Carnegie Mellon University

MELLON COLLEGE OF SCIENCE

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

TITLE: Variational Methods for Second Order Structured Deformations and Multiscale Problems

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ACCEPTED BY THE DEPARTMENT OF: Mathematical Sciences

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Variational Methods for Second Order Structured Deformations and Multiscale Problems

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Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics at the Department of Mathematical Sciences in Carnegie Mellon University

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Pittsburgh, May 3, 2019

Dedication

To my parents, Adrian and Caroline.

Acknowledgments

Thank you to Professor Irene Fonseca for supervising my Ph.D. candidacy, for her innumerable insights and support throughout the process, for her help in my professional development, and for her patience and editorial assistance throughout the preparation of this document, among others.

Thank you to the faculty at Carnegie Mellon University, in particular Professor Bill Hrusa, Professor Giovanni Leoni, Professor Dejan Slepcev, Professor Gautam Iyer and Professor Ian Tice, who instructed many wonderful classes and were always willing to go the extra distance to support my growth.

Thank you to all the graduate students and post-docs at Carnegie Mellon University, my friends, my colleagues, for always being there when I would come knocking at the door with a question, a puzzle, or just a need to blow off steam. Thank you for making an arduous process less lonely and a lot more fun.

Thank you my wonderful parents, Adrian and Caroline, for being the first people to believe in me, for always supporting my education, for letting me know that no matter where I wander I can always come home.

Thank you to my caring, beautiful fiancée Brenna Fritz, for being a source of so much joy and love, for being with me through the ups and downs of the Ph.D. process, for keeping me sane and grounded and for agreeing to share her life with me for many years to come.

Thank you all so much!

Adrian Hagerty

Abstract

In this thesis, a family of integral representation results are proved for problems involving energies with different dimensionalities and multiscale interactions. The first part is work in the framework of functions of bounded Hessian, with an eye towards application to the theory of second order structured deformations. A relaxation theorem for BH functionals is obtained in the spirit of the 1992 work of Ambrosio and Dal Maso, Fonseca and Müller within the BV context. An integral representation theorem is established for abstract second order structured deformations functionals, using proof techniques from the global method for integral representation introduced in 1998 by Bouchitté, Fonseca and Mascarenhas. The second family of results concerns systems featuring simultaneous homogenization and phase transition effects. These are studied via the technique of Γ-convergence, and multiple regimes are considered corresponding to the relative scaling of the phase transition thickness and the scale of the heterogeneity. In particular, Γ-limit results are proved in the general case of vector-valued functions when the two rates are commensurate and when the frequency of the heterogeneity is sufficiently small with respect to the thickness.

Contents

1 Introduction

This thesis consists of mathematical results in the calculus of variations motivated by questions in materials science. A recurrent feature of these problems is the presence of microstructure coupled with macroscopic effects. We seek to understand the behavior of minimizers to families of multi-scale problems by considering the microstructure to be infinitesimally small compared to the macroscopic structure. In this endeavour, we make frequent use of Γ-convergence techniques, due to De Giorgi [33]. By taking the Γ-limit of the multi-scale problems as the microstructure becomes infinitely fine, we can determine an "effective" functional, in which the multiple scales decouple, that describes the asymptotic behavior of the family of functionals.

In order to understand the limiting processes, one hopes to prove integral representation results, allowing us to work directly with the effective functional. To this end, we use so-called "blow-up" techniques developed by Fonseca and Müller [45] to characterize the pointwise behavior of *a priori* abstract functionals. Here we apply this framework and strategy to problems in the field of second order structured deformations and to problems arising from the interplay of phase transition and homogenization.

The mathematical theory of structured deformations arises as a particular model for plasticity, fracture and defects in a universal setting. Briefly, one views the formation of singularity within a material through the lens of misalignment between the macroscopic deformation and the submacroscopic crystalline structure. As there has been recent interest in a second-order theory of structured deformations, taking into account curvature and bending effects, we include work in the setting of functions of bounded Hessian, BH. The space BH, which is comprised of of $W^{1,1}$ functions with bounded variation (BV) gradient, has peculiarities which differentiate it from other second order spaces with L^1 control, such as BV^2, which consists of BV functions such that the absolutely continuous part of the gradient is itself BV. In particular, a BH function may have jumps in the gradient, corresponding to "kinks" in the function, but cannot have jumps itself. Therefore, we cannot directly apply standard BV results to BH, and thereby this goal required the development of new mathematics in order to extend certain Sobolev space constructions, such as Lipschitz extension theorems, to BH, see Chapter 3.

This thesis also contains the results of a project towards a complete theory of fluid-fluid phase transitions for materials with small scale heterogeneities. The technique of Γ-convergence has proven extremely powerful for modelling systems of asymptotic homogenization of composite materials as well as Cahn-Hilliard models of phase transitions. To understand the nature of potential interaction between the two processes, we study materials for which both properties hold. One would expect that the relative length scales of the heterogeneity versus the thickness of the phase transition would have an effect on the interaction, and we see that this is in fact true. If the scale of the heterogeneity is sufficiently small with respect to the thickness, we see that the system effectively "homogenizes" and we are left with an energy that penalizes the perimeter of the phase transition, as in the classical Modica-Mortola [62] result. However, when the length scales of the heterogeneity and the thickness are roughly commensurate, we observe interaction between the relative alignment between the heterogeneity and the orientation of the phase transition boundary, resulting in an anisotropic perimeter term. The third regime, where the phase transition is sufficiently small with respect to the heterogeneity, is part of this ongoing project, but has not been fully studied at this date.

1.1 Structured Deformations and BH

The space of $W^{1,1}$ functions whose Hessian is a Radon measure, BH (bounded Hessian) was introduced by Demengel [38]. BH is the natural setting to study second-order integral functionals with linear growths. In Chapter 3 we prove an integral representation result for relaxed functionals in BH.

In the theory of structured deformations, which model geometrical changes at microscopic and macroscopic scales, the first-order theory fails to account for the effect of microscopic jumps in the gradients. A second-order theory is introduced in [65] which uses the space BH and related spaces SBH, which consists of BH functions whose Hessian has trivial Cantor part, and SBV^2, which consists of BV^2 functions whose gradient and Hessian each have trivial Cantor part. Recent results in [11] approach a second-order theory in the SBV^2 setting and establish relaxation and integral representation theorems.

Beyond applications to second order structured deformations, the space BH appears in other areas of applied mathematics, motivating its study as a space in its own right. In the field of image processing, there has been some study of second-order energies with linear growth. The addition of second-order term in Rudin-Osher-Fatemi TV denoising can act as a regularizing factor, avoiding the so-called "staircasing effect", as discussed in [12], [13], [14]. More applications of second-order terms in regularization and denoising may be found in [50], [52], [70]. Further, second-order energies have found application in variational image fusion [58] and image colorization [53].

Another problem, this one again from materials science, in which we see the space BH as the natural setting is in models of elastic perfectly-plastic materials, where the energy is a second-order functional with linear growth, see [15], [30], [31], [37], [59], [68].

For a bounded open Lipschitz set Ω in $\mathbb{R}^N$ we define the functional

$$F(u) := \int_\Omega f(x, \nabla^2 u)dx, \quad u \in W^{2,1}(\Omega, \mathbb{R}^d).$$

where $f : \Omega \times \mathbb{R}^{d\times N\times N} \to [0, \infty)$ is a continuous function satisfying the following hypotheses:

(H1) Linear growth: $f(x, H) \leq C(1 + |H|)$ for all $x \in \Omega$, $H \in \mathbb{R}^{d\times N\times N}$ and some $C > 0$;

(H2) Modulus of continuity: $|f(x, H) - f(y, H)| \leq \omega(|x - y|)(1 + |H|)$ for all $x, y \in \Omega, H \in \mathbb{R}^{d\times N\times N}$, where $\omega(s)$ is a nondecreasing function with $w(s) \to 0$ as $s \to 0^+$.

We consider the lower-semicontinuous envelope of F in the space $BH(\Omega; \mathbb{R}^d)$,

$$\mathcal{F}(u) := \inf \Big\{ \liminf_{n\to\infty} F[u_n] : u_n \to u \text{ in } W^{1,1}(\Omega; \mathbb{R}^d), \\ \nabla^2 u_n \, \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} D(\nabla u) \text{ in } \mathcal{M}(\Omega, \mathbb{R}^{d\times N\times N}) \Big\},$$

where $\mathcal{M}(\Omega, \mathbb{R}^{d\times N\times N})$ is the set of finite Radon measures on Ω taking values in $\mathbb{R}^{d\times N\times N}$.

In Chapter 3, we will prove the following integral representation result (see Theorem 3.1).

Theorem 1.1. *If f satisfies (H1) and (H2), then for every $u \in BH(\Omega; \mathbb{R}^d)$*

$$\mathcal{F}(u) = \int_\Omega \mathcal{Q}_2 f(x, \nabla^2 u)dx + \int_\Omega (\mathcal{Q}_2 f)^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|} \right) d|D_s(\nabla u)|.$$

where $\mathcal{Q}_2 f$ is the 2-quasiconvex envelope of f, and $(\mathcal{Q}_2 f)^\infty$ is the recession function defined via

$$(\mathcal{Q}_2 f)^\infty(x, H) := \limsup_{t\to\infty} \frac{\mathcal{Q}_2 f(x, tH)}{t}$$

This result may be seen as a second-order version of the work of Fonseca and Müller [45], as well as Ambrosio and Dal Maso [3], on first-order linear growth functionals. It is inspired by recent progress in the field of $\mathcal{A}$-quasiconvexity, in the sense of Fonseca and Müller, introduced in [46]. A recent paper of Arroyo-Rabasa, De Philippis, and Rindler [7] uses a Young measure based approach to prove a relaxation result in a very general setting. Knowing that the space BH may be viewed through the lens of $\mathcal{A}$-quasiconvexity, this section adopts some of these techniques to BH relaxation, using the extra structure of BH to avoid invoking Young measures as in [7].

The author acknowledges a higher order relaxation result that is contained in the work of Amar & De Cicco [1]. However, to our knowledge, there seems to be a gap in the proof of lower semicontinuty in [1], in particular with regards to the singular part. Our proof makes use of modern results which take a completely different approach in proving lower semicontinuity.

The author is also aware of recent work by Breit, Diening and Gmeineder [23] which examines what they call $\mathbb{A}$-quasiconvexity. As they note in Section 5, with the existence of the annihilator $\mathbb{L}$, this is what Dacarogna called A-B quasiconvexity [27], where in this case we have $A = \mathbb{L}$ and $B = \mathbb{A}$. Although BH may be viewed in the frame of $\mathcal{A}$-quasiconvexity by restricting matrix-valued measures to lie in a particular subspace, it is not obvious that we can view it in the framework of A-B quasiconvexity as easily. Regardless, for their argument of lower-semicontinuity, Breit, Diening and Gmeineder make use of [7] and thus do not provide a Young measure-free argument.

The goal of this project is to establish relaxation results in the space BH using standard blow-up methods. In order to prove the upper bound, we first demonstrate an area-strict density theorem (see Section 3.2), in a very general setting, requiring no extra structure on the limiting measure μ.

We develop a direct argument with an eye towards ultimately including lower order terms in the relaxation. It should be noted that while the arguments in this section are only for second-order case, an extension to higher order derivatives should be possible using similar arguments.

The macroscopic deformation of a continuous body does not need to coincide with the submacroscopic deformation. For instance, in a crystalline body deformed beyond the plastic regime the macroscopic deformation may be simply due to several slips of the crystallographic planes. Thus, submacroscopically the lattice of the crystalline body does not deform but simply undergoes to "submacroscopic cracks" or disarrangements. This kind of multi-scale geometrical changes were addressed by Del Piero and Owen in [36] who introduced the notion of *structured deformation* (κ, u, G): κ being the macroscopic crack site, u the macroscopic deformation, and G a tensor associated with the submacroscopic geometrical changes and called deformation without disarrangements. In the example of the crystalline body, discussed above, we would have $\kappa = \emptyset$ since the submacroscopic cracks diffuse and do not generate a macroscopic crack, $G = I$ the identity tensor field since the lattice does not deform locally, and, in general the deformation gradient ∇u is different from $G = I$.

Del Piero and Owen, still in [36], showed that every structured deformation can be seen as the (appropriate) limit of sequences of piecewise-continuous "classical deformations". This result makes the theory even more interesting from a mechanical point of view, since, for instance, in the example of the crystalline body mentioned above, the "submacroscopic cracks" that form during the deformation can be thought as the jump sets of the piecewise-continuous "classical deformations" of an approximating sequence. This result also opens the way to define the energy of a structured deformation by using the "classical" energy of piecewise-continuous "classical deformations". Indeed, Choksi and Fonseca [26], following the belief that "Nature always minimizes actions", made the natural assumption that the structured deformation (κ, u, G) would be the limit, among all approximating sequences, of the approximating sequence that uses the least amount of energy. Choksi and Fonseca worked within a variational framework and described the macroscopic deformation by a function $u \in BV$ whose jump set represents the crack site κ of Del Piero and Owen, and with a deformation without disarrangements $G \in L^1$. In this framework, they proved the following approximation theorem: for any structured deformation (u, G) there exists a sequence $\{u_n\} \subset SBV$ such that

$$u_n \to u \text{ in } L^1, \qquad \nabla u_n \overset{*}{\rightharpoonup} G \text{ in the sense of measures,} \tag{1.1}$$

where ∇u_n denotes the absolutely continuous part of the distributional derivative of u_n; moreover, they defined the energy $\mathcal{E}(u, G)$ of (u, G) as

$$\mathcal{E}(u, G) := \inf_{\{u_n\}} \liminf_{n \to +\infty} \mathcal{E}_0(u_n), \tag{1.2}$$

where the inf is taken among all the sequences that generate, according to (1.1), the structured deformation (u, G), and $\mathcal{E}_0(u_n)$ is the energy associated to the "classical deformation" u_n. Thus, the energy $\mathcal{E}(u, G)$ is equal to the limit of the energies associated to the most economic approximating sequence from the energetic point of view.

The concept of structured deformation was extended in [65] by defining the *second-order structured deformation* (κ, u, G, U): where κ denotes the set of points were the fields involved are discontinuous, u, and G are as above, and U, called second-order deformation without disarrangements, is a third-order tensor field that allows to describe the submacroscopic deformation up to the second-order; for instance, it allows to describe the "bending" of the microstructure. Second-order structured deformation are important since they allow to include the effects of limits of second gradients and jumps in the first gradients of approximating deformations: these jumps play a crucial role in the mechanics of phase-transitions. In [67] two different variational frameworks for second-order structured deformation are discussed: the primary difference being the function space on which the deformation fields are defined. The first framework consider a space named SBV^2 that allows jumps of the displacement as well as its gradient. A recent paper of Barroso, Matias, Morandotti, and Owen [11] provides relaxation and integral representation results for second-order structured deformations in the framework of SBV^2. The second framework considers the space SBH of special functions of bounded Hessian. Within this framework we have $u \in W^{1,1}$ and hence $G = \nabla u$ and κ is simply the jump set of ∇u. In the SBH framework, a second-order structured deformation is therefore described by the pair (u, U). We remark that the SBH setting is more constrained than the SBV^2 setting, since the functions may not have "jumps", and hence the techniques used in [11] cannot be directly applied to the SBH setting.

Consider the family of structured deformations

$$SD_2(\Omega) := SBH(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{S}^{d\times N\times N}),$$

where $\mathbb{S}^{d\times N\times N} \subset \mathbb{R}^{d\times N\times N}$ denotes the set of tensors (M_{ijk}), $i \in \{1, \dots, d\}$, $j, k \in \{1, \dots, N\}$, such that $M_{ijk} = M_{ikj}$ for all $i, j \in \{1, \dots, N\}$, $d, N \in \mathbb{N}$. We prove a general integral representation result in the spirit of the global method of Bouchitté, Fonseca and Mascarenhas [19]. Let $\mathcal{A}(\Omega)$ be the family of open subsets of Ω. Assume that the functional

$$\mathcal{F} : SD_2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$$

satisfies the following hypotheses:

(I1) $\mathcal{F}(u, U; \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure for every $(u, U) \in SD_2(\Omega)$.

(I2) $\mathcal{F}(\cdot, \cdot; A)$ is SD_2-lower semicontinuous, in the sense that if $(u, U) \in SD_2(\Omega)$ and $\{(u_n, U_n)\} \subset SD_2(\Omega)$ with $u_n \to u$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ and $U_n \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, then

$$\mathcal{F}(u, U; A) \leq \liminf_{n\to+\infty} \mathcal{F}(u_n, U_n; A).$$

(I3) $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$ if $u = v$ and $U = V$ $\mathcal{L}^N$ a.e. $x \in A$ then $\mathcal{F}(u, U; A) = \mathcal{F}(v, V; A)$.

(I4) There exists a constant $C > 0$ such that

$$\frac{1}{C}(\|U\|_{L^1(A)} + |D^2u|(A)) \leq \mathcal{F}(u, U; A) \leq C(\mathcal{L}^N(A) + \|U\|_{L^1(A)} + |D^2u|(A))$$

for every $(u, U) \in SD_2(\Omega)$, $A \in \mathcal{A}(\Omega)$.

In Theorem 3.18 we prove an integral representation for $\mathcal{F}$ of the form

$$\mathcal{F}(u, U; A) = \int_A f(x, u, \nabla u, \nabla^2 u, U)\, dx + \int_{S(\nabla u)\cap A} h(x, u, \nabla u^+, \nabla u^-, \nu_{\nabla u})\, d\mathcal{H}^{N-1}.$$

This result is then used to define the energy of a second-order structured deformation (u, U), in the same spirit of (1.2), as the limit of the energy of the most energetically convenient approximating sequence, i.e.

$$\mathcal{F}(u,U) := \inf_{\{u_n\}} \liminf_{n\to+\infty} \mathcal{F}_0(u_n),$$

where the inf is taken among all the sequences that generate the second-order structured deformation (u,U), and $\mathcal{F}_0(u_n)$ is the energy associated to the "classical deformation" u_n, see Theorem 3.22.

The general relaxation result proved has applications also outside the framework of structured deformations. Indeed, it has an immediate corollary to any functional defined on SBH: we can show, see Theorem 3.23, that for any $\mathcal{F} : SBH(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)) \to [0,\infty)$ satisfying (H1)-(H4), we have the integral representation

$$\mathcal{F}(u;A) = \int_A f(x,u,\nabla u,\nabla^2 u)\,dx + \int_{S(\nabla u)\cap A} h(x,u,\nabla u^+,\nabla u^-,\nu_{\nabla u})\,d\mathcal{H}^{N-1}.$$

In the case of functionals defined in $BH(\Omega;\mathbb{R}^d)$, with the additional assumptions of affine invariance and area-strict continuity, our earlier BH relaxation results can leveraged along with the SBH relaxation result to yield Corollary 3.25,

$$\mathcal{F}(u;A) = \int_A f(x,\nabla^2 u)\,dx + \int_A f^\infty\left(x,\frac{dD_s(\nabla u)}{|D_s(\nabla u)|}\right) d|D_s(\nabla u)|(x).$$

The assumption of affine invariance is merely a technical detail due to the lack of a BH relaxation result involving lower order terms. We motivate the assumption of area-strict continuity by comparison to the first order global method for relaxation [19]. In this situation, although we do not assume *a priori* that our abstract lower semicontinuous functional is area-strict continuous, once we have the integral representation result, area-strict continuity follows *a posteriori*, [56]. Thus, in the first-order case, nothing is lost by adding the additional assumption that the functional is area-strict continuous. We expect that the same holds in the second-order framework.

The BH results are structured as follows. In Chapter 2 we collect some common notions and establish pointwise results about BH functions. In Section 3.4.1 we prove an approximation result in the SD_2 framework along the lines of the approximation theorems of [36] and [26]. In Section 3.4.2 we use the global method approach introduced in [19] on functionals defined on SD_2 in order to prove the main integral representation result. In Section 3.4.3 we apply the integral representation result to the problem of second order structured deformations to get a relaxation as in [26]. In Section 3.4.4 we find further application of the integral relaxation result in the spaces SBH and BH.

1.2 Homogenization and phase transition

In order to describe the behavior at equilibrium of a fluid under isothermal conditions confined in a container $\Omega \subset \mathbb{R}^N$ and having two stable phases (or a mixture of two immiscible and non-interacting fluids with two stable phases), Van der Waals in his pioneering work [73] (then rediscovered by Cahn and Hilliard in [25]) introduced the following Gibbs free energy per unit volume

$$E_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2 |\nabla u|^2 \right] \, \mathrm{d}x \,. \tag{1.3}$$

Here $\varepsilon > 0$ is a small parameter, $W : \mathbb{R} \to [0, +\infty)$ is a double well potential vanishing at two points, say $+1$ and -1 (the simplified prototype being $W(t) := (1-t^2)^2$), and $u : \Omega \to \mathbb{R}$ represents the phase of the fluid, where $u = +1$ correspond to one stable phase and $u = -1$ to the other one. According to this gradient theory for first order phase transitions, observed stable configurations minimize the energy E_ε under a mass constraint $\int_\Omega u = m$, for some fixed $m \in (-|\Omega|, |\Omega|)$.

The gradient term present in the energy (1.3) provides a selection criterion among minimizers of $I : u \mapsto \int_\Omega W(u) \, \mathrm{d}x$. If neglected then every field u such that $W(u) \equiv 0$ in Ω and satisfying the mass constraint is a minimizer of I. The singular perturbation $u \mapsto \varepsilon^2 |\nabla u|^2$ plays the role of an interfacial energy. It provides a selection criterion as it competes with the potential term in that it penalizes inhomogeneities of u and acts as a regularization for the problem. It was conjectured by Gurtin (see [51]) that for $0 < \varepsilon \ll 1$ the minimizer u_ε of the energy E_ε will approximate a piecewise constant function, u, taking values in the zero set of the potential W, and minimizing the surface area $\mathcal{H}^{N-1}(S_u)$ of the interface separating the two phases. Here $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and S_u is the set of jump points of u.

Gurtin's conjecture has been validated by Modica in [62] (see also the work of Sternberg [72]) using Γ-convergence techniques introduced by De Giorgi and Franzoni in [33]. In particular, it has been showed that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon) = \gamma \mathcal{H}^{N-1}(S_u) \,,$$

where the constant $\gamma > 0$ is the surface energy density per unit area required to make a transition from one stable phase to the other, and it is given by

$$\gamma := 2 \int_{-1}^{1} \sqrt{W(t)} \mathrm{d}t \,.$$

Several variants of the Van der Waals-Cahn-Hilliard gradient theory for phase transitions have been studied analytically. Here we recall the extension to the case of d non-interacting immiscible fluids, with a vector-valued density $u : \mathbb{R}^N \to \mathbb{R}^d$. In [48] Fonseca and Tartar treated the case of two stable phases (i.e., the potential $W : \mathbb{R}^d \to [0, \infty)$ has two zeros), while the general case of several stable phases has been solved by Baldo in [8]. In [8] and [48] it has been proved that the limit of a sequence $\{u_\varepsilon\}_{\varepsilon>0}$, where u_ε is a minimizer of E_ε, is a minimal partition of the container Ω, where each set satisfies a volume constraint and corresponds to a stable phase, i.e., a zero of W.

Other generalizations of (1.3) include the work of Bouchitté [17], who studied the case of a fluid where its two stable phases change from point to point, in order to treat the situation where the temperature of the fluid is not constant inside the container, but given *a priori*. From the mathematical point of view, this corresponds to considering the energy (1.3) with a potential of the form $W(x, u)$ vanishing on the graphs of two non constant functions $z_1, z_2 : \Omega \to \mathbb{R}^d$. Fonseca and Popovici in [47] dealt with the vectorial case of the energy (1.3) where the term $|\nabla u|$ is substituted with a more general expression of the form $h(x, \nabla u)$, while the full coupled singular perturbed problem in the vectorial case, with the energy density of the form $f(x, u, \varepsilon\nabla u)$, has been studied by Barroso and Fonseca in [10]. The case in which Dirichlet boundary conditions are considered was addressed by Owen, Rubinsten and Sternberg in [66], while in [63] Modica studied the case of a boundary contact energy. We refer to the works [72] of Sternberg and [2] of Ambrosio for the case where the zeros of the potential W are generic compact sets. Finally, in [55] Kohn and Sternberg studied the convergence of local minimizers for singular perturbation problems.

We consider the problem of fluid-fluid phase transitions in the presence of small scale heterogeneities. More precisely, for $\varepsilon, \delta > 0$ we consider the energy

$$\mathcal{F}_{\varepsilon,\delta}(u) := \int_\Omega \left[\frac{1}{\delta} W\left(\frac{x}{\varepsilon}, u(x) \right) + \delta |\nabla u(x)|^2 \right] \mathrm{d}x,$$

where $W : \mathbb{R}^N \times \mathbb{R}^d \to [0, \infty)$ is a double-well potential satisfying the following properties:

(G0) $x \mapsto W(x, p)$ is Q-periodic for all $p \in \mathbb{R}^d$,

(G1) W is a Carathéodory function, i.e.,

- (i) for all $p \in \mathbb{R}^d$ the function $x \mapsto W(x, p)$ is measurable,
- (ii) for a.e. $x \in Q$ the function $p \mapsto W(x, p)$ is continuous,

(G2) there exist $a, b \in \mathbb{R}^d$ such that $W(x, p) = 0$ if and only if $p \in \{a, b\}$, for a.e. $x \in Q$,

(G3) there exists a continuous function $W_c : \mathbb{R}^d \to [0, \infty)$ such that $W_c(p) \leq W(x, p)$ for a.e. $x \in Q$ and $W_c(p) = 0$ if and only if $p \in \{a, b\}$.

(G4) there exist $C > 0$ and $q \geq 2$ such that $\frac{1}{C}\left(|p|^q - 1\right) \leq W(x,p) \leq C(1+|p|^q)$ for a.e. $x \in Q$ and all $p \in \mathbb{R}^d$.

Here the periodicity at scale ε fixes the scaling of the heterogeneity while δ corresponds to the thickness of our transition layers.

We characterize the limiting behavior of minimizers to $\mathcal{F}_{\varepsilon,\delta}$ by identifying the Γ-limit of $\mathcal{F}_{\varepsilon,\delta}$ as $\varepsilon, \delta \to 0$ for different regimes corresponding to the relative behavior of ε and δ.

In Section 4.2, we study a scaling in which the homogenization effects occur far more rapidly than that of the phase transition, namely $\varepsilon << \delta$. For the key lemma, we require a certain quantitative control to this scale, namely

$$\frac{\varepsilon}{\delta^{\frac{3}{2}}} \to 0. \tag{1.4}$$

We will address this scaling later. Further, we also require an additional locally Lipschitz assumption on W, to be precise

(G5) W is locally Lipschitz in p, i.e., for every $K \subset \mathbb{R}^d$ compact there is a constant L such that

$$|W(x,p) - W(x,q)| \leq L|p-q|$$

for almost every $x \in Q$ and every $p, q \in K$.

Definition 1.2. We define the functional $F_0^H : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ as

$$F_0^H(u) := \begin{cases} K_H \mathcal{P}(\{u = a\}; \Omega) & \text{if } u \in BV(\Omega; \{a, b\}), \\ \\ +\infty & \text{otherwise}, \end{cases} \tag{1.5}$$

where $\mathcal{P}(\{u = a\}; \Omega)$ is the relative perimeter of $\{u = a\}$ with respect to Ω, and the transition energy density K_H is defined as

$$K_H := 2 \inf \left\{ \int_0^1 \sqrt{W_H(g(s))} |g'(s)| \mathrm{d}s : g \in C^1_{pw}([0,1]; \mathbb{R}^d; a, b) \right\}. \tag{1.6}$$

Here $C^1_{pw}([0,1]; \mathbb{R}^d; a, b)$ denotes the space of piecewise C^1 curves from $[0,1]$ to $\mathbb{R}^d$ such that $g(0) = a$ and $g(1) = b$, and the homogenized potential $W_H : \mathbb{R}^d \to [0, +\infty)$ is given by

$$W_H(p) := \int_Q W(y,p)\, \mathrm{d}y. \tag{1.7}$$

In Section 4.2 we prove the following result (see Theorem 4.4).

Theorem 1.3. *Assume that W satisfies hypotheses (G0)-(G4). Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ be two infinitesimal sequences such that*

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n^{\frac{3}{2}}} \to 0$$

Set $F_n := \mathcal{F}_{\varepsilon_n,\delta_n}$. Then the following hold:

(i) If $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ is such that

$$\sup_{n\in\mathbb{N}} F_n(u_n) < +\infty,$$

then, up to a subsequence (not relabeled), we have $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ for some $u \in BV(\Omega;\{a,b\})$.

(ii) We have $F_n \overset{\Gamma-L^1}{\longrightarrow} F_0^H$.

Next we turn to the case in which ε and δ are commensurate (see Section 4.3). For simplicity, we set $\delta = \varepsilon$, that is to say

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon|\nabla u(x)|^2 \right] \mathrm{d}x.$$

In this regime, the Γ-limit is an anisotropic perimeter caused by potential mismatch between the direction of periodicity and the orientation of the interface.

We introduce some notation. For $\nu \in \mathbb{S}^{N-1}$, with $\mathbb{S}^{N-1}$ the unit sphere of $\mathbb{R}^N$, we denote by $\mathcal{Q}_\nu$ the family of cubes Q_ν centered at the origin with two faces orthogonal to ν and with unit length sides.

Definition 1.4. Let $\nu \in \mathbb{S}^{N-1}$ and define the function $u_{0,\nu} : \mathbb{R}^N \to \mathbb{R}^d$ as

$$u_{0,\nu}(y) := \begin{cases} a & \text{if } y\cdot\nu \le 0\,, \\ b & \text{if } y\cdot\nu > 0\,. \end{cases} \tag{1.8}$$

Fix a function $\rho \in C_c^\infty(B(0,1))$ with $\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x = 1$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$. For $T > 0$, consider the family of mollifiers $\rho_T(x) := T^N \rho(Tx)$ and

$$\widetilde{u}_{\rho,T,\nu} := \rho_T * u_{0,\nu}\,. \tag{1.9}$$

When it is clear from the context, we will abbreviate $\widetilde{u}_{\rho,T,\nu}$ as $\widetilde{u}_{T,\nu}$.

Definition 1.5. We define the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ as

$$\sigma(\nu) := \lim_{T\to\infty} g(\nu, T)\,,$$

where

$$g(\nu,T) := \frac{1}{T^{N-1}} \inf\Big\{ \int_{TQ_\nu} \left[W(y,u(y)) + |\nabla u|^2\right] \mathrm{d}y \,:\, Q_\nu \in \mathcal{Q}_\nu,\ u \in \mathcal{C}(\rho, Q_\nu, T) \Big\}\,,$$

and

$$\mathcal{C}(\rho, Q_\nu, T) := \Big\{ u \in H^1(TQ_\nu;\mathbb{R}^d) \,:\, u = \widetilde{u}_{\rho,T,\nu} \text{ on } \partial\left(TQ_\nu\right) \Big\}\,.$$

Just as before, if there is no possibility of confusion, we will write $\mathcal{C}(\rho, Q_\nu, T)$ as $\mathcal{C}(Q_\nu, T)$. A treatment of the function σ, including a justification of its definition as a limit, is found in Section 4.3.2.

Consider the functional $\mathcal{F}_0 : L^1(\Omega;\mathbb{R}^d) \to [0,\infty]$ defined by

$$\mathcal{F}_0(u) := \begin{cases} \displaystyle\int_{\partial^* A} \sigma(\nu_A(x))\,\mathrm{d}\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega;\{a,b\}), \\ +\infty & \text{else,} \end{cases} \tag{1.10}$$

where $A := \{u = a\}$ and $\nu_A(x)$ denotes the measure theoretic external unit normal to the reduced boundary $\partial^* A$ of A at x (see Definition 2.9).

In Section 4.3, we prove the following result (see Theorem 4.9).

Theorem 1.6. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. Assume that (G0), (G1), (G2), (G3) and (G4) hold.*

(i) If $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ is such that

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\varepsilon_n}(u_n) < +\infty$$

then, up to a subsequence (not relabeled), $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, where $u \in BV(\Omega;\{a,b\})$,

(ii) It holds that $\mathcal{F}_{\varepsilon_n} \xrightarrow{\Gamma - L^1} \mathcal{F}_0$.

Moreover, the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ is continuous.

In the literature we can find several problems treating simultaneously phase transitions and homogenization. In [6] (see also [5]) Ansini, Braides and Zeppieri considered the family of functionals

$$\mathcal{S}_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W(u(x)) + \varepsilon f\left(\frac{x}{\delta(\varepsilon)}, Du \right) \right] \mathrm{d}x \,,$$

and identified the Γ-limit in all three regimes

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0 \,, \qquad \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} := c > 0 \,, \qquad \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = +\infty \,, \tag{1.11}$$

using abstract Γ-convergence techniques to prove the general form of the limiting functional, and more explicit arguments to derive the explicit expression in the three regimes (actually, in the first case they need to assume $\varepsilon^{3/2}\delta^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$, the same as 1.4).

Moreover, we mention the articles [39] and [40] by Dirr, Lucia and Novaga regarding a model for phase transition with an additional bulk term modeling the interaction of the fluid with a periodic mean zero external field. In [39] they considered, for $\alpha \in (0,1)$, the family of functionals

$$\mathcal{V}^{(1)}_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon^\alpha} g\left(\frac{x}{\varepsilon^\alpha} \right) u(x) \right] \mathrm{d}x \,,$$

for some $g \in L^\infty(\Omega)$, while in [40] they treated the case

$$\mathcal{V}^{(2)}_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W(u(x)) + \varepsilon |\nabla u|^2 + \nabla v\left(\frac{x}{\varepsilon} \right) \cdot \nabla u(x) \right] \mathrm{d}x \,,$$

where $v \in W^{1,\infty}(\mathbb{R}^N)$. Notice that $\mathcal{V}^{(1)}_\varepsilon$ is a particular case of $\mathcal{V}^{(2)}_\varepsilon$ when $\alpha = 1$ and $v \in H^2(\Omega)$ has vanishing normal derivative on $\partial\Omega$. An explicit expression of the Γ-limit is provided in both cases.

The work [22] by Braides and Zeppieri is similar in spirit to an ongoing project of ours where we consider the case of the wells of W depending on the space variable x. Indeed, in [22] the authors studied the asymptotic behavior of the family of functionals

$$\mathcal{G}^{(k)}_\varepsilon(u) := \int_0^1 \left[W^{(k)}\left(\frac{t}{\delta(\varepsilon)}, u(x) \right) + \varepsilon^2 |u'(t)|^2 \right] \mathrm{d}t \,,$$

for $\delta(\varepsilon) > 0$, with the potential $W^{(k)}$ defined, for $k \in [0,1)$, as

$$W^{(k)}(t,s) := \begin{cases} W(s-k) & t \in \left(0, \frac{1}{2}\right) , \\ W(s+k) & t \in \left(\frac{1}{2}, 1\right) , \end{cases}$$

with $W(t) := \min\{(t-1)^2, (t+1)^2\}$. For $k \in (0,1)$ the fact that the zeros of $W^{(k)}$ oscillate at a scale of $\delta(\varepsilon)$ leads to the formation of microscopic oscillations, whose effect is studied by identifying the zeroth, the first and the second order Γ-limit expansions (with the appropriate rescaling) in the three regimes (1.11).

In the context of the gradient theory for solid-solid phase transition, we mention the work [49] by Francfort and Müller, where the asymptotic behavior of the energy

$$\mathcal{L}_\varepsilon(u) := \int_\Omega \left[W\left(\frac{x}{\varepsilon^\gamma}, \nabla u(x)\right) + \varepsilon^2 |\triangle u|^2 \right] \, \mathrm{d}x \, .$$

for $\gamma > 0$ is studied under some growth conditions on the potential W.

1.3 Publications resulting from this thesis

1. A. Hagerty, *Relaxation of functionals in the space of vector-valued functions of bounded Hessian*, Sections 3.2 and 3.3, Calculus of Variations and Partial Differential Equations, 2019.

2. R. Cristoferi, I. Fonseca, A. Hagerty, C. Popovici, *A homogenization result in the gradient theory of phase transitions*, Section 4.3, accepted to Interfaces and Free Boundaries, 2019.

3. A. Hagerty, *A note on homogenization effects on phase transition problems*, Section 4.2, published on the Center for Nonlinear Analysis website, 2019.

4. I. Fonseca, A. Hagerty, R. Paroni, *Second order structured deformations in the space of bounded Hessian*, Section 3.4, submitted to Proceedings of the Royal Society A, 2019.

2 Preliminaries

We begin by collecting some basic notions and definitions needed throughout the thesis.

The set $\Omega \subset \mathbb{R}^N$ will always be a bounded, open domain with Lipschitz boundary. By $Q \subset \mathbb{R}^N$ we denote the unit cube centered at the origin with faces orthogonal to the coordinate axes, $Q := (-1/2, 1/2)^N$. We consider the cube of side length r centered at $x_0 \in \mathbb{R}^N$, $Q(x_0, r) := x_0 + rQ = \{x_0 + ry : y \in Q\}$.

In what follows, we fix a function $\phi \in C^\infty(\mathbb{R}^N; [0,\infty))$ such that $\mathrm{supp}(\phi) \subset B(0,1)$ and $\int_{\mathbb{R}^N} \phi(x)dx = 1$. We define the standard mollifiers ϕ_ε by $\phi_\varepsilon(x) := \frac{1}{\varepsilon^N}\phi(\varepsilon x)$, $\varepsilon > 0$.

2.1 Finite nonnegative Radon measures

The family of finite nonnegative Radon measures in an open set $U \subset \mathbb{R}^N$ will be denoted by $\mathcal{M}(U)$, and the space of finite vector valued Radon measures taking values in $\mathbb{R}^d$ will be denoted $\mathcal{M}(U;\mathbb{R}^d)$. For simplicity of notation, we will often write the Lebesgue measure of a Borel set $E \subset \mathbb{R}^N$ via the notation $|E| := \mathcal{L}^N(E)$.

For any open set $U \subset \mathbb{R}^N$ and any finite Radon measure $\mu \in \mathcal{M}(U;\mathbb{R}^d)$, by the Radon-Nikodym theorem there exist

$$\mu_{ac} \in L^1(U;\mathbb{R}^d),\ \ \mu_s \in \mathcal{M}(U;\mathbb{R}^d)$$

with $|\mu_s| \perp \mathcal{L}^N \llcorner U$, and a $|\mu_s|$-measurable function $\nu_\mu := \frac{d\mu_s}{d|\mu_s|}$ with $|\nu_\mu(x)| = 1$ for $|\mu_s|$ almost every $x \in U$, such that for every Borel set $E \subset U$ we have

$$\mu(E) = \int_E \mu_{ac}(x)dx + \int_E \nu_\mu(x)d|\mu_s|(x).$$

When the measure μ being referenced is clear in context, we will often drop the subscript and write ν_μ as ν.

Definition 2.1. We say that a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(U)$ *weakly-$*$ converges* to a finite nonnegative Radon measure μ if

$$\int_U \varphi \,\mathrm{d}\mu_n \to \int_U \varphi \,\mathrm{d}\mu$$

as $n \to \infty$, for all $\varphi \in C_0(U)$, the space of continuous functions with compact support in U. In this case we write $\mu_n \overset{*}{\rightharpoonup} \mu$.

The following compactness result for Radon measures is well known (see [42, Proposition 1.202]).

Theorem 2.2. *Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(U)$ be such that $\sup_{n\in\mathbb{N}} \mu_n(U) < \infty$. Then there exists a subsequence (not relabeled) and $\mu \in \mathcal{M}(U)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$.*

We recall a result of Reshetnyak (see [69]).

Theorem 2.3. *Let $\{\mu_n\}$ be a sequence in $\mathcal{M}(U;\mathbb{R}^d)$. If $\mu_n \overset{*}{\rightharpoonup} \mu \in \mathcal{M}(U;\mathbb{R}^d)$, then*

$$\liminf_{n\to\infty} \int_U H\left(\frac{d\mu_n}{d|\mu_n|}(x)\right) d|\mu_n| \geq \int_U H\left(\frac{d\mu}{d|\mu|}(x)\right) d|\mu|$$

for every positively 1-homogeneous and convex function $H : \mathbb{R}^d \to \mathbb{R}$ satisfying the growth condition $|H(\xi)| \leq C|\xi|$ for each $\xi \in \mathbb{R}^d$ and for some $C > 0$.

We will use a modification of a lemma which can be found in [45], Lemma 2.13. To be precise,

Lemma 2.4. *Let λ be a nonnegative Radon measure in $\mathbb{R}^N$. For λ almost every $x_0 \in \mathbb{R}^N$ and for every $0 < \sigma < 1$,*

$$\limsup_{r\to 0^+} \frac{\lambda(Q(x_0,\sigma r))}{\lambda(Q(x_0,r))} \geq \sigma^N. \tag{2.1}$$

In (2.1) we can choose $r \to 0^+$ so that, given another Radon measure μ, neither μ nor λ charge the boundary of the larger cubes. Namely, we have the following result.

Lemma 2.5. *Let λ and μ be nonnegative Radon measures in $\mathbb{R}^N$. For every $0 < \sigma < 1$, and for λ almost every $x_0 \in \mathbb{R}^N$, there exist $r_n \to 0^+$ such that $\mu(\partial Q(x_0,r_n)) = \lambda(\partial Q(x_0,r_n)) = 0$ and*

$$\lim_{n\to\infty} \frac{\lambda(Q(x_0,\sigma r_n))}{\lambda(Q(x_0,r_n))} \geq \sigma^N.$$

Proof. Fix $\sigma \in (0,1)$ and $x_0 \in \mathbb{R}^N$ so that, by Lemma 2.4, we can find $\rho_n \to 0^+$ such that

$$\lim_{n\to\infty} \frac{\lambda(Q(x_0,\sigma \rho_n))}{\lambda(Q(x_0,\rho_n))} \geq \sigma^N.$$

For every n, we can select $\delta_n < \rho_n$ such that

$$\lambda(Q(x_0,\sigma\delta_n)) \geq \frac{n}{n+1}\lambda(Q(x_0,\sigma\rho_n)).$$

Find $r_n \in (\delta_n,\rho_n)$ such that $\mu(\partial Q(x_0,r_n)) = \lambda(\partial Q(x_0,r_n)) = 0$. We obtain

$$\frac{\lambda(Q(x_0,\sigma r_n))}{\lambda(Q(x_0,r_n))} \geq \frac{\lambda(Q(x_0,\sigma\delta_n))}{\lambda(Q(x_0,\rho_n))} \geq \frac{n}{n+1}\frac{\lambda(Q(x_0,\sigma\rho_n))}{\lambda(Q(x_0,\rho_n))},$$

and we conclude that

$$\liminf_{n\to\infty} \frac{\lambda(Q(x_0,\sigma r_n))}{\lambda(Q(x_0,r_n))} \geq \sigma^N.$$

Since the sequence $\left\{\frac{\lambda(Q(x_0,\sigma r_n))}{\lambda(Q(x_0,r_n))}\right\}$ is bounded, we can extract a subsequence which is convergent. Without loss of generality we can assume that the sequence actually converges and thus

$$\lim_{n\to\infty} \frac{\lambda(Q(x_0,\sigma r_n))}{\lambda(Q(x_0,r_n))} \geq \sigma^N.$$

□

2.2 Sets of finite perimeter

We recall the definition and some well known facts about sets of finite perimeter (we refer the reader to [4] for more details).

Definition 2.6. Let $E \subset \mathbb{R}^N$ with $|E| < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set. We say that E has *finite perimeter* in Ω if

$$P(E;\Omega) := \sup\left\{ \int_E \mathrm{div}\varphi \, \mathrm{d}x \, : \, \varphi \in C^1_c(\Omega;\mathbb{R}^N) \, , \, \|\varphi\|_{L^\infty} \leq 1 \right\} < \infty \, .$$

Remark 2.7. $E \subset \mathbb{R}^N$ is a set of finite perimeter in Ω if and only if $\chi_E \in BV(\Omega)$, i.e., the distributional derivative $D\chi_E$ is a finite vector valued Radon measure in Ω, with

$$\int_{\mathbb{R}^N} \varphi \, \mathrm{d}D\chi_E = \int_E \mathrm{div}\varphi \, \mathrm{d}x$$

for all $\varphi \in C^1_c(\Omega;\mathbb{R}^N)$, and $|D\chi_E|(\Omega) = P(E;\Omega)$.

Remark 2.8. Let $\Omega \subset \mathbb{R}^N$ be an open set, let $a,b \in \mathbb{R}^d$, and let $u \in L^1(\Omega;\{a,b\})$. Then u is a function of *bounded variation* in Ω, and we write $u \in BV(\Omega;\{a,b\})$, if the set $\{u = a\} := \{x \in \Omega \, : \, u(x) = a\}$ has finite perimeter in Ω.

Definition 2.9. Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in the open set $\Omega \subset \mathbb{R}^N$. We define $\partial^* E$, the *reduced boundary* of E, as the set of points $x \in \mathbb{R}^N$ for which the limit

$$\nu_E(x) := -\lim_{r\to 0} \frac{D\chi_E(x+rQ)}{|D\chi_E|(x+rQ)}$$

exists and is such that $|\nu_E(x)| = 1$. The vector $\nu_E(x)$ is called the *measure theoretic exterior normal* to E at x.

We now recall the structure theorem for sets of finite perimeter due to De Giorgi (see [4, Theorem 3.59] for a proof).

Theorem 2.10. *Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in the open set $\Omega \subset \mathbb{R}^N$. Then*

(i) for all $x \in \partial^* E$ the set $E_r := \frac{E-x}{r}$ converges locally in $L^1(\mathbb{R}^N)$ as $r \to 0$ to the halfspace orthogonal to $\nu_E(x)$ and not containing $\nu_E(x)$,

(ii) $D\chi_E = -\nu_E \, \mathcal{H}^{N-1} \llcorner \partial^* E$,

(iii) the reduced boundary $\partial^* E$ is $\mathcal{H}^{N-1}$-rectifiable, i.e., there exist Lipschitz functions $f_i : \mathbb{R}^{N-1} \to \mathbb{R}^N$, $i \in \mathbb{N}$, such that

$$\partial^* E = \bigcup_{i=1}^{\infty} f_i(K_i) \,,$$

where each $K_i \subset \mathbb{R}^{N-1}$ is a compact set.

Remark 2.11. Using the above result it is possible to prove that (see [3, Proposition 2.2])

$$\nu_E(x) = -\lim_{r \to 0} \frac{D\chi_E(x + rQ)}{r^{N-1}}$$

for all $x \in \partial^* E$.

Finally, we state another theorem of Reshetnyak in a form specifically tailored for sets of finite perimeter, whose proof may also be found in [69].

Theorem 2.12. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets of finite perimeter in the open set $\Omega \subset \mathbb{R}^N$ such that $D\chi_{E_n} \overset{*}{\rightharpoonup} D\chi_E$ and $|D\chi_{E_n}|(\Omega) \to |D\chi_E|(\Omega)$, where E is a set of finite perimeter in Ω. Let $f : \mathbb{S}^{N-1} \to [0, \infty)$ be an upper semi-continuous bounded function. Then*

$$\limsup_{n \to \infty} \int_{\partial^* E_n \cap \Omega} f\left(\nu_{E_n}(x)\right) \, \mathrm{d}\mathcal{H}^{N-1}(x) \leq \int_{\partial^* E \cap \Omega} f\left(\nu_E(x)\right) \, \mathrm{d}\mathcal{H}^{N-1}(x) \,.$$

2.3 Γ-convergence

We refer to [20] and [28] for a complete study of Γ-convergence in metric spaces.

Definition 2.13. Let (X, m) be a metric space. We say that $F_n : X \to [-\infty, +\infty]$ Γ-converges to $F : X \to [-\infty, +\infty]$, and we write $F_n \overset{\Gamma-\mathrm{m}}{\longrightarrow} F$, if the following hold:

(i) for every $x \in X$ and every $x_n \to x$ we have

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n)\,,$$

(ii) for every $x \in X$ there exists $\{x_n\}_{n=1}^{\infty} \subset A$ (a so called *recovery sequence*) with $x_n \to x$ such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x)\,.$$

In the proof of the limsup inequality in Section 4.3.5 we will need to show that a certain set function is actually (the restriction to the family of open sets of) a finite Radon measure. The classical way to prove this is by using the De Giorgi-Letta coincidence criterion (see [34]), namely to show that the set function is inner regular as well as super and sub additive. We will use a simplified coincidence criterion due to Dal Maso, Fonseca and Leoni (see [29, Corollary 5.2]).

