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Cutting Planes and Integrality of Polyhedra: Structure and Complexity

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on various collaborations with Ahmad Abdi, Gérard Cornuéjols, Sanjeeb Dash, Oktay Günlük, Natália Guričanová and Yanjun Li. These include, but are not limited to, papers [2, 4, 5, 6, 36, 37, 52, 90].

Abstract

In this thesis, we study theoretical aspects of integer linear programming. This thesis consists of two main parts: the first part is on the theory of cutting planes for integer linear programming, while the second part is on the theory of ideal clutters in combinatorial optimization.

Cutting planes for an integer linear program are linear inequalities that are valid for all integer feasible solutions but possibly violated by some solutions to the linear programming relaxation. The Chvátal-Gomory cuts, introduced by Gomory in 1958 and further studied by Chvátal in 1973 in relation to their applications in combinatorial optimization, are one of the simplest types of cutting planes. The split cuts are another class of cutting planes that are important in modern integer linear programming. The first part of this thesis discusses our recent developments in the theory of Chvátal-Gomory cuts and split cuts. We study rational polyhedra with Chvátal rank 1, rational polyhedra with split rank 1, some sufficient conditions under which a rational polytope in the 0,1 hypercube has small Chvátal rank, and a generalization of the Chvátal closure.

Let E be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of E called *members*. We say that $\mathcal{C}$ is a *clutter* over *ground set* E if no member contains another. We say that the clutter $\mathcal{C}$ is *ideal* if the system $\left(\sum(x_e : e \in C) \geq 1 \ \forall C \in \mathcal{C},\ x_e \geq 0 \ \forall e \in E\right)$ defines an integral polyhedron. One can find rich classes of ideal clutters that arise in combinatorial optimization: the clutter of st-paths, the clutter of T-joins, the clutter of dijoins, the clutter of the odd circuits of a weakly bipartite graph, etc. As these wide range of examples suggest, characterizing when a clutter is ideal is still a major open question in integer programming and combinatorial optimization. One of the conjectures that were made to understand the question is the $\tau = 2$ Conjecture by Cornuéjols, Guenin, and Margot in 2000. In the second part of this thesis, we study and develop tools to solve the $\tau = 2$ Conjecture. We introduce intersecting clutters and multipartite clutters and study two equivalent versions of the $\tau = 2$ Conjecture stated in terms of intersecting clutters and multipartite clutters.

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Dedication

I dedicate this thesis to my parents and Jesus Christ:

> In him lie hidden all the treasures of wisdom and knowledge. (Colossians 2:3)
>
> He existed before anything else, and he holds all creation together. (Colossians 1:17)

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Chapter 1

Introduction

Integer Linear Programming is the problem of optimizing a linear function over the set of integer points satisfying a system of linear inequalities. To be precise, integer linear programming is an optimization problem of the following form:

$$\begin{aligned} \text{minimize} \quad & w^\top x \\ \text{subject to} \quad & Ax \geq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

where A is an m by n rational matrix for some positive integers m, n and w, b are rational vectors of appropriate dimension. Due to its flexibility in modeling, integer linear programming is used to formulate a wide range of practical problems in operations research. Many combinatorial optimization problems on the theoretical side of operations research can also be formulated as integer linear programs. The matching problem and the stable set problem are classic examples admitting simple integer linear programming formulations (see § 1.2).

Another important example is the so-called *Set Covering Problem*. As the formulation of the set covering problem is one of the two main topics in this thesis, let us introduce the problem and provide its most standard formulation. Let E be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of E called *members*. A *cover* of $\mathcal{C}$ is a subset of E that intersects every member of $\mathcal{C}$. The set covering problem for $\mathcal{C}$ is to find a minimum weight cover of $\mathcal{C}$ with respect to weights on elements $w \in \mathbb{Q}_+^E$. We say that $\mathcal{C}$ is a *clutter* over *ground set* E if no member contains another [59]. As it is sufficient to consider the minimal members of $\mathcal{C}$, we may assume that $\mathcal{C}$ is a clutter. The following integer linear program formulates the set covering problem:

$$\begin{aligned} \text{minimize} \quad & w^\top x \\ \text{subject to} \quad & M(\mathcal{C})x \geq \mathbf{1} \\ & x \in \mathbb{Z}_+^E \end{aligned}$$

where $M(\mathcal{C})$ denotes the member - element incidence matrix of $\mathcal{C}$, whose columns are labeled by the elements and whose rows are the characteristic vectors of the members, and $\mathbf{1}$ denotes the vector of all

ones of appropriate dimension. This version of the set covering problem is also called the *Hitting Set Problem*.

Despite its success in practical applications, inter linear programming is NP-hard in general [85, 68], as the stable set problem and the set covering problem are NP-hard [85, 68]. It is in contrast to *Linear Programming* that admits efficient polynomial time algorithms such as the *ellipsoid algorithm* by Khachiyan [86] and the *interior-point method* by Karmarkar [84]. On the other hand, integer linear programming is still closely related to linear programming, as solving the *linear programming relaxation* of an integer linear program is often the first step for solving the integer linear program. Given an integer linear program $\min\left\{w^\top x : Ax \geq b,\ x \in \mathbb{Z}^n\right\}$, its linear programming relaxation, or LP relaxation, is defined as $\min\left\{w^\top x : Ax \geq b,\ x \in \mathbb{R}^n\right\}$, obtained after relaxing the integrality constraints on variables. Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ denote the set of solutions to the linear relaxation. Then P is a rational polyhedron, which means that P is the intersection of the half-spaces defined by finitely many linear inequalities with rational coefficients. Let P_I denote the integer hull of P, namely $P_I := \operatorname{conv}(P \cap \mathbb{Z}^n)$, the convex hull of the integer points in P. Then P_I is also a rational polyhedron contained in P [99] and the integer linear program is equivalent to $\min\left\{w^\top x : x \in P_I\right\}$ as linear functions are convex. This implies that integer linear programming is equivalent to linear programming. However, a difficulty in directly applying efficient algorithms for linear programming is that computing a system of linear inequalities describing P_I is hard in general.

Cutting plane methods in combination with enumeration are commonly used to solve integer linear programming problems in practice. A basic idea of cutting plane methods is that although P is not identical to its integer hull P_I in general, we can approximate P_I better by adding linear inequalities that are valid for the points in P_I but violated by some point in P to the description of P. Such linear inequalities are called *cutting planes* or *cuts*. The *Chvátal-Gomory cuts* proposed by Gomory [71] and further studied by Chvátal [29] provide a simple way of generating cutting planes for any given polyhedron. Other simple cutting planes are the *split cuts* [12, 13, 33]. Intuitively, the difficulty of solving an integer linear program by cutting plane methods depends on its initial linear programming relaxation. The tighter P is, or the closer P is to P_I, the easier the integer program is to solve. An extreme case is when P equals P_I; then we can find an optimal integer solution in P in polynomial time. Thus one might wonder about the following question:

> **Question 1.** When is a rational polyhedron identical to its integer hull?

We say that a rational polyhedron is *integral* if it is identical to its integer hull. It is not always the case that a rational polyhedron is integral, but we are still interested in the case when a rational polyhedron is close to its integer hull.

> **Question 2.** When is it that the integer hull of a rational polyhedron can be obtained after applying some simple types of cutting planes?

These two theoretical questions are fundamental in integer linear programming, and they are the main topics of this thesis. In the first half of this thesis (Chapters 2, 3, 4, 5), we study Question 2 in terms of the Chvátal-Gomory cuts and the split cuts, with more emphasis on the former. In the second half (Chapters 6, 7, 8, 9), we study Question 1 in the context of the set covering problem. The first part focuses on the geometry of integer feasible solutions, while the second part uses more combinatorial ideas to study structures in a formulation.

The following provides a non-rigorous description of the key results in this thesis. In the first four chapters, we study the properties of the Chvátal and split closures in terms of their integer width, computational complexity and incorporating bound constraints.

- In Chapter 2, we study the problem of optimizing a linear function over the set of integer solutions contained in a rational polytope whose Chvátal rank is 1. It is observed that this problem is in the complexity class NP $\cap$ co-NP, an indication that it is probably not NP-complete. It is open whether there is a polynomial time algorithm to solve the problem. We show that any compact convex set whose Chvátal closure is empty has an integer width of at most n, and we give an example showing that this bound is tight within an additive constant of 1. This determines the time complexity of a Lenstra-type algorithm. However, the promise that a polytope has Chvátal rank 1 seems hard to verify. We extend an earlier result on the NP-hardness of determining whether the Chvátal closure of a given rational polyhedron is empty to the case when the given polyhedron is bounded in the unit hypercube. As a direct corollary, we prove that optimization and separation over the Chvátal closure of a rational polytope are NP-hard, even when the polytope is in the unit hypercube.
- In Chapter 3, we prove that deciding whether the split closure of a rational polytope is empty is NP-hard, even when the polytope is contained in the unit hypercube. This result is similar in spirit to the result for the Chvátal closure in Chapter 2. As a direct corollary, we prove that optimization and separation over the split closure of a rational polytope in the unit hypercube are NP-hard. We also show that any compact convex set whose Chvátal closure is empty has an integer width of at most $2n$.
- In Chapter 4, we provide sufficient conditions under which a polytope contained in the unit hypercube has a small Chvátal rank. Our conditions are in terms of the subgraph induced by these infeasible $0, 1$ vertices in the skeleton graph of the unit hypercube. In particular, we show that when this subgraph contains no 4-cycle, the Chvátal rank is at most 3; and when it has tree width 2, the Chvátal rank is at most 4. We also give polyhedral decomposition theorems when this graph has a vertex cutset of size one or two.
- In Chapter 5, we develop a generalization of the Chvátal closure. Integer programming problems that arise in practice and combinatorial optimization often involve nonnegative or bounded decision variables. Using information about the bounds on variables, one can generate possibly stronger cuts valid for all integer feasible solutions. In this chapter, we consider a natural extension of Chvátal-Gomory inequalities, which are obtained by strengthening Chvátal-Gomory inequalities based on the bound constraints on some variables. We prove that the closure of a rational polyhedron obtained after applying the generalized Chvátal-Gomory inequalities is also a rational polyhedron. Our technique is motivated by a result of Dunkel and Schulz.

The next four chapters make progress towards proving the $\tau = 2$ Conjecture, a well-known conjecture about ideal non-packing clutters, by introducing new characterizations and studying important special cases.

- In Chapter 6, we introduce and study *intersecting clutters*. A clutter is intersecting if the members do not have a common element yet every two members intersect. One important class of intersecting clutters comes from projective planes, namely the deltas, while another comes from graphs, namely the blockers of extended odd holes. We show that the $\tau = 2$ Conjecture, which states that every

ideal minimally non-packing clutter has covering number 2, is equivalently stated as follows: an ideal clutter has the max-flow min-cut property if and only if it has no intersecting minor. Hence, if the $\tau = 2$ Conjecture is true, we have an excluded-minor characterization for the ideal clutters that have the max-flow min-cut property. As the first step towards proving the conjecture, we provide a polynomial characterization of clutters without an intersecting minor. Using similar techniques, we provide a polynomial algorithm for finding a delta or the blocker of an extended odd hole minor in a given clutter. This result is quite surprising as the same problem is NP-hard if the input were the blocker instead of the clutter.

- In Chapter 7, we introduce *multipartite clutters*. We say that a clutter is *multipartite* if its ground set is partitioned into nonempty parts so that every member intersects each part exactly once. A multipartite clutter can be viewed as the clutter of hyperedges in a multipartite hypergraph. As each part is a cover, this construction of multipartite clutters provides a natural way of generating clutters with covering number greater than 2. In fact, the $\tau = 2$ Conjecture is equivalent to the following conjecture: if a multipartite clutter is ideal and has no intersecting minor, then it packs. In Chapter 7, we develop tools and techniques to solve this equivalent version of the $\tau = 2$ Conjecture. Using these tools, we develop a computer program to check multipartite clutters over at most 9 elements and confirm the $\tau = 2$ Conjecture for the multipartite clutters over at most 9 elements.
- In Chapter 8, we study a binary operations defined on two multipartite clutters, the *reflective product*. The reflective product preserves idealness, so it provides a way of generating ideal clutters. We observe that a lot of clutters that are known to be ideal and minimally non-packing are obtained by taking the reflective product of two multipartite clutters. Then we show that an ideal minimally non-packing clutter obtained by a reflective product must have covering number two, implying that the $\tau = 2$ Conjecture holds for the multipartite clutters obtained by a reflective product.
- In Chapter 9, we consider a special class of multipartite clutters, the multipartite clutters of vector spaces. We provide a complete excluded-minor characterization of when the multipartite clutter of a vector space is ideal and when it has the max-flow min-cut property. As a consequence, we show that Q_6 is the only ideal minimally non-packing clutter that is the multipartite clutter of a vector space, implying in turn that the $\tau = 2$ Conjecture holds for the multipartite clutters of vector spaces.

The rest of this chapter serves as an extended abstract of the thesis. Let us explain the results in greater detail and rigor in the remainder of this chapter.

1.1 Preliminaries: the Chvátal closure and the split clousre

In the first part of this thesis, the main focus in on the Chvátal-Gomory cuts and the split cuts in integer linear programming. In this section, we give an introduction to the Chvátal-Gomory cuts and the split cuts, the Chvátal closure and split closure of a polyhedron, and the Chvátal rank and split rank of a polyhedron.

Chvátal closure and rank

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, and let P_I denote its integer hull. If an inequality $cx \leq d$ with $c \in \mathbb{Z}^n$ is valid for P, then $cx \leq \lfloor d \rfloor$ is valid for all the integer solutions contained in P, and thus for P_I.

We call $cx \leq \lfloor d \rfloor$ the *Chvátal-Gomory cut or Chvátal-Gomory inequality of P obtained from $cx \leq d$.* This approach for generating cutting planes was first introduced by Gomory [71]. Chvátal [29] later introduced the following beautiful notion of closure, which is obtained by applying all possible Chvátal-Gomory inequalities.

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \leq \lfloor \max\{cx : \ x \in P\} \rfloor\}$$

It follows from the definition that $P_I \subseteq P' \subseteq P$, and we call P' the *Chvátal closure* of P. Question 2 for the Chvátal-Gomory cuts is asking when the Chvátal closure of a rational polyhedron is indentical to its integer hull.

Theorem 1.1 (Chvátal [29], Schrijver [107])**.** *The Chvátal closure of a rational polyhedron is, again, a rational polyhedron.*

As the Chvátal closure of a rational polyhedron is a rational polyhedron, we can recursively apply the operation of taking the Chvátal closure. Let $P^{(k)}$ denote $\left(P^{(k-1)}\right)'$ for $k \geq 2$, where $P^{(1)} = P'$. We say that a Chvátal-Gomory inequality of the $(k-1)^{\text{th}}$ Chvátal closure of P is a rank-k Chvátal inequality of P. In fact, there exists a finite integer k such that $P^{(k)} = P_I$ [29, 107], and the *Chvátal rank* of P is defined as the smallest such k.

Split closure and rank

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, and let P_I denote its integer hull. Given $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, any point $z \in \mathbb{Z}^n$ satisfies either $\pi z \leq \pi_0$ or $\pi z \geq \pi_0 + 1$. We call an inequality $cx \leq d$ a *split cut* if it is valid for both

$$\Pi_1 = P \cap \{x \in \mathbb{R}^n : \ \pi x \leq \pi_0\} \quad \text{and} \quad \Pi_2 = P \cap \{x \in \mathbb{R}^n : \ \pi x \geq \pi_0 + 1\}$$

for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. We call the set $S(\pi, \pi_0) := \{x \in \mathbb{R}^n : \ \pi x \leq \pi_0 \text{ or } \pi x \geq \pi_0 + 1\}$ the *split disjunction* derived from $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. Clearly, $P_I \subseteq \text{conv}(P \cap S(\pi, \pi_0)) \subseteq P$ and an inequality is a split cut if and only if it is valid for $\text{conv}(P \cap S(\pi, \pi_0))$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. Split cuts are a special case of Balas' *disjunctive cuts* [12] that can be obtained from a *split disjunction.* It is straightforward that $\{x \in \mathbb{R}^n : \ \pi_0 < \pi x < \pi_0 + 1\}$, the *split set associated with* (π, π_0), does not contain any integer point, so split cuts are also a type of *intersection cuts* introduced by Balas [13]. Note also that split cuts are a generalization of Chvátal-Gomory cuts, as a Chvátal-Gomory cut is a split cut obtained from a split disjunction where one side of the disjunction is empty.

Cook, Kannan, and Schrijver [33] introduced a notion of closure as follows.

$$P^* := \bigcap_{(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}} \text{conv}\left(P \cap S(\pi, \pi_0)\right)$$

is called the *split closure* of P. By its definition, $P_I \subseteq P^* \subseteq P$. Moreover, the split closure of P is contained in the Chvátal closure of P, as a Chvátal-Gomory cut is a split cut. The *MIR* closure of P, obtained after applying all *mixed integer rounding cuts* [100] of P, and the *MI* closure of P, obtained after applying all *Gomory's mixed integer cuts* [72] of P, are both identical to the split closure of P [100, 39].

Theorem 1.2 (Cook, Kannan, Schrijver [33])**.** *The split closure of a rational polyhedron is, again, a rational polyhedron.*

This result is the analogue of Theorem 1.1 for the split closure of a rational polyhedron. Later, Andersen, Cornuéjols, and Li [9], Dash, Günük, and Lodi [51], and Vielma [115] found different proofs. We can take the split closure recursively. Since the split closure of P is a subset of the Chvátal closure of P, there exists a finite integer k such that the k^{th} split closure of P, obtained after taking the split closure recursively k times, is identical to P_I, and the *split rank* of P is defined as the smallest such k.

1.2 Rational polytopes with rank 1

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron, and let P_I denote its integer hull. If the Chvátal rank (resp. split rank) of P is 1, then P_I can be obtained after applying the Chvátal-Gomory cuts (resp. split cuts) for P. For instance, the *fractional matching polytope* of a graph $G = (V, E)$

$$\left\{x \in \mathbb{R}^E : \sum (x_e : e \in \delta(v)) \leq 1 \ \forall v \in V, \ \ 0 \leq x_e \leq 1 \ \forall e \in E\right\}$$

has Chvátal rank 1, because the matching polytope of a graph, the convex hull of the characteristic vectors of matchings in G, is obtained after adding the *odd set inequalities* [57] and the odd set inequalities have Chvátal rank 1 [29]. Then the split rank of the fractional matching polytope is also 1, because the split rank is less than or equal to the Chvátal rank. Another example comes from the stable set problem. The *fractional stable set polytope* of $G = (V, E)$ is defined as

$$\left\{x \in \mathbb{R}^V : \ x_u + x_v \leq 1 \ \forall uv \in E, \ \ 0 \leq x_v \leq 1 \ \forall v \in V\right\}.$$

It is known that the Chvátal rank of the fractional stable set polytope is 1 if, and only if, its split rank is 1 if, and only if, G is *t-perfect* [69, 25].

In general, when is it that a rational polyhedron has Chvátal / split rank 1? Chapters 2 and 3 consider the problem of testing whether a rational polyhedron has Chvátal / split rank 1. The following theorem is for the Chvátal rank:

Theorem 1.3 ([37], proved in Chapter 2)**.** *Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ containing no integer point, it is NP-complete to test whether the Chvátal closure of P is empty, even when $P \subseteq [0, 1]^n$ or P is a rational simplex.*

This result extends an earlier result by Cornuéjols and Li [38, 40]. Analogously, the same statement for the split rank also holds:

Theorem 1.4 ([90], proved in Chapter 3)**.** *Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ containing no integer point, it is NP-complete to test whether the split closure of P is empty, even when $P \subseteq [0, 1]^n$.*

As direct consequences, we obtain the following corollaries:

Corollary 1.5 ([37])**.** *Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$, it is NP-hard to decide whether the Chvátal rank of P is 1 and it is NP-hard to optimize over the Chvátal closure of P, even when $P \subseteq [0, 1]^n$ or P is a rational simplex.*

Corollary 1.6 ([90])**.** *Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$, it is NP-hard to decide whether the split rank of P is 1 and it is NP-hard to optimize over the split closure of P, even when $P \subseteq [0, 1]^n$.*

Corollary 1.5 improves an earlier result by Eisenbrand [62] on the membership problem for the Chvátal closure of a polyhedron, while Corollary 1.6 extends a result of Caprara and Letchford [27] on the separation problem of split cuts.

We have just observed that given a rational polyhedron, optimizing over its integer hull and Chvátal closure are both NP-hard. Unlike this observation, if the integer hull and Chvátal closure of a rational polyhedron coincide, the following, which may seem at first counterintuitive, turns out to be true. Boyd and Pulleyblank [21] observed that:

Proposition 1.7 ([21]). *Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ be a rational polyhedron whose Chvátal rank is 1. Then*

(1) the problem of deciding whether $P \cap \mathbb{Z}^n = \emptyset$,

(2) given $c \in \mathbb{Q}^n$, the problem of deciding whether $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ is unbounded,

(3) given $c \in \mathbb{Q}^n$ and $x^* \in \mathbb{Z}^n$, the problem of deciding whether $cx^* = \max\{cx : x \in P \cap \mathbb{Z}^n\}$

belong to complexity class NP $\cap$ co-NP.

The same complexity statement holds for the split rank.

Proposition 1.8 ([90], proved in Chapter 3). *Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ be a rational polyhedron whose split rank is 1. Then*

(1) the problem of deciding whether $P \cap \mathbb{Z}^n = \emptyset$,

(2) given $c \in \mathbb{Q}^n$, the problem of deciding whether $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ is unbounded,

(3) given $c \in \mathbb{Q}^n$ and $x^* \in \mathbb{Z}^n$, the problem of deciding whether $cx^* = \max\{cx : x \in P \cap \mathbb{Z}^n\}$

belong to complexity class NP $\cap$ co-NP.

As it is believed that the problems in NP $\cap$ co-NP are easier than the NP-hard problems, one might wonder if there is an efficient algorithm for integer linear programming over rational polytopes with Chvátal / split rank 1. For example, Edmonds' blossom algorithm [57] finds a maximum weight matching in a graph in polynomial time, and there is a polynomial time algorithm for finding a maximum weight stable set in a t-perfect graph [69]. In Chapters 2 and 3, we study algorithms for integer linear programming over rational polytopes with Chvátal / split rank 1. In particular, we consider *Lenstra-type* algorithms and their time complexity.

Lenstra [93] found the first algorithm for integer linear programming that runs in polynomial time when there are a constant number of integer variables. An important concept in Lenstra's algorithm is the notion of *integer width*. Let $K \subset \mathbb{R}^n$ be a convex set and $d \in \mathbb{Z}^n$. The integer width of K along d is

$$w(K, d) := \lfloor \sup\{dx : x \in K\} \rfloor - \lceil \inf\{dx : x \in K\} \rceil + 1.$$

The integer width of K, denoted as $w(K, \mathbb{Z}^n)$, is the minimum of the values $w(K, d)$ over all $d \in \mathbb{Z}^n \setminus \{0\}$.

$$w(K, \mathbb{Z}^n) := \min\{w(K, d) : d \in \mathbb{Z}^n \setminus \{0\}\}.$$

It is known that if a compact convex set contains no integer point, then its integer width is bounded by a function that only depends on the ambient dimension. Banaszczyk, Litvak, Pajor, and Szarek [16] and Rudelson [105] proved that $w(K, \mathbb{Z}^n) = O(n^{4/3}\text{polylog}(n))$ for every lattice-free compact convex set $K \subseteq \mathbb{R}^n$. It is conjectured that the upper bound can be improved to $O(n)$. We show that the conjecture holds for the following two special cases.

Theorem 1.9 ([37], proved in Chapter 2)**.** *The integer width of any rational polyhedron in $\mathbb{R}^n$ whose Chvátal closure is empty is at most n.*

Theorem 1.10 (Proved in Chapter 3)**.** *The integer width of any rational polytope in $\mathbb{R}^n$ whose split closure is empty is at most $2n$.*

Based on these results on the integer width, we provide and analyze Lenstra-type algorithms for integer linear programming over rational polytopes with Chvátal / split rank 1.

1.3 Polytopes in the 0,1 hypercube that have small Chvátal rank

In Chapter 2, we have shown that it is NP-hard to test whether a rational polytope in the 0,1 hypercube has Chvátal rank 1. Although it is probably difficult to exactly characterize when a rational polytope has small Chvátal rank, understanding some sufficient conditions under which a rational polytope has small Chvátal rank is still an interesting question. In Chapter 4, we consider polytopes contained in the 0,1 hypercube and their Chvátal rank. Eisenbrand and Schulz [63] showed that the Chvátal rank of a rational polytope in the 0,1 hypercube is $O(n^2 \log n)$, while Rothvoß and Sanità [104] showed the existence of a polytope contained in the hypercube with Chvátal rank $\Omega(n^2)$. In an orthogonal direction, we study the following question:

> When does a polytope in the 0,1 hypercube have small Chvátal rank?

Equivalently, we study when the integer hull of a polytope in the 0,1 hypercube is described by Chvátal inequalities of small rank.

Take a positive integer n, and let $P \subseteq [0,1]^n$ be a polytope. Let $S := P \cap \{0,1\}^n$, and let $\bar{S} := \{0,1\}^n \setminus S$. Then $P_I = \text{conv}(S)$. We denote by G_n the *skeleton graph* of the hypercube $[0,1]^n$ whose vertices correspond to the 2^n vertices of the hypercube and whose edges correspond to its 1-dimensional faces, namely the $n2^{n-1}$ line segments joining 2 points that differ in exactly 1 coordinate. Let $G(\bar{S})$ denote the subgraph of G_n induced by the vertices in $\bar{S}$.

Theorem 1.11 ([36], proved in Chapter 4)**.** *Let $P \subseteq [0,1]^n$ be a rational polytope contained in the unit cube. Let $\bar{S} := \{0,1\}^n \setminus P$. Then the following statements hold:*

(1) if $\bar{S}$ is a stable set in G_n, then the Chvátal rank of P is at most 1,

(2) if $G(\bar{S})$ is a disjoint union of cycles of length greater than 4 and paths, then the Chvátal rank of P is at most 2,

(3) if $G(\bar{S})$ is a forest, then the Chvátal rank of P is at most 3.

(4) if $G(\bar{S})$ has tree-width 2, then the Chvátal rank of P is at most 4.

To prove this theorem, we work with a canonical polytope Q_S that has exactly the same set S of feasible 0,1 points. The description of Q_S is as follows:

$$Q_S := \left\{ x \in [0,1]^n : \sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \frac{1}{2} \text{ for } \bar{x} \in \bar{S} \right\}$$

Remark 1.12 ([36]). *Let $P \subseteq [0,1]^n$ be a rational polytope, and let $S := P \cap \{0,1\}^n$. Then the following statements hold:*

(1) P and Q_S have the same set S of feasible 0,1 solutions,

(2) the Chvátal rank of P is less than or equal to that of Q_S.

Proof. **(1)**: The inequalities defining Q_S cut off the 0,1 vectors in $\bar{S}$ and no other. Therefore $S = Q_S \cap \{0,1\}^n$. **(2)**: Note that if two polytopes P and R have the same set of integer solutions and $P \subseteq R$, then the Chvátal rank of P is always less than or equal to the Chvátal rank of R. We will construct such a polytope R from P. For each $\bar{x} \in \bar{S}$, the linear program $\min_P \sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j)$ has a positive objective value. Therefore there exists $0 < \epsilon_{\bar{x}} \leq \frac{1}{2}$ such that the inequality $\sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \epsilon_{\bar{x}}$ is valid for P. Let $R := \{x \in [0,1]^n : \sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \epsilon_{\bar{x}} \text{ for } \bar{x} \in \bar{S}\}$. Now the lemma follows by observing that R and Q_S have the same first Chvátal closure. Indeed $Q_S \subseteq R$ implies $Q_S^{(1)} \subseteq R^{(1)}$ and, applying the Chvátal procedure to the inequalities defining R, we get that $R^{(1)} \subseteq \{x \in [0,1]^n : \sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \text{ for } \bar{x} \in \bar{S}\} = Q_S^{(1)}$. □

As Q_S is structured, we have a good handle on analyzing $Q_S^{(k)}$ for $k \geq 1$. To prove Theorem 1.11, we characterize $Q_S^{(k)}$ for $k = 1, 2, 3, 4$ and we use these results on Q_S to provide sufficient conditions for a polytope in the 0,1 hypercube to have Chvátal rank at most 4.

Motivated by Theorem 1.11, Benchetrit, Fiorini, Huynh, and Weltge [17] recently proved that

Theorem 1.13 ([17]). *Let $P \subseteq [0,1]^n$ be a rational polytope contained in the unit cube. Let $\bar{S} := \{0,1\}^n \backslash P$. If $G_n[\bar{S}]$ has tree-width t, then the Chvátal rank of P is at most $t + 2t^{t/2}$.*

We consider another interesting algorithmic property of Q_S in Chapter 4. Recall Proposition 1.7 that the problem of optimizing a linear function over the set of integer solutions contained in a rational polyhedron with Chvátal rank 1 is in NP ∩ co-NP, and it is open whether there is a polynomial algorithm for the problem. We prove that with the assumption that the Chvátal rank of Q_S is a constant, stronger than the assumption that the Chvátal rank of P is a constant, one can optimize a linear function over S in polynomial time.

Theorem 1.14 ([36], proved in Chapter 4). *Let $P \subseteq [0,1]^n$ be a rational polytope, and let $S := P \cap \{0,1\}^n$. If the Chvátal rank of Q_S is at most k, then one can optimize a linear function over S in $O(n^k)$ time.*

1.4 Generalized Chvátal closure

In Chapter 5, we study a generalization of Chvátal closures. Many combinatorial optimization problems involve binary decision variables or other discrete decisions, and integer programming models in practice often impose nonnegativity constraints on variables. In these cases, the set of integer feasible solutions is contained in some proper subset S of $\mathbb{Z}^n$. Using this preliminary information about the set of integer feasible solutions, one can generate stronger inequalities than the Chvátal-Gomory inequalities, valid for the integer feasible solutions. We introduce a natural generalization of the Chvátal-Gomory inequalities as follows.

Let $S \subseteq \mathbb{Z}^n$. Given $c \in \mathbb{Z}^n$ and $d \in \mathbb{R}$, let $\lfloor d \rfloor_{S,c}$ be defined as follows:

$$\lfloor d \rfloor_{S,c} = \begin{cases} \max \{cz : \ z \in S, \ cz \leq d\} & \text{if } \{z \in S : \ cz \leq d\} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Given an inequality $cx \leq d$ with $c \in \mathbb{Z}^n$ and $d \in \mathbb{R}$ valid for P, we call $cx \leq \lfloor d \rfloor_{S,c}$ the *S-Chvátal-Gomory inequality for P obtained from $cx \leq d$*. As $\lfloor d \rfloor_{S,c} = \lfloor d \rfloor$ when $S = \mathbb{Z}^n$, the S-Chvátal-Gomory inequalities indeed generalize the Chvátal-Gomory inequalities. The *S-Chvátal closure of P* is defined as

$$P_S := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \leq \lfloor \max\{cx : \ x \in P\} \rfloor_{S,c}\}$$

In words, the S-Chvátal closure of P is what is obtained after applying all possible S-Chvátal-Gomory inequalities. We assume for convention that $\{x \in \mathbb{R}^n : cx \leq -\infty\} = \emptyset$ for any $c \in \mathbb{Z}^n$.

Recall that the Chvátal closure of a rational polyhedron is also a rational polyhedron (Theorem 1.1). A natural question is whether the S-Chvátal closure of a rational polyhedron is also a rational polyhedron. Dunkel and Schulz's unpublished manuscript [55] was the first to consider this question. Dunkel and Schulz [55] proved that

Theorem 1.15 ([55]). *Let $S = \{0,1\}^n$, and let $P \subseteq [0,1]^n$ be a rational polytope. Then the S-Chvátal closure of P is a rational polytope.*

$\{0,1\}^n$ is the set of integer points satisfying the bounds $\mathbf{0} \leq x \leq \mathbf{1}$. In [52], we extend this result to the case when S is the set of integer points satisfying arbitrary set of bound constraints on variables.

Theorem 1.16 ([52]). *Let*

$$S = \left\{ \left(z^1, z^{2\ell}, z^{2u}, z^3\right) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_{2\ell}} \times \mathbb{Z}^{n_{2u}} \times \mathbb{Z}^{n_3} : \ \ell^1 \leq z^1 \leq u^1, \ z^{2\ell} \geq \ell^2, \ z^{2u} \leq u^2 \right\}$$

where $\ell^1, u^1 \in \mathbb{R}^{n_1}$ such that $\ell^1 \leq u^1$, $\ell^2 \in \mathbb{R}^{2\ell}$, and $u^2 \in \mathbb{R}^{2u}$. Let $P \subseteq \text{conv}(S)$ be a rational polyhedron. Then the S-Chvátal closure of P is a rational polyhedron.

As $S \cong \mathbb{Z}^{n_3}$ when $n_1 = n_{2\ell} = n_{2u} = 0$, this result is an extension of Theorem 1.1.

