framework [114, 117, 131, 132] models systems with many rational players (see the surveys [95] and [99]). In finite-state MFGs, players switch between a finite number of states (see [88] for discrete-time and [25, 80, 89, 110], and [108] for continuous-time problems). Finite-state MFGs have applications in socio-economic problems, for example, in paradigm-shift and consumer choice models [37, 101, 102] or in corruption models [126]. They also arise in the approximation of continuous-state MFGs [1, 6, 12]. The MFG framework is a major paradigm change in the analysis of N-agent games. MFGs are an alternative approach to particle or agent-based models, which frequently are intractable from the analytical and numerical point of view and often provide no insight on the qualitative properties of the models. Finite-state MFGs are amenable to analytical tools and flexible enough to address a wide range of applications and to provide quantitative and qualitative information. However, in many cases of interest, they have no simple closed-form solution. Hence, the development of numerical methods is critical to applications of MFGs. Finite-state MFGs comprise systems of ordinary differential equations with initial-terminal boundary conditions. Because of these conditions, the numerical computation of solutions is challenging. Often, MFGs satisfy a monotonicity condition that was first used in [131] and [132] to study the uniqueness of solutions. Besides the uniqueness of solutions, monotonicity implies the long-time convergence of MFGs (see [80] and [89] for finite-state models and [59] and [60] for continuous-state models). Moreover, monotonicity conditions were used in [81] to prove the existence of solutions to MFGs and in [12] to construct numerical methods for stationary MFGs. Here, we consider MFGs that satisfy a monotonicity condition and develop a numerical method to compute their solutions. For stationary problems, our method is a modification of the one in [12]. Our main advance here is how we handle initial-terminal boundary conditions, to which the methods from [12] cannot be applied directly.

We consider MFGs in which each player can be at a state in $I_d = \{1, \dots, d\}$, $d \in \mathbb{N}$, $d > 1$, the players' state space. Let $\mathcal{S}^d = \{\theta \in (\mathbb{R}_0^+)^d : \sum_{i=1}^d \theta^i = 1\}$ be the probability simplex in I_d. For a time horizon, $T > 0$, the macroscopic description of the game is determined by a path

$\theta : [0, T] \to \mathcal{S}^d$ that gives the probability distribution of the players in I_d. All players seek to minimize an identical cost. Each coordinate, $u^i(t)$, of the value function, $u : [0, T] \to \mathbb{R}^d$, is the minimum cost for a typical player at state $i \in I_d$ at time $0 \le t \le T$. Finally, at the initial time, the players are distributed according to the probability vector $\theta_0 \in \mathcal{S}^d$ and, at the terminal time, are charged a cost $u_T \in \mathbb{R}^d$ that depends on their state.

In the framework presented in [88], finite-state MFGs have a Hamiltonian, $h : \mathbb{R}^d \times \mathcal{S}^d \times I_d \to \mathbb{R}$, and a switching rate, $\alpha_i^* : \mathbb{R}^d \times \mathcal{S}^d \times I_d \to \mathbb{R}_0^+$, given by

$$\alpha_j^* = \frac{\partial h(\Delta_i z, \theta, i)}{\partial z^j}, \tag{4.0.1}$$

where $\Delta_i : \mathbb{R}^d \to \mathbb{R}^d$ is the difference operator

$$(\Delta_i u)^j = u^j - u^i.$$

We suppose that h and α^* satisfy the assumptions discussed in Section 4.1. Given the Hamiltonian and the switching rate, we assemble the following system of differential equations:

$$\begin{cases} u_t^i = -h(\Delta_i u, \theta, i) \\ \theta_t^i = \sum_j \theta^j \alpha_i^*(\Delta_j u, \theta, j), \end{cases} \tag{4.0.2}$$

which, together with initial-terminal data

$$\theta(0) = \bar{\theta}_0 \text{ and } u(T) = \bar{u}_T, \tag{4.0.3}$$

with $\bar{\theta}_0 \in \mathcal{S}^d$ and $\bar{u}_T \in \mathbb{R}^d$, determines the MFG.

Solving (4.0.2) under the non-standard boundary condition (4.0.3) is a fundamental issue in time-dependent MFGs. There are several ways to address this issue, although prior approaches are not completely satisfactory. First, we can solve (4.0.2) using initial conditions $\theta(0) = \bar{\theta}_0$ and $u(0) = u_0$ and then solve for u_0 such that $u(T) = \bar{u}_T$. However, this requires solving (4.0.2) multiple times, which is computationally expensive. A more fundamental difficulty arises in the numerical approximation of continuous-state MFGs by finite-state MFGs. There, the Hamilton-Jacobi equation is a backward parabolic equation whose initial-value problem is ill-posed. Thus, a possible way to solve (4.0.2) is to use a Newton-like iteration. This idea was developed in [1, 10] and used to solve a finite-difference scheme for a continuous-state MFG. However, Newton's method involves inverting large matrices, whereas it is convenient to have algorithms that do not require matrix inversions. A second approach is to use a fixed-point iteration as in [62, 61]. Unfortunately, this iteration is not guaranteed to converge. A third approach (see [101, 102]) is to solve the master equation, which is a partial differential equation whose characteristics are given by (4.0.2). To approximate the master equation, we can use a finite-difference method constructed by solving an N-player problem. Unfortunately, even for a modest number of states, this approach is computationally expensive.

Our approach to the numerical solution of (4.0.2) relies on the monotonicity of the operator, $A : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, given by

$$A \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} h(\Delta_i u, \theta, i) \\ -\sum_j \theta^j \alpha_i^*(\Delta_j u, \theta, j) \end{bmatrix}. \tag{4.0.4}$$

More precisely, we assume that A is monotone (see Assumption 2) in the sense that

$$\left(A \begin{bmatrix} \theta \\ u \end{bmatrix} - A \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} \theta \\ u \end{bmatrix} - \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix} \right) \geq 0$$

for all $\theta, \tilde{\theta} \in \mathcal{S}^d$ and $u, \tilde{u} \in \mathbb{R}^d$. Building upon the ideas in [12] for stationary problems (also see the approaches for stationary problems in [147, 39, 148, 8]), we introduce the flow

$$\begin{bmatrix} \theta_s \\ u_s \end{bmatrix} = -A \begin{bmatrix} \theta \\ u \end{bmatrix}. \tag{4.0.5}$$

Up to the normalization of θ, the foregoing flow is a contraction provided that $\theta \in \mathcal{S}^d$. Moreover, its fixed points solve

$$A \begin{bmatrix} \theta \\ u \end{bmatrix} = 0.$$

In Section 4.2, we construct a discrete version of (4.0.5) that preserves probabilities; that is, it preserves both the total mass of θ and its non-negativity.

The time-dependent case is substantially more delicate. Our method to approximate its solutions is our main contribution. The operator associated with the time-dependent problem, $A : H^1(0, T; \mathbb{R}^d \times \mathbb{R}^d) \to L^2(0, T; \mathcal{S}^d \times \mathbb{R}^d)$, is

$$A \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} -u_t + h(\Delta_i u, \theta, i) \\ \theta_t - \sum_j \theta^j \alpha_i^*(\Delta_j u, \theta, j) \end{bmatrix}. \tag{4.0.6}$$

Under the initial-terminal condition in (4.0.3), A is a monotone operator. Thus, the flow

$$\begin{bmatrix} \theta_s \\ u_s \end{bmatrix} = -A \begin{bmatrix} \theta \\ u \end{bmatrix} \tag{4.0.7}$$

for $(\theta, u) \in L^2(0, T; \mathbb{R}^d \times \mathbb{R}^d)$ is formally a contraction. Unfortunately, even if this flow is well defined, the preceding system neither preserves probabilities nor such boundary conditions (4.0.3). Thus, in Section 4.3, we modify (4.0.7) in a way that it becomes a contraction in H^1 and preserves the boundary conditions. Finally, we discretize this modified flow and build a numerical algorithm to approximate solutions of (4.0.2)-(4.0.3). Unlike Newton-based methods, our algorithm does not need the inversion of large matrices and scales linearly with the number of states. This is particularly relevant for finite-state MFGs that arise from the discretization of continuous-state MFGs. We illustrate our results in a paradigm-shift problem introduced in [37] and studied from a numerical perspective in [102].

4.1 Framework and main assumptions

Following [89], we present the standard finite-state MFG framework and describe our main assumptions. Then, we discuss a paradigm-shift problem from [37] that we use to illustrate our methods.

4.1.1 Standard setting for finite-state MFGs

Finite-state MFGs model systems with many identical players who act rationally and non-cooperatively. These players switch between states in I_d in seeking to minimize a cost. Here, the macroscopic state of the game is a probability vector $\theta \in \mathcal{S}^d$ that gives the players' distribution in I_d. A typical player controls the switching rate, $\alpha_j(i)$, from its state, $i \in I_d$, to a new state, $j \in I_d$. Given the players' distribution, $\theta(r)$, at time r, each player chooses a non-anticipating control, α, that minimizes the cost

$$u^i(t;\alpha) = E^{\alpha}_{\mathbf{i}_t=i}\left[\int_t^T c(\mathbf{i}_r, \theta(r), \alpha(r))dr + u^{\mathbf{i}_T}(\theta(T))\right]. \tag{4.1.1}$$

In the preceding expression, $c : I_d \times \mathcal{S}^d \times (\mathbb{R}_0^+)^d \to \mathbb{R}$ is a running cost, $\Psi \in \mathbb{R}^d$ is the terminal cost, and $\mathbf{i}_s$ is a Markov process in I_d with switching rate α. The *Hamiltonian*, h, is the generalized Legendre transform of $c(i,\theta,\cdot)$:

$$h(\Delta_i z, \theta, i) = \min_{\mu \in (\mathbb{R}_0^+)^d} \{c(i,\theta,\mu) + \mu \cdot \Delta_i z\}.$$

The first equation in (4.0.2) determines the value function, u, for (4.1.1). The optimal switching rate from state i to state $j \neq i$ is given by $\alpha_j^*(\Delta_i u, \theta, i)$, where

$$\alpha_j^*(z,\theta,i) = \operatorname{argmin}_{\mu \in (\mathbb{R}_0^+)^d}\{c(i,\theta,\mu) + \mu \cdot \Delta_i z\}. \tag{4.1.2}$$

Moreover, at points of differentiability of h, we have (4.0.1). The rationality of the players implies that each of them chooses the optimal switching rate, α^*. Hence, θ evolves according to the second equation in (4.0.2).

4.1.2 Main assumptions

Because we work with the Hamiltonian, h, rather than the running cost, c, it is convenient to state our assumptions in terms of the former. For the relation between assumptions on h and c, see [89].

We begin by stating a mild assumption that ensures the existence of solutions for (4.0.2).

Assumption 1. The Hamiltonian $h(z, \theta, i)$ is locally Lipschitz in (z, θ) and differentiable in z. The map $z \mapsto h(z, \theta, i)$ is concave for each (θ, i). The function $\alpha^*(z, \theta, i)$ given by (4.0.1) is locally Lipschitz.

Under Assumption 1, there exists a solution to (4.0.2)-(4.0.3) (see [89]). This solution may not be unique as the examples in [101] and [102] show. Monotonicity conditions are commonly used in MFGs to prove the uniqueness of solutions. For finite-state MFGs, the appropriate monotonicity condition is stated in the next Assumption. Before proceeding, we define $\|v\|_\sharp = \inf_{\lambda \in \mathbb{R}} \|v + \lambda \mathbf{1}\|$.

Assumption 2. There exists $\gamma > 0$ such that the Hamiltonian, h, satisfies the following monotonicity property:

$$\theta \cdot (h(z, \tilde{\theta}) - h(z, \theta)) + \tilde{\theta} \cdot (h(\tilde{z}, \theta) - h(\tilde{z}, \tilde{\theta})) \leq -\gamma \|\theta - \tilde{\theta}\|^2.$$

Moreover, for each $M > 0$, there exist constants, γ_i, such that on the set $\|w\|, \|z\|_\sharp \leq M$, h satisfies the following concavity property:

$$h(z, \theta, i) - h(w, \theta, i) - \alpha^*(w, \theta, i) \cdot \Delta_i(z - w) \leq -\gamma_i \|\Delta_i(z - w)\|^2.$$

Under the preceding assumptions, (4.0.2)-(4.0.3) has a unique solution (see [89]). Here, the previous condition is essential to the convergence of our numerical methods, for both stationary problems in Section 4.2 and for the general time-dependent case in Section 4.3.

Remark 1. As shown in [89], Assumption 2 implies the inequality

$$\begin{aligned}
&\sum_{i=1}^{d} (u^i - \tilde{u}^i) \left(\sum_j \theta^j \alpha^*(\Delta_j u, \theta, j) - \sum_j \tilde{\theta}^j \alpha^*(\Delta_j \tilde{u}, \tilde{\theta}, j) \right) \\
&+ \sum_{i=1}^{d} (\theta^i - \tilde{\theta}^i) \left(-h(\Delta_i u, \theta, i) + k + h(\Delta_i \tilde{u}, \tilde{\theta}, i) - \tilde{k} \right) \\
&\leq -\gamma \|(\theta - \tilde{\theta})\|^2 - \sum_{i=1}^{d} \gamma_i (\theta^i + \tilde{\theta}^i) \|(\Delta_i u - \Delta_i \tilde{u})\|^2
\end{aligned}$$

for any $u, \tilde{u} \in \mathbb{R}^d$, $\theta, \tilde{\theta} \in \mathcal{S}^d$, and $k, \tilde{k} \in \mathbb{R}$.

4.1.3 Solutions and weak solutions

Because the operator, A, in (4.0.6) is monotone, we have a natural concept of weak solutions for (4.0.2)-(4.0.3). These weak solutions were considered for continuous-state MFGs in [12] and [81]. We say that $(u,\theta) \in L^2((0,T),\mathbb{R}^d) \times L^2((0,T),\mathcal{S}^d)$ is a weak solution of (4.0.2)-(4.0.3) if for all $(\tilde{u},\tilde{\theta}) \in H^1((0,T),\mathbb{R}^d) \times H^1((0,T),\mathcal{S}^d)$ satisfying (4.0.3), we have

$$\left\langle A \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} \tilde{\theta}-\theta \\ \tilde{u}-u \end{bmatrix} \right\rangle \geq 0.$$

Any solution of (4.0.2)-(4.0.3) is a weak solution, and any sufficiently regular weak solution with $\theta > 0$ is a solution.

Now, we turn our attention to the stationary problem. We recall (see [89]) that a stationary solution of (4.0.2) is a triplet, $(\bar{\theta},\bar{u},\bar{k}) \in \mathcal{S}^d \times \mathbb{R}^d \times \mathbb{R}$, satisfying

$$\begin{cases} h(\Delta_i \bar{u}, \bar{\theta}, i) = \bar{k} \\ \sum_j \bar{\theta}^j \alpha_i^*(\Delta_j \bar{u}, \bar{\theta}, j) = 0 \end{cases} \tag{4.1.3}$$

for $i = 1,\ldots,d$. As discussed in [89], the existence of solutions to (4.1.3) holds under an additional contractivity assumption. In general, as for continuous-state MFGs, solutions for (4.1.3) may not exist. Thus, we need to consider weak solutions. For a finite-state MFG, a weak solution of (4.1.3) is a triplet, $(\bar{u},\bar{\theta},\bar{k}) \in \mathbb{R}^d \times \mathcal{S}^d \times \mathbb{R}$, that satisfies

$$\begin{cases} h(\Delta_i \bar{u}, \bar{\theta}, i) \geq \bar{k} \\ \sum_j \bar{\theta}^j \alpha_i^*(\Delta_j \bar{u}, \bar{\theta}, j) = 0 \end{cases} \tag{4.1.4}$$

for $i = 1,\ldots,d$, with equality in the first equation for all indices, i, such that $\bar{\theta}^i > 0$.

4.1.4 Potential MFGs

In a potential MFG, the Hamiltonian takes the form

$$h(\nabla_i u, \theta, i) = \tilde{h}(\nabla_i u, i) + f(\theta, i),$$

where $\tilde{h} : \mathbb{R}^d \times I_d \to \mathbb{R}$, $f : \mathbb{R}^d \times I_d \to \mathbb{R}$ and f is the gradient of a convex function, $F : \mathbb{R}^d \to \mathbb{R}$; that is, $f(\theta,\cdot) = \nabla_\theta F(\theta)$. We define $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as

$$H(u,\theta) = \sum_{i=1}^{d} \theta^i \tilde{h}(\nabla_i u, i) + F(\theta). \tag{4.1.5}$$

Then, (4.0.2) can be written in Hamiltonian form as

$$\begin{cases} u_t = -D_\theta H(u,\theta) \\ \theta_t = D_u H(u,\theta). \end{cases}$$

In particular, H is conserved as follows:

$$\frac{d}{dt}H(u,\theta) = 0.$$

In Section 4.3.7, we use this last property as an additional test for our numerical method.

4.1.5 A case study – the paradigm-shift problem

A paradigm shift is a change in a fundamental assumption within a scientific theory. Scientists can simultaneous work in the context of multiple competing theories or problems. Their choice of theoretical grounding is made to maximize recognition (citations, awards, or prizes) and scientific activity (conferences or collaborations, for example). The paradigm-shift problem was formulated as a two-state MFG in [37]. Subsequently, it was studied numerically in [102] and [101] using an N-player approximation and PDE methods. Here, we present the stationary and time-dependent versions of this problem. Later, we use these versions to validate our numerical methods.

We consider the running cost, $c : I_d \times \mathcal{S}^d \times (\mathbb{R}_0^+)^2 \to \mathbb{R}$, given by

$$c(i,\theta,\mu) = f(i,\theta) + c_0(i,\mu), \text{ where } c_0(i,\mu) = \frac{1}{2}\sum_{j\neq i}^{2} \mu_j^2.$$

The functions $f = f(i,\theta)$ are *productivity functions with constant elasticity of substitution*, given by

$$\begin{cases} f(1,\theta) = \left(a_1(\theta^1)^r + (1-a_1)(\theta^2)^r\right)^{\frac{1}{r}} \\ f(2,\theta) = \left(a_2(\theta^1)^r + (1-a_2)(\theta^2)^r\right)^{\frac{1}{r}} \end{cases}$$

for $r \geq 0$ and $0 \leq a_1, a_2 \leq 1$. The Hamiltonian is

$$\begin{cases} h(u,\theta,1) = f(1,\theta) - \frac{1}{2}\left((u^1-u^2)^+\right)^2, \\ h(u,\theta,2) = f(2,\theta) - \frac{1}{2}\left((u^2-u^1)^+\right)^2, \end{cases}$$

and the optimal switching rates are

$$\begin{aligned} \alpha_2^*(u,\theta,1) &= (u^1-u^2)^+, \quad \alpha_1^*(u,\theta,1) = -(u^1-u^2)^+, \\ \alpha_1^*(u,\theta,2) &= (u^2-u^1)^+, \quad \alpha_2^*(u,\theta,2) = -(u^2-u^1)^+. \end{aligned}$$

For illustration, we examine the case where $a_1 = 1$, $a_2 = 0$, and $r = 1$ in the productivity functions above. In this case, $f = \nabla_\theta F(\theta)$ with

$$F(\theta) = \frac{(\theta^1)^2 + (\theta^2)^2}{2}.$$

Moreover, the game is potential with

$$H(u,\theta) = -\frac{1}{2}\left((u^1-u^2)^+\right)^2\theta^1 - \frac{1}{2}\left((u^2-u^1)^+\right)^2\theta^2 + F(\theta).$$

Furthermore, $(\bar{\theta}, \bar{u}, k)$ is a stationary solution if it solves

$$\begin{cases} \theta^1 - \frac{1}{2}((u^1-u^2)^+)^2 = k \\ \theta^2 - \frac{1}{2}((u^2-u^1)^+)^2 = k, \end{cases} \tag{4.1.6}$$

and

$$\begin{cases} -\theta^1(u^1-u^2)^+ + \theta^2(u^2-u^1)^+ = 0 \\ \theta^1(u^1-u^2)^+ - \theta^2(u^2-u^1)^+ = 0. \end{cases} \tag{4.1.7}$$

Since $\theta^1 + \theta^2 = 1$, and using the symmetry of (4.1.6)-(4.1.7), we conclude that

$$(\bar{\theta}, \bar{u}, k) = \left(\left(\frac{1}{2}, \frac{1}{2}\right), (p,p), \frac{1}{2}\right), \quad p \in \mathbb{R}. \tag{4.1.8}$$

The time-dependent paradigm-shift problem is determined by

$$\begin{cases} u^1_t = -\theta^1 + \frac{1}{2}((u^1-u^2)^+)^2 \\ u^2_t = -\theta^2 + \frac{1}{2}((u^2-u^1)^+)^2, \end{cases} \tag{4.1.9}$$

and

$$\begin{cases} \theta^1_t = -\theta^1(u^1-u^2)^+ + \theta^2(u^2-u^1)^+ \\ \theta^2_t = \theta^1(u^1-u^2)^+ - \theta^2(u^2-u^1)^+, \end{cases} \tag{4.1.10}$$

together with initial-terminal conditions

$$\theta^i(0) = \theta_0, \text{ and } u^i(T) = u^i_T$$

for $i = 1, 2$, $\theta_0 \in \mathcal{S}^2$, and $u_T \in \mathbb{R}^2$.

4.2 Stationary problems

To approximate the solutions of (4.1.3), we introduce a flow closely related to (4.0.5). This flow is the analog for finite-state problems of the one considered in [12]. The monotonicity in Assumption 2 gives the contraction property. Then, we construct a numerical algorithm using a Euler step combined with a projection step to ensure that θ remains a probability. Finally, we test our algorithm in the paradigm-shift model.

4.2.1 Monotone approximation

To preserve the mass of θ, we introduce the following modification of (4.0.5):

$$\begin{cases} u^i_s = \sum_j \theta^j \alpha^*_i(\Delta_j u, \theta, j) \\ \theta^i_s = -h(\Delta_i u, \theta, i) + k(s), \end{cases} \tag{4.2.1}$$

where $k : \mathbb{R}^+_0 \to \mathbb{R}$ is such that $\sum_{i=1}^d \theta^i(s) = 1$ for every $s \geq 0$. For this condition to hold, we need $\sum_{i=1}^d \theta^i_s = 0$. Therefore,

$$k(s) = \frac{1}{d}\sum_{i=1}^d h(\Delta_i u, \theta, i). \tag{4.2.2}$$

Proposition 2. Suppose that Assumptions 1-2 hold. Let (u, θ) and $(\tilde{u}, \tilde{\theta})$ solve (4.2.1)-(4.2.2). Assume that $\sum_i \theta^i(0) = \sum_i \tilde{\theta}^i(0) = 1$ and that $\theta(s), \tilde{\theta}(s) \geq 0$. Then,

$$\frac{d}{ds}\left(\|(u-\tilde{u})\|^2 + \|\theta - \tilde{\theta}\|^2\right) \\ \leq -\gamma\|(\theta-\tilde{\theta})(s)\|^2 - \sum_{i=1}^d \gamma_i(\theta^i + \tilde{\theta}^i)(s)\|(\Delta_i u - \Delta_i \tilde{u})(s)\|^2.$$

Proof. We begin with the identity

$$\frac{1}{2}\frac{d}{ds}\sum_{i=1}^d \left[(u^i - \tilde{u}^i)^2 + (\theta^i - \tilde{\theta}^i)^2\right] \\ = \sum_{i=1}^d (u^i - \tilde{u}^i)(u^i - \tilde{u}^i)_s + (\theta^i - \tilde{\theta}^i)(\theta^i - \tilde{\theta}^i)_s.$$

Using (4.2.1) in the previous equality, we obtain

$$\frac{1}{2}\frac{d}{ds}\sum_{i=1}^d \left[(u^i - \tilde{u}^i)^2 + (\theta^i - \tilde{\theta}^i)^2\right] \\ = \sum_{i=1}^d (u^i - \tilde{u}^i)\left(\sum_j \theta^j \alpha^*(\Delta_j u, \theta, j) - \sum_j \tilde{\theta}^j \alpha^*(\Delta_j \tilde{u}, \tilde{\theta}, j)\right) \\ + \sum_{i=1}^d (\theta^i - \tilde{\theta}^i)\left(-h(\Delta_i u, \theta, i) + k + h(\Delta_i \tilde{u}, \tilde{\theta}, i) - \tilde{k}\right) \\ \leq -\gamma\|(\theta-\tilde{\theta})(s)\|^2 - \sum_{i=1}^d \gamma_i(\theta^i + \tilde{\theta}^i)(s)\|(\Delta_i u - \Delta_i \tilde{u})(s)\|^2,$$

by Remark 1. □

4.2.2 Numerical algorithm

Let A be given by (4.0.4). Due to the monotonicity, for small μ, the Euler map,

$$E_\mu \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} \theta \\ u \end{bmatrix} - \mu A \begin{bmatrix} \theta \\ u \end{bmatrix},$$

is a contraction, provided that θ is nonnegative; that is, the case when θ is a probability vector. However, E_μ may not keep θ non-negative and, in general, E_μ also does not preserve the mass. Thus, we introduce the following projection operator on $\mathcal{S}^d \times \mathbb{R}^d$:

$$P \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} \varpi(\theta) \\ u \end{bmatrix},$$

where $\varpi(\theta)_i = (\theta^i + \xi)^+$ and ξ is such that

$$\sum_i \varpi(\theta)_i = 1.$$

Clearly, P is a contraction because it is a projection on a convex set. Finally, to approximate weak solutions of (4.1.3), that is solutions (4.1.4), we consider the iterative map

$$\begin{bmatrix} \theta_{n+1} \\ u_{n+1} \end{bmatrix} = PE_\mu \begin{bmatrix} \theta_n \\ u_n \end{bmatrix}. \tag{4.2.3}$$

We have the following result:

Proposition 3. Let $(\bar\theta, \bar u, \bar k)$ solve (4.1.4). Then, $(\bar\theta, \bar u)$ is a fixed point for (4.2.3). Moreover, for any fixed point of (4.2.3), there exists $\bar k$ such that $(\bar\theta, \bar u, \bar k)$ solves (4.1.4).

Finally, if μ is small enough and (4.1.4) has a weak solution, $(\bar\theta, \bar u, \bar k)$, with $\bar\theta > 0$, then the iterates in (4.2.3) are bounded and converge to $(\bar\theta, \bar u)$. Moreover, the solution is unique.

Proof. Clearly, a solution of (4.1.4) is a fixed point for (4.2.3). Conversely, let $(\bar\theta, \bar u)$ be a fixed point for (4.2.3). Then,

$$\bar u^i = \bar u^i + \mu \sum_j \bar\theta^j \alpha_i^*(\Delta_j \bar u, \bar\theta, j).$$

Hence,

$$\sum_j \bar\theta^j \alpha_i^*(\Delta_j \bar u, \bar\theta, j) = 0.$$

Additionally, we have

$$\bar\theta^i = \left(\bar\theta^i - \mu h(\Delta_i \bar u, \bar\theta, i) + \xi\right)^+$$

for some ξ. Thus, for $\bar{k} = \frac{\xi}{\mu}$,

$$h(\Delta_i \bar{u}, \bar{\theta}, i) \geq \bar{k},$$

with equality when $\bar{\theta}^i > 0$.

If μ is small enough, E_μ is a contraction because A is a monotone Lipschitz map. Thus, if there is a solution of (4.1.4), the iterates in (4.2.3) are bounded. Then, the convergence follows from the monotonicity of E_μ and the strict contraction given by $\bar{\theta} > 0$. $\square$

Remark 2. Concerning the convergence rate and the choice of the parameter μ in the preceding theorem, we observe the following. The operator A is locally Lipschitz. Thus, given bound on the initial conditions, we can assume the Lipschitz constant to be a number, $L > 0$. By selecting

$$0 < \mu < \frac{2}{L},$$

we get that E_μ is a contraction and we may assume that the initial bound on data is preserved (for example, by looking at the norm of the difference between the iterates and a given stationary solution). If there is a strictly positive stationary solution, the convergence is exponential because, for u and $\tilde{u}$ with mean 0, we have

$$\gamma \|(\theta - \tilde{\theta})(s)\|^2 + \sum_{i=1}^{d} \gamma_i (\theta^i + \tilde{\theta}^i)(s) \|(\Delta_i u - \Delta_i \tilde{u})(s)\|^2 \geq$$
$$C \sum_{i=1}^{d} \left[(u^i - \tilde{u}^i)^2 + (\theta^i - \tilde{\theta}^i)^2 \right].$$

The constant, however, depends on the lower bounds on the stationary solution and thus, we do not have a direct estimate on the rate of convergence.

4.2.3 Numerical examples

To illustrate our algorithm, we consider the paradigm-shift problem. The monotone flow in (4.2.1) is

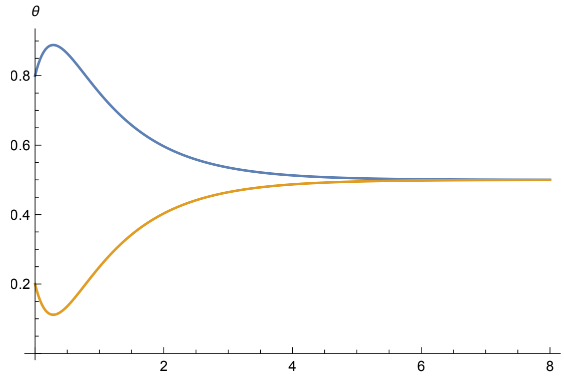
$$\begin{cases} u^1_s = -\theta^1 (u^1 - u^2)^+ + \theta^2 (u^2 - u^1)^+ \\ u^2_s = \theta^1 (u^1 - u^2)^+ - \theta^2 (u^2 - u^1)^+, \end{cases} \tag{4.2.4}$$

and

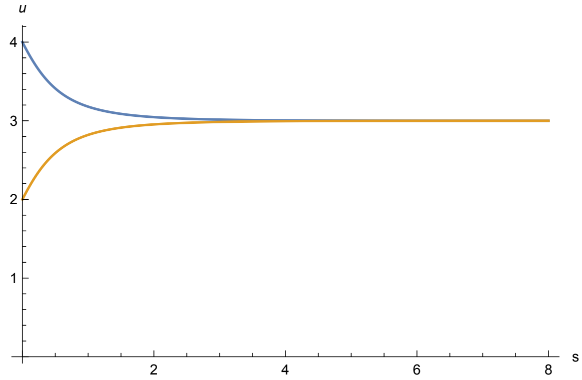
$$\begin{cases} \theta^1_s = -\theta^1 + \frac{1}{2}((u^1 - u^2)^+)^2 + k(s) \\ \theta^2_s = -\theta^2 + \frac{1}{2}((u^2 - u^1)^+)^2 + k(s). \end{cases} \tag{4.2.5}$$

According to (4.2.2),

$$k(s) = \frac{1}{2} \left(\theta^1 - \frac{1}{2}((u^1 - u^2)^+)^2 + \theta^2 - \frac{1}{2}((u^2 - u^1)^+)^2 \right).$$

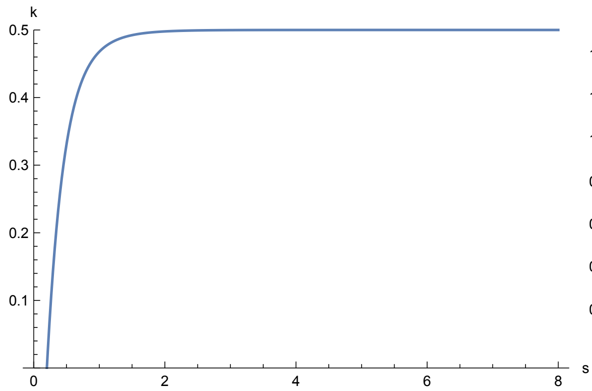


(a) Convergence of θ^i.

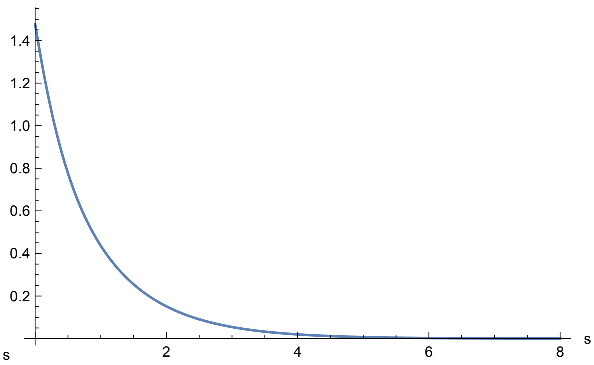


(b) Convergence of u^i.