Given $\Omega \subset \mathbb{R}^N$ an open set, we denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω.

Lemma 2.14. *Let $\lambda : \mathcal{A}(\Omega) \to [0, \infty)$ be an increasing set function such that:*

(i) for all $A, B, C \in \mathcal{A}(\Omega)$ with $\overline{A} \subset B \subset C$ it holds

$$\lambda(C) \leq \lambda(C \setminus \overline{A}) + \lambda(B)\,,$$

(ii) $\lambda(A \cup B) = \lambda(A) + \lambda(B)$, for all $A, B \in \mathcal{A}(\Omega)$ with $A \cap B = \emptyset$,

(iii) there exists a measure $\mu : \mathcal{B}(\Omega) \to [0, \infty)$ such that

$$\lambda(A) \leq \mu(A)$$

for all $A \in \mathcal{A}(\Omega)$, where $\mathcal{B}(\Omega)$ denotes the family of Borel sets of Ω.

Then λ is the restriction to $\mathcal{A}(\Omega)$ of a measure defined on $\mathcal{B}(\Omega)$.

2.4 Bounded Hessian functions and 2-quasiconvexity

We recall the space of bounded Hessian functions

$$\begin{aligned} \mathrm{BH}(\Omega; \mathbb{R}^d) \;&:=\; \{u \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^d) : D^2 u \text{ is a finite Radon measure}\} \\ &=\; \{u \in \mathrm{L}^1(\Omega; \mathbb{R}^d) : Du \in \mathrm{BV}(\Omega; \mathbb{R}^{d\times N})\}. \end{aligned}$$

We consider also the space of special functions of bounded Hessian

$$SBH(\Omega;\mathbb{R}^d) := \{u \in BH(\Omega;\mathbb{R}^d) : D_c(\nabla u) = 0\},$$

that is, BH functions with no Cantor part in the Hessian. This is distinct from the related space

$$SBV^2(\Omega;\mathbb{R}^d) := \{u \in BV(\Omega;\mathbb{R}^d) : D_c(u) = 0, \nabla u \in BV(\Omega;\mathbb{R}^{d\times N},\ D_c(\nabla u) = 0\}.$$

We now establish some basic results concerning approximate differentiability properties of functions in the setting of BH.

Theorem 2.15. *If $u \in BH(\Omega;\mathbb{R}^d)$ then*

(i) for $\mathcal{L}^N$ a.e. $x \in \Omega$

$$\lim_{r\to 0^+} \frac{1}{r^2} \fint_{Q(x,r)} \left| u(y) - u(x) - \nabla u(x)(y-x) - \frac{1}{2}\nabla^2 u(x)(y-x,y-x) \right| dy = 0, \quad (2.2)$$

and

$$\lim_{r\to 0^+} \frac{1}{r} \fint_{Q(x,r)} \left| \nabla u(y) - \nabla u(x) - \nabla^2 u(x)(y-x) \right| dy = 0; \quad (2.3)$$

(ii) for $\mathcal{H}^{N-1}$ a.e. $x \in S(\nabla u)$ we have

$$\lim_{r\to 0^+} \frac{1}{r} \fint_{Q_\nu^\pm(x,r)} \left| u(y) - u(x) - \nabla u^\pm(x)(y-x) \right| dy = 0, \quad (2.4)$$

and

$$\lim_{r\to 0^+} \fint_{Q_\nu^\pm(x,r)} \left| \nabla u(y) - \nabla u^\pm(x) \right| dy = 0, \quad (2.5)$$

where $Q_\nu^+(x,r) = Q_\nu(x,r) \cap \{y : (y-x)\cdot\nu(x) > 0\}$ and $Q_\nu^-(x,r) = Q_\nu(x,r) \cap \{y : (y-x)\cdot\nu(x) < 0\}$.

Proof. A proof of (2.3) can be found in Theorem 6.1 in [41], applied to $f = \nabla u$. Similarly, a proof of (2.4) and (2.5) can be found in Theorem 5.19 in [41], applied to $f = u$ and $f = \nabla u$ respectively.

It remains to show (2.2), which involves a second-order approximation. Its proof uses arguments similar to those found in [41], and it is included below for completeness.

Fix $x_0 \in \Omega$ such that

$$\lim_{r\to 0^+} \fint_{Q(x_0,r)} |u(x) - u(x_0)|\, dx = 0, \quad (2.6)$$

$$\lim_{r\to 0^+} \fint_{Q(x_0,r)} |\nabla u(x) - \nabla u(x_0)|\, dx = 0, \tag{2.7}$$

$$\lim_{r\to 0^+} \fint_{Q(x_0,r)} \left|\nabla^2 u(x) - \nabla^2 u(x_0)\right|\, dx = 0, \tag{2.8}$$

and

$$\lim_{r\to 0^+} \frac{|D^2_s u|(Q(x_0,r))}{r^N} = 0. \tag{2.9}$$

Since the above hold for $\mathcal{L}^N$ a.e $x_0 \in \Omega$, it suffices to show that for every such x_0 we have

$$\lim_{r\to 0^+} \frac{1}{r^2}\fint_{Q(x_0,r)} \left|u(x) - u(x_0) - \nabla u(x_0)(x-x_0) - \frac{1}{2}\nabla^2 u(x_0)(x-x_0, x-x_0)\right|\, dx = 0.$$

Without loss of generality, we take $x_0 = 0$. Define smooth functions u_ε by $u * \phi_\varepsilon$, for $0 < \varepsilon << r << \operatorname{dist}(0, \partial\Omega)$. By (2.6), (2.7) and (2.8), note that

$$\lim_{\varepsilon\to 0^+} u_\varepsilon(0) = u(0), \lim_{\varepsilon\to 0^+} \nabla u_\varepsilon(0) = \nabla u(0), \lim_{\varepsilon\to 0^+} \nabla^2 u_\varepsilon(0) = \nabla^2 u(0).$$

For $x \in Q(0,r)$, consider now the function $g_\varepsilon(t)$ defined by

$$g_\varepsilon(t) := u_\varepsilon(tx),\ \ t \in [0,1].$$

By smoothness of the u_ε , applying the fundamental theorem of calculus twice, we see that

$$g(1) = g(0) + g'(0) + \int_0^1 (1-t)g''(t)\, dt$$

and thus

$$u_\varepsilon(x) = u_\varepsilon(0) + \nabla u_\varepsilon(0)x + \int_0^1 (1-t)\nabla^2 u_\varepsilon(tx)(x,x)\, dt.$$

Rearrange these terms and subtract $\frac{1}{2}\nabla^2 u(0)(x,x)$ from both sides to obtain

$$u_\varepsilon(x) - u_\varepsilon(0) - \nabla u_\varepsilon(0)x - \frac{1}{2}\nabla^2 u_\varepsilon(0)(x,x) = \int_0^1 (1-t)(\nabla^2 u_\varepsilon(tx) - \nabla^2 u_\varepsilon(0))(x,x)\, dt,$$

and so

$$\begin{aligned}&\frac{1}{r^2}\fint_{Q(0,r)} |u_\varepsilon(x) - u_\varepsilon(0) - \nabla u_\varepsilon(0)x - \frac{1}{2}\nabla^2 u_\varepsilon(0)(x,x)|\, dx \\ &\quad\le \frac{1}{r^2}\fint_{Q(0,r)} \int_0^1 |(\nabla^2 u_\varepsilon(tx) - \nabla^2 u_\varepsilon(0))(x,x)|\, dt\, dx.\end{aligned}$$

By Fatou's lemma,

$$\begin{aligned}
&\frac{1}{r^2}\fint_{Q(0,r)}|u(x)-u(0)-\nabla u(0)x-\frac{1}{2}\nabla^2u(0)(x,x)|\,dx\\
&\leq \liminf_{\varepsilon\to0^+}\frac{1}{r^2}\fint_{Q(0,r)}|u_\varepsilon(x)-u_\varepsilon(0)-\nabla u_\varepsilon(0)x-\frac{1}{2}\nabla^2u_\varepsilon(0)(x,x)|\,dx\\
&\leq \liminf_{\varepsilon\to0^+}\frac{1}{r^2}\fint_{Q(0,r)}\int_0^1\left|\left(\nabla^2u_\varepsilon(tx)-\nabla^2u_\varepsilon(0)\right)(x,x)\right|\,dt\,dx. \qquad (2.10)
\end{aligned}$$

Thus, it suffices to bound (2.10). Applying the change of variables $z=tx$, we have

$$\int_0^1\frac{1}{t^{N+2}}\frac{1}{r^{N+2}}\int_{Q(0,tr)}\left|\left(\nabla^2u_\varepsilon(z)-\nabla^2u_\varepsilon(0)\right)(z,z)\right|\,dz\,dt$$

$$\leq\int_0^1\frac{1}{r^Nt^N}\int_{Q(0,tr)}|\nabla^2u_\varepsilon(z)-\nabla^2u_\varepsilon(0)|\,dz\,dt.$$

Using the triangle inequality, we obtain

$$\begin{aligned}
\int_0^1\frac{1}{r^Nt^N}\int_{Q(0,tr)}|\nabla^2u_\varepsilon(z)-\nabla^2u_\varepsilon(0)|\,dz &\leq\int_0^1\frac{1}{r^N}\frac{1}{t^N}\int_{Q(0,tr)}|\nabla^2u_\varepsilon(z)-\nabla^2u(z)|\,dzdt\\
&+\int_0^1\fint_{Q(0,tr)}|\nabla^2u(z)-\nabla^2u(0)|\,dzdt+\int_0^1\fint_{Q(0,tr)}|\nabla^2u(0)-\nabla^2u_\varepsilon(0)|\,dz\,dt. \quad (2.11)
\end{aligned}$$

If we let ε tend to 0^+, the second term will be unchanged and the third term will vanish. We turn our attention to the first term, namely

$$\int_0^1\frac{1}{t^N}\frac{1}{r^N}\int_{Q(0,tr)}|\nabla^2u_\varepsilon(z)-\nabla^2u(z)|\,dz\,dt.$$

Set

$$h_\varepsilon(t):=\int_{Q(0,tr)}|\nabla^2u_\varepsilon(z)-\nabla^2u(z)|\,dz,\ \text{for } t\in(0,1),$$

and note that

$$h_\varepsilon(t)\leq\int_{Q(0,tr)}|(\nabla^2u*\phi_\varepsilon)(z)-\nabla^2u(z)|\,dz+\int_{Q(0,tr)}|(D_s^2u*\phi_\varepsilon)(z)|\,dz.$$

Sending $\varepsilon\to0^+$, we have

$$\limsup_{\varepsilon\to0^+}h_\varepsilon(t)\leq\left|D_s^2u\right|(\overline{Q(0,tr)}).$$

Observe that

$$\frac{h_\varepsilon(t)}{t^N} = \frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\, dz \leq \frac{1}{t^N}\int_{Q(0,tr)} \left(|\nabla^2 u_\varepsilon(z)| + |\nabla^2 u(z)|\right)\, dz,$$

$$\frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u(z)|\, dz \leq \frac{|D^2 u|(Q(0,tr))}{t^N} \leq C$$

for some constant C by (2.8), since r is fixed. On the other hand,

$$\begin{aligned}
\frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z)|\, dz &\leq \int_{Q(0,tr)}\int_\Omega \phi_\varepsilon(z-y)\, d|D^2u|(y)\, dz \\
&= \frac{1}{t^N}\int_\Omega\int_{Q(0,tr)} \phi_\varepsilon(z-y)\, dz\, d|D^2u|(y) \\
&\leq \frac{C}{\varepsilon^N t^N}\int_{Q(0,tr+\varepsilon)}\int_{Q(0,tr)\cap B(y,\varepsilon)} dz\, d|D^2u|(y) \\
&\leq \frac{C}{\varepsilon^N t^N}\min\{\varepsilon^N, t^N\}|D^2u|(Q(0,tr+\varepsilon)).
\end{aligned}$$

Again by (2.8) and (2.9), we have

$$|D^2u|(Q(0,tr+\varepsilon)) \leq C(tr+\varepsilon)^N,$$

so we conclude that $\frac{h_\varepsilon(t)}{t^N}$ is bounded by a constant for $t \in (0,1)$, and we may apply the Reverse Fatou Lemma to deduce

$$\limsup_{\varepsilon\to 0^+}\int_0^1 \frac{1}{t^N r^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\, dz dt \leq \int_0^1 \frac{1}{t^N r^N}\left|D_s^2 u\right|(\overline{Q(0,tr)})dt.$$

Thus from (2.10) and (2.11) we have

$$\frac{1}{r^2}\fint_{Q(0,r)} |u(x) - u(0) - \nabla u(0)x - \frac{1}{2}\nabla^2 u(0)(x,x)|\, dx$$

$$\leq \int_0^1 \left(\frac{|D_s^2 u|(\overline{Q(0,tr)})}{t^N r^N} + \fint_{Q(0,tr)} |\nabla^2 u(z) - \nabla^2 u(0)|\, dz\right) dt. \tag{2.12}$$

For a given r there are only countably many $t \in (0,1)$ such that $|D_s^2 u|(\partial Q(0,tr)) > 0$. Thus, we can rewrite (2.12) as

$$\int_0^1 \left(\frac{|D_s^2 u|(Q(0,tr))}{t^N r^N} + \fint_{Q(0,tr)} |\nabla^2 u(z) - \nabla^2 u(0)|\, dz\right) dt$$

We note that by (2.8) and (2.9) we can apply the dominated convergence theorem to conclude that
$$\lim_{r\to 0^+}\int_0^1\left(\frac{|D_s^2u|(Q(0,tr))}{t^N r^N}+\fint_{Q(0,tr)}|\nabla^2u(z)-\nabla^2u(0)|\,dz\right)dt=0.$$
□

For a Borel measurable function $f:\mathbb{R}^{d\times N\times N}\to[0,\infty)$ we define the 2-quasiconvex envelope
$$\mathcal{Q}_2f(H):=\inf\left\{\int_Q f(H+\nabla^2\phi(z))dz\mid \phi\in W_0^{2,1}(Q;\mathbb{R}^d)\right\}$$
for $H\in\mathbb{R}^{d\times N\times N}$.

The notion of 2-quasiconvexity, introduced by Meyers in [61], is an extension of quasiconvexity to second-order integrands.

The BH relaxation result of Chapter 3 relies on geometric properties of Hessians and 2-quasiconvex functions. In particular, any 2-quasiconvex function is convex along certain directions- analogous to quasiconvex functions being rank-one convex, see [27]. To be precise, we will follow the notation of Ball, Currie, and Olver [9]. By $X(N,d,2)$ we denote the space of symmetric bilinear maps from $\mathbb{R}^N\times\mathbb{R}^N$ into $\mathbb{R}^d$, noting that every Hessian matrix is in $X(N,d,2)$ when viewed as a bilinear map
$$(v_1,v_2)\mapsto\frac{\partial^2u}{\partial v_1\partial v_2}(x_0).$$
We define the cone $\Lambda(N,d,2)$ as
$$\Lambda(N,d,2):=\{a\otimes b\otimes b : a\in\mathbb{R}^d, b\in\mathbb{R}^N\}.$$

Lemma 2.16. *Let $M=\dim(X(N,d,2))=\frac{d(d+1)}{2}N$. There is a basis $\{\xi_i\}_{i=1}^M\subset\Lambda(N,d,2)$ for $X(N,d,2)$ with $|\xi_i|=1$ for every $i\in\{1,\dots M\}$ and there exists $c(N,d)>0$ such that for all $H\in X(N,d,2)$ written as*
$$H=\sum_{i=1}^M a_i\xi_i,\ \ a_i\in\mathbb{R}, i=1,\dots,M,$$
it holds that
$$\frac{1}{c}|H|\le\sum_{i=1}^M|a_i|\le c|H|.$$

Proof. Since tensors of the form

$$e_k \otimes e_i \otimes e_j + e_k \otimes e_j \otimes e_i,\ k = 1, \dots, d,\ \text{and } i, j = 1, \dots, N,$$

form a basis for $X(N, d, 2)$, to see that we can form a basis contained in $\Lambda(N, d, 2)$ it will suffice to show that the span of $\Lambda(N, d, 2)$ contains these basis vectors. When $i = j$, we trivially have

$$2e_k \otimes e_i \otimes e_i \in \Lambda(N, d, 2),$$

and if $i \neq j$, we note that $\Lambda(N, d, 2)$ contains

$$e_k \otimes (e_i + e_j) \otimes (e_i + e_j) = e_k \otimes (e_i \otimes e_i + e_j \otimes e_i + e_i \otimes e_j + e_j \otimes e_j)$$

which, combined with our above observation, implies that

$$e_k \otimes e_i \otimes e_j + e_k \otimes e_j \otimes e_i \in \mathrm{Span}(\Lambda(N, d, 2)).$$

Thus we have $\mathrm{Span}(\Lambda(N, d, 2)) = X(N, d, 2)$ and we can select a basis for $X(N, d, 2)$ consisting of $\Lambda(N, d, 2)$ tensors, and by scaling these appropriately we can guarantee $|\xi_i| = 1$ for every $i \in \{1, \dots, M\}$. Note that with $H = \sum_{i=1}^{M} a_i \xi_i$, $a_i \in \mathbb{R}$, $i = 1, \dots, M$,

$$\|H\| := \sum_{i=1}^{M} |a_i|$$

defines a norm on $X(N, d, 2)$, and the existence of a constant $c > 0$ such that

$$\frac{1}{c}|H| \leq \sum_{i=1}^{M} |a_i| \leq c|H|$$

follows from the equivalence of norms on finite dimensional normed spaces.

□

Definition 2.17. We say that a function $F : X(N, d, 2) \to \mathbb{R}$ is $\Lambda(N, d, 2)$-convex if

$$F(t\xi + (1-t)\xi') \leq tF(\xi) + (1-t)F(\xi')$$

whenever $(\xi - \xi') \in \Lambda(N, d, 2)$, $t \in (0, 1)$.

Theorem 3.3 in [9] relates 2-quasiconvexity to $\Lambda(N, d, 2)$-convexity.

Lemma 2.18. *Let $F : \mathbb{R}^{N \times d \times d} \to \mathbb{R}$ be continuous and 2-quasiconvex. Then F is $\Lambda(N, d, 2)$-convex.*

Next we show that $\Lambda(N,d,2)$-convex functions with linear growth are in fact Lipschitz continuous in all of $X(N,d,2)$. This lemma is a slight modification of a similar result on separately convex functions in [42] Proposition 4.64.

Lemma 2.19. *Let $f:\mathbb{R}^{d\times N\times N}\to\mathbb{R}$ be a $\Lambda(N,d,2)$-convex function such that*

$$|f(H)|\leq C(1+|H|) \tag{2.13}$$

for some $C>0$ and all $H\in\mathbb{R}^{d\times N\times N}$. Then

$$|f(H)-f(H')|\leq \tilde{C}|H-H'|$$

for all $H,H'\in X(N,d,2)$, where $\tilde{C}$ depends only on C,N and d.

Proof. Step 1: First, consider the case where $f\in C^\infty(\mathbb{R}^{N\times d\times d})$. From Lemma 2.16 we can select a basis $\{\xi_i\}\subset\Lambda(N,d,2)$ for $X(N,d,2)$. Fix $H\in X(N,d,2)$, which can be expressed as $H=\sum_{i=1}^M a_i\xi_i, a_i\in\mathbb{R}, i=1,\dots,M$. Fix $j\in\{1,\dots,M\}$ and consider the function

$$g(t)=f\Bigg(t\xi_j+\sum_{i\neq j}a_i\xi_i\Bigg).$$

Since g is convex and smooth, it follows from [42] Theorem 4.62 that for every $t,s\in\mathbb{R}$ we have

$$g(t+s)-g(t)\geq g'(t)s.$$

In particular, letting $s=1+|H|$ and $t=a_j$,

$$\begin{aligned} g'(t)=\frac{\partial f}{\partial\xi_j}(H)\leq\frac{g(t+s)-g(t)}{s}&\leq\frac{|f(H+(1+|H|)\xi_j)|+|f(H)|}{1+|H|}\\ &=\frac{C(1+|H|+|\xi_j|(1+|H|))+C(1+|H|)}{1+|H|}\\ &\leq 3C\frac{1+|H|}{1+|H|}=3C, \end{aligned}$$

by virtue of (2.13) and the fact that $|\xi_j|=1$. Similarly,

$$g(t-s)-g(t)\geq -g'(t)s$$

so

$$-g'(t)\leq\frac{g(t-s)-g(t)}{s}\leq 3C\frac{1+|H|}{1+|H|}=3C$$

and thus

$$\left|\frac{\partial f}{\partial \xi_j}(H)\right| \leq 3C$$

for every $j = 1, \ldots, M$, and $H \in X(N, d, 2)$. Let $H, H' \in X(N, d, 2)$. By the mean value theorem, we can find $\theta \in (0, 1)$ so that

$$|f(H) - f(H')| = |\nabla f(\theta H + (1-\theta)H') \cdot (H - H')| \tag{2.14}$$

and we can decompose $H - H'$ into $\sum_{i=1}^{M} b_i \xi_i$ so that

$$\begin{aligned}|\nabla f(\theta H + (1-\theta)H') \cdot (H - H')| &= \left|\sum_{i=1}^{M}\sum_{j=1}^{M} \frac{\partial f}{\partial \xi_i}(\theta H + (1-\theta)H') b_j \xi_i \cdot \xi_j\right| \\ &\leq \sum_{i=1}^{M}\sum_{j=1}^{M} 3C|b_j| \leq 3cMC|H - H'|\end{aligned}$$

where we used Lemma 2.16. We conclude in view of (2.14).

Step 2: For an arbitrary $\Lambda(N, d, 2)$-convex function f satisfying (2.13), consider the mollified functions $f_\varepsilon := f * \phi_\varepsilon$, $\varepsilon > 0$. Each function f_ε is still $\Lambda(N, d, 2)$-convex and for every $H \in \mathbb{R}^{N\times d\times d}$ we have

$$\begin{aligned}|f_\varepsilon(H)| &\leq \left|\int_{\mathbb{R}^{N\times d\times d}} \phi_\varepsilon(S) f(H - S) dS\right| \leq C\int_{B(0,\varepsilon)} \phi_\varepsilon(S)(1 + |H - S|) dS \\ &\leq C(1 + |H|),\end{aligned}$$

and by Step 1

$$|f_\varepsilon(H) - f_\varepsilon(H')| \leq \tilde{C}|H - H'|$$

for every $H, H' \in X(N, d, 2)$ for some $\tilde{C}$ independent of ε. Since $f_\varepsilon \to f$ pointwise as $\varepsilon \to 0^+$, we have our desired result.

□

To prove the upper bound, we will establish an area-strict density result in BH. The notion of area-strict convergence is as follows.

Definition 2.20. We say that a sequence of Radon measures $\{\mu^n\} \subset \mathcal{M}(\Omega; \mathbb{R}^d)$ converges area-strictly to $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ if $\mu^n \overset{*}{\rightharpoonup} \mu$, i.e.,

$$\int_\Omega \psi \cdot d\mu_n \to \int_\Omega \psi \cdot d\mu \text{ for every } \psi \in C_c(\Omega; \mathbb{R}^d),$$

and
$$\int_\Omega \sqrt{1+|\mu_{ac}^n|^2}dx + |\mu_s^n|(\Omega) \to \int_\Omega \sqrt{1+|\mu_{ac}|^2}dx + |\mu_s|(\Omega).$$

We will make use of another Reshetnyak-type theorem found in [56] Theorem 5.

Theorem 2.21. *Let $f \in \mathbf{E}(\Omega;\mathbb{R}^{d\times N})$ and let $\{\mu^n\}$ be a sequence of matrix valued measures on Ω such that $\mu^n \to \mu$ area-strictly on Ω. Then,*

$$\int_\Omega f(x,\mu_{ac}^n)dx + \int_\Omega f^\infty\left(x,\frac{d\mu_s^n}{d|\mu_s^n|}\right)d|\mu_s^n| \to \int_\Omega f(x,\mu_{ac})dx + \int_\Omega f^\infty\left(x,\frac{d\mu_s}{d|\mu_s|}\right)d|\mu_s|$$

where $\mathbf{E}(\Omega;\mathbb{R}^{d\times N})$ is the set of all functions $f:\Omega\times\mathbb{R}^{d\times N}\to\mathbb{R}$ such that the function

$$\hat{f}(x,\xi) := (1-|\xi|)f\left(x,\frac{\xi}{1-|\xi|}\right),\ x\in\Omega,\ \xi\in B(0,1)$$

has a continuous extension to $\overline{\Omega\times B(0,1)}$.

In Chapter 3, we apply this Reshetnyak-type theorem of Kristensen and Rindler in the following form:

Theorem 2.22. *Let $f:\Omega\times\mathbb{R}^{d\times N\times N}\to[0,\infty)$ be a 2-quasiconvex continuous integrand satisfying the growth condition (H1). Then the functional*

$$\mathcal{G}(u) := \int_\Omega f(x,\nabla^2 u(x))dx + \int_\Omega f^\infty\left(x,\frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}(x)\right)d|D_s(\nabla u)|(x)$$

is continuous with respect to area-strict convergence of $D(\nabla u)$.

Proof. From Lemma 1 in [56], we know that since f is continuous and nonnegative with linear growth, we can find $g_k, h_k \in \mathbf{E}(\Omega;\mathbb{R}^{d\times N\times N})$ such that

$$g_k(x,H)\nearrow f(x,H),\ g_k^\infty(x,H)\nearrow f_\#(x,H),$$
$$h_k(x,H)\searrow f(x,H),\ h_k^\infty(x,H)\searrow f^\#(x,H),$$

for every $x\in\Omega$, $H\in X(N,d,2)$, where

$$f_\#(x,H) := \liminf\left\{\frac{f(x',tH')}{t} : x'\to x, H'\to H, t\to+\infty\right\},$$

and

$$f^\#(x,H) := \limsup\left\{\frac{f(x',tH')}{t} : x'\to x, H'\to H, t\to+\infty\right\}.$$

Let $u \in BH(\Omega;\mathbb{R}^d)$ and let $u_n \in W^{2,1}(\Omega;\mathbb{R}^d)$ be such that $u_n \to u$ in $W^{1,1}$ and $\nabla^2 u_n \mathcal{L}^N \llcorner \Omega \to D(\nabla u)$ area-strictly. For every k we apply Theorem 2.21 to obtain

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{G}(u_n) &\geq \liminf_{n\to\infty} \int_\Omega g_k(x,\nabla^2 u_n)dx \\ &= \int_\Omega g_k(x,\nabla^2 u)dx + \int_\Omega g_k^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}\right) d|D_s(\nabla u),\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \mathcal{G}(u_n) &\leq \limsup_{n\to\infty} \int_\Omega h_k(x,\nabla^2 u_n)dx \\ &= \int_\Omega h_k(x,\nabla^2 u)dx + \int_\Omega h_k^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}\right) d|D_s(\nabla u).\end{aligned}$$

Taking the supremum over k, we apply Monotone Convergence to conclude

$$\liminf_{n\to\infty} \mathcal{G}(u_n) \geq \int_\Omega f(x,\nabla^2 u)dx + \int_\Omega f_\#\left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}\right) d|D_s(\nabla u)|. \tag{2.15}$$

Similarly, since $h_1, h_1^\infty \in \mathbb{E}(\Omega;\mathbb{R}^{d\times N\times N})$ we can apply Monotone Convergence to $-h_k, -h_k^\infty$ to conclude

$$\limsup_{n\to\infty} \mathcal{G}(u_n) \leq \int_\Omega f(x,\nabla^2 u)dx + \int_\Omega f^\#\left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}\right) d|D_s(\nabla u)|. \tag{2.16}$$

A generalization of Alberti's Rank One theorem to Hessians, proved in [35], Theorem 1.6, says that

$$\frac{D_s(\nabla u)}{|D_s(\nabla u)|}(x) \in \Lambda(N,d,2) \tag{2.17}$$

for $|D_s(\nabla u)|$ almost every x. We claim that for all $H \in \Lambda(N,d,2)$

$$f_\#(x,H) = f^\#(x,H) = f^\infty(x,H). \tag{2.18}$$

To see this, as in [56], we examine the expression

$$\frac{f(x',tH')}{t} = \frac{f(x',tH') - f(x',tH)}{t} + \frac{f(x',0)}{t} + \frac{f(x',tH) - f(x',0)}{t} \tag{2.19}$$

for $x' \in \Omega$, $H' \in X(N,d,2)$ and $t > 0$. By Lemma 2.18, $f(x,\cdot)$ is $\Lambda(N,d,2)$-convex with linear growth. Hence by Lemma 2.19, $f(x, \cdot)$ is Lipschitz on all of $X(N, d, 2)$ and the Lipschitz constant is independent of x. Thus, the first term in (2.19) will vanish as $H' \to H$.

The second term clearly goes to zero as $t \to +\infty$, so we turn to the third term. We note that for all $H \in \Lambda(N, d, 2)$, $y \in \Omega$,

$$\frac{f(y, tH) - f(y, 0)}{t}$$

is an increasing function in t by $\Lambda(N, d, 2)$-convexity, and since $f(y, \cdot)$ is Lipschitz, we have

$$\lim_{t \to +\infty} \frac{f(y, tH) - f(y, 0)}{t} = \sup_{t>0} \frac{f(y, tH) - f(y, 0)}{t} = f^{\infty}(y, H).$$

As f is continuous in y for every H, we can apply Dini's Theorem to conclude that the convergence as $t \to +\infty$ is locally uniform in y. Thus, the third term converges to $f^{\infty}(x, H)$ as $t \to +\infty$ and $x' \to x$.

In view of (2.15), (2.16), (2.17) and (2.18) we conclude that

$$\lim_{n \to \infty} \mathcal{G}(u_n) = \mathcal{G}(u).$$

$\square$

3 Structured Deformations and BH

3.1 Statement of main results

We consider a functional $F : W^{2,1}(\Omega, \mathbb{R}^N) \to [0, \infty]$ given by

$$F(u) := \int_\Omega f(x, \nabla^2 u)dx, \quad u \in W^{2,1}(\Omega, \mathbb{R}^N),$$

where $f : \Omega \times \mathbb{R}^{d\times N\times N} \to [0, \infty)$ satisfies the following hypotheses:

(H1) Linear growth: $f(x, H) \leq C(1 + |H|)$ for all $x \in \Omega$, $H \in \mathbb{R}^{d\times N\times N}$ and some $C > 0$;

(H2) Modulus of continuity: $|f(x, H) - f(y, H)| \leq \omega(|x - y|)(1 + |H|)$ for all $x, y \in \Omega, H \in \mathbb{R}^{d\times N\times N}$, where $\omega(s)$ is a nondecreasing function with $w(s) \to 0$ as $s \to 0^+$.

Denoting the lower-semicontinuous envelope of F onto the space $BH(\Omega; \mathbb{R}^d)$ by

$$\mathcal{F}(u) := \inf \Big\{ \liminf_{n\to\infty} F(u_n) : u_n \to u \text{ in } W^{1,1}(\Omega; \mathbb{R}^d),$$
$$\nabla^2 u_n \, \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} D(\nabla u) \text{ in } \mathcal{M}(\Omega, \mathbb{R}^{d\times N\times N}) \Big\},$$

we will prove the following integral representation result.

Theorem 3.1. *If f satisfies (H1) and (H2), then for every $u \in BH(\Omega; \mathbb{R}^d)$ we have*

$$\mathcal{F}(u) = \int_\Omega \mathcal{Q}_2 f(x, \nabla^2 u)dx + \int_\Omega (\mathcal{Q}_2 f)^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|} \right) d|D_s(\nabla u)|.$$

The proof of Theorem 3.1 is structured as follows. In Section 3.2, we prove an area-strict density result for Radon measures and a second-order extension theorem in order to apply this theorem to BH. Section 3.3 contains the relaxation result which is achieved by a direct blow-up argument.

3.2 Density result

Here we prove a useful density result which states that we can approximate a measure in the area-strict sense via smooth functions, as long as the domain is sufficiently regular. In order to prove this, we will need the following estimate.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a convex function satisfying $|g(\xi)| \leq C(1 + |\xi|)$ for some $C > 0$ and all $\xi \in \mathbb{R}^d$ and let $\mu \in \mathcal{M}(\Omega, \mathbb{R}^d)$. For every $x \in \Omega$ and $\varepsilon < \varepsilon_0 := \operatorname{dist}(x, \partial\Omega)$,*

$$g((\mu_{ac} * \phi_\varepsilon)(x)) \leq (g(\mu_{ac}) * \phi_\varepsilon)(x),$$

and

$$g((\mu_s * 2\phi_\varepsilon)(x)) \leq \int_\Omega \frac{g(2t_\varepsilon(x)\nu(y))}{t_\varepsilon(x)} \phi_\varepsilon(y-x) d|\mu_s|(y), \tag{3.1}$$

where $t_\varepsilon \in C^\infty(B(x,\varepsilon)); [0,\infty))$ *is given by*

$$t_\varepsilon(x) = \int_\Omega \phi_\varepsilon(y-x) d|\mu_s|(y) dy$$

and (3.1) holds whenever $t_\varepsilon(x) > 0$*, a set of* $|\mu_s|$ *density 1.*

Proof. Fix $x \in \Omega$ and note that $\varepsilon < \varepsilon_0$ implies that $B(x,\varepsilon) \subset \Omega$. By Jensen's inequality,

$$g((\mu_{ac} * \phi_\varepsilon)(x)) = g\left(\int_\Omega \phi_\varepsilon(y-x)\mu_{ac}(y)dy\right) \leq \int_\Omega \phi_\varepsilon(y-x) g(\mu_{ac}(y))dy = (g(\mu_{ac}) * \phi_\varepsilon)(x).$$

where we used the fact that $\int_\Omega \phi_\varepsilon(y-x)dy = 1$.

For the singular part, we set $t_\varepsilon(x) := \int_\Omega \phi_\varepsilon(y-x)d|\mu_s|(y)dy$. Then, for $\varepsilon << 1$, we have $t_\varepsilon \in C^\infty(B(x,\varepsilon_0); [0,\infty))$, and if $t_\varepsilon > 0$ then the measure $\pi_\varepsilon := \frac{1}{t_\varepsilon}\phi_\varepsilon(\cdot - x)|\mu_s|$ is a probability measure. Thus, we can again apply Jensen's inequality to obtain

$$\begin{aligned} g((\mu_s * 2\phi_\varepsilon)(x)) &= g\left(\int_\Omega 2\phi_\varepsilon(y-x)d\mu_s(y)\right) = g\left(\int_\Omega 2t_\varepsilon(x)\nu(y)d\pi_\varepsilon(y)\right) \\ &\leq \int_\Omega g(2t_\varepsilon(x)\nu(y))d\pi_\varepsilon(y) = \int_\Omega \frac{g(2t_\varepsilon(x)\nu(y))}{t_\varepsilon(x)}\phi_\varepsilon(y-x)d|\mu_s|(y). \end{aligned}$$

□

Theorem 3.3. *If* U *is a bounded, open set in* $\mathbb{R}^N$*,* $\mu \in \mathcal{M}(U;\mathbb{R}^d)$ *is a Radon measure, and* $g : \mathbb{R}^d \to [0,\infty)$ *is a nonnegative function with the following properties:*

(A1) Linear growth: $g(p) \leq C(1+|p|)$ *for all* $p \in \mathbb{R}^d$ *and some* $C > 0$*;*

(A2) Convexity: $g(tp + (1-t)q) \leq tg(p) + (1-t)g(q)$ *for all* $p, q \in \mathbb{R}^d$ *and* $t \in (0,1)$*;*

(A3) Monotone in norm: $|p| \leq |q|$ *implies* $g(p) \leq g(q)$ *for all* $p, q \in \mathbb{R}^d$*;*

(A4) Uniform Convergence to Recession Function: For some $\alpha > 1$ *and* $C > 0$*,*

$$\left|\frac{g(tp)}{t} - g^{\infty}(p)\right| \leq \frac{C}{t^{\alpha}}$$

for all $p \in \mathbb{R}^d$, $t > 0$.

Then, for every $\Omega \subset\subset U$ *with* $|\partial\Omega| = |\mu|(\partial\Omega) = 0$, *we have*

$$\lim_{\varepsilon \to 0^+} \int_{\Omega} g(\mu_{\varepsilon})dx = \int_{\Omega} g(\mu_{ac})dx + \int_{\Omega} g(\nu)d|\mu_s|$$

where $\mu_{\varepsilon} := \mu * \phi_{\varepsilon}$.

Proof. We divide the proof into two steps, a lower and upper bound.

3.2.1 Step 1: Lower bound

We claim that

$$\int_{\Omega} g(\mu_{ac})dx + \int_{\Omega} g(\nu)d|\mu_s| \leq \liminf_{\varepsilon \to 0^+} \int_{\Omega} g(\mu_{\varepsilon})dx \tag{3.2}$$

For this inequality, we use the fact that $\{\mu_{\varepsilon}\}$ converges weakly-$*$ to μ, and that $\{|\mu_{\varepsilon}|\}$ converges weakly-$*$ to $|\mu|$ (See [4], Theorem 2.2).

We will apply the blow-up argument originally found in [45]. Choose $\varepsilon_k \to 0$ which achieve the liminf, and, for simplicity, using the notation $\mu_{\varepsilon_k} =: \mu_k$, we define the Radon measures

$$\lambda_k(E) := \int_E g(\mu_k(x))dx$$

for any Borel set $E \subset \Omega$. Due to the growth condition (A1), we have

$$\lambda_k(\Omega) = \int_{\Omega} g(\mu_k(x)) \leq \int_{\Omega} C(1 + |\mu_k(x)|)dx = C\Big(|\Omega| + |\mu_k|(\Omega)\Big), \tag{3.3}$$

and since $\{\mu_k\}$ converge weakly-$*$, the sequence $\{|\mu_k|(\Omega)\}$ is bounded. We deduce that $\{\lambda_k(\Omega)\}$ is bounded, therefore, along a subsequence (not relabeled) we have $\lambda_k \overset{*}{\rightharpoonup} \lambda$ for some finite Radon measure λ.

The growth conditions on g yield

$$\lambda << \mathcal{L}^N \llcorner \Omega + |\mu|. \tag{3.4}$$

Indeed, let E be any Borel subset of Ω with $|E| = |\mu|(E) = 0$. By inner regularity, it suffices to show $\lambda(K) = 0$ for every $K \subset E$ compact. For any such K, we have

$$|K| = |\mu|(K) = 0. \tag{3.5}$$

Define the open sets

$$K_\delta := \{x \in \Omega : \operatorname{dist}(x, K) < \delta\}. \tag{3.6}$$

Since $\partial K_\delta = \{x \in \Omega : \operatorname{dist}(x, K) = \delta\}$ are an uncountable family of disjoint sets, we can select $\delta_i \to 0^+$ such that

$$|\mu|(\partial K_{\delta_i}) = 0. \tag{3.7}$$

We have by (3.3)

$$\lambda(K) \le \lambda(K_{\delta_i}) \le \liminf_{k\to\infty} \lambda_k(K_{\delta_i}) \le \liminf_{k\to\infty} C\bigg(|K_{\delta_i}| + |\mu_k|(K_{\delta_i})\bigg)$$

$$= \lim_{k\to\infty} C\bigg(|K_{\delta_i}| + |\mu_k|(K_{\delta_i})\bigg) = C\bigg(|K_{\delta_i}| + |\mu|(K_{\delta_i})\bigg)$$

by virtue of (3.7) and the fact that $|\mu_k| \overset{*}{\rightharpoonup} |\mu|$. Since $\bigcap_{\delta>0} K_\delta = K$, letting $i \to \infty$, we get by (3.5)

$$\lambda(K) \le C\bigg(|K| + |\mu|(K)\bigg) = 0.$$

and this concludes (3.4).

We claim that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \ge g(\mu_{ac}(x_0)) \text{ for } \mathcal{L}^N a.e.\ x_0 \in \Omega, \tag{3.8}$$

and

$$\frac{d\lambda}{d|\mu_s|}(x_0) \ge g^\infty(\nu(x_0)) \text{ for } |\mu_s|\ a.e.\ x_0 \in \Omega. \tag{3.9}$$

If (3.8) and (3.9) hold, then on one hand

$$\lambda(\Omega) \le \liminf_{k\to\infty} \lambda_k(\Omega) = \lim_{k\to\infty} \int_\Omega g(\mu_k(x))dx,$$

while (3.8) and (3.9) yield

$$\int_{\Omega} g(\mu_{ac}(x))dx + |\mu_s|(\Omega) \leq \lambda(\Omega).$$

Thus, we conclude the lower bound (3.2). We begin by establishing the inequality for the absolutely continuous part, i.e. (3.8). For $\mathcal{L}^N$ almost every $x_0 \in \Omega$, we have

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0} \frac{\lambda(Q(x_0,r))}{r^N} \text{ exists and is finite,}$$

$$\lim_{r\to 0} \frac{1}{r^N} \int_{Q(x_0,r)} |\mu_{ac}(x) - \mu_{ac}(x_0)|dx = 0, \tag{3.10}$$

$$\lim_{r\to 0} \frac{|\mu_s|(Q(x_0,r))}{r^N} = 0. \tag{3.11}$$

Choose $r_n \to 0$ such that $\lambda(\partial Q(x_0,r_n)) = 0$. Then

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{n\to\infty} \frac{\lambda(Q(x_0,r_n))}{r_n^N} = \lim_{n\to\infty}\lim_{k\to\infty} \frac{\lambda_k(Q(x_0,r_n))}{r_n^N}$$

$$= \lim_{n\to\infty}\lim_{k\to\infty} \frac{1}{r_n^N} \int_{Q(x_0,r_n)} g(\mu_k(x))dx. \tag{3.12}$$

Define the functions $v_{n,k}(y) := \mu_k(x_0 + r_n y)$ for $y \in Q$. Apply a change of variables so that (3.12) becomes

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{n\to\infty}\lim_{k\to\infty} \int_Q g(v_{n,k}(y))dy.$$

Since $\mu_k \overset{*}{\rightharpoonup} \mu$, we have that for every n, the measures $v_{n,k}\ \mathcal{L}^N \llcorner Q$ converge weakly-$*$ to a measure π_n given by

$$\pi_n(E) := \frac{(T^{\#}_{x_0,r_n}\mu)(E)}{r_n^N} = \frac{\mu(x_0 + r_n E)}{r_n^N}, \text{ for every Borel set } E \subset Q \tag{3.13}$$

where $T^{\#}_{x_0,r_n}\mu$ denotes the push-forward of μ under the mapping which takes

$$x \mapsto \frac{x - x_0}{r_n}$$

Indeed, by the standard change of variables for push-forward measures (see [16] Theorem 3.6.1), for any test function $\psi \in C_c(Q)$ we have

$$\begin{aligned}\int_Q v_{n,k}(y)\psi(y)dy &= \int_Q \mu_k(x_0+r_ny)\psi(y)dy = \frac{1}{r_n^N}\int_{Q(x_0,r_n)} \mu_k(x)\psi\left(\frac{x-x_0}{r_n}\right)dx \\ &\overset{k\to 0}{\longrightarrow} \frac{1}{r_n^N}\int_{Q(x_0,r_n)} \psi\left(\frac{x-x_0}{r_n}\right)d\mu(x) = \frac{1}{r_n^N}\int_Q \psi(y)d(T^{\#}_{x_0,r_n}\mu)(y) \\ &= \int_Q \psi(y)d\pi_n(y).\end{aligned}$$

In turn, $\pi_n \overset{*}{\rightharpoonup} \mu_{ac}(x_0)\mathcal{L}^N \llcorner Q$. To see this, fix any $\psi \in C_c(Q)$. We have

$$\left|\int_Q \psi(y)d\pi_n(y) - \int_Q \psi(y)\mu_{ac}(x_0)dy\right|$$

$$\begin{aligned}&= \frac{1}{r_n^N}\left|\int_{Q(x_0,r_n)} \psi\left(\frac{x-x_0}{r_n}\right)d\mu(x) - \int_{Q(x_0,r_n)} \psi\left(\frac{x-x_0}{r_n}\right)\mu_{ac}(x_0)dx\right| \\ &= \frac{1}{r_n^N}\left|\int_{Q(x_0,r_n)} \psi\left(\frac{x-x_0}{r_n}\right)(\mu_{ac}(x)-\mu_{ac}(x_0))dx + \int_{Q(x_0,r_n)} \psi\left(\frac{x-x_0}{r_n}\right)d\mu_s(x)\right| \\ &\leq \|\psi\|_\infty\left(\frac{1}{r_n^N}\int_{Q(x_0,r_n)} |\mu_{ac}(x)-\mu_{ac}(x_0)| + \frac{|\mu_s|(Q(x_0,r_n))}{r_n^N}\right)\end{aligned}$$

which goes to 0 as $n \to \infty$ by (3.10) and (3.11). Thus, $\pi_n \overset{*}{\rightharpoonup} \mu_{ac}(x_0)\,\mathcal{L}^N \llcorner Q$.

By an identical argument, since $|\mu_k| \overset{*}{\rightharpoonup} |\mu|$ and $|\mu_{ac}| = |\mu|_{ac}$ we have $|v_{n,k}|\,\mathcal{L}^N \llcorner Q \overset{*}{\rightharpoonup} |\pi_n|$, and $|\pi_n| \overset{*}{\rightharpoonup} |\mu_{ac}|(x_0)\,\mathcal{L}^N \llcorner Q$.