The proof of Theorem 1.16 is constructive and similar in spirit to the proof of Theorem 1.15 in that we write a linear system generating non-redundant S-Chvátal-Gomory inequalities. One might wonder if

it is possible to generalize the techniques developed for proving the polyhedrality of the Chvátal closure of a rational polyhedron [29, 107] or a compact convex set [23, 43, 56] and the polyhedrality of the split closure of a rational polyhedron [9, 51, 11]. However, it is difficult to directly apply those techniques, due to a difference between the Chvátal-Gomory inequalities and the S-Chvátal-Gomory inequalities. Given $S \subseteq \mathbb{Z}^n$, $c \in \mathbb{Z}^n$, and $d \in \mathbb{R}$, it is not always the case that $d - \lfloor d \rfloor_{S,c}$ is bounded by a fixed constant, whereas $d - \lfloor d \rfloor$ is always less than 1. For example, consider the case when $c = (k+1, k, \ldots, k)$ and $d = k-1$ for an arbitrary large integer k and $S = \mathbb{Z}^n_+$. In this case, $\lfloor d \rfloor_{S,c} = 0$, so we get $d - \lfloor d \rfloor_{S,c} = k-1$, and therefore, we can make $d - \lfloor d \rfloor_{S,c}$ arbitrary large in this case. This difference between the Chvátal-Gomory inequalities and the S-Chvátal-Gomory inequalities indeed makes the Chvátal closure and S-Chvátal closure structurally different.

Proposition 1.17 ([52], proved in Chapter 5)**.** *Let $S = \{0,1\}^4$. There exists a polytope $P \subseteq [0,1]^4$ whose S-Chvátal closure has a facet that cannot be induced by an S-Chvátal-Gomory inequality.*

On the contrary, the facets of the Chvátal closure of a rational polyhedron are all defined by Chvátal-Gomory inequalities (see [32]). We will discuss this in Chapter 5 with further details.

1.5 Preliminaries: ideal clutters

The second part of this thesis focuses on the question of when the set covering polyhedron of a clutter is integral. In this section, we introduce some basics of clutter theory.

1. We define ideal clutters and the max-flow min-cut property.
2. We introduce the notion of minor and that of blocker in clutter theory.
3. The $\tau = 2$ Conjecture and the Replication Conjecture will be discussed briefly.

Ideal clutters and the max-flow min-cut property

Let $\mathcal{C}$ be a clutter over ground set E. Recall that we can formulate the set covering problem for $\mathcal{C}$ as the following integer linear program:

$$\tau(\mathcal{C}, w) = \min \left\{ w^\top x : \ M(\mathcal{C})x \geq \mathbf{1}, \ x \in \mathbb{Z}^E_+ \right\}$$

where $w \in \mathbb{Z}^E_+$ are the weights of the elements. The following integer program

$$\nu(\mathcal{C}, w) = \max \left\{ \mathbf{1}^\top y : \ M(\mathcal{C})^\top y \leq w, \ y \in \mathbb{Z}^{\mathcal{C}}_+ \right\}$$

formulates the *capacitated packing problem*, that is the problem of finding the maximum size of a *packing* of members of $\mathcal{C}$ satisfying the capacity restriction for each element. The linear programming relaxations of these two integer programs are the following primal-dual pair:

$$\begin{array}{rlll} & \text{minimize} & w^\top x & \\ \tau^*(\mathcal{C}, w) = & \text{subject to} & M(\mathcal{C})x \geq \mathbf{1} & \\ & & x \geq \mathbf{0} & \end{array} \qquad \begin{array}{rlll} & \text{maximize} & \mathbf{1}^\top y & \\ \nu^*(\mathcal{C}, w) = & \text{subject to} & M(\mathcal{C})^\top y \leq w & \\ & & y \geq \mathbf{0} & \end{array}$$

By linear programming duality, we have

$$\tau(\mathcal{C}, w) \geq \tau^*(\mathcal{C}, w) = \nu^*(\mathcal{C}, w) \geq \nu(\mathcal{C}, w).$$

However, it is not always the case that $\tau(\mathcal{C}, w) = \tau^*(\mathcal{C}, w)$ nor $\nu(\mathcal{C}, w) = \nu^*(\mathcal{C}, w)$, as the linear programming relaxation of an integer program does not always have an integer optimal solution.

We say that $\mathcal{C}$ is *ideal* if $\tau(\mathcal{C}, w) = \tau^*(\mathcal{C}, w)$ for every $w \in \mathbb{Z}_+^E$ [41], and we say that $\mathcal{C}$ has the *max-flow min-cut property* if $\nu(\mathcal{C}, w) = \nu^*(\mathcal{C}, w)$ for every $w \in \mathbb{Z}_+^E$ [113]. Observe that $\mathcal{C}$ is ideal if and only if $Q(\mathcal{C}) := \left\{x \in \mathbb{R}_+^E : M(\mathcal{C})x \geq \mathbf{1}\right\}$, the *set covering polyhedron associated with* $\mathcal{C}$, is integral. Observe also that $\mathcal{C}$ has the max-flow min-cut property if and only if the linear system $M(\mathcal{C})x \geq \mathbf{1},\ x \geq \mathbf{0}$ is total dual integral. This implies that if a clutter has the max-flow min-cut property, then it is ideal [80, 60].

When $M(\mathcal{C})$ is totally unimodular [77] or balanced [18], $\mathcal{C}$ has the max-flow min-cut property and thus is ideal. There are other rich classes of ideal clutters that can be found in the combinatorial optimization literature, and let us mention a few examples here:

- (Menger [98]) The clutter of st-paths of a graph.
- (Edmonds and Johnson [61]) The clutter of minimal T-cuts of a graft.
- (Lucchesi and Younger [96]) The clutter of minimal dicuts of a directed graph.
- (Guenin [75]) The clutter of odd circuits of a signed graph that has no odd-K_5 minor.

The first and third classes of clutters have the max-flow min-cut property [67, 96], while the second and fourth do not [111, 113]. Given that there is a variety of examples, one might expect that testing idealness is difficult.

Theorem 1.18 (Ding, Feng, Zang [53])**.** *Let $\mathcal{C}$ be a clutter over ground set E whose members are explicitly given. The problems of deciding whether*

(1) $\mathcal{C}$ is ideal and

(2) $\mathcal{C}$ has the max-flow min-cut property

are both co-NP-complete.

In fact, Ding, Feng, Zang [53] proved that even when every element is contained in exactly two members, the problems remain co-NP-complete. In spite of this hardness result, it is still important to expand our understanding of ideal clutters and clutters with the max-flow min-cut property.

Minors and blockers

The notion of *minor* is an important concept for understanding when a clutter is ideal and when a clutter has the max-flow min-cut property. Given a clutter $\mathcal{C}$ over ground set E and disjoint subsets $I, J \subseteq E$, the *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J is the clutter over $E - (I \cup J)$ whose members are

$$\text{the minimal sets of } \left\{C - J :\ C \in \mathcal{C},\ C \cap I = \emptyset\right\}.$$

We say that the minor is *proper* if $I \cup J$ is nonempty. Contracting an element $e \in E$ corresponds to setting w_e, the weight of e, to ∞, while deleting e corresponds to setting $w_e = 0$. In terms of $Q(\mathcal{C})$, deleting an element $e \in E$ is equivalent to taking the projection of $Q(\mathcal{C})$ by projecting out variable x_e and contracting e is equivalent to taking the restriction of $Q(\mathcal{C})$ by setting $x_e = 0$.

Remark 1.19 ([113]). *The following statements hold:*

(1) if a clutter is ideal, then so is every minor of it,

(2) if a clutter has the max-flow min-cut property, then so does every minor of it.

We call a clutter *minimally non-ideal* if it is not ideal but every proper minor of it is. Lehman [92] (see also Seymour [110]) proved a theorem on the structure of minimally non-ideal clutters, and the structure explains why such clutters are non-ideal. One of the most fundamental classes of minimally non-ideal clutters is the *deltas*. For $n \geq 3$, the *delta of dimension* n, denoted Δ_n, is the clutter over ground set $[n] := \{1, \ldots, n\}$ whose members are

$$\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}$$

Observe that the elements and members of Δ_n correspond to the points and lines of a degenerate projective plane. Δ_n is non-ideal [91], as $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$ is a fractional extreme point of $Q(\Delta_n)$, and it can be readily checked that every proper minor of Δ_n is ideal. We say that a clutter has Δ_n as a minor if it has a minor that is isomorphic to Δ_n. Similarly, we say that *a clutter $\mathcal{C}$ has another clutter $\mathcal{C}'$ as a minor* if a minor of $\mathcal{C}$ is isomorphic[1] to $\mathcal{C}'$.

Another important idea in clutter theory is the notion of *blocker*. Given a clutter $\mathcal{C}$ over ground set E, the *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is defined as the clutter over the same ground set E whose members are the minimal covers of $\mathcal{C}$. For instance, the blocker of the clutter of st-paths of a graph is the clutter of minimal st-cuts, the blocker of the clutter of minimal T-cuts of a graft is the clutter of minimal T-joins, and the blocker of the clutter of minimal dicuts of a directed graph is the clutter of minimal dijoins. Notice also that $b(\Delta_n) = \Delta_n$. The following theorem proved by Lehman is important, and it is often referred to as *Lehman's width-length inequality*:

Theorem 1.20 (Lehman [91]). *Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent:*

(i) $\mathcal{C}$ is ideal,

(ii) (The width-length inequality) $\min\{w(C) : C \in \mathcal{C}\} \times \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell$ *for all* $w, \ell \in \mathbb{R}^E_+$,

(iii) $b(\mathcal{C})$ is ideal,

where $w(C) := \sum(w_e : e \in C)$ *and* $\ell(B) := \sum(\ell_e : e \in B)$.

[1] Given two clutters $\mathcal{C}, \mathcal{C}'$, we say that $\mathcal{C}$ is *isomorphic* to $\mathcal{C}'$ and write $\mathcal{C} \cong \mathcal{C}'$ if $\mathcal{C}'$ can be obtained from $\mathcal{C}$ after relabeling the elements of $\mathcal{C}$.

As mentioned before, the clutter of st-paths of a graph, the clutter of minimal T-cuts of a graft, and the clutter of minimal dicuts of a directed graph are ideal. So, by Theorem 1.20, the clutter of minimal st-cuts of a graph, the clutter of minimal T-joins of a graft, and the clutter of minimal dijoins of a directed graph are all ideal as well. One can easily observe that $b(b(\mathcal{C})) = \mathcal{C}$ [81, 59] and that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ for disjoint $I, J \subseteq E$ [112]. This, together with Theorem 1.20, implies that

Remark 1.21. *A clutter is minimally non-ideal if, and only if, its blocker is minimally non-ideal.*

The $\tau = 2$ Conjecture and the Replication Conjecture

Let $\mathcal{C}$ be a clutter over ground set E. We call $\tau(\mathcal{C}) := \tau(\mathcal{C}, \mathbf{1})$ the *covering number* of $\mathcal{C}$, that is the minimum cardinality of a cover of $\mathcal{C}$. We call $\nu(\mathcal{C}) := \nu(\mathcal{C}, \mathbf{1})$ the *packing number* of $\mathcal{C}$, that is the maximum number of disjoint members in $\mathcal{C}$. We say that a clutter $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$ and say that $\mathcal{C}$ has the *packing property* if every minor of $\mathcal{C}$ packs. A direct consequence of Lehman's theorem on minimally non-ideal clutters [92] is that minimally non-ideal clutters do not pack, which implies the following:

Theorem 1.22 ([35])**.** *If a clutter has the packing property, then it is ideal.*

Notice that the packing property is a relaxed notion of the max-flow min-cut (MFMC) property. It is

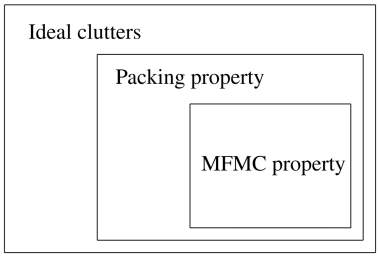


Figure 1.1: Classes of clutters

conjectured by Conforti and Cornuéjols that

The Replication Conjecture ([31])**.** *The packing property implies the max-flow min-cut property.*

Hence, the Replication Conjecture states that the packing property and the max-flow min-cut property are equivalent. The Replication Conjecture, if true, would be a set-covering analogue of the *replication lemma* by Lovász [95] for *perfect graphs*.

In an effort to prove the Replication Conjecture, Cornuéjols, Guenin, and Margot [35] came up with a stronger conjecture. We call a clutter *minimally non-packing* if it does not have the packing property but every proper minor of it does. It follows from Theorem 1.22 that a minimally non-packing clutter is either ideal or minimally non-ideal. While minimally non-ideal clutters are relatively well-understood, thanks to Lehman's theorem [92], understanding ideal minimally non-packing clutters is still a major open question. Q_6 is the clutter over ground set $\{1, \ldots, 6\}$ whose members are

$$Q_6 = \{\{1,3,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,5\}\},$$

and it is an ideal minimally non-packing clutter, studied by Seymour [113]. Schrijver [106] found an ideal minimally non-packing clutter over 9 elements. Cornuéjols, Guenin, and Margot [35] added a dozen more sporadic examples as well as an infinite class $\{Q_{r,t} : r, t \geq 1\}$. Cornuéjols, Guenin, and Margot [35] realized that all their examples have covering number two, so they conjectured the following:

The $\tau = 2$ Conjecture ([35]). *If a clutter is ideal and minimally non-packing, then its covering number is two.*

Then they showed that

Proposition 1.23 ([35]). *If the $\tau = 2$ Conjecture is true, then so is the Replication Conjecture.*

1.6 Intersecting restrictions in clutters

In Chapter 6, we consider a class of clutters, called *intersecting clutters*. A clutter $\mathcal{C}$ is *intersecting* if $\tau(\mathcal{C}) \geq 2$ and $\nu(\mathcal{C}) = 1$. In words, a clutter $\mathcal{C}$ is intersecting if $\mathcal{C} \neq \{\}, \{\emptyset\}$ and every two members of $\mathcal{C}$ intersect yet the members do not have a single common element. We call clutters $\{\}, \{\emptyset\}$ *trivial* and other clutters *nontrivial*. What are examples of intersecting clutters? We introduced Δ_n for $n \geq 3$ in § 1.5.

Remark 1.24. *The deltas, Δ_n for $n \geq 3$, are intersecting.*

Proof. Take an integer $n \geq 3$. Clearly, $\tau(\Delta_n) \geq 2$. As every two members of Δ_n intersect, $\nu(\Delta_n) = 1$, and therefore, Δ_n is intersecting. □

Another important class of intersecting clutters that will be considered in Chapter 6 are the blockers of *extended odd holes*. Take an odd integer $n \geq 5$. An *extended odd hole of dimension n* is a clutter over ground set $[n]$ whose minimum cardinality members are $\{1, 2\}$, $\{2, 3\}, \ldots, \{n-1, n\}$, $\{n, 1\}$. An extended odd hole may have a member of cardinality at least 3.

Remark 1.25. *The blockers of extended odd holes are intersecting.*

Proof. Take an odd integer $n \geq 5$, and let $\mathcal{C}$ be an extended odd hole of dimension n. Since every member of $\mathcal{C}$ has cardinality at least two and $b(b(\mathcal{C})) = \mathcal{C}$, the minimal covers of $\mathcal{C}$ do not have a common element, which means that $\tau(b(\mathcal{C})) \geq 2$. Moreover, as n is odd and a cover of $\mathcal{C}$ intersects all of $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}$, a cover of $\mathcal{C}$ has cardinality at least $\frac{n+1}{2}$. In particular, every minimal cover of $\mathcal{C}$ has cardinality at least $\frac{n+1}{2}$, implying in turn that every two minimal covers of $\mathcal{C}$ intersect. So $\nu(b(\mathcal{C})) = 1$, and therefore, the blocker of an extended odd hole is intersecting. □

We mention two other small intersecting clutters, namely, Q_6 and $\mathbb{L}_7$.

- Q_6 is the clutter over ground set $\{1, \ldots, 6\}$ whose members are $\{1, 3, 5\}$, $\{1, 4, 6\}$, $\{2, 3, 6\}$, $\{2, 4, 5\}$, and Q_6 is isomorphic to the clutter of triangles of K_4.
- $\mathbb{L}_7$ is the clutter over ground set $\{1, \ldots, 7\}$ whose members are $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 7\}$, $\{2, 5, 6\}$, $\{3, 4, 6\}$, $\{3, 5, 7\}$, and $\mathbb{L}_7$ is isomorphic to the clutter of lines of the *Fano plane* (Figure 1.2).

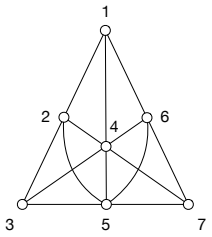


Figure 1.2: The Fano plane

Notice that Q_6 is intersecting as $\tau(Q_6) = 2$ and $\nu(Q_6) = 1$ and that $\mathbb{L}_7$ is intersecting as $\tau(\mathbb{L}_7) = 3$ and $\nu(\mathbb{L}_7) = 1$. It should also be mentioned that $\mathbb{L}_7$ is minimally non-ideal [91].

Clearly, intersecting clutters do not have the max-flow min-cut property nor the packing property, as they do not pack. So, by Remark 1.19, a clutter with the max-flow min-cut property does not contain an intersecting minor.

Conjecture 1.26. *If a clutter $\mathcal{C}$ has no intersecting minor, then the following statements are equivalent:*

(i) $\mathcal{C}$ is ideal,

(ii) $\mathcal{C}$ has the packing property,

(iii) $\mathcal{C}$ has the max-flow min-cut property.

In fact, we will see that Conjecture 1.26 is a simple restatement of the $\tau = 2$ Conjecture.

Proposition 1.27 ([4], proved in Chapter 6)**.** *The $\tau = 2$ Conjecture and Conjecture 1.26 are equivalent.*

Hence, the $\tau = 2$ Conjecture, if true, would imply that an ideal clutter has the max-flow min-cut property if and only if it has no intersecting minor, which would provide a characterization of when an ideal clutter has the max-flow min-cut property. What does it mean for a clutter not to have an intersecting minor? As a first step towards answering this question, we consider the problem of recognizing an intersecting minor in a clutter.

A *restriction* of a clutter is any minor obtained after deleting elements followed by contracting all the elements that appear in every member. Let I be a subset of the ground set E, and let J_I be defined as

$$J_I := \{e \in E - I : \ \{e\} \text{ is a cover of } \mathcal{C} \setminus I\}.$$

We call $\mathcal{C} \setminus I/J_I$ the restriction of $\mathcal{C}$ obtained after *restricting* I. Restrictions are a type of minors of a clutter. We say that the restriction $\mathcal{C} \setminus I/J_I$ is *proper* if $I \neq \emptyset$ or $I = \emptyset$ & $J_I \neq \emptyset$. Note that a restriction other than $\{\}, \{\emptyset\}$ has covering number at least 2.

Remark 1.28 ([4], proved in Chapter 6)**.** *A clutter $\mathcal{C}$ has an intersecting minor if, and only if, $\mathcal{C}$ has an intersecting restriction.*

This implies that to find an intersecting minor in a clutter, it is sufficient to consider its restrictions. What properties do clutters with an intersecting restriction have? We prove the following characterization:

Theorem 1.29 ([4], proved in Chapter 6)**.** *Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent:*

(i) $\mathcal{C}$ contains an intersecting restriction,

(ii) there exist three distinct members C_1, C_2, C_3 such that the restriction of $\mathcal{C}$ obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$ is intersecting.

In fact, this characterization of clutters containing an intersecting restriction leads to the following algorithm for recognizing them:

Input: A clutter $\mathcal{C}$

Output: Find an intersecting restriction in $\mathcal{C}$, or certify the none exists

Algorithm

1. For all distinct $C_1, C_2, C_3 \in \mathcal{C}$,
   - (a) take the restriction $\mathcal{C}'$ obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$, and
   - (b) if $\mathcal{C}'$ is intersecting, output $\mathcal{C}'$ as an intersecting minor in $\mathcal{C}$.
2. If (b) fails for every triple of distinct members, then conclude that there is no intersecting minor.

End of Algorithm

In fact, this algorithm runs in polynomial time, as proved in the following theorem:

Theorem 1.30 ([4], proved in Chapter 6)**.** *Given a clutter $\mathcal{C}$ with m members over n elements where $m, n \geq 1$, one can find an intersecting minor in $\mathcal{C}$ or certify that none exists in $O(m^5 n)$ time.*

Proof. The correctness of the above algorithm follows from Remark 1.28 and Theorem 1.29. There are $O(m^3)$ triples of three distinct members of $\mathcal{C}$. For every three distinct $C_1, C_2, C_3 \in \mathcal{C}$, it takes $O(mn)$ time to compute the restriction obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$ and $O(m^2 n)$ time to check if the restriction contains two disjoint members. Therefore, the algorithm terminates in $O(m^5 n)$ time, as required. □

This answers Conjecture 2.14 in [1] in the affirmative.

Recall that the deltas and the blockers of extended odd holes are two classes of intersecting clutters. In Chapter 6, we prove that testing whether a clutter contains one of them as a minor can also be done in polynomial time.

Theorem 1.31 ([4], proved in Chapter 6)**.** *Given a clutter $\mathcal{C}$ with m members over n elements where $m, n \geq 1$, one can find a delta or the blocker of an extended odd hole minor in $\mathcal{C}$ or certify that none exists in $O\left(n^4 m^3 (n+m)^{3.5} \log(n+m) \log\log(n+m)\right)$ time.*

This shows that Conjecture 2.13 in [1] is true.

1.7 Multipartite clutters and the $\tau = 2$ Conjecture

A counter-example to the $\tau = 2$ Conjecture, if it exists, is an ideal minimally non-packing clutter whose covering number is at least 3. In an effort to challenge the $\tau = 2$ Conjecture, we provide a systematic way of generating a certain class of clutters whose covering number can be arbitrarily large. Let $\mathcal{C}$ be a clutter whose ground set E is partitioned into nonempty parts $E_1, \ldots, E_n$. We say that $\mathcal{C}$ is *multipartite* if, for every member C,

$$|C \cap E_i| = 1 \qquad \forall i \in [n].$$

Notice that each part E_i is a cover of $\mathcal{C}$, so this construction of multipartite clutters provides a natural way of generating clutters with covering number greater than 2, as we can make $|E_1|, \ldots, |E_n| \geq 3$. A multipartite clutter can be interpreted as an n-uniform n-partite hypergraph.

In Chapter 7, we study multipartite clutters in the hope of finding a counter-example to the $\tau = 2$ Conjecture.

Multipartite clutters are a generalization of *cuboids*, multipartite clutters each of whose parts has size two, introduced by Abdi, Cornuéjols and Pashkovich [7]. Flores, Gitler and Reyes [66] also introduced cuboids and multipartite clutters whose parts have the same size, and they called them k-partitionable clutters where k is the size of each part.

Notice that if a cuboid is minimally non-packing, then its covering number is always 2. In fact, there exist cuboids that are ideal and minimally non-packing. $Q_{r,t}$ for $r, t \geq 1$, the ideal minimally non-packing clutters mentioned in § 1.5, are cuboids [7, 2], and $Q_6 = Q_{1,1}$. In [2], we reported that there are over 700 ideal minimally non-packing cuboids with at most 14 elements.

Theorem 1.32 ([6], proved in Chapter 7)**.** *The $\tau = 2$ Conjecture, if true, implies that*

every minimally non-packing multipartite clutter is a cuboid.

So, finding a minimally non-packing multipartite clutter that is not a cuboid would disprove the $\tau = 2$ Conjecture.

In fact, the $\tau = 2$ Conjecture is equivalent to the following conjecture stated in terms of multipartite clutters. In § 1.6, we defined intersecting restrictions in a clutter. We say that a clutter is *strictly polar* if it has no intersecting restriction.

Conjecture 1.33. *If a multipartite clutter is ideal and strictly polar, then it packs.*

Theorem 1.34 ([6], proved in Chapter 7)**.** *The $\tau = 2$ Conjecture and Conjecture 1.33 are equivalent.*

So, one way to refute the $\tau = 2$ Conjecture is to look for a multipartite clutter $\mathcal{C}$ such that

1. $\mathcal{C}$ strictly polar,
2. $\mathcal{C}$ is ideal, but
3. $\mathcal{C}$ does not pack.

In Chapter 7, we will provide a systematic way of searching for multipartite clutters satisfying the above three conditions. In § 1.6, we gave an efficient algorithm for determining whether a clutter is strictly polar (Theorem 1.30). Moreover, testing whether a clutter does not pack is easier than testing whether it is minimally non-packing, because in the second case one may also have to check the minors. Although it is in general difficult to test whether a clutter is ideal, we can take advantage of special structures in multipartite clutters. An *induced clutter* is any minor obtained from $\mathcal{C}$ after contracting precisely one element from each part of the ground set.

Theorem 1.35 ([6], proved in Chapter 7)**.** *A multipartite clutter is ideal if, and only if, all of its induced clutters are ideal.*

Therefore, to determine whether a multipartite clutter is ideal, we can just check its induced clutters. In fact, there is a geometric representation of multipartite clutters and we will see in Chapter 7 that the induced clutters can be interpreted in terms of the geometric representation (Proposition 7.15).

Given two graphs G and H, the *Cartesian product of G and H*, denoted $G\square H$ (we follow the notation used in [73]), is the graph over vertices $V(G)\times V(H)$, where (u_1, v_1), (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1, v_2 are adjacent, or $v_1 = v_2$ and u_1, u_2 are adjacent (see Figure 1.3 for an example). Notice that a hypercube of dimension $n \geq 1$ is simply the Cartesian product $(K_2)^{\square n}$. For integers $n \geq 1$ and $\omega_1, \ldots, \omega_n \geq 1$, an $\omega_1 \times \cdots \times \omega_n$-*rook* is the graph $H_{\omega_1,\ldots,\omega_n} := K_{\omega_1}\square \cdots \square K_{\omega_n}$. When $\omega_1 = \cdots = \omega_n = \omega$, $H_{\omega_1,\ldots,\omega_n}$ is called a *Hamming graph* $H(n, \omega)$. In particular, $H(n, 2)$ is a hypercube of dimension n.

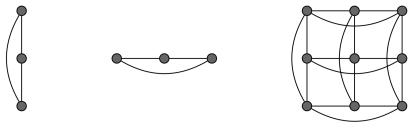


Figure 1.3: Cartesian product example

Remark 1.36 ([6])**.** *Take integers $n \geq 1$ and $\omega_1, \ldots, \omega_n \geq 1$. Then the following statements hold:*

(1) $H_{\omega_1,\ldots,\omega_n}$ has $\omega_1 \times \cdots \times \omega_n$ vertices,

(2) Every vertex has $\sum_{i=1}^{n}(\omega_i - 1)$ neighbors.

Write the vertex set of $H_{\omega_1,\ldots,\omega_n}$ as $[\omega_1] \times \cdots \times [\omega_n]$. For $v = (v_1, \ldots, v_n) \in [\omega_1] \times \cdots \times [\omega_n]$, let C_v be the set defined as follows:

$$C_v := \left\{ v_i + \sum_{j=1}^{i-1} \omega_j : \ i \in [n] \right\} \subseteq \left[\sum_{i=1}^{n} \omega_i \right].$$

Take a set $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. We define $\mathrm{mult}(S)$ as

$$\mathrm{mult}(S) := \{C_v : \ v \in S\}.$$

As the members of $\text{mult}(S)$ have the same size n, $\text{mult}(S)$ is a clutter over ground set $[\sum_{i=1}^{n} \omega_i]$. For instance, consider $R_{1,1} := \{(1,1,1),(1,2,2),(2,1,2),(2,2,1)\} \subseteq V(H_{2,2,2})$. Then

$$\text{mult}(R_{1,1}) = \{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\},$$

which means that $Q_6 = \text{mult}(R_{1,1})$ is a multipartite clutter and, in particular, a cuboid.

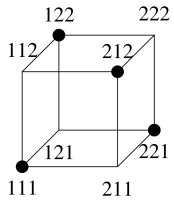


Figure 1.4: $R_{1,1}$

Remark 1.37 ([6]). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Then* $\text{mult}(S)$ *is a multipartite clutter whose ground set is partitioned into n parts $E_1,\ldots,E_n$ such that*

- $E_i = \left\{v_i + \sum_{j=1}^{i-1} \omega_j : \ v_i \in [\omega_i]\right\}$ *for* $i = 1,\ldots,n$.
- $|C \cap E_1| = \cdots = |C \cap E_n| = 1$ *for every member* $C \in \text{mult}(S)$.

Hence, the ground set of $\text{mult}(S)$ consists of n parts that correspond to the n coordinates of the points in S. We call $\text{mult}(S)$ the *multipartite clutter of* S. Conversely,

Remark 1.38 ([6]). *Let $\mathcal{C}$ be a multipartite clutter whose ground set is partitioned into parts $E_1,\ldots,E_n$ with $|E_i| = \omega_i \geq 1$ for $i \in [n]$. Then $\mathcal{C}$ is equal to* $\text{mult}(S)$ *for some $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$.*

Here, we say that S is the *Hamming representation of* $\mathcal{C}$ and $\mathcal{C}$ is the *multipartite clutter associated with* S. Therefore, by Remarks 1.37 and 1.38, we can work over $V(H_{\omega_1,\ldots,\omega_n})$.

1.8 Multipartite clutters of bounded degree

Take integers $n \geq 1$ and $\omega_1,\ldots,\omega_n \geq 1$ and a set $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. We refer to the vertices in S as *feasible*, and to the vertices in $\overline{S} := V(H_{\omega_1,\ldots,\omega_n}) - S$ as *infeasible*. Take an integer $k \geq 0$. We say that S has *degree at most k* if every infeasible vertex has at most k infeasible neighbors and that S has *degree k* if S has degree at most k and at least one infeasible vertex has k infeasible neighbors. The following theorem shows that if $\text{mult}(S)$ is non-ideal, then it has a minimally non-ideal minor whose size is bounded by the degree of S.

Theorem 1.39 ([6], proved in Chapter 7). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree at most k. Then every minimally non-ideal minor of* $\text{mult}(S)$*, if any, has at most k elements.*

So, if the degree of S is small, we can determine whether $\text{mult}(S)$ is ideal by checking minimally non-ideal clutters of small size.

Can we also find a necessary condition for a multipartite clutter to not pack? First,

Proposition 1.40 ([6], proved in Chapter 7)**.** *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Then the following statements hold:*

(1) if $n \leq 2$, then $\text{mult}(S)$ has the max-flow min-cut property, and

(2) if $\text{mult}(S)$ does not pack, then $n \geq 3$ and $\omega_n \geq 2$.

When $n \geq 3$ and $\omega_n \geq 2$, we can find bounds on the degree of a set whose multipartite clutter does not pack. By Remark 1.36, the degree of a vertex is always at most $\sum_{i=1}^{n}(\omega_i - 1)$. The following theorem gives a lower bound:

Theorem 1.41 ([6], proved in Chapter 7)**.** *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree at most k. Then the following statements hold:*

(1) if $\text{mult}(S)$ contains no Δ_3 as a minor and does not pack but all of its proper restrictions pack, then $k \geq \sum_{i=1}^{n-1}(\omega_i - 2)$, and

(2) if $\text{mult}(S)$ contains no Δ_3 as a minor and does not pack, every proper restriction of $\text{mult}(S)$ packs, and $k = \sum_{i=1}^{n-1}(\omega_i - 2)$, then $\text{mult}(S) \cong Q_6$.

The following theorem is analogous to Theorem 1.39.

Theorem 1.42 ([6], proved in Chapter 7)**.** *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree at most k. If $\text{mult}(S)$ has a restriction that does not pack, then it has one with at most $\max\left\{\frac{11}{2}k + \frac{1}{2},\ 6\right\}$ elements.*

We saw in § 1.7 that one way to refute the $\tau = 2$ Conjecture is to find an ideal strictly polar multipartite clutter that does not pack. Based on Theorem 1.41, we wrote a computer code to generate strictly polar multipartite clutters that do not pack. We will describe our algorithm in Chapter 7 with further details. Once we generate strictly polar multipartite clutters that do not pack, we check whether they are ideal. As long as their degrees are small, Theorem 1.39 implies that there is a minimally non-ideal minor of small size, and therefore, we can efficiently test idealness in that case. From our computational experiments, we came to the following conclusion:

Theorem 1.43 ([6], explained in Chapter 7)**.** *Let $\mathcal{C}$ be a multipartite clutter over at most 9 elements. If $\mathcal{C}$ is ideal and strictly polar, then $\mathcal{C}$ packs.*

1.9 The reflective product

In Chapter 8, we study two basic binary operations on pairs of multipartite clutters. Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. We define the *product*

$$S_1 \times S_2 := \{(x, y) \in V(G_1) \times V(G_2) :\ x \in S_1 \text{ and } y \in S_2\}.$$

In words, the product $S_1 \times S_2$ is obtained from S_1 after replacing each feasible point by a copy of S_2 and each infeasible point by an infeasible copy of $V(G_2)$. We will observe that if the multipartite clutters of two sets are ideal (resp. have the max-flow min-cut property), then so is (resp. does) the multipartite clutter of their product.

Define the *reflective product*

$$S_1 * S_2 := (S_1 \times S_2) \cup \left(\overline{S_1} \times \overline{S_2}\right),$$

where $\overline{S_i} := V(G_i) \setminus S_i$ for $i = 1, 2$. In words, the reflective product $S_1 * S_2$ is obtained from S_1 after replacing each feasible point by a copy of S_2 and each infeasible point by a copy of $\overline{S_2}$. For example, Figure 1.5 shows the reflective product of two sets $S_1 \subseteq V(H_{3,3})$ and $S_2 \subseteq V(H_{2,2})$ (the black round vertices).