Fig. 4.1 Evolution of θ and u with the monotone flow, for $s \in [0, 8]$. The quantities corresponding to the state 1 and 2 are depicted in blue and orange, respectively.



(a) Evolution of k.



(b) Contraction of the norm.

Fig. 4.2 Evolution of k and contraction of the norm, $\|(\theta, u) - (\bar{\theta}, \bar{u})\|$.

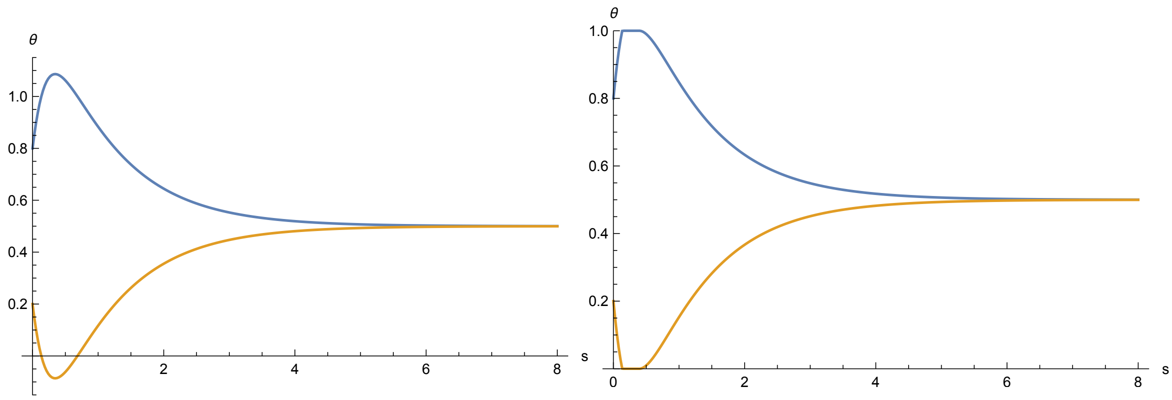
Now, we present the numerical results for this model using the iterative method in (4.2.3). We set $s \in [0, 8]$ and discretize this interval into $N = 300$ subintervals. First, we consider the following initial conditions:

$$u_0^1 = 4,\ u_0^2 = 2 \text{ and } \theta_0^1 = 0.8,\ \theta_0^2 = 0.2.$$

The convergence towards the stationary solution is illustrated in Figures 4.1a and 4.1b for θ and u. The behavior of k is shown in Figure 4.2a. In Figure 4.2b, we illustrate the contraction of the norm

$$\left\| \begin{bmatrix} \theta(s) \\ u(s) \end{bmatrix} - \begin{bmatrix} \bar{\theta} \\ \bar{u} \end{bmatrix} \right\|,$$

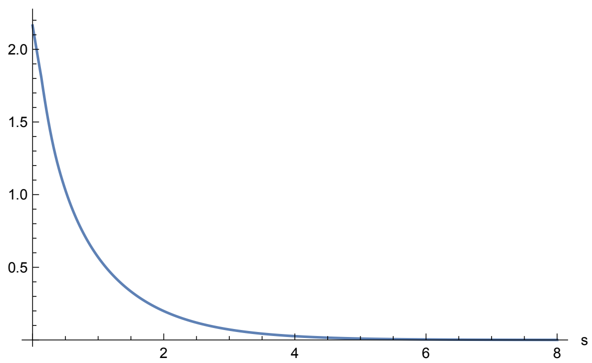
where $(\bar{\theta}, \bar{u})$ is the stationary solution in (4.1.8). Next, we consider the case in which the iterates of E_μ do not preserve positivity. In Figure 4.3, we compare the evolution of θ by



(a) Non positivity of the distribution using E_μ.

(b) Convergence using (4.2.3) while preserving the positivity of θ.

Fig. 4.3 Comparison between the iterates of E_μ and PE_μ for $\theta_0^1 = 0.8$, $\theta_0^2 = 0.2$, $u_0^1 = 5$, and $u_0^2 = 2$. The quantities corresponding to the state 1 and 2 are depicted in blue and orange, respectively.



(a) Contraction of the norm.

Fig. 4.4 Evolution of the norm, $\|(\theta, u) - (\bar{\theta}, \bar{u})\|$, using the projection method.

iterating E_μ, without the projection and using (4.2.3). In the first case, θ may not remain positive, although, in this example, convergence holds. In Figure 4.3, we plot the evolution through (4.2.3) of θ towards the analytical solution $\theta^1 = \theta^2 = 0.5$. As expected from its construction, θ is always non-negative and a probability. The contraction of the norm is similar to the previous case, see Figure 4.4.

4.3 Initial-terminal value problems

The initial-terminal conditions in (4.0.3) are the key difficulty in the design of numerical methods for the time-dependent MFG, (4.0.2). Here, we extend the strategy from the previous section to handle initial-terminal conditions. We start with an arbitrary pair of functions, $(u(t,0), \theta(t,0))$, that satisfies (4.0.3) and build a family $(u(t,s), \theta(t,s))$, $s \geq 0$, that converges to a solution of (4.0.2)-(4.0.3) as $s \to \infty$, while preserving the boundary conditions for all $s \geq 0$.

4.3.1 Representation of functionals in H^1

We begin by discussing the representation of linear functionals in H^1. Consider the Hilbert space, $H^1_T = \{\phi \in H^1([0,T], \mathbb{R}^d) : \phi(T) = 0\}$. For $\theta, u \in H^1([0,T], \mathbb{R}^d)$, we consider the variational problem

$$\min_{\phi \in H^1_T} \int_0^T \left[\frac{1}{2}(|\phi|^2 + |\dot{\phi}|^2) + \phi \cdot \left(\theta_t - \sum_j \theta^j \alpha^*(\Delta_j u, \theta, j) \right) \right] dt. \tag{4.3.1}$$

A minimizer, $\phi \in H^1_T$, of the preceding functional represents the linear functional

$$\eta \mapsto -\int_0^T \eta \cdot \left(\theta_t - \sum_j \theta^j \alpha^*(\Delta_j u, \theta, j) \right) dt$$

for $\eta \in H^1_T$, as an inner product in H^1_T; that is,

$$\int_0^T \left(\eta \cdot \phi + \dot{\eta} \cdot \dot{\phi} \right) dt = -\int_0^T \eta \cdot \left(\theta_t - \sum_j \theta^j \alpha^*(\Delta_j u, \theta, j) \right) dt$$

for $\phi, \eta \in H^1_T$. The last identity is simply the weak form of the Euler-Lagrange equation for (4.3.1),

$$-\ddot{\phi} + \phi = -\theta_t + \sum_j \theta^j \alpha^*(\Delta_j u, \theta, j), \tag{4.3.2}$$

whose boundary conditions are $\phi(T) = 0$ and $\dot{\phi}(0) = 0$. For $\theta, u \in H^1([0,T], \mathbb{R}^d)$, we define

$$\Phi(\theta, u, t) = \phi(t). \tag{4.3.3}$$

Next, let $H^1_I = \{\psi \in H^1([0,T], \mathbb{R}^d) : \psi(0) = 0\}$. For $\theta, u \in H^1([0,T], \mathbb{R}^d)$, we consider the variational problem

$$\min_{\psi \in H^1_I} \int_0^T \left[\frac{1}{2}(|\psi|^2 + |\dot{\psi}|^2) + \psi \cdot (u_t + h(\Delta_i u, \theta, i)) \right] dt. \tag{4.3.4}$$

The Euler-Lagrange equation for the preceding problem is

$$-\ddot{\psi} + \psi = -u_t - h(\Delta_i u, \theta, i), \tag{4.3.5}$$

with the boundary conditions $\psi(0) = 0$ and $\dot{\psi}(T) = 0$. Moreover, if $\psi \in H_I^1$ minimizes the functional in (4.3.4), we have

$$\int_0^T \left(\eta \cdot \psi + \dot{\eta} \cdot \dot{\psi}\right) dt = \int_0^T \eta \cdot (-u_t - h(\Delta_i u, \theta, i))\, dt$$

for $\eta, \psi \in H_I^1$. For $\theta, u \in H^1([0,T], \mathbb{R}^d)$, we define

$$\Psi(\theta, u, t) = \psi(t). \tag{4.3.6}$$

4.3.2 Solutions of the Euler-Lagrange equations

To find ϕ and ψ, we need to solve Euler-Lagrange equations, (4.3.2) and (4.3.5).

The homogeneous solutions for (4.3.2) are given by

$$\phi_{h_1} = e^t, \quad \phi_{h_2} = e^{-t}.$$

In order to compute the general solution

$$\phi(t) = y_1(t)e^t + y_2(t)e^{-t}.$$

using the variation of parameters formula, we need to calculate the Wronskian that is given by

$$W = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2.$$

Since it is non-zero we have

$$y_1(t) = C_1 + \frac{1}{2}\int_0^t e^{-r}\left(\theta_t(r) - \sum_j \theta^j(r)\alpha^*(\Delta_j u(r), \theta(r), j)\right) dr,$$

and

$$y_2(t) = C_2 - \frac{1}{2}\int_0^t e^{r}\left(\theta_t(r) - \sum_j \theta^j(r)\alpha^*(\Delta_j u(r), \theta(r), j)\right) dr.$$

Hence the solution is

$$\phi(t) = \frac{e^t}{2}\int_0^t e^{-r} f(r) dr - \frac{e^{-t}}{2}\int_0^t e^{r} f(r) dr + C_1 e^t + C_2 e^{-t},$$

where the function f is given by

$$f(r) = \theta_t(r) - \sum_j \theta^j(r)\alpha^*(\Delta_j u(r), \theta(r), j).$$

We now compute the constants C_1 and C_2, using the boundary conditions $\dot{\phi}(0) = 0$, and $\phi(T) = 0$. From the first condition we get

$$\begin{aligned}
0 = \dot{\phi}(t)\Big|_{t=0} &= \Big[\frac{1}{2}e^t \int_0^t e^{-r} f(r)dr + \frac{1}{2}f(t) + \frac{1}{2}e^{-t}\int_0^t e^r f(r)dr - \frac{1}{2}f(t) \\
&\quad + C_1 e^t - C_2 e^{-t}\Big|_{t=0} \\
&= C_1 - C_2,
\end{aligned}$$

which gives

$$C_1 = C_2. \tag{4.3.7}$$

From the second boundary condition we have

$$\begin{aligned}
0 = \phi(T) &= y_1(T)e^T + y_2(T)e^{-T} \\
&= e^T \left(C_1 + \frac{1}{2}\int_0^T e^{-r} f(r)dr \right) + e^{-T}\left(C_2 - \frac{1}{2}\int_0^T e^r f(r)dr \right).
\end{aligned}$$

We obtain using the above and (4.3.7),

$$C_1 = \frac{1}{2(e^T + e^{-T})}\int_0^T (e^{r-T} - e^{T-r}) f(r)dr.$$

Therefore, the solution of (4.3.2) is given by

$$\phi(t) = \frac{1}{2}\left(\int_0^t (e^{t-r} - e^{r-t}) f(r)dr + \frac{e^t + e^{-t}}{e^T + e^{-T}}\int_0^T (e^{r-T} - e^{T-r}) f(r)dr \right). \tag{4.3.8}$$

Similarly the general solution of (4.3.5) is given by

$$\psi(t) = z_1(t)e^t + z_2(t)e^{-t},$$

where

$$z_1(t) = D_1 + \int_0^t \frac{1}{2}e^{-r} g(r)dr,$$

and

$$z_2(t) = D_2 - \int_0^t \frac{1}{2}e^{r} g(r)dr,$$

where the function g is given by

$$g(r) = u_t(r) + h(\Delta_i u(r), \theta(r), i).$$

From $\psi(0) = 0$ we have

$$D_1 + D_2 = 0 \quad \Leftrightarrow \quad D_1 = -D_2.$$

Taking the derivative of $\psi(t)$ we get

$$\begin{aligned}\dot{\psi}(t) &= \dot{z}_1(t)e^t + z_1(t)e^t + \dot{z}_2(t)e^{-t} - z_2(t)e^{-t} \\ &= \frac{1}{2}e^{-t}g(t)e^t + z_1(t)e^t - \frac{1}{2}e^t g(t)e^{-t} - z_2(t)e^{-t} \\ &= D_1 e^t - D_2 e^{-t} + \frac{1}{2}\int_0^t (e^{t-r} + e^{r-t})g(r)dr.\end{aligned}$$

From the terminal condition $\dot{\psi}(T) = 0$ we obtain

$$D_1 e^T - D_2 e^{-T} + \frac{1}{2}\int_0^T (e^{T-r} + e^{r-T})g(r)dr = 0,$$

using $D_1 = -D_2$ we get

$$D_1 = -\frac{1}{2(e^T + e^{-T})}\int_0^T (e^{T-r} + e^{r-T})g(r)dr$$

Therefore the solution to the Euler-Lagrange equation (4.3.5) is given by

$$\psi(t) = \frac{e^{-t} - e^t}{2(e^T + e^{-T})}\int_0^T (e^{T-r} + e^{r-T})g(r)dr + \frac{1}{2}\int_0^t (e^{t-r} - e^{r-t})g(r)dr. \tag{4.3.9}$$

4.3.3 Monotone deformation flow

Next, we introduce the monotone deformation flow,

$$\begin{cases} u^i_s(t,s) = \Phi^i(\theta(\cdot,s), u(\cdot,s), t) \\ \theta^i_s(t,s) = \Psi^i(\theta(\cdot,s), u(\cdot,s), t), \end{cases} \tag{4.3.10}$$

where Φ and Ψ are given in (4.3.3) and (4.3.6). As we show in the next proposition, for smooth enough solutions, the previous flow is a contraction in H^1. Moreover, if (θ, u) solve (4.0.2)-(4.0.3), we have

$$\Phi(\theta, u, t) = \Psi(\theta, u, t) = 0.$$

Hence, solutions of (4.0.2)-(4.0.3) are fixed points for (4.3.10).

Before stating the contraction property, we recall that the H^1-norm of a pair of functions is given by

$$\|(v,\eta)\|^2_{H^1} = \int_0^T \left(|v|^2 + |\dot{v}|^2 + |\eta|^2 + |\dot{\eta}|^2\right) dt$$

for $v, \eta : [0,T] \to \mathbb{R}^d$.

Proposition 4. Let (u,θ) and $(\tilde{u},\tilde{\theta})$ be C^2 solutions of (4.3.10). Suppose that $\theta, \tilde{\theta} \geq 0$. Then,

$$\frac{d}{ds}\|(u,\theta) - (\tilde{u},\tilde{\theta})\|^2_{H^1} \leq 0,$$

with strict inequality if $(u,\theta) \neq (\tilde{u},\tilde{\theta})$.

Proof. We have

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_0^T &\left[(u-\tilde{u})^2 + (u-\tilde{u})_t^2 + (\theta-\tilde{\theta})^2 + (\theta-\tilde{\theta})_t^2\right] dt \\
&= \int_0^T \left[(u-\tilde{u})(u-\tilde{u})_s + (u-\tilde{u})_t(u-\tilde{u})_{ts}\right] dt \\
&+ \int_0^T \left[(\theta-\tilde{\theta})(\theta-\tilde{\theta})_s + (\theta-\tilde{\theta})_t(\theta-\tilde{\theta})_{ts}\right] dt.
\end{aligned}$$

Using (4.3.10), the term in the right-hand side of the previous equality becomes

$$\begin{aligned}
&\int_0^T \left[(u-\tilde{u})(\phi-\tilde{\phi}) + (u-\tilde{u})_t(\phi-\tilde{\phi})_t\right] dt \\
&+ \int_0^T \left[(\theta-\tilde{\theta})(\psi-\tilde{\psi}) + (\theta-\tilde{\theta})_t(\psi-\tilde{\psi})_t\right] dt \\
&= \int_0^T (u-\tilde{u})(\phi-\tilde{\phi})dt + \left[(u-\tilde{u})(\phi-\tilde{\phi})_t\right]\Big|_0^T \\
&\qquad - \int_0^T (u-\tilde{u})(\phi-\tilde{\phi})_{tt}dt \\
&+ \int_0^T (\theta-\tilde{\theta})(\psi-\tilde{\psi})dt + \left[(\theta-\tilde{\theta})(\psi-\tilde{\psi})_t\right]\Big|_0^T \\
&\qquad - \int_0^T (\theta-\tilde{\theta})(\psi-\tilde{\psi})_{tt}dt,
\end{aligned}$$

where we used integration by parts in the last equality. Because $u(T) = \tilde{u}(T)$, $\theta(0) = \tilde{\theta}(0)$, $\phi_t(0) = \tilde{\phi}_t(0)$, $\psi_t(T) = \tilde{\psi}_t(T)$, and using (4.3.2) and (4.3.5), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_0^T &(u-\tilde{u})^2 + (u-\tilde{u})_t^2 + (\theta-\tilde{\theta})^2 + (\theta-\tilde{\theta})_t^2 \\
&= \int_0^T (u-\tilde{u})\left(\sum \theta^j \alpha^*(\Delta_j u, \theta, j) - \sum \tilde{\theta}\alpha^*(\Delta_j \tilde{u}, \tilde{\theta}, j)\right) \\
&- \int_0^T (\theta-\tilde{\theta})\left(h(\Delta_i u, \theta, i) - h(\Delta_i \tilde{u}, \tilde{\theta}, i)\right) \\
&\leq \int_0^T -\gamma\|(\theta-\tilde{\theta})(t)\|^2 - \sum_{i=1}^d \gamma_i(\theta^i + \tilde{\theta}^i)(t)\|(\Delta_i u - \Delta_i \tilde{u})(t)\|^2 dt,
\end{aligned} \tag{4.3.11}$$

due to Remark 1. □

4.3.4 Monotone discretization

To build our numerical method, we begin by discretizing (4.3.10). We look for a time-discretization of

$$A\begin{bmatrix}\theta\\ u\end{bmatrix}=\begin{bmatrix}-\theta_t+f(u,\theta)\\ -u_t-h(u,\theta)\end{bmatrix}$$

that preserves monotonicity, where $f(u,\theta)=\sum_j \tilde{\theta}^j\alpha^*(\Delta_j\tilde{u},\tilde{\theta},j)$.

With Hamilton-Jacobi equations, implicit schemes have good stability properties. Because the Hamilton-Jacobi equation in (4.3.10) is a terminal value problem, we discretize it using an explicit forward-in-time scheme (hence, implicit backward-in-time scheme). Then, to keep the adjoint structure of A at the discrete level, we are then required to choose an implicit discretization forward in time for the first component of A. Usually, implicit schemes have the disadvantage of requiring the numerical solution of non-linear equations at each time step. Here, we discretize the operator, A, globally, and we never need to solve implicit equations.

More concretely, we split $[0,T]$ into N intervals of length $\delta t=\frac{T}{N}$. The vectors $\theta_n\in\mathcal{S}^d$ and $u_n\in\mathbb{R}^d$, $0\le n\le N$ approximate θ and u at time $\frac{nT}{N}$. We set $\mathcal{M}_N=(\mathcal{S}^d\times\mathbb{R}^d)^{N+1}$ and define

$$A^N\begin{bmatrix}\theta\\ u\end{bmatrix}_n=\begin{bmatrix}-\frac{\theta^i_{n+1}-\theta^i_n}{\delta t}+f(u^i_{n+1},\theta^i_{n+1})+k_n\\ -\frac{u^i_{n+1}-u^i_n}{\delta t}-h(u^i_n,\theta^i_n)\end{bmatrix},\tag{4.3.12}$$

where

$$k_n(s)=-\frac{1}{d}\sum_{i=1}^d\left(-\frac{\delta\theta^i_n}{\delta t}+f(u^i_{n+1},\theta^i_{n+1})\right)$$

and $\delta\theta^i_n=\theta^i_{n+1}-\theta^i_n$. Next, we show that A^N is a monotone operator in the convex subset of vectors in $\mathcal{M}$ that satisfy the initial-terminal conditions in (4.0.3). We denote by $\langle\cdot,\cdot\rangle$, the duality pairing in $(\mathcal{S}^d\times\mathbb{R}^d)^{N+1}$. More precisely, for $(\theta,u),(\tilde{\theta},\tilde{u})\in(\mathcal{S}^d\times\mathbb{R}^d)^{N+1}$

$$\left\langle\begin{bmatrix}\theta\\ u\end{bmatrix},\begin{bmatrix}\tilde{\theta}\\ \tilde{u}\end{bmatrix}\right\rangle=\sum_{k=0}^N\theta^k\cdot\tilde{\theta}^k+u^k\cdot\tilde{u}^k.$$

Proposition 5. A^N is monotone in the convex subset $\mathcal{M}_N$ of all $(\theta,u)\in(\mathcal{S}^d\times\mathbb{R}^d)^{N+1}$ such that $\theta_0=\bar{\theta}_0$ and $u_N=\bar{u}_T$. Moreover, we have the inequality

$$\begin{aligned}&\left\langle A^N\begin{bmatrix}\theta\\ u\end{bmatrix}-A^N\begin{bmatrix}\tilde{\theta}\\ \tilde{u}\end{bmatrix},\begin{bmatrix}\theta\\ u\end{bmatrix}-\begin{bmatrix}\tilde{\theta}\\ \tilde{u}\end{bmatrix}\right\rangle\\ &\le\sum_{n=1}^{N-1}\left(-\gamma\|(\theta-\tilde{\theta})(t)\|^2-\sum_{i=1}^d\gamma_i(\theta^i+\tilde{\theta}^i)(t)\|(\Delta_iu-\Delta_i\tilde{u})(t)\|^2\right).\end{aligned}$$

Proof. We begin by computing

$$
\begin{aligned}
&\left\langle A^N \begin{bmatrix} \theta \\ u \end{bmatrix} - A^N \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} \theta \\ u \end{bmatrix} - \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix} \right\rangle \\
&= \sum_{n=0}^{N-1} \Bigg[(\theta_n - \tilde{\theta}_n) \left(-\frac{u_{n+1} - u_n}{\delta t} - h(u_n, \theta_n) + \frac{\tilde{u}_{n+1} - \tilde{u}_n}{\delta t} + h(\tilde{u}_n, \tilde{\theta}_n) \right) \\
&+ (u_{n+1} - \tilde{u}_{n+1}) \Big(- \frac{\theta_{n+1} - \theta_n}{\delta t} + f(u_{n+1}, \theta_{n+1}) + k_n \\
&+ \frac{\tilde{\theta}_{n+1} - \tilde{\theta}_n}{\delta t} - f(\tilde{u}_{n+1}, \tilde{\theta}_{n+1}) - \tilde{k}_n \Big) \Bigg].
\end{aligned}
$$

With the sums developed and the indices relabeled, the preceding expression becomes

$$
\begin{aligned}
&\sum_{n=1}^{N-1} \Bigg[(\theta_n - \tilde{\theta}_n) \left(-\frac{u_{n+1} - u_n}{\delta t} - h(u_n, \theta_n) + \frac{\tilde{u}_{n+1} - \tilde{u}_n}{\delta t} + h(\tilde{u}_n, \tilde{\theta}_n) \right) \\
&+ (\theta_0 - \tilde{\theta}_0) \left(-\frac{u_1 - u_0}{\delta t} - h(u_0, \theta_0) + \frac{\tilde{u}_1 - \tilde{u}_0}{\delta t} + h(\tilde{u}_0, \theta_0) \right) \Bigg] \\
&+ \sum_{n=1}^{N-1} \Bigg[(u_n - \tilde{u}_n) \left(-\frac{\theta_n - \theta_{n-1}}{\delta t} + f(u_n, \theta_n) + \frac{\tilde{\theta}_n - \tilde{\theta}_{n-1}}{\delta t} - f(\tilde{u}_n, \tilde{\theta}_n) \right) \\
&+ (u_N - \tilde{u}_N) \left(-\frac{\theta_N - \theta_{N-1}}{\delta t} + f(u_N, \theta_N) + \frac{\tilde{\theta}_N - \tilde{\theta}_{N-1}}{\delta t} - f(\tilde{u}_N, \tilde{\theta}_N) \right) \Bigg].
\end{aligned}
$$

The second and last lines above are zero since $\theta_0 = \tilde{\theta}_0 = \bar{\theta}_0$ and $u_N = \tilde{u}_N = \bar{u}_T$. Using Remark 1, we obtain

$$
\begin{aligned}
&\left\langle A \begin{bmatrix} \theta \\ u \end{bmatrix} - A \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} \theta \\ u \end{bmatrix} - \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix} \right\rangle \\
&\leq \sum_{n=1}^{N-1} \left(-\gamma \|(\theta - \tilde{\theta})(t)\|^2 - \sum_{i=1}^{d} \gamma_i (\theta^i + \tilde{\theta}^i)(t) \|(\Delta_i u - \Delta_i \tilde{u})(t)\|^2 \right) \\
&- \sum_{n=1}^{N-1} (\theta_n - \tilde{\theta}_n) \left(\frac{u_{n+1} - \tilde{u}_{n+1}}{\delta t} - \frac{u_n - \tilde{u}_n}{\delta t} \right) \\
&- \sum_{n=1}^{N-1} (u_n - \tilde{u}_n) \left(\frac{\theta_n - \tilde{\theta}_n}{\delta t} - \frac{\theta_{n-1} - \tilde{\theta}_{n-1}}{\delta t} \right).
\end{aligned}
$$

We now show that the last two lines add to zero. Let $a_n = \theta_n - \tilde{\theta}_n$ and $b_n = u_n - \tilde{u}_n$. Accordingly, we have

$$\begin{aligned}
&-\frac{1}{\delta t}\sum_{n=1}^{N-1} a_n(b_{n+1}-b_n) - \frac{1}{\delta t}\sum_{n=1}^{N-1} b_n(a_n - a_{n-1})\\
&= -\frac{1}{\delta t}(b_N a_N - b_1 a_1) + \frac{1}{\delta t}\sum_{n=1}^{N-1} b_{n+1}(a_{n+1}-a_n) - \frac{1}{\delta t}\sum_{n=0}^{N-2} b_{n+1}(a_{n+1}-a_n)\\
&= \frac{1}{\delta t}(b_1 a_0 - b_N a_{N-1}) = 0,
\end{aligned}$$

where we summed the first term by parts and relabeled the index, n, in the last term of the first line. The last equality follows from the assumption in the statements, $a_0 = \theta_0 - \tilde{\theta}_0 = 0$ and $b_N = u_N - \tilde{u}_N = 0$. □

Using the techniques in [12], we prove the convergence of the solutions of the discretized problem as $\delta t \to 0$. As usual, we discretize the time interval, $[0,T]$, into $N+1$ equispaced points.

Proposition 6. Let $(\theta^N, u^N) \in \mathcal{M}_N$ be a solution of

$$A^N \begin{bmatrix} \theta^N \\ u^N \end{bmatrix}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

satisfying the initial-terminal conditions in (4.0.3). Suppose that u^N is uniformly bounded. Consider the step functions $\bar{u}^N$ and $\bar{\theta}^N$ taking the values $\bar{u}_n^{Ni} \in \mathbb{R}$ and $\bar{\theta}_n^{Ni} \in \mathcal{S}$ in $[\frac{(n-1)T}{N}, \frac{nT}{N}]$, with $0 \le n \le N$, for $i \in I_d$, respectively. Then, extracting a subsequence if necessary, $\bar{u}^{Ni} \rightharpoonup \bar{u}^i$ and $\bar{\theta}^{Ni} \rightharpoonup \theta^i$ weakly-* in L^∞ for $i \in I_d$. Furthermore, $(\bar{u}, \bar{\theta})$ is a weak solution of (4.0.2).

Proof. Because u^N is bounded by hypothesis and θ^N is bounded since it is a probability measure, the weak-* convergence in L^∞ is immediate. Hence, there exist $\bar{u}^i \in L^\infty([0,T])$ and $\theta^i \in L^\infty([0,T])$ as claimed.

Let $\tilde{u}^i, \tilde{\theta}^i \in C^\infty([0,T])$, with $\tilde{\theta}^i \ge 0$ for all $i \in I_d$, and $\sum_{i\in I_d} \tilde{\theta}^i = 1$. Suppose further that $\tilde{u}^i, \tilde{\theta}^i$ satisfy the boundary conditions in (4.0.3). Let $\tilde{u}_n^N = \tilde{u}\left(\frac{n}{N}T\right)$, $\tilde{\theta}_n^N = \tilde{\theta}\left(\frac{n}{N}T\right)$ be the vectors whose components are $\tilde{u}_n^{Ni}$ and $\tilde{\theta}_n^{Ni}$, respectively. By the monotonicity of A^N, we have

$$\begin{aligned}
0 &\le \left\langle A^N \begin{bmatrix} \tilde{\theta}^N \\ \tilde{u}^N \end{bmatrix}, \begin{bmatrix} \tilde{\theta}^N \\ \tilde{u}^N \end{bmatrix} - \begin{bmatrix} \theta^N \\ u^N \end{bmatrix} \right\rangle\\
&= O\left(\frac{1}{N}\right) + \left\langle A \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix} - \begin{bmatrix} \bar{\theta}^N \\ \bar{u}^N \end{bmatrix} \right\rangle,
\end{aligned}$$

and taking the limit $N \to \infty$ gives the result. □

4.3.5 Monotone discretization for the H^1 projections

Next, we discuss the representation of linear functionals for the discrete problem. For that, proceeding as in Section 4.3.1, we compute the optimality conditions of the discretized versions of (4.3.1) and (4.3.4).

Fix $(u, \theta) \in \mathcal{M}_N$ and consider the following discrete analog to (4.3.1):

$$\min_{\phi \in \tilde{H}_T^1} \delta t \sum_{n=1}^{N} \frac{1}{2}\left(\phi_n^2 + \left(\frac{\delta \phi_{n-1}}{\delta t}\right)^2\right) + \phi_n \left(\frac{\delta \theta_{n-1}}{\delta t} - f(u_n, \theta_n)\right),$$

where $\delta g_n = g_{n+1} - g_n$, and $\tilde{H}_T^1 = \{\phi = (\phi_0, \ldots, \phi_N) \in (\mathbb{R}^d)^{(N+1)} : \phi_N = 0\}$. The corresponding optimality conditions (the discrete Euler-Lagrange equation) is

$$-\frac{\delta(\delta\phi_{n-1})}{(\delta t)^2} + \phi_n = -\frac{\delta\theta_{n-1}}{\delta t} + f(u_n, \theta_n), \tag{4.3.13}$$

for $n = 1, \ldots, N-1$, coupled with the boundary conditions $\phi_N = 0$ and $\phi_1 = \phi_0$.

A minimizer of the problem above represents the following discrete linear functional

$$\eta \mapsto -\sum_{n=1}^{N} \eta_n \cdot \left(\frac{\delta\phi_{n-1}}{\delta t} - f(u_n, \theta_n)\right) \delta t$$

as an inner product in $\tilde{H}_T^1$

$$\sum_{n=1}^{N}\left(\eta_n \cdot \phi_n \delta t + \frac{1}{\delta t}\delta\eta_{n-1} \cdot \delta\phi_{n-1}\right) = -\sum_{n=1}^{N} \eta_n \cdot \left(\frac{\delta\phi_{n-1}}{\delta t} - f(u_n, \theta_n)\right)\delta t.$$

For $(\theta_n, u_n) \in \mathcal{M}_N$, we define

$$\Phi(\theta_n, u_n) = \phi_n. \tag{4.3.14}$$

We now examine a second discrete variational problem corresponding to (4.3.4). For $(u, \theta) \in \mathcal{M}_N$, we consider

$$\min_{\psi \in \tilde{H}_I^1} \delta t \sum_{n=0}^{N-1} \frac{1}{2}\left(\psi_n^2 + \left(\frac{\delta\psi_n}{\delta t}\right)^2\right) + \psi_n\left(\frac{\delta u_n}{\delta t} + h(u_n, \theta_n)\right),$$

where $\tilde{H}_I^1 = \{\psi = (\psi_0, \ldots, \psi_N) \in (\mathbb{R}^d)^{(N+1)} : \psi_0 = 0\}$.

The discrete Euler-Lagrange equation is

$$-\frac{\delta(\delta\psi_{n-1})}{(\delta t)^2} + \psi_n = -\frac{\delta u_n}{\delta t} - h(u_n, \theta_n) \tag{4.3.15}$$

for $n = 1, \ldots, N-1$, together with the conditions $\psi_0 = 0$ and $\psi_N = \psi_{N-1}$.