Recall that bounded sets in $\mathcal{M}(Q;\mathbb{R}^d)$ with the weak-$*$ topology are metrizable (see [24], Theorem 3.29). Since

$$\limsup_{n\to\infty}\limsup_{k\to\infty}\|v_{n,k}\| = \limsup_{n\to\infty}\limsup_{k\to\infty}|v_{n,k}|(Q) \leq |\mu_{ac}|(x_0)$$

we have

$$\sup_n\sup_k\|v_{n,k}\| < \infty.$$

Thus we can select a diagonal sequence $v_n := v_{n,k_n}$ such that $v_n \overset{*}{\rightharpoonup} \mu_{ac}(x_0)\,\mathcal{L}^N \llcorner Q$, $|v_n| \overset{*}{\rightharpoonup} |\mu_{ac}(x_0)|\,\mathcal{L}^N \llcorner Q$, and

$$\lim_{n\to\infty}\lim_{k\to\infty}\int_Q g(v_{n,k}(y))dy = \lim_{n\to\infty}\int_Q g(v_n(y))dy.$$

Since g is convex by (A2), consider an affine function $a + b \cdot \xi \leq g(\xi)$ (in the manner of [42] Theorem 5.14) and observe that

$$\begin{aligned} \frac{d\lambda}{d\mathcal{L}^N}(x_0) &= \lim_{n\to\infty} \int_Q g(v_n(y))dy \geq \liminf_{n\to\infty} \int_Q \Big(a + b \cdot v_n(y)\Big)dy \\ &= a + \liminf_{n\to\infty} \int_Q b \cdot v_n(y)dy \geq a + b \cdot \mu_{ac}(x_0). \end{aligned} \tag{3.14}$$

To see why the last step of (3.14) holds, for any $t < 1$ let $\psi_t \in C_c(Q;[0,1])$ be such that $\psi_t = 1$ in tQ. We have

$$\begin{aligned} \overline{\lim_{n\to\infty}} \left| \int_Q (v_n(y) - \mu_{ac}(x_0))dy \right| &\leq \overline{\lim_{n\to\infty}} \left| \int_Q (v_n(y) - \mu_{ac}(x_0))\psi_t dy \right| \\ &\quad + \overline{\lim_{n\to\infty}} \left| \int_Q (v_n(y) - \mu_{ac}(x_0))(1-\psi_t)dy \right| \\ &= \overline{\lim_{n\to\infty}} \left| \int_Q (v_n(y) - \mu_{ac}(x_0))(1-\psi_t)dy \right| \\ &\leq \overline{\lim_{n\to\infty}} \int_{Q\setminus tQ} |v_n(y) - \mu_{ac}(x_0)|dy \\ &\leq \overline{\lim_{n\to\infty}} \int_{Q\setminus tQ} |v_n(y)|dy + |\mu_{ac}(x_0)||Q\setminus tQ| \\ &\leq 2|\mu_{ac}|(x_0)|\overline{Q}\setminus tQ| = 2|\mu_{ac}|(x_0)(1-t^N). \end{aligned}$$

As this holds for any $t < 1$, we conclude that

$$\overline{\lim_{n\to\infty}} \left| \int_Q (v_n(y) - \mu_{ac}(x_0))dy \right| = 0,$$

and thus

$$\lim_{n\to\infty} \int_Q v_n(y)dy = \mu_{ac}(x_0).$$

Since (3.14) holds for any affine function below g, we conclude that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq g(\mu_{ac}(x_0)).$$

To address the singular part, we fix $\sigma \in (0,1)$. We know that for $|\mu_s|$ almost every x_0,

$$\frac{d\lambda}{d|\mu_s|}(x_0) = \lim_{r\to 0} \frac{\lambda(Q(x_0,r))}{|\mu_s|(Q(x_0,r))} = \lim_{r\to 0} \frac{\lambda(Q(x_0,r))}{|\mu|(Q(x_0,r))},$$

$$\lim_{r\to\infty} \frac{1}{|\mu_s|(Q(x_0,r))} \int_{Q(x_0,r)} |\mu_{ac}|(x)dx = 0, \tag{3.15}$$

$$⨍_{Q(x_0,r)} |\nu(x) - \nu(x_0)| d|\mu_s|(x) = 0, \tag{3.16}$$

and by Lemma 2.5 we may select $r_n \to 0$ such that $|\mu|(\partial Q(x_0, r_n)) = 0$ and

$$\lim_{n\to\infty} \frac{|\mu|(Q(x_0,\sigma r_n))}{|\mu|(Q(x_0,r_n))} \geq \sigma^N. \tag{3.17}$$

Note that in view of (3.4), $|\lambda|(\partial Q(x_0, r_n)) = 0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} \frac{d\lambda}{d|\mu_s|}(x_0) &= \lim_{n\to\infty} \frac{\lambda(Q(x_0,r_n))}{|\mu|(Q(x_0,r_n))} = \lim_{n\to\infty}\lim_{k\to\infty} \frac{\lambda_k(Q(x_0,r_n))}{|\mu|(Q(x_0,r_n))} \\ &= \lim_{n\to\infty}\lim_{k\to\infty} \frac{1}{|\mu|(Q(x_0,r_n))} \int_{Q(x_0,r_n)} g(\mu_k(x))dx. \end{aligned} \tag{3.18}$$

Let

$$t_n := \frac{|\mu|(Q(x_0,r_n))}{r_n^N},$$

and define

$$v_{n,k}(y) := \frac{\mu_k(x_0 + r_n y)}{|\mu|(Q(x_0,r_n))} r_n^N = \frac{\mu_k(x_0 + r_n y)}{t_n}.$$

We can apply a change of variables to (3.18) to get

$$\frac{d\lambda}{d|\mu_s|}(x_0) = \lim_{n\to\infty}\lim_{k\to\infty} \int_Q \frac{1}{t_n} g(t_n v_{n,k}(y))dy.$$

For a fixed n, as $k \to \infty$ we have $v_{n,k}\mathcal{L}^N \overset{*}{\rightharpoonup} \pi_n$ with

$$\pi_n(E) := \frac{(T^{\#}_{x_0,r_n}\mu)(E)}{|\mu|(Q(x_0,r_n))} = \frac{\mu(x_0 + r_n E)}{|\mu|(Q(x_0,r_n))}, \text{ for every Borel set } E \subset Q, \tag{3.19}$$

where, as in (3.13), $T^{\#}_{x_0,r_n}\mu$ denotes the push-forward of μ under the mapping which takes

$$x \to \frac{x - x_0}{r_n}$$

On the other hand, if we define measures

$$\rho_n(E) := \frac{|\mu|(x_0 + r_n E)}{|\mu|(Q(x_0, r_n))}, \text{ for every Borel Set } E \subset Q,$$

we see that $\pi_n \overset{*}{\rightharpoonup} \pi$, $\rho_n \overset{*}{\rightharpoonup} \rho$, for some Radon measures $\pi \in \mathcal{M}(Q; \mathbb{R}^d)$ and ρ a finite nonnegative Radon measure in $\mathcal{M}(Q)$, perhaps along a subsequence. We claim that

$$\pi = \nu(x_0)\rho. \tag{3.20}$$

Indeed, fix $\psi \in C_c(Q)$. We have

$$\left| \int_Q \psi(y) d\pi_n(y) - \int_Q \psi(y)\nu(x_0) d\rho_n(y) \right|$$

$$\begin{aligned}
&= \frac{1}{|\mu|(Q(x_0, r_n))} \left| \int_{Q(x_0,r_n)} \psi\left(\frac{x - x_0}{r_n}\right) d\mu(x) - \int_{Q(x_0,r_n)} \psi\left(\frac{x - x_0}{r_n}\right) \nu(x_0) d|\mu|(x) \right| \\
&\leq \frac{1}{|\mu|(Q(x_0, r_n))} \left| \int_{Q(x_0,r_n)} \psi\left(\frac{x - x_0}{r_n}\right) \nu(x) d|\mu_s|(x) - \int_{Q(x_0,r_n)} \psi\left(\frac{x - x_0}{r_n}\right) \nu(x_0) d|\mu_s|(x) \right| \\
&+ \frac{1}{|\mu|(Q(x_0, r_n))} \left(\int_{Q(x_0,r_n)} |\psi|\left(\frac{x - x_0}{r_n}\right) d|\mu_{ac}|(x) + \int_{Q(x_0,r_n)} |\psi|\left(\frac{x - x_0}{r_n}\right) \nu(x_0) d|\mu_{ac}|(x) \right) \\
&\leq \frac{\|\psi\|_\infty}{|\mu|(Q(x_0, r_n))} \int_{Q(x_0,r_n)} |\nu(x) - \nu(x_0)| d|\mu_s|(x) + \frac{2\|\psi\|_\infty}{|\mu|(Q(x_0, r_n))} |\mu_{ac}|(Q(x_0, r_n)),
\end{aligned}$$

which goes to 0 as $n \to \infty$ in view of (3.15) and (3.16). Since

$$\int_Q \psi d\pi_n \to \int_Q \psi d\pi,$$

and

$$\int_Q \psi \nu(x_0) d\rho_n = \nu(x_0) \int_Q \psi d\rho_n \to \nu(x_0) \int_Q \psi d\rho = \int_Q \psi \nu(x_0) d\rho,$$

we conclude that $\pi = \nu(x_0)\rho$. We note that $\rho(Q) \geq \sigma^N$. To see this, by (3.17) we have

$$\rho(Q) \geq \rho(\sigma\overline{Q}) \geq \limsup_{n\to\infty} \rho_n(\sigma\overline{Q}) \geq \limsup_{n\to\infty} \rho_n(\sigma Q) = \limsup_{n\to\infty} \frac{|\mu|(Q(x_0, \sigma r_n))}{|\mu|(Q(x_0, r_n))} \geq \sigma^N.$$

Diagonalizing as in the absolutely continuous case, we have

$$\frac{d\lambda}{d|\mu_s|}(x_0) = \lim_{n\to\infty} \int_Q \frac{1}{t_n} g(t_n v_n(y)) dy.$$

Fix $\eta > 0$. Since the convergence to g^∞ is uniform as in (A4), we can find $M > 0$ so that for all $p \in \mathbb{R}^d$ and $t > 0$ with $|p| > \frac{M}{t}$, we have

$$\frac{g(tp)}{t} \geq g^\infty(p) - \eta.$$

Define the sets $E_n := \{|v_n| > \frac{M}{t_n}\}$. Then,

$$\lim_{n\to\infty} \int_Q \frac{1}{t_n} g(t_n v_n(y))dy \geq \liminf_{n\to\infty} \int_{E_n} \frac{1}{t_n} g(t_n v_n(y))dy \geq \liminf_{n\to\infty} \int_{E_n} g^\infty(v_n(y))dy - \eta.$$

On the other hand,

$$\liminf_{n\to\infty} \int_{E_n} g^\infty(v_n(y))dy \geq \liminf_{n\to\infty} \int_Q g^\infty(v_n(y))dy - \limsup_{n\to\infty} \int_{Q\setminus E_n} g^\infty(v_n(y))dy$$

and, since

$$\limsup_{n\to\infty} \int_{Q\setminus E_n} g^\infty(v_n(y))dy \leq \limsup_{n\to\infty} \frac{CM}{t_n} = 0$$

we have

$$\lim_{n\to\infty} \int_Q \frac{1}{t_n} g(t_n v_n(y))dy \geq \liminf_{n\to\infty} \int_Q g^\infty(v_n(y))dy - \eta$$

for every $\eta > 0$, and thus

$$\lim_{n\to\infty} \int_Q \frac{1}{t_n} g(t_n v_n(y))dy \geq \liminf_{n\to\infty} \int_Q g^\infty(v_n(y))dy.$$

Now, by Theorem 2.3, since g^∞ is convex and 1-homogeneous with the appropriate growth condition, we have lower semicontinuity with respect to weak-$*$ convergence, and so

$$\frac{d\lambda}{d|\mu_s|}(x_0) \geq \int_Q g^\infty(\nu(x_0))d\rho(y) \geq g^\infty(\nu(x_0))\sigma^N,$$

and, letting $\sigma \to 1^-$ we conclude

$$\frac{d\lambda}{d|\mu_s|}(x_0) \geq g^\infty(\nu(x_0)).$$

3.2.2 Step 2: Upper bound

.

We claim that

$$\limsup_{\varepsilon\to\infty}\int_\Omega g(\mu_\varepsilon)dx \leq \int_\Omega g(\mu_{ac})dx + \int_\Omega g(\nu)d|\mu_s|. \tag{3.21}$$

We will use the blow-up method. Choose a sequence $\{\varepsilon_k\}$ which achieves the limsup, and define measures

$$\lambda_k(E) := \int_E g(\mu_{\varepsilon_k})dx \text{ for every Borel Set } E \subset \Omega.$$

As in Step 1, we may pass along a subsequence to a weak-$*$ limit

$$\lambda << \mathcal{L}^N + |\mu_s|. \tag{3.22}$$

To prove the upper bound, we will show that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \leq g(\mu_{ac}(x_0)) \text{ for } \mathcal{L}^N a.e.\, x_0 \in \Omega, \tag{3.23}$$

and

$$\frac{d\lambda}{d|\mu_s|}(x_0) \leq g^\infty(\nu(x_0)) \text{ for } |\mu_s|\, a.e.\, x_0 \in \Omega. \tag{3.24}$$

Assuming (3.23) and (3.24), by our boundary regularity assumption on Ω and (3.22) we have $\lambda(\partial\Omega) = 0$, and therefore

$$\lambda(\Omega) = \lambda(\overline{\Omega}) \geq \limsup_{k\to\infty}\lambda_k(\overline{\Omega}) = \limsup_{k\to\infty}\lambda_k(\Omega),$$

while

$$\lambda(\Omega) \leq \int_\Omega g(\mu_{ac}(x))dx + \int_\Omega g^\infty(\nu(x))d|\mu_s|(x),$$

and putting these together, we have (3.21). To prove (3.23), we know that for $\mathcal{L}^N$-almost every x_0, we have

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0}\frac{\lambda(\overline{Q(x_0,r)})}{r^N},$$

$$\lim_{r\to 0}\frac{|\mu_s|(\overline{Q(x_0,r)})}{r^N} = 0, \tag{3.25}$$

and

$$\lim_{r\to 0} ⨍_{Q(x_0,r)} |\mu_{ac}(x) - \mu_{ac}(x_0)|dx = 0. \tag{3.26}$$

For all such points, we have

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0} \frac{\lambda(\overline{Q(x_0,r)})}{r^N} = \lim_{r\to 0}\lim_{k\to\infty} \frac{\lambda_k(\overline{Q(x_0,r)})}{r^N},$$

and thus

$$\begin{aligned}\frac{d\lambda}{d\mathcal{L}^N}(x_0) &= \lim_{r\to 0}\lim_{k\to\infty} \frac{1}{r^N}\int_{Q(x_0,r)} g(\mu * \phi_{\varepsilon_k})dx \\ &= \lim_{r\to 0}\lim_{k\to\infty} \frac{1}{r^N}\int_{Q(x_0,r)} g(\mu_{ac} * \phi_{\varepsilon_k} + \mu_s * \phi_{\varepsilon_k})dx.\end{aligned}$$

By convexity of g, for any $p, q \in \mathbb{R}^d$ and $\theta \in (0,1)$ we have

$$g(p+q) \le \theta g\left(\frac{1}{\theta}p\right) + (1-\theta)g\left(\frac{1}{1-\theta}q\right),$$

hence

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \le \varliminf_{r\to 0}\varliminf_{k\to\infty} \frac{1}{r^N}\int_{Q(x_0,r)} \left[\theta g\left(\frac{1}{\theta}\mu_{ac} * \phi_{\varepsilon_k}\right) + (1-\theta)g\left(\frac{1}{1-\theta}\mu_s * \phi_{\varepsilon_k}\right)\right]dx$$

and in view of Lemma 3.2 we have

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \le \varliminf_{r\to 0}\varliminf_{k\to\infty} \frac{1}{r^N}\int_{Q(x_0,r)} \left[\theta g\left(\frac{1}{\theta}\mu_{ac}\right) * \phi_{\varepsilon_k} + (1-\theta)g\left(\frac{1}{1-\theta}\mu_s * \phi_{\varepsilon_k}\right)\right]dx. \tag{3.27}$$

Moreover,

$$\begin{aligned}\frac{1-\theta}{r^N}\int_{Q(x_0,r)} g\left(\frac{1}{1-\theta}\mu_s * \phi_{\varepsilon_k}\right)dx &\le \frac{1-\theta}{r^N}\int_{Q(x_0,r)} C\left(1 + \left|\frac{1}{1-\theta}\mu_s * \phi_{\varepsilon_k}\right|\right)dx \\ &\le (1-\theta)C + \frac{C}{r^N}\int_{Q(x_0,r)} |\mu_s * \phi_{\varepsilon_k}|dx,\end{aligned}$$

and this yields

$$
\begin{aligned}
\overline{\lim_{r\to 0}}\,\overline{\lim_{k\to\infty}}\frac{1-\theta}{r^N}\int_{Q(x_0,r)} g\left(\frac{1}{1-\theta}\mu_s * \phi_{\varepsilon_k}\right)dx &\leq (1-\theta)C + C\lim_{r\to 0}\frac{|\mu_s|(\overline{Q(x_0,r)})}{r^N}\\
&= (1-\theta)C, \qquad (3.28)
\end{aligned}
$$

where we have used (3.25). In turn,

$$
\begin{aligned}
\lim_{r\to 0}\lim_{k\to\infty}\frac{1}{r^N}\int_{Q(x_0,r)}\theta g\left(\frac{1}{\theta}\mu_{ac}\right) * \phi_{\varepsilon_k}dx &= \lim_{r\to 0}\frac{1}{r^N}\int_{Q(x_0,r)}\theta g\left(\frac{1}{\theta}\mu_{ac}\right)dx\\
&= \theta g\left(\frac{1}{\theta}\mu_{ac}(x_0)\right). \qquad (3.29)
\end{aligned}
$$

To see why the last step above holds, note g is convex with linear growth, and so by regularity properties of convex functions (see [42] Proposition 4.64) we have that g is Lipschitz with some constant $L > 0$. Thus,

$$
\left|\frac{1}{r^N}\int_{Q(x_0,r)}\left[g\left(\frac{1}{\theta}\mu_{ac}(x)\right) - g\left(\frac{1}{\theta}\mu_{ac}(x_0)\right)\right]dx\right| \leq \frac{1}{r^N}\int_{Q(x_0,r)}\frac{L}{\theta}|\mu_{ac}(x)-\mu_{ac}(x_0)|dx \xrightarrow{r\to 0} 0
$$

by virtue of (3.26). By (3.27), (3.28), and (3.29), we have for every $\theta \in (0,1)$ that

$$
\frac{d\lambda}{d\mathcal{L}^N}(x_0) \leq \theta g\left(\frac{1}{\theta}\mu_{ac}(x_0)\right) + (1-\theta)C,
$$

and letting $\theta \to 1^-$, by continuity of g we conclude that

$$
\frac{d\lambda}{d\mathcal{L}^N}(x_0) \leq g(\mu_{ac}(x_0))
$$

for $\mathcal{L}^N$ almost every $x_0 \in \Omega$.

Next, we tackle the singular part, i.e. (3.24) . We know that for $|\mu_s|$ almost every x_0, we have

$$
\begin{aligned}
\frac{d\lambda}{d|\mu_s|}(x_0) &= \lim_{r\to 0}\frac{\lambda(\overline{Q(x_0,r)})}{|\mu_s|(\overline{Q(x_0,r)})},\\
\lim_{r\to 0}\frac{r^N}{|\mu_s|(\overline{Q(x_0,r)})} &= 0, \qquad (3.30)
\end{aligned}
$$

$$\lim_{r\to 0}\frac{1}{|\mu_s|(\overline{Q(x_0,r)})}\int_{Q(x_0,r)}|\mu_{ac}|(x)dx=0, \tag{3.31}$$

and

$$\lim_{r\to 0}⨍_{\overline{Q(x_0,r)}} g^\infty(\nu(x))d|\mu_s|(x)=g^\infty(\nu(x_0)).$$

We choose a sequence $r_n\to 0$ such that $|\mu_s|(\partial Q(x_0,r_n))=0$. Note that by (3.22) we also have $\lambda(\partial Q(x_0,r_n))=0$. We obtain

$$\begin{aligned}\frac{d\lambda}{d|\mu_s|}(x_0)&=\lim_{n\to\infty}\frac{\lambda(Q(x_0,r_n))}{|\mu_s|(Q(x_0,r_n))}=\lim_{n\to\infty}\lim_{k\to\infty}\frac{\lambda_k(Q(x_0,r_n))}{|\mu_s|(Q(x_0,r_n))}\\&=\lim_{n\to\infty}\lim_{k\to\infty}\frac{1}{|\mu_s|(Q(x_0,r_n))}\int_{Q(x_0,r_n)}g(\mu*\phi_{\varepsilon_k})dx.\end{aligned}$$

Again appealing to convexity of g, we get

$$|\mu_s|(x_0)\le\varliminf_{n\to\infty}\varliminf_{k\to\infty}\frac{1}{|\mu_s|(Q(x_0,r_n))}\int_{Q(x_0,r_n)}\left[\frac{1}{2}g(2\mu_{ac}*\phi_{\varepsilon_k})+\frac{1}{2}g(2\mu_s*\phi_{\varepsilon_k})\right]dx. \tag{3.32}$$

Since by (A1)

$$\frac{1}{2}g(2\mu_{ac}*\phi_{\varepsilon_k})\le C(1+|\mu_{ac}*\phi_{\varepsilon_k}|),$$

we have

$$\begin{aligned}&\varlimsup_{n\to\infty}\varlimsup_{k\to\infty}\frac{1}{|\mu_s|(Q(x_0,r_n))}\int_{Q(x_0,r_n)}\frac{1}{2}g(2\mu_{ac}*\phi_{\varepsilon_k})dx\\&\quad\le\varlimsup_{n\to\infty}\varlimsup_{k\to\infty}\frac{1}{|\mu_s|(Q(x_0,r_n))}\int_{Q(x_0,r_n)}C(1+|\mu_{ac}*\phi_{\varepsilon_k}|)dx\\&\quad=\lim_{n\to\infty}\frac{C}{|\mu_s|(Q(x_0,r_n))}\left(r_n^N+\int_{Q(x_0,r_n)}|\mu_{ac}|(x)dx\right)=0,\end{aligned} \tag{3.33}$$

where we used (3.30) and (3.31).

Next, we restrict our attention to the singular part in (3.32). Since

$$|(\phi_{\varepsilon_k}*\mu_s)(x)|\le(\phi_{\varepsilon_k}*|\mu_s|)(x)=:t_{\varepsilon_k}(x), \tag{3.34}$$

and g is monotone in norm by (A3), we obtain

$$g(2\phi_{\varepsilon_k} * \mu_s(x)) \le g(2t_{\varepsilon_k}(x)).$$

Partition $Q(x_0, r_n)$ into two regions, where $t_{\varepsilon_k} < 1$ and $t_{\varepsilon_k} \ge 1$. In the first region we have

$$\begin{aligned}\int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}<1\}} g(2\phi_{\varepsilon_k} * \mu_s(x))dx &\le \int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}<1\}} g(2t_{\varepsilon_k}(x))dx \\ &\le r_n^N g(2) = Cr_n^N. \end{aligned} \tag{3.35}$$

Meanwhile, in the second region, using Lemma 3.2 we obtain

$$\begin{aligned}&\int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}\ge1\}} g(2\phi_{\varepsilon_k} * \mu_s(x))dx \\ &\qquad \le \int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}\ge1\}} \int_\Omega \frac{g(2t_{\varepsilon_k}(x)\nu(y))}{t_{\varepsilon_k}(x)} \phi_{\varepsilon_k}(x-y)d|\mu_s|(y)dx. \end{aligned} \tag{3.36}$$

Note that by (A4),

$$\left| \frac{g(2t_{\varepsilon_k}(x)\nu(y))}{t_{\varepsilon_k}(x)} - 2g^\infty(\nu(y)) \right| \le \frac{C}{t_{\varepsilon_k}(x)^\alpha},$$

and by (3.36) we have

$$\begin{aligned}\int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}\ge1\}} g(2\phi_{\varepsilon_k} * \mu_s(x)) &\le \int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}\ge1\}} \int_\Omega 2g^\infty(\nu(y))\phi_{\varepsilon_k}(x-y)d|\mu_s|(y)dx \\ &+ \int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}\ge1\}} \int_\Omega \frac{C}{t_{\varepsilon_k}(x)^\alpha} \phi_{\varepsilon_k}(x-y)d|\mu_s|(y)dx \\ &\le \int_{Q(x_0,r_n)} 2\big(g^\infty(\nu(\cdot))|\mu_s| * \phi_{\varepsilon_k}\big)(x)dx + \int_{Q(x_0,r_n)\cap\{t_{\varepsilon_k}\ge1\}} \frac{C}{t_{\varepsilon_k}(x)^{\alpha-1}}dx \\ &\le \int_{Q(x_0,r_n)} 2\big(g^\infty(\nu(\cdot))|\mu_s| * \phi_{\varepsilon_k}\big)(x)dx + Cr^N, \end{aligned} \tag{3.37}$$

In view of (3.35), (3.37), we have shown for every θ that

$$\int_{Q(x_0,r_n)} g(2\phi_{\varepsilon_k} * \mu_s(x))dx \le \int_{Q(x_0,r_n)} 2\big(g^\infty(\nu(\cdot))|\mu_s| * \phi_{\varepsilon_k}\big)(x)dx + Cr_n^N,$$

therefore,

$$\frac{1}{|\mu_s|(Q(x_0,r_n))}\overline{\lim_{k\to\infty}}\int_{Q(x_0,r_n)}\frac{1}{2}g(2\phi_{\varepsilon_k}*\mu_s(x))dx$$

$$\leq \fint_{Q(x_0,r_n)} g^{\infty}(\nu(x))d|\mu_s|(x)+C\frac{r_n^N}{|\mu_s|(Q(x_0,r_n))} \tag{3.38}$$

and by (3.30), (3.33), (3.38) we conclude that

$$\frac{d\lambda}{d|\mu_s|}(x_0)\leq g^{\infty}(\nu(x_0)).$$

□

3.2.3 A BH extension result

In our application, we are interested in the case when the measure μ is the Hessian of a BH function. It should be noted that the first-order case, when μ is the gradient of some BV function, an area-strict density theorem follows from the integral representation results of Fonseca and Müller, Ambrosio and Dal Maso, with no regularity assumption on the boundary ([45], [3]).

To apply Theorem 3.3 to a given $u\in BH(\Omega;\mathbb{R}^d)$, the main obstacle is finding an extension of u to a larger set U such that $|D(\nabla u)|(\partial\Omega)=0$. In order to achieve a fairly general class of domains, we shall borrow from the construction of Stein [71].

First we will construct the extension in the case where Ω is of type special Lipschitz. Recall that we say a set $\Omega\subset\mathbb{R}^{N+1}$ is *special Lipschitz* if there is a Lipschitz function $f:\mathbb{R}^N\to\mathbb{R}$ such that

$$\Omega=\{(x,t)\in\mathbb{R}^{N+1}:t>f(x)\}$$

where we are identifying $\mathbb{R}^{N+1}$ with $\mathbb{R}^N\times\mathbb{R}$.

We begin with a simpler approximation lemma.

Lemma 3.4. *Let $\Omega\subset\mathbb{R}^{N+1}$ be a special Lipschitz domain. For any $u\in BH(\Omega;\mathbb{R}^d)$ there exists a sequence $\{u_n\}\in W^{2,1}(\Omega;\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty}\|u_n-u\|_{W^{1,1}(\Omega;\mathbb{R}^d)}=0 \textit{ and } \sup_n\|u_n\|_{W^{1,2}(\Omega;\mathbb{R}^d)}<\infty.$$

Proof. Given any $v\in L^1(\Omega)$ and $\delta>0$, we define its translation $T_\delta v\in L^1(\Omega_\delta)$ via

$$T_\delta v(x,t):=v(x,t+\delta)$$

where $\Omega_\delta := \{(x,t) \in \mathbb{R}^{N+1} : t > f(x) - \delta\}$.

Note that if L is the Lipschitz constant of f, then for $(x,t) \in \Omega$ and $\varepsilon < \frac{\delta}{1+L}$ we have $B((x,t),\varepsilon) \subset \Omega_\delta$, so the function $\phi_\varepsilon * T_\delta v \in C^\infty(\Omega)$ is well-defined. Since $\nabla T_\delta u = T_\delta \nabla u$ and the translation is continuous in the L^1 norm, we have

$$\lim_{\delta\to 0^+} \|T_\delta u - u\|_{W^{1,1}(\Omega;\mathbb{R}^d)} = 0.$$

By standard mollification results, we must have for every $\delta > 0$ that

$$\lim_{\varepsilon\to 0^+} \|\phi_\varepsilon * T_\delta u - T_\delta u\|_{W^{1,1}(\Omega;\mathbb{R}^d)} = 0,$$

$$\limsup_{\varepsilon\to 0^+} \|\phi_\varepsilon * T_\delta u\|_{W^{2,1}(\Omega;\mathbb{R}^d)} \le \|u\|_{BH(\Omega)}$$

Thus, for any sequence $\delta_n \to 0^+$ we can choose $\varepsilon_n < \frac{\delta_n}{1+L}$ such that the smooth (and thus $W^{2,1}$) functions $\phi_{\varepsilon_n} * T_{\delta_n} u$ converge to u with bounded $W^{2,1}$ norm.

□

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a special Lipschitz domain. For any function $u \in BH(\Omega;\mathbb{R}^d)$ there is an extension $E[u] \in BH(\mathbb{R}^{N+1};\mathbb{R}^d)$ such that $E[u] = u$ in Ω and $|D(\nabla E[u])|(\partial\Omega) =$ 0.*

Proof. The general theory of extending BH functions can be reduced to the theory of extending BV functions. We recall that (see [41], 5.4.1) we since Ω is Lipschitz, given $w_1 \in BV(\Omega)$ and $w_2 \in BV(\mathbb{R}^{N+1} \setminus \overline{\Omega})$, the function

$$w(x) := \begin{cases} w_1(x) & \text{for } x \in \Omega, \\ w_2(x) & \text{for } x \in \mathbb{R}^{N+1} \setminus \overline{\Omega}, \end{cases}$$

is a BV function with

$$Dw = Dw_1|_\Omega + Dw_2|_{\mathbb{R}^{N+1}\setminus\overline{\Omega}} + (\mathrm{Trace}(w_1) - \mathrm{Trace}(w_2))\nu\ \mathcal{H}^{N-1}|_{\partial\Omega} \tag{3.39}$$

where ν indicates the inward normal vector to $\partial\Omega$.

Let $u \in BH(\Omega)$. Since ∇u is a BV function, in view of (3.39), to guarantee that our extension does not charge the boundary, it suffices to ensure that the traces of ∇u and of the extension $\nabla E[u]$ agree on the boundary $\partial\Omega$.

We will use the construction given by Stein to introduce $E[u]$. Namely, for $(x,t) \in \mathbb{R}^{N+1} \setminus \Omega$ we define

$$E[u](x,t) := \int_1^\infty u(x, t + \lambda\Delta(x,t))\psi(\lambda)d\lambda$$

where $\Delta \in C^\infty(\mathbb{R}^{N+1})$ is the regularized distance function, that is,

$$c \ \mathrm{dist}((x,t), \partial\Omega) \leq \Delta(x,t) \leq C \ \mathrm{dist}((x,t), \partial\Omega)$$

and $\psi : [1, \infty) \to \mathbb{R}$ is a continuous function with

$$\int_1^\infty \psi(\lambda)d\lambda = 1 \text{ and } \int_1^\infty \psi(\lambda)\lambda^k d\lambda = 0, \ k = 1, 2, 3...$$

The function $E[u]$ is well-defined, and it is clear that if it were sufficiently regular it would satisfy $\mathrm{Trace}(E[u]; \partial\Omega) = \mathrm{Trace}(u; \partial\Omega)$ and $\mathrm{Trace}(\nabla E[u]; \partial\Omega) = \mathrm{Trace}(\nabla u; \partial\Omega)$.

It remains to prove that $E[u] \in BH(\mathbb{R}^{N+1}; \mathbb{R}^d)$. Consider a sequence $\{u_n\} \subset W^{2,1}(\Omega; \mathbb{R}^d)$ as in Lemma 3.4. Since $\{u_n\}$ is bounded in $W^{2,1}(\Omega; \mathbb{R}^d)$ and the extension operator of Stein is a continuous linear operator

$$E : W^{2,1}(\Omega; \mathbb{R}^d) \to W^{2,1}(\mathbb{R}^{N+1}; \mathbb{R}^d),$$

then the sequence $\{E[u_n]\}$ is bounded in $W^{2,1}(\mathbb{R}^{N+1}; \mathbb{R}^d)$, and thus, along a subsequence, there is a function $v \in BH(\mathbb{R}^{N+1}; \mathbb{R}^d)$ such that

$$E[u_n] \to v \text{ in } L^1.$$

We claim that $v = E[u]$. To see this, fix any $(x,t) \in \mathbb{R}^{N+1} \setminus \Omega$. Then,

$$\begin{aligned} \left|E[u](x,t) - E[u_n](x,t)\right| &\leq \int_1^\infty C|u(x, t + \lambda\Delta(x,t)) - u_n(x, t + \lambda\Delta(x,t))|d\lambda \\ &= \int_{\Delta(x,t)}^\infty \frac{C}{\Delta(x,t)}|u(x, t+s) - u_n(x, t+s)|ds. \end{aligned}$$

For fixed t and $\ell > 0$, we can integrate both sides with respect to x over the set $\mathbb{R}^N \cap \{\Delta(x,t) \geq \ell\}$ to get

$$\begin{aligned}
&\int_{\mathbb{R}^N \cap \{\Delta(\cdot,t)\geq \ell\}} |E[u](x,t) - E[u_n](x,t)|dx \\
&\leq \frac{C}{\ell} \int_{\mathbb{R}^N \cap \{\Delta(\cdot,t)\geq \ell\}} \int_{\ell}^{\infty} |u(x,t+s) - u_n(x,t+s)|dsdx. \\
&= \frac{C}{\ell} \int_{\mathbb{R}^N \cap \{\Delta(\cdot,t)\geq \ell\}} \int_{\ell+t}^{\infty} |u(x,\tau) - u_n(x,\tau)|d\tau dx \\
&\leq \frac{C}{\ell} \int_{\Omega} |u(x,\tau) - u_n(x,\tau)|dxd\tau.
\end{aligned}$$

Then, for any $T < \infty$ we integrate over $t \in (0,T)$ to get

$$\int_0^T \int_{\mathbb{R}^N \cap \{\Delta(\cdot,t)\geq \ell\}} |E[u](x,t) - E[u_n](x,t)|dxdt \leq C\frac{T}{\ell} \int_{\Omega} |u(x,\tau) - u_n(x,\tau)|dxd\tau.$$

Since for every fixed T and ℓ, the right hand side goes to 0 as $n \to \infty$, we see that $\{E[u_n]\}$ converges to $E[u]$ in L^1_{loc}. However, we also know that $\{E[u_n]\}$ converges in L^1_{loc} to v, so we must have $E[u] = v$ and therefore $E[u] \in BH$. Since $E[u]$ is a BH function whose traces agree with u on $\partial\Omega$, it is the desired extension.

□

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^N$ be bounded and Lipschitz. For any function $u \in BH(\Omega;\mathbb{R}^d)$ there exists an extension $E[u] \in BH(\mathbb{R}^N;\mathbb{R}^d)$ such that $|D(\nabla E[u])|(\partial\Omega) = 0$.*

Proof. Since Ω is a Lipschitz domain, we can cover Ω by bounded open $U_0 \subset\subset \Omega$ and $U_1, \ldots, U_k$ such that $U_i \cap \partial\Omega$ is the graph of a Lipschitz function. We may also choose a smooth partition of unity $\psi_0, \ldots, \psi_k$ subordinate to this cover.

For $i \geq 1$, the domains U_i are the subgraphs of Lipschitz functions- So, we can find special Lipschitz domains Ω_i such that $\Omega_i \cap \Omega = U_i \cap \Omega$. Thus, by extending the functions $\psi_i u$ by zero, we can consider them to be defined on the special Lipschitz domains Ω_i. By Lemma 3.5, we can find BH functions $E[\psi_i u] \in BH(\mathbb{R}^N;\mathbb{R}^d)$ which satisfy $\text{Trace}(E[\psi_i u]) = \text{Trace}(\psi_i u)$ and $\text{Trace}(\nabla E[\psi_i u]) = \text{Trace}(\nabla(\psi_i u))$ on $U_i \cap \partial\Omega$.

Define the function $E[u]$ via

$$E[u] := \sum_{i=0}^{k} E[\psi_i u],$$

where, for the sake of notation, $E[\psi_0 u]$ is just the function $\psi_0 u$ extended by 0 to $\mathbb{R}^N$. As $E[u]$ is the sum of functions in $BH(\mathbb{R}^N;\mathbb{R}^d)$, it is clearly in $BH(\mathbb{R}^N;\mathbb{R}^d)$, and inside Ω we have

$$E[u] = \sum_{i=0}^{k} E[\psi_i u] = \sum_{i=0}^{k} \psi_i u = u.$$

It suffices to verify that $\nabla E[u]$ has the correct trace on $\partial\Omega$. To see this, note that

$$\begin{aligned}
\mathrm{Trace}(\nabla E[u];\partial\Omega) &= \sum_{i=0}^{k} \mathrm{Trace}(\nabla E[\psi_i u];\partial\Omega)\\
&= \sum_{i=0}^{k} \mathrm{Trace}(\nabla(\psi_i u);\partial\Omega\cap U_i)\\
&= \sum_{i=0}^{k} \mathrm{Trace}(u\otimes\nabla\psi_i + \psi_i\nabla u;\partial\Omega\cap U_i)\\
&= \sum_{i=0}^{k} \mathrm{Trace}(u;\partial\Omega\cap U_i)\otimes\nabla\psi_i + \psi_i\mathrm{Trace}(\nabla u;\partial\Omega\cap U_i)\\
&= \mathrm{Trace}(\nabla u;\partial\Omega),
\end{aligned}$$

where in the last line we use the fact that $\sum_{i=0}^{k}\nabla\psi_i = \nabla(\sum_{i=0}^{k}\psi_i) = \nabla(1) = 0$. Since $\nabla E[u]$ has the same trace as ∇u on $\partial\Omega$, we conclude that $|D(\nabla E[u])|(\partial\Omega) = 0$. $\square$

We now present the second-order version of Theorem 3.3.

Corollary 3.7. *Let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain. For any function $u\in BH(\Omega;\mathbb{R}^d)$ there exist smooth functions u_n such that $u_n\to u$ in L^1, $\nabla u_n\to\nabla u$ in L^1, $\nabla^2 u_n\mathcal{L}^N \overset{*}{\rightharpoonup} D(\nabla u)$, and*

$$\int_\Omega \sqrt{1+|\nabla^2 u_n|^2}dx \to \int_\Omega \sqrt{1+|\nabla^2 u|^2} + |D_s(\nabla u)|(\Omega).$$

Proof. Since Ω is Lipschitz, by Theorem 3.6 there is a function $E[u]\in BH(\mathbb{R}^N;\mathbb{R}^d)$ with $E[u] = u$ in Ω and $|D(\nabla E[u])|(\partial\Omega) = 0$.

Let $u_n := E[u]*\phi_{1/n}$. Since $\nabla^2 u_n = D(\nabla E[u])*\phi_{1/n}$, we can apply Theorem 3.3 to the measure $\mu := D(\nabla E[u])$ using the integrand

$$g(p) := \sqrt{1+|p|^2}.$$

noting that g satisfies conditions (A1)-(A4) and

$$g^\infty(p) = |p|.$$

$\square$

3.3 The integral relaxation theorem

We consider a functional

$$F(u) := \int_\Omega f(x, \nabla^2 u(x))dx.$$

where f satisfies the following hypotheses:

(H1) Linear growth: $f(x,H) \leq C(1+|H|)$ for all $x \in \Omega$, $H \in \mathbb{R}^{d\times N\times N}$ and some $C > 0$;

(H2) Modulus of continuity: $|f(x,H) - f(y,H)| \leq \omega(|x-y|)(1+|H|)$ for all $x, y \in \Omega$, $H \in \mathbb{R}^{d\times N\times N}$, where $\omega(s)$ is a nondecreasing function with $w(s) \to 0$ as $s \to 0^+$.

The relaxation of F onto the space $BH(\Omega;\mathbb{R}^d)$ is defined as

$$\mathcal{F}(u) := \inf\Big\{ \liminf_{n\to\infty} F(u_n) : u_n \to u \text{ in } W^{1,1}(\Omega;\mathbb{R}^d), \\ \nabla^2 u_n\, \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} D(\nabla u) \text{ in } \mathcal{M}(\Omega, \mathbb{R}^{d\times N\times N}) \Big\}$$

Our goal is to prove the integral representation result stated in Theorem 3.1. We will prove this in two steps. Setting

$$\mathcal{G}(u) := \int_\Omega \mathcal{Q}_2 f(x, \nabla^2 u)dx + \int_\Omega (\mathcal{Q}_2 f)^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}\right) d|D_s(\nabla u)|,$$

we will show that $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \leq \mathcal{F}$.

Theorem 3.8. *For all $u \in BH(\Omega;\mathbb{R}^d)$, we have $\mathcal{F}(u) \leq \mathcal{G}(u)$.*

Proof. We first prove this upper bound for $u \in W^{2,1}(\Omega, \mathbb{R}^d)$. By the definition of $\mathcal{F}$, it suffices to find a sequence of functions $\{u_n\} \subset W^{2,1}(\Omega, \mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,1}$, $\nabla^2 u_n\, \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \nabla^2 u\, \mathcal{L}^N \llcorner \Omega$, and

$$\liminf_{n\to\infty} \int_\Omega f(x, \nabla^2 u_n)dx \leq \int_\Omega \mathcal{Q}_2 f(x, \nabla^2 u)dx.$$

The existence of such a sequence is guaranteed by from the integral representation of the weakly lower semi-continuous envelope in $W^{2,1}(\Omega, \mathbb{R}^d)$ from [21], Theorem 1.3. In addition, necessary and sufficient conditions on lower-semicontinuity of second-order vector valued functionals can be found in [61], Theorem 4. Thus, for any $u \in W^{2,1}(\Omega, \mathbb{R}^d)$ we have

$$\mathcal{F}(u) \leq \mathcal{G}(u). \tag{3.40}$$

Next we show that for any $u \in BH(\Omega; \mathbb{R}^d)$ we have $\mathcal{F}(u) \leq \mathcal{G}(u)$. By Corollary 3.7, we know that we can find $u_n \in W^{2,1}(\Omega, \mathbb{R}^d)$ so that $u_n \to u$ in $W^{1,1}$, $\nabla^2 u_n \mathcal{L}^N \xrightharpoonup{*} D(\nabla u)$ and the convergence is area-strict, i.e.,

$$\lim_{n\to\infty} \int_\Omega \sqrt{1 + |\nabla^2 u_n|^2}\, dx = \int_\Omega \sqrt{1 + |\nabla^2 u|^2}\, dx + |D_s(\nabla u)|(\Omega).$$

Since $\mathcal{Q}_2 f$ is 2-quasiconvex with linear growth, Theorem 2.22 applies. Thus, $\mathcal{G}$ is continuous with respect to area-strict convergence, and

$$\mathcal{F}(u) \leq \liminf_{n\to\infty} \mathcal{F}(u_n) \leq \lim_{n\to\infty} \mathcal{G}(u_n) = \mathcal{G}(u)$$

where we use (3.40) on each of the u_n.

□

3.3.1 On coercivity

Before we prove the lower bound in the most general case, we will first assume that, in addition, $f(x, H)$ is coercive, i.e.,

(H3) Coercivity: $f(x, H) \geq c|H|$ for all $x \in \Omega, H \in \mathbb{R}^{d\times N\times N}$ and some $c \in (0, 1)$.

Note that under (H3), $\mathcal{Q}_2 f$ will inherit the modulus of continuity from f. Indeed, for any $x, y \in \Omega$, $H \in \mathbb{R}^{d\times N\times N}$ and $w \in W_0^{2,1}(Q; \mathbb{R}^d)$ we have

$$\begin{aligned}
&\int_Q f(x, H + \nabla^2 w(z))dz - \int_Q f(y, H + \nabla^2 w(z))dz \\
&\leq \int_Q |f(x, H + \nabla^2 w(z)) - f(y, H + \nabla^2 w(z))|dz \\
&\leq \int_Q \omega(|x - y|)(1 + |H + \nabla^2 w(z)|)dz
\end{aligned} \tag{3.41}$$

by (H2). But, if f is coercive, we have

$$\int_Q c\omega(|x - y|)(1 + |H + \nabla^2 w(z)|)dz \leq \int_Q \omega(|x - y|)(1 + f(y, H + \nabla^2 w(z))dz, \tag{3.42}$$

and so we have by (3.41) and (3.42),

$$\mathcal{Q}_2 f(x,H) - \int_Q f(y, H + \nabla^2 w(z))dz \leq \frac{1}{c}\int_Q \omega(|x-y|)(1 + f(y, H + \nabla^2 w(z))dz$$

If we choose w such that

$$\int_Q f(y, H + \nabla^2 w(z))dz \leq \mathcal{Q}_2 f(y,H) + \eta,$$

this becomes

$$\mathcal{Q}_2 f(x,H) - \mathcal{Q}_2 f(y,H) - \eta \leq \omega(|x-y|)(1 + \mathcal{Q}_2 f(y,H)) \leq \omega(|x-y|)(1+|H|)$$

where, in the last line, we may have scaled ω by some constant. Letting $\eta \to 0$ we have

$$\mathcal{Q}_2 f(x,H) - \mathcal{Q}_2 f(y,H) \leq \omega(|x-y|)(1 + \mathcal{Q}_2 f(y,H)) \leq \omega(|x-y|)(1+|H|).$$

By symmetry, the inequality holds where x and y are switched, yielding

$$|\mathcal{Q}_2 f(x,H) - \mathcal{Q}_2 f(y,H)| \leq \omega(|x-y|)(1+|H|).$$

Thus, if f is coercive, $\mathcal{Q}_2 f$ will inherit a modulus of continuity from f. I claim that if we can prove the lower bound for coercive integrands, we have it in general.

Lemma 3.9. *If we have $\mathcal{F} \geq \mathcal{G}$ for every integrand satisfying (H1), (H2), and (H3), then we have $\mathcal{F} \geq \mathcal{G}$ for every integrand satisfying (H1) and (H2).*

Proof. Let f be an arbitrary integrand satisfying (H1) and (H2), and consider the coercive integrand $f_\varepsilon := f + \varepsilon|\cdot|$. We observe that

$$\mathcal{Q}_2(f_\varepsilon) \geq \mathcal{Q}_2 f + \varepsilon|\cdot|$$

since $|\cdot|$ is convex. Furthermore, by basic properties of limits,

$$(\mathcal{Q}_2(f_\varepsilon))^\infty \geq (\mathcal{Q}_2 f + \varepsilon|\cdot|)^\infty = (\mathcal{Q}_2 f)^\infty + \varepsilon|\cdot|.$$

Now, for any sequence $\{u_n\} \subset W^{2,1}(\Omega, \mathbb{R}^d)$ with $u_n \overset{*}{\rightharpoonup} u$, we have

$$\begin{aligned}\varliminf_{n\to\infty} \int_\Omega f(x, \nabla^2 u_n(x))dx &\geq \varliminf_{n\to\infty} \int_\Omega f(x, \nabla^2 u_n(x)) + \varepsilon|\nabla^2 u_n(x)|dx - \varlimsup_{n\to\infty} \int_\Omega \varepsilon|\nabla^2 u_n(x)|dx \\ &\geq \int_\Omega \mathcal{Q}_2 f(x, \nabla^2 u(x))dx + \int_\Omega (\mathcal{Q}_2 f)^\infty\left(x, \frac{dD_s(\nabla u(x))}{d|D_s(\nabla u(x))|}\right)d|D_s(\nabla u(x))| + \varepsilon|D(\nabla)|(\Omega) - \varepsilon C\end{aligned}$$

where we have used the lower bound for f_ε and the fact that $\{\int_\Omega |\nabla^2 u_n|\}$ is bounded. Letting $\varepsilon \to 0$, we have that for any sequence $u_n \overset{*}{\rightharpoonup} u$,

$$\varliminf_{n\to\infty} F(u_n) \geq \mathcal{G}(u)$$

and, taking the infimum over all such sequences, we have

$$\mathcal{F}(u) \geq \mathcal{G}(u).$$

□

We will now prove our theorem in the case where f is coercive.

Theorem 3.10. *Assume that f satisfies (H1), (H2) and (H3). For all $u \in BH(\Omega; \mathbb{R}^d)$, we have $\mathcal{G}(u) \leq \mathcal{F}(u)$.*

Proof. Let $u \in BH(\Omega; \mathbb{R}^d)$ be given, and let $\{u_n\} \subset W^{2,1}(\Omega, \mathbb{R}^d)$ be an arbitrary sequence with $u_n \to u$ in $W^{1,1}$, $\nabla^2 u_n \, \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} D(\nabla u)$. We proceed according to the blow-up method. Define nonnegative Radon measures μ_n via

$$\mu_n(E) := \int_E f(x, \nabla^2 u_n)dx \text{ for every Borel set } E \subset \Omega.$$

Without loss of generality we may assume that $\{\mu_n(\Omega)\}$ is bounded, and so, passing to a subsequence (not relabeled), we may assume that $\{\mu_n\}$ has a weak-$*$ limit μ.

We consider the Radon-Nikodym decomposition of μ with respect to $|D(\nabla u)|$,

$$\mu = \frac{d\mu}{d\mathcal{L}^N}\mathcal{L}^N \llcorner \Omega + \frac{d\mu}{d|D_s(\nabla u)|}|D_s(\nabla u)| + \mu_s,$$

where μ_s is a nonnegative Radon measure such that $\mu_s \perp D(\nabla u)$.