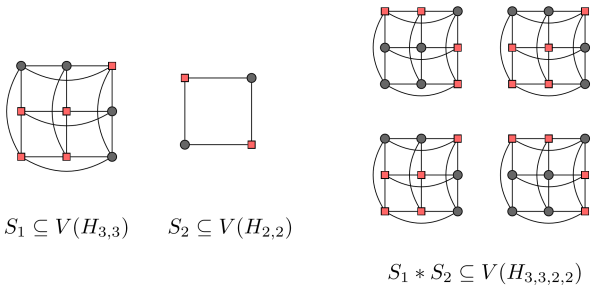


Figure 1.5: An example of taking the reflective product of two sets

Observe that $S_1 * S_2 = \overline{S_1} * \overline{S_2}$ and that $\overline{S_1 * S_2} = \overline{S_1} * S_2 = S_1 * \overline{S_2}$.

Theorem 1.44 ([6], proved in Chapter 8)**.** *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. If* $\mathrm{mult}(S_1)$, $\mathrm{mult}(\overline{S_1})$, $\mathrm{mult}(S_2)$, $\mathrm{mult}(\overline{S_2})$ *are ideal, then so are* $\mathrm{mult}(S_1 * S_2)$, $\mathrm{mult}\left(\overline{S_1 * S_2}\right)$.

Therefore, we can construct more complicated ideal multipartite clutters by the operations of taking products and reflective products. In contrast, the analogue of this theorem for the max-flow min-cut property does not hold. For example, let $S_1 := \{(1,1),(2,2)\}$ and $S_2 := \{1\}$. Then $\mathrm{mult}(S_1)$, $\mathrm{mult}(\overline{S_1})$, $\mathrm{mult}(S_2)$, $\mathrm{mult}(\overline{S_2})$ all have the max-flow min-cut property. However, $S_1 * S_2 = R_{1,1}$, and we have seen that $\mathrm{mult}(R_{1,1}) = Q_6$ does not have the max-flow min-cut property.

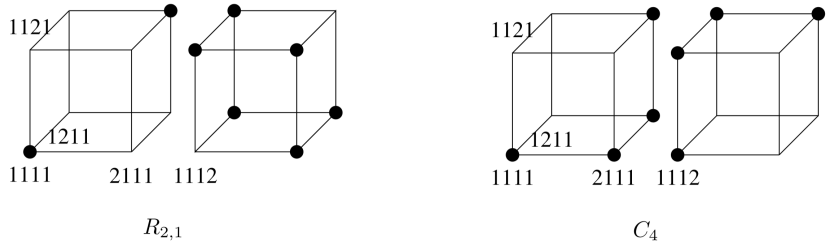
In an attempt to find a counter-example to the $\tau = 2$ Conjecture, is it possible to obtain an ideal minimally non-packing multipartite clutters with large covering number by taking the reflective product of two multipartite clutters? The following theorem answers this question in the negative:

Theorem 1.45 ([6], proved in Chapter 8)**.** *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and a set $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Assume that* $\mathrm{mult}(S)$ *contains no Δ_3 as minor and does not pack but all of its proper restrictions pack. If S is obtained by a reflective product, then $\omega_1 = \cdots = \omega_n = 2$, and therefore,* $\mathrm{mult}(S)$ *is a cuboid.*

Theorem 1.45 implies that an ideal minimally non-packing multipartite clutter obtained by a reflective product always has covering number two. In fact, there exist ideal minimally non-packing cuboids obtained by taking a reflective product. Recall that a cuboid is the multipartite of a vertex subset of the hypercube $H(n,2)$ for some $n \geq 1$. For an integer $k \geq 1$, let

$$R_{k,1} := \left\{\mathbf{1}^{k+1}, \mathbf{2}^{k+1}\right\} * \{1\} \subseteq \{1,2\}^{k+2}$$

where $\mathbf{1}^m, \mathbf{2}^m$ denote the m-dimensional vectors all of whose entries are $1, 2$, respectively. We have already seen $R_{1,1}$ in § 1.7. See Figure 1.6 for an illustration of $R_{2,1}$. We observed that $\text{mult}(R_{1,1}) = Q_{1,1}$. In



$R_{2,1}$ C_4

Figure 1.6: $R_{2,1}$ and C_4

fact, it can be readily checked that $\text{mult}(R_{k,1}) = Q_{k,1}$ for $k \geq 1$, so $\text{mult}(R_{k,1})$ is ideal and minimally non-packing.

In Chapter 8, we will prove Theorem 1.46 on the reflective product of cuboids. Take an integer $n \geq 1$. For $v \in \{1,2\}^n$, the *antipodal* of v is the vector in $\{1,2\}^n$ that differs from v in every coordinate. We say that a set $S \subseteq \{1,2\}^n$ is *antipodally symmetric* if a vector in $\{1,2\}^n$ is in S if and only if its antipodal is in S. We say that a set $S \subseteq \{1,2\}^n$ is *connected* if the subgraph of $H(n,2)$ induced by S is connected. We say that a set $S \subseteq \{1,2\}^n$ is *strictly connected* if R is connected for every *set-restriction* (will be defined in Chapter 7) R of S.

Theorem 1.46 ([2], proved in Chapter 8)**.** *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{1,2\}^{n_1}$ and $S_2 \subseteq \{1,2\}^{n_2}$, where $\text{mult}(S_1 * S_2)$ does not pack but all of its proper restrictions pack. Then one of the following statements holds:*

(i) $S_1 * S_2 \cong R_{k,1}$ *for some* $k \geq 1$,

(ii) $n_1 = 1$ *and* $S_2, \overline{S_2}$ *are antipodally symmetric and strictly connected, or*

(iii) $n_2 = 1$ *and* $S_1, \overline{S_1}$ *are antipodally symmetric and strictly connected.*

Moreover, $S_1 * S_2 \cong \overline{S_1 * S_2}$.

There is an antipodally symmetric and strictly connected set $S \subseteq \{1,2\}^4$ such that $\text{mult}(S * \{1\})$ is ideal and minimally non-packing. Consider C_4 and R_5 defined as follows:

$$C_4 := \{1111, 2111, 2211, 2221, 2222, 1222, 1122, 1112\}$$
$$R_5 := C_4 * \{1\}.$$

Notice that C_4 is antipodally symmetric and strictly connected (see Figure 1.6 for an illustration). In fact, mult(R_5) is the ideal minimally non-packing clutter Q_{10} found in [7].

1.10 Ideal vector spaces

What are some examples of multipartite clutters that are ideal? In Chapter 9, we consider a class of examples that arise as a natural generalization of the cuboids of binary spaces [2]. Let q be a prime power p^k where p is a prime number and k is a positive integer, and consider $GF(q)$, the finite field of order q. The smallest integer ℓ such that $\underbrace{a+\cdots+a}_{\ell}=0$ for all $a \in GF(q)$ is p, and we call p the *characteristic* of $GF(q)$. Throughout this section, we denote by 0 and 1 the additive and multiplicative identities of $GF(q)$, and for each $v \in GF(q)-\{0\}$, we denote by $-v$ and v^{-1} the additive and multiplicative inverses of v. Take an integer $n \geq 1$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then there exists a matrix A whose entries are in $GF(q)$ such that

$$S = \{x \in GF(q)^n : \ Ax = \mathbf{0}\}$$

where $\mathbf{0}$ denotes the vector of all zeros of appropriate dimension and all equalities in the system $Ax = \mathbf{0}$ are over $GF(q)$. We denote by $\langle v^1, \ldots, v^r \rangle$ the vector space generated by taking linear combinations of $v^1, \ldots, v^r$ over the given field.

As the element set of $GF(q)$ can be relabeled as $[q]$, $GF(q)^n \cong V(H(n,q))$, and therefore, we can define the multipartite clutter of any subset of $GF(q)^n$. For example, the element set of $GF(4)$ can be represented as $\{0,1,a,b\}$ where a and b are the numbers satisfying the following addition and multiplication tables:

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\times$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Example 1 ([5])**.** Consider $S = \langle (1,1,0), (1,0,1) \rangle \subseteq GF(4)^3$. Then

$$S = \left\{ \begin{array}{l} (0,0,0),\ (1,1,0),\ (a,a,0),\ (b,b,0),\ (1,0,1),\ (0,1,1),\ (b,a,1),\ (a,b,1), \\ (a,0,a),\ (b,1,a),\ (0,a,a),\ (1,b,a),\ (b,0,b),\ (a,1,b),\ (1,a,b),\ (0,b,b) \end{array} \right\}.$$

Then $f : GF(4) \to [4]$ defined by $f(0) = 1$, $f(1) = 2$, $f(a) = 3$, $f(b) = 4$ is a bijection. Then mult(S) can be defined as mult(S'), where

$$S' = \left\{ \begin{array}{l} (1,1,1),\ (2,2,1),\ (3,3,1),\ (4,4,1),\ (2,1,2),\ (1,2,2),\ (4,3,2),\ (3,4,2), \\ (3,1,3),\ (4,2,3),\ (1,3,3),\ (2,4,3),\ (4,1,4),\ (3,2,4),\ (2,3,4),\ (1,4,4) \end{array} \right\}.$$

Can we characterize vector spaces over $GF(q)$ whose multipartite clutters have the max-flow min-cut property or are ideal? Proposition 1.40 (1) implies that if $n \leq 2$, mult(S) for any vector space $S \subseteq GF(q)^n$ has the max-flow min-cut property. Thus, we may assume that $n \geq 3$.

Question 1.47. *Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. When does* mult(S) *have the max-flow min-cut property?*

Question 1.48. *Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. When is* mult(S) *ideal?*

Let us first look at vector spaces over $GF(2)$. In this case, we will refer to S as a *binary space*, and the points in S correspond to the cycles of M, the binary matroid represented by A. Notice that the multipartite clutter of a binary space is the cuboid of a binary space. We define the support of $v \in GF(q)^n$, denoted support(v), as $\{i \in [n] : v_i \neq 0\}$. Recall that Δ_3 and $\mathbb{L}_7$ are minimally non-ideal and that Q_6 is ideal and minimally non-packing. Let us also define two small clutters, namely, $\mathbb{O}_5$, $b(\mathbb{O}_5)$.

- $\mathbb{O}_5$ is the clutter over ground set $E(K_5)$, the edge set of K_5, whose members are the odd circuits of K_5.
- $b(\mathbb{O}_5)$ is the blocker of $\mathbb{O}_5$, and it is the clutter over ground set $E(K_5)$ whose members are the cut complements of K_5.

Seymour [113] noted that $\mathbb{O}_5$, $b(\mathbb{O}_5)$ are minimally non-ideal. The following theorem considers the $q = 2$ case.

Theorem 1.49 ([2])**.** *Let $n \geq 3$, and let $S \subseteq GF(2)^n$ be a binary space, and let M be the associated binary matroid.*

(1) mult(S) *has the max-flow min-cut property* ($\Leftrightarrow$) *$S = \langle v^1, \ldots, v^r \rangle$ where $v^1, \ldots, v^r \in GF(q)^n$ have pairwise disjoint supports* ($\Leftrightarrow$) mult(S) *has no Q_6 as a minor.*

(2) mult(S) *is ideal* ($\Leftrightarrow$) *M has the sums of circuits property* ($\Leftrightarrow$) mult(S) *has none of $\mathbb{L}_7, \mathbb{O}_5, b(\mathbb{O}_5)$ as a minor.*

Theorem 1.49 answers Questions 1.47 and 1.48 when $q = 2$. In Chapter 9, we will prove Theorems 1.50–1.52, thereby providing complete answers to Questions 1.47 and 1.48. C_5^2 is the clutter over ground set $\{1, \ldots, 5\}$ whose members are $\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}$. C_5^2 is non-ideal, because $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is a fractional extreme point of the set covering polyhedron associated with C_5^2.

Theorem 1.50 ([5], proved in Chapter 9)**.** *Let q be a prime power other than $2, 4$. Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then the following statements are equivalent:*

(i) mult(S) *contains no Δ_3, Q_6, C_5^2 as a minor,*

(ii) $S = \langle v^1, \ldots, v^r \rangle$ where $v^1, \ldots, v^r \in GF(q)^n$ have pairwise disjoint supports,

(iii) mult(S) *has the max-flow min-cut property,*

(iv) mult(S) *is ideal.*

By Theorems 1.49 and 1.50, the remaining case is when $q = 4$. The following theorem gives a characterization of vector spaces over $GF(4)$ whose multipartite clutters have the max-flow min-cut property.

Theorem 1.51 ([5], proved in Chapter 9)**.** *Let $n \geq 3$, and let $S \subseteq GF(4)^n$ be a vector space over $GF(4)$. Then the following statements are equivalent:*

(i) $\text{mult}(S)$ *contains no Δ_3, Q_6 as a minor,*

(ii) $S = \langle v^1, \ldots, v^r \rangle$ *where $v^1, \ldots, v^r \in GF(4)^n$ have pairwise disjoint supports,*

(iii) $\text{mult}(S)$ *has the max-flow min-cut property.*

Unlike the case when $q \notin \{2, 4\}$, there is a vector space over $GF(4)$ whose multipartite clutter is ideal but does not have the max-flow min-cut property.

Example 2 ([5])**.** In fact, $S = \langle (1,1,0), (1,0,1) \rangle \subseteq GF(4)^3$ in Example 1 provides an example. One can check by using PORTA [28] that $\left\{x \in \mathbb{R}^9_+ : M(\text{mult}(S))x \geq \mathbf{1}\right\}$ is an integral polyhedron, so $\text{mult}(S)$ is ideal. Notice further that $\text{mult}(S)$ does not have the max-flow min-cut property, since S contains

$$\{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\} \cong R_{1,1}$$

as a set-restriction and so $\text{mult}(S)$ has Q_6 as a minor.

Thus, the max-flow min-cut property and idealness are no longer equivalent. The following theorem provides a characterization of vector spaces over $GF(4)$ whose multipartite clutters are ideal:

Theorem 1.52 ([5], proved in Chapter 9)**.** *Let $n \geq 3$, and let $S \subseteq GF(4)^n$ be a vector space over $GF(4)$. Then the following statements are equivalent:*

(i) $\text{mult}(S)$ *contains no Δ_3 as a minor,*

(ii) $S = S_1 \times \cdots \times S_k$ *where for each $i \in [k]$,*

- $S_i = \{0\}$,
- $S_i = GF(4)$, *or*
- $S_i = \langle v^1, \ldots, v^r \rangle$ *where $r \geq 1$ and $v^1, \ldots, v^r$ are vectors of the following form, after permuting the coordinates:*

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^r \end{array} \left[\begin{array}{c|c|c|c|c} u^0 & u^1 & \mathbf{0} & \cdots & \mathbf{0} \\ u^0 & \mathbf{0} & u^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u^0 & \mathbf{0} & \mathbf{0} & \cdots & u^r \end{array}\right]$$

for some vectors $u^0, u^1 \ldots, u^r$ of nonzero entries,

(iii) $\text{mult}(S)$ *is ideal.*

As a direct consequence of the above theorems, we obtain the following:

Corollary 1.53 ([5])**.** *Let S be a multipartite vector space over $GF(q)$ for some prime power q. If mult(S) is ideal and has no intersecting restriction, then it packs.*

That is, Conjecture 1.33 holds in this case. So, the multipartite clutters obtained from vector spaces serve as evidence in support of the $\tau = 2$ Conjecture.

Chapter 2

Polytopes with Chvatal rank 1

In this chapter, we introduce the problem of deciding whether a rational polyhedron P contains an integer point under the promise that P has Chvátal rank 1, which is the main motivation of this paper. This promise on the input P very likely modifies the computational complexity of the integer feasibility problem. A result of Boyd and Pulleyblank ([21], Theorem 5.4) implies Proposition 1.7.

Proposition 1.7 ([21]). *Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ be a rational polyhedron whose Chvátal rank is 1. Then*

(1) the problem of deciding whether $P \cap \mathbb{Z}^n = \emptyset$,

(2) given $c \in \mathbb{Q}^n$, the problem of deciding whether $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ is unbounded,

(3) given $c \in \mathbb{Q}^n$ and $x^* \in \mathbb{Z}^n$, the problem of deciding whether $cx^* = \max\{cx : x \in P \cap \mathbb{Z}^n\}$

belong to complexity class NP $\cap$ co-NP.

The problems in NP $\cap$ co-NP are probably not NP-complete (since otherwise NP = co-NP), so we have the following question:

Open question 1. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron with Chvátal rank 1. Can we decide whether P contains an integer point in time polynomial in the encoding size of P?*

However, it does not seem straightforward to use the Chvátal rank 1 condition. In fact, it is NP-hard to certify that the Chvátal rank of a rational polytope given by its linear description is 1, even under some assumptions on the input polytope. We show this in § 2.2. We also note that the Chvátal rank of a polyhedron is not directly related to its geometry. In particular, the Chvátal rank is not invariant under translation. The following example illustrates that the Chvátal rank of a polyhedron may vary significantly under translation.

Example 1 ([37])**.** Let $Q_1 := \{x \in [0,1]^n : \sum_{j=1}^n v_j(1-x_j) + (1-v_j)x_j \geq \frac{1}{2}\ \forall v \in \{0,1\}^n\}$. Notice that Q_1 contains no integer point. Chvátal, Cook, and Hartmann ([30], Lemma7.2) proved that the Chvátal rank of Q_1 is exactly n. Now, let us translate Q_1 so that its center point is at the origin, and we denote by Q_2 the resulting polytope. Since $Q_2 \subseteq [-\frac{1}{2}, \frac{1}{2}]^n$, the only integer point contained in Q_2 is the origin. We can obtain both $x_i \geq 0$ and $x_i \leq 0$ as Chvátal-Gomory inequalities for Q_2 for all $i \in [n]$. Hence, the Chvátal rank of Q_2 is exactly 1.

The difficulty in understanding the Chvátal rank 1 condition is an indication that Open question 1 might not be easy to answer in general.

In § 2.1, we consider some easy cases of Open Question 1. In § 2.2, we prove Theorem 1.3 on the NP-hardness of testing whether the Chvátal closure of a polytope is empty and we explain its implications. In § 2.3, we prove Theorem 1.9 on the flatness theorem for rational polyhedra with empty Chvátal closure. The material in this chapter will be published in *Mathematical Programming A* [37].

2.1 Easy cases

In this section, we motivate Open Question 1 by presenting three special cases, which seem easier to tackle and still remain interesting.

Satisfiability problem with Chvátal rank 1

The satisfiability problem is NP-complete (see [68]), and it can be formulated as a binary integer program. Given a formula in conjunctive normal form with m clauses that consist of literals $x_1, \cdots, x_n$ and their negations, the problem of finding a satisfying assignment $x \in \{0,1\}^n$ can be equivalently formulated as the 0,1 feasibility problem over a polytope. Given a clause $\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} \neg x_j$ for some disjoint subsets I, J of $[n]$, we make a linear inequality $\sum_{i \in I} x_i + \sum_{j \in J}(1-x_j) \geq 1$. Notice that an assignment $x \in \{0,1\}^n$ satisfies all the clauses if and only if it satisfies all the corresponding inequalities. Inequalities of the form

$$\sum_{i \in I} x_i + \sum_{j \in J}(1-x_j) \geq 1 \qquad I, J \subseteq [n],\ I \cap J = \emptyset$$

are called *generalized set covering inequalities*. Then, the satisfiability problem of a given formula is equivalent to the integer feasibility problem of a polytope defined by generalized set covering inequalities and the bounds $0 \leq x \leq 1$. We call such a polytope a *SAT polytope*.

Open question 2. *Given a SAT polytope P whose Chvátal rank is 1, can we decide in polynomial time whether P contains an integer point?*

The k-satisfiability problem is a variant of the satisfiability problem where each clause in a given formula has at most k literals. It remains NP-complete for $k \geq 3$ (see [68]). On the other hand, there is a simple polynomial algorithm for the case of $k = 2$. We consider a formula whose SAT polytope has Chvátal rank 1 and each of whose clauses contains at least 3 literals. We remark that such a formula always has a satisfying assignment.

Remark 2.1 ([37]). *Let P be a SAT polytope such that each generalized set covering inequality in its description has at least 3 variables. If P has Chvátal rank 1, then P always contains an integer point.*

Proof. Observe that setting any variable to 0 or 1, and all other $n-1$ variables to $1/2$ satisfies all the constraints of P (because every generalized set covering inequality involves at least three variables). In other words, the middle point of each facet of the hypercube $[0,1]^n$ is contained in P. A result of Chvátal, Cook and Hartmann ([30], Lemma 7.2) implies that the Chvátal closure of P contains the middle point $(\frac{1}{2}, \cdots, \frac{1}{2})$ of the hypercube, so the Chvátal closure of P is always nonempty. Because the Chvátal rank of P is 1, P contains an integer point. □

A natural question is whether one can actually find an integer point in polynomial time, under the assumptions of Remark 2.1. This is open. The following example provides a positive answer when each generalized set covering inequality contains n variables.

Example 2 ([37]). Take an integer $n \geq 3$. Given $\overline{S} \subseteq \{0,1\}^n$, we construct a SAT polytope as follows:

$$P = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} ((1-v_i)x_i + v_i(1-x_i)) \geq 1, \ \forall v \in \overline{S} \right\}$$

Notice that $P \cap \{0,1\}^n = \{0,1\}^n \setminus \overline{S}$. Theorem 1.11 in [36] implies that P has Chvátal rank 1 if and only if $G(\overline{S})$, the induced subgraph of G by $\overline{S}$ where G denotes the skeleton graph of the hypercube $[0,1]^n$, has max degree 2 and has no cycle of length 4. It is easy to find a 0,1 point contained in P. First, check whether $\mathbf{0} \in P$. If not, then $\mathbf{0} \in \overline{S}$ and at least $n-2$ points among $e^1, \ldots, e^n$ (the unit vectors) are contained in P since the degree of $\mathbf{0}$ in $G[\overline{S}]$ is at most 2.

The gap between Open question 2 and Remark 2.1 is on the SAT formulas involving both clauses with 2 literals and clauses with at least 3 literals. SAT polytopes whose generalized set covering inequalities have at most 2 variables are well understood by Gerards and Schrijver [69]. They gave a characterization of the Chvátal closure in such a case, and they provided a polynomial algorithm to separate over it. Furthermore, we remark that the Chvátal rank of a SAT polytope in that case is always 1 whenever it contains no integer point. However, the Chvátal closure of a SAT polytope that includes both generalized set covering inequalities with 2 variables and 3 variables has not been studied.

When a few Chvátal-Gomory cuts are sufficient

In this section, we consider another special case of Open question 1, where we assume that the integer hull of a given polyhedron can be obtained by adding a constant number of (rank-1) Chvátal-Gomory inequalities.

Open question 3. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, and assume that the integer hull of P can be obtained by adding at most k (rank-1) Chvátal-Gomory inequalities of P to the description of P, for some constant k. Can we solve the integer feasibility problem of P in polynomial time?*

In fact, Open question 3 is open even when $k = 1$. We will show in § 2.2.4 that verifying the promise that the integer hull of a given rational polytope is obtained after adding one Chvátal-Gomory inequality is NP-hard. Thus, Open question 3 might be difficult to answer as well.

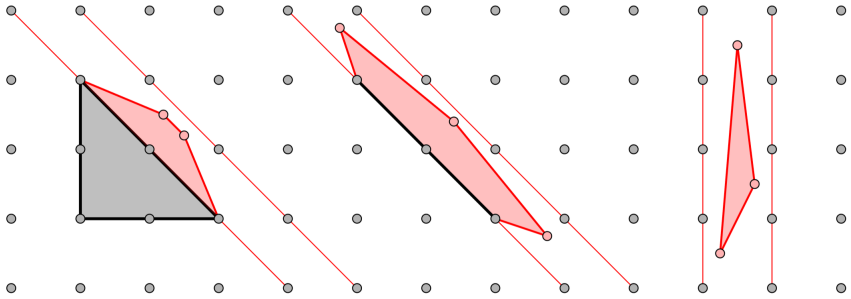


Figure 2.1: When one Chvátal-Gomory inequality is sufficient in $\mathbb{R}^2$

Remark 2.2 ([37]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polytope such that adding one Chvátal-Gomory inequality to the description of P gives its integer hull. Then there exists an algorithm for the integer feasibility problem over P which runs in time bounded by $m^n n^3 poly(L)$ where m and L denote the number of constraints in P and the encoding size of P, respectively.*

Proof. This is easy to show because a fractional vertex of P should be removed by the Chvátal-Gomory inequality. Therefore P contains an integer point if and only if an extreme point of P is integral. In this case, a trivial algorithm solves the integer feasibility problem: check all the vertices of P and conclude that $P_I \neq \emptyset$ if there exists an integral vertex or $P_I = \emptyset$ otherwise. Since there are $O(m^n)$ extreme points of P and the time complexity of the Gaussian elimination method is bounded by $n^3\text{poly}(L)$, the algorithm runs in time bounded by $m^n n^3 \text{poly}(L)$. □

In fact, Proposition 2.23 will show the existence of a $2^{O(n)}\text{poly}(L)$ time algorithm for the case of $k = 1$.

In the following, we consider a special case of Open question 3, where the input is a rational simplex. A polytope $P \subseteq \mathbb{R}^n$ is called a *simplex* of dimension ℓ for some $\ell \leq n$ if it is the convex hull of $\ell + 1$ affinely independent points. One can show that the integer feasibility problem over a rational simplex is NP-complete by the following polynomial reduction of the knapsack problem to it [109]: consider positive integers $a_1, \cdots, a_n, b$. Let $v^i := \frac{b}{a_i} e^i$ where e^i denotes the ith unit vector for $i \in [n]$. Let $v^{n+1} := \frac{b-\frac{1}{2}}{n}(\frac{1}{a_1}, \cdots, \frac{1}{a_n})$. Let $\text{conv}\{v^1, \cdots, v^{n+1}\}$ denote the convex hull of $v^1, \cdots, v^{n+1}$. Note that $av^{n+1} = b - \frac{1}{2}$ and $av^i = b$ for $i \in [n]$. Then, $\text{conv}\{v^1, \cdots, v^{n+1}\} \cap \mathbb{Z}^n = \{x \in \mathbb{Z}^n : ax = b,\ x \geq 0\}$. However, if we further assume that the integer hull of a rational simplex can be obtained by adding a constant number of (rank-1) Chvátal-Gomory inequalities, then we can solve the integer feasibility problem over the simplex in polynomial time.

Proposition 2.3 ([37]). *Let k be a positive integer. Given a rational simplex $P \subseteq \mathbb{R}^n$ such that its integer hull can be obtained from P by adding at most k (rank-1) Chvátal-Gomory inequalities, and a vector $w \in \mathbb{Q}^n$, there is an algorithm to optimize wx over P_I in time $n^{O(k)} poly(k, L)$, where L is the encoding size of P and w.*

Proof. Suppose that the dimension of P is ℓ for some $\ell \leq n$. Let $P = \{x \in \mathbb{R}^n : Ax = b,\ Cx \leq d\}$ be a minimal linear system defining P such that $Cx \leq d$ define the facets of P. We denote by $Ex \leq f$ the

set of k Chvátal-Gomory inequalities of P such that $P_I = \{x \in \mathbb{R}^n : Ax = b,\ Cx \leq d,\ Ex \leq f\}$. So the inequalities in $Ex \leq f + \epsilon\mathbf{1}$ are valid for P, where $\epsilon \in (0,1)$ and $\mathbf{1}$ denotes the vector of all ones, and $P \subseteq S$, where $S := \{x \in \mathbb{R}^n : Ax = b,\ Ex \leq f + \epsilon\mathbf{1}\}$.

We first argue that we may assume that P is full-dimensional. If not, we can find in polynomial time an unimodular matrix U such that $AU = (D, 0)$ is a Hermite normal form of A. If $D^{-1}b$ is not integral, we can just conclude that P does not contain an integer point. Thus, we may assume that $D^{-1}b$ is integral. Let U_1 and U_2 denote the two submatrices of U which consist of the first $n-\ell$ columns of U and the last ℓ columns of U, respectively. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be an unimodular transformation defined by $u(x) = U^{-1}x$. Consider the images of P, P_I, and S under u:

$$\begin{aligned} u(P) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} :\ y_1 = D^{-1}b,\ CU_2y_2 \leq d - CU_1D^{-1}b\right\}, \\ u(P_I) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} :\ y_1 = D^{-1}b,\ CU_2y_2 \leq d - CU_1D^{-1}b,\ EU_2y_2 \leq f - EU_1D^{-1}b\right\}, \\ u(S) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} :\ y_1 = D^{-1}b,\ EU_2y_2 \leq f + \epsilon\mathbf{1} - EU_1D^{-1}b\right\}. \end{aligned}$$

Note that $u(P)$ is an ℓ-dimensional simplex in $\mathbb{R}^n$, so $Q := \left\{y_2 \in \mathbb{R}^\ell : CU_2y_2 \leq d - CU_1D^{-1}b\right\}$ is an ℓ-dimensional simplex in $\mathbb{R}^\ell$. Furthermore, $u(P_I)$ is integral. Since $D^{-1}b$ is integral, $\{y_2 \in \mathbb{R}^\ell : CU_2y_2 \leq d - CU_1D^{-1}b,\ EU_2y_2 \leq f - EU_1D^{-1}b\}$ is integral and thus $Q \cap \{y_2 \in \mathbb{R}^\ell : EU_2y_2 \leq f - EU_1D^{-1}b\}$ is integral. We claim that the inequalities in the system $EU_2y_2 \leq f - EU_1D^{-1}b$ are Chvátal-Gomory inequalities of Q. In fact, we know that $u(P) \subseteq u(S)$, so $Q \subseteq \left\{y_2 \in \mathbb{R}^\ell : EU_2y_2 \leq f + \epsilon\mathbf{1} - EU_1D^{-1}b\right\}$. That means the inequalities in $EU_2y_2 \leq f + \epsilon\mathbf{1} - EU_1D^{-1}b$ are all valid for Q, so those in the system $EU_2y_2 \leq f - EU_1D^{-1}b$ are Chvátal-Gomory inequalities of Q. Now, we have obtained a full-dimensional rational simplex Q in $\mathbb{R}^\ell$ such that its integer hull Q_I can be described by adding at most k Chvátal-Gomory inequalities.

Q has $\ell+1$ inequalities in its description, so Q_I can be described by $\ell+k+1$ linear inequalities. When $\ell \leq k$, the dimension of Q is fixed and we can optimize a linear function over Q_I in polynomial time by Lenstra's algorithm [93]. Thus, we may assume that $\ell > k$. Suppose that Q_I is not empty. Then let $z \in \mathbb{Z}^\ell$ be an extreme point of Q_I. So there are ℓ linearly independent inequalities in the description of Q_I that are active at z. This means that at least $\ell - k$ inequalities among the $\ell + 1$ inequalities in the original description of Q are active at z. Thus, z belongs to a k-dimensional face of Q. Hence, if no k-dimensional face of Q contains an integer point, Q_I is empty. Since k is fixed, we can optimize a linear function over the integer hull of each k-dimensional face of Q. Notice that there are exactly $\binom{\ell+1}{k+1}$ k-dimensional faces of Q. Therefore, we can optimize a linear function over Q_I in $\ell^{O(k)}\text{poly}(L)$ time. Since we can compute the Hermite normal form of A in time polynomial in the encoding size of P and $\ell \leq n$, the result follows, as required. □

The only property of a simplex in $\mathbb{R}^n$ used in the proof of Proposition 2.3 is that the number of its facets is at most $n + 1$. The result should generalize to the case where a rational polytope $P \subseteq \mathbb{R}^n$ has $n + t$ facets, where t is a constant, and the integer hull of P is obtained by adding k (rank-1) Chvátal-Gomory inequalities.

Rounded polytopes

A full-dimensional polytope $P \subseteq \mathbb{R}^n$ is *rounded with factor* $\ell > 1$ if $B_2^n(a, r) \subseteq P \subseteq B_2^n(a, \ell r)$, where $B_2^n(p, q)$ denotes an Euclidean ball $\{x \in \mathbb{R}^n : \|x - p\|_2 \leq q\}$ centered at p with radius q. We first remark

the following:

Remark 2.4 ([37]). *Let $\ell > 1$ be a constant, and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rounded polytope with factor ℓ. We can decide whether P contains an integer point and find one if there exists any in $\ell^{O(n)}\text{poly}(L)$ time, where L is the encoding size of P.*

Proof. One can find an Euclidean ball $B_2^n(c, R) \subseteq P$ of the largest radius by solving a linear program whose encoding size is bounded above by $\text{poly}(L)$ (see Section 4.3 in [22]). If R is at least $\frac{\sqrt{n}}{2}$, an integer point that is nearest to c is contained in the ball, so we can obtain an integer point in P by rounding c. If that is not the case, we consider two Euclidean balls $B_2^n(a, r)$ and $B_2^n(a, \ell r)$ for some $a \in P$ and $0 < r < \frac{\sqrt{n}}{2}$ such that $B_2^n(a, r) \subseteq P \subseteq B_2^n(a, \ell r)$. As $c \in P$, the distance between a and c is at most ℓr, and therefore, $B_2^n(c, 2\ell r)$ contains $B_2^n(a, \ell r)$ by the triangle inequality. So, P is also contained in $B_2^n(c, 2\ell r)$. As $2\ell r < \ell\sqrt{n}$, we can enumerate all the $\ell^{O(n)}$ integer points in $B_2^n(c, 2\ell r)$ and check whether at least one of them belongs to P. □

Now, we further assume that the integer hull of P can be obtained by adding one Chvátal-Gomory inequality, which is another special case of Open question 3.