From the Euler-Lagrange equation, we obtain the following representation formula in the Hilbert space $\{\psi \in H^1_n(\{0,\dots,N\}) : \psi_0 = 0\}$:

$$\sum_{n=0}^{N-1} (\eta\cdot\psi + \delta\eta\cdot\delta\psi)\,\delta t = \sum_{0}^{N}\eta\cdot\left(-\frac{\delta u_n}{\delta t} - h(u_n,\theta_n)\right)\delta t.$$

Finally, we define

$$\Psi(\theta_n, u_n) = \psi_n, \tag{4.3.16}$$

for $(u,\theta)\in\mathcal{M}_N$.

Proposition 7. Let Φ and Ψ be given by (4.3.14) and (4.3.16). Consider the following operator:

$$Q_A\begin{bmatrix}\theta\\ u\end{bmatrix} = \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}. \tag{4.3.17}$$

Let $\mathcal{M}_N^{\bar\theta_0,\bar u_T}$ be the set of all $(\theta,u)\in\mathcal{M}_N$ that satisfy the initial condition $\theta_0=\bar\theta_0$ and the terminal condition $u_N=\bar u_T$. Then, Q_A is monotone with respect to the discrete H^1_N inner product corresponding to the norm

$$\|(\eta,\nu)\|^2_{H^1_N} = \sum_{n=0}^{N-1} |\eta_n|^2 + |\delta\eta_n|^2 + |\nu_n|^2 + |\delta\nu_n|^2. \tag{4.3.18}$$

Proof. Let $(u,\theta)\in\mathcal{M}_N^{\bar\theta_0,\bar u_T}$ and $(\tilde u,\tilde\theta)\in\mathcal{M}_N^{\bar\theta_0,\bar u_T}$. Let $\phi,\tilde\phi$ and $\psi,\tilde\psi$ be given by (4.3.14) and (4.3.16). We begin by computing

$$\begin{aligned}
&\left\langle Q_A\begin{bmatrix}\theta\\ u\end{bmatrix} - Q_A\begin{bmatrix}\tilde\theta\\ \tilde u\end{bmatrix}, \begin{bmatrix}\theta\\ u\end{bmatrix} - \begin{bmatrix}\tilde\theta\\ \tilde u\end{bmatrix}\right\rangle_{H^1_N}\\
&= \sum_{n=0}^{N-1}\left[(\theta_n-\tilde\theta_n)(\psi_n-\tilde\psi_n) + \frac{\delta(\theta_n-\tilde\theta_n)}{\delta t}\frac{\delta(\psi_n-\tilde\psi_n)}{\delta t}\right.\\
&\qquad\left. + (u_n-\tilde u_n)(\phi_n-\tilde\phi_n) + \frac{\delta(u_n-\tilde u_n)}{\delta t}\frac{\delta(\phi_n-\tilde\phi_n)}{\delta t}\right]\\
&= \sum_{n=0}^{N-1}(\theta_n-\tilde\theta_n)(\psi_n-\tilde\psi_n) + (u_n-\tilde u_n)(\phi_n-\tilde\phi_n)\\
&\qquad + \frac{1}{\delta t}\sum_{n=0}^{N-1}\left(\frac{\theta_{n+1}-\theta_n}{\delta t} - \frac{\tilde\theta_{n+1}-\tilde\theta_n}{\delta t}\right)(\delta\psi_n - \delta\tilde\psi_n)\\
&\qquad + \frac{1}{\delta t}\sum_{n=0}^{N-1}\left(\frac{u_{n+1}-u_n}{\delta t} - \frac{\tilde u_{n+1}-\tilde u_n}{\delta t}\right)(\delta\phi_n - \delta\tilde\phi_n).
\end{aligned} \tag{4.3.19}$$

Reorganizing, we see that the previous two lines are equal to

$$\frac{1}{(\delta t)^2}\sum_{n=0}^{N-1}\Bigg[\Big[(\theta_{n+1}-\tilde{\theta}_{n+1})-(\theta_n-\tilde{\theta}_n)\Big](\delta\psi_n-\delta\tilde{\psi}_n)$$
$$+\left[(u_{n+1}-\tilde{u}_{n+1})-(u_n-\tilde{u}_n)\right](\delta\phi_n-\delta\tilde{\phi}_n)\Bigg]. \tag{4.3.20}$$

Using the notation

$$a_n=\theta_n-\tilde{\theta}_n,\quad b_n=\delta\psi_n-\delta\tilde{\psi}_n,\quad c_n=u_n-\tilde{u}_n,\ \text{and}\ d_n=\delta\phi_n-\delta\tilde{\phi}_n,$$

we write (4.3.20) multiplied by $(\delta t)^2$ as

$$\begin{aligned}\sum_{n=0}^{N-1} b_n\delta a_n+d_n\delta c_n &= b_{N-1}\delta a_{N-1}+d_{N-1}\delta c_{N-1}+\sum_{n=0}^{N-2} b_n\delta a_n+d_n\delta c_n\\ &= b_{N-1}\delta a_{N-1}+d_{N-1}\delta c_{N-1}\\ &+a_{N-1}b_{N-1}-a_0b_0-\sum_{n=0}^{N-2}a_{n+1}\delta b_n\\ &+c_{N-1}d_{N-1}-c_0d_0-\sum_{n=0}^{N-2}c_{n+1}\delta d_n,\end{aligned} \tag{4.3.21}$$

where we used summation by parts in the last equality. Because $\psi_N=\psi_{N-1}$, we have $b_{N-1}=0$. Moreover, since $\theta_0=\tilde{\theta}_0$, we have $a_0=0$, and $\phi_1=\phi_0$ implies that $d_0=0$. Thus, we further have

$$\begin{aligned}d_{N-1}\delta c_{N-1} &= d_{N-1}\left(u_N-\tilde{u}_N-(u_{N-1}-\tilde{u}_{N-1})\right)\\ &= -d_{N-1}(u_{N-1}-\tilde{u}_{N-1})\\ &= -c_{N-1}d_{N-1},\end{aligned}$$

where we used the terminal condition $u_N=\tilde{u}_N$. According to these identities, (4.3.21) becomes

$$\sum_{n=0}^{N-1} b_n\delta a_n+d_n\delta c_n=-\sum_{n=0}^{N-2}a_{n+1}\delta b_n+c_{n+1}\delta d_n.$$

Therefore, (4.3.20) can be written as

$$-\sum_{n=0}^{N-2}\frac{\theta_{n+1}-\tilde{\theta}_{n+1}}{(\delta t)^2}\left(\delta^2\psi_n-\delta^2\tilde{\psi}_n\right)-\sum_{n=0}^{N-2}\frac{u_{n+1}-\tilde{u}_{n+1}}{(\delta t)^2}\left(\delta^2\phi_n-\delta^2\tilde{\phi}_n\right). \tag{4.3.22}$$

Shifting the index $n+1$ into n in (4.3.22), we obtain

$$-\sum_{n=1}^{N-1} \frac{\theta_n - \tilde{\theta}_n}{(\delta t)^2} \left(\delta^2 \psi_{n-1} - \delta^2 \tilde{\psi}_{n-1}\right) - \sum_{n=1}^{N-1} \frac{u_n - \tilde{u}_n}{(\delta t)^2} \left(\delta^2 \phi_{n-1} - \delta^2 \tilde{\phi}_{n-1}\right).$$

Using the Euler-Lagrange equations (4.3.13) and (4.3.15) in the preceding expression yields

$$\begin{aligned}
&-\sum_{n=1}^{N-1} (\theta_n - \tilde{\theta}_n) \left(\psi_n + \frac{u_{n+1} - u_n}{\delta t} + h(u_n, \theta_n) - \tilde{\psi}_n - \frac{\tilde{u}_{n+1} - \tilde{u}_n}{\delta t} - h(\tilde{u}_n, \tilde{\theta}_n)\right) \\
&-\sum_{n=1}^{N-1} (u_n - \tilde{u}_n) \left(\phi_n + \frac{\theta_n - \theta_{n-1}}{\delta t} - f(u_n, \theta_n) - \tilde{\phi}_n - \frac{\tilde{\theta}_n - \tilde{\theta}_{n-1}}{\delta t} + f(\tilde{u}_n, \tilde{\theta}_n)\right).
\end{aligned}$$

Finally, plugging the previous result into (4.3.19), we obtain

$$\begin{aligned}
&-\sum_{=1}^{N-1} (\theta_n - \tilde{\theta}_n) \left(\frac{u_{n+1} - \tilde{u}_{n+1}}{\delta t} - \frac{u_n - \tilde{u}_n}{\delta t} + h(u_n, \theta_n) - h(\tilde{u}_n, \tilde{\theta}_n)\right) \\
&-\sum_{n=1}^{N-1} (u_n - \tilde{u}_n) \left(\frac{\theta_n - \tilde{\theta}_n}{\delta t} - \frac{\theta_{n-1} - \tilde{\theta}_{n-1}}{\delta t} - f(u_n, \theta_n) + f(\tilde{u}_n, \tilde{\theta}_n)\right) \\
&\leq \sum_{n=1}^{N-1} -\gamma \|(\theta_n - \tilde{\theta}_n)\|^2 - \sum_{i=1}^{d} \gamma_i (\theta_n^i + \tilde{\theta}_n^i) \|(\Delta_i u_n - \Delta_i \tilde{u}_n)\|^2
\end{aligned}$$

by using Remark 1 and arguing as at the end of Subsection 4.3.4. □

4.3.6 Projection algorithm

As shown in Section 4.2, the monotone flow may not keep θ positive. Thus, to preserve probabilities and prevent θ from taking negative values, we define a projection operator through the following optimization problem. Given $(\eta, w) \in \mathcal{M}_N$, we solve

$$\begin{cases} \min_{\lambda_n^i} \sum_{i=1}^d (\eta_n^i - \lambda_n^i)^2 \\ \sum_{i=1}^d \lambda_n^i = 1, \quad \lambda_n^i \geq 0 \end{cases} \tag{4.3.23}$$

for $n \in \{0, \ldots, N\}$. Then, we set

$$P \begin{bmatrix} \eta \\ w \end{bmatrix}_n = \begin{bmatrix} \lambda_n \\ w_n \end{bmatrix}$$

for $0 \leq n \leq N$. We note that if η_n is a probability, then $\lambda_n = \eta_n$. Moreover, P is a contraction.

Now, we introduce the following iterative scheme:

$$w_{k+1} = P\left[w_k - \upsilon Q_A[w_k]\right], \tag{4.3.24}$$

where $w_k = (\theta_k, u_k)$, Q_A is defined in (4.3.17), and $\upsilon > 0$ is the step size.

Proposition 8. For small enough υ, the map (4.3.24) is a contraction. Moreover, if there exists a solution $(\tilde{\theta}, \tilde{u})$ of

$$\begin{cases} -\frac{\tilde{\theta}^i_{n+1} - \tilde{\theta}^i_n}{\delta t} + f(\tilde{u}^i_{n+1}, \tilde{\theta}^i_{n+1}) = 0 \\ -\frac{\tilde{u}^i_{n+1} - \tilde{u}^i_n}{\delta t} - h(\tilde{u}^i_n, \tilde{\theta}^i_n) = 0 \end{cases} \tag{4.3.25}$$

satisfying the initial-terminal conditions $\tilde{\theta}_0 = \bar{\theta}_0$ and $\tilde{u}^N = \bar{u}_T$, the iterates of (4.3.24) satisfy

$$\sum_{n=1}^{N-1} \gamma \|(\theta_{n,k} - \tilde{\theta}_{n,k})\|^2 + \sum_{i=1}^{d} \gamma_i(\theta^i_{n,k} + \tilde{\theta}^i_{n,k}) \|(\Delta_i u_{n,k} - \Delta_i \tilde{u}_{n,k})\|^2 \to 0,$$

as $k \to \infty$.

Proof. The operator E_υ is a contraction because Q_A is a monotone Lipschitz map (see Proposition 7). The convergence in the statement follows from the series

$$\sum_{k=1}^{\infty} \sum_{n=1}^{N-1} \gamma \|(\theta_{n,k} - \tilde{\theta}_{n,k})\|^2 + \sum_{i=1}^{d} \gamma_i(\theta^i_{n,k} + \tilde{\theta}^i_{n,k}) \|(\Delta_i u_{n,k} - \Delta_i \tilde{u}_{n,k})\|^2,$$

being convergent. □

Proposition 9. Let $(\bar{\theta}, \bar{u}) \in \mathcal{M}_N$ solve

$$A^N \begin{bmatrix} \theta \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with $u_N = \bar{u}_T$ and $\theta_0 = \bar{\theta}_0$. Then, $(\bar{\theta}, \bar{u})$ is a fixed point of (4.3.24).

Conversely, let $(\tilde{\theta}, \tilde{u}) \in \mathcal{M}_N$ be a fixed point of (4.3.24) with $\tilde{\theta} > 0$. Then, there exists a solution to (4.3.25), $(\bar{\theta}, \bar{u})$, with $\bar{\theta} = \tilde{\theta}$ and $\bar{u}$ given by

$$\frac{\delta \bar{u}^i_n}{\delta t} = -h(\Delta_i \tilde{u}, \theta, i) \tag{4.3.26}$$

with $\bar{u}_N = \bar{u}_T$.

Proof. The first claim of the proposition follows immediately from the definition of Q_A. To prove the second part, let $(\tilde{\theta}, \tilde{u}) \in \mathcal{M}_N$ be a fixed point of (4.3.24). For all $n \in \{0, \ldots, N\}$ and $i \in I_d$, we have

$$\tilde{u}^i_n = \tilde{u}^i_n + \upsilon \phi_n(\tilde{u}_n, \tilde{\theta}_n).$$

Therefore, $\phi_n(\tilde{u}_n, \tilde{\theta}_n) = 0$. Hence, from (4.3.13), we conclude that

$$-\frac{\delta\tilde{\theta}_{n-1}}{\delta t} + f(\tilde{u}_n, \tilde{\theta}_n) = 0.$$

Furthermore, for $\tilde{\theta}_n^i = \lambda_n^i$, where λ_n^i solves (4.3.23), we have

$$\begin{aligned}\tilde{\theta}_n^i &= P\left[\tilde{\theta}_n^i - \upsilon\psi_n(\tilde{u}_n, \tilde{\theta}_n)\right]\\ &= \left(\tilde{\theta}_n^i - \upsilon\psi_n(\tilde{u}_n, \tilde{\theta}_n) + \upsilon\kappa_n\right)^+\end{aligned}$$

for some $\kappa_n \geq 0$. If $\tilde{\theta}_n^i > 0$, $\psi_n(\tilde{u}_n, \tilde{\theta}_n) = \kappa_n$. Otherwise, $\psi_n(\tilde{u}_n, \tilde{\theta}_n) \geq \kappa_n$.

If $\tilde{\theta}_n^i > 0$, using the fact that ψ solves (4.3.15), we gather

$$\frac{\delta\tilde{u}_n^i}{\delta t} - \frac{1}{d-1}\sum_{j\neq i}\frac{\delta\tilde{u}_n^j}{\delta t} = \frac{1}{d-1}\sum_{j\neq i} h(\Delta_j\tilde{u}_n, \theta, j) - h(\Delta_i\tilde{u}_n, \theta, i).$$

Now, we define $\bar{u}$ as in the statement of the proposition. A simple computation gives

$$\frac{\delta\bar{u}_n^i}{\delta t} - \frac{\delta\bar{u}_n^j}{\delta t} = \frac{\delta\tilde{u}_n^i}{\delta t} - \frac{\delta\tilde{u}_n^j}{\delta t}.$$

Hence, $\Delta_j\bar{u}_n = \Delta_j\tilde{u}_n$. Consequently,

$$\frac{\delta\bar{u}_n^i}{\delta t} = -h(\Delta_i\bar{u}, \theta, i).$$

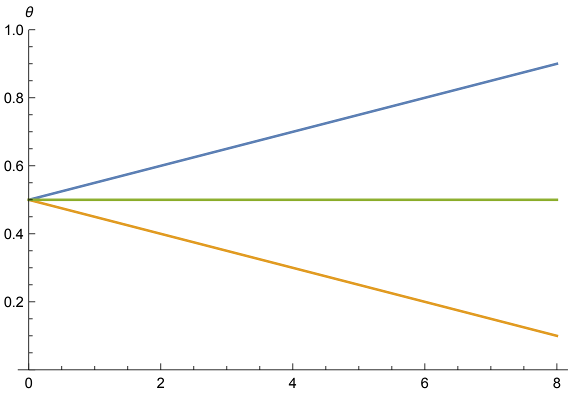
Thus, $(\bar{\theta}, \bar{u})$ solves (4.0.2). □

Remark 3. The convergence of solutions of (4.3.25) to weak solutions of (4.0.2) follows from the Minty's method and the monotonicity of the operator A as shown in Proposition 6.

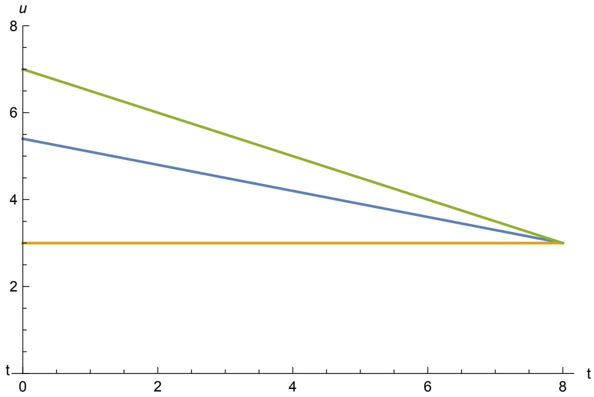
4.3.7 Numerical examples

Finally, we present numerical simulations for the time-dependent paradigm-shift problem. As explained before, we discretize the time variable, $t \in [0, T]$, into N intervals of length $\delta t = \frac{T}{N}$. We then have N equations for each state. Because $d = 2$, this system consists of $4N$ evolving equations according to (4.3.24).

To compute approximate solutions to (4.1.9)-(4.1.10), we use the projection algorithm, (4.3.24), with $N = 400$. We first consider a case in which the analytical solution can be computed explicitly. We choose $\theta^1 = \theta^2 = \frac{1}{2}$. Thus, from (4.1.9), it follows that $u^1 = u^2$ are affine functions of t with $u_t^1 = u_t^2 - \frac{1}{2}$. Our results are depicted in Figures 4.5, 4.6, and 4.7. In Figure 4.5, for $t \in [0, T]$, $T = 8$, we plot the initial guess ($s = 0$) for θ and u, and the analytical solution. In Figure 4.6, we see the evolution of the density of players and the value functions for $s \in [0, 20]$. The final results, $s = 20$, are shown in Figure 4.7. Finally, in Figure



(a) Initial condition $\theta(\cdot, 0)$ versus exact solution.



(b) Initial condition $u(\cdot, 0)$ versus exact solution.

Fig. 4.5 The blue lines correspond to the initial values ($s = 0$) for state 1, (θ^1, u^1): the orange lines correspond to the initial values for state 2, (θ^2, u^2); the green lines correspond to the analytical solution $\theta^1 = \theta^2$ and $u^1 = u^2$ for $t \in [0, 8]$.

4.8, we show the evolution of the H^1 norm of the difference between the analytical, $(\tilde{u}, \tilde{\theta})$, and computed, (u, θ), solutions. The norm $\|(\tilde{u}, \tilde{\theta}) - (u, \theta)\|^2_{H^1([0,T])}(s)$ is computed as

$$\sum_{j=0}^{N-1} \sum_{i=1}^{2} \delta t \left(|\tilde{u}^i_j - u^i_j|^2 + |\dot{\tilde{u}}^i_j - \dot{u}^i_j|^2 + |\tilde{\theta}^i_j - \theta^i_j|^2 + |\dot{\tilde{\theta}}^i_j - \dot{\theta}^i_j|^2 \right) (s)$$

for $s \geq 0$, where $v^i_j = v^i(t_j, s)$ and δt is the size of the time-discretization step.

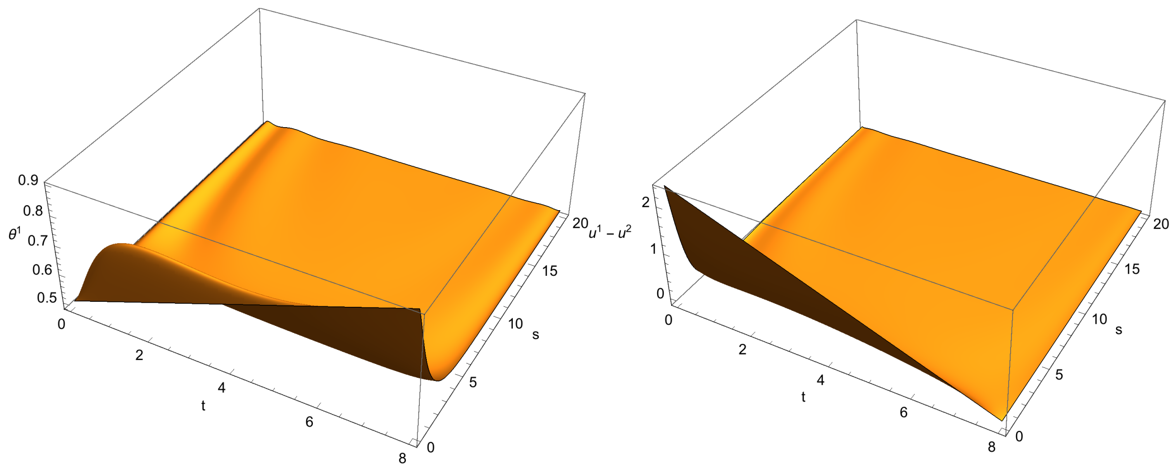
The paradigm-shift problem is a potential MFG with the Hamiltonian corresponding to

$$\tilde{h}(\Delta_i u, i) = -\frac{1}{2}((u^i - u^j)^+)^2, \text{ and } F(\theta) = \frac{\theta_1^2 + \theta_2^2}{2}$$

in (4.1.5). Thus, as a final test to our numerical method, we investigate the evolution of the Hamiltonian. In this case, as expected, the Hamiltonian converges to a constant (see Figure 4.9).

In the preceding example, while iterating (4.3.24), θ remains away from 0. In the next example, we consider a problem in which, without the projection P in (4.3.24), positivity is not preserved. We set $N = 400$ and choose initial conditions as in Figure 4.10. In Figure 4.11, we show the evolution by (4.3.24) for $s \in [0, 20]$. In Figure 4.12, we see the final result for $s = 20$. Finally, in Figure 4.13, we show the evolution of the H^1 norm of the difference $\|(\tilde{u}, \tilde{\theta}) - (u, \theta)\|^2_{H^1([0,T])}(s)$ for $s \in [0, 20]$.

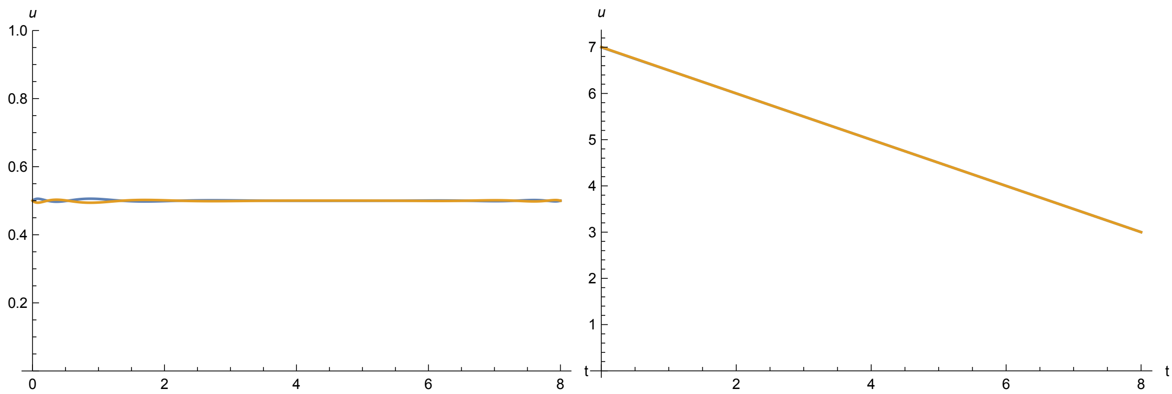
In Figure 4.14, we plot the evolution of the Hamiltonian determined using the projection method. Again, we obtain the numerical conservation of the Hamiltonian.



(a) Distribution of players $\theta^1(t, s)$.

(b) Difference $(u^1 - u^2)(t, s)$.

Fig. 4.6 Evolution, along parameter $s \in [0, 20]$, of the density of distribution of players, $\theta(\cdot, s)$, and the difference of the value functions for both sates, $(u^1 - u^2)(\cdot, s)$.



(a) Final distribution $\theta(\cdot, 20)$.

(b) Final value function $u(\cdot, 20)$.

Fig. 4.7 Final value of $u(\cdot, s)$ and $\theta(\cdot, s)$ for $s = 20$. Note that the quantities for both states superpose.

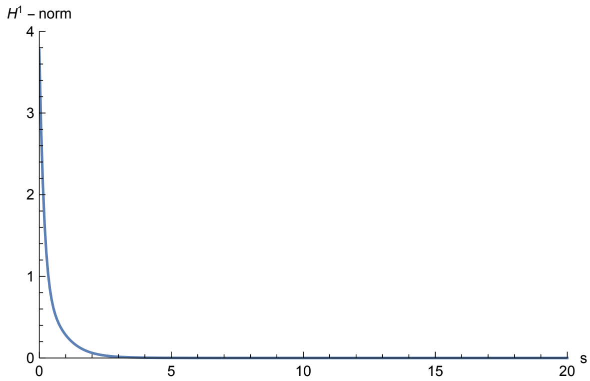


Fig. 4.8 Evolution, with the parameter s, of the H^1-norm of the difference between the computed solution $(u, \theta)(\cdot, s)$ and the analytical solution for the unconstrained probability case.

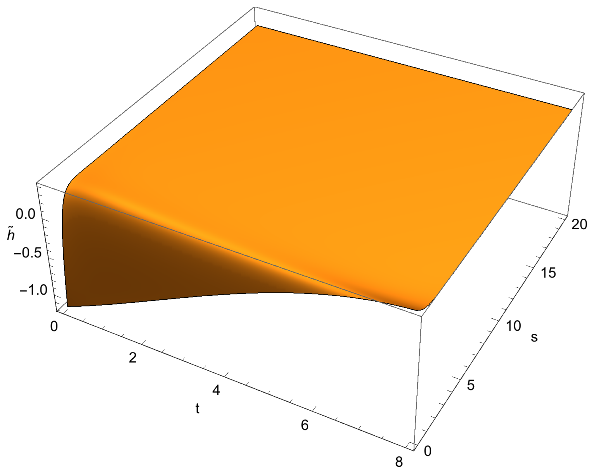
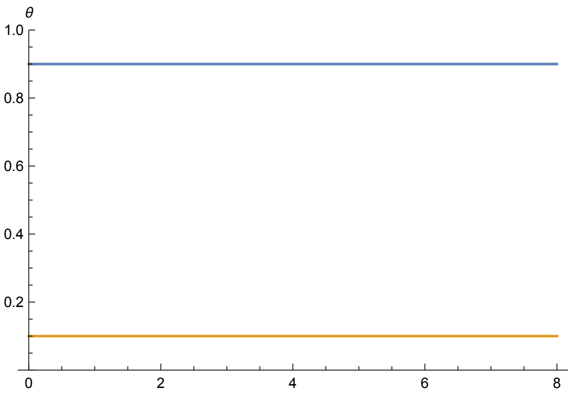
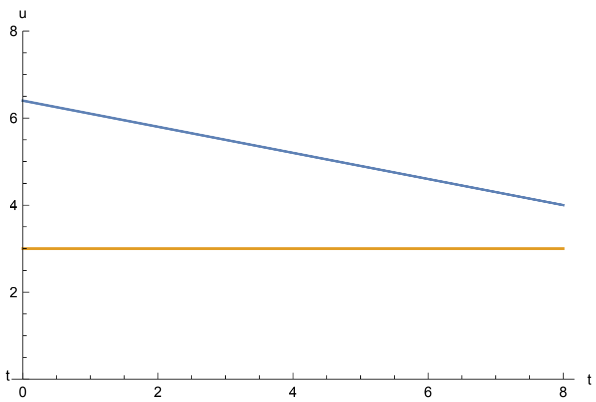


Fig. 4.9 Evolution of the Hamiltonian for $s \in [0, 20]$.

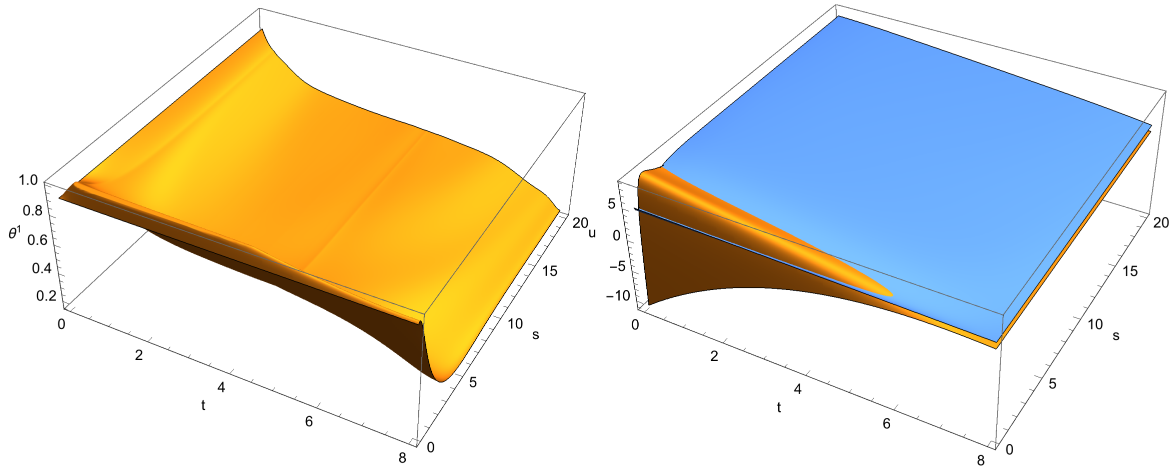


(a) Initial condition $\theta(\cdot, s = 0)$.



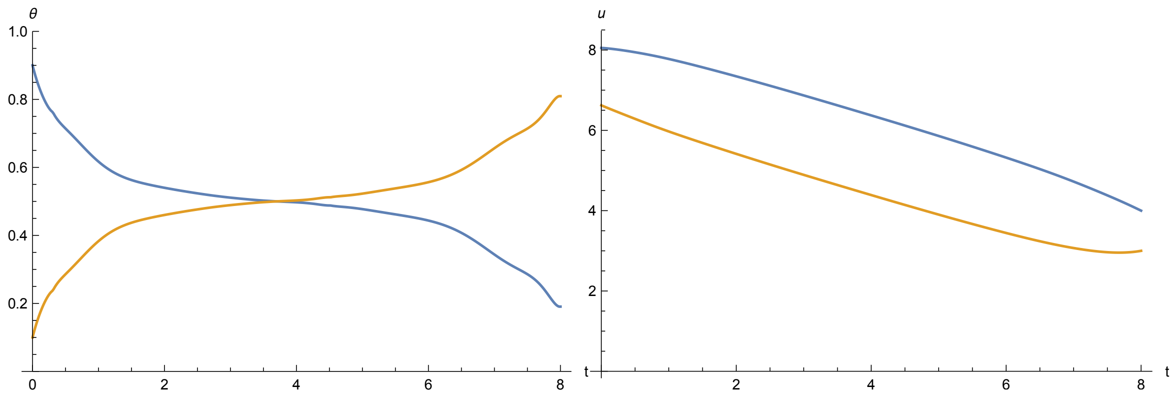
(b) Initial condition $u(\cdot, s = 0)$.

Fig. 4.10 The blue lines correspond to the initial values $(s = 0)$ for state 1, (θ^1, u^1); the orange lines correspond to the initial values for state 2, (θ^2, u^2).



(a) Distribution of players θ^1. (b) Value functions u^1 and u^2.

Fig. 4.11 Evolution of $u(\cdot, s)$ and $\theta(\cdot, s)$, for $s \in [0, 20]$. The quantities for state 1 and 2 are depicted in blue and orange, respectively.



(a) Distribution of players $\theta(\cdot, 20)$. (b) Value functions $u(\cdot, 20)$.

Fig. 4.12 Final value of $u(\cdot, s)$ and distribution $\theta(\cdot, s)$, at $s = 20$. The quantities for state 1 are depicted in blue and for state 2 in orange.