We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0)) \text{ for } \mathcal{L}^N a.e.\, x_0 \in \Omega, \tag{3.43}$$

and

$$\frac{d\mu}{d|D_s(\nabla u)|}(x_0) \geq (\mathcal{Q}_2 f)^\infty\left(x_0, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}(x_0)\right) \text{ for } |D_s(\nabla u)| \, a.e. \; x_0 \in \Omega. \tag{3.44}$$

If (3.43) and (3.44) hold, then we have

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega f(x, \nabla^2 u_n) dx &= \liminf_{n\to\infty} \mu_n(\Omega) \geq \mu(\Omega) \\ &= \int_\Omega \frac{d\mu}{d\mathcal{L}^N} dx + \int_\Omega \frac{d\mu}{d|D_s(\nabla u)|} d|D_s(\nabla u)| + \mu_s(\Omega) \geq \mathcal{G}(u).\end{aligned}$$

The arbitrariness of the sequence $\{u_n\}$ would yield $\mathcal{F}(u) \geq \mathcal{G}(u)$. The remainder of this proof is dedicated to proving (3.43) and (3.44). □

3.3.2 Step 1: $\nabla^2 u$

For $\mathcal{L}^N$ a.e. $x_0 \in \Omega$, we have

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0)).$$

Note that the measures $\{|\nabla^2 u_n| \mathcal{L}^N \llcorner \Omega\}$ are bounded in total variation, so, along a subsequence, not relabeled, we have $|\nabla^2 u_n| \overset{*}{\rightharpoonup} \nu$ for some measure ν. By the Lebesgue differentiation theorem, for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ we have

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(\overline{Q(x_0,\varepsilon)})}{\varepsilon^N}, \tag{3.45}$$

$$\nabla^2 u(x_0) = \lim_{\varepsilon\to 0^+} \frac{D(\nabla u)(\overline{Q(x_0,\varepsilon)})}{\varepsilon^N}, \tag{3.46}$$

$$0 = \lim_{\varepsilon\to 0^+} \frac{|D_s(\nabla u)|(\overline{Q(x_0,\varepsilon)})}{\varepsilon^N}, \tag{3.47}$$

$$\frac{d\nu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\nu(\overline{Q(x_0,\varepsilon)})}{\varepsilon^N} < \infty. \tag{3.48}$$

Select $\varepsilon_k \to 0$ such that $\mu(\partial Q(x_0, \varepsilon_k)) = \nu(\partial Q(x_0, \varepsilon_k)) = |D(\nabla u)|(\partial Q(x_0, \varepsilon_k)) = 0$, and write

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k\to\infty} \frac{\mu(Q(x_0,\varepsilon_k))}{\varepsilon_k^N} = \lim_{k\to\infty}\lim_{n\to\infty} \frac{\mu_n(Q(x_0,\varepsilon_k))}{\varepsilon_k^N} \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\varepsilon_k^N}\int_{Q(x_0,\varepsilon_k)} f(x, \nabla^2 u_n)dx.\end{aligned}$$

With $x = x_0 + \varepsilon_k y$, we obtain

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k\to\infty}\lim_{n\to\infty} \int_Q f(x_0+\varepsilon_k y, \nabla^2 u_n(x_0+\varepsilon_k y))dy \\ &\geq \overline{\lim_{k\to\infty}}\ \overline{\lim_{n\to\infty}} \int_Q \mathcal{Q}_2 f(x_0+\varepsilon_k y, \nabla^2 u_n(x_0+\varepsilon_k y))dy. \end{aligned} \tag{3.49}$$

Define functions $v_{n,k} \in W^{2,1}(Q;\mathbb{R}^d)$ by

$$v_{n,k}(y) := \frac{u_n(x_0+\varepsilon_k y) - P_{\varepsilon,k}y - a_{\varepsilon,k}}{\varepsilon_k^2} - \frac{1}{2}\nabla^2 u(x_0)(y,y),$$

where $a_{\varepsilon,k} := \fint_Q v_{n,k}$ and $P_{\varepsilon,k} := \fint_Q \nabla v_{n,k}$, selected so that each $v_{n,k}$ and its gradient have average zero. By (3.49) we get

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \overline{\lim_{k\to\infty}}\ \overline{\lim_{n\to\infty}} \int_Q \mathcal{Q}_2 f(x_0+\varepsilon_k y, \nabla^2 u(x_0) + \nabla^2 v_{n,k}(y))dy. \tag{3.50}$$

For fixed k, the measures $\{\nabla^2 v_{n,k}\,\mathcal{L}^N \llcorner Q\}$ converge weakly-$*$ to the measure λ_k given by

$$\lambda_k(E) := \frac{D(\nabla u)(x_0+\varepsilon_k E)}{\varepsilon_k^N} - \nabla^2 u(x_0)\mathcal{L}^N(E), \text{ for every Borel set } E \subset Q,$$

and by (3.46) $\{\lambda_k\}$ converge weakly-$*$ to 0.

To see this, fix any $\psi \in C_c(Q)$. We have

$$\begin{aligned}&\left|\int_Q \psi(y)d\lambda_k(y)\right| = \frac{1}{\varepsilon_k^N}\left|\int_{Q(x_0,\varepsilon_k)} \psi\left(\frac{x-x_0}{\varepsilon_k}\right)dD(\nabla u)(x) - \int_{Q(x_0,\varepsilon_k)} \psi\left(\frac{x-x_0}{r_n}\right)\nabla^2 u(x_0)dx\right| \\ &= \frac{1}{\varepsilon_k^N}\left|\int_{Q(x_0,\varepsilon_k)} \psi\left(\frac{x-x_0}{\varepsilon_k}\right)(\nabla^2 u(x) - \nabla^2 u(x_0))dx + \int_{Q(x_0,\varepsilon_k)} \psi\left(\frac{x-x_0}{\varepsilon_k}\right)dD_s(\nabla u)(x)\right| \\ &\leq \|\psi\|_\infty\left(\frac{1}{\varepsilon_k^N}\int_{Q(x_0,\varepsilon_k)} |\nabla^2 u(x) - \nabla^2 u(x_0)| + \frac{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))}{\varepsilon_k^N}\right)\end{aligned}$$

which goes to 0 as $k \to \infty$ by (3.46) and (3.47).

We also note that for any n, k we have

$$|\nabla^2 v_{n,k}(y)| \leq |\nabla^2 u_{n,k}(x_0 + \varepsilon_k y)| + C$$

for some $C > 0$. For fixed k we have

$$|\nabla^2 u_{n,k}(x_0 + \varepsilon_k \cdot)|\mathcal{L}^N \llcorner Q \overset{*}{\rightharpoonup} \frac{T^{\#}_{x_0,\varepsilon_k}\nu}{\varepsilon_k^N}$$

and since, by (3.48) we have

$$\lim_{k\to\infty} \frac{\nu(Q(x_0, \varepsilon_k)}{\varepsilon_k^N} < \infty,$$

we conclude that

$$\sup_{k>0} \sup_{n>0} \int_Q |\nabla^2 v_{n,k}(y)| dy < \infty.$$

Thus, along a diagonalized sequence, we can find $v_k := v_{n_k,k}$ such that $\{\nabla^2 v_k \mathcal{L}^N \llcorner Q\}$ converge weakly-$*$ to the constant measure 0 and $\{|\nabla^2 v_k|\mathcal{L}^N \llcorner Q\}$ converge weakly-$*$ to some nonnegative Radon measure π. Using the modulus of continuity of $\mathcal{Q}_2 f$, which follows from (H3), we have

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \lim_{k\to\infty} \int_Q \mathcal{Q}_2 f(x_0 + \varepsilon_k y, \nabla^2 u(x_0) + \nabla^2 v_k(y)) dy \\
&\geq \overline{\lim_{k\to\infty}} \int_Q \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 v_k(y)) dy - \int_Q \omega(\varepsilon_k)(C + |\nabla^2 v_k(y)|) dy \\
&\geq \overline{\lim_{k\to\infty}} \int_Q \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 v_k(y)) dy
\end{aligned}$$

because $\int_Q |\nabla^2 v_k(y)| dy$ is bounded and $\omega(\varepsilon_k) \to 0$.

In order to apply 2-quasiconvexity, we have a to use a $W_0^{2,1}(Q;\mathbb{R}^N)$ perturbation of $\nabla^2 u(x_0)$. For $\delta < 1$, let $\phi_\delta \in C_c^\infty(Q;[0,1])$ be such that $\phi = 1$ on $Q_\delta = Q(0, 1-\delta)$, $\mathrm{supp}(\phi) \subset Q_{\delta/2} = Q(0, 1 - \frac{\delta}{2})$, $\|\nabla\phi\|_\infty \leq \frac{C}{\delta}$, $\|\nabla^2\phi\|_\infty \leq \frac{C}{\delta^2}$ for some $C > 0$, and let $z_{k,\delta} := \phi_\delta\, v_k$. In view of the definition of 2-quasiconvexity, for every k and δ we have

$$\int_Q \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 z_{k,\delta}(y)) dy \geq \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0)). \tag{3.51}$$

On the other hand, denoting $Q_{\delta/2} \setminus \overline{Q_\delta}$ as S_δ, we obtain

$$\int_Q \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 z_{k,\delta}(y))dy = \int_{Q_\delta} \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 v_k(y))dy$$
$$+ \int_{S_\delta} \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 z_{k,\delta}(y))dy + \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0))|Q \setminus Q_{\delta/2}| \tag{3.52}$$

and, as k goes to infinity,

$$\overline{\lim_{k\to\infty}} \int_{S_\delta} \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 z_{k,\delta}(y))dy \leq C \lim_{k\to\infty} \left(\delta + \int_{S_\delta} \left(\frac{1}{\delta^2}|v_k| + \frac{1}{\delta}|\nabla v_k| + |\nabla^2 v_k|\right)dy\right) \tag{3.53}$$

where we have used the growth condition (H1) and the fact that $\mathcal{Q}_2 f \leq f$.

Since we have $\nabla^2 v_k \overset{*}{\rightharpoonup} 0$ and the average of v_k and ∇v_k are 0, $\{|v_k|\}$ and $\{|\nabla v_k|\}$ are vanishing in $L^1(Q)$. To see this, let $\{v_{k_i}\}$ be an arbitrary subsequence of $\{v_k\}$. We observe that, by the Poincaré inequality for BV functions (see [41], 5.10), we must have that $\{v_{k_i}\}$ and $\{\nabla v_{k_i}\}$ are bounded in L^1. Since we have a bounded sequence in BH, we can extract a further subsequence, not relabeled, and a function $v \in BH$ such that

$$\lim_{i\to\infty} \int_Q |v_{k_i}(x) - v(x)|dx = \lim_{i\to\infty} \int_Q |\nabla v_{k_i}(x) - \nabla v(x)|dx = 0.$$

and

$$D(\nabla v_{k_i}) \overset{*}{\rightharpoonup} D(\nabla v) \text{ in } Q$$

However, since $D(\nabla v_{k_i}) \overset{*}{\rightharpoonup} 0$, we have $D(\nabla v) = 0$ and therefore ∇v is a constant function. Since Q is connected and $\int_Q \nabla v = 0$, we must have $\nabla v = 0$. Similarly, this implies that v is a constant function, and $\int_Q v = 0$ implies $v = 0$. Thus, we have

$$\lim_{i\to\infty} \int_Q |v_{k_i}(x)|dx = \lim_{i\to\infty} \int_Q |\nabla v_{k_i}(x)|dx = 0.$$

Due to the arbitrariness of the subsequence of $\{v_k\}$, we conclude that it is true for our original sequence. Since the v_k and ∇v_k are going to 0 in L^1, (3.53) becomes

$$\overline{\lim_{k\to\infty}} \int_{S_\delta} \mathcal{Q}_2 f(x_0, \nabla^2 u(x_0) + \nabla^2 z_{k,\delta}(y))dy \leq C\delta + C\pi(\overline{S_\delta}).$$

Thus, we have that for every $\delta < 1$, using (3.51) and (3.52)

$$\begin{aligned}\varlimsup_{k\to\infty}\int_Q \mathcal{Q}_2 f(x_0,\nabla^2 u(x_0)+\nabla^2 v_k(y))dy &\geq \varlimsup_{k\to\infty}\int_{Q_\delta} \mathcal{Q}_2 f(x_0,\nabla^2 u(x_0)+\nabla^2 v_k(y))dy \\ &\geq \mathcal{Q}_2 f(x_0,\nabla^2 u(x_0)) - C\delta - C|Q\setminus Q_{\delta/2}| - C\pi(\overline{S_\delta}).\end{aligned}$$

Note that for every $\delta > 0$, $\overline{S_\delta} \subset Q \setminus Q_\delta$ and $(Q\setminus Q_\delta) \searrow \emptyset$ as $\delta \to 1^-$. Thus, as we let δ increase to 1, we have

$$\varlimsup_{k\to\infty}\int_Q \mathcal{Q}_2 f(x_0,\nabla^2 u(x_0)+\nabla^2 v_k(y))dy \geq \mathcal{Q}_2 f(x_0,\nabla^2 u(x_0)).$$

3.3.3 Step 2: $D_s(\nabla u)$

For $|D_s(\nabla u)|$ a.e. $x_0 \in \Omega$, we have

$$\frac{d\mu}{d|D_s(\nabla u)|}(x_0) \geq (\mathcal{Q}_2 f)^\infty\left(x_0, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}(x_0)\right).$$

If we note that the measures $\{|\nabla^2 u_n|\mathcal{L}^N \llcorner \Omega\}$ are bounded in total variation, along a subsequence, not relabeled, we have $|\nabla^2 u_n| \overset{*}{\rightharpoonup} \pi$ for some measure π.

Fix $\sigma \in (0,1)$. By standard properties of BV functions, we know that for $|D_s(\nabla u)|$ a.e. $x_0 \in \Omega$ we have

$$\frac{d\mu}{d|D_s(\nabla u)|}(x_0) = \lim_{\varepsilon\to 0^+}\frac{\mu(Q(x_0,\varepsilon))}{|D_s(\nabla u)|(\overline{Q(x_0,\varepsilon)})}, \tag{3.54}$$

$$\frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}(x_0) = \lim_{\varepsilon\to 0^+}\frac{D(\nabla u)(\overline{Q(x_0,\varepsilon)})}{|D_s(\nabla u)|(\overline{Q(x_0,\varepsilon)})}, \tag{3.55}$$

$$\frac{d\pi}{d|D_s(\nabla u)|}(x_0) = \lim_{\varepsilon\to 0^+}\frac{\pi(\overline{Q(x_0,\varepsilon)})}{|D_s(\nabla u)|(\overline{Q(x_0,\varepsilon)})} < \infty, \tag{3.56}$$

and

$$\lim_{\varepsilon\to 0^+}\frac{|D_s(\nabla u)|(\overline{Q(x_0,\varepsilon)})}{\varepsilon^N} = \infty. \tag{3.57}$$

By Lemma 2.5 we can select $\varepsilon_k \to 0$ such that $|D_s(\nabla u)|(\partial(Q(x_0,\varepsilon_k)) = \mu(\partial Q(x_0,\varepsilon_k)) = \pi(\partial Q(x_0,\varepsilon_k)) = 0$ and

$$\lim_{k\to\infty}\frac{|D_s(\nabla u)|(Q(x_0,\sigma\varepsilon_k))}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))} \geq \sigma^N. \tag{3.58}$$

Then, we have

$$\frac{d\mu}{d|D_s(\nabla u)|}(x_0) = \lim_{k\to\infty} \frac{\mu(Q(x_0,\varepsilon_k))}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))} = \lim_{k\to\infty}\lim_{n\to\infty} \frac{\mu_n(Q(x_0,\varepsilon_k))}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))}$$
$$= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))} \int_{Q(x_0,\varepsilon_k)} f(x,\nabla^2 u_n)dx.$$

With the change of variables $x = x_0 + \varepsilon_k y$, we obtain

$$\begin{aligned}\frac{d\mu}{d|D_s(\nabla u)|}(x_0) &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{\varepsilon_k^N}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))} \int_Q f(x_0+\varepsilon_k y, \nabla^2 u_n(x_0+\varepsilon_k y))dy \\ &\geq \overline{\lim_{k\to\infty}}\ \overline{\lim_{n\to\infty}} \frac{\varepsilon_k^N}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))} \int_Q \mathcal{Q}_2 f(x_0+\varepsilon_k y, \nabla^2 u_n(x_0+\varepsilon_k y))dy.\end{aligned} \tag{3.59}$$

Note that by (3.57)

$$t_k := \varepsilon_k^{-N}|D_s(\nabla u)|(Q(x_0,\varepsilon_k)) \to \infty$$

as $k\to\infty$ by (3.57). We define functions $V_{n,k} \in L^1(Q,\mathbb{R}^{d\times N\times N})$ defined by

$$V_{n,k}(y) := \frac{1}{t_k}\nabla^2 u_n(x_0+\varepsilon_k y)$$

and consider the associated matrix-valued measures $\Sigma_{n,k}$

$$\Sigma_{n,k}(E) := \int_E V_{n,k}(y)dy = \frac{1}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))}\int_{x_0+\varepsilon_k E} \nabla^2 u_n(x)dx$$

for every Borel set $E \subset Q$. Note that the total variation of $\Sigma_{n,k}$ is given by

$$|\Sigma_{n,k}|(E) = \frac{1}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))}\int_{x_0+\varepsilon_k E} |\nabla^2 u_n(x)|dx = \frac{(|\nabla^2 u_n|\mathcal{L}^N \llcorner \Omega)(x_0+\varepsilon_k E)}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))}.$$

Note that we can now write (3.59) as

$$\frac{d\mu}{d|D_s(\nabla u)|}(x_0) \geq \overline{\lim_{k\to\infty}}\ \overline{\lim_{n\to\infty}} \frac{1}{t_k}\int_Q \mathcal{Q}_2 f(x_0+\varepsilon_k y, t_k V_{n,k}(y))dy. \tag{3.60}$$

Now, for a fixed k, we have $\Sigma_{n,k} \overset{*}{\rightharpoonup} \Sigma_k$ as $n\to\infty$ where

$$\Sigma_k(E) := \frac{D(\nabla u)(x_0+\varepsilon_k E)}{|D_s(\nabla u)|(Q(x_0,\varepsilon_k))} \text{ for every Borel set } E\subset Q, \tag{3.61}$$

and $|\Sigma_{n,k}| \overset{*}{\rightharpoonup} \pi_k$ where

$$\pi_k(E) := \frac{\pi(x_0 + \varepsilon_k E)}{|D_s(\nabla u)|(Q(x_0, \varepsilon_k))} \text{ for every Borel set } E \subset Q$$

Letting $k \to \infty$, by (3.55) we have

$$\Sigma_k \overset{*}{\rightharpoonup} \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}(x_0)\rho \tag{3.62}$$

where ρ denotes the weak-$*$ limit of $|\Sigma_k|$. This follows from an identical argument to the claim in (3.20).

Note that

$$\rho(Q) \geq \rho(\sigma\overline{Q}) \geq \overline{\lim_k} |\Sigma_k|(\sigma\overline{Q}) = \overline{\lim_k} \frac{|D(\nabla u)|(Q(x_0, \sigma\varepsilon_k))}{|D_s(\nabla u)|(Q(x_0, \varepsilon_k))} \geq \sigma^N \tag{3.63}$$

by (3.58). Recall that we have

$$H_0 = \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}(x_0) \in \Lambda(N, d, 2). \tag{3.64}$$

by the generalized form of the Alberti rank-one theorem. We also have for any Borel set E,

$$\overline{\lim_{k\to\infty}} \pi_k(E) = \overline{\lim_{k\to\infty}} \frac{\pi(x_0 + \varepsilon_k E)}{|D_s(\nabla u)|(Q(x_0, \varepsilon_k))} \tag{3.65}$$

$$\leq \overline{\lim_{k\to\infty}} \frac{d\pi}{d|D_s(\nabla u)|}(x_0)\frac{|D_s(\nabla u)|(x_0 + \varepsilon_k E)}{|D_s(\nabla u)|(Q(x_0, \varepsilon_k))} \tag{3.66}$$

$$+ \overline{\lim_{k\to\infty}} \frac{1}{|D_s(\nabla u)|(Q(x_0, \varepsilon_k))} \int_{x_0+\varepsilon_k E} \left| \frac{d\pi}{d|D_s(\nabla u)|}(x) - \frac{d\pi}{d|D_s(\nabla u)|}(x_0) \right| d|D_s(\nabla u)|(x) \tag{3.67}$$

$$= \overline{\lim_{k\to\infty}} \frac{d\pi}{d|D_s(\nabla u)|}(x_0)\frac{|D_s(\nabla u)|(x_0 + \varepsilon_k E)}{|D_s(\nabla u)|(Q(x_0, \varepsilon_k))}. \tag{3.68}$$

Thus, taking $E = Q \setminus \sigma Q$ we have

$$\overline{\lim_{k\to\infty}} \pi_k(Q \setminus \sigma Q) \leq C(1 - \sigma^N)$$

for some $C > 0$.

Since $\sup_k \pi_k(Q) < \infty$ by (3.56), we have

$$\sup_k \sup_n |\Sigma_{n,k}|(Q) < \infty$$

and we can consider a diagonalized sequence of the $v_{n,k}$ and $\Sigma_{n,k}$ so that, using (3.62) and (3.63) $\Sigma_{n_k,k} \overset{*}{\rightharpoonup} H_0\rho$, $\overline{\lim}\,|\Sigma_{n_k,k}|(Q \setminus \sigma Q) \leq C(1-\sigma^N)$ and

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{k\to\infty}} \frac{1}{t_k}\int_Q \mathcal{Q}_2 f(x_0+\varepsilon_k y, t_k V_{n,k}(y))dy = \overline{\lim_{k\to\infty}} \frac{1}{t_k}\int_Q \mathcal{Q}_2 f(x_0+\varepsilon_k y, t_k V_{n_k,k}(y))dy.$$

Applying our modulus of continuity (H2) to (3.60), we have

$$\begin{aligned}
\frac{d\mu}{d|D_s(\nabla u)|}(x_0) &\geq \overline{\lim_{k\to\infty}} \int_Q \frac{1}{t_k}\mathcal{Q}_2 f(x_0+\varepsilon_k y, t_k V_{n_k,k}(y))dy \\
&\geq \overline{\lim_{k\to\infty}} \int_Q \frac{1}{t_k}\mathcal{Q}_2 f(x_0, t_k V_{n_k,k}(y))dy - \frac{1}{t_k}\int_Q \omega(\varepsilon_k)(1+|t_k V_{n_k,k}(y)|)dy \\
&= \overline{\lim_{k\to\infty}} \int_Q \frac{1}{t_k}\mathcal{Q}_2 f(x_0, t_k V_{n_k,k}(y))dy - \int_Q \omega(\varepsilon_k)\left(\frac{1}{t_k}+|V_{n_k,k}(y)|\right)dy \\
&= \overline{\lim_{k\to\infty}} \int_Q \frac{1}{t_k}\mathcal{Q}_2 f(x_0, t_k V_{n_k,k}(y))dy. \qquad (3.69)
\end{aligned}$$

since $\{\int_Q |V_{n_k,k}(y)|dy\}$ is bounded and $t_k \to \infty$.

For any $\eta > 0$ we can find M such that $t > M$ implies

$$\frac{\mathcal{Q}_2 f(x_0, tH_u)}{t} \geq (\mathcal{Q}_2 f)_\#(x_0, H_u) - \eta$$

for every H_u with $|H_u| = 1$. If not, there exist $\{H_{u,n}\}$ with $|H_{u,n}| = 1$ and $t_n \to \infty$ such that

$$\frac{\mathcal{Q}_2 f(x_0, t_n H_{u,n})}{t_n} < (\mathcal{Q}_2 f)_\#(x_0, H_{u,n}) - \eta. \qquad (3.70)$$

Without loss of generality, since the unit sphere is compact, we can assume $H_{u,n} \to H_u$ for some H_u with $|H_u| = 1$. Note that for any $t > 0$ fixed, we have

$$\left|\frac{\mathcal{Q}_2 f(x_0, tH)}{t} - \frac{\mathcal{Q}_2 f(x_0, tH')}{t}\right| \leq L\frac{|tH - tH'|}{t} = L|H - H'|.$$

where L is the Lipschitz constant of $\mathcal{Q}_2 f$. Since the mappings

$$H \mapsto \frac{\mathcal{Q}_2 f(x_0, tH)}{t}$$

are uniformly Lipschitz, their infimum $\mathcal{Q}_2 f_\#$ is Lipschitz. Thus, letting $n \to \infty$ in (3.70)

$$\begin{aligned}
\mathcal{Q}_2 f_\#(x_0, H_u) \leq \lim_{n\to\infty} \frac{\mathcal{Q}_2 f(x_0, t_n H_{u,n})}{t_n} &\leq \lim_{n\to\infty} \mathcal{Q}_2 f_\#(x_0, H_{u,n}) - \eta \\
&= \mathcal{Q}_2 f_\#(x_0, H_u) - \eta
\end{aligned}$$

which is impossible. Thus, for any $\eta > 0$ we can find M such that $t > M$ implies

$$\frac{\mathcal{Q}_2 f(x_0, tH_u)}{t} \geq (\mathcal{Q}_2 f)_\#(x_0, H_u) - \eta$$

for every H_u with $|H_u| = 1$. This in turn implies that for any H and t such that $|H| > \frac{M}{t}$ we have

$$\frac{\mathcal{Q}_2 f(x_0, tH)}{t} \geq (\mathcal{Q}_2 f)_\#(x_0, H) - \eta |H| \tag{3.71}$$

by letting $H_u := \frac{H}{|H|}$. Consider the set

$$E_k := \left\{ x \in Q : |V_{n_k,k}(x)| > \frac{M}{t_k} \right\}$$

We have by (3.71)

$$\begin{aligned}
\int_Q \frac{1}{t_k} \mathcal{Q}_2 f(x_0, t_k V_{n_k,k}(y)) dy &\geq \int_{E_k} \frac{1}{t_k} \mathcal{Q}_2 f(x_0, t_k V_{n_k,k}(y)) dy \\
&\geq \int_{E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) - \eta |V_{n_k,k}(y)| dy \\
&\geq \int_{E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy - \eta \int_Q |V_{n_k,k}(y)| dy
\end{aligned} \tag{3.72}$$

We can write

$$\begin{aligned}
\int_Q (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy &= \int_{E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy + \int_{Q \setminus E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy \\
&\leq \int_{E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy + \int_{Q \setminus E_k} C |V_{n_k,k}(y)| dy \\
&\leq \int_{E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy + C \frac{M}{t_k}.
\end{aligned} \tag{3.73}$$

Now (3.73) yields

$$\int_{E_k} (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy \geq \int_Q (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y)) dy - C \frac{M}{t_k} \tag{3.74}$$

As discussed in Theorem 2.22, the function $(\mathcal{Q}_2 f)_\#(x_0, \cdot)$ is positively 1-homogeneous and $\Lambda(N, d, 2)$-convex. Since $H_0 \in \Lambda(N, d, 2)$ (see (3.64)), by [54], Theorem 1.1 we can find an affine function $L(H) = b + \xi \cdot H$ such that $L \leq (\mathcal{Q}_2 f)_\#(x_0, \cdot)$ and

$$L(H_0) = (\mathcal{Q}_2 f)_\#(x_0, H_0). \tag{3.75}$$

We have

$$\limsup_{k\to\infty}\int_Q (\mathcal{Q}_2 f)_\#(x_0, V_{n_k,k}(y))dy \geq \limsup_{k\to\infty}\int_Q L(V_{n_k,k}(y))dy$$

$$= \limsup_{k\to\infty}\int_Q \Big(b+\xi\cdot V_{n_k,k}(y)\Big)dy = b+\limsup_{k\to\infty}\int_Q \xi\cdot d\Sigma_{n_k,k}(y). \tag{3.76}$$

Let $\psi_\sigma \in C_c(Q;[0,1])$ be such that $\psi_\sigma = 1$ in σQ. We have

$$\begin{aligned}\limsup_{k\to\infty}\int_Q \xi\cdot d\Sigma_{n_k,k}(y) &= \lim_{k\to\infty}\int_Q \psi_\sigma(y)\xi\cdot d\Sigma_{n_k,k}(y) + \limsup_{k\to\infty}\int_Q (1-\psi_\sigma(y))\xi\cdot d\Sigma_{n_k,k}(y)\\ &= \int_Q \xi\cdot H_0\psi_\sigma(y)d\rho(y) + \limsup_{k\to\infty}\int_Q (1-\psi_\sigma(y))\xi\cdot d\Sigma_{n_k,k}(y)\\ &\geq \xi\cdot H_0\ \rho(\sigma Q) - \limsup_{k\to\infty}\int_{Q\setminus\sigma Q}|\xi|d|\Sigma_{n_k,k}|(y)\\ &\geq \xi\cdot H_0\ \sigma^N - C\limsup_{k\to\infty}|\Sigma_{n_k,k}|(Q\setminus\sigma Q)\end{aligned} \tag{3.77}$$

for some $C>0$, where we use (3.63).

But,

$$\limsup_{k\to\infty}|\Sigma_{n_k,k}|(Q\setminus\sigma Q) \leq C(1-\sigma^N), \tag{3.78}$$

therefore, by (3.76),

$$\limsup_{k\to\infty}\int_Q \xi\cdot d\Sigma_{n_k,k}(y) \geq \xi\cdot H_0\sigma^N - C(1-\sigma^N),$$

and so by (3.76), (3.77), and (3.78)

$$\begin{aligned}\liminf_{k\to\infty}\int_Q (\mathcal{Q}_2 f)_\#(x_0, v_{n_k,k}(y))dy &\geq b+\xi\cdot H_0\sigma^N - C(1-\sigma^N)\\ &\geq \sigma^N(\mathcal{Q}_2 f)_\#(x_0,H_0) - C(1-\sigma^N).\end{aligned}$$

Putting this together with (3.69), (3.72) and (3.74), we have

$$\begin{aligned}\overline{\lim_{k\to\infty}}\int_Q \frac{1}{t_k}\mathcal{Q}_2 f(x_0, t_k v_{n_k,k}(y))dy &\geq \overline{\lim_{k\to\infty}}\left[\int_{E_k}(\mathcal{Q}_2 f)_\#(x_0, v_{n_k,k}(y))dy - \eta\int_Q |v_{n_k,k}(y)|dy\right]\\ &\geq \limsup_{k\to\infty}\left(\int_Q (\mathcal{Q}_2 f)_\#(x_0, v_{n_k,k}(y))dy - C\frac{M}{t_k}\right) - \eta C\\ &\geq \sigma^N(\mathcal{Q}_2 f)_\#(x_0,H_0) - C(1-\sigma^N) - \eta C.\end{aligned}$$

Given the arbitrariness of $\eta > 0$, we conclude that

$$\limsup_{k\to\infty} \int_Q \frac{1}{t_k} \mathcal{Q}_2 f(x_0, t_k v_{n_k,k}(y))dy \geq \sigma^N (\mathcal{Q}_2 f)_\#(x_0, H_0) - C(1-\sigma^N)$$

for every $\sigma \in (0,1)$. Thus, as we send $\sigma \to 1^-$ we have

$$\limsup_{k\to\infty} \int_Q \frac{1}{t_k} \mathcal{Q}_2 f(x_0, t_k v_{n_k,k}(y))dy \geq (\mathcal{Q}_2 f)_\#(x_0, H_0).$$

□

3.4 Global method in SD_2

3.4.1 An approximation lemma

Let Ω be an open, bounded subset of $\mathbb{R}^N$. Set

$$\mathbb{S}^{d\times N\times N} = \{U \in \mathbb{R}^{d\times N\times N} : U_{ijk} = U_{ikj}, \forall i = 1,\dots,d, \quad j,k = 1,\dots,N\}.$$

Definition 3.11. The space of second-order structured deformations $SD_2(\Omega)$ consists of pairs (u,U) with $u \in SBH(\Omega;\mathbb{R}^d)$ and $U \in L^1(\Omega;\mathbb{S}^{d\times N\times N})$,

$$SD_2(\Omega) := SBH(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathbb{S}^{d\times N\times N}).$$

The approximation result stated next can be proved by applying the generalization of Alberti's theorem to BH functions contained in [43].

Theorem 3.12. *For every $(u,U) \in SD_2(\Omega)$ there exists a sequence $\{u_n\} \subset SBH(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,1}(\Omega;\mathbb{R}^d)$ and $\nabla^2 u_n \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, with*

$$\sup_n \|u_n\|_{BH} \leq C(\|u\|_{BH} + \|U\|_{L^1})$$

for some constant $C > 0$.

For convenience of the reader, we give a self-contained proof of Theorem 3.12, for which we will make use of the following lemma.

Lemma 3.13. *Let $U \in L^1(\Omega,\mathbb{R}^{d\times N\times N})$, and for every $\delta > 0$ let $\{Q_i^\delta\}_{i\in\mathbb{N}}$ be a countable family of open sets such that $Q_i^\delta \subset \Omega$, $Q_i^\delta \cap Q_j^\delta = \emptyset$ for every $i,j \in \mathbb{N}$ with $i \neq j$, $\mathcal{L}^N(\Omega\setminus \cup_i Q_i^\delta) = 0$,* *and* $\sup_i \operatorname{diam} Q_i^\delta \leq \delta$. *For* $i \in \mathbb{N}$ *let* $V_i^\delta : Q_i^\delta \to \mathbb{R}^{d\times N\times N}$ *be such that*

$$\int_{Q_i^\delta} V_i^\delta \, dx = \int_{Q_i^\delta} U \, dx,$$

and set

$$V^\delta := \sum_i \chi_{Q_i^\delta} V_i^\delta.$$

If $\sup_\delta \|V^\delta\|_{L^1} < +\infty$ *then* $V^\delta \mathcal{L}^N \overset{*}{\rightharpoonup} U\mathcal{L}^N$.

Proof. Arguing component-wise, it suffices to prove the lemma for scalar fields, hence we suppose that $U \in L^1(\Omega)$. Define

$$\overline{V}^\delta := \sum_i \chi_{Q_i^\delta} \fint_{Q_i^\delta} V_i^\delta \, dx = \sum_i \chi_{Q_i^\delta} \fint_{Q_i^\delta} U \, dx.$$

Fix $\varepsilon > 0$ and choose $W \in L^1(\Omega) \cap C_0^\infty(\Omega)$ such that $\|U - W\|_{L^1(\Omega)} < \varepsilon/3$. Define

$$\overline{W}^\delta := \sum_i \chi_{Q_i^\delta} \fint_{Q_i^\delta} W \, dx.$$

Since W is uniformly continuous, there exists $\eta > 0$ such that if $|x - y| < \eta$ then $|W(x) - W(y)| \leq \varepsilon/(3\mathcal{L}^N(\Omega))$. For $0 < \delta \leq \eta$ we have

$$\begin{aligned} \|\overline{W}^\delta - W\|_{L^1(\Omega)} &= \sum_i \int_{Q_i^\delta} \left| W(x) - \fint_{Q_i^\delta} W(y) \, dy \right| dx \\ &\leq \sum_i \int_{Q_i^\delta} \fint_{Q_i^\delta} |W(x) - W(y)| \, dy \, dx \leq \frac{\varepsilon}{3}, \end{aligned}$$

and

$$\|\overline{W}^\delta - \overline{V}^\delta\|_{L^1(\Omega)} = \sum_i \mathcal{L}^N(Q_i^\delta) \left| \fint_{Q_i^\delta} (W(y) - U(y)) \, dy \right| \leq \|U - W\|_{L^1(\Omega)} \leq \frac{\varepsilon}{3}.$$

Thus

$$\|\overline{V}^\delta - U\|_{L^1(\Omega)} \leq \|U - W\|_{L^1(\Omega)} + \|\overline{W}^\delta - W\|_{L^1(\Omega)} + \|\overline{W}^\delta - \overline{V}^\delta\|_{L^1(\Omega)} \leq \varepsilon,$$

and we conclude that $\overline{V}^\delta \to U$ in $L^1(\Omega)$. For $\psi \in C_0(\Omega)$ we have

$$\lim_{\delta\to 0^+} \int_\Omega (V^\delta - U)\psi \, dx = \lim_{\delta\to 0^+} \int_\Omega (V^\delta - \overline{V}^\delta)\psi \, dx,$$

and hence it suffices to show that $\lim_{\delta\to 0^+}\int_\Omega (V^\delta - \overline{V}^\delta)\psi\,dx = 0$. Note that

$$\begin{aligned}\int_\Omega \overline{V}^\delta \psi\,dx &= \sum_i \int_{Q_i^\delta} \fint_{Q_i^\delta} V^\delta(y)\,dy\,\psi(x)\,dx = \sum_i \int_{Q_i^\delta} V^\delta(y)\,dy \fint_{Q_i^\delta}\psi(x)\,dx \\ &= \sum_i \int_{Q_i^\delta} V^\delta(y)\psi(y)\,dy + \sum_i \int_{Q_i^\delta} V^\delta(y) \fint_{Q_i^\delta} (\psi(x)-\psi(y))\,dx\,dy,\end{aligned}$$

and therefore

$$\left|\int_\Omega (V^\delta - \overline{V}^\delta)\psi\,dx\right| = \left|\sum_i \int_{Q_i^\delta} V^\delta(y) \fint_{Q_i^\delta} (\psi(x)-\psi(y))\,dx\,dy\right| \le \sup_\delta \|V^\delta\|_{L^1(\Omega)} O(\delta),$$

since ψ is uniformly continuous. This concludes the proof. □

We now proceed to establish the approximation theorem.

Proof of Theorem 3.12 We claim that it suffices to prove that for every $V \in L^1(\Omega;\mathbb{S}^{d\times N\times N})$ there exists a sequence $\{f^\varepsilon\}\subset SBH(\Omega;\mathbb{R}^d)$ such that $f^\varepsilon \to 0$ in $W^{1,1}(\Omega;\mathbb{R}^d)$, $\nabla^2 f^\varepsilon \overset{*}{\rightharpoonup} V$ in $\mathcal{M}(\Omega)$ and $\sup_\varepsilon |D^2 f^\varepsilon|(\Omega) \le C\|V\|_{L^1(\Omega)}$. In fact, if the claim holds then we can define $u_n := u + f^{\varepsilon_n}$ where the sequence $\{f^\varepsilon\}$ is the one obtained by applying the claim to $V := U - \nabla^2 u$.

We now prove the claim. For simplicity of notation we will consider $N = 2$, however the same argument works for a generic N. Extend V outside Ω by 0 and denote this extension still by V. Fix $\varepsilon > 0$ and let $\{Q^{\varepsilon,l}\}_l$ be the family of open cubes whose side length is ε and whose centers $y^{\varepsilon,l}$ belong to the lattice $(\varepsilon\mathbb{Z})^2$. Let

$$\phi^\varepsilon(x) := \left(1 - \frac{2|x_1|}{\varepsilon}\right)\chi_{\{|x_2|<\varepsilon/2, |x_1|<|x_2|\}} + \left(1 - \frac{2|x_2|}{\varepsilon}\right)\chi_{\{|x_1|<\varepsilon/2, |x_2|<|x_1|\}}$$

i.e., ϕ^ε is the function whose graph is the pyramid over the cube $Q(0,\varepsilon)$ of height one. Let $\{A^{\varepsilon,l}\}_l$ be a family of tensors in $\mathbb{S}^{d\times 2\times 2}$ to be defined later and let $f^\varepsilon \in SBH(\Omega;\mathbb{R}^d)$ be given by

$$f^\varepsilon(x) := \sum_l \frac{1}{2}\phi^\varepsilon(x - y^{\varepsilon,l})A^{\varepsilon,l}(x - y^{\varepsilon,l}, x - y^{\varepsilon,l}).$$

We now define $A^{\varepsilon,l}$ as the tensor for which

$$\int_{Q^{\varepsilon,l}} \nabla^2 f^\varepsilon\,dx = \int_{Q^{\varepsilon,l}} V\,dx. \tag{3.79}$$

Note that, since $\nabla^2\phi^\varepsilon = 0$ and A must be symmetric,

$$(\nabla^2 f^\varepsilon)_{irs} = (\nabla\phi^\varepsilon)_s A^{\varepsilon,l}_{ijr}(x_j - y^{\varepsilon,l}_j) + (\nabla\phi^\varepsilon)_r A^{\varepsilon,l}_{ijs}(x_j - y^{\varepsilon,l}_j) + \phi^\varepsilon A^{\varepsilon,l}_{irs},$$

where the summation convention is adopted throughout this proof. Define

$$Z^\varepsilon_{js} := \int_{Q(0,\varepsilon)} (\nabla\phi^\varepsilon)_s(x)x_j\,dx = \int_{Q^{\varepsilon,l}} (\nabla\phi^\varepsilon)_s(x - y^{\varepsilon,l})(x_j - y^{\varepsilon,l}_j)\,dx,$$

$$\widetilde{z}^\varepsilon := \int_{Q(0,\varepsilon)} \phi^\varepsilon(x)\,dx = \int_{Q^{\varepsilon,l}} \phi^\varepsilon(x - y^{\varepsilon,l})\,dx, \qquad \text{and} \qquad \widetilde{V}^{\varepsilon,l} := \int_{Q^{\varepsilon,l}} V\,dx,$$

and rewrite (3.79) as

$$A^{\varepsilon,l}_{ijr} Z^\varepsilon_{js} + A^{\varepsilon,l}_{ijs} Z^\varepsilon_{jr} + A^{\varepsilon,l}_{irs}\widetilde{z}^\varepsilon = \widetilde{V}^{\varepsilon,l}_{irs}. \tag{3.80}$$

It turns out that $Z^\varepsilon = -\varepsilon^2 I$, where I is the identity matrix. Indeed,

$$Z^\varepsilon_{11} = \int_{Q(0,\varepsilon)} x_1 \frac{-2\,\mathrm{sgn}(x_1)}{\varepsilon}\chi_{\{|x_1|<|x_2|\}}\,dx = 2\int_{\varepsilon/2}^{\varepsilon/2}\int_{-x_1}^{x_1} \frac{-2x_1}{\varepsilon}\,dx_2 dx_1 = -\varepsilon^2$$

and, similarly,

$$Z^\varepsilon_{22} = \int_{Q(0,\varepsilon)} x_2 \frac{-2\,\mathrm{sgn}(x_2)}{\varepsilon}\chi_{\{|x_2|<|x_1|\}}\,dx = -\varepsilon^2.$$

On the other hand,

$$Z^\varepsilon_{12} = \int_{Q(0,\varepsilon)} x_2 \frac{-2\,\mathrm{sgn}(x_1)}{\varepsilon}\chi_{\{|x_1|<|x_2|\}}\,dx = 0$$

since the integrand is odd in x_2 and x_1 and the region of integration is symmetric in both variables, and the same is true for Z^ε_{21}. We can also calculate $\widetilde{z}^\varepsilon$ as the volume of a pyramid with base ε^2 and height 1 to find $\widetilde{z}^\varepsilon = \frac{1}{3}\varepsilon^2$.

From this, (3.80) becomes

$$-\frac{5}{3}\varepsilon^2 A^{\varepsilon,l} = \widetilde{V}^{\varepsilon,l}, \tag{3.81}$$

We now prove that $f^\varepsilon \to 0$ in $W^{1,1}(\Omega;\mathbb{R}^d)$. We have

$$\begin{aligned}
\int_\Omega |f^\varepsilon|\,dx &= \frac{1}{2}\sum_l \int_{Q^{\varepsilon,l}\cap\Omega} |\phi^\varepsilon(x - y^{\varepsilon,l})A^{\varepsilon,l}(x - y^{\varepsilon,l}, x - y^{\varepsilon,l})|\,dx \\
&\le C\sum_l |A^{\varepsilon,l}|\varepsilon^2\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega) \le C\varepsilon^2\sum_l |\widetilde{V}^{\varepsilon,l}| \\
&\le C\varepsilon^2\sum_l \int_{Q^{\varepsilon,l}} |V|\,dx \le C\varepsilon^2\|V\|_{L^1(\Omega)}
\end{aligned}$$

where we have used (3.81). Furthermore, again by (3.81) we obtain

$$\begin{aligned}\int_\Omega |\nabla f^\varepsilon|\,dx &\leq C\left[\sum_l \|\nabla\phi^\varepsilon\|_{L^\infty}|A^{\varepsilon,l}|\varepsilon^2\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)+\|\phi^\varepsilon\|_{L^\infty}|A^{\varepsilon,l}|\varepsilon\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)\right]\\ &\leq C\left[\sum_l \frac{1}{\varepsilon}\varepsilon^2\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)+|A^{\varepsilon,l}|\varepsilon\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)\right]\leq C\varepsilon\|V\|_{L^1(\Omega)}.\end{aligned}$$

Next, we show that $\sup_\varepsilon |D^2 f^\varepsilon|(\Omega)\leq C\|V\|_{L^1(\Omega)}$. Indeed, by (3.81)

$$\int_\Omega |\nabla^2 f^\varepsilon|\,dx \leq C\sum_l\left[|A^{\varepsilon,l}|\varepsilon\frac{1}{\varepsilon}\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)+|A^{\varepsilon,l}|\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)\right]\leq C\|V\|_{L^1(\Omega)},$$

and

$$\begin{aligned}\int_{\Omega\cap S(\nabla f^\varepsilon)}|[\nabla f^\varepsilon]|\,d\mathcal{H}^1 &\leq \sum_l\int_{\partial Q^{\varepsilon,l}\cap\Omega}|[\nabla f^\varepsilon]|\,d\mathcal{H}^1+\int_{d^{\varepsilon,l}}|[\nabla f^\varepsilon]|\,d\mathcal{H}^1\\ &\leq C\sum_l\left(\frac{1}{\varepsilon}\left(|A^{\varepsilon,l}|\varepsilon^2\right)\varepsilon+|A^{\varepsilon,l}|\varepsilon^2\right)\\ &\leq C\sum_l |A^{\varepsilon,l}|\mathcal{L}^2(Q^{\varepsilon,l})\leq C\|V\|_{L^1(\Omega)},\end{aligned}$$

where $d^{\varepsilon,l}$ is the union of the diagonals of $Q^{\varepsilon,l}$, and we used the estimate

$$\int_{d^{\varepsilon,l}}|[\nabla f^\varepsilon]|\,d\mathcal{H}^1\leq C\int_0^\varepsilon \frac{1}{\varepsilon}|A^{\varepsilon,l}||(t,t)|^2\,dt\leq C|A^{\varepsilon,l}|\varepsilon^2.$$

That $\nabla^2 f^\varepsilon \overset{*}{\rightharpoonup} V$ in $\mathcal{M}(\Omega)$ follows from (3.79), the inequalities above and from Lemma 3.13. $\square$

3.4.2 The global method

Let $\mathcal{A}(\Omega)$ be the family of open subsets of Ω. Consider a functional

$$\mathcal{F}: SD_2(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty] \tag{3.82}$$

satisfying the following hypotheses:

(I1) $\mathcal{F}(u,U;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure for every $(u,U)\in SD_2(\Omega)$.

(I2) $\mathcal{F}(\cdot,\cdot;A)$ is SD_2-lower semicontinuous, in the sense that if $(u,U)\in SD_2(\Omega)$, $\{(u_n,U_n)\}\subset$

$SD_2(\Omega)$, $u_n \to u$ in $W^{1,1}(\Omega;\mathbb{R}^d)$ and $U_n \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, then

$$\mathcal{F}(u,U;A) \le \liminf_{n\to+\infty} \mathcal{F}(u_n,U_n;A).$$

(I3) $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$, if $u = v$ and $U = V$ $\mathcal{L}^N$ a.e. $x \in A$ then $\mathcal{F}(u,U;A) = \mathcal{F}(v,V;A)$.

(I4) There exists a constant $C > 0$ such that

$$\frac{1}{C}(\|U\|_{L^1(A)} + |D^2u|(A)) \le \mathcal{F}(u,U;A) \le C(\mathcal{L}^N(A) + \|U\|_{L^1(A)} + |D^2u|(A))$$

for every $(u,U) \in SD_2(\Omega)$, $A \in \mathcal{A}(\Omega)$.

In the spirit of the global method for relaxation [18, 19], given $(u,U;A) \in SD_2(\Omega)\times\mathcal{A}(\Omega)$ we define

$$\mathfrak{A}(u,U;A) := \left\{(v,V) \in SD_2(\Omega) : \text{spt } (u-v) \subset\subset A, \int_A (U-V)\;dx = 0\right\}, \tag{3.83}$$

and

$$m(u,U;A) := \inf\left\{\mathcal{F}(v,V;A) : (v,V) \in \mathfrak{A}(u,U;A)\right\}. \tag{3.84}$$

Lemma 3.14. *If (I1) and (I4) hold, then for every $(u,U) \in SD_2(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$\limsup_{\delta\to0^+} m(u,U;A_\delta) \le m(u,U;A),$$

where $A_\delta = \{x \in A : \text{dist}(x,\partial A) > \delta\}$.