Proposition 2.5 ([37]). *Let $\ell > 1$ be a constant, and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rounded polytope with factor ℓ. If the integer hull of P can be obtained by adding one Chvátal-Gomory inequality to the description of P, then we can decide whether P contains an integer point in $n^{O(\ell)}\text{poly}(L)$ time, where L is the encoding size of P.*

To prove this, we use the notion of integer width defined in § 1.2. Take an integer $n \geq 1$ and a convex set $K \subseteq \mathbb{R}^n$. Recall that the integer width of K, denoted $w(K, \mathbb{Z}^n)$ is defined as

$$w(K, \mathbb{Z}^n) := \inf_{d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} w(K, d).$$

Lemma 2.6 ([37]). *Let $P \subseteq \mathbb{R}^n$ be a rounded polytope with factor $\ell > 1$. If there exists a direction $d \in \mathbb{Z}^n$ such that $w(P, d) \leq k$ for some nonnegative integer k, then either $\|d\|_2 \leq (k+1)\ell$ or $w(P, e^i) \leq 1$ for all $i \in [n]$.*

Proof. Since P is rounded with factor ℓ, P satisfies $B_2^n(a, r) \subseteq P \subseteq B_2^n(a, \ell r)$ for some $r > 0$ and $a \in \mathbb{R}^n$. Assume that $\|d\|_2 > (k+1)\ell$. Since $w(P, d) \leq k$, there exists $d_0 \in \mathbb{Z}$ such that $d_0 < dx < d_0 + k + 1$ for all $x \in P$. Notice that $B_2^n(a, r) \subseteq P \subseteq \{x \in \mathbb{R}^n : d_0 < dx < d_0 + k + 1\}$ and the distance between two hyperplanes $\{x \in \mathbb{R}^n : dx = d_0\}$ and $\{x \in \mathbb{R}^n : dx = d_0 + k + 1\}$ is exactly $(k+1)/\|d\|_2$. This implies that $2r$ is at most $(k+1)/\|d\|_2$. Hence, we get $r \leq \frac{k+1}{2\|d\|_2} < \frac{1}{2\ell}$, *i.e.*, $2\ell r < 1$. Suppose that there is some i such that $w(P, e^i) \geq 2$. Then there are two points $u, v \in P$ such that $u_i \leq b$ and $v_i \geq b + 1$ for some $b \in \mathbb{Z}$. So $\|u - v\|_2 \geq |u_i - v_i| \geq 1$. Since $B_2^n(a, \ell r)$ contains P, the distance between any two points in P is at most $2\ell r$ and thus we get $2\ell r \geq 1$. However, this contradicts the previous observation that $2\ell r < 1$. Therefore, $w(P, e^i) \leq 1$ for all $i \in [n]$. □

Proof of Proposition 2.5. Consider the following algorithm:

(1) For each $d \in \mathbb{Z}^n$ with $\|d\|_2 \leq \ell$, compute $w(P, d)$. If $w(P, d) = 0$ for some d with $\|d\|_2 \leq \ell$, then $P_I = \emptyset$. Otherwise, go to step (2).

(2) Compute $w(P, e^i)$ for $i \in [n]$. If there exists $i \in [n]$ such that $w(P, e^i) \geq 2$, then $P_I \neq \emptyset$. If there exists $i \in [n]$ such that $w(P, e^i) = 0$, then $P_I = \emptyset$. Otherwise, go to step (3).

(3) Let $z_j := \lfloor \max\{x_j : x \in P\} \rfloor$ for $j \in [n]$. If $(z_1, \cdots, z_n) \in P$, then $P_I \neq \emptyset$. Otherwise, $P_I = \emptyset$.

Step (1) can be done in polynomial time, because there are at most $\binom{n}{\ell} 2^\ell \binom{2\ell-1}{\ell}$ integral vectors d with $\|d\|_2 \leq \ell$. By assumption, there exists a Chvátal-Gomory inequality $dx \leq \bar{d}_0$ such that $\{x \in P : \bar{d}x \leq \bar{d}_0\} = P_I$. Note that P_I is empty if and only if $w(P, \bar{d}) = 0$. Going into Step (2), we have $w(P, d) \geq 1$ for all $d \in \mathbb{Z}^n$ with $\|d\|_2 \leq \ell$ and $\|\bar{d}\|_2 > \ell$. If $w(P, e^i) \geq 2$ for some $i \in [n]$, then $w(P, \bar{d}) \geq 1$ by Lemma 2.6 (when $k = 0$) and thus $P_I \neq \emptyset$. If $w(P, e^i) = 0$ for some $i \in [n]$, then P_I is empty. Therefore, going into Step (3), we have $w(P, e^i) = 1$ for all $i \in [n]$, and P can have at most one integer point. z is the only possibility and we can compute z by solving n linear programs, therefore, in polynomial time. □

2.2 Recognizing rational polytopes with an empty Chvátal closure is NP-hard

Recently, Cornuéjols and Li [38, 37] proved that it is NP-complete to decide whether the Chvátal closure of a rational polytope is empty. In this section, we prove Theorem 1.3 that states that the problem remains NP-complete, even when the input polytope is contained in the unit hypercube or is a simplex. We prove this in § 2.2.1 and 2.2.2. This hardness result has some nice consequences. In particular, the result implies that both optimizing and separating over the Chvátal closure of a rational polytope given by its linear description are NP-hard, even when the polytope is contained in the unit cube or is a simplex (Corollary 1.5). This extends an earlier result of Eisenbrand [62], and we explain this in § 2.2.3. Another consequence is that for any positive integer k, it is NP-hard to decide whether adding at most k (rank-1) Chvátal-Gomory cuts is sufficient to describe the integer hull of a rational polytope given by its linear description, and we derive this in § 2.2.4.

2.2.1 The case of polytopes contained in the unit hypercube

The next theorem is the main result of this section, and it is a half of Theorem 1.3.

Theorem 2.7 ([37]). *Let $P = \{x \in [0, 1]^n : Ax \leq b\}$ be a nonempty rational polytope contained in the unit hypercube. It is NP-complete to decide whether the Chvátal closure of P is empty, even when P contains no integer point.*

We reduce the equality knapsack problem, which is formally stated below, to the problem of deciding emptiness of the Chvátal closure of a rational polytope given by its linear description.

Equality Knapsack Problem (see [68]). Given positive integers $a_1, \ldots, a_n, b$, is there a set of nonnegative integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = b$?

Without loss of generality, we assume that $a_1, \ldots, a_n$ are relatively prime. We follow the idea behind Cornuéjols and Li's construction ([38, 37], Lemma 1), where they first construct some points using the

input data for an instance of the equality knapsack problem and then take their convex hull to construct a rational polytope. Although the polytopes generated from their construction are not necessarily contained in the unit hypercube, we are able to refine their idea and choose our points in the unit hypercube as described in the next lemma. Theorem 2.7 immediately follows from it.

Lemma 2.8 ([37]). *Given an equality knapsack instance of n positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time generate the linear description of a rational polytope $P \subseteq [0,1]^{n+4}$ contained in the unit hypercube satisfying the following:*

(a) P can be chosen to be the convex hull of $n+10$ points in $[0,1]^{n+4}$.

(b) $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \in P$ but P contains no integer point.

(c) P is full-dimensional.

(d) There exists a solution to the equality knapsack instance if and only if there exists a Chvátal-Gomory inequality of P that separates $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.

(e) There exists a solution to the equality knapsack instance if and only if the Chvátal closure of P is empty and the number of Chvátal-Gomory inequalities to certify this is exactly 2.

Proof. Let a rational polytope $P \subseteq [0,1]^{n+4}$ be defined as the convex hull of the following $n+10$ points $v^1, \cdots, v^{n+10} \in [0,1]^{n+4}$:

$$
\begin{array}{rcccccccccc}
v^1 & := & (& \frac{1}{2b}, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^2 & := & (& 0, & \frac{1}{2b}, & \cdots, & 0, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
 & & & & & & & \vdots & & & & & \\
v^n & := & (& 0, & 0, & \cdots, & 0, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^{n+1} & := & (& 0, & 0, & \cdots, & 0, & 0, & 0, & 1/2, & 1/2, & 1/2 &) \\
v^{n+2} & := & (& 1, & 1, & \cdots, & 1, & 1, & 1, & 1/2, & 1/2, & 1/2 &) \\
v^{n+3} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1, & 1 &) \\
v^{n+4} & := & (& 1/4, & 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 &) \\
v^{n+5} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1, & 1/2 &) \\
v^{n+6} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0, & 1/2 &) \\
v^{n+7} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1/2, & 1, & 1 &) \\
v^{n+8} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1/2, & 0, & 0 &) \\
v^{n+9} & := & (& \frac{a_1}{2b}, & \frac{a_2}{2b}, & \cdots, & \frac{a_{n-1}}{2b}, & \frac{a_n}{2b}, & 0, & 0, & \frac{1}{2} - \frac{1}{4b}, & 0 &) \\
v^{n+10} & := & (& 1 - \frac{a_1}{2b}, & 1 - \frac{a_2}{2b}, & \cdots, & 1 - \frac{a_{n-1}}{2b}, & 1 - \frac{a_n}{2b}, & 1, & \frac{1}{2} + \frac{1}{4b}, & 0, & 0 &)
\end{array}
$$

Let $u := \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Notice that $u = \frac{1}{2}v^{n+1} + \frac{1}{2}v^{n+2}$, so u is contained in P. In addition, none of $v^1, \ldots, v^{n+10}$ is contained in $\{0,1\}^{n+4}$, so P contains no integer point. This shows that P satisfies (b).

Claim 1. *P is full-dimensional.*

Proof of Claim. It is easy to show that the $n+4$ vectors in $\{v^i - v^{n+1} : i = 1, \ldots, n, n+2, n+3, n+5, n+7\}$ are linearly independent. Then the $n+5$ points $v^1, \ldots, v^n, v^{n+1}, v^{n+2}, v^{n+3}, v^{n+5}, v^{n+7}$ are affinely independent, thereby proving that the dimension of P is $n+4$, as required. ◇

By Claim 1, we know that P satisfies (c). Claim 1 also implies that we can compute the linear description of P in polynomial time, as stated in the following claim.

Claim 2. *The linear description of P can be obtained in polynomial time.*

Proof of Claim. Since P is full-dimensional, the number of facets of P is at most $\binom{n+10}{n+4} \leq n^6$. Given $n+4$ affinely independent points among $v^1, \cdots, v^{n+10}$, we can compute the hyperplane containing these $n+4$ points using the Gaussian elimination method. Since the encoding size of each v^i is polynomial in $\log a_1, \cdots, \log a_n, \log b$, and n, the complexity of the hyperplane is also polynomially bounded by the input encoding size. Therefore, we can find each facet of P in polynomial time. $\diamond$

To prove that P satisfies (d) and (e), we need the following two claims:

Claim 3. *If there exists a solution to the equality knapsack instance, then the Chvátal closure of P is empty and the number of Chvátal-Gomory inequalities to certify this is exactly 2.*

Proof of Claim. Let $(w_1, \cdots, w_n)$ be a solution to the knapsack instance. Then $\sum_{i=1}^n a_i w_i = b$ and $w_i \geq 0$ for $i \in [n]$. Let $d := (w_1, \cdots, w_n, -\sum_{i=1}^n w_i, 1, -1, 1) \in \mathbb{Z}^{n+4}$. Notice that $w_k \leq a_k w_k \leq \sum_{i=1}^n a_i w_i = b$, so we get $\frac{w_k}{2b} \leq \frac{1}{2}$. Since $b > 1$, we know that $0 < \frac{1}{2b} \leq \frac{1}{4}$. Thus, $0 < dv^k = \frac{w_k}{2b} + \frac{1}{2b} < 1$ for $k \in [n]$. It is easy to show that $dv^{n+1} = dv^{n+2} = dv^{n+5} = dv^{n+6} = dv^{n+7} = dv^{n+8} = \frac{1}{2}$, $dv^{n+4} = \frac{1}{4}$, and $dv^{n+3} = 1$. In addition, $dv^{n+9} = dv^{n+10} = \frac{1}{4b}$. That means $0 < dv^i < 1$ for $i \neq n+3$ and $dv^{n+3} = 1$. Then, $dx > 0$ is valid for P, and we obtain its corresponding Chvátal-Gomory inequality $dx \geq 1$. In fact, $P \cap \{x \in \mathbb{R}^{n+4} : dx \geq 1\} = \{v^{n+3}\}$, because v^{n+3} is the only vertex of P that is not cut off by $dx \geq 1$. Notice that $x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} \leq \frac{7}{2}$ is also valid for P. Then $x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} \leq 3$ is valid for P', and v^{n+3} violates this inequality. Therefore, $P \cap \{x \in \mathbb{R}^n : dx \geq 1,\ x_{n+3} + x_{n+2} + x_{n+3} + x_{n+4} \leq 3\}$ is empty. Hence, the Chvátal closure of P is empty and the number of Chvátal-Gomory inequalities to certify this is 2. $\diamond$

Claim 4. *If there exists a Chvátal-Gomory inequality separating $u = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, then there exists a solution to the equality knapsack instance.*

Proof of Claim. There is a valid inequality $dx \leq d_0 + \epsilon$ for P such that $(d, d_0) \in \mathbb{Z}^{n+5}$, $0 < \epsilon < 1$, and $du > d_0$. We claim that d and d_0 satisfy the following five properties:

1) $d_{n+1} = -\sum_{i=1}^n d_i$.
2) $d_0 = -1$.
3) $d_{n+2} = d_{n+4} = -1$ and $d_{n+3} = 1$.
4) $\sum_{i=1}^n a_i d_i = -b$.
5) $d_i \leq 0$ for $i \in [n]$.

Then, $(-d_1, \cdots, -d_n)$ is a solution to the equality knapsack instance.

Since $d_0 < du \leq d_0 + \epsilon < d_0 + 1$, we get $d_0 < \frac{1}{2}\sum_{i=1}^{n+4} d_i < d_0 + 1$. In addition, we know that $dv^k \leq d_0 + \epsilon < d_0 + 1$ for $k \in [n + 10]$. The integrality of $\sum_{i=1}^{n+4} d_i$ implies that $\frac{1}{2}\sum_{i=1}^{n+4} d_i$ should be equal to $d_0 + \frac{1}{2}$, and thus we get $\sum_{i=1}^{n+4} d_i = 2d_0 + 1$ and $du = d_0 + \frac{1}{2}$. Consider dv^{n+1} and dv^{n+2}:

$$d_0 + 1 > dv^{n+1} = du - \frac{1}{2}\sum_{i=1}^{n+1} d_i = d_0 + \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{n+1} d_i, \tag{2.1}$$

$$d_0 + 1 > dv^{n+2} = du + \frac{1}{2}\sum_{i=1}^{n+1} d_i = d_0 + \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n+1} d_i. \tag{2.2}$$

By (2.1) and (2.2), we get $-1 < \sum_{i=1}^{n+1} d_i < 1$. Since $\sum_{i=1}^{n+1} d_i$ is an integer, $\sum_{i=1}^{n+1} d_i = 0$ and the first property is satisfied. Then we know that $d_{n+2} + d_{n+3} + d_{n+4} = 2d_0 + 1$. Now, consider dv^{n+3} and dv^{n+4}:

$$d_0 + 1 > dv^{n+3} = du + \frac{1}{2}(d_{n+2} + d_{n+3} + d_{n+4}) = 2d_0 + 1, \tag{2.3}$$

$$d_0 + 1 > dv^{n+4} = \frac{1}{2}du = \frac{1}{2}d_0 + \frac{1}{4}. \tag{2.4}$$

By (2.3) and (2.4), we obtain $-\frac{3}{2} < d_0 < 0$ and thus $d_0 = -1$. So the second property holds and $d_{n+2} + d_{n+3} + d_{n+4} = -1$. Consider dv^{n+5} and dv^{n+6}:

$$d_0 + 1 > dv^{n+5} = du + \frac{1}{2}(d_{n+2} + d_{n+3}) = d_0 + \frac{1}{2} + \frac{1}{2}(d_{n+2} + d_{n+3}), \tag{2.5}$$

$$d_0 + 1 > dv^{n+6} = du - \frac{1}{2}(d_{n+2} + d_{n+3}) = d_0 + \frac{1}{2} - \frac{1}{2}(d_{n+2} + d_{n+3}). \tag{2.6}$$

By (2.5) and (2.6), we know that $-1 < d_{n+2} + d_{n+3} < 1$. So, $d_{n+2} + d_{n+3} = 0$. Similarly, we get $d_{n+3} + d_{n+4} = 0$ by considering dv^{n+7} and dv^{n+8}. Together with the observation $d_{n+2} + d_{n+3} + d_{n+4} = -1$, we get $d_{n+3} = 1$ and $d_{n+2} = d_{n+4} = -1$. Hence, the third property is satisfied. To prove the fourth property, we consider dv^{n+9} and dv^{n+10}:

$$dv^{n+9} = \frac{1}{2b}\sum_{i=1}^{n} a_i d_i + (\frac{1}{2} - \frac{1}{4b}) < d_0 + 1 = 0, \tag{2.7}$$

which implies that $\sum_{i=1}^{n} a_i d_i < -b + \frac{1}{2}$, so $\sum_{i=1}^{n} a_i d_i \leq -b$ since the sum is an integer;

$$dv^{n+10} = \sum_{i=1}^{n+1} d_i - \frac{1}{2b}\sum_{i=1}^{n} a_i d_i - (\frac{1}{2} + \frac{1}{4b}) = -\frac{1}{2b}\sum_{i=1}^{n} a_i d_i - (\frac{1}{2} + \frac{1}{4b}) < d_0 + 1 = 0, \tag{2.8}$$

which implies that $\sum_{i=1}^{n} a_i d_i > -b - \frac{1}{2}$, so $\sum_{i=1}^{n} a_i d_i \geq -b$ since the sum is an integer. Therefore, $\sum_{i=1}^{n} a_i d_i = -b$. Lastly, consider dv^k for $k \in [n]$:

$$dv^k = \frac{1}{2b}d_k - \frac{1}{2b} < d_0 + 1 = 0. \tag{2.9}$$

By (2.9), $d_k < 1$ and thus $d_k \leq 0$. $\diamond$

Claim 3 proves one direction of (d) and that of (e), and Claim 4 proves the other directions of (d) and (e). Therefore, (d) and (e) are also satisfied, as required. This finishes the proof. $\square$

2.2.2 The case of simplices

We proved Theorem 2.7 in § 2.2.1 for the polytopes in the unit hypercube. The next theorem is for the case of Theorem 1.3 when P is a simplex.

Theorem 2.9 ([37]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational simplex. It is NP-complete to decide whether the Chvátal closure of P is empty, even when P contains no integer point.*

To prove Theorem 2.7, we constructed a polytope that is the convex hull of $n+10$ points in $[0,1]^{n+4}$, but a simplex in $\mathbb{R}^{n+4}$ has less vertices. By allowing to choose some points sitting outside the hypercube, we are able to reduce the number of points so that we can construct rational simplices as described in the following lemma. Lemma 2.10 is very similar to Lemma 2.8, but its proof is more technical and involves a longer argument.

Lemma 2.10 ([37]). *Given an equality knapsack instance of n positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time generate the linear description of a rational polytope $P \subseteq \mathbb{R}^{n+1}$ and a point $u \in P$ satisfying the following:*

(a) P is a full-dimensional simplex.

(b) P contains no integer point.

(c) There exists a solution to the equality knapsack instance if and only if there exists a Chvátal-Gomory inequality of P that separates u.

(d) There exists a solution to the equality knapsack instance if and only if the Chvátal closure of P is empty and the number of Chvátal-Gomory inequalities to certify this is exactly 2.

Proof. Let $P \in \mathbb{R}^{n+1}$ be a rational polytope defined as the convex hull of the following $n+2$ points $v^1, \ldots, v^{n+2} \in \mathbb{R}^{n+1}$:

$$\begin{array}{rcccccc}
v^1 := & (& \frac{1}{2rB}, & 0, & \cdots, & 0, & \frac{1}{2r} - \frac{b}{2rBA} \quad) \\
v^2 := & (& 0, & \frac{1}{2rB}, & \cdots, & 0, & \frac{1}{2r} - \frac{b}{2rBA} \quad) \\
 & & & & \vdots & & \\
v^n := & (& 0, & 0, & \cdots, & \frac{1}{2rB}, & \frac{1}{2r} - \frac{b}{2rBA} \quad) \\
v^{n+1} := & (& ra_1, & ra_2, & \cdots, & ra_n & -rb + \frac{1}{2} \quad) \\
v^{n+2} := & (& -ra_1, & -ra_2, & \cdots, & -ra_n & rb + 1 \quad)
\end{array}$$

where A and B denote $\sum_{i=1}^n a_i$ and the smallest integer greater than $\frac{b}{A}$, respectively and $r := 2019b + \frac{1}{2b}$.

Claim 1. *P is a full-dimensional simplex.*

Proof of Claim. It is easy to show that $v^1, \ldots, v^{n+2}$ are affinely independent, thereby proving that P is full-dimensional a rational simplex. $\diamond$

By Claim 1, P satisfies (a).

Claim 2. *The linear description of P can be obtained in polynomial time.*

Proof of Claim. Since P is a full-dimensional rational simplex in $\mathbb{R}^{n+1}$, it contains exactly $n+2$ facets. One can obtain each facet-defining inequality of P by the Gaussian elimination method, and the complexity of each facet-defining inequality is polynomially bounded by $\log a_1, \cdots, \log a_b$, and $\log b$, as required. $\diamond$

Let $u := (\frac{a_1}{6rBA}, \cdots, \frac{a_n}{6rBA}, \frac{1}{6r} + \frac{1}{2} - \frac{b}{6rBA})$. We will show that P satisfies (c) and (d) for this choice fo u. We need the following two claims:

Claim 3. *If there exists a solution to the equality knapsack instance, then the Chvátal closure of P is empty and the number of Chvátal-Gomory inequalities to certify this is exactly 2.*

Proof of Claim. Let $(w_1, \cdots, w_n)$ be a solution to the knapsack instance. Then $\sum_{i=1}^n a_i w_i = b$ and $w_i \geq 0$ for $i \in [n]$. Let $d := (w_1, \cdots, w_n, 1) \in \mathbb{Z}^n$. So $dv^k = \frac{w_k}{2rB} + \frac{1}{2r} - \frac{b}{2rBA}$ for $k = 1, \cdots, n$. Since B is the smallest integer greater than $\frac{b}{A}$, we have $0 < B - \frac{b}{A} \leq 1$, so we get that $0 < \frac{1}{2r} - \frac{b}{2rBA} \leq \frac{1}{2rB}$. This implies $0 < dv^k \leq \frac{w_k+1}{2rB}$. As $w_k \leq a_k w_k \leq b$, we have $w_k + 1 \leq 2b$. Hence, $0 < dv^k < 1$ for $k \in [n]$. Moreover, we have $dv^{n+1} = \frac{1}{2}$ and $dv^{n+2} = 1$. Since $dv^1, \cdots, dv^{n+2}$ are all positive, it follows that $dx \geq 1$ is valid for P'. In addition, $x_{n+1} \leq rb + 1 = 2019b^2 + \frac{3}{2}$ is valid for P, so $x_{n+1} \leq rb + \frac{1}{2} = 2019b^2 + 1$ is valid for P'. Since $P \cap \{x \in \mathbb{R}^{n+1} : \ dx \geq 1\} = \{v^{n+2}\}$ and the last component of v^{n+2} is greater than $rb + \frac{1}{2}$, it follows that $P \cap \{x \in \mathbb{R}^{n+1} : \ dx \geq 1, \ x_{n+1} \leq rb + \frac{1}{2}\} = \emptyset$. Therefore $P' = \emptyset$, as required. $\diamond$

Claim 4. *If there exists a Chvátal-Gomory cut separating u, then there exists a solution to the equality knapsack instance.*

Proof of Claim. Let $u^1 := \frac{1}{A} \sum_{i=1}^n a_i v^i$. Then $u^1 = (\frac{a_1}{2rBA}, \cdots, \frac{a_n}{2rBA}, \frac{1}{2r} - \frac{b}{2rBA}) \in P$. Let $u^2 := \frac{1}{2}v^{n+1} + \frac{1}{2}v^{n+2} = (0, \cdots, 0, \frac{3}{4})$. Then $\frac{1}{3}u^1 + \frac{2}{3}u^2 = (\frac{a_1}{6rBA}, \cdots, \frac{a_n}{6rBA}, \frac{1}{6r} + \frac{1}{2} - \frac{b}{6rBA}) = u$. So, both u^2 and u are in P. If $P' = \emptyset$, at least one Chvátal-Gomory inequality is violated by u. In other words, there exists an inequality $dx \leq d_0 + \alpha$ valid for P such that $(d, d_0) \in \mathbb{Z}^{n+2}$, $0 < \alpha < 1$, and $d_0 < du$. We claim that d and d_0 satisfy the following four properties:

1) $\sum_{i=1}^n a_i d_i = b d_{n+1}$.

2) $d_{n+1} = -1$.

3) $d_0 = -1$.

4) $d_i \leq 0$ for $i = 1, \cdots, n$.

Then, $(-d_1, \cdots, -d_{n+1})$ is a solution to the equality knapsack instance.

Let $\Delta := \sum_{i=1}^n a_i d_i - b d_{n+1}$. Then Δ is an integer. Note that $r\Delta + \frac{1}{2}d_{n+1} - 1 < \lfloor dv^{n+1} \rfloor \leq d_0$, so $r\Delta + \frac{1}{2}d_{n+1} - 1 < d_0 < du = \frac{1}{6rBA}\Delta + (\frac{1}{2} + \frac{1}{6r})d_{n+1}$, implying in turn that

$$6r\left(r - \tfrac{1}{6rBA}\right)\Delta - 6r \quad < \quad d_{n+1}. \tag{2.10}$$

Observe that $-r\Delta + d_{n+1} - 1 < \lfloor dv^{n+1} \rfloor \leq d_0$. Since $-r\Delta + d_{n+1} - 1 < d_0 < du = \frac{1}{6rBA}\Delta + (\frac{1}{2} + \frac{1}{6r})d_{n+1}$, we get

$$\left(\tfrac{1}{2} - \tfrac{1}{6r}\right) d_{n+1} \quad < \quad 1 + \left(r + \tfrac{1}{6rBA}\right) \Delta. \tag{2.11}$$

Suppose for a contradiction that $\Delta \neq 0$. There are four cases to consider; $\Delta > 0$ and $d_{n+1} \geq 0$; $\Delta > 0$ and $d_{n+1} < 0$; $\Delta < 0$ and $d_{n+1} \geq 0$; and $\Delta < 0$ and $d_{n+1} < 0$.

(Case 1: $\Delta > 0$ and $d_{n+1} \geq 0$): We know $(\frac{1}{2} - \frac{1}{6r}) > \frac{1}{3}$. By (2.11), we get $d_{n+1} < (3r + \frac{1}{2rBA})\Delta + 3$. Together with (2.10), we have

$$6\Delta r^2 - (3\Delta + 6)r - \tfrac{1}{BA}\Delta - \tfrac{1}{2rBA}\Delta \quad < \quad 3.$$

As $\Delta \geq 1$, it follows that $-6\Delta r \leq -6r$ and $-2\Delta \leq -\frac{1}{BA}\Delta - \frac{1}{2rBA}\Delta$. Hence, we obtain the following:

$$\Delta(6r^2 - 9r - 2) \quad < \quad 3,$$

which cannot be true, because $r \geq 2019$ and $\Delta \geq 1$. Therefore, Case 1 is not possible.

(Case 2: $\Delta > 0$ and $d_{n+1} < 0$): By (2.10), $6\Delta r^2 - \frac{1}{BA}\Delta - 6r < d_{n+1}$. Notice tht $\Delta r^2 \geq \frac{1}{BA}\Delta$, so $5\Delta r^2 - 6r < d_{n+1}$. As $\Delta \geq 1$ and $r \geq 2019$, it follows that $d_{n+1} > 0$, contradicting the assumption that $d_{n+1} < 0$.

(Case 3: $\Delta < 0$ and $d_{n+1} \geq 0$): Since $\Delta \leq -1$ and $\frac{1}{6rBA} > 0$, the right hand side (2.11) is less than $1 - r$, a negative number. As $\frac{1}{2} > \frac{1}{6r}$, (2.11) implies that $d_{n+1} < 0$, contradicting the assumption $d_{n+1} \geq 0$.

(Case 4: $\Delta < 0$ and $d_{n+1} < 0$): Notice that $\frac{1}{2rBA}\Delta + \frac{1}{2r}d_{n+1} - 1 < \lfloor du^1 \rfloor \leq d_0$, so $\frac{1}{2rBA}\Delta + \frac{1}{2r}d_{n+1} - 1 < d_0 < du = \frac{1}{6rBA}\Delta + (\frac{1}{2} + \frac{1}{6r})d_{n+1}$. It follows that $\frac{1}{3rBA}\Delta - 1 < (\frac{1}{2} - \frac{1}{3r})d_{n+1} < \frac{1}{3}d_{n+1}$, and thus $\frac{1}{rBA}\Delta - 3 < d_{n+1}$. Observe that (2.11) and the assumption $d_{n+1} < 0$ imply that $d_{n+1} < 1 + (r + \frac{1}{6rBA})\Delta$. So, we obtain

$$-4 \quad < \quad \left(r - \tfrac{5}{6rBA}\right) \Delta.$$

Since $\Delta \leq -1$, we have $\left(r - \frac{5}{6rBA}\right) \Delta \leq -r + \frac{5}{6rBA} < 1 - r$. Then we get $-4 < 1 - r$, a contradiction as $r \geq 2019$.

Therefore, each of the four cases is not possible, implying in turn that $\Delta = 0$. So, (d, d_0) satisfies the first property. Moreover $\Delta = 0$ implies that $du^1 = \frac{1}{2r}d_{n+1}$, $du = (\frac{1}{2} + \frac{1}{6r})d_{n+1}$, $dv^{n+1} = \frac{1}{2}d_{n+1}$, and $dv^{n+2} = d_{n+1}$. Suppose for a contradiction that $d_{n+1} \geq 0$. If $d_{n+1} = 0$, then d_0 satisfies $d_0 < du = 0 < d_0 + 1$, which is not possible as d_0 is an integer. This implies that $d_{n+1} \geq 1$. Then the following holds.

$$\lfloor du \rfloor \quad = \quad \lfloor (\tfrac{1}{2} + \tfrac{1}{6r})d_{n+1} \rfloor \quad < \quad d_{n+1} \quad = \quad \lfloor d_{n+1} \rfloor \quad = \quad \lfloor dv^{n+2} \rfloor \quad \leq \quad \lfloor d_0 + \alpha \rfloor \quad = \quad d_0.$$

However, we assumed that $d_0 < du$, and this implies $d_0 \leq \lfloor du \rfloor$, a contradiction. Thus, $d_{n+1} \leq -1$. Note that $\frac{1}{2r}d_{n+1} - 1 < \lfloor du^1 \rfloor \leq d_0$. Since $d_0 < du = (\frac{1}{2} + \frac{1}{6r})d_{n+1}$, it follows that $-1 < (\frac{1}{2} - \frac{1}{3r})d_{n+1}$ and thus $-2 \leq d_{n+1}$. If $d_{n+1} = -2$, $\lfloor dv^{n+1} \rfloor = -1$ and $\lfloor du \rfloor = -2$. Then $\lfloor du \rfloor < \lfloor dv^{n+1} \rfloor \leq d_0$, but this contradicts the observation $d_0 \leq \lfloor du \rfloor$. Therefore, $d_{n+1} = -1$, so (d, d_0) satisfies the second property.

Since $d_{n+1} = -1$, it follows that $du = -\frac{1}{2} - \frac{1}{6r}$, implying that $-1 < du < 0$ and thus $d_0 = -1$ which is the third property. To prove the fourth property, let us consider dv^k for $k \in [n]$. $dv^k = \frac{d_k}{2rB} - (\frac{1}{2r} - \frac{b}{2rBA}) < d_0 + 1 = 0$. Then, $d_k < B - \frac{b}{A}$. Since B is the smallest integer greater than $\frac{b}{A}$, $B \leq \frac{b}{A} + 1$. Therefore, $d_k < 1$ and thus $d_k \leq 0$ for $k \in [n]$. $\diamond$

By Claims 3 and 4, P satisfies (c) and (d).

To complete the proof, we need to show that P has no integer point. Notice that if P contains an integer point, then there exists an integer $d \in [-rb + \frac{1}{2}, rb + 1]$ such that $P(d) := \{x \in P : x_{n+1} = d\}$ contains an integer point.

Claim 5. *$P(d) = \{x \in P : x_{n+1} = d\}$ has no integer point if $d > 0$.*

Proof of Claim. Suppose for a contradiction that $P(d)$ has an integer point z_0. For $i \in [n+1]$, let p_0^i denote the intersection point of $\{x \in \mathbb{R}^{n+1} : \ x_{n+1} = d\}$ and the line segment between v^{n+2} and v^i for $i \in [n+1]$. Notice that $P(d) = \operatorname{conv}\left(p_0^1, \ldots, p_0^{n+1}\right)$. Let a and p^i for $i \in [n+1]$ denote the vectors obtained from $(a_1, \cdots, a_n, 0)$ and p_0^i after projecting out the last coordinate, respectively. Then it can be checked that $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ and

$$\begin{aligned} p^i &= \tfrac{-rd+\frac{1}{2}-\frac{b}{2BA}}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} a \quad + \left(\tfrac{1}{2rB} - \tfrac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rB(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) e^i \quad \text{for } i \in [n], \\ p^{n+1} &= \tfrac{-rd+\frac{3}{4}r}{rb+\frac{1}{4}} a. \end{aligned}$$

As the $(n+1)^{\text{st}}$ coordinate of p_0^i is d for each i, $\operatorname{conv}\left(p^1, \ldots, p^{n+1}\right)$ is precisely the projection of $P(d)$ onto the space defined by the first n coordinates. As z_0 is an integer point in $P(d)$, it follows that $z \in \mathbb{Z}^n$, the vector obtained from $z_0 \in \mathbb{Z}^{n+1}$ after projecting out its last coordinate, is an integer point in $\operatorname{conv}\left(p^1, \cdots, p^{n+1}\right)$.