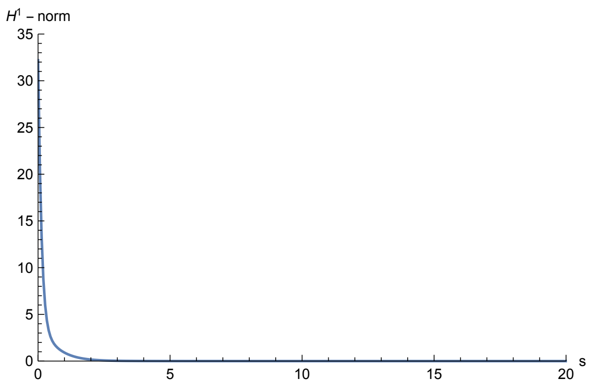


Fig. 4.13 Evolution, with respect to the parameter s, of the H^1-norm of the difference of the solution $(u, \theta)(\cdot, s)$ and the solution obtained at $s = 20$: $\|(u, \theta)(\cdot, s) - (u, \theta)(\cdot, 20)\|_{H^1}$.

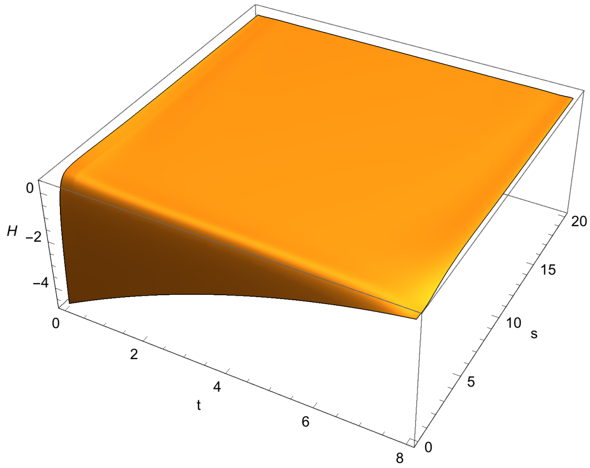


Fig. 4.14 Evolution of the Hamiltonian with the s-dynamics that preserves the probability and the positivity of the distribution of players.

4.4 Conclusions

As the examples in the preceding sections illustrate, we have developed an effective method for the numerical approximation of monotonic finite-state MFGs. As observed previously, [10, 1, 6, 3], monotonicity properties are essential for the construction of effective numerical methods to solve MFGs and were used explicitly in [12]. Here, in contrast with earlier approaches, we do not use a Newton-type iteration as in [1, 6] nor do we require the solution of the master equation as in [101, 102]. While for $d = 2$, the master equation can be handled numerically as in the preceding references, this approach becomes numerically prohibitive when there is a large number of states. The master equation determines the value function $U(\theta, i, t)$, where $\theta \in \mathcal{S}^d$. A direct approach to the master equation requires a grid in $\mathcal{S}^d$, or equivalently in a subset of $\mathbb{R}^{d-1}$. However, when d is moderately large, a direct approach requires the storage of an extremely large number of points. With our approach, we only need $2d$ values for each time step. The key contribution of this chapter is the projection method that makes addressing the initial-terminal value problem possible. This was an open problem since the introduction of monotonicity-based methods in [12]. Our methods can be applied to discretizing continuous-state MFGs, and we foresee additional extensions. The first one concerns the planning problem considered in [3]. A second extension regards boundary value problems, which are natural in many applications of MFGs. Finally, our methods may be improved by using higher-order integrators in time, provided that monotonicity is preserved. These matters will be the subject of future research.

Chapter 5

Nonlocal mean-field games

Here, we discuss numerical methods for mean-field games systems with nonlocal dependence on the measure (distribution of the players).

5.1 Introduction

We start by introducing the Fourier approximation techniques for *first-order nonlocal MFG models*. More precisely, we consider the system

$$\begin{cases} -\partial_t u + H(x, \nabla u) = F[x, m], \\ \partial_t m - \operatorname{div}(m \nabla_p H(x, \nabla u)) = 0, \ (x,t) \in \mathbb{T}^d \times [0,1], \\ u(x,1) = U(x), \ m(x,0) = M(x), \quad x \in \mathbb{T}^d. \end{cases} \tag{5.1.1}$$

Here, $u : \mathbb{T}^d \times [0,1] \to \mathbb{R}$ and $m : \mathbb{T}^d \times [0,1] \to \mathbb{R}_+$ are the unknown functions. Furthermore, $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$ is the Hamiltonian, and $F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is a *nonlocal coupling* term between the Hamilton-Jacobi and Fokker-Planck equations. Next, $U \in C^2(\mathbb{T}^d)$ and $M \in L^\infty(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$ (with a slight abuse of notation we identify the absolutely continuous measures with their densities) are terminal-initial conditions for u and m, respectively.

In (5.1.1), u is the *value function* of a generic agent from a continuum population of players, whereas m represents the *density* of this population. Each agent aims at solving the optimization problem

$$u(x,t) = \inf_{\gamma \in H^1([t,1]), \gamma(t)=x} \int_t^1 L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(\cdot, s)) ds + U(\gamma(1)), \tag{5.1.2}$$

where U is a terminal cost function and L is the Legendre transform of H; that is,

$$L(x,v) = \sup_p -v \cdot p - H(x,p), \ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

Since the contribution of a generic agent is small, its actions on the population distribution can be neglected. We assume that m is fixed, but unknown, in (5.1.2). Consequently, u must solve a Hamilton-Jacobi equation; that is, the first PDE in (5.1.1) with terminal data U.

Furthermore, given u, the optimal trajectories of agents are determined by

$$\dot{\gamma}(s) = -\nabla_p H(\gamma(s), \nabla u(\gamma(s), s)).$$

Therefore, m, being the population density, must satisfy the Fokker-Planck equation; that is, the second PDE in (5.1.1) with initial data M, the *population density* at time $t = 0$.

The existence, uniqueness and stability theories for (5.1.1) are well understood [135, 57, 56]. Here, we are specifically interested in approximation methods for the solutions of (5.1.1) that can be useful for numerical solution and modeling purposes.

Currently, there are number of efficient approximation methods for solutions of MFG systems. We refer to [2, 5, 11, 7] for finite-difference schemes, [29, 16, 31, 30, 40] for convex optimization techniques and [13, 106] for monotone flows. Although general, the aforementioned methods are particularly advantageous when F in (5.1.1) depends locally on m. The reason is that the local coupling F yield analytic pointwise formulas for infinite-dimensional operators involved in the algorithms. On the other hand, the nonlocal coupling case, F, do not yield such formulas. Hence, we are interested in developing approximation methods that specifically suit nonlocal coupling case, F.

Our approach is based on a Fourier approximation of F and is inspired by the methods in [153]. Here, we use the classical trigonometric polynomials as an approximation basis. Nevertheless, our method is flexible and allows more general bases. For instance, one may consider (5.1.1) on different domains and boundary conditions and choose a basis accordingly.

Additionally, our approach yields a mesh-free numerical approximation of u and m. More precisely, we directly recover the optimal trajectories of the agents rather than the values of u and m on a given mesh. In particular, our methods may blend well with recently developed ideas for fast and curse-of-the-dimensionality-resistant solution approach for first-order Hamilton-Jacobi equations [70, 139]. Hence, our techniques may lead to numerical schemes for nonlocal MFG that are efficient in high dimensions.

To avoid technicalities, we consider a *linear nonlocal coupling*, F. More precisely, we assume that

$$F(x, m) = \int_{\mathbb{T}^d} K(x, y) m(y, t) dy, \; x \in \mathbb{T}^d, \; m \in \mathcal{P}(\mathbb{T}^d),$$

where the kernel is twice differentiable $K \in C^2(\mathbb{T}^d \times \mathbb{T}^d)$. Thus, for this particular coupling, we deal with the following system

$$\begin{cases} -\partial_t u + H(x, \nabla u) = \fint_{\mathbb{T}^d} K(x,y) m(y,t) dy, \\ \partial_t m - \operatorname{div}(m \nabla_p H(x, \nabla u)) = 0, & (x,t) \in \mathbb{T}^d \times [0,1], \\ m(x,0) = M(x), \ u(x,1) = U(x), & x \in \mathbb{T}^d. \end{cases} \tag{5.1.3}$$

Our basic idea is to show that when K is a generalized polynomial in a given basis, then (5.1.3) is equivalent to a fixed point problem, in a space of continuous curves, that has good structural properties. In particular, when K is symmetric and positive semi-definite, (5.1.3) is equivalent to a convex optimization problem in the space of continuous curves.

Furthermore, we discuss how to construct generalized polynomial kernels that approximate a given K. Additionally, we observe that for translation invariant K the approximating kernels have a particularly simple structure. Consequently, for such K the aforementioned optimization problem is much simpler to solve.

This chapter is organized as follows. In Section 5.2, we present standing assumptions and some preliminary results. In Section 5.3, we prove the equivalence of (5.1.3) to a fixed point problem over the space of continuous curves when K is a generalized polynomial. Next, in Section 5.4, we discuss approximation methods for a general kernel. In Section 5.5, we construct a discretization for the optimization problem from Section 5.3 and devise a variant of a primal dual hybrid gradient algorithm for the discrete problem. Finally, in Section 5.6, we study several numerical examples.

5.2 Assumptions and preliminary results

We denote by $\mathbb{T}^d$ the d-dimensional flat torus. Furthermore, throughout the chapter, we assume that $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$, and

$$\begin{aligned} \frac{1}{C} I_d \leq& \nabla^2_{pp} H(x,p) \leq C I_d, \ \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d, \\ -C(1+|p|^2) \leq& \nabla_x H(x,p) \cdot p, \ \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d, \end{aligned} \tag{5.2.1}$$

for some constant $C > 0$. We denote by $\mathcal{P}(\mathbb{T}^d)$ the space of Borel probability measures on the d-dimensional torus, $\mathbb{T}^d$. Next, we assume that $M \in L^\infty(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$ is an essentially bounded probability measure, $U \in C^2(\mathbb{T}^d)$ a twice continuously differentiable function, $K \in C^2(\mathbb{T}^d \times \mathbb{T}^d)$ a twice differentiable kernel, and

$$\|M\|_{L^\infty(\mathbb{T}^d)}, \ \|U\|_{C^2(\mathbb{T}^d)}, \ \|K\|_{C^2(\mathbb{T}^d \times \mathbb{T}^d)} \leq C, \tag{5.2.2}$$

bounded by some constant $C > 0$. Additionally, we suppose that the kernel K is positive semi-definite; that is,

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} K(x, y) f(x) f(y) dx dy \geq 0, \ \forall f \in L^\infty(\mathbb{T}^d). \tag{5.2.3}$$

The kernel K is symmetric if

$$K(x, y) = K(y, x), \ \forall x, y \in \mathbb{T}^d. \tag{5.2.4}$$

We equip $\mathcal{P}(\mathbb{T}^d)$ with the *Monge-Kantorovich distance* that is given by

$$\|m_2 - m_1\|_{MK} = \sup \left\{ \int_{\mathbb{T}^d} \phi(x)(m_2(x) - m_1(x)) dx \text{ s.t. } \|\phi\|_{\text{Lip}} \leq 1 \right\}. \tag{5.2.5}$$

Now, we introduce the definition of the generalized differential from non-smooth analysis. Let v be a real-valued function defined on an open set $\Omega \in \mathbb{R}^d$. For any $x \in \mathbb{R}^d$, the *Frèchet super-differential and sub-differential* of v at x are given by the sets

$$\nabla^- v(x) = \left\{ p \in \mathbb{R}^d : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\},$$
$$\nabla^+ v(x) = \left\{ p \in \mathbb{R}^d : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\},$$

respectively. If a function v is differentiable at a point $x \in \mathbb{R}^d$ then the above sets are singletons and coincide

$$\nabla v(x) = \nabla^- v(x) = \nabla^+ v(x).$$

In the rest of this section, we present some preliminary results and formulas. For the optimal control and related Hamilton-Jacobi equations theory we refer to [82, 21]. We begin by the defining a solution for (5.1.3).

Definition 5.2.1. A pair (u, m) is a solution of (5.1.3) if $u \in W^{1,\infty}(\mathbb{T}^d \times [0, 1])$ is a viscosity solution of

$$\begin{cases} -\partial_t u + H(x, \nabla u) = \int_{\mathbb{T}^d} K(x, y) m(y, t) dy, & (x, t) \in \mathbb{T}^d \times [0, 1]), \\ u(x, 1) = U(x), & x \in \mathbb{T}^d, \end{cases} \tag{5.2.6}$$

and $m \in L^\infty(\mathbb{T}^d \times [0, 1]) \cap C\left([0, 1]; \mathcal{P}(\mathbb{T}^d)\right)$ is a distributional solution of

$$\begin{cases} \partial_t m - \operatorname{div}(m \nabla_p H(x, \nabla u)) = 0, \ (x, t) \in \mathbb{T}^d \times [0, 1], \\ m(x, 0) = M(x), \ x \in \mathbb{T}^d. \end{cases} \tag{5.2.7}$$

The following theorem asserts that (5.1.3) is well-posed.

Theorem 5.2.2. *i) Under assumptions* (5.2.1) *and* (5.2.2)*, system* (5.1.3) *admits a solution* (u, m)*. Moreover, there exists a constant* $C_1(C) > 0$ *such that*

$$\nabla^2_{xx} u,\ \|u\|_{W^{1,\infty}},\ \|m\|_{L^\infty} \leq C_1, \tag{5.2.8}$$

for any solution (u, m)*. Additionally, if* K *is positive semi-definite,* i.e. (5.2.3) *holds, then* (u, m) *is unique.*

ii) Solutions of (5.1.3) *are stable with respect to variations of* U, M *and* K *in respective norms. Particularly, suppose that* $\{K_r\}_{r=1}^{\infty} \subset C^2(\mathbb{T}^d \times \mathbb{T}^d)$ *is such that*

$$\lim_{r\to\infty} \|K - K_r\|_{C^2(\mathbb{T}^d\times\mathbb{T}^d)} = 0, \tag{5.2.9}$$

and $\{(u_r, m_r)\}_{r=1}^{\infty}$ *are solutions of* (5.1.3) *corresponding to kernel* K_r*. Then, the sequence* $\{(u_r, m_r)\}_{r=1}^{\infty}$ *is precompact in* $C(\mathbb{T}^d \times [0, 1]) \times C\left([0, 1]; \mathcal{P}(\mathbb{T}^d)\right)$ *with all accumulation points being solutions of* (5.1.3)*. Consequently, if* (5.2.3) *holds then*

$$\begin{aligned} &\lim_{r\to\infty} u_r(x,t) = u(x,t),\ \textit{uniformly in } (x,t) \in \mathbb{T}^d \times [0,1],\\ &\lim_{r\to\infty} \|m_r(\cdot,t) - m(\cdot,t)\|_{MK} = 0,\ \textit{uniformly in } t \in [0,1], \end{aligned} \tag{5.2.10}$$

where (u, m) *is the unique solution of* (5.1.3)*.*

Proof. See [135, 57, 56]. □

Next, consider an arbitrary basis of smooth functions

$$\Phi = \{\phi_1, \phi_2, \cdots, \phi_r\} \subset C^2(\mathbb{T}^d). \tag{5.2.11}$$

For $a = (a_1, a_2, \cdots, a_r) \in C([0, 1]; \mathbb{R}^r)$ we denote by u_a the viscosity solution of

$$\begin{cases} -\partial_t u(x,t) + H(x, \nabla u(x,t)) = \sum\limits_{i=1}^{r} a_i(t)\phi_i(x),\ (x,t) \in \mathbb{T}^d \times [0,1]\\ u(x,1) = U(x),\ x \in \mathbb{T}^d. \end{cases} \tag{5.2.12}$$

From the optimal control theory, we have that

$$u_a(x,t) = \inf_{\gamma\in H^1([t,1]),\gamma(t)=x} \int_t^1 \left(L\left(\gamma(s), \dot\gamma(s)\right) + \sum_{i=1}^{r} a_i(s)\phi_i(\gamma(s)) \right) ds + U(\gamma(1)), \tag{5.2.13}$$

for all $(x, t) \in \mathbb{T}^d \times [0, 1]$, where γ is an differentiable trajectory in $(t, 1]$ starting at $\gamma(t) = x$, and

$$L(x, v) = \sup_{p\in\mathbb{R}^d} -v \cdot p - H(x, p). \tag{5.2.14}$$

Moreover, for all $(x,t)\in\mathbb{T}^d\times[0,1]$ there exists $\gamma_{x,t,a}\in C^2([t,1];\mathbb{T}^d)$ such that

$$u_a(x,t)=\int_t^1\left(L\left(\gamma_{x,t,a}(s),\dot\gamma_{x,t,a}(s)\right)+\sum_{i=1}^r a_i(s)\phi_i(\gamma_{x,t,a}(s))\right)ds+U(\gamma_{x,t,a}(1)),\tag{5.2.15}$$

and satisfies the Euler-Lagrange equation:

$$\begin{aligned}&\frac{d}{ds}\nabla_v L\left(\gamma_{x,t,a}(s),\dot\gamma_{x,t,a}(s)\right)\\ =&\nabla_x L\left(\gamma_{x,t,a}(s),\dot\gamma_{x,t,a}(s)\right)+\sum_{i=1}^r a_i(t)\nabla\phi_i(\gamma_{x,t,a}(s)),\ \ s\in[t,1].\end{aligned}\tag{5.2.16}$$

Additionally, we have

$$\begin{aligned}-\nabla_v L\left(x,\dot\gamma_{x,t,a}(t)\right)\in&\nabla_x^+u_a(x,t),\\ -\nabla_v L\left(\gamma_{x,t,a}(s),\dot\gamma_{x,t,a}(s)\right)=&\nabla_x u_a(\gamma_{x,t,a}(s),s),\ \ s\in(t,1],\\ \dot\gamma_{x,t,a}(s)=&\nabla_p H(\gamma_{x,t,a}(s),\nabla_x u_a(\gamma_{x,t,a}(s),s)),\ \ s\in(t,1].\end{aligned}\tag{5.2.17}$$

In fact, this previous equation is also sufficient for (5.2.15) to hold. For lighter notation, we denote $\gamma_{x,0,a}$ by $\gamma_{x,a}$.

In general, u_a is not everywhere differentiable. Nevertheless, u_a is semiconcave and hence $\nabla^+u_a(x,t)\neq\emptyset$ for all (x,t), and $\nabla^+u_a(x,t)=\{\nabla u_a(x,t)\}$ for a.e. (x,t). In fact, points (x,t) where u_a is not differentiable are precisely those for which (5.2.13) admits multiple minimizers. Thus, at points $x\in\mathbb{T}^d$ where $u_a(x,0)$ is not differentiable we choose $\gamma_{x,a}$ in such a way that the map $(x,t)\mapsto\gamma_{x,a}(t)$ is Borel measurable.

Furthermore, we denote by m_a the distributional solution of

$$\begin{cases}\partial_t m-\operatorname{div}\left(m\nabla_p H(x,\nabla u_a)\right)=0,\ \ (x,t)\in\mathbb{T}^d\times[0,1],\\ m(x,0)=M(x),\ \ x\in\mathbb{T}^d.\end{cases}\tag{5.2.18}$$

One can show that m_a is given by the *push-forward* of the measure M by the map $\gamma_{\cdot,a}(t)$

$$m_a(\cdot,t)=\gamma_{\cdot,a}(t)\sharp M,\tag{5.2.19}$$

where the above *push-forward* is a measure defined by $\gamma_{\cdot,a}(t)\sharp M(A)=M(\gamma_{\cdot,a}^{-1}(t)(A))$, for any Borel set $A\subset\mathbb{R}^d$.

We equip $C([0,1];\mathbb{R}^r)$ with the L^∞ norm

$$\|a\|_\infty=\max_i\sup_{t\in[0,1]}|a_i(t)|.$$

Then, one has that

$$\lim_{n\to\infty} \|m_{a_n}(\cdot,t) - m_a(\cdot,t)\|_{MK} = 0, \text{ uniformly in } t \in [0,1], \tag{5.2.20}$$

if $\lim\limits_{n\to\infty} \|a_n - a\|_\infty = 0$. For a detailed discussion on m_a see Chapter 4 in [57].

Finally, we denote by

$$G(a) = \int_{\mathbb{T}^d} u_a(x,0)M(x)dx, \ a \in C\left([0,1];\mathbb{R}^r\right), \tag{5.2.21}$$

the averaged value function at initial time $t = 0$.

Our first theorem addresses the properties of G.

Theorem 5.2.3. *The functional $a \mapsto G(a)$ is concave and everywhere Fréchet differentiable. Moreover,*

$$\partial_{a_i} G = \int_{\mathbb{T}^d} \phi_i(x)m_a(x,\cdot)dx, \ 1 \le i \le r. \tag{5.2.22}$$

Proof. We denote by

$$p(a) = \left(\int_{\mathbb{T}^d} \phi_i(x)m_a(x,\cdot)dx\right)_{i=1}^r, \ a \in C([0,1];\mathbb{R}^r).$$

We prove that for every $a \in C([0,1];\mathbb{R}^r)$

$$0 \ge G(b) - G(a) - (b-a)\cdot p(a) \ge o\left(\|b-a\|_\infty\right).$$

We have that

$$\begin{aligned}
&G(b) - G(a) - (b-a)\cdot p(a) \\
=& \int_{\mathbb{T}^d} \left[\int_0^1 \left(L\left(\gamma_{x,b}(t), \dot\gamma_{x,b}(t)\right) + \sum_{i=1}^r b_i(t)\phi_i(\gamma_{x,b}(t))\right) dt + U(\gamma_{x,b}(1))\right] M(x)dx \\
&- \int_{\mathbb{T}^d} \left[\int_0^1 \left(L\left(\gamma_{x,a}(t), \dot\gamma_{x,a}(t)\right) + \sum_{i=1}^r a_i(t)\phi_i(\gamma_{x,a}(t))\right) dt + U(\gamma_{x,a}(1))\right] M(x)dx \\
&- \sum_{i=1}^r \int_0^1 (b_i(t) - a_i(t))dt \int_{\mathbb{T}^d} \phi_i(x)m_a(x,t)dx.
\end{aligned}$$

From (5.2.19) we have that

$$\int_{\mathbb{T}^d} \phi_i(x)m_a(x,t)dx = \int_{\mathbb{T}^d} \phi_i(\gamma_{x,a}(t))M(x)dx, \ t \in [0,1], \ 1 \le i \le r.$$

Using the previous identify in the above equation we get

$$\begin{aligned}G(b)-G(a)-(b-a)\cdot p(a) &= \int_{\mathbb{T}^d} M(x)dx\int_0^1 L\left(\gamma_{x,b}(t),\dot\gamma_{x,b}(t)\right)-L\left(\gamma_{x,a}(t),\dot\gamma_{x,a}(t)\right)dt\\ &+\int_{\mathbb{T}^d} M(x)dx\int_0^1\sum_{i=1}^r b_i(t)(\phi_i(\gamma_{x,b}(t))-\phi_i(\gamma_{x,a}(t)))dt\\ &+\int_{\mathbb{T}^d} M(x)\left(U(\gamma_{x,b}(1))-U(\gamma_{x,a}(1))\right)dx.\end{aligned}$$

By definition, we have that

$$\begin{aligned}&\int_0^1 L\left(\gamma_{x,b}(t),\dot\gamma_{x,b}(t)\right)+\sum_{i=1}^r b_i(t)\phi_i(\gamma_{x,b}(t))dt+U(\gamma_{x,b}(1))\\ \leq&\int_0^1 L\left(\gamma_{x,a}(t),\dot\gamma_{x,a}(t)\right)+\sum_{i=1}^r b_i(t)\phi_i(\gamma_{x,a}(t))dt+U(\gamma_{x,a}(1)),\ \forall x\in\mathbb{T}^d.\end{aligned}$$

Hence,

$$G(b)-G(a)-(b-a)\cdot p(a)\leq 0,\ \forall a,b\in C([0,1];\mathbb{T}^d).$$

This previous inequality yields the concavity of G. On the other hand, we have that

$$\begin{aligned}G(b)-G(a)-(b-a)\cdot p(a) &= \int_{\mathbb{T}^d} M(x)dx\int_0^1 L\left(\gamma_{x,b}(t),\dot\gamma_{x,b}(t)\right)-L\left(\gamma_{x,a}(t),\dot\gamma_{x,a}(t)\right)dt\\ &+\int_{\mathbb{T}^d} M(x)dx\int_0^1\sum_{i=1}^r a_i(t)(\phi_i(\gamma_{x,b}(t))-\phi_i(\gamma_{x,a}(t)))dt\\ &+\int_{\mathbb{T}^d} M(x)\left(U(\gamma_{x,b}(1))-U(\gamma_{x,a}(1))\right)dx\\ &+\int_{\mathbb{T}^d} M(x)dx\int_0^1\sum_{i=1}^r (b_i(t)-a_i(t))(\phi_i(\gamma_{x,b}(t))-\phi_i(\gamma_{x,a}(t)))dt.\end{aligned}$$

Therefore, again by the definition of $\gamma_{x,a}$ and $\gamma_{x,b}$, we have that

$$\begin{aligned}&G(b)-G(a)-(b-a)\cdot p(a)\\ \geq&\int_{\mathbb{T}^d} M(x)dx\int_0^1\sum_{i=1}^r (b_i(t)-a_i(t))(\phi_i(\gamma_{x,b}(t))-\phi_i(\gamma_{x,a}(t)))dt\\ \geq&-\|b-a\|_\infty\sum_{i=1}^r\int_0^1\left|\int_{\mathbb{T}^d}\phi_i(\gamma_{x,b}(t))M(x)dx-\int_{\mathbb{T}^d}\phi_i(\gamma_{x,a}(t))M(x)dx\right|dt\\ =&-\|b-a\|_\infty\sum_{i=1}^r\int_0^1\left|\int_{\mathbb{T}^d}\phi_i(x)m_b(x,t)dx-\int_{\mathbb{T}^d}\phi_i(x)m_a(x,t)dx\right|dt\\ \geq&-\|b-a\|_\infty\sum_{i=1}^r \mathrm{Lip}(\phi_i)\int_0^1\|m_b(\cdot,t)-m_a(\cdot,t)\|_{MK}dt.\end{aligned}$$

Hence, by (5.2.20) the proof is complete. □

5.3 The optimization problem

In this section, we assume that K is a generalized polynomial in the basis Φ; that is,

$$K(x,y) = \sum_{i,j=1}^{r} k_{ij}\phi_i(x)\phi_j(y),\ x,y \in \mathbb{T}^d, \tag{5.3.1}$$

where $\mathbf{K} = (k_{ij})_{i,j=1}^r \in M_{r,r}(\mathbb{R})$ is a matrix of coefficients. For such K, (5.1.3) takes form

$$\begin{cases} -\partial_t u + H(x,\nabla u) = \sum\limits_{i=1}^{r} \phi_i(x) \sum\limits_{j=1}^{r} k_{ij} \int_{\mathbb{T}^d} \phi_j(y) m(y,t)dy, \\ \partial_t m - \operatorname{div}(m \nabla_p H(x,\nabla u)) = 0,\ (x,t) \in \mathbb{T}^d \times [0,1], \\ m(x,0) = M(x),\ u(x,1) = U(x),\ x \in \mathbb{T}^d. \end{cases} \tag{5.3.2}$$

Our main result is the following theorem.

Theorem 5.3.1. *i) A pair (u,m) is a solution of* (5.3.2) *if and only if $(u,m) = (u_{a^*}, m_{a^*})$ for some $a^* \in C([0,1];\mathbb{R}^r)$ such that*

$$a^* = \mathbf{K}\partial_a G(a^*). \tag{5.3.3}$$

ii) If $\mathbf{K}$ is positive-definite then (5.3.3) *is equivalent to finding a* 0 *of a monotone operator $a \mapsto \mathbf{K}^{-1}a - \partial_a G(a),\ a \in C([0,1];\mathbb{R}^r)$.*

iii) Additionally, if $\mathbf{K}$ is symmetric, (5.3.3) *is equivalent to the convex optimization problem*

$$\begin{aligned} &\inf_{a \in C([0,1];\mathbb{R}^r)} \frac{1}{2}\langle \mathbf{K}^{-1}a, a\rangle - G(a) \\ =& \inf_{a \in C([0,1];\mathbb{R}^r)} \frac{1}{2}\langle \mathbf{K}^{-1}a, a\rangle - \int_{\mathbb{T}^d} u_a(x,0)M(x)dx. \end{aligned} \tag{5.3.4}$$

Proof. Items *ii)* and *iii)* follow immediately from *i)* by the concavity of G. Thus, we just prove *i)*.

By Theorem 5.2.2 the system (5.3.2) admits a solution (u,m). We define a^* as

$$a_i^*(t) = \sum_{j=1}^{r} k_{ij} \int_{\mathbb{T}^d} \phi_j(y)m(y,t)dy,\ t \in [0,1]. \tag{5.3.5}$$

Then $a^* \in C([0,1];\mathbb{R}^r)$, and by the definition of u_a and m_a we have that $(u,m) = (u_{a^*}, m_{a^*})$. Hence, by Theorem 5.2.3, we have that

$$\partial_{a_i} G(a^*) = \int_{\mathbb{T}^d} \phi_i(x)m(x,\cdot)dx,\ 1 \le i \le r.$$

Consequently, from (5.3.5) we obtain

$$a_i^* = \sum_{j=1}^{r} k_{ij} \partial_{a_j} G(a^*).$$

□

Remark 5.3.2. The optimization problem (5.3.4) is equivalent to the optimal control of Hamilton-Jacobi PDE pointed out in [135] (equation (59) in Section 2.6). One can think of (5.3.4) as the aforementioned problem written in Fourier coordinates.

5.4 Approximating the kernel

In this section, we show that we can construct suitable approximations for an arbitrary K. We begin by a the following lemma.

Lemma 5.4.1. *Suppose that K is given by* (5.3.1)*. Then K is positive semi-definite if and only if $\mathbf{K} = (k_{ij})_{ij,=1}^r$ is positive semi-definite.*

Proof. Fix an arbitrary vector $(\xi_i)_{i=1}^r \in \mathbb{R}^r$. Then there exists a unique vector $(\lambda_i)_{i=1}^r \in \mathbb{R}^r$ such that

$$\xi_i = \sum_{j=1}^{r} \lambda_j \int_{\mathbb{T}^d} \phi_i(x)\phi_j(x)dx,\ 1 \leq i \leq r,$$

because $\{\phi_i\}$ are linearly independent. Therefore, for

$$f = \sum_{j=1}^{r} \lambda_j \phi_j$$

we have that

$$\xi_i = \int_{\mathbb{T}^d} f(x)\phi_i(x)dx,\ 1 \leq i \leq r.$$

Hence,

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} K(x,y) f(x) f(y) dx dy = \sum_{i,j=1}^{r} k_{ij} \xi_i \xi_j,$$

that yields the proof. □

Now, we now choose a trigonometric basis:

$$\phi_\alpha(x) = e^{2i\pi\alpha\cdot x},\ x \in \mathbb{T}^d,\ \alpha \in \mathbb{Z}^d. \tag{5.4.1}$$

For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}^d$, we denote by

$$|\alpha| = (|\alpha_1|, |\alpha_2|, \cdots, |\alpha_d|),$$

and for $\alpha, r \in \mathbb{Z}^d$

$$\alpha \leq r \iff \alpha_j \leq r_j,\ 1 \leq j \leq d.$$

For $r_1, r_2 \in \mathbb{N}_0^d$ we denote by

$$K_{r_1 r_2}(x, y) = \sum_{|\alpha| \leq r_1, |\beta| \leq r_2} \hat{K}_{\alpha\beta} e^{2i\pi(\alpha \cdot x + \beta \cdot y)},\ x, y \in \mathbb{T}^d,$$

the *rectangular partial Fourier sum* of K, where

$$\hat{K}_{\alpha\beta} = \int_{\mathbb{T}^d} K(x, y) e^{-2i\pi(\alpha \cdot x + \beta \cdot y)} dx dy,\ \alpha, \beta \in \mathbb{Z}^d.$$

Furthermore, for $r_1, r_2 \in \mathbb{N}_0^d$ we denote by

$$\Sigma_{r_1 r_2}(x, y) = \frac{1}{\prod_{j=1}^d (1 + r_{1j})(1 + r_{2j})} \sum_{|\alpha| \leq r_1, |\beta| \leq r_2} K_{r_1 r_2}(x, y),\ x, y \in \mathbb{T}^d,$$

the *rectangular Fejér average* of K.