Proof. Let $\varepsilon > 0$. Choose $(\widetilde{u},\widetilde{U}) \in \mathfrak{A}(u,U;A)$ such that

$$\mathcal{F}(\widetilde{u},\widetilde{U};A) \le m(u,U;A) + \varepsilon.$$

Let $\delta_0 :=$ dist(spt$(u-\widetilde{u}),\partial A) > 0$. For $0 < \delta < \delta_0/2$ define

$$\widehat{U} = \begin{cases} \widetilde{U} & \text{in } A_{2\delta}, \\ (\mathcal{L}^N(A_\delta\backslash A_{2\delta}))^{-1}(\int_{A_\delta} U\,dx - \int_{A_{2\delta}} \widetilde{U}\,dx) & \text{on } A_\delta \setminus A_{2\delta}. \end{cases}$$

Since $(\widetilde{u},\widehat{U}) \in \mathfrak{A}(u,U;A_\delta)$, for every compact set $K \subset A_{2\delta}$ we have by (I1) and (I3),

$$\begin{aligned}
m(u,U;A_\delta) &\le \mathcal{F}(\widetilde{u},\widehat{U};A_\delta) \\
&\le \mathcal{F}(\widetilde{u},\widetilde{U};A_{2\delta}) + \mathcal{F}(u,\widehat{U};A_\delta \backslash K) \\
&\le \mathcal{F}(\widetilde{u},\widetilde{U};A) + C\left(\mathcal{L}^N(A\backslash K) + \int_{A_\delta\backslash K} |\widehat{U}|\,dx + |D^2u|(A\backslash K)\right) \\
&\le m(u,U;A) + \varepsilon + C\Bigg(\mathcal{L}^N(A\backslash K) + |D^2u|(A\backslash K) \\
&\qquad + \frac{\mathcal{L}^N(A_\delta\backslash K)}{\mathcal{L}^N(A_\delta\backslash A_{2\delta})}\left|\int_{A_\delta} U\,dx - \int_{A_{2\delta}} \widetilde{U}\,dx\right|\Bigg).
\end{aligned}$$

Using inner regularity and letting $K \nearrow A_{2\delta}$, we have

$$\begin{aligned}
m(u,U;A_\delta) &\le m(u,U;A) + \varepsilon \\
&\quad + C\left(\mathcal{L}^N(A\backslash A_{2\delta}) + |D^2u|(A\backslash A_{2\delta}) + \left|\int_{A_\delta} U\,dx - \int_{A_{2\delta}} \widetilde{U}\,dx\right|\right)
\end{aligned}$$

and since $\int_A U\,dx = \int_A \widetilde{U}\,dx$, we obtain

$$\limsup_{\delta\to 0^+} m(u,U;A_\delta) \le m(u;A) + \varepsilon$$

and by letting ε go to zero we finish the proof. $\square$

Again by analogy with [18, 19], for a fixed $(u,U) \in SD_2(\Omega)$ we set $\mu := \mathcal{L}^N \llcorner \Omega + |D^2_s u|$, we define

$$\mathcal{A}^*(\Omega) := \{Q_\nu(x,\varepsilon) : x \in \Omega, \nu \in S^{N-1}, \varepsilon > 0\},$$

and for $A \in \mathcal{A}(\Omega)$ and $\delta > 0$,

$$\begin{aligned}
m^\delta(u,U;A) := \inf\Bigg\{ &\sum_{i=1}^{\infty} m(u,U;Q_i) : Q_i \in \mathcal{A}^*(\Omega),\ Q_i \cap Q_j = \emptyset,\ Q_i \subset A, \\
&\operatorname{diam}(Q_i) < \delta,\ \mu(A \setminus \cup_{i=1}^{\infty} Q_i) = 0\Bigg\}.
\end{aligned}$$

Since m^δ increases as δ goes to 0, we can define

$$m^*(u,U;A) := \sup_{\delta>0} m^\delta(u,U;A) = \lim_{\delta\to 0^+} m^\delta(u,U;A).$$

Lemma 3.15. *Assume that hypotheses (I1)-(I4) hold. Then for all $A \in \mathcal{A}(\Omega)$*

$$\mathcal{F}(u, U; A) = m^*(u, U; A).$$

Proof. Fix $A \in \mathcal{A}(\Omega)$. For every $\delta > 0$ and every collection of cubes $\{Q_i\}_{i=1}^\infty$ admissible in the definition of m^δ we obtain

$$m^\delta(u, U; A) \leq \sum_{i=1}^\infty m(u, U; Q_i) \leq \sum_{i=1}^\infty \mathcal{F}(u, U; Q_i) \leq \mathcal{F}(u, U; A)$$

where we used (I1) in the last inequality. Hence $m^*(u, U; A) \leq \mathcal{F}(u, U; A)$.

Conversely, fix $\delta > 0$ and choose a family $\{Q_i^\delta\}_{i=1}^\infty$ such that

$$\sum_{i=1}^\infty m(u, U; Q_i^\delta) \leq m^\delta(u, U; A) + \delta.$$

For each Q_i^δ let $(v_i^\delta, V_i^\delta) \in \mathfrak{A}(u, U; Q_i^\delta)$ be such that

$$\mathcal{F}(v_i^\delta, V_i^\delta; Q_i^\delta) \leq m(u, U; Q_i^\delta) + \delta \mathcal{L}^N(Q_i^\delta).$$

Now, we stitch together these v_i^δ and V_i^δ to define

$$v^\delta := \sum_{i=1}^\infty v_i^\delta \chi_{Q_i^\delta} + u\chi_{N_\delta}, \ \ V^\delta := \sum_{i=1}^\infty V_i^\delta \chi_{Q_i^\delta} + U\chi_{N_\delta},$$

where $N_\delta := \Omega \setminus \cup_{i=1}^\infty Q_i^\delta$. By the coercivity hypothesis (I4), we have $v^\delta \in BH(\Omega)$ and $V^\delta \in L^1(\Omega)$. By (I1) and (I3),

$$\mathcal{F}(v^\delta, V^\delta; A) = \sum_{i=1}^\infty \mathcal{F}(v_i^\delta, V_i^\delta; Q_i^\delta) + \mathcal{F}(u, U; N_\delta \cap A),$$

and since $\mu(N_\delta \cap A) = 0$, by (I4) we have $\mathcal{F}(u, U; N_\delta \cap A) = 0$, and so

$$\mathcal{F}(v^\delta, V^\delta; A) \leq \sum_{i=1}^\infty \left[m(u, U; Q_i^\delta) + \delta \mathcal{L}^N(Q_i^\delta)\right] \leq m^\delta(u, U; A) + \delta + \delta \mathcal{L}^N(A).$$

If we prove that $v^\delta \to u$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ and $V^\delta \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, then by lower semicontinuity of $\mathcal{F}$ (see (I2)), we will have

$$\mathcal{F}(u, U; A) \leq \liminf_{\delta \to 0^+} \mathcal{F}(v^\delta, V^\delta; A) \leq \liminf_{\delta \to 0^+} m^\delta(u, U; A) = m^*(u, U; A),$$

thus proving the lemma. To see that $v^\delta \to u$ in $W^{1,1}$, by the BV Poincaré inequality (see Theorem 5.10 in [41]) applied to $\left(\nabla u - \nabla v^\delta\right)$ we obtain

$$\begin{aligned}\|\nabla u - \nabla v^\delta\|_{L^1(\Omega)} = \sum_{i=1}^{\infty} \|\nabla u - \nabla v^\delta\|_{L^1(Q_i^\delta)} &\leq \sum_{i=1}^{\infty} C\delta |D^2 u - D^2 v^\delta|(Q_i^\delta) \\ &\leq C\delta(|D^2 u|(A) + |D^2 v^\delta|(A)).\end{aligned}$$

By coercivity of $\mathcal{F}$ (see (I4)) we have that $\{|D^2 v^\delta|(A)\}$ is bounded, so this term goes to 0 with δ. By Poincaré's inequality applied now to $u - v$, we see that since $\|\nabla u - \nabla v^\delta\|_{L^1(\Omega)} \to 0$ we have that $\|u - v^\delta\|_{W^{1,1}(\Omega)} \to 0$. Finally, again by (I4)

$$\sup_\delta \|V^\delta\|_{L^1(\Omega)} < \infty$$

and applying Lemma 3.13 we conclude that $V^\delta \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$. □

Theorem 3.16. *If (I1), (I2) and (I4) hold then for every $(u, U) \in SD_2(\Omega)$ and for all $\nu \in S^{N-1}$*

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} = \lim_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

for μ a.e. $x_0 \in \Omega$ where $\mu := \mathcal{L}^N \llcorner \Omega + |D^2_s u|$.

Proof. By (I4), $F(u, U; \cdot)$ is absolutely continuous with respect to μ. Therefore, by Besicovitch's derivation theorem,

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

exist for μ-almost every $x_0 \in \Omega$. Since $m(u, U; \cdot) \leq \mathcal{F}(u, U; \cdot)$, we have trivially that

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \geq \limsup_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

whenever the left-hand limit exists. Thus, it suffices to show that

$$\liminf_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \geq \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

for μ-almost every $x_0 \in \Omega$. Fix $t > 0$ and let

$$\begin{aligned} E_t := \{x \in \Omega : \exists\, \varepsilon_n \to 0 \text{ such that } &\mu(\partial Q_\nu(x, \varepsilon_n)) = 0 \text{ and} \\ &\mathcal{F}(u, U; Q_\nu(x, \varepsilon_n)) > m(u, U; Q_\nu(x, \varepsilon_n)) + t\mu(Q_\nu(x, \varepsilon_n)) \text{ for every } n\}.\end{aligned}$$

First, we observe that the condition that μ does not charge the boundary of the cubes is innocuous: for every $x \in \Omega$ such that there is a sequence $\{\varepsilon_n\}$ converging to 0 with

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon_n)) > m(u, U; Q_\nu(x, \varepsilon_n)) + t\mu(Q_\nu(x, \varepsilon_n))$$

we can find another sequence $\{\varepsilon'_n\}$ such that

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon'_n)) > m(u, U; Q_\nu(x, \varepsilon'_n)) + t\mu(Q_\nu(x, \varepsilon'_n)),\ \mu(\partial Q_\nu(x, \varepsilon'_n)) = 0. \tag{3.85}$$

Indeed, for every n we can find $\varepsilon_n^k \nearrow \varepsilon_n$ so that $\mu(\partial Q_\nu(x, \varepsilon_n^k)) = 0$. By inner regularity we have

$$\lim_{k\to\infty} \mathcal{F}(u, U; Q_\nu(x, \varepsilon_n^k)) = \mathcal{F}(u, U; Q_\nu(x, \varepsilon_n)),\ \lim_{k\to\infty} \mu(Q_\nu(x, \varepsilon_n^k)) = \mu(Q_\nu(x, \varepsilon_n)),$$

and by Lemma 3.14

$$\limsup_{k\to\infty} m(u, U; Q_\nu(x, \varepsilon_n^k)) \leq m(u, U; Q_\nu(x, \varepsilon_n)).$$

Hence for k large enough we have

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon_n^k)) > m(u, U; Q_\nu(x, \varepsilon_n^k)) + t\mu(Q_\nu(x, \varepsilon_n^k)).$$

Extracting a diagonal subsequence of $\{\varepsilon_n^k\}$ we obtain a suitable subsequence $\{\varepsilon'_n := \varepsilon_n^{k(n)}\}$ for which (3.85) holds. Thus we see that without loss of generality we can take the ε_n so that μ does not charge the boundary.

Fix a compact set $K \subset \Omega$ such that $K \subset E_t$. For $\delta > 0$, define the families of cubes

$$\begin{aligned} X^\delta := \{Q_\nu(x, \varepsilon) : \varepsilon < \delta,\ \overline{Q_\nu(x, \varepsilon)} \subset \Omega,\ \mu(\partial Q_\nu(x, \varepsilon)) = 0, \\ \mathcal{F}(u, U; Q_\nu(x, \varepsilon)) > m(u, U; Q_\nu(x, \varepsilon)) + t\mu(Q_\nu(x, \varepsilon))\}, \end{aligned}$$

$$Y^\delta = \{Q_\nu(x, \varepsilon) : \varepsilon < \delta,\ \overline{Q_\nu(x, \varepsilon)} \subset \Omega \setminus K,\ \mu(\partial Q_\nu(x, \varepsilon)) = 0\}.$$

Since $K \subset E_t$, for every $x \in K$ there exists $Q_\nu(x, \varepsilon) \in X^\delta$ for some $\varepsilon < \delta$, and, similarly, if $x \in \Omega \setminus K$ there exists a cube $Q_\nu(x, \varepsilon) \in Y^\delta$. Hence we can write

$$\Omega = \bigcup_{Q\in X^\delta} Q \cup \bigcup_{Q'\in Y^\delta} Q'$$

and applying the Vitali-Besicovitch covering theorem, we can find a countable collection of

$Q_i^{X^\delta} \in X^\delta, Q_j^{Y^\delta} \in Y^\delta$, all mutually disjoint, such that

$$\Omega = \bigcup_{i=1}^{\infty} Q_i^{X^\delta} \cup \bigcup_{j=1}^{\infty} Q_j^{Y^\delta} \cup E$$

where $\mu(E) = 0$, and, as a consequence $\mathcal{F}(u, U; E) = 0$. Note that since $Q_j^{Y^\delta} \subset \Omega \setminus K$ for all j, we have

$$\mu(K) = \mu(\Omega \cap K) = \mu\left(\bigcup_{i=1}^{\infty} Q_i^{X^\delta}\right),$$

and thus

$$\begin{aligned}
\mathcal{F}(u, U; \Omega) &= \sum_{i=1}^{\infty} \mathcal{F}(u, U; Q_i^{X^\delta}) + \sum_{j=1}^{\infty} \mathcal{F}(u, U; Q_j^{Y^\delta}) \\
&\geq \sum_{i=1}^{\infty} \left[m(u, U; Q_i^{X^\delta}) + t\mu(Q_i^{X^\delta})\right] + \sum_{j=1}^{\infty} m(u, U; Q_j^{Y^\delta}) \\
&\geq m^\delta(u, U; \Omega) + t\sum_{i=1}^{\infty} \mu(Q_i^{X^\delta}) = m^\delta(u, U; \Omega) + t\mu(K).
\end{aligned}$$

Sending $\delta \to 0$, we can apply Lemma 3.15 to obtain

$$\mathcal{F}(u, U; \Omega) \geq m^*(u, U, \Omega) + t\mu(K) = \mathcal{F}(u, U; \Omega) + t\mu(K)$$

and so $\mu(K) = 0$ for every compact $K \subset E_t$. By inner regularity we conclude that $\mu(E_t) = 0$, i.e., for μ-almost every $x \in \Omega$, if ε is sufficiently small,

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon)) \leq m(u, U; Q_\nu(x, \varepsilon)) + t\mu(Q_\nu(x, \varepsilon))$$

and thus

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \leq \liminf_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} + t.$$

Sending $t \to 0$, we assert our claim. □

Lemma 3.1. Assume that hypotheses (I1), (I3) and (I4) hold. Let $\{(v_\varepsilon, V_\varepsilon)\} \subset SD_2(\Omega)$, $(u, U) \in SD_2(\Omega), x_0 \in \Omega, \nu \in S^{N-1}$, and let λ be a nonnegative Radon measure on Ω. Let $x_0 \in \Omega$ and suppose that

$$\lim_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\lambda(Q_\nu(x_0, \varepsilon))}$$

exists. Then,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \frac{m(v_\varepsilon, V_\varepsilon; Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))} - \lim_{\varepsilon\to 0^+} \frac{m(u, U; Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))} \leq \\
&\limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \frac{C}{\lambda(Q_\nu(x_0,\varepsilon))} \Big\{ \varepsilon^{N+1} + \varepsilon^N(1-\delta^N) + |D^2 u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\delta\varepsilon)}) \\
&+ |D^2 v_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\delta\varepsilon)}) + \frac{1}{\varepsilon^2(1-\delta)^2} \int_{Q_\nu(x_0,\varepsilon)} |u(x) - v_\varepsilon(x)|\, dx \\
&+ \frac{1}{\varepsilon(1-\delta)} \int_{Q_\nu(x_0,\varepsilon)} |\nabla u(x) - \nabla v_\varepsilon(x)|\, dx + \left| \int_{Q_\nu(x_0,\varepsilon)} V_\varepsilon\, dx - \int_{Q_\nu(x_0,\delta\varepsilon)} U\, dx \right| \Big\}.
\end{aligned}$$

Proof. Fix $\delta \in (0,1)$ and let $\varepsilon > 0$ be so small that $Q_\nu(x_0,\varepsilon) \subset \Omega$. Choose a cut-off function $\phi \in C_c^\infty(Q_\nu(x_0,\varepsilon))$ such that $\phi = 1$ in a neighborhood of $Q_\nu(x_0,\varepsilon\delta)$,

$$\|\nabla\phi\|_{L^\infty} \leq \frac{2}{\varepsilon(1-\delta)}, \qquad \text{and} \qquad \|\nabla^2\phi\|_{L^\infty} \leq \frac{4}{\varepsilon^2(1-\delta)^2}.$$

Define

$$w_\varepsilon := \begin{cases} \phi u + (1-\phi) v_\varepsilon & \text{in } Q_\nu(x_0,\varepsilon), \\ v_\varepsilon & \text{otherwise}, \end{cases}$$

and choose $(\widetilde{u}, \widetilde{U}) \in \mathfrak{A}(u, U; Q_\nu(x_0,\varepsilon\delta))$ such that

$$\frac{1}{2}\varepsilon^{N+1} + m(u, U; Q_\nu(x_0,\varepsilon\delta)) \geq \mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0,\varepsilon\delta)).$$

By outer regularity of $\mathcal{F}(\widetilde{u}, \widetilde{U}; \cdot)$ (see (I1)) we can find $\delta' \in (\delta, 1)$ such that

$$\mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0,\varepsilon\delta')) - \frac{1}{2}\varepsilon^{N+1} \leq \mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0,\varepsilon\delta)).$$

Set

$$\widetilde{v}_\varepsilon := \begin{cases} \widetilde{u} & \text{in } Q_\nu(x_0,\varepsilon\delta), \\ w_\varepsilon & \text{on } \Omega \setminus Q_\nu(x_0,\varepsilon\delta), \end{cases}$$

and

$$\widetilde{V}_\varepsilon := \begin{cases} \widetilde{U} & \text{in } Q_\nu(x_0,\varepsilon\delta), \\ (\mathcal{L}^N(Q_\nu(x_0,\varepsilon) \setminus Q_\nu(x_0,\varepsilon\delta))^{-1} (\int_{Q_\nu(x_0,\varepsilon)} V_\varepsilon\, dx - \int_{Q_\nu(x_0,\varepsilon\delta)} U\, dx) & \text{on } \Omega \setminus Q_\nu(x_0,\varepsilon\delta). \end{cases}$$

Recalling that $\int_{Q_\nu(x_0,\varepsilon\delta)} U\, dx = \int_{Q_\nu(x_0,\varepsilon\delta)} \widetilde{U}\, dx$, we have $(\widetilde{v}_\varepsilon, \widetilde{V}_\varepsilon) \in \mathfrak{A}(v_\varepsilon, V_\varepsilon; Q_\nu(x_0,\varepsilon))$, and by

(I3) and (I4) we obtain

$$
\begin{aligned}
m(v_\varepsilon, V_\varepsilon; Q_\nu(x_0,\varepsilon)) &\leq \mathcal{F}(\widetilde{v}_\varepsilon, \widetilde{V}_\varepsilon; Q_\nu(x_0,\varepsilon)) \\
&\leq \mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0,\varepsilon\delta')) + \mathcal{F}(w_\varepsilon, \widetilde{V}_\varepsilon; Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \\
&\leq \varepsilon^{N+1} + m(u, U; Q_\nu(x_0,\varepsilon\delta)) + \\
&\qquad C\Bigg(\varepsilon^N(1-\delta^N) + |D^2 w_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) + \\
&\qquad \left|\int_{Q_\nu(x_0,\varepsilon)} V_\varepsilon \, dx - \int_{Q_\nu(x_0,\varepsilon\delta)} U \, dx\right|\Bigg).
\end{aligned} \tag{3.86}
$$

Since

$$\nabla w_\varepsilon = (u - v_\varepsilon) \otimes \nabla\phi + \phi \nabla u + (1-\phi)\nabla v_\varepsilon,$$

we obtain

$$
\begin{aligned}
|D^2 w_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) &\leq C\Bigg\{|D^2 u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \\
&\quad + |D^2 v_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \\
&\quad + \frac{1}{\varepsilon^2(1-\delta)^2}\int_{Q_\nu(x_0,\varepsilon)} |u(x) - v_\varepsilon(x)| \, dx \\
&\quad + \frac{1}{\varepsilon(1-\delta)}\int_{Q_\nu(x_0,\varepsilon)} |\nabla u(x) - \nabla v_\varepsilon(x)| \, dx.\Bigg\}
\end{aligned} \tag{3.87}
$$

From Lemma 2.4 we deduce that

$$
\begin{aligned}
\overline{\lim_{\delta\to 1^-}}\ \overline{\lim_{\varepsilon\to 0^+}} \frac{m(u, U; Q_\nu(x_0,\varepsilon\delta))}{\lambda(Q_\nu(x_0,\varepsilon))} &= \overline{\lim_{\delta\to 1^-}}\ \overline{\lim_{\varepsilon\to 0^+}} \left(\frac{m(u, U; Q_\nu(x_0,\varepsilon\delta))}{\lambda(Q_\nu(x_0,\varepsilon\delta))}\frac{\lambda(Q_\nu(x_0,\varepsilon\delta))}{\lambda(Q_\nu(x_0,\varepsilon))}\right) \\
&\leq \lim_{\varepsilon\to 0^+} \frac{m(u, U; Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))},
\end{aligned}
$$

and hence to complete the proof it suffices to substitute (3.87) into (3.86), divide the resulting inequality by $\lambda(Q_\nu(x_0,\varepsilon))$ and take the $\limsup$ as $\varepsilon \to 0^+$ and $\delta \to 1^-$. $\square$

The next corollary is an immediate consequence of the previous lemma.

Corollary 3.17. *Assume that hypotheses (I1), (I3) and (I4) hold. Let $(v,V),(u,U) \in SD_2(\Omega), x_0 \in \Omega, \nu \in S^{N-1}$, and let λ be a nonnegative Radon measure on Ω be given. Let $x_0 \in \Omega$ and suppose that*

$$\lim_{\varepsilon\to 0^+} \frac{m(u, U; Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))} \quad \textit{and} \quad \lim_{\varepsilon\to 0^+} \frac{m(v, V; Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))}$$

exist. Then

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0^+}\frac{m(v,V;Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))}-\lim_{\varepsilon\to 0^+}\frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))}\right| \\
&\leq \limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+}\frac{C}{\lambda(Q_\nu(x_0,\varepsilon))}\Big\{\varepsilon^{N+1}+\varepsilon^N(1-\delta^N)+|D^2u|(Q_\nu(x_0,\varepsilon)\setminus\overline{Q_\nu(x_0,\delta\varepsilon)}) \\
&+|D^2v|(Q_\nu(x_0,\varepsilon)\setminus\overline{Q_\nu(x_0,\delta\varepsilon)})+\frac{1}{\varepsilon^2(1-\delta)^2}\int_{Q_\nu(x_0,\varepsilon)}|u(x)-v(x)|\,dx \\
&+\frac{1}{\varepsilon(1-\delta)}\int_{Q_\nu(x_0,\varepsilon)}|\nabla u(x)-\nabla v(x)|\,dx+\left|\int_{Q_\nu(x_0,\varepsilon)}V\,dx-\int_{Q_\nu(x_0,\varepsilon)}U\,dx\right| \\
&+\left|\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\delta\varepsilon)}V\,dx\right|+\left|\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\delta\varepsilon)}U\,dx\right|\Big\}.
\end{aligned}$$

Theorem 3.18. *Under hypotheses (I1), (I2), (I3),and (I4), for every $(u,U)\in SD_2(\Omega)$ and $A\in\mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u,U;A)=\int_A f(x,u,\nabla u,\nabla^2 u,U)\,dx+\int_{S(\nabla u)\cap A}h(x,u,\nabla u^+,\nabla u^-,\nu_{\nabla u})\,d\mathcal{H}^{N-1},$$

where

$$f(x_0,r,\xi,G,H):=\lim_{\varepsilon\to 0^+}\frac{m(r+\xi(\cdot-x_0)+1/2G(\cdot-x_0,\cdot-x_0),H;Q(x_0,\varepsilon))}{\varepsilon^N},$$

$$g(x_0,r,\eta,\zeta,\nu):=\lim_{\varepsilon\to 0^+}\frac{m(r+u_{\eta,\zeta,\nu}(\cdot-x_0),O;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}},$$

for all $x_0\in\Omega, r\in\mathbb{R}^N,\xi,\eta,\zeta\in\mathbb{R}^{d\times N}, G,H\in\mathbb{R}^{d\times N\times N},\nu\in S^{N-1}$, with $O\in\mathbb{R}^{d\times N\times N}$ being the matrix with all entries equal to zero, and

$$u_{\eta,\zeta,\nu}(y):=\begin{cases}\eta y & \text{if } y\cdot\nu>0,\\ \zeta y & \text{otherwise.}\end{cases}$$

Proof. We first show that

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{L}^N}(x_0)=f(x_0,u(x_0),\nabla u(x_0),\nabla^2 u(x_0),U(x_0)) \tag{3.88}$$

for $\mathcal{L}^N$ a.e. $x_0\in\Omega$. Define

$$v_a(x):=u(x_0)+\nabla u(x_0)(x-x_0)+\frac{1}{2}\nabla^2 u(x_0)(x-x_0,x-x_0).$$

By Theorem 2.15 and Theorem 3.16, for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2} \fint_{Q(x_0,\varepsilon)} |u(x) - v_a(x)| \, dx = 0, \qquad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \fint_{Q(x_0,\varepsilon)} |\nabla u(x) - \nabla v_a(x)| \, dx = 0, \tag{3.89}$$

$$\lim_{\varepsilon\to 0^+} \frac{|D^2 u|(Q(x_0;\varepsilon))}{\mathcal{L}^N(Q(x_0;\varepsilon))} = |\nabla^2 u(x_0)|, \tag{3.90}$$

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{m(u,U;Q(x_0;\varepsilon))}{\mathcal{L}^N(Q(x_0;\varepsilon))}, \tag{3.91}$$

$$\frac{d\mathcal{F}(v_a,U(x_0);\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{m(v_a,U(x_0);Q(x_0;\varepsilon))}{\mathcal{L}^N(Q(x_0;\varepsilon))}, \tag{3.92}$$

$$\lim_{\varepsilon\to 0^+} \fint_{Q(x_0,\varepsilon)} |U(x) - U(x_0)| \, dx = 0 \tag{3.93}$$

Select a point $x_0 \in \Omega$ with the above properties. Apply Corollary 3.17 with $v := v_a, V := U(x_0)$ and $\lambda := \mathcal{L}^N \llcorner \Omega$ to find

$$\left| \lim_{\varepsilon\to 0^+} \frac{m(v_a,U(x_0);Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))} - \lim_{\varepsilon\to 0^+} \frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))} \right| \leq C \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \mathcal{G}(\varepsilon,\delta,u,v_a,U),$$

where

$$\begin{aligned}
\mathcal{G}(\varepsilon,\delta,u,v_a,U) \quad := \quad & C\Bigg\{ \varepsilon + (1-\delta^N) + \frac{|D^2 u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^N} + |\nabla^2 u(x_0)|(1-\delta^N) \\
& + \frac{1}{(1-\delta)^2} \frac{1}{\varepsilon^2} \fint_{Q(x_0,\varepsilon)} |u(x) - v_a(x)| \, dx \\
& + \frac{1}{(1-\delta)} \frac{1}{\varepsilon} \fint_{Q(x_0,\varepsilon)} |\nabla u(x) - \nabla v_a(x)| \, dx \\
& + \left| \fint_{Q(x_0,\varepsilon)} U \, dx - U(x_0) \right| + \left| \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\setminus Q(x_0,\varepsilon\delta)} U \, dx \right| + (1-\delta^N) U(x_0). \Bigg\}
\end{aligned}$$

By (3.90) we find

$$0 \leq \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \frac{|D^2 u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^N} \leq \limsup_{\delta\to 1^-} |\nabla^2 u(x_0)|(1-\delta^N) = 0,$$

and by (3.93) we obtain

$$\begin{aligned}\limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+}\left|\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\setminus Q(x_0,\varepsilon\delta)} U\,dx\right| &= \limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+}\left|\fint_{Q(x_0,\varepsilon)} U\,dx - \delta^N\fint_{Q(x_0,\varepsilon\delta)} U\,dx\right| \\ &= \limsup_{\delta\to 1^-}|U(x_0)-\delta^N U(x_0)| = 0,\end{aligned}$$

which, together with (3.89), yields

$$\limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+}\mathcal{G}(\varepsilon,\delta,u,v_a,U)=0$$

and, consequently,

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{L}^N}(x_0)=\lim_{\varepsilon\to 0^+}\frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))}=\lim_{\varepsilon\to 0^+}\frac{m(v_a,U(x_0);Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))},$$

concluding the proof of (3.88).

Now we show that

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(\nabla u)}(x_0)=g(x_0,u(x_0),\nabla u^+(x_0),\nabla u^-(x_0),\nu_{\nabla u}(x_0)),$$

for $\mathcal{H}^{N-1}\lfloor S(\nabla u)$ a.e. $x_0\in\Omega$. Hereafter, for simplicity, we will just write ν in place of $\nu_{\nabla u}$. Define

$$v_J(x):=u(x_0)+\begin{cases}\nabla u^+(x_0)(x-x_0) & \text{if } (x-x_0)\cdot\nu(x_0)>0,\\ \nabla u^-(x_0)(x-x_0) & \text{if } (x-x_0)\cdot\nu(x_0)<0.\end{cases}$$

Again by Theorem 2.15 and Theorem 3.16, for $\mathcal{H}^{N-1}$ a.e. $x_0\in S(\nabla u)$ we have

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\fint_{Q_\nu(x_0,\varepsilon)}|u(x)-v_J(x)|\,dx=0,\qquad \lim_{\varepsilon\to 0^+}\fint_{Q_\nu(x_0,\varepsilon)}|\nabla u(x)-\nabla v_J(x)|\,dx=0, \tag{3.94}$$

$$\lim_{\varepsilon\to 0^+}\frac{|D^2u|(Q_\nu(x_0;\varepsilon))}{\varepsilon^{N-1}}=|[\nabla u](x_0)|, \tag{3.95}$$

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(\nabla u)}(x_0)=\lim_{\varepsilon\to 0^+}\frac{m(u,U;Q_\nu(x_0;\varepsilon))}{\varepsilon^{N-1}}, \tag{3.96}$$

$$\frac{d\mathcal{F}(v_J,O;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(\nabla u)}(x_0)=\lim_{\varepsilon\to 0^+}\frac{m(v_J,O;Q_\nu(x_0;\varepsilon))}{\varepsilon^{N-1}}, \tag{3.97}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0;\varepsilon)} |U| dx = 0. \tag{3.98}$$

Select a point $x_0 \in S(\nabla u)$ such that the above properties hold. Apply Corollary 3.17 with $v := v_J, V := O$ and $\lambda := \mathcal{H}^{N-1} \llcorner S(\nabla u)$ to deduce that

$$\left| \lim_{\varepsilon\to 0^+} \frac{m(v_J, O; Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} - \lim_{\varepsilon\to 0^+} \frac{m(u, U; Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \right| \le C \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \mathcal{G}_J(\varepsilon,\delta,u,v_J,U),$$

where

$$\begin{aligned} \mathcal{G}_J(\varepsilon,\delta,u,v_J,U) \;=\; & C\Bigg\{ \varepsilon^2 + \varepsilon(1-\delta^N) + \frac{|D^2u|(Q_\nu(x_0,\varepsilon)\setminus \overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^{N-1}} + |[\nabla u](x_0)|(1-\delta^{N-1}) \\ & + \frac{1}{(1-\delta)^2}\frac{1}{\varepsilon} \fint_{Q_\nu(x_0,\varepsilon)} |u(x) - v_J(x)|\; dx \\ & + \frac{1}{(1-\delta)} \fint_{Q_\nu(x_0,\varepsilon)} |\nabla u(x) - \nabla v_J(x)|\; dx \\ & + \varepsilon \left| \fint_{Q_\nu(x_0,\varepsilon)} U\, dx \right| + \left| \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\varepsilon\delta)} U\, dx \right| \Bigg\}. \end{aligned}$$

By (3.95) we find

$$0 \le \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \frac{|D^2u|(Q_\nu(x_0,\varepsilon)\setminus \overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^{N-1}} \le \limsup_{\delta\to 1^-} |[\nabla u](x_0)|(1-\delta^{N-1}) = 0,$$

while from (3.98) we obtain

$$\limsup_{\varepsilon\to 0^+} \left| \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\varepsilon\delta)} U\, dx \right| \le \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0,\varepsilon)} |U|\, dx = 0,$$

and thus, using Eq. (3.94), we conclude that

$$\limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \mathcal{G}_J(\varepsilon,\delta,u,v_J,U) = 0,$$

and hence the proof is completed. $\square$

3.4.3 Applications (SD_2 integral representation)

We consider the functional defined for each $A \in \mathcal{A}(\Omega)$ by

$$\mathcal{F}_0(u;A) := \begin{cases} \int_A f_0(x,u,\nabla u,\nabla^2 u)\,dx & \\ \qquad + \int_{S(\nabla u)\cap A} g_0(x,u,\nabla u^+,\nabla u^-,\nu_{\nabla u})\,d\mathcal{H}^{N-1} & \text{if } u \in SBH(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.99}$$

where the densities f_0 and g_0 satisfy the following hypotheses:

(J1) $f_0 : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N} \to [0,+\infty)$ continuous and

$$\frac{1}{C}|\Lambda| \leq f_0(x,u,\xi,\Lambda) \leq C(1+|\Lambda|)$$

for all $(x,u,\xi,\Lambda) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N}$ and for some $C > 0$;

(J2) the function $g_0 : \Omega \times \mathbb{R}^d \times (\mathbb{R}^{d\times N})^2 \times S^{N-1} \to [0,+\infty)$ is continuous and

$$\frac{1}{C}|\xi-\eta| \leq g_0(x,u,\xi,\eta,\nu) \leq C(1+|\xi-\eta|)$$

for all $(x,u,\xi,\eta,\nu) \in \Omega \times \mathbb{R}^d \times (\mathbb{R}^{d\times N})^2 \times S^{N-1}$ and for some $C > 0$;

The functional $\mathcal{F} : SD_2(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty)$ is defined by

$$\mathcal{F}(u,U;A) := \inf\{\liminf_{n\to+\infty} \mathcal{F}_0(u_n;A) : u_n \to u, \text{ in } L^1(\Omega;\mathbb{R}^d), \nabla^2 u_n \overset{*}{\rightharpoonup} U\}. \tag{3.100}$$

Lemma 3.19. *For every $(u,U) \in SD_2(\Omega)$, $A \in \mathcal{A}(\Omega)$ and every sequence $\{(u_n,U_n)\} \subset SD_2(\Omega)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and $U_n \overset{*}{\rightharpoonup} U$ in in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N\times N})$,*

$$\mathcal{F}(u,U;A) \leq \liminf_{n\to\infty} \mathcal{F}(u_n,U_n;A).$$

Proof. Fix a sequence $\{(u_n,U_n)\} \subset SD_2(\Omega)$ such that $u_n \to u$ in L^1 and $U_n \overset{*}{\rightharpoonup} U$. For every (u_n,U_n) we can pick a sequence $\{(u_{n,k},U_{n,k})\} \subset SD_2(\Omega)$ such that $u_{n,k} \to u_n$ in L^1 and $U_{n,k} \overset{*}{\rightharpoonup} U_n$ as $k \to \infty$ and

$$\liminf_{k\to\infty} \mathcal{F}_0(u_{n,k},U_{n,k};A) \leq \mathcal{F}(u_n,U_n;A) + \frac{1}{n}.$$

We can extract diagonalized sequences $v_n := u_{n,k_n}$ and $V_n := v_{n,k_n}$ such that $v_n \to u$ in L^1, $V_n \overset{*}{\rightharpoonup} U$ as $n \to \infty$, and

$$\liminf_{n\to\infty} \mathcal{F}_0(v_n,V_n;A) \leq \liminf_{n\to\infty} \mathcal{F}(u_n,U_n;A)$$

and thus

$$\mathcal{F}(u,U;A) \leq \liminf_{n\to\infty} \mathcal{F}(u_n,U_n;A).$$

□

Lemma 3.20. *The functional $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$, if $u = v$ and $U = V$ $\mathcal{L}^N$ a.e. $x \in A$ then $\mathcal{F}(u,U;A) = \mathcal{F}(v,V;A)$.*

Proof. Let A, u,U, and v,V be as in the statement of the lemma. For every sequence $\{(u_n,U_n)\} \subset SD_2(\Omega)$ such that $u_n \to u$ in $L^1(A)$ and $U_n \overset{*}{\rightharpoonup} U$, we also have $u_n \to v$ in $L^1(A)$ and $V_n \overset{*}{\rightharpoonup} V$. Thus

$$\mathcal{F}(u,U;A) \geq \mathcal{F}(v,V;A),$$

and by symmetry we conclude that

$$\mathcal{F}(u,U;A) = \mathcal{F}(v,V;A).$$

□

Lemma 3.21. *Assume hypotheses (J1) and (J2) hold. For every $(u,U) \in SD_2(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$ we have*

$$\frac{1}{C}\left(\|U\|_{L^1(A)} + |D^2u|(A)\right) \leq \mathcal{F}(u;A) \leq C\left(\mathcal{L}^N(A) + \|U\|_{L^1(A)} + |D^2u|(A)\right)$$

where $C > 0$. Moreover, for every $u \in BH(\Omega;\mathbb{R}^d)$ the functional $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure.

Proof. We note that hypotheses (J1) and (J2) imply that

$$\frac{1}{C}|D^2u|(A) \leq \mathcal{F}_0(u;A) \leq C(\mathcal{L}^N(A) + |D^2u|(A))$$

for every $u \in SBH(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}(\Omega)$. For any $(u,U) \in SD_2(\Omega)$ and any $\delta > 0$, we can find $u_n \in SBH(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in L^1, $\nabla^2 u_n \overset{*}{\rightharpoonup} U$, and

$$\mathcal{F}(u,U;A) \geq \liminf_{n\to\infty} \mathcal{F}_0(u_n;A) - \delta.$$

On one hand, this implies that

$$\mathcal{F}(u,U;A) \geq \liminf_{n\to\infty} \frac{1}{C}|D^2u_n|(A) - \delta \geq \frac{1}{C}|D^2u|(A) - \delta,$$

and letting $\delta \to 0$ we have

$$\mathcal{F}(u,U;A) \geq \frac{1}{C}|D^2 u|(A). \tag{3.101}$$

On the other hand, we obtain

$$\mathcal{F}(u,U;A) \geq \liminf_{n\to\infty} \frac{1}{C}|D^2 u_n|(A) - \delta \geq \liminf_{n\to\infty} \frac{1}{C}\|\nabla^2 u_n\|_{L^1(A)} - \delta \geq \frac{1}{C}\|U\|_{L^1(A)} - \delta$$

and, again letting $\delta \to 0$, we have

$$\mathcal{F}(u,U;A) \geq \frac{1}{C}\|U\|_{L^1(A)}.$$

Averaging this with (3.101), we deduce that

$$\mathcal{F}(u,U;A) \geq \frac{1}{C}\left(\|U\|_{L^1(A)} + |D^2 u|(A)\right).$$

To prove the upper bound, we consider the sequence $\{u_n\}$ constructed in Theorem 3.12 which satisfies $u_n \to u$ in L^1, $\nabla^2 u_n \overset{*}{\rightharpoonup} U$ and

$$\sup_n |D(\nabla u_n)|(A) \leq C\left(|D(\nabla u)|(A) + \|U\|_{L^1(A)}\right).$$

Then we have

$$\begin{aligned}\mathcal{F}(u,U;A) &\leq \liminf_{n\to\infty} \mathcal{F}_0(u_n;A) \leq \liminf_{n\to\infty} C\left(\mathcal{L}^N(A) + |D(\nabla u_n)|(A)\right)\\ &\leq C\left(\mathcal{L}^N(A) + |D(\nabla u)|(A) + \|U\|_{L^1(A)}\right).\end{aligned}$$

Finally, we will prove that for $(u,U) \in SD_2(\Omega)$, $\mathcal{F}(u,U;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure. We will apply the coincidence criterion, Lemma 2.14. Since item (ii) follows directly from the fact that $\mathcal{F}_0(u,\cdot)$ is a Radon measure and item (iii) follows from the growth condition that we have just proved, it only remains to prove that for any open sets $A, B, C \in \mathcal{A}(\Omega)$ with $\overline{A} \subset B \subset C$ we have

$$\mathcal{F}(u,U;C) \leq \mathcal{F}(u,U;C \setminus \overline{A}) + \mathcal{F}(u,U;B).$$

To see this, for $\varepsilon > 0$ we choose $v_n \in BH(\Omega;\mathbb{R}^d)$ and $w_n \in BH(\Omega;\mathbb{R}^d)$ as in the definition of $\mathcal{F}(u,U;\cdot)$ (perhaps along a subsequence) so that

$$\lim_{n\to\infty} \mathcal{F}_0(v_n, C \setminus \overline{A}) \leq \mathcal{F}(u,U;C \setminus \overline{A}) - \varepsilon \tag{3.102}$$

and

$$\lim_{n\to\infty} \mathcal{F}_0(w_n, B) \leq \mathcal{F}(u, U; B) - \varepsilon.$$

We will use a slicing argument in order to construct (up to a subsequence) a sequence $\{u_n\} \subset BH(C;\mathbb{R}^d)$ as in the definition of $\mathcal{F}(u,U;\cdot)$ so that

$$\liminf_{n\to\infty} \mathcal{F}_0(u_n; C) \leq \lim_{n\to\infty} \mathcal{F}_0(v_n, C\setminus \overline{A}) + \lim_{n\to\infty} \mathcal{F}_0(w_n, B).$$

Let $\delta > 0$ be so small that

$$S_\delta := \{x \in B : \mathrm{dist}(x, A) < \delta\} \subset\subset B.$$

Given $k \in \mathbb{N}$ we can decompose $S_\delta \setminus A$ into a disjoint union of strips, to be precise we write

$$S_\delta \setminus A = \bigcup_{i=1}^{k} L_{i,k},$$

where

$$L_{i,k} = \left\{ x \in S_\delta : \frac{(i-1)\delta}{k} < \mathrm{dist}(x, A) \leq \frac{i\delta}{k} \right\}.$$

By coercivity of $\mathcal{F}_0$, we have

$$\sup_n |D(\nabla v_n)|(C\setminus \overline{A}) + \sup_n |D(\nabla w_n)|(B) \leq M$$

for some $M < \infty$, and thus

$$\sup_n \sum_{i=1}^{k} \left(|D(\nabla v_n)| + |D(\nabla w_n)|\right)(L_{i,k}) \leq M.$$

We remark that since there are only finitely many values of i and infinitely many values of n, there must be some fixed i such that

$$\left(|D(\nabla v_n)| + |D(\nabla w_n)|\right)(L_{i,k}) \leq \frac{M}{k}$$

for infinitely many $n \in \mathbb{N}$. Thus for any k, there is a $i_k \in \{1,\dots,k\}$ and a subsequence $\{n_j^{(k)}\} \subset \{n\}$ such that

$$\left(|D(\nabla v_{n_j^{(k)}})| + |D(\nabla w_{n_j^{(k)}})|\right)(L_{i_k,k}) \leq \frac{M}{k}, \quad \forall j,k \in \mathbb{N}.$$

We consider a smooth cutoff function $\phi_k \in C_c^\infty(B;[0,1])$ such that $\{0 < \phi_k < 1\} \subset L_{i_k,k}$,

$\phi_k(x) = 0$ if $\operatorname{dist}(x, A) \leq \frac{i_k - 1}{k}\delta$, $\phi_k(x) = 1$ if $\operatorname{dist}(x, A) \geq \frac{i_k}{k}\delta$ and

$$\|\nabla \phi_k\|_\infty \leq Ck, \ \|\nabla^2 \phi_k\|_\infty \leq Ck^2.$$

For $x \in C$, we define

$$u_{j,k} = \phi_k v_{n_j^{(k)}} + (1 - \phi_k) w_{n_j^{(k)}}.$$

Then we have

$$\mathcal{F}_0(u_{j,k}; C) \leq \mathcal{F}_0(v_{n_j^{(k)}}; C \setminus \overline{A}) + \mathcal{F}_0(w_{n_j^{(k)}}; B) + \mathcal{F}_0(u_{j,k}; L_{i_k,k}),$$

and the last term is bounded by

$$\begin{aligned}\mathcal{F}_0(u_{j,k}; L_{i_k,k}) \leq & C\Big(\mathcal{L}^N(L_{i_k,k}) + k^2 \int_{L_{i_k,k}} |v_{n_j^{(k)}} - w_{n_j^{(k)}}| dx + k \int_{L_{i_k,k}} |\nabla v_{n_j^{(k)}} - \nabla w_{n_j^{(k)}}| dx \\ & + |D(\nabla v_{n_j^{(k)}})|(L_{i_k,k}) + |D(\nabla w_{n_j^{(k)}})|(L_{i_k,k})\Big) \\ \leq & C\Big(\frac{1}{k} + k^2 \int_{L_{i_k,k}} |v_{n_j^{(k)}} - w_{n_j^{(k)}}| dx + k \int_{L_{i_k,k}} |\nabla v_{n_j^{(k)}} - \nabla w_{n_j^{(k)}}| dx\Big).\end{aligned}$$

Since $v_n \to u$ and $w_n \to u$ in $W^{1,1}(B \setminus \overline{A})$, for any k we can choose an element $n_{j_k}^{(k)}$ of $n_j^{(k)}$ so that the map $k \mapsto n_{j_k}^{(k)}$ is increasing and

$$\int_{B \setminus \overline{A}} |v_{n_{j_k}^{(k)}} - w_{n_{j_k}^{(k)}}| dx = o(1/k^2)$$

and

$$\int_{B \setminus \overline{A}} |\nabla v_{n_{j_k}^{(k)}} - \nabla w_{n_{j_k}^{(k)}}| dx = o(1/k).$$

With this choice we have that

$$\liminf_{k \to \infty} \mathcal{F}_0(u_{j_k,k}; L_{i_k,k}) = 0.$$

Since $v_n \to u$ in $L^1(C \setminus \overline{A})$, $w_n \to u$ in $L^1(B)$ and $\nabla^2 v_n \overset{*}{\rightharpoonup} U$ in $C \setminus \overline{A}$, $\nabla^2 w_n \overset{*}{\rightharpoonup} U$ in B, we must have that $u_{j_k,k} \to u$ in $L^1(C)$ and $\nabla^2 u_{j_k,k} \overset{*}{\rightharpoonup} U$ in C. Thus, by definition of $\mathcal{F}(u, U; \cdot)$, we conclude

$$\begin{aligned}\mathcal{F}(u, U; C) \leq \liminf_{k \to \infty} \mathcal{F}_0(u_{j_k,k}; C) &\leq \lim_{k \to \infty} \mathcal{F}_0(v_{n_{j_k}^{(k)}}; C \setminus \overline{A}) + \lim_{k \to \infty} \mathcal{F}_0(w_{n_{j_k}^{(k)}}; B) \\ &\leq \mathcal{F}(u, U; C \setminus \overline{A}) + \mathcal{F}(u, U; B) - 2\varepsilon.\end{aligned}$$

Sending $\varepsilon \to 0$, we are done. □

Theorem 3.22. *Under hypotheses (J1) and (J2) the functional $\mathcal{F}$, defined by Eq.* (3.100)*, there exist functions $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N} \times \mathbb{R}^{d\times N\times N} \to [0,\infty)$ and $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times S^{N-1} \to [0,\infty)$ such that*

$$\begin{aligned}\mathcal{F}(u,U;A) \quad := \quad & \int_A f(x,u(x),\nabla u(x),\nabla^2 u(x),U)\,dx \\ & + \int_{S(\nabla u)\cap A} g(x,u(x),\nabla u^+(x),\nabla u^-(x),\nu_{\nabla u}(x))\,d\mathcal{H}^{N-1}(x),\end{aligned}$$

for all $u \in SBH(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$.

Proof. We note that by Lemmas 3.19, 3.20 and 3.21, the functional $\mathcal{F}$ satisfies the hypotheses of Theorem 3.18, and so the integral representation result follows immediately. □

3.4.4 Applications (SBH, BH integral representation)

In this section we obtain integral representation results for abstract lower semicontinuous functionals on SBH and BH. Consider a functional

$$\mathcal{F} : BH(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty] \tag{3.103}$$

satisfying the following hypotheses:

(K1) $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure,
(K2) $\mathcal{F}(\cdot;A)$ is $L^1(A,\mathbb{R}^d)$-lower semicontinuous,
(K3) $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$ if $u = v$ $\mathcal{L}^N$ a.e. in A then $\mathcal{F}(u;A) = \mathcal{F}(v;A)$,
(K4) there exists a constant $C > 0$ such that

$$\frac{1}{C}|D^2u|(A) \le \mathcal{F}(u;A) \le C(\mathcal{L}^N(A) + |D^2u|(A)).$$

Given $(u,A) \in BH(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)$ we introduce

$$\mathfrak{A}(u;A) := \left\{ v \in BH(\Omega;\mathbb{R}^d) : \text{spt } (u-v) \subset\subset A \right\}, \tag{3.104}$$

and

$$m(u;A) := \inf\{\mathcal{F}(v,A) : v \in \mathfrak{A}(u;A)\}. \tag{3.105}$$

As a corollary of Theorem 3.18, we have the following SBH representation theorem.