Notice that $p^{n+1} = C_1 a$ and $p := \frac{1}{A}\sum_{i=1}^n a_i p^i = C_2 a$. where

$$C_1 := \tfrac{-rd+\frac{3}{4}r}{rb+\frac{1}{4}} \quad \text{and} \quad C_2 := \left(\tfrac{-rd+\frac{1}{2}-\frac{b}{2BA}}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \tfrac{1}{2rBA} - \tfrac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right).$$

Thus, p^{n+1} and p are on the line through $\mathbf{0}$ and a. Moreover, it can be readily shown that the intersection of $\operatorname{conv}\left(p^1, \cdots, p^{n+1}\right)$ and the line through $\mathbf{0}$ and a is the line segment between p^{n+1} and p. We will first show that the line segment between p^{n+1} and p.

We will show that $\operatorname{conv}\left(p^{n+1}, p\right)$ contains no integer point, thereby showing that $z \notin \operatorname{conv}\left(p^{n+1}, p\right)$. Since $a_1, \cdots, a_n$ are relatively prime, there is no integer point strictly between ℓa and $(\ell+1)a$ for any $\ell \in \mathbb{Z}$. As $p^{n+1} = C_1 a$ and $p = C_2 a$, it is sufficient to argue that $C_1, C_2 \in (\ell, \ell+1)$ for some $\ell \in \mathbb{Z}$. Notice that d can be expressed as $kb + h$ for some $0 \le k \le 2019b$ and $0 \le h < b$. Then we can rewrite both p^{n+1} and p as follows:

$$p^{n+1} = \left(-k + \tfrac{-rh+\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}}\right) a = \left(-k-1 + \tfrac{r(b-h)+\frac{1}{4}+\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}}\right) a,$$

$$\begin{aligned} p &= \left(-k + \tfrac{-rh+(r-k)(\frac{1}{2r}-\frac{b}{2rBA})+k}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \tfrac{1}{2rBA} - \tfrac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) a \\ &= \left(-k-1 + \tfrac{r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1)}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \tfrac{1}{2rBA} - \tfrac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) a. \end{aligned}$$

In the following, we consider three possible cases: (1) $h = 0$, (2) $h = 1$ & $k = 2019b = r - \frac{1}{2b}$, and (3) $h \ge 1$ & $k \le 2019b - 1 = r - 1 - \frac{1}{2b}$.

(Case 1: $h = 0$): In this case, the integer part of C_1 is $-k$, while its fractional part is $\frac{\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}}$ since it is certainly positive and less than 1. Notice that $\frac{1}{2r} - \frac{b}{2rBA} = \frac{B-\frac{b}{A}}{2rB} \le \frac{1}{2rB}$, because B is the smallest

integer greater than $\frac{b}{A}$. Then $(r-k)(\frac{1}{2r}-\frac{b}{2rBA})+k \leq \frac{1}{2B}+r$. In addition, $0 < 1-\frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} < 1$, because $0 < d < rb+1$. Therefore, $\frac{-rh+(r-k)(\frac{1}{2r}-\frac{b}{2rBA})+k}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}$ is positive and at most $\frac{1}{2rB}+\frac{1}{2rBA}$ which is less than 1. That means the integer part of C_2 is $-k$ and its fractional part is positive. In this case, $C_1, C_2 \in (-k, -k+1)$.

(Case 2: $h = 1$ and $k = 2019b = r - \frac{1}{2b}$): Since $k < r$, $0 < r(b-1)+\frac{1}{4}+\frac{1}{4}k+\frac{3}{4}r < rb+\frac{1}{4}$. Thus, we get $0 < \frac{r(b-1)+\frac{1}{4}+\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}} < 1$. Then $C_1 \in (-k-1, -k)$. Note that

$$\begin{aligned} r(b-h)+(r-k-1)(\tfrac{1}{2r}-\tfrac{b}{2rBA})+(k+1) &= r(b-1)+(\tfrac{1}{2b}-1)(\tfrac{1}{2r}-\tfrac{b}{2rBA})+2019b+1 \\ &= rb+1-\tfrac{1}{2r}+\tfrac{b}{2rBA}+\tfrac{1}{2b}(-1+\tfrac{1}{2r}-\tfrac{b}{2rBA}). \end{aligned}$$

In addition,

$$\frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} = \frac{1}{4rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}.$$

In this case,

$$\frac{r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1)}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} = 1 + \frac{\frac{1}{2b}(-1+\frac{1}{2r}-\frac{b}{2rBA})+\frac{1}{4rBA}}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}}.$$

is less than 1, because $\frac{1}{2r}-\frac{b}{2rBA}+\frac{1}{4rBA} \leq \frac{1}{2rB}+\frac{1}{4rBA} < \frac{1}{2b}$. Therefore, we get that $C_2 \in (-k-1, -k)$.

(Case 3: $h \geq 1$ and $k \leq 2019b - 1 = r - 1 - \frac{1}{2b}$): As in the previous case, we can show that $C_1 \in (-k-1, -k)$. Notice that

$$\begin{aligned} r(b-h)+(r-k-1)(\tfrac{1}{2r}-\tfrac{b}{2rBA})+(k+1) &\leq rb-\tfrac{1}{2b}+\tfrac{1}{2b}(\tfrac{1}{2r}-\tfrac{b}{2rBA}) \\ &= rb+1-\tfrac{1}{2r}+\tfrac{b}{2rBA}-(1+\tfrac{1}{2b})(1-\tfrac{1}{2r}+\tfrac{b}{2rBA}). \end{aligned}$$

We also have the following:

$$\frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} \leq \frac{1}{2rBA} \leq \frac{1}{2rb} \leq \frac{1}{rb+1-\frac{1}{2r}-\frac{b}{2rBA}}.$$

Since $1-(1+\frac{1}{2b})(1-\frac{1}{2r}+\frac{b}{2rBA}) < 0$, we get

$$\frac{r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1)}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} < 1.$$

It is obvious that $r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1) > 0$, so $C_2 \in (-k-1, -k)$.

Therefore, the line segment between $p^{n+1} = C_1 a$ and $p = C_2 a$ cannot contain an integer point, implying in turn that $z \notin \operatorname{conv}\left(p^{n+1}, p\right)$.

Using our observation that $z \notin \operatorname{conv}\left(p^{n+1}, p\right)$, we will show that $z \notin \operatorname{conv}\left(p^1, \cdots, p^{n+1}\right)$, thereby leading to a contradiction. Although z is not on the line through $\mathbf{0}$ and a, we can argue that z is close to the line. By our supposition, $z \in \operatorname{conv}\left(p^1, \cdots, p^{n+1}\right)$, so we can check that $\delta \in \mathbb{R}$ such that

$$\|z - \delta a\|_\infty \leq \frac{1}{rB}.$$

Moreover, as z is not on the line through $\mathbf{0}$ and a in $\mathbb{R}^n$, there exists an index $j \in \{1, \cdots, n-1\}$ such that (z_j, z_n) is not on the line through $\mathbf{0}$ and (a_j, a_n) in $\mathbb{R}^2$. The following can be proved with a simple geometric analysis in $\mathbb{R}^2$:

($\star$) Let $p := (p_1, p_2) \neq (0,0)$ and $q := (q_1, q_2)$ be two points in $\mathbb{Z}^2$. Then, for any $\delta \in \mathbb{R}^n$, $\|q - \delta p\|_\infty \geq \frac{|p_1 q_2 - p_2 q_1|}{|p_1| + |p_2|}$.

By ($\star$), it follows that $\|(z_j, z_n) - \delta(a_j, a_n)\|_\infty \geq \frac{|a_j z_n - a_n z_j|}{|a_j| + |a_n|}$ for any $\delta \in \mathbb{R}$. Since (z_j, z_n) is not on the line through $\mathbf{0}$ and (a_j, a_n), we have $|a_j z_n - a_n z_j| \geq 1$. Since $0 < a_j, a_n \leq b$, it follows that $\|(z_j, z_n) - \delta(a_j, a_n)\|_\infty \geq \frac{1}{2b}$. As $\|z - \delta a\|_\infty \geq \|(z_j, z_n) - \delta(a_j, a_n)\|_\infty$, we obtain

$$\|z - \delta a\|_\infty \geq \frac{1}{2b}$$

for all $\delta \in \mathbb{R}$. However, this contradicts our earlier observation that $\|z - \delta a\|_\infty \leq \frac{1}{rB}$ for some $\delta \in \mathbb{R}$. Therefore, $\operatorname{conv}\left(p^1, \ldots, p^{n+1}\right)$ contains no integer point, implying in turn that $P(d)$ contains no integer point, as required. ◇

We now consider the case $d \leq 0$.

Claim 6. *$P(d) = \{x \in P : \ x_{n+1} = d\}$ for $d \leq 0$ contains no integer point.*

Proof of Claim. For $i = 1, \ldots, n, n+2$, let w_0^i denote the intersection point of the line segment between v^{n+1} and v^i and the hyperplane $\left\{x \in \mathbb{R}^{n+1} : x_{n+1} = d\right\}$. Then $P(d)$ is the convex hull of $w_0^1, \ldots, w_0^n, w_0^{n+2}$. Let w^i for $i = 1, \cdots, n, n+2$ denote the vector obtained from w_0^i after projecting out the last coordinate. Then

$$\begin{aligned} w^i &:= \frac{rd - \frac{1}{2} + \frac{b}{2BA}}{-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA}} a \quad + \left(\frac{1}{2rB} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rB(-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA})}\right) e^i \quad \text{for } i \in [n], \\ w^{n+2} &:= \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}} a. \end{aligned}$$

Then $\operatorname{conv}\left(w^1, \ldots, w^n, w^{n+2}\right)$ is precisely the projection of $P(d)$ onto the space of the first n coordinates. Therefore, it is sufficient to show that $\operatorname{conv}\left(w^1, \ldots, w^n, w^{n+2}\right)$ has no integer point. Let w denote $\frac{1}{A}\sum_{i=1}^n a_i w^i$. Then w can be written as

$$w = \left(\frac{rd - \frac{1}{2} + \frac{b}{2BA}}{-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rBA(-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA})}\right) a.$$

In fact, the line though $\mathbf{0}$ and a in $\mathbb{R}^n$ intersects with $\operatorname{conv}\left(w^1, \cdots, w^n, w^{n+2}\right)$ in the line segment between w^{n+2} and w. As the case when $d > 0$, we can argue that $\operatorname{conv}\left(w^{n+2}, w\right)$ contains no integer point. Using this, we can also prove that $\operatorname{conv}\left(w^1, \cdots, w^n, w^{n+2}\right)$ contains no integer point. ◇

Claims 5 and 6 imply that P contains no integer point. This finishes the proof. □

Theorem 2.9 follows Lemma 2.10. Putting Theorem 2.7 and 2.9 together, we obtain Theorem 1.3.

Theorem 1.3 ([37]). *Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ containing no integer point, it is NP-complete to test whether the Chvátal closure of P is empty, even when $P \subseteq [0,1]^n$ or P is a rational simplex.*

2.2.3 Optimization and separation over Chvátal closure

Eisenbrand [62] showed that the separation problem over the Chvátal closure of a rational polyhedron given by its linear description is NP-hard, answering an early question of Schrijver [108]. He derived this result as an extension of a result by Caprara and Fischetti [26].

Separation problem over the Chvátal closure. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, and let $\bar{x} \in \mathbb{Q}^n$ be a rational point. Then either show that $\bar{x} \in P'$ or find a valid Chvátal-Gomory inequality $dx \leq d_0$ for P' such that $d\bar{x} > d_0$.

According to a general result given by Grötschel, Lovász and Schrijver [74], this problem is equivalent to its optimization version up to a polynomial time overhead.

Optimization problem over the Chvátal closure. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, and let $c \in \mathbb{Q}^n$ be a rational objective coefficient vector. Then find a point $x^* \in P'$ satisfying $cx^* = \max\{cx : x \in P'\}$, or show $P' = \emptyset$, or find a ray z of the recession cone of P' for which cz is positive.

As an immediate corollary of Theorem 2.7 and Theorem 2.9, we obtain the following, which answers an open question raised by Letchford, Pokutta, and Schulz [94].

Theorem 2.11 ([37]). *The optimization and separation problems over the Chvátal closure of a rational polytope given by its linear description are NP-hard, even when the input polytope is contained in the unit hypercube or is a rational simplex.*

2.2.4 Deciding whether adding a certain number of Chvátal-Gomory cuts can yield the integer hull

Theorem 2.11 indicates that the number of Chvátal-Gomory cuts of a rational polytope to obtain its Chvátal closure can be, in general, super-polynomial in the encoding size of the polytope. It seems rare that the Chvátal closure of a rational polytope is obtained by adding a constant number of (rank-1) Chvátal-Gomory cuts. Besides, we know that the Chvátal rank of a rational polytope can be larger than 1, so it seems rarer that we can obtain the integer hull of a rational polytope by adding a constant number of Chvátal-Gomory cuts. Given a rational polytope, can we easily decide whether its integer hull 'cannot' be obtained by adding a fixed number of (rank-1) Chvátal-Gomory cuts? The answer to this question is probably 'no'. We remark the following, which can be derived from Lemma 2.8 and a result of Mahajan and Ralphs ([97], Proposition 3.4).

Remark 2.12 ([37]). *Let $P = \{x \in [0,1]^n : Ax \leq b\}$ be a rational polytope contained in the unit hypercube, and let k be a positive integer. Deciding whether we can obtain the integer hull of P by adding at most k (rank-1) Chvátal-Gomory inequalities to the linear description of P is NP-hard.*

Proof. If $k \geq 2$, we know from Lemma 2.8 that the decision problem is NP-hard. To prove that the problem is still NP-hard even when $k = 1$, we borrow the construction of Mahajan and Ralphs [97]. They constructed a polytope using the data for an instance of the partition problem, which is NP-hard and stated below.

Partition Problem (see [68]). Given positive integers $a_1, \cdots, a_n$, is there a subset K of the set of indices $[n]$ such that $\sum_{i \in K} a_i = \sum_{j \in [n] \setminus K} a_j$?

Let $a_1, \cdots, a_n$ be the input for an instance of the partition problem. Let $\widetilde{a_k} := \frac{1}{\sum_{j=1}^n a_j} a_k$ for $k \in [n]$. Let P be the convex hull of the following $n+4$ points in $[0,1]^{n+2}$:

$$
\begin{array}{rcccccccc}
v^1 := & (& \frac{1}{2} + \frac{1}{2(n+1)}, & \frac{1}{2(n+1)}, & \cdots, & \frac{1}{2(n+1)}, & 0, & 0 &) \\
v^2 := & (& \frac{1}{2(n+1)}, & \frac{1}{2} + \frac{1}{2(n+1)}, & \cdots, & \frac{1}{2(n+1)}, & 0, & 0 &) \\
 & & & & \vdots & & & & \\
v^n := & (& \frac{1}{2(n+1)}, & \frac{1}{2(n+1)}, & \cdots, & \frac{1}{2} + \frac{1}{2(n+1)}, & 0, & 0 &) \\
v^{n+1} := & (& \widetilde{a_1}, & \widetilde{a_2}, & \cdots, & \widetilde{a_n}, & 1, & 1 &) \\
v^{n+2} := & (& \widetilde{a_1}, & \widetilde{a_2}, & \cdots, & \widetilde{a_n}, & \frac{1}{2} - \frac{1}{2\sum_{j=1}^n a_j}, & 0 &) \\
v^{n+3} := & (& \widetilde{a_1}, & \widetilde{a_2}, & \cdots, & \widetilde{a_n}, & 0, & \frac{1}{2} - \frac{1}{2\sum_{j=1}^n a_j} &) \\
v^{n+4} := & (& 0, & 0, & \cdots, & 0, & \frac{1}{2}, & 0 &)
\end{array}
$$

We show that the Chvátal closure of P is empty, meaning that the integer hull of P is empty. Let $d := (1, \cdots, 1, 1, -1)$. Then $dv^i = 1 - \frac{1}{2(n+1)}$ for $i \in [n]$. Besides, we get $dv^{n+1} = 1$, $dv^{n+2} = \frac{3}{2} - \frac{1}{2\sum_{j=1}^n a_j}$, $dv^{n+3} = \frac{1}{2} + \frac{1}{2\sum_{j=1}^n a_j}$, and $dv^{n+4} = \frac{1}{2}$. Then $0 < dx < 2$ is valid for all $x \in P$, and thus $dx = 1$ is valid for P'. Since $0 < a_1 < \sum_{j=1}^n a_j$, $0 < \widetilde{a_1} < 1$. This implies that the first component of each v^i be less than 1, so $x_1 \leq 0$ is valid for P'. Notice that $P \cap \{x \in [0,1]^{n+2} : x_1 \leq 0\} = \{v^{n+4}\}$. Besides, $dv^{n+4} = \frac{1}{2} \neq 1$. Since $P' \subseteq P \cap \{x \in [0,1]^{n+2} : dx = 1,\ x_1 \leq 0\} = \emptyset$, we have that $P' = \emptyset$, as required.

The integer hull of P, which is empty, is obtained by adding a Chvátal-Gomory inequality $\pi x \leq \pi_0$ if and only if $\pi x < \pi_0 + 1$ is valid for P and every point in P violates $\pi x \leq \pi_0$ (or equivalently, $P \subseteq \{x \in \mathbb{R}^{n+2} : \pi_0 < \pi x < \pi_0 + 1\}$). Mahajan and Ralphs ([97], Proposition 3.4) proved that there is $(\pi, \pi_0) \in \mathbb{Z}^{n+3}$ such that $P \subseteq \{x \in \mathbb{R}^{n+2} : \pi_0 < \pi x < \pi_0 + 1\}$ if and only if there exists a subset K of $[n]$ such that $\sum_{i \in K} a_i = \sum_{j \in [n] \setminus K} a_j$. Therefore, the problem of deciding if we can obtain the integer hull of a rational polytope by adding at most k Chvátal-Gomory inequalities to the linear description of P is NP-hard, even when $k = 1$. □

Note from the proof of Remark 2.12 that k is not necessarily a constant. Observe that the construction of Mahajan and Ralphs used to prove Remark 2.12 is in the spirit of our constructions in Lemmas 2.8 and 2.10, but one difference is that the Chvátal closure of a polytope from their construction is always empty.

The decision problem remains NP-hard, even when the input polytope is a rational simplex, as stated in the following remark. It follows from Lemma 2.10 and Proposition 3.2 in [97].

Remark 2.13 ([37]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational simplex, and let k be a positive integer. Deciding if we can obtain the integer hull of P by adding at most k Chvátal-Gomory inequalities to the linear description of P is NP-hard.*

2.3 Flatness theorem for closed convex sets with empty Chvátal closure

Recall the definition of integer width of a convex set K given in § 2.1. When K is unbounded or has a large volume, there exists a direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ where $w(K, d)$ is large. On the other hand, it is possible that there is a direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $w(K, d)$ is relatively small if K does not contain any integer point. In fact, the famous flatness theorem by Khinchine [87] states that $w(K, \mathbb{Z}^n)$ for any compact convex set K containing no integer point is bounded by $f(n)$, a function that depends only on the ambient dimension n. Khinchine's flatness theorem [87] shows that $f(n) \leq (n+1)!$. A crucial component of Lenstra's algorithm [93] is to find a flat direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ of a polyhedron $P \subseteq \mathbb{R}^n$ containing no integer point. Lenstra [93] gave a polynomial algorithm to find a direction $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $w(K, d) \leq 2^{O(n^2)}$ for a given lattice-free compact convex set K. Then, it generates $2^{O(n^2)}$ subproblems in $\mathbb{R}^{n-1}$ by intersecting K with $2^{O(n^2)}$ parallel hyperplanes orthogonal to d. Hence, the algorithm works recursively, and the number of total steps required is $2^{O(n^3)}$.

Over the last few decades there have been huge improvements on the upper bound $f(n)$ (see [15, 16, 82, 83, 87, 105]). The current best known asymptotic upper bound is $f(n) = O(n^{4/3}\text{polylog}(n))$ given by Banaszczyk, Litvak, Pajor, and Szarek [16] and Rudelson [105]. It has been even conjectured that $f(n) = O(n)$. However, the existence of a polynomial algorithm to find a direction $d \in \mathbb{Z}^n$ such that $w(K, d) = O(n^{4/3}\text{polylog}(n))$ for a convex set K containing no integer point is not known. Dadush, Peikert and Vempala [46] and Dadush and Vempala [47] developed an algorithm to find all vectors $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $w(K, d) = w(K, \mathbb{Z}^n)$ in $2^{O(n)}\text{poly}(L)$ time and space.

In this section, we first prove that $f(n) \leq n$ if K is a compact convex set whose Chvátal closure is empty. The Chvátal closure of a closed convex set is defined similarly to that of a polyhedron [44, 43, 48]. For a closed convex set K, $\sigma_K(d) := \sup\{dx : x \in K\}$ for $d \in \mathbb{R}^n$ is its support function. It is known that any closed convex set K can be expressed as $K = \bigcap_{d \in \mathbb{R}^n} \{x \in \mathbb{R}^n : dx \leq \sigma_K(d)\}$, which is the set of solutions satisfying the system of linear inequalities given by its support function (see Theorem C.2.2.2 in [79]). Dadush, Dey, and Vielma later showed that the inequalities with integer coefficients are sufficient to describe K (Proposition 2.1 in [44]). In other words, $K = \bigcap_{d \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : dx \leq \sigma_K(d)\}$. The Chvátal closure of K is defined as what is obtained after rounding down their right hand side values. More precisely, given a closed convex set K, the Chvátal closure of K is defined as

$$K' := \bigcap_{d \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ dx \leq \lfloor \sigma_K(d) \rfloor\}.$$

By its definition, K' is contained in K and it is also clear that $K \cap \mathbb{Z}^n \subseteq K'$.

Let $K \subseteq \mathbb{R}^n$ be a convex set and $a \in \mathbb{R}^n$ be a point. We denote by $K - a := \{x - a : x \in K\}$ the translation of K by $-a$. Let ℓK for some real number ℓ be defined as $\ell K := \{\ell x : x \in K\}$.

Proposition 2.14 ([37]). *Let $K \subseteq \mathbb{R}^n$ be a compact convex set whose Chvátal closure is empty. If $K - a \subseteq -\ell(K - a)$ for some $a \in K$ and $\ell > 0$, then the integer width of K is at most $\lceil \ell \rceil$.*

Proof. Since the Chvátal closure of K is empty, $a \in K$ should be cut off by a Chvátal-Gomory inequality of K. In other words, there exists $(d, d_0) \in \mathbb{Z}^{n+1}$ such that $\max\{dx : x \in K\} < d_0$ and $da > d_0 - 1$. Then, we get $\max\{dx : x \in K - a\} = \max\{dx : x \in K\} - da < 1$, and this implies $\min\{dx : x \in -\ell(K - a)\} = -\max\{dx : x \in \ell(K - a)\} > -\ell$. We assumed that $K - a \subseteq -\ell(K - a)$, so $\min\{dx : x \in K - a\} \geq \min\{dx : x \in -\ell(K - a)\} > -\ell$. Hence, we have $\max\{dx : x \in K\} < d_0$ and $\min\{dx : x \in K\} > da - \ell > d_0 - \ell - 1$. Therefore, the integer width of K (along d) is at most $\lceil \ell \rceil$. □

If $K \subseteq \mathbb{R}^n$ is a centrally symmetric compact convex set, then $K - a = -(K - a)$ for some $a \in K$. Although an asymmetric convex set K does not contain such a point $a \in K$, Süss [114] and Hammer [76] proved the following:

Theorem 2.15 ([76], Theorem 2, see also [114]). *Let $K \subseteq \mathbb{R}^n$ be a full-dimensional compact convex set, then there exists $a \in K$ such that $K - a \subseteq -n(K - a)$.*

Combining Proposition 2.14 and Theorem 2.15, we can prove the following theorem:

Theorem 2.16 ([37]). *Let $K \subseteq \mathbb{R}^n$ be a compact convex set whose Chvátal closure is empty. Then the integer width of K is at most n.*

Proof. If K is full-dimensional, then Proposition 2.14 and Theorem 2.15 imply that the integer width of K is at most n. Thus we may assume that K is not full-dimensional. Then $K \subseteq \{x \in \mathbb{R}^n : cx = c_0\}$ for some $c \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $d \in \mathbb{R}$. If c is rational, then the integer width of K is either 0 or 1, depending on c_0. Thus we may assume that c is irrational. Since $c = (c_1, \ldots, c_n)$ is nonzero, we may further assume that $c_n \neq 0$ without loss of generality. Then we can approximate c with a rational vector, based on the Simultaneous Diophantine Approximation Theorem due to Dirichlet [54]:

> *Given any real numbers $r_1, \ldots, r_{n-1}$ and $0 < \varepsilon < 1$, there exist integers $d_1, \ldots, d_n$ such that $\left| r_i - \frac{d_i}{d_n} \right| < \frac{\varepsilon}{d_n}$ for $i = 1, \ldots, n-1$ and $1 \leq d_n \leq \left(\frac{1}{\varepsilon}\right)^{n-1}$.*

As K is compact, there exists a sufficiently large integer $M > 0$ such that $K \subseteq [-M, M]^n$. Let $\varepsilon = \frac{1}{3Mn}$. Then by the Simultaneous Diophantine Approximation Theorem, there exist integers $d_1, \ldots, d_n$ such that $\left| \frac{c_i}{c_n} - \frac{d_i}{d_n} \right| < \frac{\varepsilon}{d_n}$ for $i = 1, \ldots, n-1$. Let $z \in K$. Then $\sum_{i=1}^n c_i z_i = c_0$, and this implies that

$$\sum_{i=1}^n d_i z_i \in \left[-\varepsilon M n + \frac{d_n c_0}{c_n},\ \varepsilon M n + \frac{d_n c_0}{c_n} \right].$$

As $2\varepsilon M n < 1$, the integer width of K is at most 1. □

The upper bound given by Theorem 2.16 turns out to be very tight as shown in the following proposition.

Proposition 2.17 ([37]). *There exists a polytope in $\mathbb{R}^n$ such that its Chvátal closure is empty and its integer width is $n - 1$.*

Proof. Let $P_n := \{x \in \mathbb{R}^n : x \geq \frac{1}{n+1}\mathbf{1},\ \sum_{i=1}^n x_i \leq n - 1 + \frac{n}{n+1}\}$. Figure 2.2 depicts P_n when $n = 2$. Then P_n is the convex hull of $(n-1)e^i + \frac{1}{n+1}\mathbf{1}$ for $i \in [n]$ and $\frac{1}{n+1}\mathbf{1}$. Since $x_i \geq 1$ is valid for P'_n for each i, $\sum_{i=1}^n x_i \geq n$ is valid for P'_n. Together with $\sum_{i=1}^n x_i \leq n - 1 + \frac{n}{n+1}$, this shows the emptiness of P'_n.

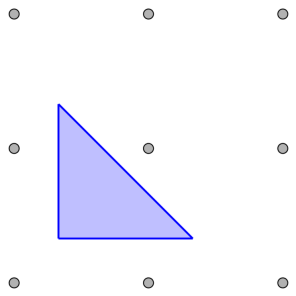


Figure 2.2: P_2 in $\mathbb{R}^2$

Now we show that the integer width of P_n is $n-1$. Let $d \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Since the integer width of P_n along d is the same as that along $-d$, we may assume $\sum_{i=1}^n d_i \geq 0$. Notice that $\max\{dx : x \in P_n\} = (n-1)\max\{d_1, \cdots, d_n\} + \frac{1}{n+1}\sum_{i=1}^n d_i$ and $\min\{dx : x \in P_n\} = (n-1)\min\{0, d_1, \cdots, d_n\} + \frac{1}{n+1}\sum_{i=1}^n d_i$. Then the integer width of P_n along d is either $(n-1)(\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\})$ or $(n-1)(\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\}) + 1$. Clearly, $\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\}$ is at least 1. Hence, the integer width of P_n along d is at least $n-1$. It is easy to show that the integer width of P_n along $\mathbf{1}$ is exactly $n-1$. □

2.3.1 Flatness result

Can we bound the integer width of a closed convex set whose Chvátal closure is empty, even when it is unbounded? The answer is no; let us elaborate with the following example.

Example 3 ([37])**.** Let $P := \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1\}$. P can be rewritten as $P = \{\alpha(1, \sqrt{2}) : \alpha \geq 1\}$. It is clear that P does not contain an integer point. For every $d = (d_1, d_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $d_1 + d_2\sqrt{2} \neq 0$ and thus either $\max\{dx : x \in P\}$ or $\min\{dx : x \in P\}$ is unbounded. Therefore, the integer width of P is unbounded.

In fact, we can prove that the Chvátal closure of P is empty. It is sufficient to show that for any $z \geq 1$, there is a Chvátal-Gomory inequality that cuts off the line segment between $(1, \sqrt{2})$ and $z(1, \sqrt{2})$. By the Dirichlet approximation theorem, we can find $(d_1, d_2) \in \mathbb{Z}^2$ such that

$$\left|\sqrt{2} - \frac{d_1}{d_2}\right| < \frac{1}{2zd_2}.$$

Then, we get $|d_1 - d_2\sqrt{2}| < \frac{1}{2z}$. Since $d_1 - d_2\sqrt{2} \neq 0$, we may assume without loss of generality that $-\frac{1}{2z} < d_1 - d_2\sqrt{2} < 0$. In this case, $d_1x_1 - d_2x_2 \leq d_1 - d_2\sqrt{2}$ is a valid inequality for P. We then obtain a Chvátal-Gomory inequality $d_1x_1 - d_2x_2 \leq -1$ from it, because $-1 < d_1 - d_2\sqrt{2} < 0$. Notice that $d_1z - d_2z\sqrt{2} = z(d_1 - d_2\sqrt{2})$ and $z(d_1 - d_2\sqrt{2}) > -\frac{1}{2}$, so both $(1, \sqrt{2})$ and $z(1, \sqrt{2})$ are cut off by the

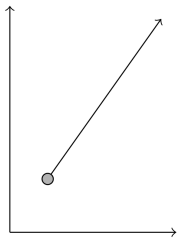


Figure 2.3: P in $\mathbb{R}^2$

Chvátal-Gomory inequality. In this case, we need infinitely many Chvátal-Gomory inequalities to certify that the Chvátal closure of P is empty.

As explained in this example, there is no global bound on the integer width of an unbounded closed convex set whose Chvátal closure is empty. What made the integer width unbounded in the previous example was an irrational ray $(1, \sqrt{2})$ that is not contained in a proper rational linear subspace. We say that an irrational vector r is *fully irrational* if there is no proper rational linear subspace containing r. In general, we can show that

Remark 2.18 ([37])**.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex set. If K contains a fully irrational ray $r \in \mathbb{R}^n$, then the integer width of K is unbounded.*

Proof. Let $d \in \mathbb{Z}^n \setminus \{0\}$. Notice that dr is nonzero. Otherwise, r is contained in a proper rational linear subspace $\{x \in \mathbb{R}^n : dx = 0\}$, a contradiction to the assumption. Then either $\sup\{dx : x \in K\}$ or $\inf\{dx : x \in K\}$ is unbounded, so we have that $w(K, d)$ is unbounded. Therefore, $w(K, d)$ is unbounded for each $d \in \mathbb{Z}^n \setminus \{0\}$, and the integer width of K is unbounded. □

Hence, a closed convex set with bounded integer width does not contain a fully irrational ray. Let K be a closed convex set that does not contain a fully irrational ray, and consider its recession cone C, that is, the collection of all the rays contained in K. Let $\text{lin}(C)$ denote the linear hull of C, that is, the smallest linear subspace containing C. Then $\text{lin}(C)$ is a rational linear subspace. In fact, we can generalize Theorem 2.16 as the following:

Theorem 2.19 ([37])**.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex set that can be expressed as $K = Q + C$ where Q is a compact convex set and C is a cone such that $\text{lin}(C)$ is rational. If the Chvátal closure of K is empty, then the integer width of K is at most n.*

It turns out that Theorem 2.19 cannot be generalized to a closed convex set K that can be expressed as $K = Q + C$ where Q is not necessarily bounded, as shown by the following example.

Example 4 ([37])**.** Let $K := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1,\ x_3 \geq x_1^2\right\}$. The recession cone C of K is simply $\{\alpha(0, 0, 1) : \alpha \geq 0\}$, so $\text{lin}(C)$ is rational and $K = K + C$. Notice that K is contained in $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{2}x_1 - x_2 = 0,\ x_1 \geq 1\right\}$, and we saw in in Example 3 that its Chvatal closure is empty. That means the Chvátal closure of K is empty as well. However, the integer width of K is unbounded.