Remark 5.4.2. If K is real valued then $K_{r_1 r_2}$ and $\Sigma_{r_1 r_2}$ are real valued for any $r_1, r_2 \in \mathbb{N}_0^d$.

Proposition 5.4.3. *If K is positive semi-definite (symmetric) then, K_{rr} and Σ_{rr} are also positive semi-definite (symmetric) for all $r \in \mathbb{N}_0^d$. Moreover,*

$$\lim_{\min_j r_j \to \infty} \|\Sigma_{rr} - K\|_{C^2(\mathbb{T}^d \times \mathbb{T}^d)} = 0, \tag{5.4.2}$$

Additionally, if $K \in C^3(\mathbb{T}^d \times \mathbb{T}^d)$ then

$$\lim_{\min_j r_j \to \infty} \|K_{rr} - K\|_{C^2(\mathbb{T}^d \times \mathbb{T}^d)} = 0. \tag{5.4.3}$$

Proof. The convergence properties (5.4.2), (5.4.3) are classical results in Fourier analysis. Thus, we will just prove that K_{rr} and Σ_{rr} are positive semi-definite (symmetric). For that, we use the representation formulas

$$\begin{aligned} K_{rr}(x, y) &= \int_{\mathbb{T}^d \times \mathbb{T}^d} K(z, w) D_{rr}(x - z, y - w) dz dw, \\ \Sigma_{rr}(x, y) &= \int_{\mathbb{T}^d \times \mathbb{T}^d} K(z, w) F_{rr}(x - z, y - w) dz dw,\ x, y \in \mathbb{T}^d, \end{aligned}$$

where D_{rr} and F_{rr} are, respectively, the $2d$*-dimensional rectangular Dirichlet and Fejér kernels*. A crucial feature of D_{rr} and F_{rr} is that they are symmetric and decompose into lower dimensional kernels:

$$D_{rr}(z, w) = D_r(z) D_r(w),\ F_{rr}(z, w) = F_r(z) F_r(w),\ z, w \in \mathbb{T}^d,$$

where D_r and F_r are the corresponding d-dimensional kernels. In particular, K_{rr}, Σ_{rr} are symmetric if K is such. Furthermore, for an arbitrary $f \in L^\infty(\mathbb{T}^d)$ we have that

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{T}^d} K_{rr}(x,y)f(x)f(y)dxdy\\
=&\int_{\mathbb{T}^d\times\mathbb{T}^d} K(z,w)dzdw\int_{\mathbb{T}^d\times\mathbb{T}^d} f(x)f(y)D_{rr}(x-z,y-w)dxdy\\
=&\int_{\mathbb{T}^d\times\mathbb{T}^d} K(z,w)dzdw\int_{\mathbb{T}^d\times\mathbb{T}^d} f(x)f(y)D_r(x-z)D_r(y-w)dxdy\\
=&\int_{\mathbb{T}^d\times\mathbb{T}^d} K(z,w)f_r(z)f_r(w)dzdw \geq 0.
\end{aligned}$$

Thus, K_{rr} is positive semi-definite if K is such. The proof for Σ_{rr} is identical. □

Remark 5.4.4. By Proposition 5.4.3, kernels K_{rr}, Σ_{rr} are positive semi-definite. Therefore, their coefficients matrices with respect to basis $\{\cos(2\pi\alpha\cdot x), \sin(2\pi\alpha\cdot x)\}$ are also positive semi-definite by Lemma 5.4.1. Nevertheless, to take full advantage of Theorem 5.3.1 one would need these matrices to be positive definite (invertible). To solve this problem one can add a regularization term, εI, where I is the identity matrix of the suitable dimension and $\varepsilon > 0$ is a small constant. However, as discussed below, this regularization is not necessary for translation invariant kernels.

Suppose that

$$K(x,y) = \eta(x-y),\ \ x,y\in\mathbb{T}^d,$$

where η is a periodic function. Then, we have

$$\begin{aligned}
\int_{\mathbb{T}^d} K(x,y)\cos(2\pi\alpha\cdot y)dy &= \int_{\mathbb{T}^d}\eta(x-y)\cos(2\pi\alpha\cdot y)dy\\
&= \int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot(x-y))dy\\
&= \cos(2\pi\alpha\cdot x)\int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy\\
&\quad+\sin(2\pi\alpha\cdot x)\int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy.
\end{aligned}\tag{5.4.4}$$

Similarly, we obtain that

$$\begin{aligned}
\int_{\mathbb{T}^d} K(x,y)\sin(2\pi\alpha\cdot y)dy &= \int_{\mathbb{T}^d}\eta(x-y)\sin(2\pi\alpha\cdot y)dy\\
&= \int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot(x-y))dy\\
&= \sin(2\pi\alpha\cdot x)\int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy\\
&\quad-\cos(2\pi\alpha\cdot x)\int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy.
\end{aligned}\tag{5.4.5}$$

Therefore, we have that

$$\begin{aligned}
\int_{\mathbb{T}^d} K(x,y)\cos(2\pi\alpha\cdot x)\cos(2\pi\alpha\cdot y)dxdy &= \int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy,\\
\int_{\mathbb{T}^d} K(x,y)\sin(2\pi\alpha\cdot x)\cos(2\pi\alpha\cdot y)dxdy &= \int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy,\\
\int_{\mathbb{T}^d} K(x,y)\cos(2\pi\alpha\cdot x)\sin(2\pi\alpha\cdot y)dxdy &= -\int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy,\\
\int_{\mathbb{T}^d} K(x,y)\sin(2\pi\alpha\cdot x)\sin(2\pi\alpha\cdot y)dxdy &= \int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy.
\end{aligned}$$

Hence, the coefficients matrices of partial Fourier sums (and their linear combinations) of K consist of 2×2 blocks that correspond to expansion terms with a frequency $\alpha\in\mathbb{Z}^d$; that is,

$$\Delta_\alpha = \begin{pmatrix} \int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy & \int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy \\ -\int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy & \int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy \end{pmatrix}. \tag{5.4.6}$$

Thus, the coefficient matrix will be degenerate if $\det(\Delta_\alpha)=0$ for some α. But we have that

$$\det(\Delta_\alpha) = \left(\int_{\mathbb{T}^d}\eta(y)\cos(2\pi\alpha\cdot y)dy\right)^2 + \left(\int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy\right)^2.$$

Hence, $\det(\Delta_\alpha)=0$ if and only if $\Delta_\alpha=0$ or, equivalently, there are no expansion terms with frequency α. But then, we can simply remove these terms in our basis and obtain a non-degenerate matrix.

Moreover, to invert the coefficients matrix one just has to invert the 2×2 blocks. Additionally, if K is symmetric; that is, $\eta(y)=\eta(-y)$, we have that

$$\int_{\mathbb{T}^d}\eta(y)\sin(2\pi\alpha\cdot y)dy = 0,\ \forall\alpha\in\mathbb{Z}^d.$$

Hence, the coefficient matrices are simply diagonal. Therefore, we have proved the following proposition.

Proposition 5.4.5. *If K is translation invariant then all partial Fourier sums of K and their linear combinations, such as K_{rr} and Σ_{rr}, contain only* $\cos(2\pi\alpha\cdot x)\cos(2\pi\alpha\cdot y), \cos(2\pi\alpha\cdot x)\sin(2\pi\alpha\cdot y), \sin(2\pi\alpha\cdot x)\cos(2\pi\alpha\cdot y), \sin(2\pi\alpha\cdot x)\sin(2\pi\alpha\cdot y)$ *expansion terms. Therefore, coefficient matrices of such approximations with respect to trigonometric basis consist of* 2×2 *blocks that are multiples of Δ_α in* (5.4.6)*. If, additionally, K is symmetric these coefficient matrices are diagonal.*

Remark 5.4.6. In general, if $\{\phi_1,\phi_2,\cdots,\phi_r,\cdots\}$ is an orthonormal basis consisting of eigenfunctions of *Hilbert-Schmidt integral operator* $f(\cdot)\mapsto\int_{\mathbb{T}^d}K(\cdot,y)f(y)dy$; that is,

$$\int_{\mathbb{T}^d}K(x,y)\phi_\alpha(y)dx = \lambda_\alpha\phi_\alpha(x),\ x\in\mathbb{T}^d,\ \alpha\in\mathbb{N},$$

for some $\{\lambda_\alpha\} \subset \mathbb{R}$. Then, one has that

$$k_{\alpha\beta} = \int_{\mathbb{T}^d} K(x,y)\phi_\alpha(x)\phi_\beta(y)dxdy = \lambda_\beta \delta_{\alpha\beta}.$$

Consequently, for arbitrary $I \subset \mathbb{N} \times \mathbb{N}$ we have that

$$K_I(x,y) = \sum_{(\alpha,\beta)\in I} k_{\alpha\beta}\phi_\alpha(x)\phi_\beta(y) = \sum_{(\alpha,\alpha)\in I} \lambda_\alpha \phi_\alpha(x)\phi_\alpha(y).$$

Therefore, all partial Fourier sums of K in basis $\{\phi_\alpha(x)\phi_\beta(y)\}$ contain only terms $\phi_\alpha(x)\phi_\alpha(y)$ and yield diagonal coefficient matrices consisting of corresponding eigenvalues of the Hilbert-Schmidt integral operator.

In general, it is not easy to calculate the eigenfunctions of a given Hilbert-Schmidt integral operator. Nevertheless, as we saw above, for translation invariant symmetric periodic K these eigenfunctions are precisely the trigonometric functions.

5.5 A numerical method

In this section we propose a numerical method to solve (5.1.3) for a symmetric and positive semi-definite K. We assume that an approximation K_r of the form (5.3.1) is already constructed with a symmetric and positive definite kernel $\mathbf{K}$. Thus, we devise an algorithm for the solution of (5.3.2).

By Theorem 5.3.1 we have that (5.3.2) is equivalent to (5.3.4). We rewrite latter as

$$\inf_{a\in C([0,1];\mathbb{R}^r)} S(a), \tag{5.5.1}$$

where

$$S(a) = \frac{1}{2}\langle \mathbf{J}a, a\rangle - G(a),$$

and $\mathbf{J} = \mathbf{K}^{-1}$. Therefore, in what follows, we present a suitable discretization of (5.3.4).

5.5.1 Discretization of u_a

We start with the discretization of u_a. For that, we discretize the representation formula (5.2.13), that we can rewrite as

$$u_a(x,0) = \inf_{\mathbf{u}} \int_0^1 L_a(\mathbf{x}(s), \mathbf{u}(s), s)ds + U(\mathbf{x}(1)), \tag{5.5.2}$$

where $\mathbf{x}$ satisfies the following controlled ODE

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{u}(s), \quad s \in [0,1] \\ \mathbf{x}(0) = x. \end{cases} \tag{5.5.3}$$

We choose a uniform discretization of the time interval:

$$0 = s_0 < s_1 < s_2 < \ldots < s_N = 1,$$

with a step size $h_t = \frac{1}{N}$, hence $s_i = ih_t = \frac{i}{N}$, with $i = 0, \ldots, N$. We denote the values of $\mathbf{x}$ and $\mathbf{u}$ at time s_i by $\mathbf{x}(s_i) = x_i$, $\mathbf{u}(s_i) = u_i$. Using a backward Euler discretization of (5.5.3) we have

$$u_i = \frac{x_i - x_{i-1}}{h_t}, \; i \in \{1, \ldots, N\}.$$

Discretizing the integral (5.5.2) with a right point quadrature rule and using the above discretization we get

$$\begin{cases} \quad [u_a](x,0) & = \inf_{\{x_i\}_0^N} h_t \sum_{i=1}^N L_a\left(x_i, \frac{x_i - x_{i-1}}{h_t}, s_i\right) + U(x_N), \\ \text{subject to:} & x_0 = x, \end{cases} \tag{5.5.4}$$

where

$$L_a(x,u,s) = L(x,u) + \sum_{k=1}^r a_k(s)\phi_k(x), \; (x,u,s) \in \mathbb{T}^d \times \mathbb{R}^d \times [0,1].$$

5.5.2 Discretization of G

We start by discretizing the initial measure M using a convex combination of Dirac δ distributions. Denoting the discretized measure by $[M]$, we have

$$[M] = \sum_{\alpha=1}^Q c_\alpha \delta_{y_\alpha}$$

or, in the distributional sense,

$$\int_{\mathbb{T}^d} \psi(y) d[M](y) = \sum_{\alpha=1}^Q c_\alpha \psi(y_\alpha), \quad \psi \in C(\mathbb{T}^d), \tag{5.5.5}$$

for some $\{y_\alpha\}_{\alpha=1}^Q \subset \mathbb{T}^d$ and $\{c_\alpha \geq 0\}_{\alpha=1}^Q$ such that $\sum_{\alpha=1}^Q c_\alpha = 1$, where ψ is a test function. Then, G is discretized as follows

$$[G](a) = \sum_{\alpha=1}^Q c_\alpha \; [u_a](y_\alpha, 0). \tag{5.5.6}$$

5.5.3 Discretization of S

Now, we discretize the operator S in (5.5.1). We first discretize a_k-s by taking their values at times s_i, that we denote by:

$$[a]_k = (a_k(s_0), \cdots, a_k(s_N)) = (a_{k0}, \cdots, a_{kN}),\ k = 1, 2, \cdots, r.$$

Recall that

$$\langle \mathbf{J}a, a\rangle = \sum_{k,l=1}^{r} \mathbf{J}_{kl} \int_0^1 a_k(s)a_l(s)ds.$$

We discretize this previous quadratic form by a simple right point quadrature rule.

$$[\langle \mathbf{J}a, a\rangle] = h_t \sum_{k,l=1}^{r} \mathbf{J}_{kl} \sum_{i=1}^{N} a_{ki}a_{li}.$$

So the discretization of S is

$$\begin{aligned} [S](a) &= \frac{h_t}{2} \sum_{k,l=1}^{r} \mathbf{J}_{kl} \sum_{i=1}^{N} a_{ki}a_{li} - [G](a) \\ &= \frac{h_t}{2} \sum_{k,l=1}^{r} \mathbf{J}_{kl} \sum_{i=1}^{N} a_{ki}a_{li} - \sum_{\alpha=1}^{Q} c_\alpha\ [u_a](y_\alpha, 0), \end{aligned} \tag{5.5.7}$$

where we used (5.5.6). Therefore, the discretization of (5.5.1) is

$$\begin{aligned} \inf_{\{a_{ki}\}} [S](a) = \inf_{\{a_{ki}\}} \sup_{\{x_{\alpha i}:\ x_{\alpha 0}=y_\alpha\}} & \frac{h_t}{2} \sum_{k,l=1}^{r} \mathbf{J}_{kl} \sum_{i=1}^{N} a_{ki}a_{li} - h_t \sum_{\alpha=1}^{Q} \sum_{i=1}^{N} c_\alpha L\left(x_{\alpha i}, \frac{x_{\alpha i} - x_{\alpha(i-1)}}{h_t}\right) \\ & - h_t \sum_{\alpha=1}^{Q} \sum_{i=1}^{N} \sum_{k=1}^{r} c_\alpha a_{ki}\phi_k(x_{\alpha i}) - \sum_{\alpha=1}^{Q} c_\alpha U(x_{\alpha N}). \end{aligned} \tag{5.5.8}$$

5.5.4 Primal-dual hybrid-gradient method

Now, we specify the Lagrangian to be quadratic and devise a primal-dual hybrid-gradient algorithm [68] to solve (5.5.1). More precisely, we assume that

$$L(x, u) = \frac{|u|^2}{2},\ (x, u) \in \mathbb{T}^d \times \mathbb{R}^d,$$

and therefore (5.5.8) becomes

$$\begin{aligned}\inf_{\{a_{ki}\}}[S](a) = \inf_{\{a_{ki}\}} \sup_{\{x_{\alpha i}\,:\,x_{\alpha 0}=y_\alpha\}} &\frac{h_t}{2}\sum_{k,l=1}^{r}\mathbf{J}_{kl}\sum_{i=1}^{N}a_{ki}a_{li} - \sum_{\alpha=1}^{Q}\sum_{i=1}^{N}c_\alpha\frac{|x_{\alpha i}-x_{\alpha(i-1)}|^2}{2h_t}\\ &- h_t\sum_{\alpha=1}^{Q}\sum_{i=1}^{N}\sum_{k=1}^{r}c_\alpha a_{ki}\phi_k(x_{\alpha i}) - \sum_{\alpha=1}^{Q}c_\alpha U(x_{\alpha N}).\end{aligned} \tag{5.5.9}$$

Now, we describe the algorithm. For each iteration time $\nu \geq 0$ we have three groups of variables: $a^\nu = \{a^\nu_{ki}\}_{k,i=1,1}^{r,N}$, $x^\nu = \{x^\nu_{\alpha i}\}_{\alpha,i=1,0}^{Q,N}$, and $z^\nu = \{z^\nu_{\alpha i}\}_{\alpha,i=1,0}^{Q,N}$. Furthermore, we the proximal step parameters, $\lambda, \omega > 0$, for variables a and x, respectively. Additionally, we take $0 \leq \theta \leq 1$.

Step 1. Given a^ν, x^ν, z^ν the first step of the algorithm is to solve the proximal problem

$$\begin{aligned}\inf_{\{a_{ki}\}} &\frac{h_t}{2}\sum_{k,l=1}^{r}\mathbf{J}_{kl}\sum_{i=1}^{N}a_{ki}a_{li} - \sum_{\alpha=1}^{Q}\sum_{i=1}^{N}c_\alpha\frac{|z^\nu_{\alpha i}-z^\nu_{\alpha(i-1)}|^2}{2h_t}\\ &- h_t\sum_{\alpha=1}^{Q}\sum_{i=1}^{N}\sum_{k=1}^{r}c_\alpha a_{ki}\phi_k(z^\nu_{\alpha i}) - \sum_{\alpha=1}^{Q}c_\alpha U(z^\nu_{\alpha N}) + \frac{1}{2\lambda}\sum_{k=1}^{r}\sum_{i=1}^{N}(a_{ki}-a^\nu_{ki})^2,\end{aligned}$$

that is equivalent to

$$\inf_{\{a_{ki}\}}\frac{h_t}{2}\sum_{k,l=1}^{r}\mathbf{J}_{kl}\sum_{i=1}^{N}a_{ki}a_{li} - h_t\sum_{\alpha=1}^{Q}\sum_{i=1}^{N}\sum_{k=1}^{r}c_\alpha a_{ki}\phi_k(z^\nu_{\alpha i}) + \frac{1}{2\lambda}\sum_{k=1}^{r}\sum_{i=1}^{N}(a_{ki}-a^\nu_{ki})^2.$$

Thus, we obtain the following update of the a-variable.

$$\begin{pmatrix} a^{\nu+1}_{1i}\\ a^{\nu+1}_{2i}\\ \vdots\\ a^{\nu+1}_{ri}\end{pmatrix} = (\lambda h_t\mathbf{J} + \mathrm{Id}_r)^{-1}\begin{pmatrix} a^\nu_{1i} + \lambda h_t\sum_{\alpha=1}^{Q}c_\alpha\phi_1(z^\nu_{\alpha i})\\ a^\nu_{2i} + \lambda h_t\sum_{\alpha=1}^{Q}c_\alpha\phi_2(z^\nu_{\alpha i})\\ \vdots\\ a^\nu_{ri} + \lambda h_t\sum_{\alpha=1}^{Q}c_\alpha\phi_r(z^\nu_{\alpha i})\end{pmatrix}, \quad 1 \leq i \leq N. \tag{5.5.10}$$

Remark 5.5.1. Note that although the number of variables $\{a_{ki}\}_{k,i=1,1}^{r,N}$ is $r \times N$, the calculations of $\{a_{ki}\}$ for different i-s are mutually independent. Therefore, the only complexity is in the inversion of an $r \times r$ matrix $\lambda\sigma\mathbf{J} + \mathrm{Id}_r$ that can be computed beforehand and used throughout the scheme. Moreover, as seen in Section 5.4, translation invariant symmetric kernels yield diagonal matrices that extremely simplify the calculations.

Step 2. Given $a^{\nu+1}, x^\nu, z^\nu$ we update x-variable by solving the proximal problem

$$\inf_{\{x_{\alpha i}\colon\, x_{\alpha 0}=y_\alpha\}} \sum_{\alpha=1}^{Q}\sum_{i=1}^{N} c_\alpha \frac{|x_{\alpha i}-x_{\alpha(i-1)}|^2}{2h_t} + h_t \sum_{\alpha=1}^{Q}\sum_{i=1}^{N}\sum_{k=1}^{r} c_\alpha a_{ki}^{\nu+1}\phi_k(x_{\alpha i})$$
$$+ \sum_{\alpha=1}^{Q} c_\alpha U(x_{\alpha N}) + \frac{1}{2\omega}\sum_{\alpha=1}^{Q}\sum_{i=1}^{N}|x_{\alpha i}-x_{\alpha i}^{\nu}|^2.$$

Solving this previous problem may be a costly operation. Hence, we just perform a one step gradient descent. Therefore, we obtain

$$\begin{aligned}
x_{\alpha 1}^{\nu+1} &= x_{\alpha 1}^{\nu} - \frac{\omega c_\alpha}{h_t}(x_{\alpha 1}-y_\alpha) - \frac{\omega c_\alpha}{h_t}(x_{\alpha 1}-x_{\alpha 2}) - \omega c_\alpha h_t \sum_{k=1}^{r} a_{k1}^{\nu+1}\nabla\phi_k(x_{\alpha 1}),\\
x_{\alpha i}^{\nu+1} &= x_{\alpha i}^{\nu} - \frac{\omega c_\alpha}{h_t}(x_{\alpha i}-x_{\alpha(i-1)}) - \frac{\omega c_\alpha}{h_t}(x_{\alpha i}-x_{\alpha(i+1)}),\\
&\quad - \omega c_\alpha h_t \sum_{k=1}^{r} a_{ki}^{\nu+1}\nabla\phi_k(x_{\alpha i}), \quad 1\le i\le N-1,\\
x_{\alpha N}^{\nu+1} &= x_{\alpha N}^{\nu} - \frac{\omega c_\alpha}{h_t}(x_{\alpha N}-x_{\alpha(N-1)}) - \omega c_\alpha \nabla U(x_{\alpha N}) - \omega c_\alpha h_t \sum_{k=1}^{r} a_{kN}^{\nu+1}\nabla\phi_k(x_{\alpha N}).
\end{aligned} \tag{5.5.11}$$

Step 3. In the final step we update the z-variable by

$$z_{\alpha i}^{\nu+1} = x_{\alpha i}^{\nu+1} + \theta(x_{\alpha i}^{\nu+1} - x_{\alpha i}^{\nu}), \quad 1\le\alpha\le Q,\ 1\le i\le N. \tag{5.5.12}$$

Remark 5.5.2. Note that the updates for $\{x_{\alpha i}\}, \{z_{\alpha i}\}$ variables are mutually independent for different α-s. Therefore, our a-updates are parallel in time, and x, z-updates are parallel in space.

Remark 5.5.3. Strictly speaking, one cannot simply apply the primal-dual hybrid gradient method to (5.5.9) because the coupling between a and x is not bilinear, and there is no concavity in x. Nevertheless, our calculations always yield solid results. Therefore, there is a natural problem of rigorously understanding the convergence properties of the aforementioned algorithm. We plan to address this problem in our future work.

5.6 Numerical examples

In this section, we present several numerical experiments. We first look into one-dimensional case, in Section 5.6.1, and after we consider the two-dimensional case, in Section 5.6.2.

For our calculations, we choose the periodic Gaussian kernel that is given by

$$K_{\sigma,\mu}^{d}(x,y) = \prod_{i=1}^{d} K_{\sigma,\mu}^{1}(x_i,y_i),\ x,y\in\mathbb{T}^d, \tag{5.6.1}$$

where

$$K^1_{\sigma,\mu}(x,y) = \frac{\mu}{\sqrt{2\pi(\frac{\sigma}{2})^2}} \sum_{k=-\infty}^{\infty} e^{-\frac{(x-y-k)^2}{2(\frac{\sigma}{2})^2}}, \ x, y \in \mathbb{T}, \tag{5.6.2}$$

and $\sigma, \mu > 0$ are given parameters. Here, σ models how spread is the kernel. The smaller σ the more weight agents assign to their immediate neighbors – this translates into crowd-aversion in the close neighborhood only. Furthermore, μ is the total weight of the agents. Therefore, μ measures how sensitive is a generic agent to the total population, the bigger the more averse is the agent to others. As we observe in the numerical experiments, for smaller σ and larger μ the more separated are the agents. This phenomenon was also observed in [17].

Throughout the section we denote by

$$\phi_k(x) = \begin{cases} 1, \ \text{if } k = 1, \\ \sqrt{2}\cos \pi(k-1)x, \ \text{if } k \text{ is odd, and } k > 0, \\ \sqrt{2}\sin \pi k x, \ \text{if } k \text{ is even}, \ x \in \mathbb{T}. \end{cases} \tag{5.6.3}$$

Therefore, we have

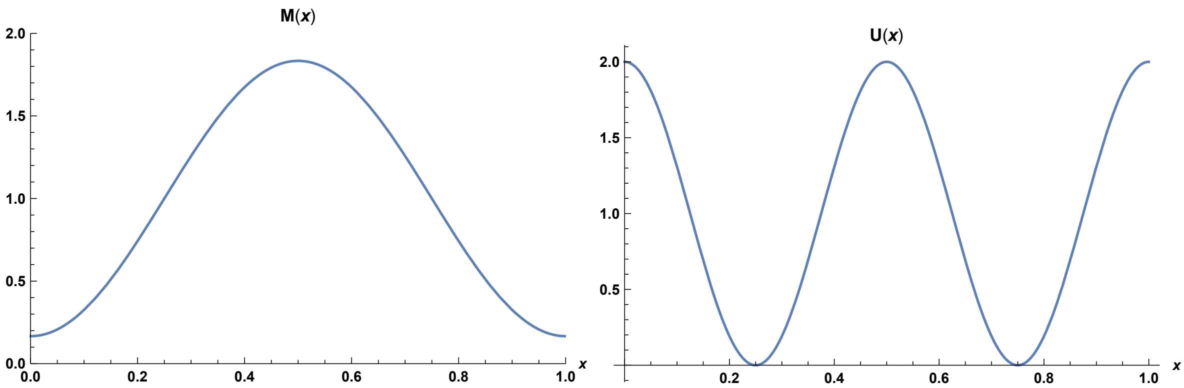
$$\{\phi_1, \phi_2, \phi_3, \cdots\} = \{1, \sqrt{2}\sin 2\pi x, \sqrt{2}\cos 2\pi x, \cdots\}.$$

5.6.1 One-dimensional examples

For all simulations we use the same initial-terminal conditions

$$M(x) = \frac{1}{6} + \frac{5}{3}\sin^2 \pi x, \quad U(x) = 1 + \sin\left(4\pi x + \frac{\pi}{2}\right), \ x \in \mathbb{T},$$

that are depicted in Figure 5.1. We also use the same time and space discretization for all one



(a) Initial distribution of agents, $M(x)$. (b) Terminal cost function, $U(x)$.

Fig. 5.1 Initial-terminal conditions.

dimensional experiments, and the same parameters for the numerical scheme. We discretize the time interval using a step size $\Delta t = \frac{1}{N}$. For the discretization of M we use

$$y_\alpha = \frac{\alpha}{Q+1}, \quad c_\alpha = \frac{M(y_\alpha)}{\sum_{\beta=1}^{Q} M(y_\beta)} \; 1 \le \alpha \le Q.$$

We choose $N = 20,\ Q = 50$ and use eight basis functions, $r = 8$. Additionally, we set the numerical scheme parameters to $\lambda = 3,\ \omega = \frac{1}{12}$ and $\theta = 1$.

Remark 5.6.1. For the standard primal-dual hybrid gradient method, we must have $\omega\lambda < \frac{1}{A^2}$, where A is the norm of the bilinear-form matrix. As we mentioned in Remark 5.5.3, here we do not have a bilinear coupling between a and x. Thus, we estimate A by an upper bound on the (l_2, l_2) Lipschitz norm of the mapping

$$F_{ki}(x) = h_t \sum_{\alpha=1}^{Q} c_\alpha \phi_k(x_{\alpha i}),\ 1 \le l \le r,\ 1 \le i \le N.$$

More precisely, we have that

$$\begin{aligned}
\mathrm{Lip}(F)^2 &= \sup_{\{x_{\beta j}\}} \sup_{\|w_{\beta j}\|_2 \le 1} \sum_{k,i} \left(\sum_{\beta,j} \frac{\partial F_{ki}}{\partial x_{\beta j}} w_{\beta j} \right)^2 \\
&= \sup_{\{x_{\beta j}\}} \sup_{\|w_{\beta j}\|_2 \le 1} \sum_{k,i} \left(\sum_{\beta} h_t c_\beta \nabla\phi_k(x_{\beta i}) w_{\beta i} \right)^2 \\
&\le h_t^2 \sup_{\{x_{\beta j}\}} \sup_{\|w_{\beta j}\|_2 \le 1} \sum_{k,i} \left(\sum_{\beta} c_\beta^2 \|\nabla\phi_k(x_{\beta i})\|_2^2 \cdot \sum_{\beta} w_{\beta i}^2 \right) \\
&\le h_t^2 \sup_{\|w_{\beta j}\|_2 \le 1} \sum_{k,i} \mathrm{Lip}(\phi_k)^2 \left(\sum_{\beta} c_\beta^2 \cdot \sum_{\beta} w_{\beta i}^2 \right) \\
&= h_t^2 \sup_{\|w_{\beta j}\|_2 \le 1} \sum_{k} \mathrm{Lip}(\phi_k)^2 \sum_{\beta} c_\beta^2 \sum_{\beta,i} w_{\beta i}^2 \\
&= h_t^2 \sum_{k} \mathrm{Lip}(\phi_k)^2 \sum_{\beta} c_\beta^2.
\end{aligned}$$

Thus, we take

$$A^2 = h_t^2 \sum_{k=1}^{r} \mathrm{Lip}(\phi_k)^2 \sum_{\beta=1}^{Q} c_\beta^2.$$

The trigonometric expansion of $K^1_{\sigma,\mu}$ is given by

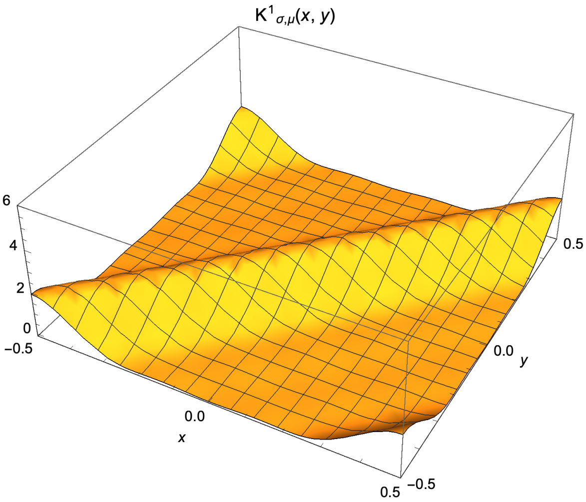
$$K^1_{\sigma,\mu}(x,y) = \mu \left(1 + 2 \sum_{n=1}^{\infty} e^{-\frac{(\pi n \sigma)^2}{2}} \cos 2\pi n(x-y) \right),\ x, y \in \mathbb{T}, \tag{5.6.4}$$

or

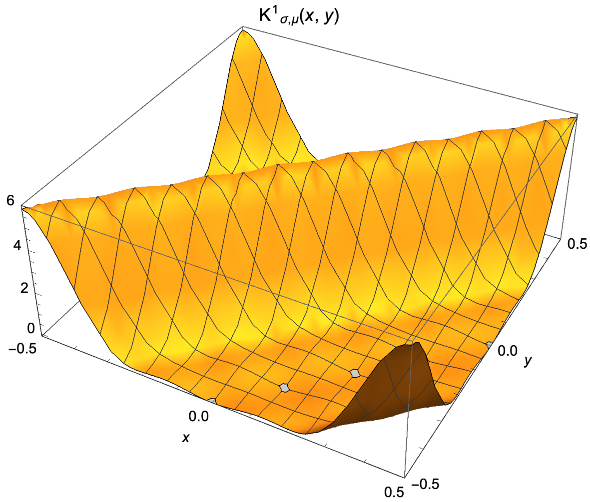
$$K^1_{\sigma,\mu}(x,y) = \sum_{k=1}^{\infty} \mu e^{-\frac{1}{2}\left(\pi\sigma\left[\frac{k}{2}\right]\right)^2} \phi_k(x)\phi_k(y), \;\; x,y \in \mathbb{T}, \tag{5.6.5}$$

in our notation. Therefore, for a given r, the matrices $\mathbf{K}, \mathbf{J}$ are given by

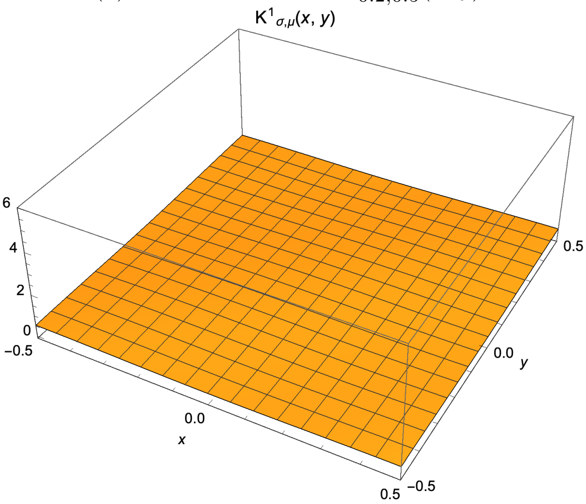
$$\begin{aligned} \mathbf{K} =& \mathrm{diag}\left(\mu e^{-\frac{1}{2}\left(\pi\sigma\left[\frac{k}{2}\right]\right)^2}\right)_{k=1}^{r}, \\ \mathbf{J} =& \mathrm{diag}\left(\mu^{-1} e^{\frac{1}{2}\left(\pi\sigma\left[\frac{k}{2}\right]\right)^2}\right)_{k=1}^{r}. \end{aligned} \tag{5.6.6}$$



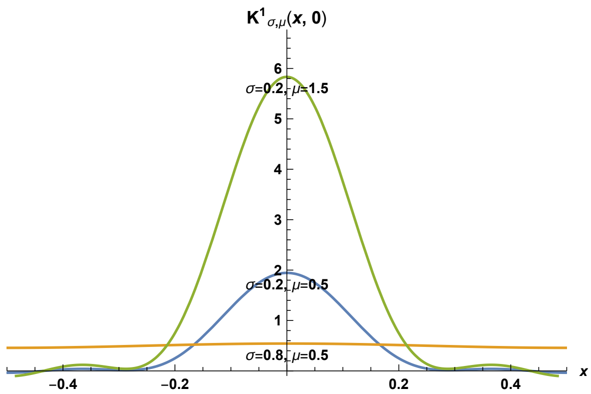
(a) Gaussian kernel, $K^1_{0.2,0.5}(x,y)$.