Theorem 3.23. *Under hypotheses (K1), (K2), (K3) and (K4), for every $u \in SBH(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u; A) = \int_A f(x, u, \nabla u, \nabla^2 u)\, dx + \int_{S(\nabla u)\cap A} h(x, u, \nabla u^+, \nabla u^-, \nu_{\nabla u})\, d\mathcal{H}^{N-1},$$

where

$$f(x_0, g, G, \Sigma) := \lim_{\varepsilon\to 0^+} \frac{m(g + G(\cdot - x_0) + 1/2\Sigma(\cdot - x_0, \cdot - x_0); Q(x_0, \varepsilon))}{\varepsilon^N},$$

$$h(x_0, g, L, H, \nu) := \lim_{\varepsilon\to 0^+} \frac{m(g + u_{L,H,\nu}(\cdot - x_0); Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}},$$

for all $x_0 \in \Omega, g \in \mathbb{R}^d, G, H, L \in \mathbb{R}^{d\times N}, \Sigma \in \mathbb{R}^{d\times N\times N}, \nu \in S^{N-1}$, and where

$$u_{L,H,\nu}(y) := \begin{cases} Ly & \text{if } y\cdot\nu > 0, \\ Hy & \text{otherwise.} \end{cases}$$

In the case where the functional $\mathcal{F}$ is invariant under affine translations of u, we can leverage this result to upper bound $\mathcal{F}$ on the space BH.

Corollary 3.24. *Let $\mathcal{F}$ satisfy hypotheses $(K1), (K2), (K3), (K4)$, and further assume that for every affine function*

$$v(x) := p + Ax$$

for $p \in \mathbb{R}^d, A \in \mathbb{R}^{d\times N}$, we have

$$\mathcal{F}(u; \cdot) = \mathcal{F}(u + v; \cdot).$$

Then for every $u \in SBH(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}(u; A) = \int_A f(x, \nabla^2 u)\, dx + \int_{S(\nabla u)\cap A} h(x, \nabla u^+ - \nabla u^-, \nu_{\nabla u})\, d\mathcal{H}^{N-1}$$

where, with an abuse of notation, we write $f(x, \Sigma) := f(x, 0, 0, \Sigma)$ and $h(x, J, \nu) := h(x, 0, 0, J, \nu)$. Moreover, for $u \in BH(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}(u, A) \le \int_A f(x, \nabla^2 u)\, dx + \int_A f^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right) d|D_s(\nabla u)|$$

where f^∞ is the recession function defined by

$$f^\infty(x, \Sigma) = \lim_{t\to\infty} \frac{f(x, t\Sigma)}{t}.$$

Proof. The assumption that $\mathcal{F}$ is affine invariant implies that m is also affine invariant. Thus for any $x_0 \in \Omega$, $g \in \mathbb{R}^d$, $G \in \mathbb{R}^{d\times N}$, $\Sigma \in \mathbb{R}^{d\times N\times N}$, $\nu \in S^{N-1}$, we have

$$f(x_0, g, G, \Sigma) = f(x_0, 0, 0, \Sigma)$$

and for any $x_0 \in \Omega, g \in \mathbb{R}^d, L, H \in \mathbb{R}^{d\times N}, \nu \in S^{N-1}$ we have

$$h(x_0, g, L, H, \nu) = g(x_0, 0, 0, H - L, \nu).$$

In particular, we deduce that for every $u \in W^{2,1}(\Omega; \mathbb{R}^d)$

$$\mathcal{F}(u; A) = \int_A f(x, \nabla^2 u)\; dx.$$

The relaxation of such functionals to BH was the subject of Section 3.3, where we get an integral representation of the relaxation, to be precise

$$\begin{aligned} \inf &\left\{ \liminf_{n\to\infty} \mathcal{F}(u_n; A) :,\; u_n \in W^{2,1}(\Omega;\mathbb{R}^d), u_n \to u, \sup_n \|u_n\|_{W^{2,1}} < \infty \right\} \\ &= \int_A \mathcal{Q}_2 f(x, \nabla^2 u)\, dx + \int_A (\mathcal{Q}_2 f)^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)} \right) d|D_s(\nabla u)| \end{aligned} \tag{3.106}$$

for every $u \in BH(\Omega; \mathbb{R}^d)$, $A \in \mathcal{A}(\Omega)$, where $\mathcal{Q}_2 f$ is the 2-quasiconvex envelope of f. In this case, since $\mathcal{F}$ is lower semicontinuous, we must have that f is 2-quasiconvex as shown in [9], [46], [61], , and thus $\mathcal{Q}_2 f = f$. Thus for every $u \in BH(\Omega; \mathbb{R}^d)$ we may take a recovery sequence for the relaxation $u_n \in W^{2,1}(\Omega; \mathbb{R}^d)$ such that $u_n \to u$ in L^1 and

$$\lim_{n\to\infty} \mathcal{F}(u_n; A) = \int_A f(x, \nabla^2 u)\, dx + \int_A f^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)} \right) d|D_s(\nabla u)|$$

to conclude from lower semicontinuity of $\mathcal{F}$ that

$$\mathcal{F}(u; A) \le \int_A f(x, \nabla^2 u)\, dx + \int_A f^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)} \right) d|D_s(\nabla u)|.$$

$\square$

We can push this further under a stronger continuity assumption on $\mathcal{F}$. If $\mathcal{F}$ is *continuous* with respect to *area-strict* convergence, (see Definition 2.20), then this upper bound is actually sharp.

This condition is very natural for BH lower semicontinuous integral functionals. Indeed, Theorem 2.22 shows that 2-quasiconvex potentials along with their recession function of the form 3.106 are automatically area-strict continuous. In the first order global method result [19], the area-strict continuity assumption is not needed, but once we have the integral representation, we can see that it is automatically area-strict continuous by [56]. Thus in the first order case an assumption of area-strict continuity is innocuous, which motivates our assumption here. With the assumption of area-strict continuity, we have the following:

Corollary 3.25. *Let $\mathcal{F}$ satisfy hypotheses $(K1), (K2), (K3), (K4)$, and further assume that for every affine function*

$$v(x) := p + Ax$$

for $p \in \mathbb{R}^d, A \in \mathbb{R}^{d\times N}$, we have

$$\mathcal{F}(u; \cdot) = \mathcal{F}(u + v; \cdot)$$

and that for every $u \in BH(\Omega; \mathbb{R}^d)$ and every sequence $\{u_n\} \subset BH(\Omega; \mathbb{R}^d)$ so that $u_n \to u$ in L^1 and $D(\nabla u_n) \to D(\nabla u)$ area-strictly,

$$\lim_{n\to\infty} \mathcal{F}(u_n; \Omega) = \mathcal{F}(u; \Omega).$$

Then for every $u \in BH(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}(u, A) = \int_A f(x, \nabla^2 u)\, dx + \int_A f^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)} \right) d|D_s(\nabla u)|.$$

Proof. Following the proof of Corollary 3.25, (K1), (K2), (K3), (K4) and the affine invariance property give us a representation of $\mathcal{F}$ on $W^{2,1}(\Omega; \mathbb{R}^d)$. For any $u \in BH$, we can use Corollary 3.7 to construct a sequence $u_n \in W^{2,1}(\Omega; \mathbb{R}^d)$ so that $u_n \to u$ in L^1 and $D(\nabla u_n) \to D(\nabla u)$ area-strictly. Thus, by area-strict continuity, we have

$$\mathcal{F}(u; \Omega) = \lim_{n\to\infty} \mathcal{F}(u_n; \Omega). \tag{3.107}$$

On the other hand, the functional

$$u \in BH(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(x, \nabla^2 u)\, dx + \int_\Omega f^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)} \right) d|D_s(\nabla u)| =: I(u; \Omega)$$

is area-strict continuous on BH by Theorem 2.22 and agrees with $\mathcal{F}$ on $W^{2,1}$, therefore

$$\lim_{n\to\infty} \mathcal{F}(u_n; \Omega) = \lim_{n\to\infty} I(u_n; \Omega) = I(u; \Omega).$$

This, together with (3.107) yields

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x, \nabla^2 u)\, dx + \int_\Omega f^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right) d|D_s(\nabla u)|$$

for every $u \in BH(\Omega;\mathbb{R}^d)$.

4 Phase Transitions and Homogenization

4.1 Statement of main results

Consider a double well potential $W : \mathbb{R}^N \times \mathbb{R}^d \to [0,\infty)$ satisfying the following properties:

(G0) $x \mapsto W(x,p)$ is Q-periodic for all $p \in \mathbb{R}^d$,

(G1) W is a Carathéodory function, i.e.,

- (i) for all $p \in \mathbb{R}^d$ the function $x \mapsto W(x,p)$ is measurable,
- (ii) for a.e. $x \in Q$ the function $p \mapsto W(x,p)$ is continuous,

(G2) there exist $a, b \in \mathbb{R}^d$ such that $W(x,p) = 0$ if and only if $p \in \{a,b\}$, for a.e. $x \in Q$,

(G3) there exists a continuous function $W_c : \mathbb{R}^d \to [0,\infty)$ such that $W_c(p) \leq W(x,p)$ for a.e. $x \in Q$ and $W_c(p) = 0$ if and only if $p \in \{a,b\}$.

(G4) there exist $C > 0$ and $q \geq 2$ such that $\frac{1}{C}|p|^q - C \leq W(x,p) \leq C(1+|p|^q)$ for a.e. $x \in Q$ and all $p \in \mathbb{R}^d$.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary.

Definition 4.1. For $\varepsilon, \delta > 0$ we define the potential $\mathcal{F}_{\varepsilon,\delta} : H^1(\Omega;\mathbb{R}^d) \to [0,\infty)$ by

$$\mathcal{F}_{\varepsilon,\delta}(u) := \int_\Omega \left[\frac{1}{\delta} W\left(\frac{x}{\varepsilon}, u(x)\right) + \delta |\nabla u(x)|^2 \right] \, \mathrm{d}x.$$

Remark 4.2. Hypotheses (G1), (G2) (G3) and (G4) conform with the prototypical potential

$$W(x,p) := \sum_{i=1}^k \chi_{E_i}(x) W_i(p)\,,$$

where $E_i \subset Q$ are measurable pairwise disjoint sets with $Q = \cup_{i=1}^k E_i$, and $W_i : \mathbb{R}^d \to [0,\infty)$ are continuous functions with quadratic growth at infinity and such that $W_i(p) = 0$ if and only if $p \in \{a,b\}$, modeling the case of a heterogeneous mixture composed of k different compositions. Here $\widetilde{W}$ in (G3) may be taken as $\widetilde{W} := \min\{W_1, \ldots, W_k\}$.

4.1.1 The case $\varepsilon << \delta$

In the case where $\varepsilon << \delta$, the homogenization effects occur so rapidly that the system is essentially homogenized before interacting with the phase transition problem. In this case, we prove that the Γ-limit of $\mathcal{F}_{\varepsilon,\delta}$ coincides with the interfacial energy associated with a homogenized potential. In this regime, we require an additional regularity assumption on W:

(G5) W is locally Lipschitz in p, that is, for every $K \subset \mathbb{R}^d$ compact there is a constant L such that

$$|W(x,p) - W(x,q)| \leq L|p-q|$$

for almost every $x \in Q$ and every $p, q \in K$.

Definition 4.3. We define the functional $F_0^H : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ as

$$F_0^H(u) := \begin{cases} K_H \mathcal{P}(\{u=a\};\Omega) & \text{if } u \in BV(\Omega;\{a,b\}), \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

Here the transition energy density K_H is defined as

$$K_H := 2 \inf \left\{ \int_0^1 \sqrt{W_H(g(s))}|g'(s)| \mathrm{d}s : g \in C^1_{pw}([0,1];\mathbb{R}^d;a,b) \right\}, \tag{4.2}$$

where $C^1_{pw}([0,1];\mathbb{R}^d;a,b)$ denotes the space of piecewise C^1 curves from $[0,1]$ to $\mathbb{R}^d$ such that $g(0) = a$ and $g(1) = b$, and the homogenized potential $W_H : \mathbb{R}^d \to [0,+\infty)$ is given by

$$W_H(p) := \int_Q W(y,p) \,\mathrm{d}y \tag{4.3}$$

When the length scale of homogenization is sufficiently small with respect to the transition thickness, we have the following result.

Theorem 4.4. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ $\{\delta_n\}_{n\in\mathbb{N}}$ be two infinitesimal sequences such that*

$$\lim_{n\to\infty} \frac{\delta_n^{\frac{3}{2}}}{\varepsilon_n} \to +\infty.$$

Set $F_n := \mathcal{F}_{\varepsilon_n,\delta_n}$. Assume that W satisfies hypotheses (G0)-(G4). Then the following hold:

1. *If $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ is such that*

$$\sup_{n\in\mathbb{N}} F_n(u_n) < +\infty,$$

then, up to a subsequence (not relabeled), we have $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ for some $u \in BV(\Omega;\{a,b\})$.

2. As $n \to \infty$, we have $F_n \overset{\Gamma - L^1}{\longrightarrow} F_0^H$.

4.1.2 The case $\delta = \varepsilon$

For $\varepsilon > 0$ consider the energy $\mathcal{F}_\varepsilon : H^1(\Omega;\mathbb{R}^d) \to [0,\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) := \mathcal{F}_{\varepsilon,\varepsilon}(u) = \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon |\nabla u(x)|^2 \right] \mathrm{d}x \,, \tag{4.4}$$

We introduce some definitions. For $\nu \in \mathbb{S}^{N-1}$, with $\mathbb{S}^{N-1}$ the unit sphere of $\mathbb{R}^N$, we denote by $\mathcal{Q}_\nu$ the family of cubes Q_ν centered at the origin with two faces orthogonal to ν and with unit length sides.

Definition 4.5. Let $\nu \in \mathbb{S}^{N-1}$ and define the function $u_{0,\nu} : \mathbb{R}^N \to \mathbb{R}^d$ as

$$u_{0,\nu}(y) := \begin{cases} a & \text{if } y \cdot \nu \leq 0 \,, \\ b & \text{if } y \cdot \nu > 0 \,. \end{cases} \tag{4.5}$$

Fix a function $\rho \in C_c^\infty(B(0,1))$ with $\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x = 1$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$. For $T > 0$, set $\rho_T(x) := T^N \rho(Tx)$ and

$$\widetilde{u}_{\rho,T,\nu} := \rho_T * u_{0,\nu} \,. \tag{4.6}$$

When it is clear from the context, we will abbreviate $\widetilde{u}_{\rho,T,\nu}$ as $\widetilde{u}_{T,\nu}$.

Definition 4.6. We define the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ as

$$\sigma(\nu) := \lim_{T\to\infty} g(\nu, T) \,,$$

where

$$g(\nu, T) := \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u|^2 \right] \mathrm{d}y \,:\, Q_\nu \in \mathcal{Q}_\nu,\, u \in \mathcal{C}(\rho, Q_\nu, T) \right\},$$

and

$$\mathcal{C}(\rho, Q_\nu, T) := \left\{ u \in H^1(TQ_\nu;\mathbb{R}^d) \,:\, u = \widetilde{u}_{\rho,T,\nu} \text{ on } \partial\,(TQ_\nu) \right\}.$$

Just as before, if there is no possibility of confusion, we will write $\mathcal{C}(\rho, Q_\nu, T)$ as $\mathcal{C}(Q_\nu, T)$.

Remark 4.7. For every $\nu \in \mathbb{S}^{N-1}$, $\sigma(\nu)$ is well defined and finite (see Lemma 4.23) and its definition does not depend on the choice of the mollifier ρ (see Lemma 4.25). Moreover, the function $\nu \mapsto \sigma(\nu)$ is upper semi-continuous on $\mathbb{S}^{N-1}$ (see Proposition 4.26).

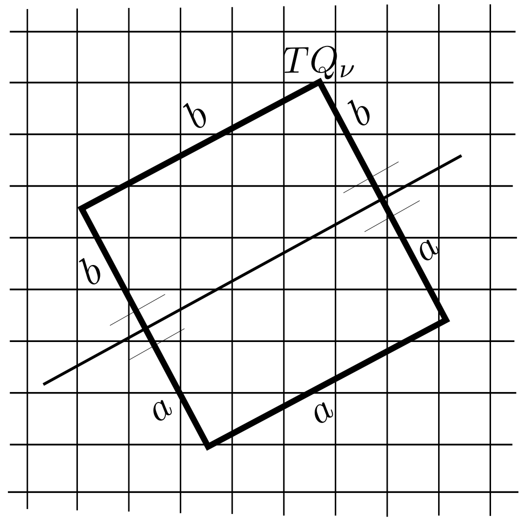

Figure 1: The misalignment between a square Q_ν with two faces orthogonal to ν and the directions of periodicity of W (the grid in the picture) is the reason for the anisotropy character of the limiting surface energy.

Using [19], it is possible to prove that the infimum in the definition of $g(\nu, T)$ may be taken with respect to one fixed cube $Q_\nu \in \mathcal{Q}_\nu$. Namely, given $\nu \in \mathbb{S}^{N-1}$ and $Q_\nu \in \mathcal{Q}_\nu$ it holds

$$\sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \left[W(y, u(y)) + |Du|^2\right] \mathrm{d}y \,:\, u \in \mathcal{C}(Q_\nu, T) \right\}.$$

Remark 4.8. In the context of homogenization when dealing with nonconvex potentials W it is natural to consider, in the cell problem for the limiting density function σ, the infimum over all possible cubes TQ_ν. For instance, this was observed by Müller in [64], where the asymptotic behavior as $\varepsilon \to \infty$ of the family of functionals

$$G_\varepsilon(u) := \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u\right)\, dx,$$

defined for $u \in H^1(\Omega; \mathbb{R}^d)$, is studied. The limiting energy is of the form

$$\int_\Omega \overline{W}(\nabla u(x))\, \mathrm{d}x,$$

with

$$\overline{W}(\lambda) := \inf_{k\in\mathbb{N}} \inf_{\psi\in W_0^{1,p}(kQ)} \frac{1}{k^N} \int_{kQ} W(y, \lambda + \nabla\psi(y))\, \mathrm{d}y.$$

In the case where W is convex, the infimum over $k \in \mathbb{N}$ is not needed (see [60]).

Consider the functional $\mathcal{F}_0 : L^1(\Omega;\mathbb{R}^d) \to [0,\infty]$ defined by

$$\mathcal{F}_0(u) := \begin{cases} \displaystyle\int_{\partial^* A} \sigma(\nu_A(x))\,\mathrm{d}\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega;\{a,b\}), \\ +\infty & \text{else,} \end{cases} \tag{4.7}$$

where $A := \{u = a\}$ and $\nu_A(x)$ denotes the measure theoretic external unit normal to the reduced boundary $\partial^* A$ of A at x (see Definition 2.9).

We now state the main Γ-convergence result in the case $\varepsilon = \delta$.

Theorem 4.9. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. Assume that (G0), (G1), (G2), (G3) and (G4) hold.*

(i) If $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ is such that

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\varepsilon_n}(u_n) < +\infty$$

then, up to a subsequence (not relabeled), $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, where $u \in BV(\Omega;\{a,b\})$,

(ii) As $n \to \infty$, it holds $\mathcal{F}_{\varepsilon_n} \stackrel{\Gamma - L^1}{\longrightarrow} \mathcal{F}_0$.

Moreover, the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ is continuous.

The proof of the Theorem 4.9 will be divided into several parts. We would like to briefly comment on the main ideas we will use.

After establishing auxiliary technical results in Section 4.3.1, we will prove the compactness result of Theorem 4.9 (i) (see Proposition 4.29) by reducing our functional to the standard Cahn-Hilliard energy (1.3).

In Section 4.3.4 we will obtain the liminf inequality by using the blow-up method introduced by Fonseca and Müller in [44] (see also [45]). Although this strategy can nowadays can be considered standard, for clarity and completeness we include the argument.

The limsup inequality is presented in Section 4.3.5 and requires new geometric ideas. This is due to the fact that the periodicity of W in the first variable is an essential ingredient to build a recovery sequence. It turns out (see Proposition 4.19) that there exists a dense set $\Lambda \subset \mathbb{S}^{N-1}$ such that, for every $v_1 \in \Lambda$ there exists $T_{v_1} \in \mathbb{N}$ and $v_2,\dots,v_N \in \Lambda$ for which $W(x + T_{v_1}v_i, p) = W(x,p)$ for a.e. $x \in \Omega$, all $p \in \mathbb{R}^N$ and all $i = 1,\dots,N$, and such that $\{v_1,\dots,v_N\}$ is an orthonormal basis of $\mathbb{R}^N$. Using this fact, in the first step of the proof of Proposition 4.31 we obtain a recovery sequence for the special class of functions

$u \in BV(\Omega;\{a,b\})$ for which the normals to the interface $\partial^* A$, where $A := \{u = a\}$, belong to Λ. We decided to construct a recovery sequence only locally, in order to avoid the technical problem of gluing together optimal profiles for different normal directions to the transition layer. For this reason, we first prove that the localized version of the Γ-limit is a Radon measure absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner \partial^* A$, and then we show that its density, identified using cubes whose faces are orthogonal to elements of Λ, is bounded above by σ. Finally, in the second step we conclude using a density argument that will invoke Reshetnyak's upper semi-continuity theorem (see Theorem 2.12) and the upper semi-continuity of σ (see Proposition 4.26).

4.2 The case $\varepsilon << \delta$

We proceed to prove the Γ convergence result in the case where the homogenization occurs at a much smaller scale than the phase transition. To be precise, we consider the scaling

$$\frac{\varepsilon}{\delta^{\frac{3}{2}}} \to 0$$

Remark 4.10. The reason that this scaling is necessary as opposed to the more general case without a factor of $\frac{3}{2}$ is not yet clear. Indeed, if one could show that a sequence $\{u_n\}$ with bounded energy satisfies

$$\lim_{n\to\infty} F_n(u_n; Q \setminus \{|x_N| > \delta\}) = 0$$

then Theorem 4.4 would follow in the more general scaling $\varepsilon << \delta$.

First, in order to rule out possible pathological behavior corresponding to large values of u, we will introduce a truncated potential $\widetilde{W}$.

Definition 4.11. Let $R > 0$ be given such that every minimizing curve $g \in C^1_{pw}([0,1];\mathbb{R}^d;a,b)$ for the minimization problem defining K_H (see (4.2)) is such that $|g(t)| \leq R$ for every $t \in [-1,1]$. Let

$$M := \operatorname*{ess\,sup}_{x\in\Omega} \max_{|p|\leq R} W(x,p),$$

and define the *truncated potential* $\widetilde{W} : \Omega \times \mathbb{R}^d \to [0,\infty)$ as

$$\widetilde{W}(x,p) := \min\{W(x,p), M\}.$$

Remark 4.12. The truncated potential $\widetilde{W}$ is Lipschitz (not only locally) in p, uniformly in x. Moreover, note that $0 < M < +\infty$ by the upper bound given by (G4).

The proof of Theorem 4.4 is based on a convergence result (Lemma 4.14) stating that in the functional F_n it is possible to *substitute* the (truncated) energy with the homogenized energy, which leads to the following definition.

Definition 4.13. For a sequence $\delta_n \to 0$, we define the *homogenized energy* $F_n^H : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ by

$$F_n^H(u) = \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{\delta_n} \widetilde{W}_H(u(x)) + \delta_n |\nabla u(x)|^2 \right] \mathrm{d}x & u \in H^1(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widetilde{W}_H$ is defined as

$$\widetilde{W}_H(p) := \int_Q \widetilde{W}(y,p)\, \mathrm{d}y.$$

Note that this definition of $\widetilde{W}_H$ coincides with that in (4.3).

4.2.1 A homogenization lemma

We prove that as $n \to \infty$, the limiting behavior of F_n^H captures the limiting behavior the truncated problem.

Lemma 4.14. *Let $\{\delta_n\}, \{\varepsilon_n\}$ be sequences converging to 0 and let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ be such that*

$$\sup_{n\in\mathbb{N}} \int_\Omega \delta_n |\nabla u_n|^2 \,\mathrm{d}x < \infty. \tag{4.8}$$

Then

$$\lim_{n\to\infty} \left| \frac{1}{\delta_n} \int_\Omega \left[\widetilde{W}\left(\frac{x}{\varepsilon_n}, u_n(x)\right) - \widetilde{W}_H(u_n(x)) \right] \mathrm{d}x \right| = 0.$$

Proof. Let

$$T := \sup_{n\in\mathbb{N}} \int_\Omega \delta_n |\nabla u_n|^2 \,\mathrm{d}x < \infty. \tag{4.9}$$

Write

$$\Omega = \bigcup_{i=1}^{M_n} Q(p_i,\varepsilon_n) \cup R_n,$$

where $p_i \in \varepsilon_n\mathbb{Z}^N$, R_n is the set of cubes $Q(z,\varepsilon_n)$ with $z \in \varepsilon_n\mathbb{Z}^N$ such that $Q(z,\varepsilon_n) \cap \partial\Omega \neq \emptyset$, and $M_n \in \mathbb{N}$. Note that

$$\begin{aligned}
&\left|\frac{1}{\delta_n}\int_{\bigcup_{i=1}^{M_n} Q(p_i,\varepsilon_n)} \left(\widetilde{W}\left(\frac{x}{\varepsilon_n}, u_n\right) - \widetilde{W}_H(u_n)\right)\, \mathrm{d}x\right| \\
&\qquad\qquad \leq \frac{1}{\delta_n}\sum_{i=1}^{M_n}\left|\int_{Q(p_i,\varepsilon_n)} \left(\widetilde{W}\left(\frac{x}{\varepsilon_n}, u_n\right) - \widetilde{W}_H(u_n)\right)\, \mathrm{d}x\right| \\
&\qquad\qquad = \frac{\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M}\left|\int_{Q} \left(\widetilde{W}(y, u_n(p_i+\varepsilon_n y)) - \widetilde{W}_H(u_n(p_i+\varepsilon_n y))\right)\, \mathrm{d}y\right|, \qquad (4.10)
\end{aligned}$$

where in the last step we have used the change of variables $y := \frac{x-p_i}{\varepsilon_n}$ and we used the fact that $\widetilde{W}\left(y - \frac{p_i}{\varepsilon_n}, \cdot\right) = \widetilde{W}(y,\cdot)$ by periodicity. From here, we can rewrite (4.10) as

$$\begin{aligned}
&\frac{\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M_n}\left|\int_Q\int_Q \left(\widetilde{W}(y, u_n(p_i+\varepsilon_n y)) - \widetilde{W}(z, u_n(p_i+\varepsilon_n y))\right) \mathrm{d}z\, \mathrm{d}y\right| \\
&\quad = \frac{\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M_n}\left|\int_Q\int_Q \left(\widetilde{W}(y, u_n(p_i+\varepsilon_n y)) - \widetilde{W}(y, u_n(p_i+\varepsilon_n z))\right) \mathrm{d}z\, \mathrm{d}y\right| \\
&\quad \leq \frac{\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M_n}\int_Q\int_Q \left|\widetilde{W}(y, u_n(p_i+\varepsilon_n y)) - \widetilde{W}(y, u_n(p_i+\varepsilon_n z))\right| \mathrm{d}z\, \mathrm{d}y \\
&\quad \leq \frac{L\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M_n}\int_Q\int_Q |u_n(p_i+\varepsilon_n y)) - u_n(p_i+\varepsilon_n z))|\, \mathrm{d}z\, \mathrm{d}y \\
&\quad \leq \frac{L\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M_n}\left(\int_Q\int_Q |u_n(p_i+\varepsilon_n y) - \overline{u}_{i,n}|\, \mathrm{d}z\, \mathrm{d}y + \int_Q\int_Q |\overline{u}_{i,n} - u_n(p_i+\varepsilon_n z)|\, \mathrm{d}z\, \mathrm{d}y\right),
\end{aligned} \tag{4.11}$$

where in the second to last step $L > 0$ is the Lipschitz constant of $\widetilde{W}$, and we define

$$\overline{u}_{i,n} := \int_Q u_n(p_i+\varepsilon_n z)\mathrm{d}z.$$

By symmetry, the last term in (4.11) can be written as

$$\frac{2L\varepsilon_n^N}{\delta_n}\sum_{i=1}^{M_n}\int_Q\int_Q |u_n(p_i+\varepsilon_n y) - \overline{u}_{i,n}|\, \mathrm{d}z\, \mathrm{d}y = \frac{2L}{\delta_n}\sum_{i=1}^{M_n}\int_Q |u_n(p_i+\varepsilon_n y) - \overline{u}_{i,n}|\, \mathrm{d}y.$$

By the Poincaré inequality and (4.9), we have

$$\begin{aligned}\frac{2L}{\delta_n}\sum_{i=1}^{M_n}\int_Q |u_n(p_i+\varepsilon_n y)-\overline{u}_{i,n}|\,\mathrm{d}y &\leq \frac{C\varepsilon_n^{N+1}}{\delta_n}\sum_{i=1}^{M_n}\int_Q |\nabla u_n(p_i+\varepsilon_n y)|\,\mathrm{d}y\\ &= \frac{C\varepsilon_n}{\delta_n}\sum_{i=1}^{M_n}\int_{Q(p_i,\varepsilon_n)}|\nabla u_n(x)|\,\mathrm{d}x\\ &\leq \frac{C\varepsilon_n}{\delta_n}\int_\Omega |\nabla u_n|\,\mathrm{d}x\\ &\leq \frac{C\varepsilon_n}{\delta_n}|\Omega|^{\frac{1}{2}}\left(\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &= \frac{C\varepsilon_n}{\delta_n^{\frac{3}{2}}}\left(\delta_n\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq \frac{C\varepsilon_n}{\delta_n^{\frac{3}{2}}}T^{\frac{1}{2}}.\end{aligned} \tag{4.12}$$

Using (4.10), (4.11) and (4.12), we conclude that

$$\lim_{n\to\infty}\left|\frac{1}{\delta_n}\int_{\bigcup_{i=1}^{M_n}Q(p_i,\varepsilon_n)}\left(\widetilde{W}\left(\frac{x}{\varepsilon_n},u_n\right)-\widetilde{W}_H(u_n)\right)\,\mathrm{d}x\right| = 0. \tag{4.13}$$

Noticing that $\widetilde{W}$ and $\widetilde{W_H}$ are bounded and $|R_n|\leq C\varepsilon_n$ we get

$$\lim_{n\to\infty}\left|\frac{1}{\delta_n}\int_{R_n}\left(\widetilde{W}\left(\frac{x}{\varepsilon_n},u_n\right)-\widetilde{W}_H(u_n)\right)\,\mathrm{d}x\right| = 0. \tag{4.14}$$

Thus, from (4.13) and (4.14) we conclude. □

4.2.2 The Γ-convergence result

With Lemma 4.14, we may proceed to prove the Γ-convergence result stated in Theorem 4.4.

Proof of Theorem 4.4. Step 1: Compactness. Let $\{u_n\}_{n\in\mathbb{N}}\subset H^1(\Omega;\mathbb{R}^d)$ be a sequence such that

$$\sup_{n\in\mathbb{N}} F_n(u_n) < +\infty.$$

Then we have

$$\sup_{n\in\mathbb{N}}\int_\Omega \delta_n|\nabla u_n|^2\,\mathrm{d}x < \infty,$$

and thus, since $\widetilde{W} \leq W$, we can apply Lemma 4.14 to conclude

$$\sup_{n\in\mathbb{N}} F_n^H(u_n) < +\infty.$$

Invoking classical results (see, for instance, [48, Theorem 4.1]) we get that, up to a subsequence (not relabeled) $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ for some $u \in BV(\Omega;\{a,b\})$.

Step 2: Liminf inequality. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ and $u \in BV(\Omega;\{a,b\})$ be such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. In order to prove that

$$F_0^H(u) \leq \liminf_{n\to\infty} F_n(u_n),$$

without loss of generality we restrict ourselves to the case in which

$$\lim_{n\to\infty} F_n(u_n) = \liminf_{n\to\infty} F_n(u_n) < +\infty. \tag{4.15}$$

Using Step 1, we get that $u \in BV(\Omega;\{a,b\})$. Moreover, noticing that by definition of M (see Definition 4.11) we have

$$K_H = 2\inf\left\{\int_0^1 \sqrt{\widetilde{W}_H(g(s))}|g'(s)|ds : g \in C^1_{pw}([0,1];\mathbb{R}^d;a,b)\right\}, \tag{4.16}$$

where $\widetilde{W}$ is defined in (4.11). Using standard results (see, for instance, [48, Theorem 3.4]), we obtain

$$F_0^H(u) \leq \liminf_{n\to\infty} F_n^H(u_n) \leq \liminf_{n\in\mathbb{N}} F_n(u_n),$$

where in the last step we used Lemma 4.14 noting that (4.15) yields the validity of (4.8).

Step 3: Limsup inequality. Let $u \in BV(\Omega;\{a,b\})$. We want to find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ such that

$$F_0^H(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

Since F_0^H is the Γ-limit of F_n^H (again, because the constant K_H is the same regardless of truncation) we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ such that

$$F_0^H(u) \geq \limsup_{n\to\infty} F_n^H(u_n).$$

Moreover, by our choice of truncation, $|u_n| \leq R$, so that $W(x, u_n(x)) = \widetilde{W}(x, u_n(x))$ for a.e. $x \in \Omega$. Note that

$$\sup_{n\in\mathbb{N}} \int_\Omega \delta_n |\nabla u_n|^2 \,\mathrm{d}x < +\infty$$

and thus, we can apply Lemma 4.14 to conclude

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n) = \limsup_{n\to\infty} F^H_{\varepsilon_n}(u_n)$$

In particular, we have

$$F^H_0(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

□

4.3 The case $\delta = \varepsilon$

Before proving the Γ-convergence result in the case where $\delta = \varepsilon$, we must establish preliminary technical results.

4.3.1 Some technical results

The first result relies on De Giorgi's slicing method (see [32]), and it allows to adjust the boundary conditions of a given sequence of functions without increasing the energy, by carefully selecting where to make the transition from the given function to one with the right boundary conditions. Although the argument is nowadays considered to be standard, we include it here for the convenience of the reader.

For $\varepsilon > 0$, we localize the functional $\mathcal{F}_\varepsilon$ by setting

$$\mathcal{F}_\varepsilon(u, A) := \int_A \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon |Du(x)|^2 \right] \mathrm{d}x\,,$$

where $A \in \mathcal{A}(\Omega)$ and $u \in H^1(A; \mathbb{R}^d)$. Also, for $j \in \mathbb{N}$, we define

$$A^{(j)} := \{x \in A \,:\, \mathrm{d}(x, \partial A) < 1/j\}\,.$$

Lemma 4.15. *Let $D \in \mathcal{A}(\Omega)$ be a cube with $0 \in D$ and let $\nu \in \mathbb{S}^{N-1}$. Let $\{D_k\}_{k\in\mathbb{N}} \subset \mathcal{A}(\Omega)$ with $D_k \subset D$ be cubes, let $\{\eta_k\}_{k\in\mathbb{N}}$ with $\eta_k \to 0$ as $k \to \infty$, and let $u_k \in H^1(D_k; \mathbb{R}^d)$, with $k \in \mathbb{N}$, satisfy*

(i) $\chi_{D_k} \to \chi_D$ *in* $L^1(\mathbb{R}^N)$,

(ii) $u_k \chi_{D_k} \to u_{0,\nu}$ *in* $L^1(D; \mathbb{R}^d)$,

(iii) $\sup_{k\in\mathbb{N}} \mathcal{F}_{\eta_k}(u_k, D_k) < \infty$.

Let $\rho \in C_c^\infty(B(0,1))$ with $\int_{\mathbb{R}^N} \rho(x)\mathrm{d}x = 1$. Then there exists a sequence $\{w_k\}_{k\in\mathbb{N}} \subset H^1(D;\mathbb{R}^d)$, with $w_k = \widetilde{u}_{\rho,1/\eta_k,\nu}$ in $D_k^{(j_k)}$, where $\widetilde{u}_{\rho,1/\eta_k,\nu}$ is defined as in (4.6), for some $\{j_k\}_{k\in\mathbb{N}}$ with $j_k \to \infty$ as $k \to \infty$, such that

$$\liminf_{k\to\infty} \mathcal{F}_{\eta_k}(u_k, D_k) \geq \limsup_{k\to\infty} \mathcal{F}_{\eta_k}(w_k, D)\,.$$

Moreover, $w_k \to u_{0,\nu}$ in $L^q(D;\mathbb{R}^d)$ as $k \to \infty$, where $q \geq 2$ is as in (G4).

Proof. Assume, without loss of generality, that

$$\liminf_{k\to\infty} \mathcal{F}_{\eta_k}(u_k, D_k) = \lim_{k\to\infty} \mathcal{F}_{\eta_k}(u_k, D_k) < +\infty \tag{4.17}$$

and that, as $n \to \infty$, $u_n(x)\chi_{D_k}(x) \to u_{0,\nu}(x)$ for a.e. $x \in D$.

Step 1. We claim that

$$\lim_{k\to\infty} \|u_k - u_{0,\nu}\|_{L^q(D_k;\mathbb{R}^d)} = 0. \tag{4.18}$$

Indeed, using (G4), we get

$$|u_k(x) - u_{0,\nu}(x)|^q \leq C\left(W\left(\frac{x}{\eta_k}, u_k(x)\right) + 1\right), \tag{4.19}$$

for $x \in D_k$. From (4.17) we have $\chi_{D_k}(x)W(\frac{x}{\eta_k}, u_k(x)) \to 0$ as $k \to \infty$ for a.e. $x \in D$, and thus

$$\begin{aligned}
C|D| &- \limsup_{k\to\infty} \|u_k - u_{0,\nu}\|^q_{L^q(D_k;\mathbb{R}^d)} \\
&= \liminf_{k\to\infty} \int_{D_k} \left[CW\left(\frac{x}{\eta_k}, u_k(x)\right) + C - |u_k(x) - u_{0,\nu}(x)|^q \right] \mathrm{d}x \\
&\geq \int_D \liminf_{k\to\infty} \chi_{D_k}(x) \left[CW\left(\frac{x}{\eta_k}, u_k(x)\right) + C - |u_k(x) - u_{0,\nu}(x)|^q \right] \mathrm{d}x \\
&\geq C|D|,
\end{aligned}$$

where we used Fatou's lemma and (4.19).

Step 2. Here we abbreviate $\widetilde{u}_{\rho,1/k,\nu}$ as $\widetilde{u}_{1/k,\nu}$. Set $v_k := \widetilde{u}_{1/\eta_k,\nu}$ and $\lambda_k := \|u_k\chi_{D_k} - v_k\|_{L^2(D;\mathbb{R}^d)}$. Using Step 1, since $q \geq 2$ we get $\lim_{k\to\infty} \lambda_k = 0$. For every $k, j \in \mathbb{N}$ divide $D_k^{(j)}$

into $M_{k,j}$ equidistant layers $L^i_{k,j}$ of width $\eta_k\lambda_k$, for $i = 1, \dots, M_{k,j}$. It holds

$$M_{k,j}\eta_k\lambda_k = \frac{1}{j}\,. \tag{4.20}$$

For every $k, j \in \mathbb{N}$ let $L^{i_0}_{k,j}$, with $i_0 \in \{1, \dots, M_{k,j}\}$, be such that

$$\int_{L^{i_0}_{k,j}} A_k(x)\,\mathrm{d}x \le \frac{1}{M_{k,j}} \int_{D^{(j)}_k} A_k(x)\,\mathrm{d}x\,, \tag{4.21}$$

where

$$A_k(x) := \frac{1}{\eta_k}(1 + |u_k - v_k|^q + |v_k|^q) + \frac{1}{\lambda_k^2\eta_k}|u_k(x) - v_k(x)|^2 + \eta_k\left(|\nabla u_k(x)|^2 + |\nabla v_k(x)|^2\right)\,.$$

Further, consider cut-off functions $\varphi_{k,j} \in C^\infty_c(D)$ with

$$0 \le \varphi_{k,j} \le 1\,, \qquad \|\nabla\varphi_{k,j}\| \le \frac{C}{\eta_k\lambda_k}\,, \tag{4.22}$$

such that

$$\varphi_{k,j}(x) = 1\,, \qquad \text{for } x \in \left(\bigcup_{i=1}^{i_0-1} L^i_{k,j}\right) \cup (D_k \setminus D^{(j)}_k)\,, \tag{4.23}$$

$$\varphi_{k,j}(x) = 0\,, \qquad \text{for } x \in \left(\bigcup_{i=i_0+1}^{M_{k,j}} L^i_{k,j}\right) \cup (D \setminus D_k)\,. \tag{4.24}$$

Set

$$\widetilde{w}_{k,j} := \varphi_{k,j}u_k + (1 - \varphi_{k,j})v_k\,.$$

It holds that $\lim_{j\to\infty}\lim_{k\to\infty}\|\widetilde{w}_{k,j} - u_{0,\nu}\|_{L^q(D;\mathbb{R}^d)} = 0$. Let $j_k \in \mathbb{N}$ be such that $D^{(j_k)}_k \subset \bigcup_{i=i_0+1}^{M_{k,j}} L^i_{k_j}$. Then $\widetilde{w}_{k,j} = v_k$ in $D^{(j_k)}_k$. We claim that

$$\liminf_{k\to\infty}\mathcal{F}_{\eta_k}(u_k, D_k) \ge \limsup_{j\to\infty}\limsup_{k\to\infty}\mathcal{F}_{\eta_k}(\widetilde{w}_{k,j}, D)\,. \tag{4.25}$$

Indeed

$$\begin{aligned}\mathcal{F}_{\eta_k}(\widetilde{w}_{k,j}, D_k) = \mathcal{F}_{\eta_k}&\left(u_k, \left(\bigcup_{i=1}^{i_0-1} L^i_{k,j} \right) \cup (D_k \setminus D_k^{(j)}) \right) + \mathcal{F}_{\eta_k}\left(\widetilde{w}_{k,j}, L^{i_0}_{k,j} \right) \\ &+ \mathcal{F}_{\eta_k}\left(v_k, \bigcup_{i=i_0+1}^{M_{k,j}} L^i_{k,j} \right) \\ &=: A_{k,j} + B_{k,j} + C_{k,j}\,. \end{aligned} \tag{4.26}$$

To estimate the first term in (4.26) we notice that

$$\liminf_{k\to\infty} \mathcal{F}_{\eta_k}(u_k, D_k) \geq \limsup_{j\to\infty} \limsup_{k\to\infty} A_{k,j}\,. \tag{4.27}$$

Consider the term $B_{k,j}$. Using (G4) together with (4.22) we have that

$$\begin{aligned} B_{k,j} &\leq C \int_{L^{i_0}_{k,j}} \left[\frac{1}{\eta_k}(1 + |\widetilde{w}_{k,j}|^q) + \eta_k \left(|\nabla \varphi_{k,j}|^2 |u_k - v_k|^2 + |\nabla u_k|^2 + |\nabla v_k|^2 \right) \right] \mathrm{d}x \\ &\leq C \int_{L^{i_0}_{k,j}} \left[\frac{1}{\eta_k}(1 + |u_k - v_k|^q + |v_k|^q) + \frac{1}{\eta_k \lambda_k^2} |u_k - v_k|^2 + \eta_k \left(|\nabla u_k|^2 + |\nabla v_k|^2 \right) \right] \mathrm{d}x \\ &\leq \frac{C}{M_{k,j}} \int_{D_k^{(j)}} \left[\frac{1 + |u_k - v_k|^q}{\eta_k} + \frac{|u_k - v_k|^2}{\eta_k \lambda_k^2} + \eta_k \left(|\nabla u_k|^2 + |\nabla v_k|^2 \right) \right] \mathrm{d}x\,, \end{aligned} \tag{4.28}$$

where in the last step we used (4.21) and the fact that $\sup_{k\in\mathbb{N}} \|v_k\|_{L^\infty(D;\mathbb{R}^d)} < \infty$. Since for a cube rQ with side length r we have

$$|(rQ)^{(j)}| \leq \frac{2N r^{N-1}}{j},$$

and the cubes D_k are all contained in the bounded cube D, we can find $\bar{j} \in \mathbb{N}$ such that for all $j \geq \bar{j}$ and $k \in \mathbb{N}$ we get

$$\frac{|D_k^{(j)}|}{M_{k,j}\eta_k} \leq \frac{C}{j M_{k,j} \eta_k} = C\lambda_k\,. \tag{4.29}$$

Step 1 (see (4.18)) yields

$$\frac{C}{M_{k,j}\eta_k} \int_{D_k^{(j)}} \left[1 + |u_k - v_k|^q \right] \mathrm{d}x \leq Cj\lambda_k \left[\|u_k - v_k\|^q_{L^q(D_k;\mathbb{R}^d)} + 1 \right] \leq Cj\lambda\,. \tag{4.30}$$

Moreover, by (4.20) we obtain

$$\frac{1}{M_{k,j}\eta_k\lambda_k^2}\int_{D_k^{(j)}} |u_k - v_k|^2 \,\mathrm{d}x \le Cj\lambda_k\,, \tag{4.31}$$

$$\eta_k\int_{D_k} |\nabla u_k|^2 \,\mathrm{d}y \le \limsup_{k\to\infty}\mathcal{F}_{\eta_k}(u_k, D_k) < \infty\,, \tag{4.32}$$

and, since

$$\|\nabla v_k\|_{L^\infty} \le \frac{C}{\eta_k}, \tag{4.33}$$

$$\frac{\eta_k}{M_{k,j}}\int_{D_k^{(j)}} |\nabla v_k|^2 \,\mathrm{d}y \le \frac{C}{M_{k,j}\eta_k} = Cj\lambda_k\,. \tag{4.34}$$

From (4.28), (4.29), (4.30), (4.31), (4.32) and (4.34) we get

$$\lim_{j\to\infty}\lim_{k\to\infty} B_{k,j} = 0\,. \tag{4.35}$$

We now estimate the term $C_{k,j}$. Using (4.33), we obtain

$$\begin{aligned} C_{k,j} &\le \frac{1}{\eta_k}\int_{\bigcup_{i=i_0+1}^{M_{k,j}} L^i_{k,j}} \left[\, W(v_k(y)) + \eta_k^2|\nabla v_k(y)|^2 \,\right] \mathrm{d}y \\ &\le \frac{C}{\eta_k}\left|\, D_k^{(j)} \cap \{x\in D \,:\, |x\cdot\nu| < \eta_k\} \,\right|\,, \end{aligned}$$

and so

$$\lim_{j\to\infty}\lim_{k\to\infty} C_{k,j} = 0\,. \tag{4.36}$$

Similarly, it holds that

$$\lim_{k\to\infty}\mathcal{F}_{\eta_k}(\widetilde{w}_{k,j}, D\setminus D_k) \le \lim_{k\to\infty}\frac{C}{\eta_k}|(D\setminus D_k)\cap\{x\in D \,:\, |x\cdot\nu| < \eta_k\}| = 0\,. \tag{4.37}$$

Using (4.26), (4.27), (4.35), (4.36) and (4.37) we obtain (4.25).

Applying a diagonalizing argument, it is possible to find an increasing sequence $\{j(k)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}[B_{k,j(k)} + C_{k,j(k)} + \mathcal{F}_{\eta_k}(\widetilde{w}_{k,j(k)}, D\setminus D_k)] = 0\,,$$

and $\lim_{k\to\infty}\|\widetilde{w}_{k,j(k)} - u_{0,\nu}\|_{L^1(D;\mathbb{R}^d)} = 0$. Thus, the sequence $\{w_k\}_{k\in\mathbb{N}}$, with $w_k := \widetilde{w}_{k,j(k)}$ satisfies the claim of the lemma. $\square$

Remark 4.16. We will make use of the basic idea behind the proof of Lemma 4.15 in several occasions. In particular, it is possible to see that the result of Lemma 4.15 still holds true if the set $D \subset \mathbb{R}^N$ is a finite union of cubes, and $D_k = D$ for all $k \in \mathbb{N}$.