Let $d = (d_1, d_2, d_3) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$. Notice that $(z, \sqrt{2}z, z^2) \in K$ for any positive integer z. As $d_1 z + d_2\sqrt{2}z \neq 0$ for any integer z, $d_1 z + d_2\sqrt{2}z + d_3 z^2 = d_3 z^2 + (d_1 + d_2\sqrt{2})z$ becomes unbounded as z goes to infinity. So, either $\sup\{dx : x \in K\}$ or $\inf\{dx : x \in K\}$ is unbounded. Therefore, the integer width of K is unbounded.

As a direct consequence of Theorem 2.19, we obtain Theorem 1.9.

Theorem 1.9 ([37]). *The integer width of any rational polyhedron in $\mathbb{R}^n$ whose Chvátal closure is empty is at most n.*

Theorem 1.9 will be useful in developing an algorithm for solving the integer feasibility problem over the rational polyhedra with Chvátal rank 1 in the later part of this section.

2.3.2 Proof of Theorem 2.19

To prove Theorem 2.19, we show Lemma 2.21 and Lemma 2.22 in this section. For Lemma 2.21, we need the following result due to Dadush, Dey, and Vielma [43].

Theorem 2.20 ([43], Theorem 1). *If $K \subseteq \mathbb{R}^n$ is a compact convex set, then the Chvátal closure of K is a rational polytope.*

Lemma 2.21 ([37]). *Let $K \subseteq \mathbb{R}^n$ be a closed convex set that can be expressed as $K = Q + C$ where Q is a compact convex set and C is a cone such that $\text{lin}(C)$ is rational. If the Chvátal closure of K is empty, then there exists a finite list of Chvátal-Gomory inequalities such that the intersection of their corresponding half-spaces is empty.*

Proof. By Theorem 2.20, we may assume that K is unbounded, so C has a nontrivial ray. If $\text{lin}(C)$ is a rational linear subspace, there exists a rational matrix A with full row rank such that $\text{lin}(C) = \{x \in \mathbb{R}^n : Ax = 0\}$. We remark that we may assume $A = (I, 0)$ where I is the identity matrix with the same number of rows as A, which means $\text{lin}(C) = \{x = (x^1, x^2) \in \mathbb{R}^{n_1+n_2} : Ix^1 + 0x^2 = x^1 = 0\}$ where $n_1 + n_2 = n$. When $A \neq (I, 0)$, we can find an unimodular matrix U such that $AU = (H, 0)$ is a Hermite normal form of A. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be an unimodular transformation defined as $u(x) = U^{-1}x$ for $x \in \mathbb{R}^n$. Notice that

$$u(K') = \bigcap_{dU \in \mathbb{Z}^n} \{y \in \mathbb{R}^n : \ dUy \leq \lfloor \sup\{dUy : y \in u(K)\} \rfloor\}.$$

Hence, $u(K') = (u(K))'$. Then it is sufficient to show that there is a finite list of Chvátal-Gomory inequalities of $u(K)$ whose corresponding half-spaces have empty intersection. Moreover, the recession cone of $u(K)$ is $u(C)$, and notice that $\text{lin}(u(C)) = \left\{y = (y^1, y^2) \in \mathbb{R}^{n_1+n_2} : Hy^1 = 0\right\}$ and it is equal to $\left\{y = (y^1, y^2) \in \mathbb{R}^{n_1+n_2} : y^1 = 0\right\}$. Thus, we may indeed assume that $A = (I, 0)$.

We will first show that if the Chvátal closure of K is empty, then it suffices to look at the Chvátal-Gomory inequalities obtained from the directions orthogonal to $\text{lin}(C)$. Since $\text{lin}(C)$ is a rational linear

subspace, the relative interior of C contains a ray $\bar{r}$ whose components are integers. Let us consider $K + \bar{r}$, the translation of K by $\bar{r}$. Notice that $K + \bar{r} \subseteq K$. Since the Chvátal closure of K is empty, there are some Chvátal-Gomory inequalities of K that remove all points in $K + \bar{r}$. Let's pick a direction $d \in \mathbb{Z}^n \setminus \{0\}$ that is not orthogonal to $\text{lin}(C)$. We may assume that $\sup\{dx : x \in K\}$ has some finite value f. Otherwise, we can ignore the Chvátal-Gomory inequality obtained from d. Then, $dr \leq 0$ for all $r \in C$. If $d\bar{r} = 0$, then $dr = 0$ for all $r \in C$, a contradiction to the assumption that d is not orthogonal to $\text{lin}(C)$. Hence, $d\bar{r} < 0$. In fact, we know that $d\bar{r} \leq -1$, because both d and r have integer components. Notice that $\sup\{dx : x \in K + \bar{r}\} = f + d\bar{r}$. Since $d\bar{r} \leq -1$, the Chvátal-Gomory inequality $dx \leq \lfloor \sup\{dx : x \in K\} \rfloor = \lfloor f \rfloor$ obtained from d does not cut off any point in $K + \bar{r}$. This implies that the points in $K + \bar{r}$ are cut off by only the Chvátal-Gomory inequalities obtained from directions orthogonal to $\text{lin}(C)$. So, we have

$$(K + \bar{r}) \cap \bigcap_{d \in \text{lin}(C)^\perp \cap \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ dx \leq \lfloor \sup\{dx : \ x \in K\} \rfloor\} = \emptyset,$$

where $\text{lin}(C)^\perp$ denotes the orthogonal complement of $\text{lin}(C)$. Let $\bar{x} \in K + \text{lin}(C)$. Then $\bar{x} + r \in K + \bar{r}$ for some $r \in \text{lin}(C)$, so there exists a direction $d \in \text{lin}(C)^\perp \cap \mathbb{Z}^n$ such that $d(\bar{x} + r) > \lfloor \sup\{dx : x \in K\} \rfloor$. As $r \in \text{lin}(C)$, we know that $dr = 0$. Then we get $d\bar{x} > \lfloor \sup\{dx : x \in K\} \rfloor$, so $\bar{x}$ is also cut off by the same Chvátal-Gomory inequality. Therefore, we have that

$$(K + \text{lin}(C)) \cap \bigcap_{d \in \text{lin}(C)^\perp \cap \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ dx \leq \lfloor \sup\{dx : \ x \in K\} \rfloor\} = \emptyset. \qquad (\star)$$

To complete the proof, we look at $\widetilde{K}$, that is the projection of K onto $\text{lin}(C)^\perp$. Since $K = Q + C$, $\widetilde{K}$ is the same as the projection of Q onto $\text{lin}(C)^\perp$. Then $\widetilde{K}$ is a compact convex set and $K + \text{lin}(C)$ is the same as $\widetilde{K} + \text{lin}(C)$. Recall that $\text{lin}(C) = \left\{x = (x^1, x^2) \in \mathbb{R}^{n_1+n_2} : x^1 = 0\right\}$, so $\text{lin}(C)^\perp = \left\{x = (x^1, x^2) \in \mathbb{R}^{n_1+n_2} : x^2 = 0\right\}$. Then $\text{lin}(C)^\perp \cap \mathbb{Z}^n = \left\{d = (d^1, d^2) \in \mathbb{Z}^{n_1+n_2} : d^2 = 0\right\}$, so $dx \leq \lfloor \sup\{dx : x \in K\} \rfloor$ for $d \in \text{lin}(C)^\perp \cap \mathbb{Z}^n$ is equivalent to $d^1x^1 \leq \lfloor \sup\{d^1x^1 : x^1 \in \widetilde{K}\rfloor$. Then, $(\star)$ is equivalent to

$$\widetilde{K} \cap \bigcap_{d^1 \in \mathbb{Z}^{n_1}} \left\{x^1 \in \mathbb{R}^{n_1} : \ d^1x^1 \leq \lfloor \sup\{d^1x^1 : \ x^1 \in \widetilde{K}\} \rfloor\right\} = \emptyset.$$

Since $\widetilde{K}$ is a compact convex set, its Chvátal closure is a rational polytope due to Theorem 2.20. Therefore, the Chvátal closure of $\widetilde{K}$ is described by a finite number of Chvátal-Gomory inequalities. In turn, there is a finite subset $D \subseteq \mathbb{Z}^{n_1}$ such that $\bigcap_{d^1 \in D} \left\{x^1 \in \mathbb{R}^{n_1} : d^1x^1 \leq \lfloor \sup\{d^1x^1 : x^1 \in \widetilde{K}\} \rfloor\right\} = \emptyset$. This implies

$$\bigcap_{d \in D \times \{\mathbf{0}\}} \{x \in \mathbb{R}^n : \ dx \leq \lfloor \sup\{dx : \ x \in K\} \rfloor\} = \emptyset,$$

so the Chvátal-Gomory inequalities obtained from directions in a finite list $D \times \{\mathbf{0}\}$ are sufficient to show that the Chvátal closure of K is empty, as required. □

To prove Theorem 2.19, we introduce the concept of a *simplicial cylinder*. Let $P \subseteq \mathbb{R}^n$ be a full-dimensional rational polyhedron. We denote by L and $L^\perp$ the lineality space of P and its orthogonal

complement, respectively. We say that P is a *simplicial cylinder* if $P \cap L^{\perp}$ is a simplex. Observe that a simplicial cylinder $P \subseteq \mathbb{R}^n$ whose lineality space L has dimension $n - \ell$ can be described by $\ell + 1$ linear inequalities.

Let P be a rational polyhedron given by its linear description $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where each row of A has relatively prime integers and b has integer components. We call P a *thin simplicial cylinder* if it is a simplicial cylinder and $Ax \leq b - \mathbf{1}$, where $\mathbf{1}$ denotes the vector of all ones, is an infeasible system. Note that a thin simplicial cylinder is a lattice-free set, which does not contain an integer point in its interior but might include one on its boundary (see Figure 2.4).

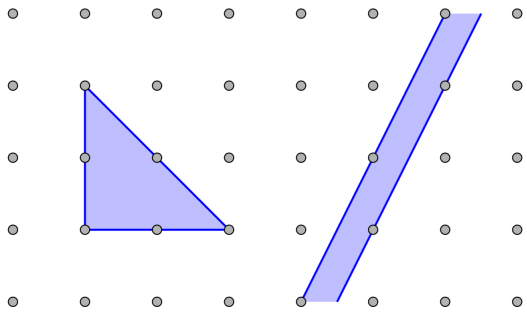


Figure 2.4: Thin simplicial cylinders in $\mathbb{R}^2$

Lemma 2.22 ([37]). *Let K be a closed convex set. If there exists a finite list of Chvátal-Gomory inequalities of K such that the intersection of their corresponding half-spaces is empty, K is contained in the interior of a thin simplicial cylinder.*

Proof. Helly's theorem implies that there are $\ell + 1$ Chvátal-Gomory inequalities of K for some $\ell \leq n$ such that the intersection of the corresponding linear half-spaces is empty. Then, there exists a system $Ax \leq b - \epsilon\mathbf{1}$ of $\ell + 1$ linear inequalities valid for K, where (A, b) has integer entries and $0 < \epsilon < 1$, such that $Ax \leq b - \mathbf{1}$ is an infeasible system. We may assume that each row of A has relatively prime integer entries. We may also assume that the system is minimal in a sense that $\{x \in \mathbb{R}^n : a^i x \leq b_i - 1 \text{ for } i \in I\}$ is not empty for any proper subset I of $[\ell + 1]$. Now, consider the polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$. We claim that its recession cone $C := \{x : Ax \leq 0\}$ has empty interior. Otherwise, the polyhedron P contains points in the form of $x + kr$ for some $x \in P$ and some ray vector $r \in \mathbb{R}^n$ in the interior of C, where $k \in \mathbb{R}_+$. For k large enough, the points of the form are also in the polyhedron $S := \{x \in \mathbb{R}^n : Ax \leq b - \mathbf{1}\}$, which is empty, a contradiction. Therefore, the linear space $C - C$ has dimension strictly less than n. By the Minkowski-Weyl theorem, we can write the polyhedron P as $P = Q + C$ where Q is a polytope. Consider the cylinder $R := Q + C - C$. Consider all the inequalities $a^i x \leq b_i$, $i = 1, \cdots, t$, in the description of P that are valid for R. Then for $i = t + 1, \ldots, \ell + 1$, there exists $r^i \in C$ such that $a^i r^i < 0$. Consider $r = \sum_{i=t+1}^{\ell+1} r^i$. Then $a^i r \leq a^i r^i < 0$ for $i = t + 1, \ldots, \ell + 1$. We claim that the linear system $a^i x \leq b_i - 1$, $i = 1, \cdots, t$, is infeasible. If $a^i x \leq b_i - 1$, $i = 1, \cdots, t$, were feasible, then, by the same argument as given above, S would be nonempty, a contradiction. Thus $a^i x \leq b_i - 1$, $i = 1, \cdots, t$, is infeasible. By the minimality of the system, this implies $t = \ell + 1$, and therefore Q is a simplex of dimension ℓ. That means $R = P$ and P is a simplicial cylinder containing K in its interior. □

Proof of Theorem 2.19. Lemma 2.21 implies that there exists a finite list of Chvátal-Gomory inequalities of K such that the intersection of their corresponding half-spaces is empty. Then, we know by Lemma 2.22 that there exists a thin simplicial cylinder $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ containing K in its interior. Let $\ell+1$ be the number of rows in A for some $\ell \leq n$. We denote by $a^1, \cdots, a^{\ell+1}$ the rows of A. Notice that $P \cap L^\perp$ is an ℓ-dimensional simplex, where L and $L^\perp$ denote the lineality space of P and its orthogonal complement, respectively.

We will show that the integer width of P along some a^i is at most $\ell + 1$. Then the integer width of K is at most ℓ, because the hyperplane defined by $a^i x = b_i$ does not go through K. Suppose that the integer width of P along each a^i is at least $\ell + 2$ for the sake of contradiction. Then, the width of P along each a^i is at least $\ell + 1$. Using an affine transformation, we can transform P to $\{x \in \mathbb{R}^n : x_1, \cdots, x_\ell \geq 0, \sum_{i=1}^{\ell} x_i \leq 1\}$. Under the same affine transformation, we know that $\{x \in \mathbb{R}^n : Ax \leq b - \mathbf{1}\}$ is transformed to $\{x \in \mathbb{R}^n : x_i \geq \epsilon_i \ \forall\ i \in [\ell],\ \sum_{i=1}^{\ell} x_i \leq 1 - \epsilon\}$ for some $0 < \epsilon_i \leq \frac{1}{\ell+1}$ for $i \in [\ell]$ and $0 < \epsilon \leq \frac{1}{\ell+1}$. Notice that $\left(\frac{1}{\ell+1}, \ldots, \frac{1}{\ell+1}\right) \in \mathbb{R}^n$ is contained in $\{x \in \mathbb{R}^n : x_i \geq \epsilon_i \ \forall\ i \in [\ell],\ \sum_{i=1}^{\ell} x_i \leq 1-\epsilon\}$. However, $\{x \in \mathbb{R}^n : Ax \leq b - \mathbf{1}\}$ is empty by the assumption that P is a thin simplicial cylinder, and it cannot be transformed to a nonempty set under any affine transformation. With this contradiction, we have proved that the integer width of K is at most $\ell \leq n$. □

2.3.3 A Lenstra-type algorithm

Recently Hildebrand and Köppe [78], Dadush, Peikert, and Vempala (see [42, 46, 47]) improved Lenstra-type algorithms for integer programming. Their algorithms are similar to Lenstra's algorithm in spirit in that a main step consists in finding a flat direction of a lattice-free convex body. In particular, Dadush, Peikert, and Vempala (see [42, 46, 47]) used a $2^{O(n)}\text{poly}(L)$ time algorithm to find a flattest direction for a convex body containing no integer point, and they proved that the time complexity of their Lenstra-type algorithm is bounded by $2^{O(n)}\,(f(n))^n\,\text{poly}(L)$, where $f(n)$ is the upper bound on the integer width of a compact convex set with no integer point. Together with the current tightest upper bound $f(n) = O(n^{4/3}\text{polylog}(n))$ [16, 105], the time complexity of the algorithm is bounded by $2^{O(n)}\left(n^{4/3}\text{polylog}(n)\right)^n\text{poly}(L)$. Theorem 1.9 implies that there exists a $2^{O(n)}n^n\text{poly}(L)$ time Lenstra-type algorithm for the integer feasibility problem over Chvátal rank 1 rational polyhedra. On the other hand, Proposition 2.17 indicates that we cannot improve this time complexity if we use a Lenstra-type procedure. Note that this does not improve the current best algorithm for integer programming. Dadush [42] provided a $2^{O(n)}n^n\text{poly}(L)$ time Kannan-type algorithm for integer programming over general convex compact sets in his doctoral dissertation, and we remark that it is the fastest algorithm for integer programming. Instead of finding one flat direction at a time, his algorithm finds many flat directions at each step, thereby reducing the number of recursive steps from $\left(n^{4/3}\text{polylog}(n)\right)^n$ to $(3n)^n$.

Based on Theorem 2.19 and Proposition 2.22, we can state the following proposition.

Proposition 2.23. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be rational polyhedron with Chvátal rank 1. Assume that if P contains no integer point, then P is contained in the interior of a thin simplicial cylinder defined by $\ell + 1$ inequalities for some $\ell \leq n$. Then, there exists a $2^{O(n)}\ell^n\text{poly}(L)$ time Lenstra-type algorithm that decides whether P contains an integer point, where L is the encoding size of P.*

Since any rational polyhedron with empty Chvátal closure in $\mathbb{R}^n$ is always contained in the interior of a

thin simplicial cylinder which is defined by at most $n+1$ inequalities, Proposition 2.23 directly implies the following:

Remark 2.24. *There is a Lenstra-type algorithm that can decide in $2^{O(n)}n^n poly(L)$ time, where L is the encoding size of P, whether a given rational polyhedron $P \subseteq \mathbb{R}^n$ with Chvátal rank 1 contains an integer point.*

Although our algorithm correctly decides whether a given rational polyhedron with Chvátal rank 1 contains an integer point, it does not find an integer point when one exists. In order to provide an algorithm that actually finds an integer point when exists, we believe that it is necessary to analyze some properties of integer feasible rational polyhedra with Chvátal rank 1, which is widely open.

Chapter 3

Polytopes with split rank 1

Split cuts are the most commonly used general-purpose cutting planes, and it is known that Gomory's mixed integer (GMI) cuts, the mixed integer rounding (MIR) cuts, and the Chvátal-Gomory cuts are all split cuts. Recall that $S(\pi, \pi_0) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : \ \pi x \leq \pi_0 \text{ or } \pi x \geq \pi_0 + 1\}$ the split or the split disjunction derived from $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. We call an inequality a split cut if it is valid for $\text{conv}(P \cap S(\pi, \pi_0))$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. Recall that

$$P^* := \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} \text{conv}\,(P \cap S(\pi, \pi_0))$$

is the split closure of a rational polyhedron P.

There are many computational results [64, 14] showing that split cuts are effective in practice, Caprara and Letchford [27] showed that optimizing over the split closure of a rational polyhedron is NP-hard. In addition, Mahajan and Ralphs [97] showed that it is NP-complete to decide whether there exists a split $S(\pi, \pi_0)$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $P \cap S(\pi, \pi_0)$ is empty, which implies that selecting an optimal split in terms of the gap closed is NP-hard. In § 3.1, we prove Theorem 1.4, and we will argue that our reduction for proving this NP-hardness result extends the result of Caprara and Letchford [27]. The reduction also generalizes the result of Mahajan and Ralphs [97] to an arbitrary number of split disjunctions. § 3.1.2 contains more precise statements. In § 3.2, we prove Theorem 1.10, stating that if a rational polyherdon has split rank 1 and contains no integer point, then its integer width is at most $2n$. In fact, we prove Theorem 3.8 that is more general than Theorem 1.10. The material in Section 3.1 will be published in *Discrete Optimization* [90].

3.1 Deciding whether the split closure of a rational polytope is empty is NP-hard

In this section, we give a proof of Theorem 1.4.

3.1.1 Reduction from Equality Knapsack

As we mentioned earlier, Mahajan and Ralphs [97] considered the problem of deciding whether there exists a single split disjunction that can certify that the split closure of a rational polytope is empty, and they proved that the problem is NP-complete. We prove Theorem 1.4 by providing a polynomial reduction from the Equality Knapsack Problem (see [68]):

Partition Problem. Given n positive integer weights $a_1, \cdots, a_n$, either find a set of binary integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = \frac{1}{2}\sum_{i=1}^n a_i$ or show that none exists.

Equality Knapsack Problem. Given n positive integer weights $a_1, \ldots, a_n$ and a capacity b, either find a set of nonnegative integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = b$ or show that none exists.

Our reduction is similar to Lemma 2.8.

Lemma 3.1 ([90]). *The problem of deciding whether the split closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ given by its linear description is empty is in complexity class NP.*

Proof. Theorem 13 in [51] by Dash, Günlük, and Lodi implies that the split closure of P can be described by finitely many split inequalities whose encoding sizes are polynomially bounded by the encoding size of P. When the split closure is empty, then the intersection of the half-spaces defined by finitely many split inequalities is empty. Then by Helly's theorem, for some $k \leq n+1$, there are k split inequalities of polynomially bounded encoding size that certify that the split closure of P is empty. Therefore, we have a polynomial size NP certificate for the problem. □

Now that we know the problem is in NP, what remains is to show that the problem is NP-hard, even when the input polytope is contained in the unit hypercube.

Lemma 3.2 ([90]). *Given an equality knapsack instance of n positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time generate the linear description of a rational polytope $P \subseteq [0,1]^{n+4}$ contained in the unit hypercube that satisfies the following:*

(a) $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ *is contained in P, but P contains no integer point.*

(b) There exists a solution to the equality knapsack instance if and only if there exists a split cut for P that separates $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.

(c) There exists a solution to the equality knapsack instance if and only if the split closure of P is empty and there is a single split disjunction to certify this.

Proof. We may assume that b is sufficiently large so that $b > n+2$, while the knapsack problem still remains NP-hard. We may also assume that $0 < a_1, \ldots, a_n < b$. Consider the following $n+6$ points

$v^1, \ldots, v^{n+6}$ in $[0,1]^{n+4}$.

$$
\begin{array}{rccccccccc}
v^1 := \Big(& \frac{a_1}{4b}, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} & \Big) \\
v^2 := \Big(& 0, & \frac{a_2}{4b}, & \cdots, & 0, & 0, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} & \Big) \\
 & & & & \vdots & & & & & & \\
v^{n-1} := \Big(& 0, & 0, & \cdots, & \frac{a_{n-1}}{4b}, & 0, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} & \Big) \\
v^n := \Big(& 0, & 0, & \cdots, & 0, & \frac{a_n}{4b}, & 0, & \frac{1}{8b}, & 0, & \frac{1}{8b} & \Big) \\
v^{n+1} := \Big(& \frac{a_1}{4b}, & \frac{a_2}{4b}, & \cdots, & \frac{a_{n-1}}{4b}, & \frac{a_n}{4b}, & 0, & 0, & \frac{1}{4} - \frac{1}{8b}, & 0 & \Big) \\
v^{n+2} := \Big(& 1 - \frac{a_1}{2b}, & 1 - \frac{a_2}{2b}, & \cdots, & 1 - \frac{a_{n-1}}{2b}, & 1 - \frac{a_n}{2b}, & 1, & \frac{1}{4} + \frac{1}{8b}, & 0, & \frac{1}{4} + \frac{1}{8b} & \Big) \\
v^{n+3} := \Big(& 0, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{2} + \frac{1}{8b}, & 0, & 0 & \Big) \\
v^{n+4} := \Big(& 0, & 0, & \cdots, & 0, & 0, & 0, & 0, & \frac{1}{2}, & 0 & \Big) \\
v^{n+5} := \Big(& 0, & 0, & \cdots, & 0, & 0, & 0, & 0, & 0, & \frac{1}{2} + \frac{1}{8b} & \Big) \\
v^{n+6} := \Big(& 0, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{4} - \frac{n+2}{8b}, & \frac{1}{4} + \frac{1}{8b}, & \frac{1}{4} - \frac{n+2}{8b} & \Big)
\end{array}
$$

Let P be a rational polytope defined as follows:

$$
P := \left\{ x = \sum_{i=1}^{n+6} v^i y_i : \begin{array}{rcccl} \frac{4b}{4b+1} & \leq & \sum_{i=1}^{n+6} y_i & \leq & n+6-\frac{4b}{4b+1} \\ y_{n+3}+y_{n+5}-1 & \leq & y_{n+4} & \leq & y_{n+3}+y_{n+5} \\ 0 & \leq & y_i & \leq & 1, \quad \forall i \in [n] \end{array} \right\}
$$

Claim 1. *The linear description of P that involves only x variables can be obtained in polynomial time.*

Proof of Claim. We can rewrite P as $P = \{x \in \mathbb{R}^{n+4} : x = Vy,\ Ay \leq b\}$ where V is the matrix whose columns are $v^1, \ldots, v^{n+6}$ and $Ay \leq b$ is the system of the other constraints in P. Notice that $v^1, \ldots, v^n$, v^{n+2}, v^{n+3}, v^{n+4}, and v^{n+5} are linearly independent, and let B denote the column submatrix of V that consists of these vectors. Let N denote the column submatrix of the remaining columns. Then $x = Vy$ is equivalent to $y_B = B^{-1}x - B^{-1}Ny_N$, where y_B and y_N consist of the components of y that correspond to B and N, respectively. Let A be decomposed into its two column submatrices C and D so that $Ay = Cy_B + Dy_N$. Then, P can be written as $P = \{x \in \mathbb{R}^{n+4} : CB^{-1}x + (D - CB^{-1}N)y_N \leq b\}$. y_N consists of only two variables y_{n+1} and y_{n+6}, so projecting away y_N from P can be done in polynomial time by the Fourier-Motzkin elimination method. Therefore, we can find a linear system describing P that involves x variables only in polynomial time. ◇

To complete the proof, we show that P satisfies properties (a), (b), and (c). Let u denote $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. To show that (a) is satisfied, we need the following two claims.

Claim 2. *$u \in P$ and P is centrally symmetric with respect to u.*

Proof of Claim. Notice that $\sum_{i=1}^{n+6} v^i = (1, \ldots, 1)$. Then $u = \sum_{i=1}^{n+6} \frac{1}{2} v^i \in P$, because $y_i = \frac{1}{2}$ for $i \in [n+6]$ satisfy the constraints. In addition, given $x = \sum_{i=1}^{n+6} v^i y_i$, observe that $2u - x = \sum_{i=1}^{n+6} v^i (1 - y_i)$ as $2u = \sum_{i=1}^{n+6} v^i$. Therefore, $x \in P$ if and only if $2u - x \in P$, so P is centrally symmetric with respect to u, as required. ◇

Claim 3. *$P \subseteq [0,1]^{n+4}$ and $P \cap \{0,1\}^{n+4} = \emptyset$.*

Proof of Claim. For $x = \sum_{i=1}^{n+6} v^i y_i \in P$, we know that $0 \leq \sum_{i=1}^{n+6} v^i y_i \leq \sum_{i=1}^{n+6} v^i = (1, \ldots, 1)$, because $v^1, \ldots, v^{n+6} \geq 0$. That means P is contained in $[0,1]^{n+4}$. Let $z = \sum_{i=1}^{n+6} v^i y_i \in P$. We would like to

show that $z \notin \{0,1\}^n$. Suppose otherwise. If $z_j = 1$ for some $1 \leq j \leq n$, then it must be the case that $y_j = y_{n+1} = y_{n+2} = 1$ because $z_j = \frac{a_j}{4b} y_j + \frac{a_j}{4b} y_{n+1} + \frac{2b-a_j}{2b} y_{n+2} \leq 1$ and the equality holds only if $y_j = y_{n+1} = y_{n+2} = 1$. In fact, $y_{n+1} = y_{n+2} = 1$ implies that $z_j > 0$ for each $j \in [n+4]$ and thus $z = (1, \ldots, 1)$ and $y_i = 1$ for each $i \in [n+6]$. However, this violates constraint $\sum_{i=1}^{n+6} y_i < n+6$, a contradiction. Thus, $z_j = 0$ for all $1 \leq j \leq n$. This implies $y_i = 0$ for $1 \leq i \leq n+2$, so $z = (0, \ldots, 0)$ is the only possibility. However, we observed that $(1, \ldots, 1) \notin P$, so $(0, \ldots, 0) \notin P$ by Claim 2. This contradicts the assumption that $z \in P$. Therefore, we get that $P \cap \{0,1\}^{n+4} = \emptyset$, as required. ◇

By Claim 2 and Claim 3, we know that P satisfies (a). To prove that P also satisfies (b) and (c), we show the following two claims:

Claim 4. *If there exists a solution to the equality knapsack instance, then the split closure of P is empty and there is a single split disjunction to certify this.*

Proof of Claim. Let $(d_1, \ldots, d_n)$ be a solution to the equality knapsack instance, so $\sum_{i=1}^{n} a_i d_i = b$ and $d_i \geq 0$ for $i \in [n]$. Let $\pi := (d_1, \ldots, d_n, -\sum_{i=1}^{n} d_i, 1, -1, 1) \in \mathbb{Z}^{n+4}$. Observe that

$$\pi v^i = \frac{a_i d_i}{4b} + \frac{1}{4b} \quad i = 1, \ldots, n, \quad \pi v^{n+1} = \frac{1}{8b}, \quad \pi v^{n+2} = \frac{1}{4b},$$

$$\pi v^{n+3} = \frac{1}{2} + \frac{1}{8b}, \quad \pi v^{n+4} = -\frac{1}{2}, \quad \pi v^{n+5} = \frac{1}{2} + \frac{1}{8b}, \quad \pi v^{n+6} = \frac{1}{4} - \frac{n}{4b} - \frac{5}{8b}.$$

Let $x \in P$. Then $x = \sum_{i=1}^{n+6} v^i y_i$ for some y satisfying the constraints for P. Notice that $\sum_{i=n+3}^{n+5} y_i \pi v^i = \frac{1}{8b}(y_{n+3} + y_{n+5}) + \frac{1}{2}(y_{n+3} - y_{n+4} + y_{n+5})$. Then we have

$$0 \leq \sum_{i=n+3}^{n+5} y_i \pi v^i \leq \frac{1}{4b} + \frac{1}{2} \tag{3.1}$$

where the first equality holds only if $y_{n+3} = y_{n+4} = y_{n+5} = 0$ and the second equality holds only if $y_{n+3} = y_{n+4} = y_{n+5} = 1$. Now, consider $y_{n+6}\pi v^{n+6} + \sum_{i=1}^{n+2} y_i \pi v^i$. Clearly, $\pi v^i \geq 0$ for $1 \leq i \leq n+2$ and $\pi v^{n+6} \geq 0$ as we assumed that $b \geq n+3$. This implies

$$0 \leq y_{n+6}\pi v^{n+6} + \sum_{i=1}^{n+2} y_i \pi v^i \leq \pi v^{n+6} + \sum_{i=1}^{n+2} \pi v^i = \frac{1}{2} - \frac{1}{4b} \tag{3.2}$$

where the first equality holds only when $y_1 = \cdots = y_{n+2} = y_{n+6} = 0$ and the second equality holds only when $y_1 = \cdots = y_{n+2} = y_{n+6} = 1$. From (3.1) and (3.2), we get that $0 \leq \pi x \leq 1$ where $\pi x = 0$ only if $y_i = 0$ for all $i \in [n+6]$ and $\pi x = 1$ only if $y_i = 1$ for all $i \in [n+6]$. As $0 < \sum_{i=1}^{n+6} y_i < n+6$, we know that πx can be neither 0 nor 1. That means $P \subseteq \{x : 0 < \pi x < 1\}$. Therefore, $P \cap S(\pi, 0) = \emptyset$ and thus the split closure of P is empty, as required. ◇

Claim 4 proves one direction of each of (b) and (c). The other direction of each can be shown by the following claim.

Claim 5. *If there exists a split cut separating $u = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, then there exists a solution to the equality knapsack instance.*

Proof of Claim. Since there is a split cut that separates u, there exist $\pi \in \mathbb{Z}^{n+4}$ and $\pi_0 \in \mathbb{Z}$ such that $u \notin \operatorname{conv}(P \cap S(\pi, \pi_0))$, Then $\pi_0 < \pi u < \pi_0 + 1$. As $S(-\pi, -\pi_0 - 1)$ is identical to $S(\pi, \pi_0)$, we may assume that $\pi_0 \geq 0$ without loss of generality. We will show that π and π_0 satisfy the following five properties.

(1) $\pi_{n+1} = -\sum_{i=1}^{n} \pi_i$.

(2) $\pi_{n+2} = \pi_{n+4} = 1$ and $\pi_{n+3} = -1$.

(3) $\pi_0 = 0$.

(4) $\sum_{i=1}^{n} a_i \pi_i = b$.

(5) $\pi_i \geq 0$ for $i = 1, \ldots, n$.

(1) – (5) imply that $(\pi_1, \ldots, \pi_n)$ is a solution to the equality knapsack instance. Since $\sum_{i=1}^{n+4} \pi_i$ is an integer and $\pi u = \frac{1}{2} \sum_{i=1}^{n+4} \pi_i$ is strictly between two consecutive integers π_0 and $\pi_0 + 1$, we get $\pi u = \pi_0 + \frac{1}{2}$. Let $x \in P$. Then $2u - x \in P$ by Claim 2. If $x, 2u - x \in S(\pi, \pi_0)$, then $u = \frac{1}{2}x + \frac{1}{2}(2u - x) \in \operatorname{conv}(P \cap S(\pi, \pi_0))$, a contradiction. Hence, for every $x \in P$, either $\pi_0 < \pi x < \pi_0 + 1$ or $\pi_0 < \pi(2u - x) < \pi_0 + 1$ holds.