(b) Gaussian kernel, $K^1_{0.2,1.5}(x,y)$.



(c) Gaussian kernel, $K^1_{0.8,0.5}(x,y)$.



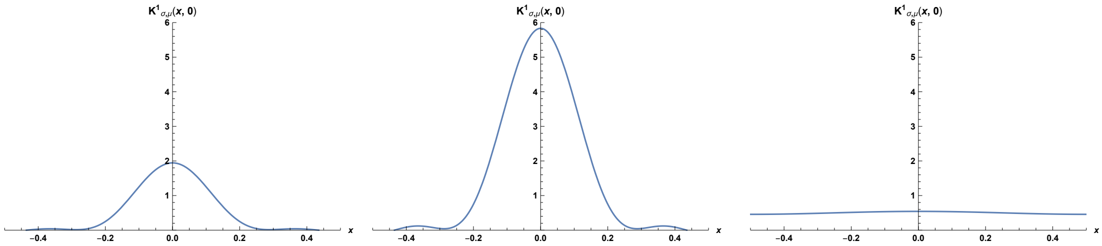
(d) Comparison of kernels on $K^1_{\sigma,\mu}(x,0)$.

Fig. 5.2 Plots of the three Gaussian kernels in (a)-(c), and a comparison of their sections in (d).

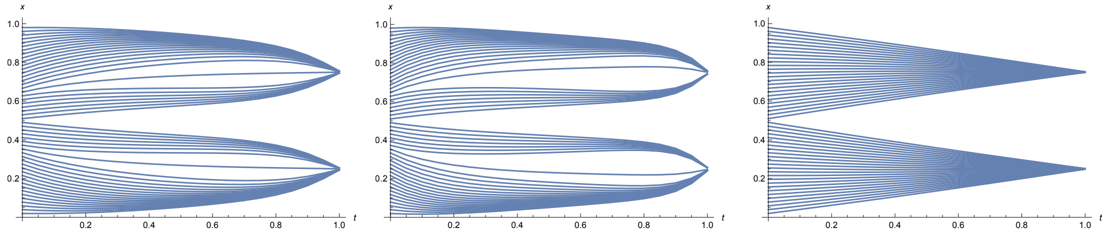
In Figure 5.2 we plot the Gaussian kernels we used, for $r = 8$ and different values of μ and σ. We see the influence of these values in Figure 5.3. In the first column of Figure 5.3 we compare the results regarding different values of μ and σ.

Comparing the first and the second columns of Figure 5.3, we see that the trajectories of the agents in the first column are closer than in the second one. This is due to the fact that $\mu = 0.5$ in the first kernel and $\mu = 1.5$ in the second one, hence the second kernel (higher value of μ) penalizes higher density of agents more than the former. Therefore, the agents spread out more before the final time when they converge to the points of low-cost near minima of the terminal cost function, U, see Figure 5.1 (b).

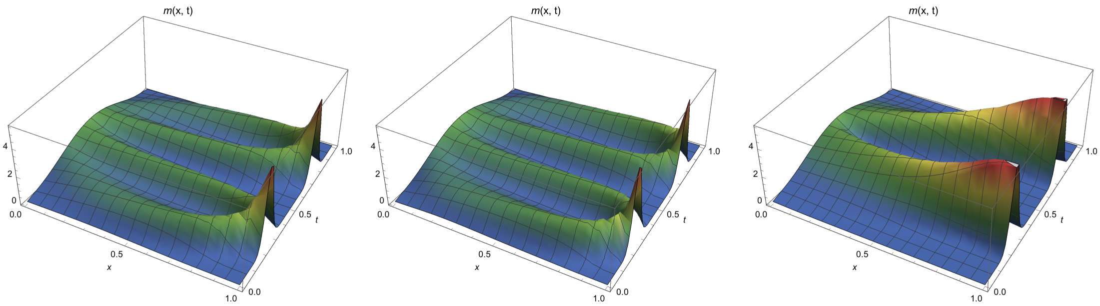
In the last column the value of $\sigma = 0.8$ is higher, this means that agents are indifferent to the distances between them – they take into account the total mass. Hence, they minimize the travel distances from initial positions to low-cost locations of U ignoring the population density. In fact, in this case $K^1_{\sigma,\mu} \approx \mu$, and therefore $\int_{\mathbb{T}} K^1_{\mu,\sigma}(x,y)m(y,t)dy \approx \mu$. Thus, in this case (5.1.3) approximates a decoupled system of Hamilton-Jacobi and Fokker-Planck equations. The optimal trajectories of the decoupled system are straight lines by Hopf-Lax formula. As we can see in Figure 5.3 (d), this fact is consistent with the straight-line trajectories that we obtain.



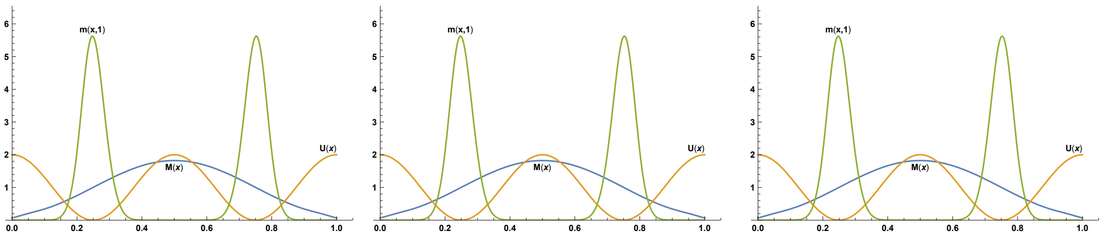
(a) $K^1_{0.2,0.5}(x,0)$. (b) $K^1_{0.2,1.5}(x,0)$. (c) $K^1_{0.8,0.5}(x,0)$.



(d) Trajectories, $\mathbf{x}(t, y_\alpha)$.



(e) Density, $m(x,t)$.



(f) $M(x)$ – blue, $m(x,1)$ – green, $U(x)$ – orange.

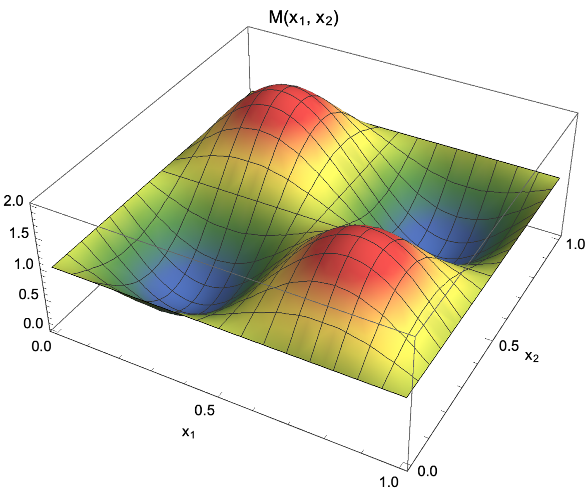
Fig. 5.3 Simulations using Gaussian kernels with different parameters, for each column, $(\sigma, \mu) \in \{(0.2, 0.5), (0.2, 1.5), (0.8, 0.5)\}$. In the first row, we show a section of each kernel. In the second row, we plot the trajectories of the agents, $\{\mathbf{x}(t, y_\alpha)\}_{\alpha=1}^{Q}$, at time $t \in [0,1]$ and initial positions $\{y_\alpha\}_{\alpha=1}^{Q} \subset \mathbb{T}$. In the third row, we plot the time evolution of the distribution of players, $m(t,x)$. Each plot of the last row displays the initial-terminal conditions, $M(x)$ and $U(x)$, and the final distribution, $m(x,1)$.

5.6.2 Two-dimensional examples

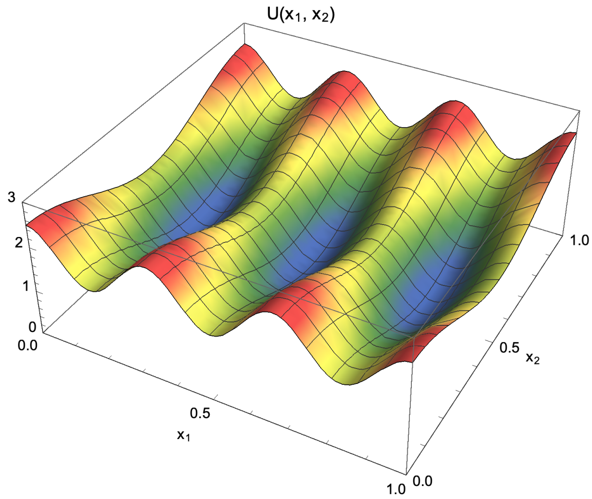
Here, we consider the case of two-dimensional state space. The initial distribution of players and the terminal cost function are given by

$$
\begin{aligned}
M(x_1,x_2) =&1+\frac{1}{2}\cos\left(\pi+2\pi\left(x_1-x_2\right)\right)+\frac{1}{2}\sin\left(\frac{\pi}{2}+2\pi\left(x_1+x_2\right)\right),\\
U(x_1,x_2) =&\frac{3}{2}+\frac{1}{2}\left(\cos\left(6\pi x_1\right)+\cos\left(2\pi x_2\right)\right),\ \ (x_1,x_2)\in\mathbb{T}^2,
\end{aligned}
$$

that are depicted in Figure 5.4.



(a) Initial distribution of agents, $M(x_1,x_2)$.



(b) Terminal cost function, $U(x_1,x_2)$.

Fig. 5.4 Initial-terminal conditions.

The corresponding expansion of the kernel is given by

$$
\begin{aligned}
&K^2_{\sigma,\mu}(x_1,x_2;y_1,y_2)\\
&=\sum_{k,k'=1}^{\infty}\mu^2 e^{-\frac{\pi^2\sigma^2}{2}\left(\left[\frac{k}{2}\right]^2+\left[\frac{k'}{2}\right]^2\right)}\phi_k(x_1)\phi_k(y_1)\phi_{k'}(x_2)\phi_{k'}(y_2)\\
&=\sum_{k,k'=1}^{\infty}\mu^2 e^{-\frac{\pi^2\sigma^2}{2}\left(\left[\frac{k}{2}\right]^2+\left[\frac{k'}{2}\right]^2\right)}\phi_{k,k'}(x_1,x_2)\phi_{k,k'}(y_1,y_2),
\end{aligned}
\tag{5.6.7}
$$

where

$$
\phi_{k,k'}(x_1,x_2)=\phi_k(x_1)\phi_{k'}(x_2),\ \ x_1,x_2\in\mathbb{T},\ \ k,k'\in\mathbb{N}. \tag{5.6.8}
$$

Thus, for a fixed r we take as a basis functions the set:

$$
\{\phi_{1,1},\phi_{1,2},\cdots,\phi_{1,r-1},\phi_{2,1},\cdots,\phi_{2,r-2},\cdots,\phi_{r-1,1}\}=\{\psi_1,\psi_2,\cdots,\psi_{\frac{r(r-1)}{2}}\}.
$$

Therefore, we take all functions $\phi_{k,k'}$ such that $k+k' \leq r$ and order them in the lexicographic order. The corresponding matrices will be of size $\frac{r(r-1)}{2} \times \frac{r(r-1)}{2}$:

$$\begin{aligned} \mathbf{K} =& \mathrm{diag}\left(\mu^2 e^{-\frac{\pi^2\sigma^2}{2}\left(\left[\frac{k}{2}\right]^2+\left[\frac{k'}{2}\right]^2\right)}\right)_{k+k'\leq r}, \\ \mathbf{J} =& \mathrm{diag}\left(\mu^{-2} e^{\frac{\pi^2\sigma^2}{2}\left(\left[\frac{k}{2}\right]^2+\left[\frac{k'}{2}\right]^2\right)}\right)_{k+k'\leq r}, \end{aligned} \tag{5.6.9}$$

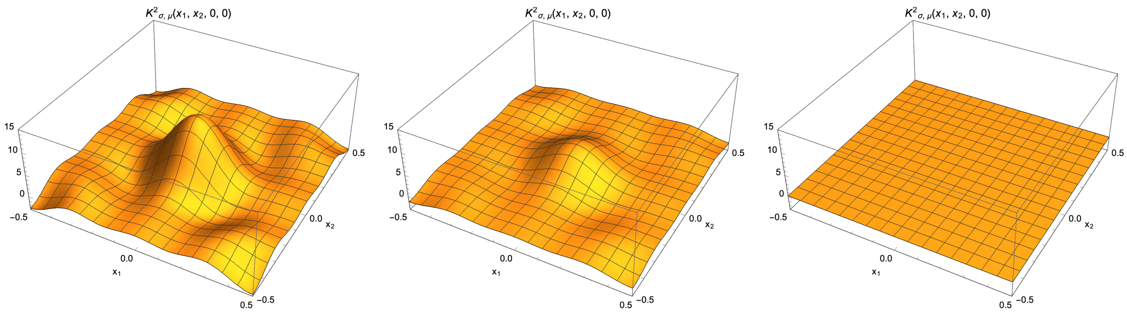
where the order is again lexicographic.

To compare the results, we use the same time and space discretization in our 2−dimensional experiments, as well as the same parameters for the numerical scheme. We discretize the time using a step size $\Delta t = \frac{1}{N}$. For the discretization of M we use

$$y_{\alpha\alpha'} = \left(\frac{\alpha}{Q+1}, \frac{\alpha'}{Q+1}\right), \quad c_{\alpha\alpha'} = \frac{M(y_{\alpha\alpha'})}{\sum_{\beta,\beta'=1}^{Q} M(y_{\beta\beta'})}, \; 1 \leq \alpha, \alpha' \leq Q.$$

We choose $N = 20$, $Q = 20$ and use eight basis functions, $r = 8$. Furthermore, we set the numerical scheme parameters to $\lambda = 1$, $\omega = \frac{1}{12}$ and $\theta = 1$.

In Figure 5.5, we plot the Gaussian kernels used in the simulations, with different values of μ and σ. We see that the bigger μ is the higher the peak of the kernel, see (a) and (b) in Figure 5.5. This means that each agent in (a) is more adverse of being in crowded areas than agents is (b), $\mu = 0.75$ and $\mu = 0.5$ respectively. For higher values of σ we see that the kernel becomes flat, compare (b) with (c) in Figure 5.5, for $\sigma = 0.1$ and $\sigma = 1$ respectively. As before, this means that the agents penalize others independent of mutual distances.



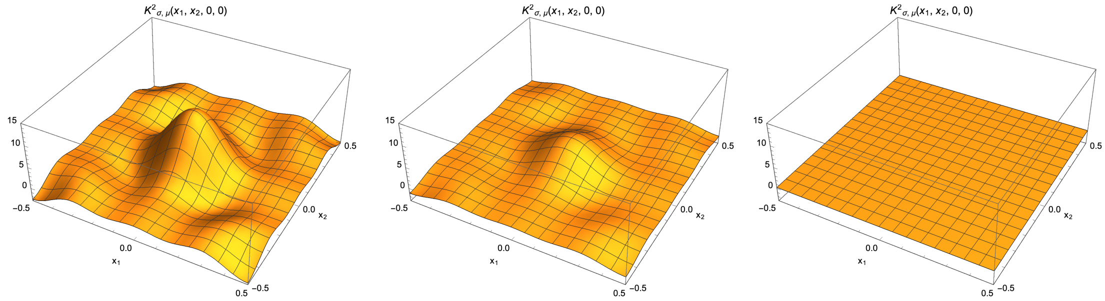
(a) $K^2_{0.1,0.75}(x_1, x_2; 0, 0)$. (b) $K^2_{0.1,0.5}(x_1, x_2; 0, 0)$. (c) $K^2_{1,0.5}(x_1, x_2; 0, 0)$.

Fig. 5.5 Plots of the Gaussian kernels for $(\sigma, \mu) \in \{(0.1, 0.75), (0.1, 0.5), (1, 0.5)\}$.

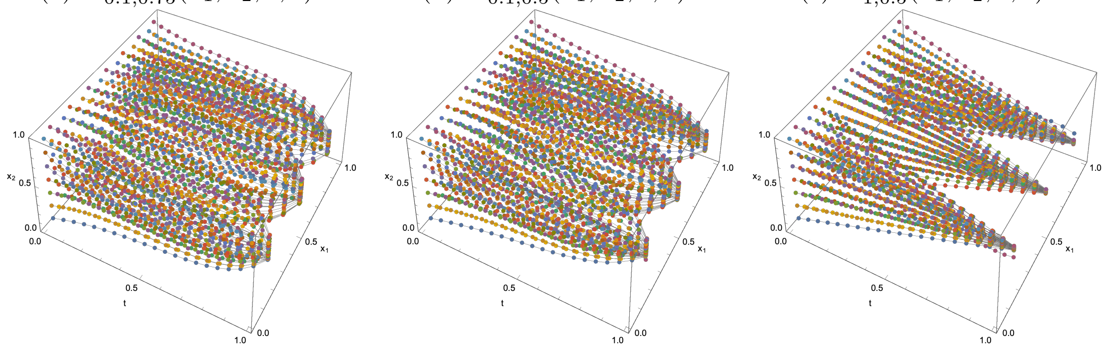
In Figure 5.6, we compare the simulation results using the same initial-terminal conditions, see Figure 5.4, but different kernel functions (plotted in the first row of Figure 5.6). In the last row of Figure 5.6 we have the final distribution of agents.

We see that for larger values of μ, left column compared with the middle one, the agents' concentration near low-cost regions of terminal cost, U, is less dense. We also see that when σ is bigger the the agents become more indifferent to the density of the crowd, and concentrate more densely near low-cost values of U – see the right column in Figure 6 (f).

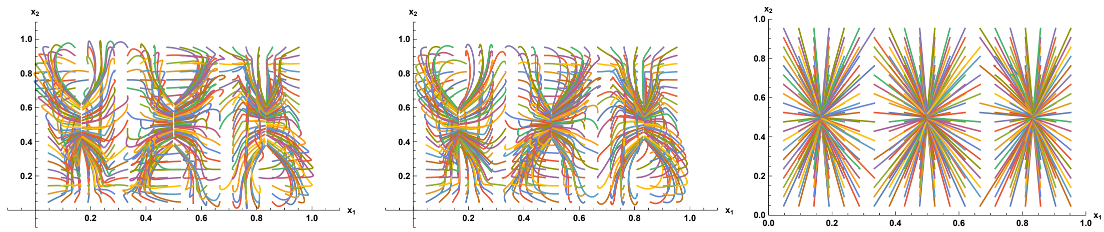
As in the 1-dimensional case, looking to the projected trajectories in the 2-dimensional plane we observe that for flat kernel agents follow straight lines from the initial positions to closest low-cost regions of the terminal cost function.



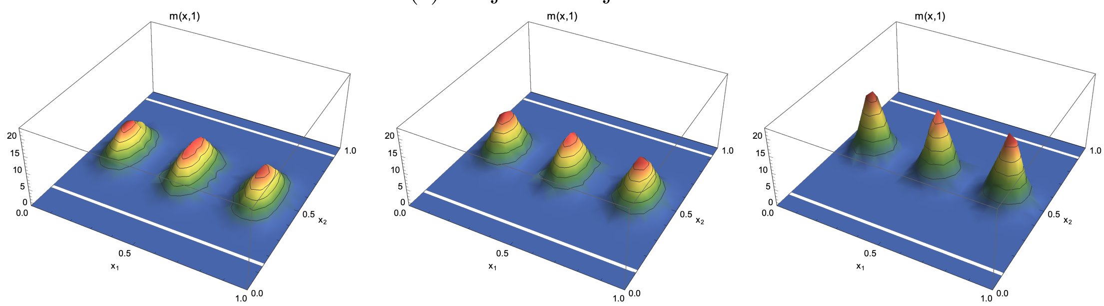
(a) $K^2_{0.1,0.75}(x_1, x_2; 0, 0)$. (b) $K^2_{0.1,0.5}(x_1, x_2; 0, 0)$. (c) $K^2_{1,0.5}(x_1, x_2; 0, 0)$.



(d) Trajectories, $\mathbf{x}(t, y_\alpha)$.



(e) Projected trajectories.



(f) Final density, $m(\mathbf{x}, 1)$.

Fig. 5.6 Simulations using Gaussian kernels with different parameters for each column, $(\sigma, \mu) \in \{(0.1, 0.75), (0.1, 0.5), (1, 0.5)\}$. In the first row we show a section of each kernel. In the second row we show the trajectories of the agents, $\{\mathbf{x}(t, y_{\alpha\alpha'})\}_{\alpha,\alpha'=1}^{Q}$, $t \in [0, 1]$, with initial positions $\{y_{\alpha\alpha'}\}_{\alpha,\alpha'=1}^{Q} \in \mathbb{T}^2$. In the third row, we plot the 2D projection of the trajectories. And in the last row, we plot the final distribution of the agents, $m(\mathbf{x}, 1)$.

Part III

Mean-field dynamic price model

Chapter 6

Dynamic price formation model

6.1 Introduction

The mean-field game (MFG) framework [114, 117, 131, 132] models systems with many rational players (see, e.g., the surveys [95] and [99]). Here, we are interested in the price formation in electricity markets. In our model, a large number of agents owns storage devices that can be charged and later supply the grid with electricity. Agents seek to maximize profit by trading electricity at a price $\varpi(t)$, which is set by a supply versus demand balance condition.

With the advent of electric cars, a large number of network-connected batteries are already available, and their number is only likely to increase. Moreover, energy can be stored as heat or cold, using space or water heaters and air-conditioning units [119, 120, 118]. With new small network-capable devices, appliances can be connected to the grid and use smart algorithms to control their energy usage. These algorithms can balance supply and demand and, thus, are particularly relevant when combined with solar and wind energy production, where power demand seldom matches production.

Price formation models were some of the first MFG models [133]. This line of research was pursued by several authors, see [48, 47, 145, 42, 41, 87] and the monograph [104]. Some of these models are formulated as free boundary problems [48, 47]; others as a load control problem [144, 143]. For example, using mean-field control and MFG, the load-control problem through switching on and off space heaters was studied in [119, 120, 118]. Previous authors addressed the price issue by assuming that the demand is a given function of the price [137] or that the price is a given function of the demand, see [72], [71], and [73]. In particular, in these references, the authors use a price function to study mean-field equilibrium in electricity markets in a setting that is similar to ours.

Here, we pursue a different approach: often, in economic models, prices of goods and services are determined by the balance between supply and demand rather than by a given function of the supply. Therefore, the price as a function of the supply or demand is not known a priori and a key unknown in the problem. This observation motivated the approach in [87], where price arises from supply versus demand constraints. However, that model is more complex than the one discussed here and was only studied from a numerical perspective. Thus, mathematical issues such as the existence and uniqueness of a price, the well-posedness of the model, and the convergence of numerical methods were left unanswered and are settled here.

Our model comprises three quantities of interest: a price $\varpi \in C([0,T])$, a value function $u \in C(\mathbb{R} \times [0,T])$, and a path describing the statistical distribution of the agents, $m \in C([0,T],\mathcal{P})$, where $\mathcal{P}$ is the set of probability measures in $\mathbb{R}$ with bounded first moment, endowed with the 1-Wasserstein distance. These quantities are determined by the following problem.

Problem 1 (Price-formation model)**.** Given $\epsilon \geq 0$, a Hamiltonian, $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $H \in C^\infty$, an energy production rate $\mathbf{Q} : [0,T] \to \mathbb{R}$, $\mathbf{Q} \in C^\infty([0,T])$, a terminal cost $\bar{u} : \mathbb{R} \to \mathbb{R}$, $\bar{u} \in C^\infty(\mathbb{R})$ and an initial probability distribution $\bar{m} \in \mathcal{P} \cap C_c^\infty(\mathbb{R})$, find $u : \mathbb{R} \times [0,T] \to \mathbb{R}$, $m \in C([0,T],\mathcal{P})$, and $\varpi : [0,T] \to \mathbb{R}$ solving

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = \epsilon u_{xx} \\ m_t - (D_pH(x, \varpi(t) + u_x)m)_x = \epsilon m_{xx} \\ \int_\Omega D_pH(x, \varpi(t) + u_x) dm = -\mathbf{Q}(t), \end{cases} \tag{6.1.1}$$

and satisfying the initial-terminal conditions

$$\begin{cases} u(x,T) = \bar{u}(x), \\ m(x,0) = \bar{m}(x). \end{cases} \tag{6.1.2}$$

In the previous problem, $x \in \mathbb{R}$ represents the state of a typical agent; that is, the energy stored by the agent. The function $u(x,t)$ is the value function for an agent whose charge is x at time t. The Hamiltonian, $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is determined by the optimization problem that each agent seeks to solve, as described in Section 6.2. We require u to be a viscosity of the first equation in (6.1.1). However, if $\epsilon > 0$, parabolic regularity theory gives additional regularity for u. For each $t \in [0,T]$, m determines the distribution of the energy storage of the agents. Here, we assume that m is a weak solution of the second equation in (6.1.1); that is, for every $\psi \in C_c^2(\mathbb{R} \times [0,T])$, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} (\psi_t + \psi_x D_pH(x, \varpi + u_x) - \epsilon \psi_{xx})\, m dx dt \\ &\quad = \int_{\mathbb{R}} \psi(x,T) m(x,T) dx - \int_{\mathbb{R}} \psi(x,0) \bar{m}(x) dx. \end{aligned}$$

The parameter ϵ corresponds to random fluctuation in the storage of the agents. Finally, the spot price, $\varpi(t)$, is selected so that the total energy used balances the supply, $\mathbf{Q}(t)$, the condition imposed by the last equation in (6.1.1).

In the current model, agents have a time horizon $T > 0$, and, at time T, they incur in the terminal cost $\bar{u}(x)$ that depends on their state at the terminal time. For example, agents may prefer to have the batteries fully charged at the end of the day. Moreover, the initial distribution of agents, $\bar{m}$, is known. These two facts are encoded in the initial-terminal boundary conditions, (6.1.2). This model can easily be modified to address periodic in time boundary conditions and the infinite horizon discounted problem.

First, in Section 6.2, we present a derivation of our model and examine some of its mathematical properties. Then, after a brief discussion of the main assumptions, in Section 6.3, we prove our main result given by the following theorem.

Theorem 15. Suppose that Assumptions 3–7 (see Section 6.3) hold. Then, there exists a solution (u, m, ϖ) of Problem 1 where u is a viscosity solution of the first equation, Lipschitz and semiconcave in x, and differentiable almost everywhere with respect to m, $m \in C([0,T], \mathcal{P})$, and ϖ is Lipschitz continuous. Moreover, if $\epsilon > 0$ this solution is unique.

If $\epsilon = 0$ and Assumption 8 holds, then there is a unique solution (u, m, ϖ). Moreover, u is differentiable in x for every x, and u_{xx} and m are bounded.

Remark 4. In the case $\epsilon > 0$, the regularity of the solutions can be improved using parabolic regularity.

There are two main contributions on this chapter. First, is the existence part of the preceding theorem which is proved in Section 6.4 using a fixed-point argument. The key step is establishing an ordinary differential equation satisfied by the price, ϖ. Using this equation, we obtain Lipschitz bounds and then apply Schauder's fixed-point theorem. To prove the uniqueness part of the theorem, we use the monotonicity method. This is achieved in Section 6.5 where we identify a new monotonicity structure for mean-field games with constraints. Finally, we discuss linear-quadratic models, that can be solved explicitly and compare our model with the ones in [73]. Our results suggest that a price determined by a supply versus demand condition may help stabilize the oscillations of the price in particular in peak-demand situations.

6.2 A mean-field model for price formation

Here, we present the derivation of our price model. To simplify the discussion, we examine the deterministic case, $\epsilon = 0$. We consider an electricity grid connecting consumers to producers of energy. In our model, each consumer has a storage device connected to the network, for example, an electric car battery. We assume that all devices are similar. Consumers trade electricity, charging the batteries when the price is low and selling electricity to the market when the price is high. A typical consumer has a battery whose charge at time $t \in [0,T]$ is $\mathbf{x}(t)$. This charge changes according to an energy flow rate, the control variable selected by each consumer, which is a bounded measurable function of $\alpha : [0,T] \to A$, where $A \subset \mathbb{R}$. Positive values of α correspond to buying energy from the grid, and negative values to selling to the grid. Accordingly, each consumer charge, $\mathbf{x}$, changes according to the dynamics:

$$\dot{\mathbf{x}}(t) = \alpha(t).$$

Each consumer seeks to select α to minimize its cost, thus maximizing profit. This cost is determined by a terminal cost and by the integral of the *running cost*, $\ell(\alpha, x, t)$, where $\alpha(t)$ is the energy traded with the electricity grid at time t, and ℓ depends in time through $\varpi(t)$, the spot electricity price and is of the form

$$\ell(\alpha, x, t) = \ell_0(\alpha, x) + \varpi(t)\alpha(t). \tag{6.2.1}$$

In the preceding expression, the term $\varpi(t)\alpha(t)$ is the instantaneous cost corresponding to a charging current $\alpha(t)$. The current (or more precisely power), α is measured in Watt, W, and the price, ϖ, in $\$\mathrm{W}^{-1}s^{-1}$. The function ℓ_0 accounts for non-linear effects of the current usage, for example, battery wear and tear, and for state preferences. For example, we often take

$$\ell_0(\alpha, x, t) = \frac{c}{2}\alpha^2(t) + V(x), \tag{6.2.2}$$

where c is a constant that accounts for the battery's wear off, typically given in $\$\mathrm{W}^{-2}s^{-1}$, and $V(x)$ is a potential that takes into account battery constraints and charge preferences. The singular case where

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

corresponds to the case where the battery charges satisfies $0 \leq x \leq 1$. To avoid singularities, we work with smooth potentials growing as $x \to \pm\infty$; this behaviour correspond to a penalty on the battery charge rather than a hard constraint. The nonlinear term, $\frac{c}{2}\alpha^2(t)$, models battery wear and tear, which is large in high-current regimes. The particular quadratic form in (6.2.2) simplifies the mathematical treatment. However, it can be replaced by a convex function of α without any major change in the discussion.