The proof of the limsup inequality, Proposition 4.31, uses periodicity properties of the potential energy W. In particular, we will show that W is periodic in the first variable not only with respect to the canonical set of orthogonal direction, but also with respect to a dense set of orthogonal directions. In the sequel we will use the notation $\Lambda := \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ and $\{e_1, \dots, e_N\}$ will denote the standard orthonormal basis for $\mathbb{R}^N$. We first recall the following classical extension theorem for isometries (for a proof see, for instance, [57, Theorem 10.2]).

Theorem 4.17. (Witt's Extension Theorem) *Let V be a finite dimensional vector space over a field $\mathbb{K}$ with characteristic different from 2, and let B be a symmetric bilinear form on V with $B(u,u) > 0$ for all $u \neq 0$. Let U, W be subspaces of V and let $T : U \to W$ be an isometry, that is, $B(u,v) = B(Tu, Tv)$ for all $u, v \in U$. Then T can be extended to an isometry from V to V.*

Lemma 4.18. *Let $\nu \in \Lambda$. Then there exist a rotation $R_\nu : \mathbb{R}^N \to \mathbb{R}^N$ and $\lambda_\nu \in \mathbb{N}$ such that $R_\nu e_N = \nu$ and $\lambda_\nu R_\nu e_i \in \mathbb{Z}^N$ for all $i = 1, \dots, N$.*

Proof. Let $\nu \in \Lambda$ be fixed. Consider the spaces

$$U := \mathrm{Span}(e_N), \qquad W := \mathrm{Span}(\nu)$$

as subspaces of $V := \mathbb{Q}^N$ over the field $\mathbb{K} := \mathbb{Q}$, with B being the standard Euclidean inner product. Then, the linear map $T : U \to W$ defined by $T(e_N) := \nu$ is an isometry. Apply Theorem 4.17 to extend T as a linear isometry $T : \mathbb{Q}^N \to \mathbb{Q}^N$. In particular, $T(e_i) \cdot T(e_j) = \delta_{ij}$. Up to redefining the sign of $T(e_1)$ so that $\det T > 0$, we can assume T to be a rotation. Let $\lambda_\nu \in \mathbb{N}$ be such that $\lambda_\nu T(e_i) \in \mathbb{Z}^N$ for all $i = 1, \dots, N$. Finally, define $R_\nu : \mathbb{R}^N \to \mathbb{R}^N$ to be the unique continuous extension of T to all of $\mathbb{R}^N$, which is well defined as isometries are uniformly continuous. □

Proposition 4.19. *Let $\nu_N \in \Lambda$. Then there exist $\nu_1, \dots, \nu_{N-1} \in \Lambda$ and $T \in \mathbb{N}$ such that $\nu_1, \dots, \nu_{N-1}, \nu_N$ is an orthonormal basis of $\mathbb{R}^N$, and for a.e. $x \in Q$ it holds $W(x + T\nu_i, p) = W(x, p)$ for all $p \in \mathbb{R}^d$ and $i = 1, \dots, N$.*

Proof. Let $R : \mathbb{R}^N \to \mathbb{R}^N$ be a rotation and let $T := \lambda_{\nu_N} \in \mathbb{N}$ be given by Lemma 4.18 relative to ν_N. Set $\nu_i := Re_i$ for $i = 1, \dots, N-1$. We have that $T\nu_i \in \mathbb{Z}^N$ for all $i = 1, \dots, N$. Fix $i \in \{1, \dots, N\}$ and write $T\nu_i = \sum_{j=1}^N \lambda_j e_j$, for some $\lambda_j \in \mathbb{Z}$. For $p \in \mathbb{R}^d$, using the periodicity of $W(\cdot, p)$ with respect to the canonical directions, for a.e. $x \in Q$ we have that

$$W(x + T\nu_i, p) = W\left(x + \sum_{j=1}^{N} \lambda_j e_j, p\right) = W(x, p).$$

□

In the following, given a linear map $L; \mathbb{R}^N \to \mathbb{R}^N$, we will denote by $\|L\|$ the Euclidean norm of L, i.e., $\|L\|^2 := \sum_{i,j=1}^{N} [L(e_i) \cdot e_j]^2$. For the sake of notation, we will also define the set of rational rotations $SO(N; \mathbb{Q}) \subset SO(N)$ as the rotations $R \in SO(N)$ such that $Re_i \in \mathbb{Q}^N$ for $i \in \{1, \ldots, N\}$.

Lemma 4.20. *Let $\varepsilon > 0$, $\nu \in \Lambda$, and let $S : \mathbb{R}^N \to \mathbb{R}^N$ be a rotation with $S(e_N) = \nu$. Then there exists a rotation $R \in SO(N; \mathbb{Q})$ such that $R(e_N) = \nu$ and $\|R - S\| < \varepsilon$.*

Proof. Step 1 We claim that $SO(N; \mathbb{Q})$ is dense in $SO(N)$ for every $N \geq 1$.

We proceed by induction on N. When $N = 1$, $SO(N)$ consists of the identity, so the claim is trivial. Let $N > 1$ be fixed and let $\varepsilon > 0$ and $S \in SO(N)$ be arbitrary. By density of $\mathbb{Q}^N \cap \mathbb{S}^{N-1}$, we can find a sequence $\{q_n\}_{n\in\mathbb{N}} \in \Lambda$ with $|q_n| = 1$ such that $q_n \to S(e_N)$ as $n \to \infty$. By Lemma 4.18 we can find $R_n \in SO(N; \mathbb{Q})$ such that $R_n(e_N) = q_n$. Since $SO(N)$ is a compact set, we can extract a convergent subsequence (not relabeled) of $\{R_n\}$ such that $R_n \to R \in SO(N)$, with $R(e_N) = \lim_{n\to\infty} R_n(e_N) = S(e_N)$.

Thus, the rotation $R^{-1} \circ S$ fixes e_N and may be identified with a rotation $T \in SO(N-1)$, i.e., writing $e_i =: (e_i', 0), i = 1, \ldots, N-1$, it follows that $Re_i = (Te_i', 0), i = 1, \ldots, N-1$. By the induction hypotheses, we can find $T' \in SO(N-1; \mathbb{Q})$ such that

$$\|T - T'\| < \frac{\varepsilon}{2}.$$

Define $R' \in SO(N; Q_\nu)$ by

$$R'e_i := \begin{cases} (T'e_i', 0) & i = 1, \ldots, N-1, \\ e_N & i = N. \end{cases}$$

Let n_0 be so large that

$$\|R - R_{n_0}\| < \frac{\varepsilon}{2}.$$

We claim that our desired rotation is $R_{n_0} \circ R' \in SO(N;\mathbb{Q})$. Indeed,

$$\begin{aligned}\|R_{n_0} \circ R' - S\| &\leq \|R_{n_0} \circ R' - R_{n_0} \circ R^{-1} \circ S\| + \|R_{n_0} \circ R^{-1} \circ S - S\| \\ &= \|R' - R^{-1} \circ S\| + \|R_{n_0} - R\| \\ &= \|T' - T\| + \|R_{n_0} - R\| < \varepsilon.\end{aligned}$$

Step 2 Let $S \in SO(N)$ with $S(e_N) = \nu$ be given. If $N = 1$, there is nothing else to prove, so we proceed with $N > 1$.

By Lemma 4.18 we can find a rotation $R_1 \in SO(N;\mathbb{Q})$ such that $R_1(e_N) = \nu$. Since $R_1^{-1} \circ S$ is a rotation with $(R_1^{-1} \circ S)(e_N) = e_N$, as in Step 1 we can identify $R^{-1} \circ S$ with a rotation $T_1 \in SO(N-1)$. Also by Step 1, $SO(N-1;\mathbb{Q})$ is dense in $SO(N-1)$, so we can find $T_2 \in SO(N-1;\mathbb{Q})$ such that $\|T_2 - T_1\| < \varepsilon$. As before, identifying T_2 with a rotation $R_2 \in SO(N;\mathbb{Q})$ that fixes e_N, we set $R := R_1 \circ R_2 \in SO(N;\mathbb{Q})$. We have that $(R_1 \circ R_2)(e_N) = R_1(e_N) = \nu$ and

$$\|R_1 \circ R_2 - S\| = \|R_2 - R_1^{-1} \circ S\| = \|T_2 - T_1\| < \varepsilon.$$

□

Definition 4.21. Let $V \subset \mathbb{S}^{N-1}$. We say that a set $E \subset \mathbb{R}^N$ is a *V-polyhedral set* if ∂E is a Lipschitz manifold contained in the union of finitely many affine hyperplanes each of which is orthogonal to an element of V.

A variant of well known approximation results of sets of finite perimeter by polyhedral sets yields the following (see [4, Theorem 3.42]).

Lemma 4.22. *Let $V \subset \mathbb{S}^{N-1}$ be a dense set. If E is a set with finite perimeter in Ω, then there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of V-polyhedral sets such that*

$$\lim_{n\to\infty} \|\chi_{E_n} - \chi_E\|_{L^1(\Omega)} = 0\,, \qquad \lim_{n\to\infty} |P(E_n;\Omega) - P(E;\Omega)| = 0\,.$$

Proof. Using [4, Theorem 3.42] it is possible to find a family $\{F_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ of polyhedral sets such that

$$\|\chi_{F_n} - \chi_E\|_{L^1(\Omega)} \leq \frac{1}{n}\,, \qquad |P(F_n;\Omega) - P(E;\Omega)| \leq \frac{1}{n}\,.$$

For every $n \in \mathbb{N}$, let $\Gamma_1^{(n)}, \ldots, \Gamma_{s_n}^{(n)}$ be the hyperplanes whose union contains the boundary of F_n. Let $\nu_1^{(n)}, \ldots, \nu_{s_n}^{(n)} \in \mathbb{S}^{N-1}$ be such that $\Gamma_i = (\nu_i^{(n)})^\perp$. Then it is possible to find rotations $R_i^{(n)} : \mathbb{R}^N \to \mathbb{R}^N$ such that $R_i^{(n)}\nu_i^{(n)} \in \Lambda$ and, denoting by $E_n \subset \mathbb{R}^N$ the set *enclosed* by the hyperplanes $(R_i^{(n)}\nu_i^{(n)})^\perp$, we get

$$\|\chi_{E_n} - \chi_E\|_{L^1(\Omega)} \le \frac{2}{n}, \qquad |P(E_n;\Omega) - P(E;\Omega)| \le \frac{2}{n}.$$

□

4.3.2 Properties of the function σ

The aim of this section is to study properties of the function σ introduced in Definition 4.6 that we will need in the proof of Proposition4.31 in order to prove the limsup inequality.

Lemma 4.23. *Let $\nu \in \mathbb{S}^{N-1}$. Then $\sigma(\nu)$ is well defined and is finite.*

Proof. Let $\nu \in S^{N-1}$. For $T > \sqrt{N}$ let $Q_T \in \mathcal{Q}_\nu$ and $u_T \in \mathcal{C}(Q_T, T)$ be such that

$$\frac{1}{T^{N-1}} \int_{TQ_T} W(y, u_T(y)) + |\nabla u_T(y)|^2 dy \le g(T) + \frac{1}{T}, \tag{4.38}$$

where, for simplicity of notation, we write $g(T)$ for $g(\nu, T)$. Let $\{\nu_T^{(1)}, \dots, \nu_T^{(N)}\}$ be an orthonormal basis of $\mathbb{R}^N$ normal to the faces of Q_T such that $\nu = \nu_T^{(N)}$. We define an oriented rectangular prism centered at 0 via

$$P(\alpha, \beta) := \{x \in \mathbb{R}^N : |x \cdot \nu| \le \beta \text{ and } |x \cdot \nu_T^{(i)}| \le \alpha \text{ for } 1 \le i \le N-1\}.$$

Let $S > T + 3 + \sqrt{N}$. We claim that for all $m \in \mathbb{N}$ with $2 \le m < T$, we have

$$g(S) \le g(T) + R(m, S, T), \tag{4.39}$$

where the quantity $R(m, S, T)$ does not depend on ν and is such that

$$\lim_{m\to\infty} \lim_{T\to\infty} \lim_{S\to\infty} R(m, S, T) = 0.$$

Note that if this holds then

$$\limsup_{S\to\infty} g(S) \le \liminf_{T\to\infty} g(T),$$

and this ensures the existence of the limit in the definition of σ. Therefore, the remainder of Step 1 is dedicated to proving 4.39.

The idea is to construct a competitor u_S for the infimum problem defining $g(S)$ by taking $\lfloor \frac{S}{T} \rfloor^{N-1}$ copies of $TQ_\nu \cap \nu^\perp$ centered on $\nu^\perp \cap SQ_\nu$ in each of which we define u_S to be (a translation of) u_T. In order to compare the energy of u_S to the energy of u_T, we need the copies of the cube TQ_ν to be integer translations of the original. Moreover, we also have to ensure that the boundary conditions render u_S admissible for the infimum problem defining $g(S)$. For this reason, we need the centers of the translated copies of $TQ_\nu \cap \nu^\perp$ to be close to $\nu^\perp \cap SQ_\nu$ (recall that the mollifiers $\rho_{T,\nu}$ and $\rho_{S,\nu}$ only depend on the direction ν).

Set

$$M_{T,S} := \left\lfloor \frac{S - \frac{1}{T}}{T + \sqrt{N} + 2} \right\rfloor^{N-1},$$

and notice that

$$\lim_{T\to\infty} \lim_{S\to\infty} \frac{T^{N-1}}{S^{N-1}} M_{T,S} = 1\,. \tag{4.40}$$

We can tile $\left(S - \frac{1}{T}\right) Q_T$ with disjoint prisms $\left\{p_i + P\left(T + \sqrt{N} + 2, S - \frac{1}{T}\right)\right\}_{i=1}^{M_{S,T}}$ so that

$$p_i + P\left(T + \sqrt{N} + 2, S - \frac{1}{T}\right) \subset \left(S - \frac{1}{T}\right) Q_T, \qquad p_i \in \nu^\perp,$$

for each $i \in \{1, \dots, M_{S,T}\}$. In each cube $p_i + \sqrt{N} Q_T$ we can find $x_i \in \mathbb{Z}^N$ since $\mathrm{dist}(\cdot, \mathbb{Z}^N) \leq \sqrt{N}$ in $\mathbb{R}^N$, and we have

$$x_i + (T+2)Q_T \subset p_i + (T + \sqrt{N} + 2)Q_T.$$

Consider cut-off functions $\varphi_{S,T} \in C_c(SQ_T; [0,1])$ and, for $m \in \mathbb{N}$ with $2 \leq m < T$, $i \in \{1, \dots, M_{S,T}\}$, let $\varphi_{m,i} \in C_c(x_i + (T + \frac{1}{m})Q_T; [0,1])$ be such that

$$\varphi_{S,T}(x) = \begin{cases} 0 & \text{if } x \in \partial(SQ_T), \\ \\ 1 & \text{if } x \in \left(S - \frac{1}{T}\right) Q_T, \end{cases} \qquad \|\nabla \varphi_{S,T}\|_{L^\infty} \leq CT, \tag{4.41}$$

and

$$\varphi_{m,i}(x) = \begin{cases} 0 & \text{if } x \in \partial\left(x_i + \left(T + \frac{1}{m}\right) Q_T\right), \\ \\ 1 & \text{if } x \in x_i + TQ_T, \end{cases} \qquad \|\nabla \varphi_{m,i}\|_{L^\infty} \leq Cm, \tag{4.42}$$

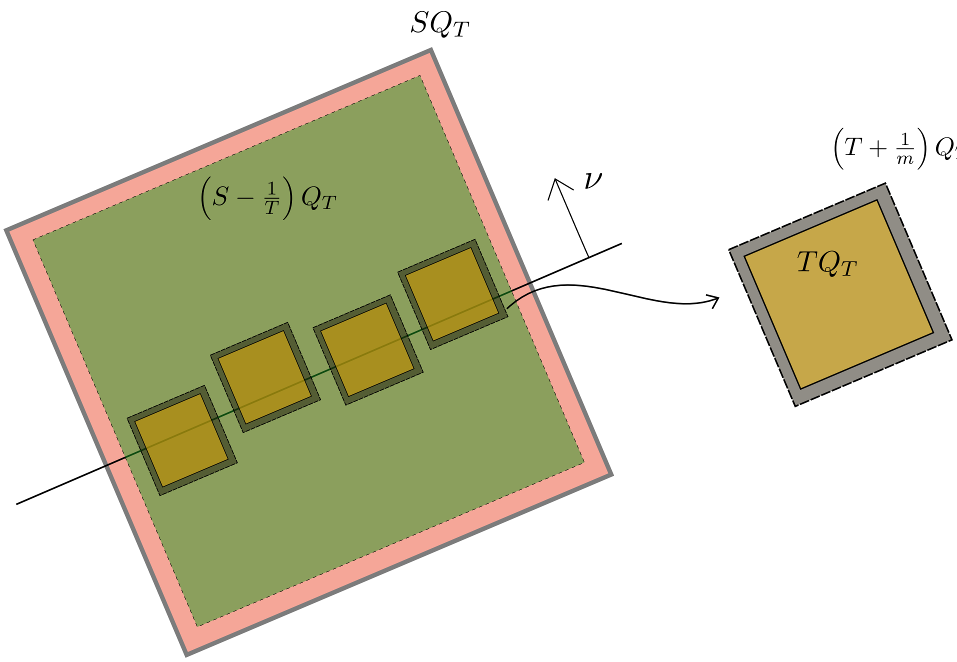

Figure 2: Construction of the function u_S: in each yellow cube $x_i + TQ_T$ we defined it as a copy of u_T and we use the grey region $(x_i + (T + \frac{1}{m})Q_T) \setminus (x_i + TQ_T)$ around it to adjust the boundary conditions and make them match the value of u_S in the green region. Finally, in the pink region $SQ_T \setminus (S - \frac{1}{T})Q_T$ we make the transition in order for u_S to be an admissible competitor for the infimum problem defining $g(S)$.

for some $C > 0$. Define $u_S : SQ_T \to \mathbb{R}^d$ by

$$u_S(x) := \begin{cases} u_T(x - x_i) & \text{if } x \in x_i + TQ_T \\ \varphi_{m,i}(x)(\rho_T * u_{0,\nu})(x + p_i - x_i) + (1 - \varphi_{m,i}(x))(\rho_m * u_{0,\nu})(x) & \\ & \text{if } x \in (x_i + (T + \frac{1}{m})Q_T) \setminus (x_i + TQ_T) \\ \varphi_{S,T}(x)(\rho_m * u_{0,\nu})(x) + (1 - \varphi_{S,T}(x))(\rho_S * u_{0,\nu})(x) & \\ & \text{if } x \in SQ_T \setminus (S - \frac{1}{T})Q_T \\ (\rho_m * u_{0,\nu})(x) & \text{otherwise.} \end{cases}$$

Notice that since $p_i \cdot \nu = 0$, if $x \in \partial(x_i + TQ_T)$ we have

$$u_T(x - x_i) = (\rho_T * u_{0,\nu})(x - x_i) = (\rho_T * u_{0,\nu})(x + p_i - x_i).$$

Thus $u_S \in H^1(SQ_T;\mathbb{R}^d)$ and, if $x \in \partial SQ_T$ then $u_S(x) = (\rho_S * u_{0,\nu})(x)$, so u_S is admissible for the infimum in the definition of $g(S)$. In particular,

$$\begin{aligned} g(S) &\leq \frac{1}{S^{N-1}} \int_{SQ_T} \left[W(x, u_S(x)) + |\nabla u_S(x)|^2 \right] dx \\ &= \frac{1}{S^{N-1}} \mathcal{F}_1(u_S, SQ_T) \\ &=: I_1(T,S) + I_2(T,S,m) + I_3(T,S,m) + I_4(T,S,m), \end{aligned} \tag{4.43}$$

where

$$\begin{aligned} I_1(T,S) :=& \frac{1}{S^{N-1}} \sum_{i=1}^{M_{T,S}} \mathcal{F}_1(u_S, x_i + TQ_T), \\ I_2(T,S,m) :=& \frac{1}{S^{N-1}} \sum_{i=1}^{M_{T,S}} \mathcal{F}_1 \left(u_S, \left(x_i + \left(T + \frac{1}{m} \right) Q_T \right) \setminus (x_i + TQ_T) \right), \\ I_3(T,S,m) :=& \frac{1}{S^{N-1}} \mathcal{F}_1(u_S, E^{(1)}_{T,S,m}), \\ I_4(T,S,m) :=& \frac{1}{S^{N-1}} \mathcal{F}_1(u_S, E^{(2)}_{T,S}), \end{aligned}$$

and we set

$$E^{(1)}_{T,S,m} := \left(S - \frac{1}{T} \right) Q_T \setminus \bigcup_{i=1}^{M_{T,S}} \left(x_i + \left(T + \frac{1}{m} \right) Q_T \right)$$

and

$$E^{(2)}_{T,S} := SQ_T \setminus \left(S - \frac{1}{T} \right) Q_T.$$

It is worth pointing out the following properties of $\rho_L * u_{0,\nu}$ for $L > 0$. We will demonstrate that

$$\nabla(\rho_L * u_{0,\nu})(x) = 0 \text{ if } |x \cdot \nu| \geq \frac{1}{L} \tag{4.44}$$

and that

$$\|\nabla(\rho_L * u_{0,\nu})\|_\infty \leq CL. \tag{4.45}$$

To prove these, we note that $u_{0,\nu}$ is a jump function and hence its distributional derivative is the vector measure $(b-a) \otimes \nu \mathcal{H}^{N-1} \llcorner \nu^\perp$. Then we see

$$\nabla(\rho_L * u_{0,\nu})(x) = \int_{B\left(x, \frac{1}{L}\right) \cap \nu^\perp} \rho_L(y)(b-a) \otimes \nu d\mathcal{H}^{N-1}(y).$$

Thus, if $|x \cdot \nu| \geq \frac{1}{L}$, we have $B\left(x, \frac{1}{L}\right) \cap \nu^\perp = \emptyset$ and thus $\nabla(\rho_L * u_{0,\nu})(x) = 0$. To see (4.45), we can estimate

$$\mathcal{H}^{N-1}\left(B\left(x, \frac{1}{L}\right) \cap \nu^\perp\right) \leq C\frac{1}{L^{N-1}}.$$

On the other hand, since $\|\nabla \rho_L\|_\infty \leq L^N$, we have for every x that

$$|\nabla(\rho_L * u_{0,\nu})(x)| \leq CL^N \mathcal{H}^{N-1}\left(B\left(x, \frac{1}{L}\right) \cap \nu^\perp\right) \leq CL$$

and thus $\|\nabla(\rho_L * u_{0,\nu})\|_\infty \leq CL$.

We now bound each of terms $I_1, \ldots, I_4$ separately. We start with $I_1(T,S)$. Since $x_i \in \mathbb{Z}^N$, the periodicity of W together with (4.38) yield

$$\begin{aligned} I_1(T,S) &= \frac{1}{S^{N-1}} M_{S,T} \int_{TQ_T} \left[W(x, u_T(x)) + |\nabla u_T(x)|^2\right] dx \\ &\leq \frac{1}{S^{N-1}} M_{T,S} T^{N-1}\left(g(T) + \frac{1}{T}\right). \end{aligned} \tag{4.46}$$

In order to estimate $I_2(T,S,m)$, notice that by (4.44)

$$(\rho_m * u_{0,\nu})(x) = \begin{cases} a & \text{if } x \cdot \nu \leq -\frac{1}{m}, \\ \\ b & \text{if } x \cdot \nu \geq \frac{1}{m}, \end{cases} \tag{4.47}$$

and that

$$(\rho_T * u_{0,\nu})(x + p_i - x_i) = \begin{cases} a & \text{if } x \cdot \nu \leq -\frac{1}{T} - \frac{\sqrt{N}}{2}, \\ \\ b & \text{if } x \cdot \nu \geq \frac{1}{T} + \frac{\sqrt{N}}{2}, \end{cases}$$

since $x_i \in p_i \sqrt{N} Q_T$. Furthermore, since for every $x \in \mathbb{R}^d$, the function $t \mapsto (\rho_T * u_{0,\nu})(x + t\nu)$ is constant outside of an interval of size $1/T$, we have, for every $i \in \{1, \ldots, M_{T,S}\}$, that

$$\begin{aligned} &\int_{\left(x_i + \left(T + \frac{1}{m}\right)Q_T\right) \setminus (x_i + TQ_T)} |\nabla(\rho_T * u_{0,\nu})(x + p_i - x_i)|^2 \, \mathrm{d}x \\ &\qquad \leq \frac{1}{T} \|\nabla(\rho_T * u_{0,\nu})\|^2_{L^\infty} \left[\left(T + \frac{1}{m}\right)^{N-1} - T^{N-1}\right]. \end{aligned} \tag{4.48}$$

Thus, using (4.42) and (4.48) we obtain

$$\begin{aligned} I_2(T,S,m) &\le \frac{C}{S^{N-1}} M_{T,S} \Bigg[\left(\frac{\sqrt{N}}{2} + \frac{1}{m} \right) (1 + \|\nabla \varphi_{m,i}\|^2_{L^\infty}) + \frac{1}{m} \|\nabla(\rho_m * u_{0,\nu})\|^2_{L^\infty} \\ &\qquad + \frac{1}{T} \|\nabla(\rho_T * u_{0,\nu})\|^2_{L^\infty} \Bigg] \left[\left(T + \frac{1}{m} \right)^{N-1} - T^{N-1} \right] \\ &\le \frac{C}{S^{N-1}} M_{T,S} \left(1 + m^2 + T\right) \left[\left(T + \frac{1}{m} \right)^{N-1} - T^{N-1} \right] \\ &\le C \frac{T^{N-1}}{S^{N-1}} M_{T,S} \left(1 + m^2 + T\right) \left[\left(1 + \frac{1}{Tm} \right)^{N-1} - 1 \right] \\ &\le C \frac{T^{N-1}}{S^{N-1}} M_{T,S} \left(1 + m^2 + T\right) \left(\frac{N-1}{Tm} \right) =: J_2(T,S,m) \end{aligned} \tag{4.49}$$

where in the last step we used the inequality

$$(1+t)^{N-1} \le 1 + C(N-1)t \tag{4.50}$$

for $t \ll 1$, that is valid here when $T \gg 1$.

Using (4.47), we can estimate $I_3(T,S,m)$ as

$$\begin{aligned} I_3(T,S,m) &= \frac{1}{S^{N-1}} \int_{E^{(1)}_{T,S,m}} \left[W(x, \rho_m * u_{0,\nu}) + |\nabla(\rho_m * u_{0,\nu})|^2 \right] \, \mathrm{d}x \\ &\le \frac{C}{S^{N-1}} \left| E^{(1)}_{T,S,m} \cap \left\{ |x \cdot \nu| < \frac{1}{m} \right\} \right| \left(1 + \|\nabla(\rho_m * u_{0,\nu})\|^2_{L^\infty} \right) \\ &\le \frac{C}{S^{N-1}} \left[\left(S - \frac{1}{T} \right)^{N-1} - M_{T,S} T^{N-1} \right] \frac{1+m^2}{m} \\ &= C \left[\left(1 - \frac{1}{ST} \right)^{N-1} - \frac{T^{N-1}}{S^{N-1}} M_{T,S} \right] \frac{(1+m^2)}{m} =: J_3(T,S,m). \end{aligned} \tag{4.51}$$

Finally, we get

$$
\begin{aligned}
I_4(T,S,m) \le \frac{C}{S^{N-1}} \Bigg[& \left| E^{(2)}_{T,S} \cap \left\{ |x\cdot\nu| < \frac{1}{S} \right\} \right| (1 + \|\nabla(\rho_S * u_{0,\nu})\|^2_{L^\infty}) \\
& + \left| E^{(2)}_{T,S} \cap \left\{ |x\cdot\nu| < \frac{1}{m} \right\} \right| (1 + \|\nabla\varphi_{S,T}\|^2_{L^\infty} + \|\nabla(\rho_m * u_{0,\nu})\|^2_{L^\infty}) \Bigg] \\
\le{}& C\frac{1+S^2}{S^{N-1}} \left[S^{N-1} - \left(S - \frac{1}{T} \right)^{N-1} \right] \frac{1}{S} \\
& + C\frac{1+m^2}{S^{N-1}} \left[S^{N-1} - \left(S - \frac{1}{T} \right)^{N-1} \right] \frac{1}{m} \\
\le{}& C\frac{1+S^2}{S}\frac{N-1}{TS} + C\frac{1+m^2}{Tm}\frac{N-1}{TS} =: J_4(T,S,m). \qquad (4.52)
\end{aligned}
$$

where in the last step we used (4.50), assuming $T \gg 1$.

Taking into account (4.49), (4.51), and (4.52), we obtain

$$
\lim_{m\to\infty} \lim_{T\to\infty} \lim_{S\to\infty} [\, J_2(T,S,m) + J_3(T,S,m) + J_4(T,S,m) \,] = 0. \qquad (4.53)
$$

Thus, in view of (4.43), (4.46), (4.40) and (4.38), we conclude (4.39) with

$$
R(m,S,T) := J_2(T,S,m) + J_3(T,S,m) + J_4(T,S,m). \qquad (4.54)
$$

Notice that $R(m,S,T)$ does not depend on ν nor on Q_T.

Finally, to prove that $\sigma(\nu) < \infty$ for all $\nu \in \mathbb{S}^{N-1}$ we notice that, by sending $S \to \infty$ in (4.39) we get

$$
\sigma(\nu) \le g(T) + \lim_{S\to\infty} R(m,S,T).
$$

Since $g(T) < \infty$ and, by (4.53) and (4.54), $\lim_{S\to\infty} R(m,S,T) < \infty$ for all $T > 0$, we conclude. $\square$

Remark 4.24. The proof of Lemma 4.23 shows, in particular, that

$$
\lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ} \left[W(y,u(y)) + |Du|^2 \right] \mathrm{d}y \,:\, u \in \mathcal{C}(Q,T) \right\}
$$

exists, for every $\nu \in \mathbb{S}^{N-1}$ and every $Q \in \mathcal{Q}_\nu$. This will be used later in the proof of Lemma 4.28.

Next we show that the definition of $\sigma(\nu)$ does not depend on the choice of the mollifier ρ we choose to impose the boundary conditions.

Lemma 4.25. *For every $\nu \in \mathbb{S}^{N-1}$ the definition of $\sigma(\nu)$ does not depend on the choice of the mollifier ρ.*

Proof. Fix $\nu \in \mathbb{S}^{N-1}$ and let $\{T_n\}_{n\in\mathbb{N}}$ be such that $T_n \to \infty$ as $n \to \infty$. Let $\rho^{(1)}, \rho^{(2)} \in C_c^\infty(B(0,1))$ be two mollifiers and let us denote by $\sigma(\nu, \rho^{(1)})$ and $\sigma(\nu, \rho^{(2)})$ the functions defined as in Definition 4.6 using $\rho^{(1)}$ and $\rho^{(2)}$, respectively, to impose the boundary conditions for the admissible class of functions. Let $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu$ and $\{u_n^{(1)}\}_{n\in\mathbb{N}} \subset H^1(T_nQ_n;\mathbb{R}^d)$ with $u_n^{(1)} := \rho_{T_n}^{(1)} * u_{0,\nu}$ on ∂T_nQ_n be such that

$$\lim_{n\to\infty} \frac{1}{T_n^{N-1}} \mathcal{F}_1(u_n, T_nQ_n) = \sigma(\nu, \rho^{(1)}). \tag{4.55}$$

For every $m, n \in \mathbb{N}$, consider the cubes $\left(T_n + \frac{1}{m}\right) Q_n$ and a function $\varphi_{n,m} \in C^\infty((T_n + \frac{1}{m})Q_n)$ with $0 \leq \varphi_{n,m} \leq 1$ such that $\varphi_{n,m} \equiv 1$ in T_nQ_n, $\varphi_{n,m} \equiv 0$ on $\partial[(T_n + \frac{1}{m})Q_n]$ and $\|\nabla\varphi_{n,m}\|_\infty \leq Cm$. For every $n \in \mathbb{N}$ define $u_n^{(2)} \in H^1((T_n + \frac{1}{m})Q_n;\mathbb{R}^d)$ as

$$u_n^{(2)}(x) := \begin{cases} u_n^{(1)}(x) & \text{if } x \in T_nQ_n, \\ \varphi_{n,m}(x)(\rho_{T_n}^{(1)} * u_{0,\nu})(x) + (1 - \varphi_{n,m}(x))\,(\rho_{T_n}^{(2)} * u_{0,\nu})(x) & \text{otherwise .} \end{cases}$$

Then $u_n^{(2)} = \rho_{T_n}^{(2)} * u_{0,\nu}$ on $\partial[(T + \frac{1}{m})Q_n]$ and $u_n^{(2)}$ is constant (taking values a or b) outside the set $\{(x', x_\nu) \in \mathbb{R}^N : x' \in Q_n', x_\nu = s\nu \text{ for } s \in [-\frac{1}{T_n}, \frac{1}{T_n}]\}$ where $Q_n' := [(T_n + \frac{1}{m})Q_n \setminus T_nQ_n] \cap \nu^\perp$. We have

$$\frac{1}{(T_n + \frac{1}{m})^{N-1}} \mathcal{F}_1\left(u_n^{(2)}, \left(T_n + \frac{1}{m}\right) Q_n\right) \leq \frac{1}{T_n^{N-1}} \mathcal{F}_1(u_n, T_nQ_n) + R_{n,m}, \tag{4.56}$$

where

$$\begin{aligned} R_{n,m} &:= \frac{1}{T_n^{N-1}} \int_{(T_n + \frac{1}{m})Q \setminus T_nQ_n} [\, W(y, u_n^{(2)}(y)) + |\nabla u_n^{(2)}(y)|^2 \,] \, \mathrm{d}y \\ &\leq \frac{C}{T_n^N} \left[\left(T_n + \frac{1}{m}\right)^{N-1} - T_n^{N-1} \right] (1 + T_n^2 + m^2) \\ &\leq \frac{C}{mT_n^2}(1 + T_n^2 + m^2)\,, \end{aligned}$$

where in the last inequality we use (4.50). Using (4.55) and (4.56) we get

$$\sigma(\nu, \rho^{(2)}) \leq \liminf_{n\to\infty} \frac{1}{(T_n + \frac{1}{m})^{N-1}} \mathcal{F}_1\left(u_n^{(2)}, \left(T_n + \frac{1}{m}\right) Q_n\right) \leq \sigma(\nu, \rho^{(1)}) + \frac{C}{m}.$$

Using the arbitrariness of m we get the result. $\square$

We now prove a regularity property for the function σ.

Proposition 4.26. *The function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ is upper semi-continuous.*

Proof. Step 1. Fix $\nu \in \mathbb{S}^{N-1}$ and let $\{\nu_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^{N-1}$ be such that $\nu_n \to \nu$ as $n \to \infty$. We first prove that, for fixed $T > 0$, the function $\nu \mapsto g(\nu, T)$ is continuous. We claim that $\limsup_{n\to\infty} g(\nu_n, T) \leq g(\nu, T)$. Fix $\varepsilon > 0$. Let $Q_\nu \in \mathcal{Q}_\nu$ and $u \in \mathcal{C}(TQ_\nu, \nu)$ be such that

$$\left| T^{N-1} g(\nu, T) - \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u|^2 \right] \mathrm{d}y \right| < \varepsilon \,. \tag{4.57}$$

Without loss of generality, by density, we can assume that $u \in L^\infty(\Omega; \mathbb{R}^d)$. For every $n \in \mathbb{N}$, let $\mathcal{R}_n : \mathbb{R}^N \to \mathbb{R}^N$ be a rotation such that $\mathcal{R}_n \nu_n = \nu$ and $\mathcal{R}_n \to \mathrm{Id}$ as $n \to \infty$, where $\mathrm{Id} : \mathbb{R}^N \to \mathbb{R}^N$ is the identity map. Define $u_n \in \mathcal{C}(TQ_{\nu_n}, \nu_n)$ as $u_n(y) := u(R_n y)$. By (4.57) we have

$$\begin{aligned} T^{N-1} g(\nu_n, T) &\leq \int_{TQ_{\nu_n}} \left[W(y, u_n(y)) + |\nabla u_n|^2 \right] \mathrm{d}y \\ &\leq \int_{TQ_\nu} \left[W(y, u(y)) + |\nabla u|^2 \right] \mathrm{d}y + \delta_n \\ &\leq T^{N-1} g(\nu, T) + \varepsilon + \delta_n \,, \end{aligned} \tag{4.58}$$

where

$$\delta_n := \left| \int_{TQ_{\nu_n}} W(y, u_n(y)) \mathrm{d}y - \int_{TQ_\nu} W(y, u(y)) \mathrm{d}y \right| .$$

We claim that $\delta_n \to 0$ as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary in (4.58), this would confirm the claim.

Fix $\eta > 0$ and let $M := C(1 + \|u\|^q_{L^\infty})$, where $C > 0$ and $q \geq 2$ are given by (G4). Let $K \subset \mathbb{R}^N$ be a compact set such that $TQ_\nu \subset K$ and $TQ_{\nu_n} \subset K$ for every $n \in \mathbb{N}$. Notice that $W(x, u(x)) \leq M$ for all $x \in TQ_\nu$. Using the Scorza-Dragoni theorem (see [42, Theorem 6.35]) and the Tietze extension theorem (see [42, Theorem A.5]), we can find a compact set $E \subset K$ with $|E| < \eta$ and continuous map $\widetilde{W} : K \times \mathbb{R}^d \to [0, \infty)$ such that $\widetilde{W}(x, \cdot) = W(x, \cdot)$ for all $x \in K \setminus E$ and $|\widetilde{W}(x, u(x))| \leq M$ for every $x \in K$. We claim that

$$\int_{TQ_\nu} \left| W(y, u(y)) - \widetilde{W}(y, u(y)) \right| \mathrm{d}y \leq C\eta \,, \tag{4.59}$$

and that

$$\int_{TQ_{\nu_n}} \left| W(y, u_n(y)) - \widetilde{W}(y, u_n(y)) \right| \mathrm{d}y \leq C\eta \,. \tag{4.60}$$

Indeed

$$\begin{aligned}\int_{TQ_\nu} \left| W(y,u(y)) - \widetilde{W}(y,u(y)) \right| \, \mathrm{d}y &= \int_E \left| W(y,u(y)) - \widetilde{W}(y,u(y)) \right| \, \mathrm{d}y \\ &\leq 2M|E| \\ &\leq 2M\eta \, .\end{aligned}$$

A similar argument yields (4.60). Since TQ_ν is bounded

$$\int_{TQ_\nu} \left| \widetilde{W}(R_n y, u(y)) - \widetilde{W}(y,u(y)) \right| \, \mathrm{d}y \to 0 \, , \tag{4.61}$$

as $n \to \infty$. Thus, from (4.59), (4.60) and (4.61) we obtain

$$\limsup_{n\to\infty} \delta_n \leq 2C\eta \, .$$

The claim follows from the arbitrariness of η.

In an analogous way it is possible to show that $\liminf_{n\to\infty} g(\nu_n, T) \geq g(\nu, T)$, and thus we conclude that the function $\nu \to g(\nu, T)$ is continuous.

Step 2. Fix $\nu \in S^{N-1}$, $\varepsilon > 0$, and let $T > 0$ be such that

$$|g(\nu,T) - \sigma(\nu)| < \varepsilon \, . \tag{4.62}$$

Let $\{\nu_n\}_{n\in\mathbb{N}}$ be a sequence converging to ν. By Step 1 we have that

$$\lim_{n\to\infty} g(\nu_n, T) = g(\nu, T). \tag{4.63}$$

Then, for $S > T + 3 + \sqrt{N}$, using (4.39) and (4.62) we get, for $m \in \{1, \ldots, T\}$,

$$\begin{aligned} g(\nu_n, S) &\leq g(\nu_n, T) + R(m,S,T) \\ &= g(\nu,T) + g(\nu_n,T) - g(\nu,T) + R(m,S,T) \\ &\leq \sigma(\nu) + \varepsilon + g(\nu_n,T) - g(\nu,T) + R(m,S,T). \end{aligned}$$

Taking the limit as $S \to \infty$ we obtain

$$\sigma(\nu_n) \leq \sigma(\nu) + \varepsilon + g(\nu_n,T) - g(\nu,T) + \lim_{S\to\infty} R(m,S,T).$$

Letting $n \to \infty$, by (4.63)

$$\limsup_{n\to\infty} \sigma(\nu_n) \leq \sigma(\nu) + \varepsilon + \lim_{S\to\infty} R(m,S,T).$$

Finally, taking $T \to \infty$ and then $m \to \infty$, using (4.54), we conclude that

$$\limsup_{n\to\infty} \sigma(\nu_n) \leq \sigma(\nu) + \varepsilon$$

for every $\varepsilon > 0$, and thus we obtain upper-semicontinuity. $\square$

The following technical results, that will be fundamental in the proof of the limsup inequality (see Proposition 4.31), aim at providing two different ways to obtain, for $\nu \in \mathbb{S}^{N-1}$, the value $\sigma(\nu)$.

Lemma 4.27. *Let $\nu \in \Lambda$. Then*

$$\sigma(\nu) = \lim_{T\to\infty} g^\Lambda(\nu, T)\,, \tag{4.64}$$

where

$$g^\Lambda(\nu,T) := \frac{1}{T^{N-1}} \inf\Big\{ \int_{TQ_\nu} \big[W(y,u(y)) + |Du|^2\big]\,\mathrm{d}y \,:\, Q_\nu \in \mathcal{Q}^\Lambda_\nu,\ u \in \mathcal{C}(Q_\nu,T) \Big\}\,,$$

and $\mathcal{Q}^\Lambda_\nu$ is the family of cubes with unit length side centered at the origin with two faces orthogonal to ν and the other faces orthogonal to elements of Λ.

Proof. Fix $\nu \in \Lambda$. From the definition of $\sigma(\nu)$ it follows that

$$\sigma(\nu) \leq \liminf_{T\to\infty} g^\Lambda(\nu,T)\,. \tag{4.65}$$

Let $\{T_n\}_{n\in\mathbb{N}}$ with $T_n \to \infty$ as $n \to \infty$. By Lemma 4.23, let $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu$ and $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in \mathcal{C}(Q_n,T_n) \cap L^\infty(T_nQ_n;\mathbb{R}^d)$ be such that

$$\lim_{n\to\infty} \frac{1}{T_n^{N-1}} \mathcal{F}_1(u_n, T_nQ_n) = \sigma(\nu). \tag{4.66}$$

For every fixed T_n, an argument similar to the one used in Step 1 of the proof of Proposition 4.26 together with Lemma 4.20 ensure that it is possible to find rotations $\mathcal{R}_n : \mathbb{R}^N \to \mathbb{R}^N$ with $\mathcal{R}_n(e_N) = \nu$ and $\mathcal{R}_n(e_i) \in \Lambda$ for all $i = 1, \dots, N-1$ such that

$$|\mathcal{F}_1(u_n, T_nQ_n) - \mathcal{F}_1(\widetilde{u}_n, T_n\mathcal{R}_n(Q_n))| < \frac{1}{n}, \tag{4.67}$$

where $\widetilde{u}_n(x) := u_n(R_n^{-1}x)$. Thus

$$\begin{aligned}\limsup_{n\to\infty} g^\Lambda(\nu,T) &\leq \limsup_{n\to\infty} \frac{1}{T_n^{N-1}}\mathcal{F}_1(\widetilde{u}_n, T_n\mathcal{R}_n(Q_n))\\ &\leq \limsup_{n\to\infty} \frac{1}{T_n^{N-1}}\mathcal{F}_1(u_n, T_nQ_n)\\ &= \sigma(\nu),\end{aligned} \tag{4.68}$$

where the last step follows from (4.66), while in the second to last step we used (4.67). By (4.65) and (4.68) and the arbitrariness of the sequence $\{T_n\}_{n\in\mathbb{N}}$, we conclude (4.64). $\square$

Lemma 4.28. *For $\nu \in \mathbb{S}^{N-1}$ and $Q \in \mathcal{Q}_\nu$ define*

$$\sigma^Q(\nu) := \lim_{T\to\infty} g^Q(\nu,T)\,,$$

where

$$g^Q(\nu,T) := \frac{1}{T^{N-1}}\inf\Big\{\int_{TQ}\big[W(y,u(y)) + |Du|^2\big]\,\mathrm{d}y \,:\, u\in\mathcal{C}(Q,T)\Big\}.$$

Then there exists $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu$ such that $\sigma^{Q_n}(\nu) \to \sigma(\nu)$ as $n\to\infty$. In particular, if $\nu\in\Lambda$ it is possible to take $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu^\Lambda$.

Proof. First of all notice that, in view of Remark 4.24, $\sigma^Q(\nu)$ is well defined. By definition, we have $\sigma(\nu) \leq \sigma^Q(\nu)$ for all $Q\in\mathcal{Q}_\nu$. Thus, it suffices to prove that it is possible to find a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu$ such that $\sigma^{Q_n}(\nu) \leq \sigma(\nu) + R_n$, where $R_n \to 0$ as $n\to\infty$. Let $\{T_n\}_{n\in\mathbb{N}}$ be an increasing sequence with $T_n\to\infty$ as $n\to\infty$ such that

$$g(\nu,T_n) \leq \sigma(\nu) + \frac{1}{n}.$$

It is then possible to find $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu$ (or, using Lemma 4.27, $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}_\nu^\Lambda$ in case $\nu\in\Lambda$) such that for all $n\in\mathbb{N}$ it holds

$$g^{Q_n}(\nu,T_n) \leq g(\nu,T_n) + \frac{1}{n}. \tag{4.69}$$

An argument similar to the one used in Lemma 4.23 to establish (4.39) shows that for every $\nu\in\mathbb{S}^{N-1}$, $Q\in\mathcal{Q}_\nu$, $T>0$, $S > T+3+\sqrt{N}$ and $m\in\{1,\dots,T\}$, it holds

$$g^Q(\nu,S) \leq g^Q(\nu,T) + R(m,S,T), \tag{4.70}$$

where $R(m, S, T)$ is independent of $\nu \in \mathbb{S}^{N-1}$ and of $Q \in \mathcal{Q}_\nu$ (see (4.54)), and is such that

$$\lim_{m\to\infty} \lim_{T\to\infty} \lim_{S\to\infty} R(m, S, T) = 0.$$

In particular, for all $n \in \mathbb{N}$, it is possible to choose $m_n \in \{1, \ldots, T_n\}$ such that

$$\lim_{n\to\infty} \lim_{S\to\infty} R(m_n, S, T_n) = 0. \tag{4.71}$$

Thus, we get

$$g^{Q_n}(\nu, S) \leq g^{Q_n}(\nu, T_n) + R(m_n, S, T_n). \tag{4.72}$$

From (4.69) and (4.72), sending $S \to \infty$, we get

$$\sigma^{Q_n}(\nu) \leq \sigma(\nu) + \frac{2}{n} + \lim_{S\to\infty} R(m_n, S, T_n).$$

Using (4.71) we conclude that

$$\sigma(\nu) = \lim_{n\to\infty} \sigma^{Q_n}(\nu)\,.$$

□

4.3.3 Compactness

Proposition 4.29. *Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega; \mathbb{R}^d)$ be a sequence with $\sup_{n\in\mathbb{N}} \mathcal{F}_{\varepsilon_n}(u_n) < +\infty$, where $\varepsilon_n \to 0^+$. Then there exists $u \in BV(\Omega; \{a, b\})$ such that, up to a subsequence (not relabeled), $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$.*

Proof. Let $\widetilde{W} : \mathbb{R}^d \to [0, \infty)$ be the continuous function given by (G3). Let $R > 0$ be such that $\frac{1}{C}|p|^q - C > 0$ for $|p| > R$, where $C > 0$ and $q \geq 2$ are as in (G4), and $|a|, |b| < R$. Take a function $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\varphi \equiv 1$ in $B_R(0)$ and $\varphi \equiv 0$ in $B_{2R}(0)$. Define the function $\mathcal{W} : \mathbb{R}^d \to [0, \infty)$ by

$$\mathcal{W}(p) := \varphi(p)\widetilde{W}(p) + (1 - \varphi(p)) \left(\frac{1}{C}|p|^q - C \right),$$

for $p \in \mathbb{R}^d$. Notice that $\mathcal{W}(p) = 0$ if and only if $p \in \{a, b\}$. Since $\widetilde{W}(p) \leq W(x, p)$ for a.e. $x \in Q$, we get

$$\mathcal{F}_{\varepsilon_n}(u_n) \geq \int_\Omega \left[\frac{1}{\varepsilon}\mathcal{W}(u(x)) + \varepsilon|\nabla u(x)|^2 \right] \,\mathrm{d}x =: \widetilde{\mathcal{F}}_{\varepsilon_n}(u_n)\,,$$

and, in turn, we have that $\sup_{n\in\mathbb{N}} \widetilde{\mathcal{F}}_{\varepsilon_n}(u_n) < +\infty$. We now proceed as in [48] to obtain a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ and $u \in BV(\Omega;\{a,b\})$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. □

4.3.4 Liminf inequality

This section is devoted to the proof of the liminf inequality.