(1): Consider $w^1 := \left(0, \ldots, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{4b}{4b+1} v^{n+3} + v^{n+4} + \frac{4b}{4b+1} v^{n+5} \in P$. Then $\pi w^1 = \pi u - \frac{1}{2} \sum_{i=1}^{n+1} \pi_i$ and $\pi\left(2u - w^1\right) = \pi u + \frac{1}{2} \sum_{i=1}^{n+1} \pi_i$. We know that $\pi u = \pi_0 + \frac{1}{2}$ and that either $\pi_0 < \pi w^1 < \pi_0 + 1$ or $\pi_0 < \pi\left(2u - w^1\right) < \pi_0 + 1$ holds, and we get $-1 < \sum_{i=1}^{n+1} \pi_i < 1$ in each case. Since $\sum_{i=1}^{n+1} \pi_i$ is an integer strictly between -1 and 1, it is equal to 0. Hence, (1) is satisfied.

(2) & (3): By (1), we obtain $\frac{1}{2} \sum_{i=n+2}^{n+4} \pi_i = \pi u$. Consider $w^2 := \left(0, \ldots, 0, 0, \frac{1}{2}, 0, 0\right) = \frac{4b}{4b+1} v^{n+3} \in P$. By symmetry, $2u - w^2 = \left(1, \ldots, 1, 1, \frac{1}{2}, 1, 1\right) \in P$. Notice that $\pi w^2 = \pi u - \frac{1}{2}(\pi_{n+3} + \pi_{n+4})$ and $\pi\left(2u - w^2\right) = \pi u + \frac{1}{2}(\pi_{n+3} + \pi_{n+4})$. As we argued before, we get $\pi_{n+3} + \pi_{n+4} = 0$. By considering $w^3 := \left(0, \ldots, 0, 0, 0, 0, \frac{1}{2}\right) = \frac{4b}{4b+1} v^{n+5} \in P$, we can similarly argue that $\pi_{n+2} + \pi_{n+3} = 0$. Next, consider $w^4 := \left(0, \ldots, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \frac{1}{2} w^1 \in P$. Then, $\pi w^4 = \pi u - \frac{1}{4} \sum_{i=n+2}^{n+4} \pi_i$ and $\pi\left(2u - w^4\right) = \pi u + \frac{1}{4} \sum_{i=n+2}^{n+4} \pi_i$. Since we know that $\pi u = \pi_0 + \frac{1}{2}$ and that either $\pi_0 < \pi w^4 < \pi_0 + 1$ or $\pi_0 < \pi\left(2u - w^4\right) < \pi_0 + 1$ holds, we obtain $-1 \leq \sum_{i=n+2}^{n+4} \pi_i \leq 1$. We observed that $\pi u = \frac{1}{2} \sum_{i=n+2}^{n+4} \pi_i = \pi_0 + \frac{1}{2}$ and assumed earlier that $\pi_0 \geq 0$, so we get $\sum_{i=n+2}^{n+4} \pi_i \geq 1$. Then $\sum_{i=n+2}^{n+4} \pi_i = 1$ and this means $\pi_{n+2} = \pi_{n+4} = 1$ and $\pi_{n+3} = -1$, because we already have $\pi_{n+2} + \pi_{n+3} = \pi_{n+3} + \pi_{n+4} = 0$. As a result, $\pi_0 = \pi u - \frac{1}{2} = 0$. Therefore, (2) and (3) are satisfied.

(4): By (3) and $\pi u = \pi_0 + \frac{1}{2}$, we have $\pi u = \frac{1}{2}$. We first consider $v^{n+1} \in P$. We have that $\pi v^{n+1} = -(\frac{1}{4} - \frac{1}{8b}) + \frac{1}{4b} \sum_{i=1}^{n} a_i \pi_i$. As $\pi_0 = 0$, either $0 < \pi v^{n+1} < 1$ or $0 < \pi\left(2u - v^{n+1}\right) < 1$ should hold. Since $\pi\left(2u - v^{n+1}\right) = 1 - \pi v^{n+1}$, we in fact have $0 < \pi v^{n+1} < 1$. In particular, $\pi v^{n+1} > 0$ implies that $\sum_{i=1}^{n} a_i \pi_i > b - \frac{1}{2}$ and thus we obtain $\sum_{i=1}^{n} a_i \pi_i \geq b$. Next, consider $v^{n+2} \in P$. Notice that $\pi v^{n+2} = (\frac{1}{2} + \frac{1}{4b}) - \frac{1}{2b} \sum_{i=1}^{n} a_i \pi_i$ and $\pi\left(2u - v^{n+2}\right) = 1 - \pi v^{n+2}$. Similarly, we get $\pi v^{n+2} > 0$, and this implies $\sum_{i=1}^{n} a_i \pi_i < b + \frac{1}{2}$. Since $\sum_{i=1}^{n} a_i \pi_i$ is an integer, it is indeed at most b. Consequently, $\sum_{i=1}^{n} a_i \pi_i = b$, as required.

(5): Let $i \in [n]$. To show that $\pi_i \geq 0$, we consider $v^i \in P$. Notice that $\pi v^i = \frac{1}{4b} a_i \pi_i + \frac{1}{4b}$ and $\pi\left(2u - v^i\right) = 1 - \pi v^i$. As we know that either $0 < \pi v^i < 1$ or $0 < \pi\left(2u - v^i\right) < 1$, we get $0 < \pi v^i < 1$. Then, $\pi v^i > 0$ implies that $a_i \pi_i > -1$. Since $a_i \pi_i$ is an integer, $a_i \pi_i \geq 0$ and thus $\pi_i \geq 0$, as required. $\diamond$

Claim 4 and Claim 5 finally prove that P satisfies (b) and (c), as required. □

As a direct consequence of Lemmas 3.1 and 3.2, we obtain Theorem 1.4.

Theorem 1.4 ([90]). *Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ containing no integer point, it is NP-complete to test whether the split closure of P is empty, even when $P \subseteq [0,1]^n$.*

3.1.2 Implications

In this section, we note some consequences of Theorem 1.4 and Lemma 3.2. The separation problem over the split closure of a rational polyhedron is defined as follows.

Separation Problem. Given a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and a rational vector $\bar{x} \in \mathbb{Q}^n$, either show that $\bar{x}$ is contained in the split closure of P or a split cut that is violated by $\bar{x}$.

Theorem 3.3 (Separation [90]). *The separation problem over the split closure of a rational polyhedron is NP-hard, even when P is contained in the unit hypercube.*

Proof. Lemma 3.2 implies that, given an equality knapsack instance of $n-4$ positive weights $a_1, \ldots, a_n$ and a positive capacity b, one can in polynomial time construct a rational polytope $P \subseteq [0,1]^n$ such that there exists a split cut separating $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ from P if and only if the equality knapsack instance has a solution. Therefore, the separation problem over the split closure of a rational polytope in the unit hypercube is NP-hard. □

We remark that Theorem 1.4 also trivially implies Theorem 3.3, as the separation problem over the split closure considers a rational polytope whose split closure is empty as a special case. Furthermore, due to Grötschel, Lovász, and Schrijver [74]'s theorem on the equivalence between optimization and separation, we also get the hardness result for the optimization problem over the split closure.

Corollary 3.4 (Optimization [90]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron and $c \in \mathbb{Q}^n$ be a rational vector. Optimizing linear function cx over the split closure of P is NP-hard, even when P is contained in the unit hypercube $[0,1]^n$.*

Mahajan and Ralphs [97] proved that selecting a split disjunction certifying that a rational polytope has empty split closure is NP-hard. Lemma 3.2, in particular, part (c) generalizes this result.

Theorem 3.5 ([90]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polytope and k be an any arbitrary integer. It is NP-hard to decide whether there exist k split disjunctions $S(\pi^i, \pi_0^i)$ where $(\pi^i, \pi_0^i) \in \mathbb{Z}^n \times \mathbb{Z}$ for $i = 1, \ldots, k$ such that $\bigcap_{i=1}^k \operatorname{conv}\left(P \cap S(\pi^i, \pi_0^i)\right) = \emptyset$.*

When P contains no integer point, deciding emptiness of the split closure of P is the same as checking whether the split closure of P coincides with its integer hull and is the same as checking whether the split rank of P is 1. As a result, we obtain another direct corollary of Theorem 1.4.

Theorem 3.6 ([90]). *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron. It is NP-hard to decide whether the split rank of P is exactly 1, even when P is contained in the unit hypercube $[0,1]^n$ and P contains no integer point.*

3.2 Flatness theorem for rational polytopes of split rank 1

Corollary 3.4 indicates that it is difficult to optimize over the split closure of a rational polyhedron. On the other hand, when we assume that the split closure of a rational polyhedron is identical to its integer hull, optimizing over the split closure seems to become easier. In fact, we can show that

Proposition 1.8 ([90]). *Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ be a rational polyhedron whose split rank is 1. Then*

(1) the problem of deciding whether $P \cap \mathbb{Z}^n = \emptyset$,

(2) given $c \in \mathbb{Q}^n$, the problem of deciding whether $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ is unbounded,

(3) given $c \in \mathbb{Q}^n$ and $x^* \in \mathbb{Z}^n$, the problem of deciding whether $cx^* = \max\{cx : x \in P \cap \mathbb{Z}^n\}$

belong to complexity class NP $\cap$ co-NP.

Proof. **(1)**: Lemma 3.1 implies that the problem is in NP. The problem is also in co-NP, because we can exhibit a point in $P \cap \mathbb{Z}^n$ whose encoding size is polynomially bounded if $P \cap \mathbb{Z}^n \neq \emptyset$. **(2)**: Notice that $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ is unbounded if, and only if, $P \cap \mathbb{Z}^n \neq \emptyset$ and $\max\{cx : x \in P\}$ is unbounded. Therefore, it follows from part (1) that the problem is in NP$\cap$co-NP. **(3)**: If $cx^* \neq \max\{cx : x \in P \cap \mathbb{Z}^n\}$, then either $P \cap \mathbb{Z}^n = \emptyset$, $\max\{cx : x \in P \cap \mathbb{Z}^n\}$ is unbounded, or there exists $z \in P \cap \mathbb{Z}^n$ such that $cz > cx^*$. If $cz > cx^*$ for some $z \in P \cap \mathbb{Z}^n$, we can pick one whose encoding size is polynomially bounded. So, it follows from parts (1)&(2) that the problem is in co-NP. If $cx^* = \max\{cx : x \in P \cap \mathbb{Z}^n\}$, then $x^* \in P$ and $cx \leq cx^*$ is valid for $\operatorname{conv}(P \cap \mathbb{Z}^n)$, and as the split rank of P is 1, $cx \leq cx^*$ can be written as a consequence of at most $n+1$ (rank-1) split inequalities. Therefore, the problem is in NP, as required. □

One might wonder whether there is a polynomial time algorithm to solve integer programming over a rational polytope that has split rank 1. We studied the same question for the Chvátal rank in Chapter 2. We saw in § 1.2 that the matching problem [57] is an example where there exists a polynomial time algorithm. However, as Theorem 3.6 suggests, it seems hard to use the split rank 1 condition when trying to find an efficient algorithm.

We have seen that the notion of integer width is important in Lenstra's algorithm for integer linear programming. Recall that it is conjectured that the integer width of a lattice-free compact convex set is $O(n)$. In this section, we prove Theorem 3.8, and we obtain Theorem 1.10 as a corollary.

As the split closure of a rational polyhedron, one can define the split closure of a closed convex set (see [45]). Given a closed convex set $K \subseteq \mathbb{R}^n$, the split closure of K is defined as follows:

$$K^* := \bigcap_{(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Z}} \operatorname{conv}\left(K \cap S(\pi,\pi_0)\right).$$

Take an integer $n \geq 1$. Given a positive definite matrix C and a vector $a \in \mathbb{R}^n$, let $E(C,a)$ denote the ellipsoid $\{x \in \mathbb{R}^n : \|C(x-a)\|_2 \leq 1\}$.

Proposition 3.7. *Let $K \subset \mathbb{R}^n$ be a full-dimensional compact convex set whose split closure is empty. If $E(C,a) \subseteq K \subseteq E(\frac{1}{\ell}C,a)$, then the integer width of K is at most $\lceil 2\ell \rceil$.*

Proof. Since the split closure of K is empty, $a \in K$ should be cut off by a split cut of K. In other words, there exists $(\pi,\pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $a \notin \operatorname{conv}(K \cap S(\pi,\pi_0))$, which implies that $\pi_0 < \pi a < \pi_0 + 1$. Let $x_{\min} := \operatorname{argmin}\{\pi x : x \in E(C,a)\}$ and $x_{\max} := \operatorname{argmax}\{\pi x : x \in E(C,a)\}$. By the geometry of $E(C,a)$, we have that $a = \frac{1}{2}x_{\min} + \frac{1}{2}x_{\max}$. Notice that either $\pi x_{\min} > \pi_0$ or $\pi x_{\max} < \pi_0 + 1$ is satisfied, since if not, $\pi x_{\min} \leq \pi_0$ and $\pi x_{\max} \geq \pi_0 + 1$ implying that $a \in S(\pi,\pi_0)$, a contradiction. Thus we may assume that $\pi x_{\min} > \pi_0$ without loss of generality. Moreover, the geometry of $E(C,a)$ and $E(\frac{1}{\ell}C,a)$ implies that the minimum and maximum of πx over $E(\frac{1}{\ell}C,a)$ are obtained at $a + \ell(x_{\min} - a)$ and $a - \ell(x_{\min} - a)$, respectively. As $K \subset E(\frac{1}{\ell}C,a)$, it follows that

$$\pi\left(a + \ell(x_{\min} - a)\right) \leq \min\{\pi x : x \in K\} \leq \max\{\pi x : x \in K\} \leq \pi\left(a - \ell(x_{\min} - a)\right),$$

implying in turn that the integer width of K is at most $\lceil 2\ell\pi(a - x_{\min})\rceil$. As we observed that $\pi x_{\min} > \pi_0$ and $\pi a < \pi_0 + 1$, we have that $2\ell\pi(a - x_{\min}) < 2\ell$. Therefore, the integer width of K is at most $\lceil 2\ell \rceil$. □

Using Proposition 3.7, we can prove the following theorem:

Theorem 3.8. *Let $K \subseteq \mathbb{R}^n$ be a compact convex set whose split closure is empty. Then the integer width of K is at most $2n$.*

Proof. First, consider the case when K is full-dimensional. It was proved by Löner (reported by Danzer, Grünbaum, and Klee [49]) and John [89] that

> *For every full-dimensional compact convex set K, there exists an ellipsoid $E(C,a)$ such that $E(C,a) \subseteq K \subseteq E\left(\frac{1}{n}C,a\right)$.*

So, this theorem and Proposition 3.7 imply that the integer width of K is at most $2n$. Thus we may assume that K is not full-dimensional. Then $K \subseteq \{x \in \mathbb{R}^n : cx = c_0\}$ for some $c \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $d \in \mathbb{R}$. If c is rational, then the integer width of K is either 0 or 1, depending on c_0. Thus we may assume that c is irrational. Since $c = (c_1,\ldots,c_n)$ is nonzero, we may further assume that $c_n \neq 0$ without loss of generality. Then we can approximate c with a rational vector, based on the Simultaneous Diophantine Approximation Theorem due to Dirichlet [54]:

Given any real numbers $r_1, \ldots, r_{n-1}$ and $0 < \varepsilon < 1$, there exist integers $d_1, \ldots, d_n$ such that $\left| r_i - \frac{d_i}{d_n} \right| < \frac{\varepsilon}{d_n}$ for $i = 1, \ldots, n-1$ and $1 \leq d_n \leq \left(\frac{1}{\varepsilon}\right)^{n-1}$.

As K is compact, there exists a sufficiently large integer $M > 0$ such that $K \subseteq [-M, M]^n$. Let $\varepsilon = \frac{1}{3Mn}$. Then by the Simultaneous Diophantine Approximation Theorem, there exist integers $d_1, \ldots, d_n$ such that $\left| \frac{c_i}{c_n} - \frac{d_i}{d_n} \right| < \frac{\varepsilon}{d_n}$ for $i = 1, \ldots, n-1$. Let $z \in K$. Then $\sum_{i=1}^n c_i z_i = c_0$, and this implies that

$$\sum_{i=1}^n d_i z_i \in \left[-\varepsilon M n + \frac{d_n c_0}{c_n},\ \varepsilon M n + \frac{d_n c_0}{c_n} \right].$$

As $2\varepsilon M n < 1$, the integer width of K is at most 1. □

Theorem 1.10 is a direct corollary of Theorem 3.8

Theorem 1.10. *The integer width of any rational polytope in $\mathbb{R}^n$ whose split closure is empty is at most $2n$.*

3.3 Further notes

An interesting question is whether we can prove a theorem similar to Theorem 1.4 for *t-branch split cuts* introduced by Dash and Günlük [50]. It is also an open question whether the separation of the t-branch split cuts of a rational polyhedron is NP-hard. Unfortunately, the same argument as the reduction shown in Lemma 3.2 might not work, because it is possible that there exist two split disjunctions such that the union of the corresponding split sets contain P, even when there is no solution to the equality knapsack instance.

Chapter 4

Polytopes in the 0,1 hypercube that have small Chvátal rank

Let $S \subseteq \{0,1\}^n$, and let $\bar{S} := \{0,1\}^n \setminus S$. Recall that Q_S is defined as

$$Q_S := \left\{ x \in [0,1]^n : \sum_{j=1}^{n} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq \frac{1}{2} \text{ for } \bar{x} \in \bar{S} \right\}.$$

By Remark 1.12, we know that the Chvátal rank of any polytope $P \subseteq [0,1]^n$ such that $P \cap \{0,1\}^n = S$ is bounded above by that of Q_S. Indeed, the proof of Remark 1.12 shows that $P^{(k)} \subseteq Q_S^{(k)}$ for $k \geq 1$. In fact, we have a good handle on $Q_S^{(k)}$, thanks to the following lemma. The middle point of a k-dimensional 0,1 hypercube $[0,1]^k$ is defined as the vector in $\mathbb{R}^k$ all of whose entries are equal to $\frac{1}{2}$.

Lemma 4.1 (Chvátal, Cook, Hartmann [30])**.** *Let $S \subseteq \{0,1\}^n$. The middle points of all $(k+1)$-dimensional faces of $[0,1]^n$ belong to $Q_S^{(k)}$ for $0 \leq k \leq n-1$.*

Chvátal, Cook and Hartmann [30] proved this result when $S = \emptyset$. The result clearly follows for general $S \subseteq \{0,1\}^n$ since $Q_\emptyset \subseteq Q_S$ implies $Q_\emptyset^{(k)} \subseteq Q_S^{(k)}$.

Recall that G_n denotes the skeleton graph of $[0,1]^n$ and that $G(\bar{S})$ denotes the subgraph of G_n induced by $\bar{S}$. The goal of this chapter is to provide conditions on $G(\bar{S})$ under which the Chvátal rank of any polytope $P \subseteq [0,1]^n$ with $P \cap \{0,1\}^n = S$ is small.

In §4.1, we provide some tools that are frequently used to the results of this chapter. We characterize the descriptions of $Q_S^{(1)}$, $Q_S^{(2)}$, $Q_S^{(3)}$, $Q_S^{(4)}$ in §4.2. In §4.3, we give polyhedral decomposition theorems for $\text{conv}(S)$ when $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2. We will see that these decomposition theorems are useful to prove Theorem 1.11. In §4.4, we give a proof of Theorem 1.11. Finally, in §4.5, we give a proof of Theorem 1.14. The material in this chapter is published in *Mathematical Programming B* [36].

4.1 Basic tools

In this section, we present some basic tools and some notation that will be used later in this chapter.

Lemma 4.2 ([36]). *Consider a half-space $D := \{x \in \mathbb{R}^n : dx \geq d_0\}$. Let $T := D \cap \{0,1\}^n$ and $\bar{T} := \{0,1\}^n \setminus T$. For every face F of $[0,1]^n$, the graph $G(F \cap \bar{T})$ is connected. In particular $G(\bar{T})$ is a connected graph.*

Proof. Suppose that $G(F \cap \bar{T})$ is disconnected. Let $\bar{x}$ and $\bar{y}$ be vertices in distinct connected components of $G(F \cap \bar{T})$ with the property that the number of distinct coordinate values in the vectors $\bar{x}$ and $\bar{y}$ is as small as possible. Let j be a coordinate in which $\bar{x}$ and $\bar{y}$ differ and assume that $\bar{x}_j = 0$ and $\bar{y}_j = 1$. If $d_j < 0$, then $\bar{x} + e_j \in \bar{T}$ and is contained in the same component as $\bar{x}$. Besides, it has one more component in common with $\bar{y}$ than $\bar{x}$. Similarly, if $d_j \geq 0$, then $\bar{y} - e_j \in \bar{T}$ and has one more component in common with $\bar{x}$ than $\bar{y}$. In either case, we get a contradiction. □

We defined the skeleton graph of the 0,1 hypercube, but the skeleton graph of any general polytope can be defined similarly. Formally, the skeleton graph of a polytope is a graph whose vertices correspond to the extreme points of the polytope and whose edges correspond to the 1-dimensional faces containing two extreme points of the polytope.

Theorem 4.3 (Angulo, Ahmad, Dey, Kaibel [10]). *Let P be a polytope and let $G = (V, E)$ be its skeleton graph. Let $S \subset V$, $\bar{S} = V \setminus S$, and $\bar{S}_1, \ldots, \bar{S}_t$ be a partition of $\bar{S}$ such that there are no edges of G connecting $\bar{S}_i, \bar{S}_j$ for all $1 \leq i < j \leq t$. Then $\text{conv}(S) = \bigcap_{i=1}^{t} \text{conv}(V \setminus \bar{S}_i)$.*

Theorem 4.3 shows that we can consider each connected component of $G(\bar{S})$ separately when studying $\text{conv}(S)$. In Sections 4.3.1 and 4.3.2, we give similar theorems in the case where $P \subset [0,1]^n$ and $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2.

A matrix A is totally unimodular if every square submatrix has determinant -1, 0, or 1. It is known that both duplicating a row and multiplying a row by -1 preserve totally unimodularity. If A is totally unimodular, it is easy to observe that $P := \{x \in \mathbb{R}^n : Ax \geq b\}$ for any vector b with integer entries is always integral. In fact, replacing an inequality $a^i x \geq b_i$ of the system $Ax \geq b$ by either $a^i x \leq b_i$ or $a^i x = b_i$ preserves the integrality of P. We can easily observe the following, using a characterization of totally unimodular matrices due to Ghouila-Houri [70].

Remark 4.4. *Let A be a 0,1 matrix.*

- *If A has at most 2 rows, then A is totally unimodular.*
- *If $A = \begin{pmatrix} 1 & 0/1 & 0/1 & 0 \cdots 0 & 1 \cdots 1 \\ 0 & 1 & 0 & 0 \cdots 0 & 1 \cdots 1 \\ 0 & 0 & 1 & 0 \cdots 0 & 1 \cdots 1 \end{pmatrix}$, then A is totally unimodular.*
- *If $A = \begin{pmatrix} 1 & 1 & 0 & 0 \cdots 0 & 1 \cdots 1 \\ 0 & 1 & 0 & 1 \cdots 1 & 1 \cdots 1 \\ 0 & 0 & 1 & 1 \cdots 1 & 1 \cdots 1 \end{pmatrix}$, then A is totally unimodular.*

- *If A is totally unimodular, then so is* $\begin{pmatrix} A \\ I \\ -I \end{pmatrix}$.

In particular, if a system of linear inequalities consists of $0 \leq x \leq 1$ plus two additional constraints which have only 0,1 coefficients, then its constraint matrix is totally unimodular by Remark 4.4 and thus the linear system defines an integral polyhedron.

Throughout the paper, we will use the following notation. Let $N := \{1, \ldots, n\}$. For a 0,1 vector $\bar{x}$, we denote by $\bar{x}^i$ the 0,1 vector that differs from $\bar{x}$ only in coordinate $i \in N$, and more generally, for $J \subseteq N$, we denote by $\bar{x}^J$ the 0,1 vector that differs from $\bar{x}$ exactly in the coordinates J. Besides, let e^i denote the ith unit vector for $i \in N$.

4.2 The Chvátal rank of Q_S

4.2.1 Chvátal rank 1

For each $\bar{x} \in \bar{S}$, we call

$$\sum_{j=1}^{n} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \tag{4.1}$$

the *vertex inequality* corresponding to $\bar{x}$. For example, when $\bar{x} = 0$, the corresponding vertex inequality is $x_1 + x_2 + \cdots + x_n \geq 1$. Note that each vertex inequality cuts off exactly the vertex $\bar{x}$ and it goes through all the neighbors of $\bar{x}$ on $[0,1]^n$.

Theorem 4.5 ([36])**.** *$Q_S^{(1)}$ is the intersection of $[0,1]^n$ with the half-spaces defined by the vertex inequalities (4.1) for $\bar{x} \in \bar{S}$.*

Proof. Let e be an 1-dimensional face of G_n. Because the middle point of e belongs to Q_S by Lemma 4.1, any valid inequality $dx \geq d_0$ for Q_S cuts off at most one of the two vertices of e. Let $\bar{T}$ denote the set of $0,1$ vectors that satisfy $dx < d_0$. Since $G(\bar{T})$ is a connected graph by Lemma 4.2, it follows that every valid inequality $dx \geq d_0$ for Q_S cuts off at most one vertex $\bar{x}$ of $[0,1]^n$. The Chvátal-Gomory inequality obtained from $dx \geq d_0$ cannot cut off a vertex of $[0,1]^n$ other than $\bar{x}$. In particular, it cannot cut off the neighbors of $\bar{x}$ on $[0,1]^n$. The inequalities that cut off $\bar{x}$ but none of its neighbors on $[0,1]^n$ are implied by $\sum_{j=1}^{n} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1$ and $0 \leq x \leq 1$. Furthermore, this inequality is a rank 1 Chvátal-Gomory cut for Q_S since it is obtained from rounding $\sum_{j=1}^{n} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq \frac{1}{2}$. This shows that $Q_S^{(1)} = \{x \in [0,1]^n : \sum_{j=1}^{n} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \text{ for } \bar{x} \in \bar{S}\}$. □

Theorem 4.6 ([36])**.** *The polytope Q_S has Chvátal rank 1 if, and only if, $\bar{S}$ is a nonempty stable set in G_n.*

Proof. (⇐): Assume all connected components of $G(\bar{S})$ have cardinality 1. By Theorem 4.3, $\text{conv}(S) = \bigcap_{\bar{x} \in \bar{S}} \left\{ x \in [0,1]^n : \sum_{j=1}^{n} \bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j \geq 1 \right\}$, which is equal to $Q_S^{(1)}$ by Theorem 4.5. (⇒): Assume some connected component of $G(\bar{S})$ has at least 2 vertices, i.e. $G(\bar{S})$ contains at least 1 edge.

Without loss of generality, we may assume that $\{0, e^1\} \subseteq \bar{S}$ where e^1 denotes the first unit vector. Then the point $\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right)$ belongs to $Q_S^{(1)}$ by Lemma 4.1 but not to $\text{conv}(S)$ since $\sum_{j=2}^{n} x_j \geq 1$ is valid for $\text{conv}(S)$. This shows $Q_S^{(1)} \neq \text{conv}(S)$. □

In particular, Theorem 4.6 implies that if S contains all the 0,1 vertices of $[0,1]^n$ with an even (odd resp.) number of 1s, then $P \subseteq [0,1]^n$ with $P \cap \{0,1\}^n = S$ has Chvátal rank at most 1.

4.2.2 Chvátal rank 2

First we provide an explicit characterization of $Q_S^{(2)}$. Let $\bar{x}, \bar{y} \in \bar{S}$ be two adjacent vertices of $G(\bar{S})$. Using the notation introduced in Section 4.1, we write $\bar{y} = \bar{x}^i$, where i indexes the coordinate where $\bar{x}$ and $\bar{y}$ differ. The inequality

$$\sum_{j \in N \setminus \{i\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1 \tag{4.2}$$

is called the *edge inequality* corresponding to edge $\bar{x}\bar{y}$ in $G(\bar{S})$. For example, when $\bar{x} = 0$ and $\bar{y} = e^1$, the corresponding edge inequality is $x_2 + x_3 + \cdots + x_n \geq 1$. The inequality (4.2) is the strongest inequality that cuts off $\bar{x}$ and $\bar{y}$ but no other vertex of $[0,1]^n$. Indeed, its boundary contains all $2(n-1)$ neighbors of $\bar{x}$ or $\bar{y}$ on $[0,1]^n$ (other than $\bar{x}$ and $\bar{y}$ themselves). The next theorem states that vertex and edge inequalities are sufficient to describe the second Chvátal closure of Q_S.

Theorem 4.7 ([36]). *$Q_S^{(2)}$ is the intersection of $Q_S^{(1)}$ with the half-spaces defined by the edge inequalities (4.2) for $\bar{x}, \bar{y} \in \bar{S}$ such that $\bar{x}\bar{y}$ is an edge of G_n.*

Proof. The 2-dimensional faces of $[0,1]^n$ correspond to the 4-cycles of G_n, namely, squares. Because the center of each 2-dimensional face belongs to $Q_S^{(1)}$ by Lemma 4.1, any valid inequality for $Q_S^{(1)}$ cuts off at most two vertices of it from $[0,1]^n$, and these two vertices are adjacent. Indeed, by Lemma 4.2, the graph induced by the vertices that are cut off is connected and this graph cannot contain a subpath of length 2 since any such path belongs to a square of G_n. This proves the claim. The tightest such valid inequalities are the edge inequalities.

Next we show that they are valid for $Q_S^{(2)}$. The edge inequalities can be obtained from vertex inequalities valid for $Q_S^{(1)}$ as follows. Let $\bar{x}\bar{y}$ be an edge in $G(\bar{S})$. Say $\bar{x}_i = 0$ and $\bar{y}_i = 1$. Then $x_i + \sum_{j \in N \setminus \{i\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1$ and $-x_i + \sum_{j \in N \setminus \{i\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 0$ are valid for $Q_S^{(1)}$. Adding them and multiplying by $\frac{1}{2}$, it follows that the inequality $\sum_{j \in N \setminus \{i\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq \frac{1}{2}$ is valid for $Q_S^{(1)}$. After rounding it, we obtain $\sum_{j \in N \setminus \{i\}} (\bar{x}_j(1 - x_j) + (1 - \bar{x}_j)x_j) \geq 1$, valid for $Q_S^{(2)}$. □

Note that the edge inequality (4.2) dominates the vertex inequalities for $\bar{x} \in \bar{S}$ and for $\bar{y} \in \bar{S}$. Thus vertex inequalities are only needed for the isolated vertices of $G(\bar{S})$. A characterization for Q_S to have Chvátal rank 2 will be proved in Theorem 4.10.

4.2.3 Chvátal rank 3

Squares of $G(\bar{S})$ correspond to 2-dimensional faces of $[0,1]^n$. If $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell} \in \bar{S}$, then we say that $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$ is a square (see Figure 4.1). Note that

$$\sum_{j \in N \setminus \{i,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{4.3}$$

is the strongest inequality cutting off exactly the four points of the square $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$. Indeed, the $4(n-2)$ neighbors of $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}$ in $[0,1]^n$ (other than $\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}$ themselves) all satisfy (4.3) at equality. We call (4.3) a *square inequality*. As an example, if $(0, e^1, e^2, e^1+e^2)$ is a square contained in $G(\bar{S})$, the corresponding square inequality is $x_3 + x_4 + \cdots + x_n \geq 1$.

If $\bar{x}$ and $t \geq 3$ of its neighbors $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$ all belong to $\bar{S}$, then we say that $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ is a *star* (see Figure 4.1). The following *star inequality* is valid for $\text{conv}(S)$.

$$\sum_{r=1}^{t} (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2 \sum_{j \neq i_1, \ldots, i_t} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2. \tag{4.4}$$

Indeed, it cuts off exactly the vertices of the star, and goes through the other $n-t$ neighbors of $\bar{x}$ in $[0,1]^n$ and the $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$. For example, if $(0, e^1, \ldots, e^t)$ is a star, then (4.4) is $x_1 + \cdots + x_t + 2(x_{t+1} + \cdots + x_n) \geq 2$.

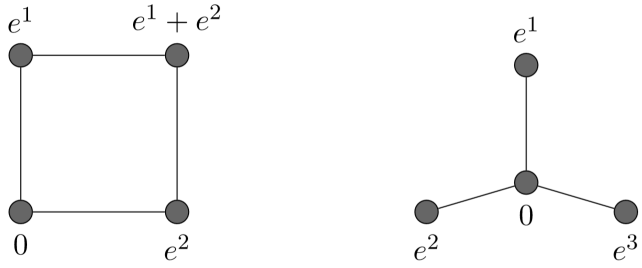


Figure 4.1: Square and star with $\bar{x} = 0$

The description of $Q_S^{(3)}$ is given in Theorem 4.9. To prove Theorem 4.9, we need the following lemma that consider the case when $\bar{S}$ is the vertex set of a star.