Each consumer minimizes the functional

$$J(x, t, \alpha) = \int_t^T \ell(\alpha(s), \mathbf{x}(t), t)ds + \bar{u}(\mathbf{x}(T)), \tag{6.2.3}$$

where $\bar{u}$ is the *terminal cost* and $\alpha \in \mathcal{A}_t$, where $\mathcal{A}_t$ is the set of bounded measurable functions $\alpha : [t,T] \to A$.

The *value function*, u, is the infimum of J over all controls in $\mathcal{A}_t$; that is,

$$u(x,t) = \inf_{\alpha \in \mathcal{A}_t} J(x,t,\alpha).$$

The *Hamiltonian*, H, for the preceding control problem is

$$H(x,p) = \sup_{a \in A} \left(-pa - \ell_0(x,a)\right).$$

For example, for ℓ_0 as in (6.2.2), we have

$$H(x,p) = \frac{p^2}{2c} + V(x).$$

From standard optimal control theory, u is a viscosity solution (see [20]) of the *Hamilton-Jacobi equation*

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = 0 \\ u(x,T) = \bar{u}(x). \end{cases} \tag{6.2.4}$$

For ℓ_0 as in (6.2.2), the prior equation becomes

$$-u_t + \frac{1}{2c}(u_x + \varpi(t))^2 - V(x) = 0.$$

Finally, at points of differentiability of u, the optimal control is given by

$$\alpha^*(t) = -D_pH(x, \varpi(t) + u_x(\mathbf{x}(t),t)).$$

The associated *transport equation* is the adjoint of the linearized Hamilton-Jacobi equation:

$$\begin{cases} m_t - \left(D_pH(x, u_x + \varpi(t))m\right)_x = 0, \\ m(x,0) = \bar{m}(x), \end{cases} \tag{6.2.5}$$

where $\bar{m}$ is the initial distribution of the agents.

Taking ℓ_0 as in (6.2.2), the transport equation above becomes

$$m_t - \frac{1}{c}(m(\varpi + u_x))_x = 0.$$

Finally, we fix an *energy production function* $\mathbf{Q}(t)$ and require that the production balances demand. Mathematically, this constraint corresponds to the identity

$$\int_{\mathbb{R}} \alpha^*(t) m(x,t) dx = \mathbf{Q}(t);$$

that is,

$$\int_{\mathbb{R}} D_p H(x, u_x + \varpi(t)) m(x,t) dx = -\mathbf{Q}(t). \tag{6.2.6}$$

This foregoing equality is the balance equation that forces the consumed energy to match the production; this constraint determines the price, $\varpi(t)$.

Combining (6.2.4), (6.2.5) and (6.2.6), we obtain (6.1.1) with $\epsilon = 0$ and the initial-terminal conditions (6.1.2).

Now, we consider the case where the agents are subject to independent random consumption. In this case $\epsilon > 0$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a *sample space*, $\mathcal{F}$ a *σ-algebra* on Ω and P a *probability measure*. Let W_t be a Brownian motion on Ω and $\{\mathcal{F}_t\}_{t\geq 0}$ the associated *filtration*. In this case, we model the agent's motion by the stochastic differential equation

$$d\mathbf{x}(t) = \alpha(t)dt + \sqrt{2\epsilon} dW_t,$$

where the control, α, is a bounded progressively measurable real-valued process. Following the previous steps and using standard arguments in stochastic optimal control, we arrive again at (6.1.1).

6.3 Main Assumptions

We begin by discussing our main assumptions. First, we suppose that H is the Legendre transform of a Lagrangian that is the sum of an "energy flow cost", $\ell_0(\alpha)$, and a "charge preference cost", $V(x)$, as follows:

Assumption 3. The Hamiltonian H is the Legendre transform of a convex Lagrangian:

$$H(x,p) = \sup_{\alpha \in \mathbb{R}} -p\alpha - \ell_0(\alpha) - V(x), \tag{6.3.1}$$

where $\ell_0 \in C^2(\mathbb{R})$ is a uniformly convex function and $V \in C^2(\mathbb{R})$ is bounded from below.

Remark 5. The preceding hypothesis implies that the map $p \mapsto H(x,p)$ is (strictly) convex. Moreover, the Hamiltonian in (6.3.1) can be written as

$$H(x,p) = H_0(p) - V(x). \tag{6.3.2}$$

Thus,

$$D^2_{xp} H(x,p) = 0$$

for all $x, p \in \mathbb{R}$.

To obtain a fixed point, we need several a priori estimates. These depend on convexity and regularity properties of the data. The following two assumptions lay out our requirements on the potential, V.

Assumption 4. The potential V in (6.3.1) and the terminal data $\bar{u}$ are globally Lipschitz.

Assumption 5. The potential V in (6.3.1) and the terminal data $\bar{u}$ satisfy

$$|D^2_{xx}V| \leq C, \qquad |D^2_{xx}\bar{u}| \leq C$$

for some positive constant C.

Next, we state an additional regularity for the initial-terminal data that is used to prove second-order estimates.

Assumption 6. There exists a constant, $C > 0$, such that

$$|\bar{m}_{xx}|, |\bar{u}_{xx}| \leq C.$$

The next two assumptions are used to ensure the solvability of the demand-supply relation; that is, given $\mathbf{Q}$ that we can determine a suitable price.

Assumption 7. There exists $\theta > 0$ such that

$$D^2_{pp}H(x, p) > \theta$$

for all $x, p \in \mathbb{R}$. In addition, there exists $C > 0$ such that

$$|D^3_{ppp}H| \leq C.$$

Remark 6. Using (6.3.2) in Remark 5, the preceding assumption combined with Assumption 3 implies that the function $p \mapsto D_pH(p, x)$ is strictly increasing and

$$\lim_{p\to-\infty} D_pH(p, x) = -\infty \qquad \lim_{p\to+\infty} D_pH(p, x) = +\infty,$$

uniformly in x.

Remark 7. The uniform convexity of ℓ_0 in Assumption 3 gives an upper bound for $D^2_{pp}H$. Thus, Assumption 3 and 7 imply

$$|D^2_{pp}H(x, p)| \leq C$$

for all $x, p \in \mathbb{R}$.

The following hypothesis gives regularity and uniqueness of solutions in the first-order case.

Assumption 8. The potential, V, and the terminal cost, $\bar{u}$, are convex.

6.4 Existence of a solution

Here, we establish the existence of a solution for the price model, (6.1.1), using a fixed-point argument on ϖ. In the following two propositions, we examine the Hamilton-Jacobi equation

$$\begin{cases} -u_t + H(x, \varpi + u_x) = \epsilon u_{xx} \\ u(x,T) = \bar{u}(x). \end{cases} \tag{6.4.1}$$

First, using Assumption 4, we prove the Lipschitz continuity of u. Next, using Assumption 5, we obtain the semiconcavity of u. The proofs follow standard arguments in optimal control theory. However, we present them here to make it evident that the Lipschitz and semiconcavity constants are uniform in ϖ and ϵ, both essential points in our argument.

Proposition 10. Consider the setting of Problem 1 and suppose that Assumptions 3 and 4 hold. Let u solve (6.4.1). Then, $u(x,t)$ is locally bounded and the map $x \mapsto u(x,t)$ is Lipschitz for $0 \le t \le T$. Moreover, the Lipschitz bound on u does not depend on ϖ nor on ϵ.

Proof. The proof follows from the representation of u as a solution to a stochastic control problem (or deterministic if $\epsilon = 0$). We fix a filtered probability space $(\Omega, \mathcal{F}_t, P)$ that supports a one-dimensional Brownian motion W_t. Then,

$$u(x,t) = \inf E\left[\int_t^T \ell_0(\alpha) + \varpi\alpha + V(\mathbf{x})ds + \bar{u}(\mathbf{x}(T))\right],$$

where the infimum is taken over bounded progressively measurable controls $\alpha : [t,T] \to \mathbb{R}$ and $\mathbf{x}$ solves the stochastic differential equation

$$d\mathbf{x} = \alpha dt + \sqrt{2\epsilon} dW_t.$$

To prove local boundedness, we use the sub-optimal control $\alpha \equiv 0$ to get an upper bound, and the fact that V is bounded by below to obtain the lower bound. We observe, however, that the lower bound depends on bounds on ϖ.

Then, we fix an optimal control, α^*, for (x,t); that is,

$$u(x,t) = E\left[\int_t^T \ell_0(\alpha^*) + \varpi\alpha^* + V(\mathbf{x}^*)ds + \bar{u}(\mathbf{x}(T)^*)\right].$$

Then, for any $h \in \mathbb{R}$, we have

$$u(x+h,t) \le E\left[\int_t^T \ell_0(\alpha^*) + \varpi\alpha^* + V(\mathbf{x}^* + h)ds + \bar{u}(\mathbf{x}(T)^* + h)\right],$$

from which the Lipschitz bound follows. Note that this Lipschitz bound does not depend on ϖ, only on T and on the Lipschitz estimates for V and $\bar{u}$. □

Proposition 11. Consider the setting of Problem 1 and suppose that Assumptions 3 and 5 hold. Then, $x \mapsto u(x,t)$ is semiconcave with a semiconcavity constant that does not depend on ϵ nor on ϖ.

Proof. As before, we fix an optimal control α^* for (x,t); that is,

$$u(x,t) = E\left[\int_t^T \ell_0(\alpha^*) + \varpi\alpha^* + V(\mathbf{x}^*)ds + \bar{u}(\mathbf{x}(T)^*)\right].$$

Then, for any $h \in \mathbb{R}$, we have

$$u(x \pm h, t) \le E\left[\int_t^T \ell_0(\alpha^*) + \varpi\alpha^* + V(\mathbf{x}^* \pm h)ds + \bar{u}(\mathbf{x}(T)^* \pm h)\right].$$

Therefore,

$$u(x+h,t) - 2u(x,t) + u(x-h,t) \le Ch^2.$$

Note that C does not depend on ϖ, only on T and on the semiconcavity estimates for V and $\bar{u}$. □

We have the following stability properties for the solutions of (6.4.1).

Proposition 12. Consider the setting of Problem 1 and suppose that Assumptions 3–5 hold. Suppose that $\varpi_n \to \varpi$ uniformly on $[0,T]$, then $u^n \to u$ locally uniformly and $u_x^n \to u_x$ almost everywhere.

Proof. The local uniform convergence of u^n follows from the stability of viscosity solutions. Because u^n is semiconcave and converges uniformly to u, $u_x^n \to u_x$ almost everywhere. □

Now, we examine the Fokker-Planck equation.

$$\begin{cases} m_t - \operatorname{div}(mD_pH(x, \varpi + u_x)) = \epsilon\Delta m, \\ m(x,0) = \bar{m}(x). \end{cases} \tag{6.4.2}$$

Let $\mathcal{P}$ denote the set of probability measures on $\mathbb{R}$ with finite second-moment and endowed with the 1-Wasserstein distance.

Proposition 13. Consider the setting of Problem 1 with $\epsilon > 0$ and suppose that Assumptions 3–4 hold. Then, (6.4.2) has a solution $m \in C([0,T], \mathcal{P})$. Moreover,

$$d_1(m(t), m(t+h)) \le Ch^{1/2}. \tag{6.4.3}$$

In addition, if Assumptions 5 and 6 hold, for any sequence $\varpi_n \to \varpi$ uniformly on $[0,T]$ and corresponding solutions u_n of (6.4.1) and m_n of (6.4.2), we have $m_n \to m$ in $C([0,T], \mathcal{P}_1)$.

Proof. The existence of a solution in $C([0,T],\mathcal{P}_1)$ and the estimate in (6.4.3) were proven in [54]. We note that, for $\epsilon \leq \epsilon_0$, the constant C can be chosen to depend only on ϵ_0, on the problem data, and on $\|\varpi\|_{L^\infty}$. Thus, by the Ascoli-Arzela theorem, we have that $m_n \to m$ in $C([0,T],\mathcal{P}_1)$ for some $m \in C([0,T],\mathcal{P}_1)$. Because $\epsilon > 0$, $m_n \to m$ in, for example, $L^2(\mathbb{R}\times[0,T])$. Moreover, (6.4.2) has a unique solution. Thus, it suffices to check that m solves (6.4.2). Because $u_x^n \to u_x$, almost everywhere, by semiconcavity, we have for any $\psi \in C_c^\infty(\mathbb{R}\times[0,T])$

$$\int_0^T\int_{\mathbb{R}} \psi_x D_pH(x,\varpi^n+u_x^n)m^n dxdt \to \int_0^T\int_{\mathbb{R}} \psi_x D_pH(x,\varpi+u_x)mdxdt,$$

which gives that m is a weak solution of (6.4.2). □

Next, we prove an estimate for solutions of the system comprising (6.4.1) and (6.4.2).

Proposition 14. Consider the setting of Problem 1 with $\epsilon > 0$ and suppose that Assumptions 3 and 6 hold. Let (u,m) solve 6.4.1 and 6.4.2. Then

$$\int_0^T\int_{\mathbb{R}} D_{pp}^2 H u_{xx}^2 mdxdt \leq C \tag{6.4.4}$$

Proof. We begin by differentiating (6.4.1) twice with respect to x, multiply by m, and integrate by parts using (6.4.2). □

Remark 8. Formally, the previous estimates hold for $\epsilon = 0$. However, the above proof requires that u is three times differentiable, which is not usually the case. Nevertheless, the estimate in (6.4.4) is uniform in ϵ.

Finally, we consider the price-supply relation. Due to Remark 6 and to the Lipschitz continuity of u given by Proposition 10, there exists a unique ϑ_0 such that

$$\int_{\mathbb{R}} D_pH(x,\vartheta_0+u_x(x,0))\bar{m}dx = -\mathbf{Q}(0). \tag{6.4.5}$$

Moreover, ϑ_0 is bounded by a constant that depends only on the problem data.

Next, we differentiate

$$\int_{\mathbb{R}} D_pH(x,\varpi+u_x)mdx = -\mathbf{Q}(t)$$

in time to get the identity

$$\dot{\varpi}\int_{\mathbb{R}} D_{pp}^2 Hmdx + \int_{\mathbb{R}}\left[D_{pp}^2 Hu_{xt}m + D_pHm_t\right]dx = -\dot{\mathbf{Q}}. \tag{6.4.6}$$

Differentiating (6.4.1) in x and substituting (6.4.2) both quantities on the second term of the left hand side of (6.4.6), we get the following identity

$$\begin{aligned}\int_{\mathbb{R}} D^2_{pp}Hu_{xt}m + D_pHm_t = &\int_{\mathbb{R}} D^2_{pp}H\left(-\epsilon\Delta u_x + D_pHu_{xx} + D_xH\right)m\\ &+\int_{\mathbb{R}} D_pH\left(\epsilon\Delta m + (mD_pH)_x\right).\end{aligned}$$

If Assumption 3 holds, we have by Remark 5 that $D^2_{xp}H = 0$. Hence,

$$\int_{\mathbb{R}} D^2_{pp}Hu_{xt}m + D_pHm_t = \int_{\mathbb{R}} D^2_{pp}HD_xHm + \epsilon D^3_{ppp}Hu^2_{xx}m. \tag{6.4.7}$$

Accordingly, we have the identity

$$\dot{\varpi}\int_{\mathbb{R}} D^2_{pp}Hm = -\dot{\mathbf{Q}} - \int_{\mathbb{R}}\left(D^2_{pp}HD_xH + \epsilon D^3_{ppp}Hu^2_{xx}\right)m. \tag{6.4.8}$$

Thus, given ϖ, we solve (6.4.1) and (6.4.2) and define the following ordinary differential equation

$$\begin{cases}\dot{\vartheta} = \frac{-\dot{\mathbf{Q}}-\int_{\mathbb{R}} D^2_{pp}H(x,\varpi+u_x)D_xH(x,\varpi+u_x)m+\epsilon D^3_{ppp}H(x,\varpi+u_x)u^2_{xx}m}{\int_{\mathbb{R}} D^2_{pp}H(x,\varpi+u_x)m}\\ \vartheta(0) = \vartheta_0,\end{cases} \tag{6.4.9}$$

where ϑ_0 is determined by (6.4.5). Then, (u, m, ϖ) solves (6.1.1) if ϖ solves (6.4.9).

Proposition 15. Consider the setting of Problem 1 with $\epsilon > 0$ and suppose that Assumptions 3–7 hold. Suppose that $\varpi^n \to \varpi$ uniformly in $C([0,T])$. Let u^n, m^n, and ϑ^n be the solutions to (6.4.1), (6.4.2), and 6.4.9 with ϖ replaced by ϖ^n. Then, ϑ^n converges to ϑ, uniformly in $C([0,T])$, where ϑ solves (6.4.9). Moreover, there exists a constant C that depends only on the problem data but not on ϖ such that $\|\vartheta\|_{W^{1,\infty}([0,T])} \leq C$.

Proof. The bound in $W^{1,\infty}([0,T])$ for ϑ is a consequence of Remark 6 and of the bounds in Assumption 7, in Remark 7, and in Proposition 14.

According to Proposition 12, the uniform convergence of $\varpi_n \to \varpi$ gives the convergence of $u^n_x \to u_x$, almost everywhere. In addition, Proposition 13 gives the convergence $m_n \to m$ in $C([0,T],\mathcal{P})$. Because $D^2_{pp}H$ is bounded from below by Assumption 7, we have the convergence of the right-hand side of (6.4.9) as follows, for any $\psi \in C([0,T])$,

$$\int_0^T \psi\dot{\vartheta}_n ds \to \int_0^T \psi\dot{\vartheta}ds.$$

Also, because the family ϑ_n is equicontinuous, any subsequence has a further convergent subsequence that must converge to ϑ. Thus, $\vartheta^n \to \vartheta$, uniformly. □

With the preceding estimates, we can now prove a fixed-point result and show the existence of a solution for $\epsilon > 0$.

Proof of Theorem 15 - part 1, existence for $\epsilon \geq 0$. We begin by addressing the case $\epsilon > 0$. According to Proposition 15, the map $\varpi \to \vartheta$ determined by (6.4.1), (6.4.2), and (6.4.9) is continuous in $C([0,T])$, bounded, and compact due to the $W^{1,\infty}$ bound for ϖ. Thus, by Schauder's fixed-point theorem, it has a fixed point.

Now, we examine the case $\epsilon = 0$. The key difficulty is the continuity of the map $\varpi \to m$ in the case $\epsilon = 0$. To overcome this difficulty, we use the vanishing viscosity method and the techniques in [79].

Let $(u^\epsilon, m^\epsilon, \varpi^\epsilon)$ solve (6.1.1) with $\epsilon > 0$. By the above, we have that ϖ^ϵ is uniformly bounded. Moreover, by Proposition 10, u^ϵ is uniformly locally bounded and Lipschitz. Therefore, as $\epsilon \to 0$, extracting a subsequence if necessary, $\varpi^\epsilon \to \varpi$ and $u^\epsilon \to u$ where u is a viscosity solution of (6.4.1).

Now, we introduce a phase-space measure μ^ϵ as follows

$$\int_0^T \int_{\mathbb{R}^2} \psi(x,p,t) d\mu^\epsilon(x,p,t) = \int_0^T \int_{\mathbb{R}} \psi(x, \varpi^\epsilon + u_x^\epsilon, t) m^\epsilon dx dt$$

for all $\psi \in C_b(\mathbb{R} \times \mathbb{R} \times [0,T])$. Because $m^\epsilon \in C([0,T], \mathbb{R})$ with a modulus of continuity that is uniform in ϵ, as $\epsilon \to 0$, we have $\mu^\epsilon \rightharpoonup \mu$; that is

$$\int_0^T \int_{\mathbb{R}^2} \psi d\mu^\epsilon \to \int_0^T \int_{\mathbb{R}^2} \psi d\mu.$$

Moreover, due to the strict convexity of the Hamiltonian, arguing as in [79], we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^2} \psi_t - D_p H(x,p) D_x \psi d\mu \\ &\quad = \int_{\mathbb{R}} \psi(x,T) m(x,T) dx - \int_{\mathbb{R}} \psi(x,0) \bar{m}(x) dx. \end{aligned}$$

Next, we fix $\delta > 0$ and consider a standard mollifier η_δ. We define

$$v^\delta = \eta_\delta * u,$$

as the convolution.

We note that $|D^2 v^\delta| \leq \frac{C}{\delta^2}$. Then, using the uniform convexity of the Hamiltonian, we get

$$-v_t^\delta + \eta_\delta * |u_x - v_x^\delta|^2 + H(x, \varpi + v_x^\delta) \leq O(\delta).$$

Therefore, $w = v^\delta - u^\epsilon$ satisfies

$$\begin{aligned} &- w_t + D_p H(x, \varpi^\epsilon + u_x^\epsilon) w_x - \epsilon w_{xx} \\ &+ \eta_\delta * |u_x - v_x^\delta|^2 + \gamma |\varpi + v_x^\delta - \varpi^\epsilon - u_x^\epsilon|^2 \leq O(\delta) + O(\frac{\epsilon}{\delta^2}). \end{aligned}$$

Integrating with respect to m^ϵ, we conclude that

$$\int_0^T \int_{\mathbb{R}^2} \eta_\delta * |u_x - v_x^\delta|^2 + \gamma|\varpi + v_x^\delta - p|^2 d\mu^\epsilon \leq O(\delta) + O(\frac{\epsilon}{\delta^2}) + \|v^\delta - u^\epsilon\|_{L^\infty}.$$

Next, we let $\epsilon \to 0$, to get

$$\int_0^T \int_{\mathbb{R}^2} \eta_\delta * |u_x - v_x^\delta|^2 + \gamma|\varpi + v_x^\delta - p|^2 d\mu \leq O(\delta).$$

Finally, by letting $\delta \to 0$, we conclude that m-almost every point is a point of approximate continuity of u_x. Therefore, $v_x^\delta \to u_x$ almost everywhere. Hence, $p = \varpi + u_x$ μ-almost everywhere. Therefore, we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} (\psi_t - D_pH(x, \varpi + u_x)D_x\psi)\, d\mu \\
&= \int_0^T \int_{\mathbb{R}} (\psi_t - D_pH(x, \varpi + u_x)D_x\psi)\, mdxdt \\
&= \int_{\mathbb{R}} \psi(x, T)m(x, T)dx - \int_{\mathbb{R}} \psi(x, 0)\bar{m}(x)dx,
\end{aligned}$$

which gives that m solves (6.4.2) with $\epsilon = 0$.

Note also, that the preceding reasoning implies that u is differentiable almost everywhere with respect to m. □

Finally, we record two additional results for (6.1.1). The first is an energy estimate that is similar to other results in MFG.

Proposition 16. Let (u, m, ϖ) be the solution of Problem 1 constructed in Theorem 15. Suppose that there exists $C > 0$ such that

$$pD_pH(x, p) - H(x, p) \geq \frac{1}{C}H(x, p) - C.$$

Then,

$$\int_0^T \int_{\mathbb{R}} H(x, \varpi + u_x)(m_0 + m)dxdt \leq C.$$

Proof. We take the first equation in (6.1.1) and multiply it by $\bar{m} - m$, and the second equation by $u - \bar{u}$. Adding the resulting expressions and integrating by parts results in the desired estimate. □

The last result in this section concerns the regularity of the solutions (6.4.1) in the case where both the potential and terminal data are convex.

Proposition 17. Suppose that $\epsilon = 0$, that Assumptions 3, 5, and 8 hold and let ϖ be a Lipschitz function. Then, the solution to (6.4.1) is differentiable in x for every $x \in \mathbb{R}$. Moreover, u_{xx} is bounded.

Proof. Due to Assumption 8, we see that $u(x,t)$ is convex in x by direct inspection of the variational problem (6.2.3). By Proposition 11, u is semiconcave in x. This gives the bound for u_{xx} and the differentiability of u in x. □

The preceding proposition implies the regularity of the solutions of Problem 1, as stated in the next Corollary.

Corollary 1. Suppose that Assumptions 3–8 hold and that $\epsilon = 0$. Then, there exists a solution (u, m, ϖ) of Problem 1 with u differentiable in x for every x and u_{xx} bounded. Moreover, m is also bounded.

Proof. The result follows by combining Proposition 17 with the fact that the transport equation with locally Lipschitz coefficients has a unique weak solution in L^∞. □

Proof of Theorem 15 - part 2, additional regularity for $\epsilon = 0$. Additional regularity for the case where Assumption 8 holds and $\epsilon = 0$ follows from Corollary 1. □

6.5 Uniqueness

Now, we examine the uniqueness of solutions. We begin by observing that (6.1.1) can be written as a monotone operator. As a consequence, we obtain a uniqueness result.

We set

$$\Omega_T = \mathbb{R} \times [0, T],$$

and

$$\begin{aligned}
D =&(C^\infty(\Omega_T) \cap C([0,T], \mathcal{P})) \times (C^\infty(\Omega_T) \cap W^{1,\infty}(\Omega_T)) \times C^\infty([0,T]),\\
D_+ =&\{(m, u, \varpi) \in D \text{ s.t. } m > 0\},\\
D^b =&\{(m, u, \varpi) \in D \text{ s.t. } m(x,0) = \bar{m}(x),\ u(x,T) = \bar{u}(x)\},\\
D^b_+ =&D^b \cap D_+,
\end{aligned}$$

Then, we define $A : D^b_+ \to D$ as

$$\begin{aligned}
A\begin{bmatrix} m \\ u \\ \varpi \end{bmatrix} =&A_1\begin{bmatrix} m \\ u \\ \varpi \end{bmatrix} + A_2\begin{bmatrix} m \\ u \\ \varpi \end{bmatrix}\\
=&\begin{bmatrix} u_t + \epsilon u_{xx} \\ m_t - \epsilon m_{xx} \\ 0 \end{bmatrix} + \begin{bmatrix} -H(x, Du + \varpi) \\ -\operatorname{div}(m D_p H(x, \varpi + u_x)) \\ \int_\Omega m D_p H(x, \varpi + u_x) dx + \mathbf{Q}(t) \end{bmatrix}.
\end{aligned} \tag{6.5.1}$$

Furthermore, for $w=(m,u,\varpi),\ \tilde{w}=(\tilde{m},\tilde{u},\tilde{\varpi})\in D$, we set

$$\langle w,\tilde{w}\rangle=\int\limits_{\Omega_T}(m\tilde{m}+u\tilde{u})\,dxdt+\int\limits_0^T\varpi\tilde{\varpi}dt.$$

Then, A is a *monotone operator* if

$$\langle A[w]-A[\tilde{w}],w-\tilde{w}\rangle\geq 0\quad\text{for all}\quad w,\tilde{w}\in D_+^b.$$

Under the convexity of the map $p\mapsto H(x,p)$, A is a monotone operator.

Proposition 18. Suppose the map $p\mapsto H(x,p)$ is convex. Then A is a monotone operator.

Proof. Let $w=(m,u,\varpi),\ \tilde{w}=(\tilde{m},\tilde{u},\tilde{\varpi})\in D_+^b$. Then, integrating by parts, we obtain

$$\begin{aligned}&\langle A_1[w]-A_1[\tilde{w}],w-\tilde{w}\rangle\\=&\int\limits_{\Omega_T}((u-\tilde{u})_t+\epsilon\Delta(u-\tilde{u}))(m-\tilde{m})+\int\limits_{\Omega_T}((m-\tilde{m})_t-\epsilon\Delta(m-\tilde{m}))(u-\tilde{u})\\=&0,\end{aligned}$$

because $u-\tilde{u}$ and $m-\tilde{m}$ vanish at $t=0$ and $t=T$. Furthermore, we have that

$$\begin{aligned}
&\langle A_2[w]-A_2[\tilde{w}],w-\tilde{w}\rangle\\
=&-\int\limits_{\Omega_T}(H(x,u_x+\varpi)-H(x,\tilde{u}_x+\tilde{\varpi}))(m-\tilde{m})dxdt\\
&-\int\limits_{\Omega_T}\operatorname{div}(mD_pH(x,u_x+\varpi)-\tilde{m}D_pH(x,\tilde{u}_x+\tilde{\varpi}))(u-\tilde{u})dxdt\\
&+\int\limits_0^T(\varpi-\tilde{\varpi})\int_{\mathbb{R}}(mD_pH(x,u_x+\varpi)-\tilde{m}D_pH(x,\tilde{u}_x+\tilde{\varpi}))dxdt\\
=&-\int\limits_{\Omega_T}(H(x,u_x+\varpi)-H(x,\tilde{u}_x+\tilde{\varpi}))(m-\tilde{m})dxdt\\
&+\int\limits_{\Omega_T}(mD_pH(x,u_x+\varpi)-\tilde{m}D_pH(x,\tilde{u}_x+\tilde{\varpi}))(u_x-\tilde{u}_x)dxdt\\
&+\int\limits_{\Omega_T}(mD_pH(x,u_x+\varpi)-\tilde{m}D_pH(x,\tilde{u}_x+\tilde{\varpi}))(\varpi-\tilde{\varpi})dxdt\\
=&\int\limits_{\Omega_T}m\Big(H(x,\tilde{u}_x+\tilde{\varpi})-H(x,u_x+\varpi)-(\tilde{u}_x+\tilde{\varpi}-u_x-\varpi)D_pH(x,u_x+\varpi)\Big)dxdt\\
&+\int\limits_{\Omega_T}\tilde{m}\Big(H(x,u_x+\varpi)-H(x,\tilde{u}_x+\tilde{\varpi})-(u_x+\varpi-\tilde{u}_x-\tilde{\varpi})D_pH(x,\tilde{u}_x+\tilde{\varpi})\Big)dxdt\\
\geq&0,
\end{aligned}$$

by the convexity of $p\mapsto H(x,p)$. Combining the previous inequalities, we conclude that

$$\begin{aligned}
&\langle A[w]-A[\tilde{w}],w-\tilde{w}\rangle\\
=&\langle A_1[w]-A_1[\tilde{w}],w-\tilde{w}\rangle+\langle A_2[w]-A_2[\tilde{w}],w-\tilde{w}\rangle\geq 0.
\end{aligned}$$

□

Now, we discuss the last part of the proof of Theorem 15.

Proof of Theorem 15 - part 3, uniqueness. Let (m,u,ϖ) and $(\tilde{m},\tilde{u},\tilde{\varpi})$ solve Problem 1. If $\epsilon>0$ or if $\epsilon=0$ and Assumption 8 holds, we have m and $\tilde{m}$ are absolutely continuous with respect to the Lebesgue measure. Thus, the computations in the proof of Proposition 18, combined with the uniform convexity of H in Assumption 7, give

$$\int_0^T\int_{\mathbb{R}}|\varpi+u_x-\tilde{\varpi}-\tilde{u}_x|^2(\tilde{m}+m)=0.$$

Therefore, $\varpi + u_x = \tilde{\varpi} + \tilde{u}_x$ almost everywhere. In both cases, this implies

$$u_t = \tilde{u}_t,$$

almost everywhere and, thus, $u = \tilde{u}$. Finally, the uniqueness of the Fokker-Planck equation, for $\epsilon > 0$ or for the transport equation, when $\epsilon = 0$ and Assumption 8 holds, give $m = \tilde{m}$. $\square$

6.6 Linear-quadratic models

Here, we consider linear-quadratic price models. First, we examine the case without a potential and determine an explicit solution. Then, we introduce a quadratic potential that accounts for charge level preferences. In this last case, we describe a procedure to solve the problem, up to the inversion of Laplace transforms and solution of ordinary differential equations.

6.6.1 State-independent quadratic cost

First, we consider the quadratic state-independent cost

$$\ell(t,\alpha) = \frac{c}{2}\alpha^2 + \alpha\varpi(t), \tag{6.6.1}$$

where c is a constant that accounts for the usage-depreciation of the battery. The corresponding MFG is

$$\begin{cases} -u_t + \frac{(\varpi(t)+u_x)^2}{2c} = 0 \\ m_t - \frac{1}{c}(m(\varpi(t)+u_x))_x = 0 \\ \frac{1}{c}\int_{\mathbb{R}}(\varpi(t)+u_x)m dx = -\mathbf{Q}(t). \end{cases} \tag{6.6.2}$$

The stored energy by each agent follows optimal trajectories that solve the Euler Lagrange equation:

$$c\ddot{\mathbf{x}} + \dot{\mathbf{p}} = 0.$$

Integrating the previous equation in time, we get

$$\dot{\mathbf{x}}(t) = \frac{1}{c}\left(\theta - \varpi(t)\right), \tag{6.6.3}$$

where θ is time independent. Next, by differentiating the Hamilton-Jacobi equation, we get

$$-(u_x)_t + (u_x + \varpi)\frac{u_{xx}}{c} = 0.$$

Using the previous equation, taking into account the transport equation, and integrating by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} u_x m dx &= \int_{\mathbb{R}} u_{xt} m + u_x m_t = \int_{\mathbb{R}} u_{xt} m + \frac{1}{c} u_x \left(m(\varpi + u_x)\right)_x \\ &= \frac{1}{c}\int_{\mathbb{R}} (\varpi + u_x) u_{xx} m - u_{xx} m(\varpi + u_x) dx = 0,\end{aligned}$$

assuming that m has fast enough decay at infinity.