Proposition 4.30. *Given a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0^+$, let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ be such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. Then*

$$\mathcal{F}_0(u) \le \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n)\,.$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$. Without loss of generality, and possibly up to a subsequence, we can assume that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) < \infty\,. \tag{4.73}$$

By Proposition 4.29, we get $u \in BV(\Omega;\{a,b\})$. Set $A := \{u = a\}$. Consider, for every $n \in \mathbb{N}$, the finite nonnegative Radon measure

$$\mu_n := \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u_n(x)\right) + \varepsilon |Du_n(x)|^2\right] \mathcal{L}^N \llcorner \Omega\,.$$

From (4.73) we have that $\sup_{n\in\mathbb{N}} \mu_n(\Omega) < \infty$. Thus, up to a subsequence (not relabeled), $\mu_n \stackrel{w^*}{\rightharpoonup} \mu$, for some finite nonnegative Radon measure μ in Ω. In particular,

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) = \liminf_{n\to\infty} \mu_n(\Omega) \ge \mu(\Omega)\,. \tag{4.74}$$

We claim that for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \partial^* A$ it holds

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x_0) \ge \sigma(\nu_A(x_0))\,, \tag{4.75}$$

where $\lambda := \mathcal{H}^{N-1} \llcorner \partial^* A$. The liminf inequality follows from (4.74) and (4.75). The rest of the proof is devoted at showing the validity of (4.75).

Step 1. For $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* A$ we have

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) < \infty\,. \tag{4.76}$$

Fix $x_0 \in \partial^* A$ satisfying (4.76) and a cube $Q_\nu \in \mathcal{Q}_\nu$, with $\nu := \nu_A(x_0)$. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence with $\delta_k \to 0$ as $k \to \infty$, such that $\mu(\partial Q_\nu(x_0,\delta_k)) = 0$, where $Q_\nu(x_0,\delta_k) := x_0 + \delta_k Q_\nu$

for all $k \in \mathbb{N}$. Then it holds

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x_0) = \lim_{k\to\infty} \frac{\mu(Q_\nu(x_0,\delta_k))}{\delta_k^{N-1}} = \lim_{k\to\infty}\lim_{n\to\infty} \frac{\mu_n(Q_\nu(x_0,\delta_k))}{\delta_k^{N-1}}\,. \tag{4.77}$$

We have

$$\begin{aligned}
\frac{\mu_n(Q_\nu(x_0,\delta_k))}{\delta_k^{N-1}} &= \frac{1}{\delta_k^{N-1}} \int_{Q_\nu(x_0,\delta_k)} \left[\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |Du_n(x)|^2 \right] \,\mathrm{d}x \\
&= \delta_k \int_{Q_\nu} \left[\frac{1}{\varepsilon_n} W\left(\frac{x_0+\delta_k z}{\varepsilon_n}, u_n(x_0+\delta_k z)\right) + \varepsilon_n |Du_n(x_0+\delta_k z)|^2 \right] \mathrm{d}z \\
&= \int_{Q_\nu - \frac{\varepsilon_n}{\delta_k}s_n} \left[\frac{\delta_k}{\varepsilon_n} W\left(\frac{\delta_k}{\varepsilon_n} y, u_n(y_k^n)\right) + \varepsilon_n |Du_n(y_k^n)|^2 \right] \mathrm{d}y\,,
\end{aligned} \tag{4.78}$$

where in the last step, for the sake of simplicity, we set $y_k^n := x_0 + \delta_k y + \varepsilon_n s_n$, we wrote $\frac{x_0}{\varepsilon_n} = m_n - s_n$, with $m_n \in \mathbb{Z}^N$ and $|s_n| \le \sqrt{N}$, and we used the periodicity of W to simplify, for $z = y + \frac{\varepsilon_n}{\delta_k}s_n$, $z \in Q_\nu$,

$$W\left(\frac{x_0+\delta_k z}{\varepsilon_n}, \cdot\right) = W\left(\frac{x_0+\delta_k(y+\frac{\varepsilon_n}{\delta_k}s_n)}{\varepsilon_n}, \cdot\right) = W\left(m_n + \frac{\delta_k}{\varepsilon_n} y, \cdot\right) = W\left(\frac{\delta_k}{\varepsilon_n} y, \cdot\right)\,.$$

Consider the functions $u_{k,n}(x) := u_n(x_0+\delta_k x)$, for $n,k \in \mathbb{N}$. We claim that

$$\lim_{k\to\infty}\lim_{n\to\infty} \|u_{k,n} - u_{0,\nu_A(x_0)}\|_{L^1(Q_\nu;\mathbb{R}^d)} = 0\,, \tag{4.79}$$

where $u_{0,\nu_A(x_0)}$ is defined as in (4.5). Set $Q_\nu^+ := Q_\nu \cap \{x \in \mathbb{R}^N \,:\, x\cdot\nu > 0\}$ and Q_ν^- its complement in Q_ν. We get

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} \|u_{k,n} - u_{0,\nu_A(x_0)}\|_{L^1(Q_\nu;\mathbb{R}^d)} \\
&\qquad = \lim_{k\to\infty}\lim_{n\to\infty} \left[\int_{Q_\nu^-} |u_n(x_0+\delta_k x) - a| \,\mathrm{d}x + \int_{Q_\nu^+} |u_n(x_0+\delta_k x) - b| \,\mathrm{d}x \right] \\
&\qquad = \lim_{k\to\infty} \left[\int_{Q_\nu^-} |u(x_0+\delta_k x) - a| \,\mathrm{d}x + \int_{Q_\nu^+} |u(x_0+\delta_k x) - b| \,\mathrm{d}x \right] \\
&\qquad = \lim_{k\to\infty} \frac{1}{\delta_k^N} \left[\int_{Q_\nu(x_0,\delta_k)\cap H_\nu^-} |u(y) - a| \,\mathrm{d}y + \int_{Q_\nu(x_0,\delta_k)\cap H_\nu^+} |u(y) - b| \,\mathrm{d}y \right] \\
&\qquad = |b-a| \lim_{k\to\infty} \left[\frac{|Q_\nu(x_0,\delta_k)\cap H_\nu^- \cap B|}{\delta_k^N} + \frac{|Q_\nu(x_0,\delta_k)\cap H_\nu^+ \cap A|}{\delta_k^N} \right] \\
&\qquad = 0\,,
\end{aligned}$$

where $H_\nu^+ := \{x \in \mathbb{R}^N \,:\, x\cdot\nu > x_0\cdot\nu\}$, H_ν^- is its complement in $\mathbb{R}^N$ and $B := \Omega \setminus A$. The last step follows from (i) of Theorem 2.10.

Step 2. Using a diagonal argument, and (4.79), it is possible to find an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that, setting

$$\eta_k := \frac{\varepsilon_{n_k}}{\delta_k}\,, \qquad x_k := \eta_k s_{n_k}\,, \qquad w_k(x) := u_{k,n_k}(x - x_k)\,,$$

the following hold:

(i) $\lim_{k\to\infty} \eta_k = 0$;

(ii) $\lim_{k\to\infty} x_k = 0$;

(iii) $w_k \to u_{0,\nu}$ in $L^q(Q_\nu;\mathbb{R}^d)$ for all $q \geq 1$;

(iv) we have

$$\lim_{k\to\infty} \delta_k \int_{Q_\nu - x_k} \left[\frac{1}{\varepsilon_{n_k}} W\left(\frac{y}{\eta_k}, u_{n_k}(y_k^{n_k}) \right) + \varepsilon_{n_k} |Du_{n_k}(y_k^{n_k})|^2 \right] \mathrm{d}y$$
$$= \lim_{k\to\infty} \lim_{n\to\infty} \delta_k \int_{Q_\nu - \frac{\varepsilon_n}{\delta_k} s_n} \left[\frac{1}{\varepsilon_n} W\left(\frac{\delta_k}{\varepsilon_n} y, u_n(y_k^n) \right) + \varepsilon_n |Du_n(y_k^n)|^2 \right] \mathrm{d}y\,.$$

From (4.77), (4.78) and (iv) we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x_0) = \lim_{k\to\infty} \int_{Q_\nu - x_k} \left[\frac{1}{\eta_k} W\left(\frac{y}{\eta_k}, w_k(y) \right) + \eta_k |Dw_k(y)|^2 \right] \mathrm{d}y\,.$$

Let Q_k be the largest cube contained in $Q_\nu - x_k$ centered at zero and having the same principal axes of Q_ν. Since $x_k \to 0$ as $k \to \infty$, $Q_k \subset Q_\nu - x_k$ for k large and the integrand is nonnegative, we have that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x_0) \geq \limsup_{k\to\infty} \int_{Q_k} \left[\frac{1}{\eta_k} W\left(\frac{y}{\eta_k}, w_k(y) \right) + \eta_k |Dw_k(y)|^2 \right] \mathrm{d}y\,. \tag{4.80}$$

Step 3. Finally we modify w_k close to ∂Q_k in order to render it an admissible function for the infimum problem defining $\sigma(\nu)$ as in Definition 4.6. Using Lemma 4.15 we find a sequence $\{\bar{w}_k\}_{k\in\mathbb{N}} \subset H^1(Q_\nu;\mathbb{R}^d)$ such that

$$\liminf_{k\to\infty} \mathcal{F}_{\eta_k}(w_k, Q_k) \geq \limsup_{k\to\infty} \mathcal{F}_\eta(\bar{w}_k, Q_\nu)\,, \tag{4.81}$$

and with $\bar{w}_k = (\widetilde{u_k})_{T,\nu}$ on ∂Q_ν, where $(\widetilde{u_k})_{T,\nu}$ is defined as in (4.6). Hence, by (4.80) and (4.81)

$$
\begin{aligned}
\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x_0) &\geq \limsup_{k\to\infty} \int_{Q_\nu} \left[\frac{1}{\eta_k} W\left(\frac{y}{\eta_k}, \bar{w}_k(y) \right) + \eta_k |D\bar{w}_k(y)|^2 \right] \mathrm{d}y \\
&= \limsup_{k\to\infty} \int_{\frac{1}{\eta_k}Q_\nu} \left[\eta_k^{N-1} W\left(z, \bar{w}_k(\eta_k z)\right) + \eta_k^{N+1} |D\bar{w}_k(\eta_k z)|^2 \right] \mathrm{d}z \\
&= \limsup_{k\to\infty} \eta_k^{N-1} \int_{\frac{1}{\eta_k}Q_\nu} \left[W\left(z, v_k(z)\right) + |Dv_k(z)|^2 \right] \mathrm{d}z \\
&\geq \sigma(\nu)\,,
\end{aligned}
$$

since $\bar{w}_k \in C(Q_\nu, \frac{1}{\eta_k})$, where $v_k(z) := \bar{w}_k(\eta_k z)$, and this concludes the proof. □

4.3.5 Limsup inequality

In this section we construct a recovery sequence.

Proposition 4.31. *Let $u \in BV(\Omega; \{a,b\})$. Given a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0^+$ as $n \to \infty$, there exist $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ as $n\to\infty$ such that*

$$
\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \leq \mathcal{F}_0(u)\,. \tag{4.82}
$$

Proof. Notice that it is enough to prove the following: given any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n\to\infty$, it is possible to extract a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}}$ for which there exists $\{u_k\}_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u_k \to u$ in $L^1(\Omega;\mathbb{R}^d)$ as $k\to\infty$ such that

$$
\limsup_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(u_k) \leq \mathcal{F}_0(u)\,.
$$

Since $L^1(\Omega;\mathbb{R}^d)$ is separable, we conclude using the Urysohn property of the Γ-limit (see [28, Proposition 8.3]).

Case 1. Assume that the set $A := \{u = a\}$ is a Λ-polyhedral set (see Definition 4.21). We need to localize the Γ-limit of our sequence of functionals. For $\{\delta_n\}_{n\in\mathbb{N}}$ with $\delta_n \to 0$, $v \in L^1(\Omega;\mathbb{R}^d)$ and $U \in \mathcal{A}(\Omega)$ we set

$$
\mathcal{W}_{\{\delta_n\}}(v;U) := \inf\left\{ \liminf_{n\to\infty} \mathcal{F}_{\delta_n}(v_n, U) : v_n \to v \text{ in } L^1(U;\mathbb{R}^d), v_n \in H^1(U;\mathbb{R}^d) \right\}.
$$

Let $\mathcal{C}$ be the family of all open cubes in Ω with faces parallel to the axes, centered at points

$x \in \Omega \cap Q_\nu$ and with rational edge length. Denote by $\mathcal{R}$ the countable subfamily of $\mathcal{A}(\Omega)$ whose elements are Ω and all finite unions of elements of $\mathcal{C}$, i.e.,

$$\mathcal{R} := \Omega \cup \left\{ \bigcup_{i=1}^{k} C_i : k \in \mathbb{N}, C_i \in \mathcal{C} \right\}.$$

Let $\varepsilon_n \to 0^+$. We will select a suitable subsequence in the following manner. We enumerate the elements of $\mathcal{R}$ by $\{R_i\}_{i\in\mathbb{N}}$. First considering R_1, by a diagonalization argument, we can find a subsequence $\{\varepsilon_{n_j}\}_{j\in\mathbb{N}} \subset \{\varepsilon_n\}_{n\in N}$ and functions $\{u_j^{R_1}\}_{j\in N} \subset H^1(R_1;\mathbb{R}^d)$ such that

$$u_j^{R_1} \to u \quad \text{in } L^1(R_1;\mathbb{R}^d),$$

and

$$\mathcal{W}_{\{\varepsilon_{n_j}\}}(u; R_1) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_{n_j}}(u_j^{R_1}, R_1).$$

Now, considering R_2, we can extract a further subsequence $\{\varepsilon_{n_{j_k}}\}_{k\in\mathbb{N}}$ and functions $\{u_k^{R_2}\} \subset H^1(R_2;\mathbb{R}^d)$ such that

$$u_k^{R_2} \to u \quad \text{in } L^1(R_2;\mathbb{R}^d), \quad u_{j_k}^{R_1} \to u \quad \text{in } L^1(R_1;\mathbb{R}^d),$$

and

$$\mathcal{W}_{\left\{\varepsilon_{n_{j_k}}\right\}}(u; R_2) = \lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_{j_k}}}(u_k^{R_2}, R_1), \quad \mathcal{W}_{\left\{\varepsilon_{n_{j_k}}\right\}}(u; R_1) = \lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_{j_k}}}(u_{j_k}^{R_1}, R_1).$$

Continuing along the $\{R_i\}$ in this fashion and employing a further diagonalization argument, we can assert the existence of a subsequence $\{\varepsilon_n^{\mathcal{R}}\}_{n\in\mathbb{N}}$ of $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with the following property: for every $C \in \mathcal{R}$, there exists a sequence $\{u_{\varepsilon_n^{\mathcal{R}}}^C\}_{n\in\mathbb{N}} \subset H^1(C;\mathbb{R}^d)$ such that

$$u_{\varepsilon_n^{\mathcal{R}}}^C \to u \quad \text{in } L^1(C;\mathbb{R}^d),$$

and

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; C) = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_{\varepsilon_n^{\mathcal{R}}}^C, C)\,. \tag{4.83}$$

We claim that

(C1) the set function $\lambda : \mathcal{A}(\Omega) \to [0,\infty)$ given by

$$\lambda(B) := \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; B)$$

is a positive finite Radon measure absolutely continuous with respect to $\mu := \mathcal{H}^{N-1} \llcorner \partial^* A$,

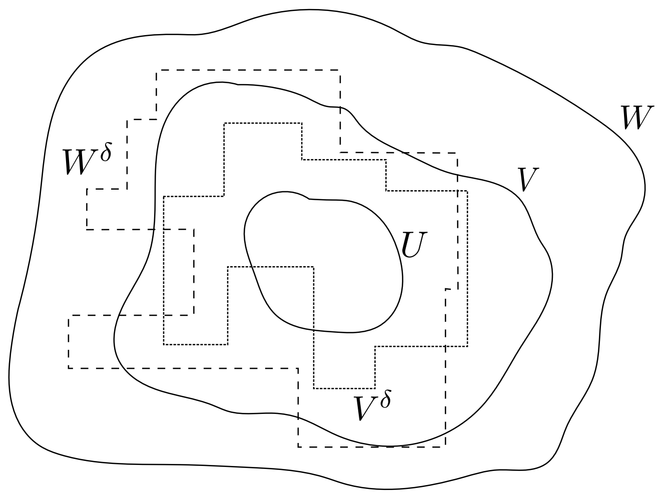

Figure 3: The sets $\overline{U} \subset V \subset W$ and $V^\delta \subset V$, $W^\delta \subset W \setminus \overline{U}$. Notice that $V^\delta \setminus W^\delta$ can be non-empty.

(C2) for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \partial A$, it holds

$$\frac{d\lambda}{d\mu}(x_0) \leq \sigma(\nu(x_0))\,. \tag{4.84}$$

This allows us to conclude. Indeed, we have that

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u^{\Omega}_{\varepsilon_n^{\mathcal{R}}}, \Omega) &= \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) \\
&= \int_{\partial A_0} \frac{d\lambda}{d\mu}(x)\, \mathrm{d}\mathcal{H}^{N-1}(x) \\
&\leq \int_{\partial A_0} \sigma(\nu(x))\, \mathrm{d}\mathcal{H}^{N-1}(x) \\
&= \mathcal{F}_0(u).
\end{aligned}$$

Step 1. We first prove claim (C1).

We use the coincidence criterion in Lemma 2.14 to show that $\lambda(B)$ is the restriction of a positive finite measure to $\mathcal{A}(\Omega)$.

We will first prove (i) in Lemma 2.14. Let $U, V, W \in \mathcal{A}(\Omega)$ be such that $\overline{U} \subset\subset V \subset W$. For $\delta > 0$, let V^δ and W^δ be two elements of $\mathcal{R}$ such that $V^\delta \subset V$, $W^\delta \subset W \setminus \overline{U}$, and

$$\mathcal{H}^{N-1}\left(\partial^* A_0 \cap (W \setminus (V^\delta \cup W^\delta))\right) < \delta. \tag{4.85}$$

Let $\{v_n\}_{n\in\mathbb{N}} \subset H^1(V^\delta;\mathbb{R}^d)$ and $\{w_n\}_{n\in\mathbb{N}} \subset H^1(W^\delta;\mathbb{R}^d)$ be such that $v_n \to u$ in $L^1(V^\delta;\mathbb{R}^d)$,

$w_n \to u$ in $L^1(W^\delta;\mathbb{R}^d)$, and (see (4.83))

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;V^\delta) = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(v_n, V^\delta)\,, \tag{4.86}$$

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;W^\delta) = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, W^\delta)\,. \tag{4.87}$$

Let $\rho:\mathbb{R}^N \to [0,+\infty)$ be a symmetric mollifier, and define

$$\xi_n(x) := \frac{1}{(\varepsilon_n^{\mathcal{R}})^N}\rho\Big(\frac{x}{\varepsilon_n^{\mathcal{R}}}\Big)\,. \tag{4.88}$$

From Remark 4.16 we can assume that $w_n = \xi_n * u$ on ∂W^δ and $v_n = \xi_n * u$ on ∂V^δ. Using a similar argument to the one found in Lemma 4.15 applied to the sets $E_n := (W^\delta\setminus V^\delta)\setminus(W^\delta\setminus V^\delta)^{(n)}$, and $E := W^\delta \setminus V^\delta$ with boundary data $\xi_n * u$, it is possible to find functions $\{\varphi_n\}\subset C^\infty(W^\delta)$ with $\operatorname{supp}\nabla\varphi_n \subset L_n^{(i_0)}$ (here we are using the notation of the proof of Lemma 4.15) such that, if we define the function $u_n : W \to \mathbb{R}^d$ as

$$u_n := \chi_{V^\delta\cup W^\delta}\left(\varphi_n v_n + (1-\varphi_n)w_n\right) + \chi_{(W\setminus(V^\delta\cup W^\delta)}(\xi_n * u)\,,$$

we have that $u_n \in H^1(W;\mathbb{R}^d)$ and

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, L_n^{(i_0)}) = 0\,. \tag{4.89}$$

Notice that $u_n \to u$ in $L^1(W;\mathbb{R}^d)$ as $n\to\infty$. Moreover, we get

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;W) &\le \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, W)\\
&\le \liminf_{n\to\infty}\Big[\,\mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, V^\delta) + \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, W^\delta)\\
&\qquad\qquad + \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, W\setminus(V^\delta\cup W^\delta)) + \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, L_n^{(i_0)})\Big]\\
&\le \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;V^\delta) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;W^\delta)\\
&\qquad\qquad + \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, W\setminus(V^\delta\cup W^\delta))
\end{aligned} \tag{4.90}$$

where in the last step we used (4.86), (4.87) and (4.89). We see that

$$\begin{aligned}
\liminf_{n\to\infty} & \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, W \setminus (V^\delta \cup W^\delta)) \\
&= \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}\left(\xi_n * u, \{x \in W \setminus (V^\delta \cup W^\delta) \,:\, \operatorname{dist}(x, \partial A) \le \varepsilon_n^{\mathcal{R}}\}\right) \\
&\le C \liminf_{n\to\infty} \frac{\mathcal{L}^N(\{x \in W \setminus (V^\delta \cup W^\delta) \,:\, \operatorname{dist}(x, \partial A) \le \varepsilon_n^{\mathcal{R}}\})}{\varepsilon_n^{\mathcal{R}}} \\
&= C\mathcal{H}^{N-1}\left(\partial A \cap (W \setminus (V^\delta \cup W^\delta))\right) \\
&\le C\delta \,, \qquad (4.91)
\end{aligned}$$

where in the last step we used (4.85). Using (4.90), (4.91) and the fact that $V^\delta \subset V$ and $W^\delta \subset W \setminus \overline{U}$, we get

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; W) &\le C\delta + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; V^\delta) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; W^\delta) \\
&\le C\delta + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; V) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; W \setminus \overline{U}) \,.
\end{aligned}$$

Letting $\delta \to 0^+$, we obtain (i).

We proceed to proving (ii) in Lemma 2.14. Let $U, V \in \mathcal{A}(\Omega)$ be such that $U \cap V = \emptyset$. Fixing $\eta > 0$, we can find $u_n \in H^1(U \cup V; \mathbb{R}^d)$ such that $u_n \to u$ and

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; U \cup V) \ge \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, U \cup V) - \eta.$$

Then, since the restriction of u_n to U and V converges to u in these sets,

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; U) \le \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, U)$$

and

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; V) \le \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, V)$$

by definition, we have

$$\begin{aligned}
\lambda(U) + \lambda(V) &\le \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, U) + \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, V) \\
&\le \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, U \cup V) \le \lambda(U \cup V) + \eta.
\end{aligned}$$

Sending $\eta \to 0^+$, we conclude

$$\lambda(U) + \lambda(V) \le \lambda(U \cap V).$$

To prove the opposite inequality, as in the proof of (i), we select $U^\delta \subset U$, $V^\delta \subset V$ with $U^\delta, V^\delta \in \mathcal{R}$ and

$$\mathcal{H}^{N-1}\left(\partial^* A_0 \cap \left((U \cup V) \setminus \left(U^\delta \cup V^\delta\right)\right)\right) < \delta. \tag{4.92}$$

Again we may select $v_n \in H^1(V^\delta; \mathbb{R}^d)$ and $u_n \in H^1(U^\delta; \mathbb{R}^d)$ such that $v_n \to u$ in $L^1(V^\delta; \mathbb{R}^d)$, $u_n \to u$ in $L^1(U^\delta; \mathbb{R}^d)$ and

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u_n; U^\delta) \leq \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, U^\delta), \tag{4.93}$$

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(v_n; V^\delta) \leq \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(v_n, V^\delta). \tag{4.94}$$

As in (i), we may assume without loss of generality that $u_n = \xi_n * u$ on ∂U^δ, $v_n = \xi_n * v$ on ∂V^δ, and we can find functions $\varphi_n \in C^\infty(U \cap V; [0,1])$ so that, defining

$$w_n := \chi_{U^\delta \cup V^\delta}(\varphi_n u_n + (1-\varphi_n) v_n) + \chi_{(U\cup V)\setminus(U^\delta \cup V^\delta)} \xi_n * u$$

we have $w_n \in H^1(U \cup V; \mathbb{R}^d)$ and

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, L_n^{(i_0)}) = 0, \tag{4.95}$$

where $\nabla \varphi_n \subset L_n^{(i_0)}$, again using the notation of Lemma 4.15. Observing that $w_n \to u$ in $L^1(U \cup V; \mathbb{R}^d)$, we get

$$\begin{aligned}
\lambda(U \cup V) &\leq \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, U \cup V) \\
&\leq \Big[\mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_n, U^\delta) + \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(v_n, V^\delta) \\
&\qquad + \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, (U \cup V) \setminus (U^\delta \cup V^\delta)) + \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, L_n^{(i_0)})\Big] \\
&\leq \lambda(U^\delta) + \lambda(V^\delta) + \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, (U \cup V) \setminus (U^\delta \cup V^\delta))
\end{aligned}$$

where in the last step we used (4.93), (4.94), and (4.95). Noticing as in (ii) that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(w_n, (U \cup V) \setminus (U^\delta \cup V^\delta)) \leq C\mathcal{H}^{N-1}\left(\partial^* A_0 \cap \left((U \cup V) \setminus \left(U^\delta \cup V^\delta\right)\right)\right)$$

and by (4.92) we have

$$\lambda(U \cup V) \leq \lambda(U^\delta) + \lambda(V^\delta) + C\delta \leq \lambda(U) + \lambda(V) + C\delta$$

and, letting $\delta \to 0$, we conclude (ii).

We prove (iii) in Lemma 2.14. Let $\Omega' \subset\subset \Omega$. Recalling (4.88), we know that $u * \xi_n$ is constant outside a tubular neighborhood of width $\varepsilon_n^{\mathcal{R}}$ around $\partial^* A$ and that $\|\nabla(u * \xi_n)\|_{L^\infty} \leq \frac{C}{\varepsilon_n^{\mathcal{R}}}$. Thus

$$\lambda(\Omega') = \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; \Omega') \leq \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u * \xi_n, \Omega') \leq C\mathcal{H}^{N-1}(\Omega' \cap \partial^* A) = C\mu(\Omega'). \tag{4.96}$$

This shows, by the coincidence criterion Lemma 2.14, that $\lambda \llcorner \Omega'$ is a Radon measure. Since μ is a finite Radon measure in Ω and (4.96) holds for every $\Omega' \subset\subset \Omega$, we conclude that λ is a finite Radon measure in Ω absolutely continuous with respect to μ, which was the claim (C1).

Step 2. We now prove (C2). Let $x_0 \in \Omega \cap \partial^* A$ be on a face of $\partial^* A$ (since the set is polyhedral) and write $\nu := \nu_A(x_0)$. Using Proposition 4.19 it is possible to find a rotation R_ν and $T \in \mathbb{N}$ such that, setting $Q_\nu := R_\nu Q$, we get $Q_\nu \in \mathcal{Q}_\nu$ and

$$W(x + nTv, p) = W(x, p)\,,$$

for a.e. $x \in \Omega$, every $v \in \mathbb{S}^{N-1}$ that is orthogonal to one face of Q_ν, every $p \in \mathbb{R}^M$ and $n \in \mathbb{N}$. By Remark 2.11 it follows that for μ-almost every $x_0 \in \Omega$,

$$\frac{d\lambda}{d\mu}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\lambda(Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}}, \tag{4.97}$$

where $Q_\nu(x_0, \varepsilon) := x_0 + \varepsilon Q$. In view of Lemma 4.28, it is possible to find $\{T_k\}_{k\in\mathbb{N}} \subset T\mathbb{N}$ with $T_k \to \infty$ as $k \to \infty$, and $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{C}(Q_\nu, T_k)$ such that

$$\begin{aligned}
\sigma^{Q_\nu}(\nu) &= \lim_{k\to\infty} \frac{1}{T_k^{N-1}} \int_{T_k Q_\nu} \Big[W(y, u_k(y)) + |\nabla u_k(y)|^2 \Big] dy \\
&= \lim_{k\to\infty} \int_{Q_\nu} \Big[T_k W(T_k x, v_k(x)) + \frac{1}{T_k} |\nabla v_k(x)|^2 \Big] dx,
\end{aligned} \tag{4.98}$$

where $v_k : Q_\nu \to \mathbb{R}^d$ is defined as $v_k(x) := u_k(T_k x)$ and $\sigma^{Q_\nu}(\nu)$ is defined as in Lemma 4.28. Without loss of generality, by density, we can assume $u_k \in \mathcal{C}(Q_\nu, T_k) \cap L^\infty(T_k Q_\nu; \mathbb{R}^d)$. Since the choice of mollifier $\rho \in C_c^\infty(B(0, 1))$ is arbitrary by Lemma 4.25, we will assume here that $\operatorname{supp} \rho \subset B(0, \frac{1}{2})$ and thus

$$u_k\,(T_k x) = u_{0,\nu}(x) \quad \text{if} \quad |T_k x| \geq \frac{1}{2}\,.$$

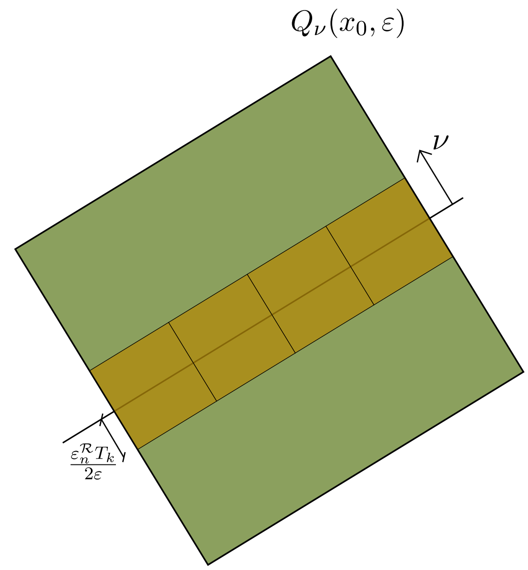

Figure 4: The construction of the recovery sequence $v_{n,k}^{(\varepsilon)}$: for every $\varepsilon > 0$ and $k \in \mathbb{N}$ fixed, we defined it as $u_{0,\nu}$ in the green region and, in each yellow square of side length $\frac{\varepsilon_n^{\mathcal{R}} T_k}{\varepsilon}$, as a rescaled version of the function u_k.

For $x \in \mathbb{R}^N$ let $x_\nu := x \cdot \nu$ and $x' := x - x_\nu \nu$. Moreover, set $Q'_\nu := Q_\nu \cap \nu^\perp$.

For $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, extend the function $x' \mapsto v_k(x' + t\nu)$ to the whole $\nu^\perp$ by periodicity, and define

$$v_{n,k}^{(\varepsilon)}(x) := \begin{cases} u_{0,\nu}(x) & \text{if } |x_\nu| > \frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}, \\ \\ v_k\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}} T_k}\right) & \text{if } |x_\nu| \leq \frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}. \end{cases} \tag{4.99}$$

The idea behind the definition of the function $v_{n,k}^{(\varepsilon)}$ is the following (see Figure 4): for every fixed $\varepsilon > 0$ and $k \in \mathbb{N}$ we are tiling the face of A orthogonal to ν with $\varepsilon_n^{\mathcal{R}}$-rescaled copies of the optimal profile u_k. The fact that A is a Λ-polyhedral set and that $T_k \in T\mathbb{N}$ ensure that it is possible to use the periodicity of W to estimate the energy in each cube of edge length $\frac{\varepsilon_n^{\mathcal{R}} T_k}{\varepsilon}$. The presence of the factor ε in (4.99) localizes the function around the point x_0 and accommodates the blow-up method we are using to prove the limsup inequality and, because of periodicity, will play no essential role in the fundamental estimate (4.106).

Let $m_n \in R_\nu\left(T\mathbb{Z}^N\right)$ and $s_n \in [0, T)^N$ be such that $\frac{x_0}{\varepsilon_n^{\mathcal{R}}} = m_n + s_n$, and let

$$x_{\varepsilon,n} := -\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} s_n \,.$$

Note that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} x_{\varepsilon,n} = 0. \tag{4.100}$$

Define the functions $u_{n,\varepsilon,k} \in H^1(Q_\nu(x_0,\varepsilon);\mathbb{R}^d)$ by

$$u_{n,\varepsilon,k}(x) := v^{(\varepsilon)}_{n,k}\left(\frac{x-x_0}{\varepsilon} - x_{\varepsilon,n}\right).$$

We claim that there is $\varepsilon'(x_0)$ such that for every $0 < \varepsilon < \varepsilon'(x_0)$ and any $k \in \mathbb{N}$

$$\lim_{n\to\infty} \|u_{n,\varepsilon,k} - u\|_{L^1(Q_\nu(x_0,\varepsilon);\mathbb{R}^d)} = 0. \tag{4.101}$$

Since x_0 is on a face of $\partial^* A$, we can find ε' such that $u = u_{0,\nu}(x - x_0)$ in $Q_\nu(x_0,\varepsilon')$. Changing variables,

$$\begin{aligned}\int_{Q_\nu(x_0,\varepsilon)} &|u_{n,\varepsilon,k}(x) - u(x)|\mathrm{d}x \\ &= \int_{(\varepsilon Q_\nu - \varepsilon x_{\varepsilon,n})\cap\{z:|z_\nu|\le\frac{\varepsilon_n^{\mathcal{R}} T_k}{2}\}} \left|v^{(\varepsilon)}_{n,k}\left(\frac{z}{\varepsilon}\right) - u(x_0 + z + \varepsilon x_{\varepsilon,n})\right|\mathrm{d}z.\end{aligned}$$

Since the functions $v^{(\varepsilon)}_{n,k}$ are uniformly bounded with respect to $n \in \mathbb{N}$, we prove our claim by noticing that $|(\varepsilon Q_\nu - \varepsilon x_{\varepsilon,n}) \cap \{z : |z_\nu| \le \frac{\varepsilon_n^{\mathcal{R}} T_k}{2}\}| \to 0$ as $n \to \infty$.

Thus, using the definition of λ and (4.101), we get

$$\frac{\lambda(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \le \liminf_{n\to\infty} \frac{1}{\varepsilon^{N-1}} \mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_{n,\varepsilon,k}, Q_\nu(x_0,\varepsilon)). \tag{4.102}$$

We want to rewrite the right-hand side of (4.102) in terms of the functions $v^{\varepsilon}_{n,k}$. To do so, changing variables, we write

$$\begin{aligned}
&\frac{1}{\varepsilon^{N-1}}\mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_{n,\varepsilon,k}, Q_\nu(x_0,\varepsilon)) \\
&\quad = \int_{Q_\nu} \left[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon y}{\varepsilon_n^{\mathcal{R}}}, u_{n,\varepsilon,k}(x_0+\varepsilon y) \right) + \varepsilon\varepsilon_n^{\mathcal{R}} |\nabla u_{n,\varepsilon,k}(x_0+\varepsilon y)|^2 \right] \mathrm{d}y \\
&\quad = \int_{Q_\nu} \left[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon y}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(y-x_{\varepsilon,n}) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(y-x_{\varepsilon,n})|^2 \right] \mathrm{d}y \\
&\quad = \int_{Q_\nu - x_{\varepsilon,n}} \left[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon (y+x_{\varepsilon,n})}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(y) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(y)|^2 \right] \mathrm{d}y \\
&\quad = \int_{Q_\nu - x_{\varepsilon,n}} \left[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(m_n + \frac{\varepsilon y}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(y) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(y)|^2 \right] \mathrm{d}y \\
&\quad = \int_{Q_\nu - x_{\varepsilon,n}} \left[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon y}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(y) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(y)|^2 \right] \mathrm{d}y \\
&\quad = \mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}}(v_{n,k}^{(\varepsilon)}, Q_\nu - x_{\varepsilon,n}),
\end{aligned}$$

where in the second to last step we used the periodicity of W.

We claim that

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}} \left(v_{n,k}^{\varepsilon}, (Q_\nu - x_{\varepsilon,n}) \setminus Q_\nu \right) = 0. \tag{4.103}$$

Indeed, using Fubini's Theorem and a change of variables, we have

$$\begin{aligned}
&\mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}} \left(v_{n,k}^{\varepsilon}, (Q_\nu - x_{\varepsilon,n}) \setminus Q_\nu \right) \\
&\quad = \int_{-\frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}}^{\frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}} \int_{(Q'_\nu - x_{\varepsilon,n})\setminus Q'_\nu} \left[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon y}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(y) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(y)|^2 \right] \mathrm{d}y \\
&\quad = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{(Q'_\nu - x_{\varepsilon,n})\setminus Q'_\nu} \left[T_k W\left(\left(T_k \frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k} + T_k x_\nu \nu \right), v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k} + x_\nu \nu \right) \right) \right. \\
&\qquad\qquad \left. + \frac{1}{T_k} \left| \nabla v_k \left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k} + x_\nu \nu \right) \right|^2 \right] \mathrm{d}\mathcal{H}^{N-1}(x')\mathrm{d}x_\nu .
\end{aligned}$$

Fix $k \in \mathbb{N}$. By (4.100), for each $\varepsilon > 0$, let $n(\varepsilon) \in \mathbb{N}$ be such that $|x_{\varepsilon,n}| < \varepsilon$ for all $n \geq n(\varepsilon)$. In particular, we have $(Q'_\nu - x_{\varepsilon,n}) \setminus Q'_\nu \subset (1+\varepsilon)Q'_\nu \setminus Q'_\nu$. Set $\mu_n^{\varepsilon,k} := \frac{\varepsilon}{\varepsilon_n^{\mathcal{R}} T_k}$. For every $x_\nu \in (-\frac{1}{2}, \frac{1}{2})$, the functions $f, g : Q'_\nu \to \mathbb{R}$ defined by

$$f(x') := W\left((T_k x' + T_k x_\nu \nu), v_k(x' + x_\nu \nu)\right), \quad g(x') := \left| \nabla v_k \left(x' + \nu x_\nu \right) \right|^2$$

are Q'_ν periodic. The Riemann-Lebesgue Lemma yields

$$\lim_{n\to\infty}\int_U f(\mu_n^{\varepsilon,k}x')\mathrm{d}\mathcal{H}^{N-1}(x') \\ = |U|\int_{Q'_\nu} W\left((T_kx'+T_kx_\nu\nu),v_k(x'+x_\nu\nu)\right)\mathrm{d}\mathcal{H}^{N-1}(x') \tag{4.104}$$

and

$$\lim_{n\to\infty}\int_U g(\mu_n^{\varepsilon,k}x')\mathrm{d}\mathcal{H}^{N-1}(x') = |U|\int_{Q'_\nu}\left|\nabla v_k\left(x'+x_\nu\nu\right)\right|^2\mathrm{d}\mathcal{H}^{N-1}(x')\,, \tag{4.105}$$

for every open and bounded set $U\subset\mathbb{R}^N$. Thus we get

$$\begin{aligned}
&\limsup_{n\to\infty}\mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}}\left(v^\varepsilon_{n,k},(Q_\nu-x_{\varepsilon,n})\setminus Q_\nu\right)\\
&\quad\le\limsup_{n\to\infty}\int_{-\frac12}^{\frac12}\int_{(1+\varepsilon)Q'_\nu\setminus Q'_\nu}\Bigg[T_kW\bigg(\bigg(T_k\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+T_kx_\nu\nu\bigg),v_k\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+x_\nu\nu\bigg)\bigg)\\
&\qquad\qquad+\frac{1}{T_k}\bigg|\nabla v_k\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+x_\nu\nu\bigg)\bigg|^2\Bigg]\mathrm{d}\mathcal{H}^{N-1}(x')\mathrm{d}x_\nu\\
&\quad\le|(1+\varepsilon)Q'_\nu\setminus Q'_\nu|\left(\int_{Q_\nu}\left[T_kW\left(T_kx,v_k(x)\right)+\frac{1}{T_k}\left|\nabla v_k(x)\right|^2\right]dx\right).
\end{aligned}$$

Sending $\varepsilon\to0$ we obtain (4.103).

Finally, we claim that

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to0^+}\limsup_{n\to\infty}\mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}}\left(v^\varepsilon_{n,k},Q_\nu\right)=\sigma^{Q_\nu}(\nu)\,. \tag{4.106}$$

Recalling the definition of the functions $v^{(\varepsilon)}_{n,k}$ (see (4.99)) and using Fubini's Theorem we can write

$$\begin{aligned}
\mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}}\left(v^\varepsilon_{n,k},Q_\nu\right)&=\mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}}\left(v^\varepsilon_{n,k},Q_\nu\cap\left\{|x_\nu|\le\frac{\varepsilon_n^{\mathcal{R}}T_k}{2\varepsilon}\right\}\right)\\
&=\int_{-\frac{\varepsilon_n^{\mathcal{R}}T_k}{2\varepsilon}}^{\frac{\varepsilon_n^{\mathcal{R}}T_k}{2\varepsilon}}\int_{Q'_\nu}\Bigg[\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}}W\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}}+\frac{\varepsilon x_\nu\nu}{\varepsilon_n^{\mathcal{R}}},v_k\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+\frac{\varepsilon x_\nu\nu}{\varepsilon_n^{\mathcal{R}}T_k}\bigg)\bigg)\\
&\qquad\qquad+\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}T_k^2}\bigg|\nabla v_k\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+\frac{\varepsilon x_\nu\nu}{\varepsilon_n^{\mathcal{R}}T_k}\bigg)\bigg|^2\Bigg]\mathrm{d}x'\mathrm{d}x_\nu\\
&=\int_{-\frac12}^{\frac12}\int_{Q'_\nu}\Bigg[T_kW\bigg(T_k\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+T_ky_\nu\nu,v_k\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+y_\nu\nu\bigg)\bigg)\\
&\qquad\qquad+\frac{1}{T_k}\bigg|\nabla v_k\bigg(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}T_k}+y_\nu\nu\bigg)\bigg|^2\Bigg]\mathrm{d}x'\mathrm{d}y_\nu.
\end{aligned}$$

Thus, using (4.104) and (4.105) (that are independent of ε), we obtain

$$\begin{aligned}\lim_{k\to\infty}\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\mathcal{F}_{\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}}\left(v^{\varepsilon}_{n,k},Q_\nu\right) &= \lim_{k\to\infty}\int_{Q_\nu}\left(T_kW(T_kx,v_k(x))+\frac{1}{T_k}|\nabla v_k(x)|^2\right)\mathrm{d}x\\ &= \sigma^{Q_\nu}(\nu).\end{aligned}$$

From (4.102), (4.103) and (4.106) we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\lambda(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} &\le \limsup_{k\to\infty}\limsup_{\varepsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{\varepsilon^{N-1}}\mathcal{F}_{\varepsilon_n^{\mathcal{R}}}(u_{n,\varepsilon,k},Q_\nu(x_0,\varepsilon))\\ &\le \sigma^{Q_\nu}(\nu)\,. \end{aligned}\tag{4.107}$$

In order to conclude, we use Lemma 4.28 to find a sequence $\{Q_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}^{\Lambda}_\nu$ such that $\sigma^{Q_n}(\nu) \to \sigma(\nu)$ as $n\to\infty$. Using (4.107) we obtain for every $n\in\mathbb{N}$

$$\frac{d\lambda}{d\mu}(x_0) = \lim_{\varepsilon\to 0}\frac{\lambda(Q_n(x_0,\varepsilon))}{\varepsilon^{N-1}} \le \sigma^{Q_n}(\nu)$$

and, letting $n\to\infty$ we have

$$\frac{d\lambda}{d\mu}(x_0) \le \sigma(\nu).$$

Using the Urysohn property, we conclude that if the set $A := \{u = a\}$ is Λ-polyhedral, then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ such that

$$\limsup_{n\to\infty}\mathcal{F}_{\varepsilon_n}(u_n) \le \mathcal{F}_0(u).$$

Case 2. We now consider the general case of a function $u \in BV(\Omega;\{a,b\})$. Using Lemma 4.22 it is possible to find a sequence of functions $\{v_k\}_{k\in\mathbb{N}} \subset BV(\Omega;\{a,b\})$ with the following properties: the set $A_k := \{v_k = a\}$ is a Λ-polyhedral set and, setting $A := \{u = a\}$, we have

$$\lim_{k\to\infty}\|\chi_{A_k}-\chi_A\|_{L^1(\Omega)} = 0\,, \qquad \lim_{k\to\infty}|P(A_k;\Omega)-P(A;\Omega)| = 0\,.$$

From the result of Case 1, for every $k\in\mathbb{N}$ it is possible to find a sequence $\{u^k_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ with $u^k_n \to v_k$ as $n\to\infty$, such that

$$\limsup_{n\to\infty}\mathcal{F}_{\varepsilon_n}(u^k_n) \le \mathcal{F}_0(v_k)\,.$$

Choose an increasing sequence $\{n(k)\}_{k\in\mathbb{N}}$ such that, setting $u_k := u^k_{n(k)}$,

$$\|u_k - u\|_{L^1} \leq \frac{1}{k}\,, \qquad \mathcal{F}_{\varepsilon_n}(u^k_n) \leq \mathcal{F}_0(v_k) + \frac{1}{k}\,. \tag{4.108}$$

Recalling that the function σ is upper semi-continuous on $\mathbb{S}^{N-1}$ (see Proposition 4.26), from Theorem 2.12 and (4.108) we get

$$\limsup_{k\to\infty} \mathcal{F}_{\varepsilon_{n(k)}}(u^k) \leq \limsup_{k\to\infty} \mathcal{F}_0(v_k) \leq \mathcal{F}_0(u)\,.$$

This concludes the proof of the limsup inequality. □

4.3.6 Continuity of σ

To prove that the function $\nu \mapsto \sigma(\nu)$ is continuous, notice that Theorem 4.9 implies, in particular, that the functional $\mathcal{F}_0$ is lower semi-continuous with respect to the L^1 convergence. It then follows from [4, Theorem 5.11] that the function σ, when extended 1-homogeneously to the whole $\mathbb{R}^N$, is convex. Since $\sigma(\nu) < \infty$ for every $\nu \in \mathbb{S}^{N-1}$ (see Lemma 4.23), we also deduce that σ is continuous.

For the convenience of the reader, we recall here the argument used in [4, Theorem 5.11] to prove convexity. Take $v_0, v_1, v_2 \in \mathbb{R}^N$ such that $v_0 = v_1 + v_2$. We claim that $\sigma(v_0) \leq \sigma(v_1) + \sigma(v_2)$. Using the 1-homogeneity of σ, this is equivalent to convexity. To prove the claim, let $E := \{x \in \Omega \,:\, x \cdot \nu_0 \leq \alpha\}$, where $\alpha \in \mathbb{R}$ is such that $\Omega \setminus E \neq \emptyset$, $\Omega \cap E \neq \emptyset$. Consider a cube $z + rQ \subset \Omega \setminus E$, where $z \in \mathbb{R}^N$ and $r > 0$, and a triangle $T \subset rQ$ with outer normals $-\frac{\nu_0}{|\nu_0|}$, $\frac{\nu_1}{|\nu_1|}$ and $\frac{\nu_2}{|\nu_2|}$. For $n \in \mathbb{N}$, let

$$E_n := E \cup \bigcup_{i=1}^{n^{N-1}} \left(z_i + \frac{1}{n} T \right),$$

where the z_i's are such that $z_i + \frac{1}{n}T \subset z + rQ$ and $(z_i + \frac{1}{n}T) \cap (z_j + \frac{1}{n}T) \neq \emptyset$ if $i \neq j$. It can be shown that $\chi_{E_n} \to \chi_E$, so by lower semi-continuity of $\mathcal{F}_0$ we obtain

$$\begin{aligned}\mathcal{H}^{N-1}(E)\sigma(\nu_0) = \mathcal{F}_0(\chi_E) &\leq \liminf_{n\to\infty} \mathcal{F}_0(\chi_{E_n}) \\ &= \mathcal{H}^{N-1}(E)\sigma(\nu_0) + L[\sigma(\nu_1) + \sigma(\nu_2) - \sigma(\nu_0)],\end{aligned}$$

where $L > 0$ is the length of the side of T orthogonal to ν_0. This proves the claim.

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