Thus, the supply vs demand balance condition becomes

$$\mathbf{Q}(t) = -\frac{1}{c}\int_{\mathbb{R}} (u_x + \varpi) m dx = \frac{1}{c}\left(\Theta - \varpi\right),$$

where

$$\Theta = -\int_{\mathbb{R}} u_x m dx \tag{6.6.4}$$

is constant. From the above, we obtain the following linear price-supply relation

$$\varpi = \Theta - c\mathbf{Q}(t). \tag{6.6.5}$$

Integrating (6.6.3) in time and taking into account the linear price-supply relation (6.6.5), we gather

$$\mathbf{x}(T) = \mathbf{x}(t) + \frac{1}{c}\int_t^T (\theta - \varpi(s)) ds = x + \frac{T-t}{c}(\theta - \Theta) + \int_t^T \mathbf{Q}(s) ds. \tag{6.6.6}$$

Accordingly, u is given by the optimization problem

$$\begin{aligned}u(x,t) = &\inf_{\theta} \int_t^T \left[\frac{(\theta - \Theta + c\mathbf{Q}(s))^2}{2c} + \frac{1}{c}(\theta - \Theta + c\mathbf{Q}(s))(\Theta - c\mathbf{Q}(s))\right] ds \\ &+ \bar{u}\left(x + \frac{(\theta - \Theta)}{c}(T-t) + K(t)\right),\end{aligned}$$

where

$$K(t) = \int_t^T Q(s) ds.$$

By setting $\mu = \theta - \Theta$, we get

$$\begin{aligned}u(x,t) = &\inf_{\mu} \int_t^T \left[\frac{(\mu + c\mathbf{Q}(s))^2}{2c} + \frac{1}{c}(\mu + c\mathbf{Q}(s))(\Theta - c\mathbf{Q}(s))\right] ds \\ &+ \bar{u}\left(x + \frac{\mu}{c}(T-t) + K(t)\right).\end{aligned}$$

Thus, given Θ, we determine a function, u^Θ, solving the preceding minimization problem. For that, we expand the integral to get

$$u^\Theta(x,t) = \inf_\mu \left[\frac{T-t}{2c}\mu^2 + \frac{1}{c}(T-t)\Theta\mu + \int_t^T \left(\Theta - c\frac{\mathbf{Q}(s)}{2}\right)\mathbf{Q}(s)ds \right.$$
$$\left. + \bar{u}\left(x + \frac{\mu}{c}(T-t) + K(t)\right)\right].$$

Next, we take the derivative of the right-hand side of the prior identity with respect to μ and obtain the relation

$$\mu + \bar{u}_x\left(\mathbf{x}(T)\right) = -\Theta. \tag{6.6.7}$$

If $\bar{u}$ is a convex function, the preceding equation has a unique solution, $\mu(\Theta)$ for each given Θ. Thus, given Θ, we obtain a solution, u^Θ for the Hamilton-Jacobi equation. Finally, we use the resulting expression for u^Θ in (6.6.4) at $t=0$ to obtain the following condition for Θ:

$$\Theta = -\int_{\mathbb{R}} u_x^\Theta(x,0)m_0(x)dx. \tag{6.6.8}$$

Solving the preceding equation, we obtain Θ and hence ϖ using the price-supply relation, (6.6.5).

As an example, we consider the terminal cost

$$\bar{u}(y) = \frac{\gamma}{2}\left(y-\zeta\right)^2.$$

Solving(6.6.7), we obtain

$$\mu = -\frac{\gamma(K(t)+x-\zeta)+\Theta}{1+\gamma\frac{T-t}{c}}. \tag{6.6.9}$$

Accordingly, we have

$$u^\Theta(x,t) = \frac{\gamma(K(t)+x-\zeta)^2 + \frac{(t-T)}{c}\Theta(2\gamma(K(t)+x-\zeta)+\Theta)}{2\left(1+\gamma\frac{T-t}{c}\right)} + \Theta K(t) - c\int_t^T \frac{\mathbf{Q}^2(s)}{2}ds.$$

Therefore,

$$u_x(x,t) = \gamma\frac{K(t)+x-\zeta-\frac{(T-t)}{c}\Theta}{1+\gamma\frac{T-t}{c}}$$

Using the previous expression for $t=0$ in (6.6.8), we obtain the following equation for Θ

$$\Theta = -\gamma\frac{K(0)+\bar{x}-\zeta-\frac{T}{c}\Theta}{1+\gamma\frac{T}{c}}$$

where

$$\bar{x} = \int_{\mathbb{R}} x m_0 dx.$$

Thus,

$$\Theta = -\gamma(K(0) + \bar{x} - \zeta). \tag{6.6.10}$$

Therefore, using (6.6.5), we obtain

$$\varpi = -\gamma(K(0) + \bar{x} - \zeta) - c\mathbf{Q}.$$

Finally, we use the above results and conclude that each agent dynamics is

$$\begin{cases} \dot{\mathbf{x}} = \frac{(\bar{x}-x)\gamma}{1+\frac{T}{c}\gamma} + \mathbf{Q} \\ \mathbf{x}(0) = x. \end{cases}$$

In alternative, using

$$\dot{\mathbf{x}}(t) = -\frac{\varpi + u_x(\mathbf{x}(t), t)}{c}$$

we have

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{(\bar{\mathbf{x}}(t)-\mathbf{x}(t))\gamma}{1+\frac{T-t}{c}\gamma} + \mathbf{Q} \\ \mathbf{x}(0) = x, \end{cases} \tag{6.6.11}$$

where

$$\bar{\mathbf{x}}(t) = \int_{\mathbb{R}} x m(x,t) dx.$$

Averaging (6.6.11) with respect to m, we obtain

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}(t), \tag{6.6.12}$$

which is simply the conservation of energy. Thus, the trajectory of an individual agent can be computed by combining (6.6.11) with (6.6.12) into the system

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{(\bar{\mathbf{x}}(t)-\mathbf{x}(t))\gamma}{1+\frac{T-t}{c}\gamma} + \mathbf{Q}(t) \\ \dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}. \end{cases}$$

The previous system is a closed system of ordinary differential equations that only involves $\mathbf{Q}$ and the parameters of the problem. Surprisingly, it also does not depend on ζ. This is due to the fact that the average of the position of the agents is determined by $\mathbf{Q}$. Hence, the only way agents can improve their value function is by getting close to each other. This is seen in the mean-reverting structure in (6.6.11).

6.6.2 Quadratic cost with potential

Now, we consider a running cost with a quadratic potential. This potential penalizes the agents when the charge or stored energy deviates too much from a set point, κ. This penalty has the form of $\frac{\eta}{2}(x-\kappa)^2$, where η measures the strength of the penalty. Thus, we have

$$\ell(t,x,\alpha) = c\frac{\alpha^2}{2} + \alpha\varpi(t) + \frac{\eta}{2}(x-\kappa)^2.$$

The corresponding MFG is

$$\begin{cases} -u_t + \frac{(\varpi(t)+u_x)^2}{2c} - \frac{\eta}{2}(x-\kappa)^2 = 0 \\ m_t - \frac{1}{c}(m(\varpi(t)+u_x))_x = 0 \\ \frac{1}{c}\int(\varpi(t)+u_x)m = -\mathbf{Q}(t). \end{cases} \tag{6.6.13}$$

Differentiating the Hamilton-Jacobi equation, we conclude that

$$-(u_x)_t + (u_x+\varpi)u_{xx} - \eta(x-\kappa) = 0.$$

We define the following quantities

$$\Pi = \int_{\mathbb{R}} u_x m \text{ and } \Xi = \int_{\mathbb{R}} xm.$$

Taking the time derivative on the first quantity and using the transport equation, we get

$$\begin{aligned} \dot{\Pi} &= \int_{\mathbb{R}} u_{xt}m + u_x m_t \\ &= \int_{\mathbb{R}} (\varpi+u_x)u_{xx}m - \eta\int_{\mathbb{R}}(x-\kappa)m + \int_{\mathbb{R}} u_x(m(\varpi+u_x))_x \\ &= \int_{\mathbb{R}} (\varpi+u_x)u_{xx}m - \int_{\mathbb{R}} u_{xx}m(\varpi+u_x) - \eta\int_{\mathbb{R}}(x-\kappa)m. \end{aligned}$$

Simplifying the preceding expression, we obtain

$$\dot{\Pi} = -\eta(\Xi-\kappa).$$

Next, we take the transport equation, multiply it by x, and integrate by parts, to get

$$\begin{aligned} \dot{\Xi} &= \frac{d}{dt}\int_{\mathbb{R}} xm = \int_{\mathbb{R}} xm_t = \int_{\mathbb{R}} x\left(m(\varpi+u_x)\right)_x \\ &= -\int_{\mathbb{R}} m(\varpi+u_x) + [x(m(\varpi+u_x)|_{\Omega} \\ &= -\varpi - \int_{\mathbb{R}} u_x m. \end{aligned}$$

Thus, we conclude that

$$\dot{\Xi} = -\varpi - \Pi.$$

Therefore, we obtain the following averaged dynamics

$$\begin{cases} \dot{\Xi} = -\varpi - \Pi \\ \dot{\Pi} = -\eta(\Xi - \kappa). \end{cases}$$

Taking the time derivative of the second equation and using the first equation, we get

$$\ddot{\Pi} - \eta\Pi = \eta(\varpi + \kappa).$$

The preceding equation has the following solution

$$\Pi = -\kappa + e^{\sqrt{\eta}t}C_1 + e^{-\sqrt{\eta}t}C_2 + \frac{\sqrt{\eta}}{2}\int_0^t \left(e^{\sqrt{\eta}(t-s)} - e^{-\sqrt{\eta}(t-s)}\right)\varpi(s)ds.$$

Moreover, at $t = 0$, we have

$$\dot{\Pi}(0) = -\eta(\Xi(0) - \kappa) = -\eta(\bar{x} - \kappa),$$

where

$$\bar{x} = \int_{\mathbb{R}} x\bar{m}.$$

Thus, we need an additional constant to determine $\Pi(0)$. Given this constant, from the constraint equation in (6.6.13), we get

$$\begin{aligned} \varpi_{\Pi(0)}(t) &= -\Pi - \mathbf{Q}(t) \\ &= f_{\Pi(0)}(t) - \frac{\sqrt{\eta}}{2}\int_0^t \left(e^{\sqrt{\eta}(t-s)} - e^{-\sqrt{\eta}(t-s)}\right)\varpi_{\Pi(0)}(s)ds, \end{aligned}$$

where $f_{\Pi(0)}(t) = \kappa - e^{\sqrt{\eta}t}C_1 - e^{-\sqrt{\eta}t}C_2 - \mathbf{Q}(t)$, and C_1 and C_2 are determined by the value of $\dot{\Pi}(0)$ and by the unknown value $\Pi(0)$.

The preceding equation is a Volterra integral equation of the second kind with a separable kernel. In principle, we can solve this equation using Laplace's transform. The previous equation is of the form

$$\varpi_{\Pi(0)}(t) = f_{\Pi(0)}(t) - \lambda(k * \varpi_{\Pi(0)})(t), \tag{6.6.14}$$

where

$$k(t) = -\frac{\sqrt{\eta}}{2}\left(e^{\sqrt{\eta}t} - e^{-\sqrt{\eta}t}\right)$$

and $(k * \varpi) = \int_0^t k(t-s)\varpi(s)ds$ denotes the convolution product of the kernel k with ϖ.

Let $\mathcal{L}$ denote the Laplace transform. Because $\mathcal{L}\{(k * \varpi)(t)\} = \mathcal{L}\{k(t)\}\mathcal{L}\{\varpi(t)\}$, applying the Laplace transform to (6.6.14) yields

$$\mathcal{L}\{\varpi(t)\} = \mathcal{L}\{f_{\Pi(0)}(t)\} + \lambda\mathcal{L}\{k(t)\}\mathcal{L}\{\varpi(t)\}.$$

Simplifying the above equation, we obtain

$$\varpi_{\Pi(0)}(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f_{\Pi(0)}(t)\}}{1 - \lambda\mathcal{L}\{k(t)\}}\right\},$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

Finally, we take the resulting expression for $\varpi_{\Pi(0)}$ into the Hamilton-Jacobi equation, solve it and obtain a function $u_{\Pi(0)}(x,t)$. Then, the value Π_0 is determined implicitly by the equation

$$\Pi(0) = \int_{\mathbb{R}} (u_{\Pi(0)})_x(x,0)m_0 dx. \tag{6.6.15}$$

In the case of quadratic terminal data,

$$u(x,T) = \frac{\gamma}{2}(x-\zeta)^2,$$

we can reduce the solution of the Hamilton-Jacobi equation into solving ordinary differential equations. For that, we look for a solution

$$u(x,t) = \theta_0(t) + \theta_1(t)x + \theta_2(t)x^2$$

satisfying

$$u(x,T) = \frac{\gamma}{2}(x-\zeta)^2 = \theta_0(T) + \theta_1(T)x + \theta_2(T)x^2.$$

Then, the first equation in (6.6.13) becomes

$$-(\dot{\theta}_0 + \dot{\theta}_1(t)x + \dot{\theta}_2(t)x^2) + \frac{(\varpi(t) + \theta_1(t) + 2\theta_2(t)x)^2}{2c} - \frac{\eta}{2}(x-\kappa)^2 = 0.$$

Thus, by matching powers of x, we obtain differential equations for θ_i, $0 \leq i \leq 2$. The resulting expression can be used in (6.6.15) to obtain the solution.

6.7 Real Data

In this section, we use real data of daily energy consumption in the UK, during a twenty-four hour period. The data is available online at https://www.nationalgrid.com/uk/. In Figure 6.1, we plot the power supply oscillation $\mathbf{Q}$ (which is simply the negative of the demand) normalized to have mean zero over 24 hours.

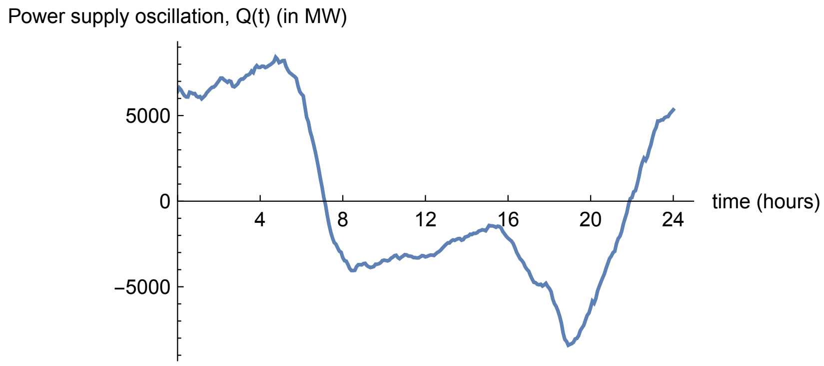


Fig. 6.1 Normalized electricity production $\mathbf{Q}$.

We compare our price-formation model with the MFG model presented in [73]. In that model, the energy price is a function of the aggregate consumption. In that case, the price is not determined by a supply vs demand condition and, thus, there may be an energy imbalance. Here, we consider the state-independent quadratic cost model from Section 6.6.1. In our model, the price depends only on the constant that accounts for battery's wear and tear. This constant can be empirically estimated, but, here, we calibrate our model against the model in [73] using a least squares approach. Let ϑ be the priced computed in [73]. According to (6.6.5), the price given is $\varpi^{c,\Theta} = \Theta - c\mathbf{Q}$. Thus, we estimate the value of c, by solving the minimization problem

$$\min_{c,\Theta\in\mathbb{R}} \|\varpi^{c,\Theta} - \vartheta\|_2^2 = \min_{c,\Theta\in\mathbb{R}} \|\Theta - c\mathbf{Q} - \vartheta\|_2^2, \tag{6.7.1}$$

and, using $N = 10^6$ agents, we obtained $c = 0.00172\$(\mathrm{kW})^{-2}h^{-1}$.

The price given by our model is plotted in Figure 6.2. We predict smaller peak oscillations and thus, our methods may help stabilize the market.

6.8 Conclusions and extensions

Here, we described a model for price formation in electricity markets, proved the well-posedness of the problem, and developed methods to compute the solutions. Our model has a minimal number of features and fits well real data. In addition, our model may have stabilizing properties of the price at peak consumption.

Several extensions of our model are of interest. First, we can consider the case where the supply $\mathbf{Q}(\varpi, t)$ depends on price. Provided the supply increases with the price, which is a natural assumption from the economic point of view, the solvability conditions are similar. In particular, (6.4.8) becomes

$$\dot{\varpi}\left[\frac{\partial \mathbf{Q}}{\partial \varpi} + \int_{\mathbb{R}} D^2_{pp} H m dx\right] = -\frac{\partial \mathbf{Q}}{\partial t} - \int_{\mathbb{R}} \left[D^2_{pp} H D_x H m + \epsilon D^3_{ppp} H u^2_{xx} m\right] dx.$$

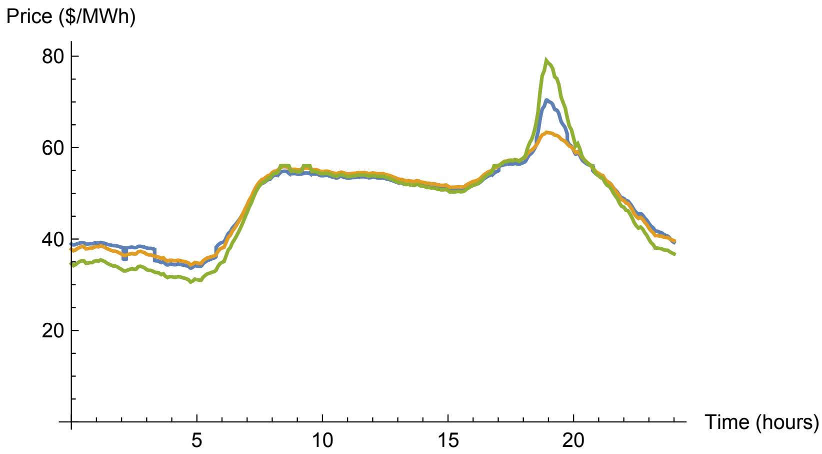


Fig. 6.2 Evolution of electricity price during a twenty-four hour period. In green, we plot the energy's price when no batteries are connected to the grid. In blue, we plot the price with batteries connected to the grid and the price is given by the model in [73]. In yellow, we plot the price corresponding to our model.

Thus, we obtain similar bounds for $\dot{\varpi}$ if $\frac{\partial \mathbf{Q}}{\partial \varpi} \geq 0$. Therefore, the existence theory follows a similar argument. Moreover, if $\frac{\partial \mathbf{Q}}{\partial \varpi} \geq 0$, the operator A in section (6.5) is monotone and, therefore, uniqueness of solution holds.

In real applications, $\mathbf{Q}$ may depend on delayed prices. While this does not fit directly into our framework, we can consider a Taylor expansion:

$$\begin{aligned}\mathbf{Q}(\varpi(t-\tau),t) \simeq & \mathbf{Q}(\varpi(t),t) - \tau \frac{\partial \mathbf{Q}(\varpi(t),t)}{\partial \varpi}\dot{\varpi}(t) \\ & + \frac{\tau^2}{2}\left[\frac{\partial \mathbf{Q}(\varpi(t),t)}{\partial \varpi}\ddot{\varpi} + \frac{\partial^2 \mathbf{Q}(\varpi(t),t)}{\partial \varpi^2}\dot{\varpi}^2(t)\right] + \dots\end{aligned}$$

Thus, it is natural to look at the case where $\mathbf{Q}$ depends on the price and its derivatives.

Finally, a natural extension is the case where $\mathbf{Q}$ has random fluctuations. This is particularly relevant if the energy production is subject to unpredictable changes - this is the case of wind energy. For the case where $\mathbf{Q}$ is random, we need to use the master equation as in [140].

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Appendix A

Viscosity solutions

In this appendix we present a brief introduction to viscosity solutions. We recall its definition and present some main results in the viscosity solutions' theory.

A.1 Definition

In this section we present the definition of viscosity solution for a particular equation, namely the Initial Value Problem for the Hamilton-Jacobi Equation,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + H(x, Du(x,t)) = 0 \quad (x,t) \in \mathbb{R}^n \times]0,T[\\ u(x,0) = u_0(x) \quad \in \mathbb{R}^n \end{cases} \tag{A.1.1}$$

where u_0 is any initial condition.

Definition A.1.1. Let u be a bounded, uniformly continuous function on $\mathbb{R}^n \times]0,T[$.
We call u a *viscosity sub-solution* of the initial value problem (A.1.1) if and only if:

- for each $\phi \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$, if $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ is a local maximizing point of $u - \phi$ then:
$$\frac{\partial \phi}{\partial t}(x_0,t_0) + H(x_0, D\phi(x_0,t_0)) \leq 0, \tag{A.1.2}$$

and *viscosity super-solution* if and only if:

- for each $\phi \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$, if $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ is a local minimizing point of $u - \phi$ then we have:
$$\frac{\partial \phi}{\partial t}(x_0,t_0) + H(x_0, D\phi(x_0,t_0)) \geq 0. \tag{A.1.3}$$

In the case u verifies (A.1.2) and (A.1.3) we say that u is a *viscosity solution*, provided that the equality $u(x,0) = u_0(x)$ is satisfied, in both cases.

A.2 Comparison Principle

Here we present a very useful result, for uniqueness and convergence results of homogenized equations, and its proof. For this result we need to assume the following properties:

$$H(x,u(x,t),p) - H(x,v(x,t),p) \geq \gamma_R(u-v), \quad \gamma_R > 0 \tag{A.2.1}$$

for all $(x,t) \in (\Omega \times (0,T])$, $-R \leq u \leq v \leq R$ and $p \in \mathbb{R}^n (\forall_{0<R<+\infty})$.

$$|H(x,u(x,t),p) - H(y,u(x,t),p)| \leq m_R\left(|x-y|(1+|p|)\right) \tag{A.2.2}$$

where $m_R(s) \to 0$ as $s \to 0$, for all $x,y \in \Omega$, $-R \leq u \leq R$ and $p \in \mathbb{R}^N$ $(\forall_{0<R<+\infty})$.
Before we state the Comparison Principle, there are some remarks to be done. Since we have defined the notion of viscosity solutions only for open intervals, we need the following result.

Lemma A.2.1. *If $u \in \mathcal{C}(\bar{\Omega} \times [0,T])$ is a sub-solution (respective super-solution) of* (A.1.1) *on $\Omega \times (0,T)$, then u is a sub-solution (respective super-solution) of*

$$\frac{\partial u}{\partial t} + H(x, Du(x,t)) = 0 \quad (x,t) \in (\Omega \times (0,T]). \tag{A.2.3}$$

Here note that we are now allowing that the maximum point may occur at $t = T$.

Proof. This lemma states that the boundary points $(x,t) \in (\Omega \times \{T\})$ can be treated as interior points. Let us suppose that $u(x,t)$ is a sub-solution of the problem, then we must have:
$\forall \phi \in \mathcal{C}^2(\Omega \times (0,T])$ if (x_0,T) is a local maximisation point of $u - \phi$, then we have:

$$\frac{\partial \phi}{\partial t}(x_0,T) + H(x_0, Du(x_0,T)) \leq 0.$$

Considering the function

$$\bar{\phi}(x,t) := \phi(x,t) + \frac{\epsilon}{T-t} \quad x \in \Omega, \quad 0 < t < T,$$

then for a sufficiently small $\epsilon > 0$, $u - \bar{\phi}$ has a local maximum at the point (x_ϵ, t_ϵ) where $0 < t_\epsilon < T$ and $(x_\epsilon, t_\epsilon) \to (x_0, T)$. Hence

$$\frac{\partial \bar{\phi}(x_\epsilon, t_\epsilon)}{\partial t} + H(x_\epsilon, D\bar{\phi}(x_\epsilon, t_\epsilon)) \leq 0,$$

and so

$$\frac{\partial \phi(x_\epsilon, t_\epsilon)}{\partial t} + \frac{\epsilon}{(T-t_\epsilon)^2} + H(x_\epsilon, D\bar{\phi}(x_\epsilon, t_\epsilon)) \leq 0.$$

Taking $\epsilon \to 0$ we get

$$\frac{\partial \phi(x_0, T)}{\partial t} + H(x_0, D\phi(x_0, T)) \leq 0.$$

Which proves (A.2.3) to be a sub-solution, provided $u - \phi$ has a maximum at (x_0, T). By doing the correct changes in the inequalities occurring in this proof, we conclude the same for super-solutions of (A.2.3) when $u - \phi$ as a minimum at (x_0, T). □

We can now state the main result in this section.

Theorem 16 (Comparison Principle)**.** Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that $u(x,t), v(x,t) \in (\Omega \times (0,T])$ are sub and super-solutions of (A.1.1). Furthermore if $u \leq v$ on $(\partial\Omega \times (0,T])$ and the last properties hold, then $u \leq v$ in $\bar{\Omega} \times (0,T]$.

Proof. We want to prove that if u, v are, respectively, sub and super-solutions, on $(\Omega \times (0,T])$ then $M := \max_{\bar{\Omega}\times(0,T]}(u(x,t) - v(x,t)) \leq 0$. We suppose, by contradiction, that $M > 0$, since $u \leq v$ in $\partial\Omega \times (0,T]$ the maximum is not attained at the boundary, since $u \leq v$ in $(\partial\Omega \times (0,T])$. Since u and v are viscosity solutions, their first derivative may not exist at some point, and so, we must rely on a technique called doubling of variables, in order to prove the result. This technique is performed by introducing the following test function

$$\psi_{\epsilon,\eta}(x,y,t,s) = u(x,t) - v(y,s) - \frac{|x-y|^2}{\epsilon^2} - \frac{|t-s|^2}{\eta^2}.$$

This function resembles $u - v$ when the penalization terms $\frac{|x-y|^2}{\epsilon^2}$ and $\frac{|x-y|^2}{\epsilon^2}$ are close to zero. This terms also impose that the maximization point (x,t,y,s) of $\psi_{\epsilon,\eta}$ must be attained when $(x,t) \to (y,s)$ for sufficiently small ϵ and η. Let us denote this maximum point of $\psi_{\epsilon,\eta}$ as $M_{\epsilon,\eta}$. Taking into account the results of Lemma A.2.2, stated after the end of this proof, we may proceed. For sufficiently small ϵ, η then $(x_\epsilon, t_\eta), (y_\epsilon, s_\eta) \in (\Omega \times (0,T])$. Since $(x_\epsilon, y_\epsilon, t_\eta, s_\eta)$ is a maximization point of $\psi_{\epsilon,\eta}$, then (x_ϵ, t_ϵ) is a maximization point of

$$(x,t) \mapsto u(x,t) - \phi^1_{\epsilon,\eta}(x,t)$$

where

$$\phi^1_{\epsilon,\eta}(x,t) = v(y_\epsilon, s_\eta) + \frac{|x-y_\epsilon|^2}{\epsilon^2} + \frac{|t-s_\eta|^2}{\eta^2}$$

since u is a viscosity sub-solution of (A.1.1) and $(x_\epsilon, t_\eta) \in \Omega \times (0,T]$ we get:

$$\begin{aligned} &\frac{\partial \phi^1_{\epsilon,\eta}}{\partial t}(x_\epsilon, t_\eta) + H(x_\epsilon, u(x_\epsilon, t_\eta), D\phi^1_{\epsilon,\eta}(x_\epsilon, t_\eta)) \\ &= 2\frac{(t_\eta - s_\eta)}{\eta^2} + H\Big(x_\epsilon, u(x_\epsilon, t), 2\frac{x_\epsilon - y_\epsilon}{\epsilon^2}\Big) \leq 0 \end{aligned} \tag{A.2.4}$$

and in the same way (y_ϵ, s_η) is a maximization point of

$$(y, s) \mapsto -v(y, s) + \phi^2_{\epsilon,\eta}(y)$$

where

$$\phi^2(y, s)_{\epsilon,\eta} = u(x_\epsilon, t_\eta) - \frac{|x_\epsilon - y|^2}{\epsilon^2} - \frac{|t_\eta - s|^2}{\eta^2}$$

hence (y_ϵ, s_η) is a point of minimum of $v - \phi^2_{\epsilon,\eta}$, since v is a viscosity super-solution of (A.1.1) and $(y_\epsilon, s_\eta) \in (\Omega \times (0, T])$ and so:

$$\begin{aligned} &\frac{\partial \phi^2_{\epsilon,\eta}}{\partial t}(y_\epsilon, s_\eta) + H(y_\epsilon, v(y_\epsilon, s_\eta), D\phi^1_{\epsilon,\eta}(y_\epsilon, s_\eta)) \\ &= -2\frac{(t_\eta - s_\eta)}{\eta^2}(-1) + H\Big(y_\epsilon, v(y_\epsilon, t), 2\frac{(x_\epsilon - y_\epsilon)}{\epsilon^2}\Big) \geq 0. \end{aligned} \tag{A.2.5}$$

By subtracting (A.2.5) from (A.2.4) we get

$$H\Big(x_\epsilon, u(x_\epsilon, t_\eta), 2\frac{(x_\epsilon - y_\epsilon)}{\epsilon^2}\Big) - H\Big(y_\epsilon, v(y_\epsilon, s_\eta), 2\frac{|x_\epsilon - y_\epsilon|}{\epsilon^2}\Big) \leq 0 \tag{A.2.6}$$

We note that $p := 2\frac{|x_\epsilon - y_\epsilon|}{\epsilon^2}$, the gradients of the sub and super-viscosity solutions are equal, and also $\frac{\partial \phi^1}{\partial t}(x_\epsilon, t_\eta) = \frac{\partial \phi^2}{\partial t}(y_\epsilon, s_\eta)$, this last equality let us reduce the problem to the one where we have only $H(x, w, Dw) = 0$.
By adding and subtracting $H(x_\epsilon, v(y_\epsilon, s_\eta), p_\epsilon)$ in (A.2.6) we get

$$H(x_\epsilon, u(x_\epsilon, t_\eta), p_\epsilon) - H(x_\epsilon, v(y_\epsilon, s_\eta), p_\epsilon) \tag{A.2.7}$$

$$\leq H(y_\epsilon, v(x_\epsilon, s_\eta), p_\epsilon) - H(x_\epsilon, v(y_\epsilon, s_\eta), p_\epsilon) \tag{A.2.8}$$

Applying now, (A.2.1) to the term in the left and (A.2.2) to the one in the right side, we are lead to

$$\gamma_R(u(x_\epsilon, t_\eta) - v(y_\epsilon, s_\eta)) \leq m(|x_\epsilon - y_\epsilon|(1 + |p|))$$

and from this

$$\gamma_R M_{\epsilon,\eta} \leq m\left(|x_\epsilon - y_\epsilon| + 2\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2}\right) - \gamma_R\left(\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} + \frac{|t_\eta - s_\eta|^2}{\eta^2}\right)$$

by taking ϵ and η to 0 and using the lemma we get

$$\gamma_R M \leq 0$$

which is a contradiction to our hypotheses, since $\gamma_R > 0$, so we may conclude that $M < 0$ which is what we want to prove. $\square$

In the proof above, we have used a result that we state, and refer to its proof now.

Lemma A.2.2. *The following properties hold.*

1. $M_{\epsilon,\eta} \to M$ *as* $(\epsilon, \eta) \to (0, 0)$

2. *If* $(x_\epsilon, y_\epsilon, t_\eta, s_\eta)$ *is a point of maximum of* $\psi_{\epsilon,\eta}$, *then we have:*

$$\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} \to 0 \ \textit{as}\ \epsilon \to 0$$

$$\frac{|t_\eta - s_\eta|^2}{\eta^2} \to 0 \ \textit{as}\ \eta \to 0$$

$$u(x_\epsilon, t_\eta) - v(y_\epsilon, s_\eta) \to M \ \textit{as}\ (\epsilon, \eta) \to (0, 0)$$

3. $(x_\epsilon, y_\epsilon, t_\eta, s_\eta) \in (\Omega \times (0, T])$ *if* ϵ *and* η *are sufficiently small.*

Proof. See [20]. □

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