Lemma 4.8 ([36]). *Assume $\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t} \in \bar{S}$ for $t \geq 1$. If $t \geq 3$, i.e., $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ is a star, then $\text{conv}(S)$ is completely defined by the corresponding star inequality together with the edge inequalities and the bounds $0 \leq x \leq 1$. If $t = 1$ or 2, then $\text{conv}(S)$ is defined by edge inequalities and the bounds $0 \leq x \leq 1$.*

Proof. We may assume that $\bar{x} = 0$, $\bar{S} = \{0, e^1, \ldots, e^t\}$ and $I := \{1, \ldots, t\}$. If $t = n$, then S is the set of 0,1 vectors satisfying the system $\sum_{j=1}^n x_j \geq 2$ with $0 \leq x \leq 1$. This constraint matrix is totally unimodular by Remark 4.4. Therefore it defines an integral polytope, which must be $\text{conv}(S)$. Notice that the constraint matrix of $\{x \in [0,1]^n : \sum_{j \in N \setminus \{r\}} x_j \geq 1 \text{ for } r = 1\}$ and that of $\{x \in [0,1]^n : \sum_{j \in N \setminus \{r\}} x_j \geq 1 \text{ for } r = 1, 2\}$

are also totally unimodular by Remark 4.4, implying in turn that these two polytopes are integral. Hence, if $t = 1$ or 2, then $\text{conv}(S)$ is defined by edge inequalities and the bounds $0 \leq x \leq 1$, as required.

If $3 \leq t < n$, it is sufficient to show that $R := \{x \in [0,1]^n : \sum_{i \in I} x_i + 2\sum_{j \in N \setminus I} x_j \geq 2,\ \sum_{j \in N \setminus \{r\}} x_j \geq 1 \text{ for } 1 \leq r \leq t\}$ is an integral polytope. Let v be an extreme point of R. We will show that v is an integral vector. Since we assumed $n \geq 3$, R has dimension n and there exist n linearly independent inequalities active at v.

Claim 1. *If the star inequality is active at v, then v is integral.*

Proof of Claim. If no edge inequality is active at v, then $n-1$ inequalities among $0 \leq x \leq 1$ are active at v. Since $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j = 2$, it follows that all coordinates of v are integral. Thus we may assume that an edge inequality $\sum_{j \in N \setminus \{1\}} x_j \geq 1$ is active at v. Consider the face F of R defined by setting this edge inequality and the star inequality as equalities. Clearly v is a vertex of F. Observe that the two equations defining F can be written equivalently as $\sum_{j \in N \setminus \{1\}} x_j = 1$ and $x_1 + \sum_{j \in N \setminus I} x_j = 1$. Furthermore, any other edge inequality $\sum_{j \in N \setminus \{r\}} x_j \geq 1$ is implied by $x \geq 0$ since it can be rewritten as $\sum_{j \in I \setminus \{1,r\}} x_j \geq 0$ using $x_1 + \sum_{j \in N \setminus I} x_j = 1$. This means that F is entirely defined by $0 \leq x \leq 1$ and the two equations $x_1 + \sum_{j \in N \setminus I} x_j = 1$ and $\sum_{j \in N \setminus \{1\}} x_j = 1$. This constraint matrix is totally unimodular by Remark 4.4, showing that v is an integral vertex, as required. ◇

By Claim 1, we may assume that the star inequality is not active at v.

Claim 2. *If the star inequality is not active at v, then at most one edge inequality is tight at v.*

Proof of Claim. As the star inequality is not active, we have $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j > 2$. Suppose for contradiction that $k \geq 2$ edge inequalities are tight at v, say $\sum_{j \in N \setminus \{r\}} x_j \geq 1$ for $1 \leq r \leq k$. Then $v_1 = \cdots = v_k$. If v_1 is fractional, v has at least k fractional coordinates. We assumed that only k inequalities other than $0 \leq x \leq 1$ are active at v, so the other coordinates are integral. If $v_j = 1$ for some $j \geq k+1$, then $\sum_{j \in N \setminus \{r\}} v_j > 1$ for each $1 \leq r \leq k$, which contradicts the assumption that $\sum_{j \in N \setminus \{r\}} v_j = 1$. Hence, $v_j = 0$ for $j \notin \{1, \ldots, k\}$ and $v_1 = \cdots = v_k = \frac{1}{k-1}$. Then $\sum_{r=1}^{t} v_r + 2\sum_{j \in N \setminus I} v_j = \frac{k}{k-1} \leq 2$. However, this contradicts our observation that $\sum_{i \in I} v_i + 2\sum_{j \in N \setminus I} v_j > 2$. Hence at most one edge inequality is tight at v. ◇

By Claim 2, we may assume that at most one edge inequality is tight at v. Then this case reduces to the $t = 1$ case, implying in turn that v is integral, as required. □

We are now ready to prove the following theorem:

Theorem 4.9 ([36])**.** *$Q_S^{(3)}$ is the intersection of $Q_S^{(2)}$ with the half-spaces defined by the square inequalities (4.3) and the star inequalities (4.4).*

Proof. Applying the Chvátal procedure to inequalities defining $Q_S^{(2)}$, it is straightforward to show the validity of the inequalities (4.3) and (4.4) for $Q_S^{(3)}$. To complete the proof of the theorem, we need to show that all other valid inequalities for $Q_S^{(3)}$ are implied by those defining $Q_S^{(2)}$, (4.3) and (4.4). Consider a valid inequality for $Q_S^{(3)}$ and let $\bar{T}$ denote the set of 0,1 vectors cut off by this inequality. If $\bar{T} = \emptyset$, then the inequality is implied by $0 \leq x \leq 1$. Thus, we assume that $\bar{T} \neq \emptyset$. Let $T := \{0,1\}^n \setminus \bar{T}$. By the

definition of a Chvátal inequality, there exists a valid inequality $ax \geq b$ for $Q_S^{(2)}$ that cuts off exactly the vertices in $\bar{T}$. By Lemma 4.1, the center points of the 3-dimensional faces of $[0,1]^n$ all belong to $Q_S^{(2)}$. This means $ax \geq b$ does not cut off any of them. By Lemma 4.2, $G(\bar{T})$ is a connected graph. We claim that the distance between any 2 vertices in $G(\bar{T})$ is at most 2. Indeed, otherwise $G(\bar{T})$ contains two opposite vertices of a cube, and therefore its center satisfies $ax < b$, a contradiction.

We consider 3 cases: $|\bar{T}| \leq 3$, $G(\bar{T})$ contains a square, and $G(\bar{T})$ contains no square. First, if $|\bar{T}| \leq 3$, then $G(\bar{T})$ is either an isolated vertex, an edge, or a path of length two. Then vertex and edge inequalities with the bounds $0 \leq x \leq 1$ are sufficient to describe $\text{conv}(T)$ by Lemma 4.8. Then we may assume that $|\bar{T}| \geq 4$. If $G(\bar{T})$ contains a square $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$, it cannot cut off any other vertex of $[0,1]^n$ (otherwise, by Lemma 4.2 there would be another vertex of $\bar{T}$ adjacent to the square, and thus in a cube, a contradiction). Thus, $\bar{T} = \{\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell}\}$. Since $\text{conv}(T) = \{x \in [0,1]^n : \sum_{j \in N \setminus \{i,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1\}$, a Chvátal inequality derived from $ax \geq b$ will therefore be implied by the square inequality that corresponds to $(\bar{x}, \bar{x}^i, \bar{x}^\ell, \bar{x}^{i\ell})$ and the bounds $0 \leq x \leq 1$.

Thus, we may assume that $G(\bar{T})$ contains no square and $|\bar{T}| \geq 4$. Note that a cycle of G_n that is not a square has length at least six. Since the distance between any two vertices in $G(\bar{T})$ is at most two, $G(\bar{T})$ contains no cycle of G_n. Thus, $G(\bar{T})$ is a tree. In fact, $G(\bar{T})$ is a star since the distance between any two of its vertices is at most two. Thus $\bar{T} = \{\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t}\}$ for some $t \geq 3$. By Lemma 4.8, $\text{conv}(T)$ is described by edge and star inequalities with the bounds $0 \leq x \leq 1$. □

Note that, if an edge $\bar{x}\bar{y}$ of $G(\bar{S})$ belongs to a square of $G(\bar{S})$, the corresponding inequality is not needed in the description of $Q_S^{(3)}$ since it is dominated by the square inequality. On the other hand, if an edge belongs to a star $(\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ of $G(\bar{S})$ with $t < n$, there is no domination relationship between the corresponding edge inequality and the star inequality. Lastly, combining Theorems 4.7 and 4.9, we obtain the following result:

Theorem 4.10 ([36]). *For $n \geq 3$, the Chvátal rank of Q_S is 2 if, and only if, $G(\bar{S})$ contains a connected component of cardinality at least 2, and each connected component of $G(\bar{S})$ is either a cycle of length greater than 4 or a path.*

Proof. (⇐): Since $G(\bar{S})$ contains neither a 4-cycle nor a star, Theorem 4.9 implies that $Q_S^{(3)} = Q_S^{(2)}$. It follows that $Q_S^{(2)} = \text{conv}(S)$. Since $G(\bar{S})$ contains a connected component of size greater than 1, $Q_S^{(1)} \neq \text{conv}(S)$ by Theorem 4.6. Thus Q_S has Chvátal rank exactly 2. (⇒): Suppose a connected component of $G(\bar{S})$ contains a cycle of length 4 or a vertex of degree greater than 2. Consider first the 4-cycle case, say $\{0, e^1, e^2, e^1 + e^2\} \subseteq \bar{S}$. Then the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right)$ belongs to $Q_S^{(2)}$ by Lemma 4.1 but not to $\text{conv}(S)$ since $\sum_{j=3}^n x_j \geq 1$ is valid for $\text{conv}(S)$. Now consider a vertex of degree greater than 2, say $\{0, e^1, e^2, e^3\} \subseteq \bar{S}$ where e^1, e^2, e^3 denote the first 3 unit vectors. Then the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right)$ belongs to $Q_S^{(2)}$ by Lemma 4.1 but not to $\text{conv}(S)$ since $\sum_{j=1}^3 x_j + 2\sum_{j=4}^n x_j \geq 2$ is valid for $\text{conv}(S)$. □

4.2.4 Chvátal rank 4

In this section, we give the characterization of $Q_S^{(4)}$. It is somewhat more involved than the results for $Q_S^{(1)}$, $Q_S^{(2)}$ and $Q_S^{(3)}$, but it is in the same spirit.

Consider any cube with vertices in $G(\bar{S})$. Specifically, for $\bar{x} \in \{0,1\}^n$, recall that we use the notation $\bar{x}^i$ to denote the 0,1 vertex that differs from $\bar{x}$ only in coordinate i, and more generally, for $J \subseteq N$, let $\bar{x}^J$ denote the 0,1 vector that differs from $\bar{x}$ exactly in the coordinates J. If the 8 points $\bar{x}, \bar{x}^i, \bar{x}^k, \bar{x}^\ell, \bar{x}^{ik}, \bar{x}^{i\ell}, \bar{x}^{k\ell}, \bar{x}^{ik\ell}$ all belong to $\bar{S}$, then we say that these points form a *cube* (see Figure 4.2). Note that

$$\sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1 \tag{4.5}$$

is a valid inequality for $\text{conv}(S)$ and that it cuts off exactly 8 vertices of $[0,1]^n$, namely the 8 corners of the cube. In fact, it is the strongest such inequality since it is satisfied at equality by all $8(n-3)$ of their neighbors in $[0,1]^n$. We call (4.5) a *cube inequality*.

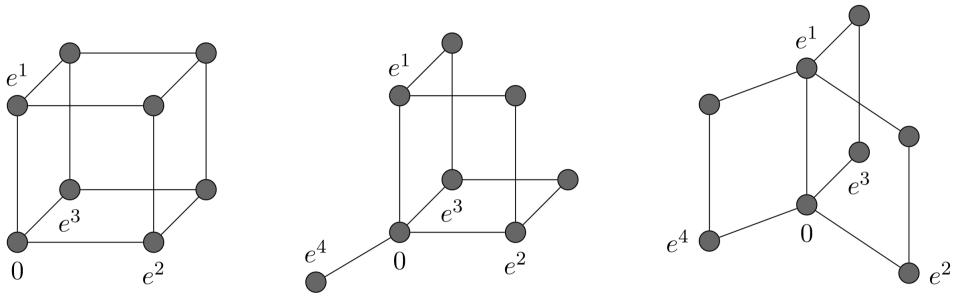


Figure 4.2: Cube, tulip, and propeller with $\bar{x} = 0$

If $\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \bar{x}^{i_3}, \bar{x}^{i_1 i_2}, \bar{x}^{i_2 i_3}, \bar{x}^{i_3 i_1}, \bar{x}^{i_4}, \ldots, \bar{x}^{i_t}$ all belong to $\bar{S}$ for some $t \geq 4$, then we say that these points form a *tulip* (see Figure 4.2). Let $I_T := \{i_1, \ldots, i_t\}$. Note that

$$\sum_{k=1}^{3}(\bar{x}_{i_k}(1-x_{i_k}) + (1-\bar{x}_{i_k})x_{i_k}) + 2\sum_{r=4}^{t}(\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 3\sum_{j \notin I_T}(\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 3 \tag{4.6}$$

is a valid inequality for $\text{conv}(S)$ that cuts off exactly these points. We call it a *tulip inequality*. For example, if $\bar{x} = 0$, and $\bar{x}^{i_k} = e^k$ for $k = 1,2,3$, (4.6) is $x_1 + x_2 + x_3 + 2\sum_{r=4}^{t} x_{i_r} + 3\sum_{j \notin I_T} x_j \geq 3$.

If $\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \ldots, \bar{x}^{i_t}, \bar{x}^{i_{t+1}}, \bar{x}^{i_1 i_{t+1}}, \bar{x}^{i_2 i_{t+1}}, \ldots, \bar{x}^{i_t i_{t+1}}$ all belong to $\bar{S}$ for some $t \geq 3$, then we say that these points form a *propeller* (see Figure 4.2). Besides, we say that the edge $\bar{x}\bar{x}^{i_{t+1}}$ is the *axis* of the propeller. Let $I_P := \{i_1, \ldots, i_{t+1}\}$. Note that

$$\sum_{r=1}^{t}(\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P}(\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2 \tag{4.7}$$

is a valid inequality that cuts off exactly the above points. We call it a *propeller inequality*. It goes through $2(n-t-1)$ neighbors of $\bar{x}$ and $\bar{x}^{i_{t+1}}$, $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1}, \ldots, \bar{x}^{i_t}$, and another $t(t-1)/2$ neighbors of two vertices among $\bar{x}^{i_1 i_{t+1}}, \ldots, \bar{x}^{i_t i_{t+1}}$. For example, if $\bar{x} = 0$, $\bar{x}^{i_{t+1}} = e^1$ and $\bar{x}^{i_k} = e^{k+1}$ for $k = 1, \ldots, t$, the propeller inequality is $x_2 + \cdots + x_{t+1} + 2(x_{t+2} + \cdots + x_n) \geq 2$.

A characterization of $Q_S^{(4)}$ is given in Theorem 4.15. To prove the theorem, we need the following 4

technical lemmas that consider 4 basic cases, and their proofs are similar to that of Lemma 4.8. Although we omit the proofs of these lemmas, we will show how the general case can be reduced to one of these basic cases.

Lemma 4.11 ([36]). *Let $\bar{S} = \{0, e^1, \dots, e^k, e^1+e^2\}$ for some $k \geq 3$. Then $\text{conv}(S)$ is described by a square inequality for the square $(0, e^1, e^2, e^1+e^2)$, a star inequality for the star $(0, e^1, \dots, e^k)$, edge inequalities for the edges connecting 0 to $e^3, \dots, e^k$ and the bounds $0 \leq x \leq 1$.*

Lemma 4.12 ([36]). *Let $\bar{S} = \{0, e^1, \dots, e^k, e^1+e^2, e^1+e^3\}$ for some $k \geq 4$. Then $\text{conv}(S)$ is described by two square inequalities for $(0, e^1, e^2, e^1+e^2)$ and $(0, e^1, e^3, e^1+e^3)$, a star inequality for the star $(0, e^1, \dots, e^k)$, edge inequalities for the edges connecting 0 to $e^4, \dots, e^k$ and the bounds $0 \leq x \leq 1$.*

Lemma 4.13 ([36]). *Consider the tulip $\bar{S} = \{0, e^1, e^2, e^3, \dots, e^k, e^1+e^2, e^2+e^3, e^3+e^1\}$ for some $k \geq 4$. Then $\text{conv}(S)$ is described by the tulip inequality, the three square inequalities, a star inequality for the star $(0, e^1, e^2, \dots, e^k)$, edge inequalities for the edges connecting 0 to $e^4, \dots, e^k$, and the bounds $0 \leq x \leq 1$.*

Lemma 4.14 ([36]). *Let $\bar{S} = \{0, e^1, \dots, e^\ell, e^1+e^2, \dots, e^1+e^k\}$ for some $k \geq 4$ and $\ell \geq k+1$. Note that $\bar{S}$ is a propeller which consists of k squares and $(0, e^1, e^2, \dots, e^\ell)$ is a star. Then $\text{conv}(S)$ is described by the star inequality for the star $(0, e^1, e^2, \dots, e^\ell)$, edge inequalities for the edges connecting 0 to $e^{k+1}, \dots, e^\ell$, the square and propeller inequalities that correspond to the propeller $(0, e^1, \dots, e^k, e^1+e^2, \dots, e^1+e^k)$, and the bounds $0 \leq x \leq 1$.*

With Lemmas 4.11, 4.12, 4.13, 4.14, we are ready to prove Theorem 4.15.

Theorem 4.15 ([36]). *$Q_S^{(4)}$ is the intersection of $Q_S^{(3)}$ and the half spaces defined by all cube, tulip, and propeller inequalities.*

Proof. We first show that the inequalities stated in the theorem are valid for $Q_S^{(4)}$.

Claim 1. *The cube, tulip, and propeller inequalities are valid for $Q_S^{(4)}$.*

Proof of Claim. A cube can be decomposed into two vertex-disjoint squares, and $x_\ell + \sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$ and $-x_\ell + \sum_{j \in N \setminus \{i,k,\ell\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 0$ are the corresponding square inequalities which are valid for $Q_S^{(3)}$. Adding them, dividing by 2, and applying the Chvátal procedure generates the cube inequality, so it is valid for $Q_S^{(4)}$.

A tulip contains a star with $\bar{x}$ as its root, and the corresponding star inequality is $\sum_{r=1}^t (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_T} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2$. In addition, it has three squares containing $\bar{x}$, and the corresponding square inequalities are $\sum_{j \in N \setminus \{i_1,i_2\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$, $\sum_{j \in N \setminus \{i_2,i_3\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$, and $\sum_{j \in N \setminus \{i_1,i_3\}} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$. These four inequalities are all valid for $Q_S^{(3)}$. Adding them, dividing by 2, and applying the Chvátal procedure shows the validity of the tulip inequality for $Q_S^{(4)}$.

A propeller contains two stars with $\bar{x}$, $\bar{x}^{i_{t+1}}$ as their roots, respectively, and the corresponding star inequalities are $x_{i_{t+1}} + \sum_{r=1}^t (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 2$ and $-x_{i_{t+1}} + \sum_{r=1}^t (\bar{x}_{i_r}(1-x_{i_r}) + (1-\bar{x}_{i_r})x_{i_r}) + 2\sum_{j \notin I_P} (\bar{x}_j(1-x_j) + (1-\bar{x}_j)x_j) \geq 1$. These are valid for $Q_S^{(3)}$. Adding them, dividing by 2, and applying the Chvátal procedure shows the validity of the propeller inequality for $Q_S^{(4)}$. ◇

To complete the proof of the theorem, we need to show that every valid inequality for $Q_S^{(4)}$ is a consequence of the inequalities defining $Q_S^{(3)}$ and cube, tulip and propeller inequalities. Consider any valid inequality for $Q_S^{(4)}$ and let $\bar{T}$ denote the set of 0,1 vectors cut off by this inequality. Let $T := \{0,1\}^n \setminus \bar{T}$. We will show that vertex, edge, square, star, cube, tulip and propeller inequalities are sufficient to describe $\text{conv}(T)$. It follows from the definition of a Chvátal inequality that there exists a valid inequality $ax \geq b$ for $Q_S^{(3)}$ that cuts off the same set $\bar{T}$. We know that $G(\bar{T})$ is a connected graph by Lemma 4.2. We claim the following three for $\bar{T}$.

Claim 2. *If a path of length three appears in $G(\bar{T})$, then either the square of G_n containing the first three vertices of the path or the square containing the last three vertices belongs to $G(\bar{T})$*

Proof of Claim. Consider a path of length three in $G(\bar{T})$. We may assume without loss of generality that the path is $(e^1, 0, e^2, e^2 + e^3)$. Suppose both $e^1 + e^2$ and e^3 satisfy $ax \geq b$. Then their middle point m in $[0,1]^n$ also satisfies $ax \geq b$, contradicting the fact that e^1 and $e^2 + e^3$ (and therefore their middle point, which is m) satisfy $ax < b$. Therefore $e^1 + e^2$ or e^3 is in $\bar{T}$, forming a square with either $e^1, 0, e^2$ or $0, e^2, e^2 + e^3$. ◇

Claim 3. *The maximum distance in G_n between two vertices in $G(\bar{T})$ is at most three.*

Proof of Claim. Let $u, v \in \bar{T}$. Since u and v are connected in $G(\bar{T})$, there is a path between u and v in $G(\bar{T})$. If the distance between u and v in G_n is at least 4, then there exists a vertex w on the path such that the distance in G_n between u and w is 4. Their middle point in $[0,1]^n$ is also cut off by $ax \geq b$. Since they are opposite vertices of a 4-dimensional face of $[0,1]^n$, the middle point of the face is cut off by the inequality. However, this contradicts Lemma 4.1 for $Q_S^{(3)}$. Hence, the maximum distance in G_n between two points in $\bar{T}$ is at most three, as required. ◇

Claim 4. *If $G(\bar{T})$ contains two squares, then either they share a common edge or $G(\bar{T})$ is a 3-dimensional cube and the two squares are opposite 2-dimensional faces of it.*

Proof of Claim. Assume that $G(\bar{T})$ contains two squares. Without loss of generality, we may assume that one of them is $(0, e^1, e^2, e^1 + e^2)$. Suppose that the second square does not share an edge with it. If they share a vertex, we may assume that the second square is $(0, e^3, e^4, e^3 + e^4)$. Note that the distance in G_n between $e^1 + e^2$ and $e^3 + e^4$ is 4, contradicting Claim 3. Thus, the two squares do not share any vertex. Because $G(\bar{T})$ is connected and no path of length greater than three exists, it easy to verify that $G(\bar{T})$ must be a 3-dimensional cube. ◇

We now consider different cases according to the number of squares contained in $G(\bar{T})$. We first consider the case when $G(\bar{T})$ has no square. Then the distance in G_n between any two vertices in $G(\bar{T})$ is at most two by Claim 2. Then $G(\bar{T})$ can be a single vertex, an edge, two consecutive edges, or a star. Hence, by Lemma 4.8, vertex, edge, and star inequalities with the bounds $0 \leq x \leq 1$ are sufficient to describe $\text{conv}(T)$. Thus, we may assume that $G(\bar{T})$ contains at least one square.

Claim 5. *If $G(\bar{T})$ contains exactly one square, edge, star, and square inequalities are sufficient.*

Proof of Claim. Without loss of generality, we may assume that it is $(0, e^1, e^2, e^1 + e^2)$. If $\bar{T}$ consists of just this square, then the square inequality $\sum_{j=3}^{n} x_j \geq 1$ suffices. If not, the square is adjacent to at least one 0,1 point in $\bar{T}$ and thus we may assume that e^3 is in $\bar{T}$. Note that the other points in $\bar{T}$ (if any) are not adjacent to any of $e^1, e^2, e^1 + e^2$, by the first claim and the assumption that only one square exists in

$G(\bar{T})$. Therefore, we may assume that $\bar{T}$ is $\{0, e^1, e^2, \dots, e^k, e^1 + e^2\}$ for some $k \geq 3$. In this case, edge, star, and square inequalities are sufficient by Lemma 4.11. ◇

Claim 6. *If $G(\bar{T})$ contains exactly two squares, edge, star, and square inequalities are sufficient.*

Proof of Claim. We may assume that $\bar{T}$ contains $0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3$. If no other vertex belongs to $\bar{T}$, then $x_3 + \sum_{j=4}^n x_j \geq 1$ and $x_2 + \sum_{j=4}^n x_j \geq 1$ together with $0 \leq x \leq 1$ suffice since the constraint matrix for this system is totally unimodular by Remark 4.4. So we may assume that there exists $v \in \bar{T} \setminus \{0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3\}$. By connectivity of $G(\bar{T})$ we may assume that v is adjacent to at least one of $0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3$. Since $G(\bar{T})$ contains only two squares, v is adjacent to exactly one of these vertices. If v is adjacent to e^2, then v can be written as $e^2 + e^k$ for some $k \geq 4$. However, this is impossible by the second claim since the distance in G_n between $e^2 + e^k$ and $e^1 + e^3$ is 4. Thus, v cannot be adjacent to e^2. Likewise, v cannot be adjacent to e^3, $e^1 + e^2$, and $e^1 + e^3$. Without loss of generality, v is adjacent to 0. If there exists $u \in \bar{T}$ adjacent to e^1, then $G(\bar{T})$ should contain an additional square containing either u or v by the first claim. Therefore, all the vertices in $\bar{T} \setminus \{0, e^1, e^2, e^3, e^1 + e^2, e^1 + e^3\}$ are adjacent to 0. Namely $\bar{T} = \{0, e^1, e^2, e^3, \dots, e^k, e^1 + e^2, e^1 + e^3\}$ for some $k \geq 4$. In this case, edge, star, and square inequalities are sufficient by Lemma 4.12. ◇

By Claims 5 and 6, we may assume that $G(\bar{T})$ contains at least three squares. If $G(\bar{T})$ contains a cube, then $G(\bar{T})$ contains no other vertex by Claim 4, and therefore, we may assume that $\bar{T} = \{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1, e^1 + e^2 + e^3\}$. In this case $\sum_{j=4}^n x_j \geq 1$ together with $0 \leq x \leq 1$ suffices. So, we can further assume that $G(\bar{T})$ contains no cube. Any two of the squares should share a common edge by Claim 4. There are two possibilities: all squares share a common edge or three squares are the three 2-dimensional faces incident to a vertex of $[0, 1]^n$. Thus we may assume that $\bar{T}$ contains either $\{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$ or $\{0, e^1, e^2, e^3, e^4, e^1 + e^2, e^1 + e^3, e^1 + e^4\}$.

Claim 7. *If $\{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\} \subseteq \bar{T}$, edge, star, square, and tulip inequalities are sufficient.*

Proof of Claim. If $\bar{T} = \{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$, then $x_1 + \sum_{j=4}^n x_j \geq 1$, $x_2 + \sum_{j=4}^n x_j \geq 1$, and $x_3 + \sum_{j=4}^n x_j \geq 1$ together with $0 \leq x \leq 1$ gives $\mathrm{conv}(T)$. This is because the constraint matrix of the system is totally unimodular by Remark 4.4. Thus, we may assume that there exists $v \in \bar{T} \setminus \{0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$, and by the connectivity of $G(\bar{T})$ we may assume that v is adjacent to at least one of vertices $0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3$, and $e^3 + e^1$. If v is adjacent to e^1, then v can be written as $e^1 + e^k$ for some $k \geq 4$. Then the distance in G_n between v and $e^2 + e^3$ is 4. If v is adjacent to $e^1 + e^2$, then v is $e^1 + e^2 + e^k$ for some $k \geq 4$. Then the distance in G_n between v and e^3 is 4. Therefore, v is adjacent to 0. Hence, $\bar{T}$ is a tulip $\{0, e^1, e^2, e^3, \dots, e^k, e^1 + e^2, e^2 + e^3, e^3 + e^1\}$ for some $k \geq 4$. In this case, edge, star, square, and tulip inequalities are sufficient by Lemma 4.13. ◇

Claim 8. *If $\{0, e^1, e^2, e^3, e^4, e^1 + e^2, e^1 + e^3, e^1 + e^4\} \subseteq \bar{T}$, edge, star, square, and propeller inequalities are sufficient.*

Proof of Claim. By Claim 4, all the squares contain $\{0, e^1\}$. As shown in the case when $G(\bar{T})$ contains exactly two squares, all vertices which are not in any square but in $\bar{T}$ should be adjacent to a single common vertex which can be either 0 or e^1. Hence, we may assume that $\bar{T} = \{0, e^1, e^2, e^3, \dots, e^k, e^{k+1}, \dots, e^\ell, e^1 + e^2, \dots, e^1 + e^k\}$ for some $k \geq 3$ and $\ell \geq k + 1$. In this case, edge, star, square, and propeller inequalities are sufficient by Lemma 4.14. ◇

By Claims 5, 6, 7, 8, conv(T) can be described by vertex, edge, square, star, cube, tulip and propeller inequalities, as required. □

As a consequence, we can characterize when the Chvátal rank of Q_S is 3.

Theorem 4.16 ([36]). *The Chvátal rank of Q_S is 3 if, and only if, $G(\bar{S})$ contains no cube, tulip or propeller but it contains a star or a square.*

Proof. This follows from Theorems 4.9, 4.10, 4.15. □

We can now prove the first three statements of Theorem 1.11.

> **Theorem 1.11 ([36]).** *Let $P \subseteq [0,1]^n$ be a rational polytope contained in the unit cube. Let $\bar{S} := \{0,1\}^n \setminus P$. Then the following statements hold:*
>
> *(1) If $\bar{S}$ is a stable set in G_n, then the Chvátal rank of P is at most 1.*
>
> *(2) If $G(\bar{S})$ is a disjoint union of cycles of length greater than 4 and paths, then the Chvátal rank of P is at most 2.*
>
> *(3) If $G(\bar{S})$ is a forest, then the Chvátal rank of P is at most 3.*

Proof. Let $S := \{0,1\}^n \setminus \bar{S}$. **(1)**: If $\bar{S}$ is a stable set in G_n, then the Chvátal rank of Q_S is 1 by Theorem 4.6, implying in turn that the Chvátal rank of P is at most 1 by Remark 1.12. **(2)**: If $G(\bar{S})$ is a disjoint union of cycles of length greater than 4 and paths, then the Chvátal rank of Q_S is at most 2 by Theorem 4.10, and by Remark 1.12, that of P is at most 2. **(3)**: If $G(\bar{S})$ is a forest, then it has no cycle, and in particular, it contains no square. If the maximum degree of $G(\bar{S})$ is smaller than 3, then it is a disjoint union of paths, meaning that the Chvátal rank of Q_S is at most 2 by part (2). If the max degree is at least 3, then $G(\bar{S})$ contains a star. Moreover, as $G(\bar{S})$ contains no sqaure, $G(\bar{S})$ contains none of cube, tulip, and propeller. Therefore, by Theorem 4.16, the Chvátal rank of Q_S is at most 3. Then the Chvátal rank of P is at most 3 by Remark 1.12, as required. □

4.3 Vertex cutsets

In this section, we give polyhedral decomposition theorems for conv(S) when the graph $G(\bar{S})$ contains a vertex cutset of cardinality 1 or 2.

4.3.1 Cut vertex

Theorem 4.17 below shows that conv(S) can be decomposed when $G(\bar{S})$ contains a vertex cut. This result is in the spirit of the theorem of Angulo, Ahmed, Dey and Kaibel (Theorem 4.3) but it is specific to

polytopes contained in the unit hypercube. At the end of this section, we give an example showing that the result does not extend to general polytopes. Before we state Theorem 4.17, let us illustrate an example first.

Let $G = (V, E)$ be a graph and let $X \subseteq V$. For $v \in X$, let $N_X[v]$ denote the closed neighborhood of v in the graph $G(X)$. That is $N_X[v] := \{v\} \cup \{u \in X : uv \in E\}$.

Example ([36])**.** Let $S = \{e^2, e^1 + e^2, e^1 + e^3\} \subset \{0,1\}^3$, and we consider $\text{conv}(S) \subset [0,1]^3$. In Figure 4.3, $\text{conv}(S)$ is a triangle which can be viewed as the intersection of the two tetrahedrons in the figure. Notice that e^3 is a cut vertex in $G(\bar{S})$, whose deletion leaves $\bar{S}_1 := \{0, e^1\}$ and $\bar{S}_2 := \{e^2 + e^3, e^1 + e^2 + e^3\}$ as two separate components. The set of 0,1 points that do not belong to the left tetrahedron is exactly $N_{\bar{S}}[e^3] \cup \bar{S}_2$, whereas that of the right one is $N_{\bar{S}}[e^3] \cup \bar{S}_1$.

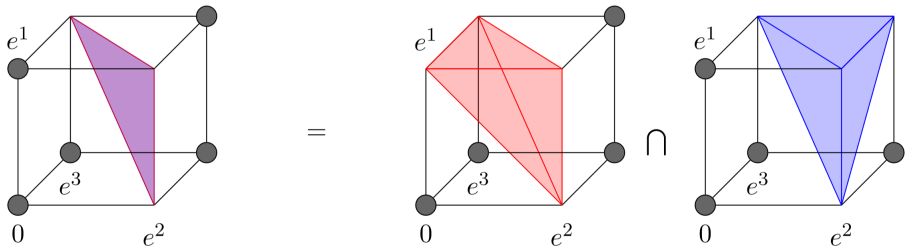


Figure 4.3: An example of decomposition around a cut vertex in $\mathbb{R}^3$

Theorem 4.17 ([36])**.** *Let $S \subseteq \{0,1\}^n$ and $\bar{S} := \{0,1\}^n \setminus S$. Let v be a cut vertex in $G(\bar{S})$ and let $\bar{S}_1, \ldots, \bar{S}_t$ denote the connected components of $G(\bar{S} \setminus \{v\})$. Then*

$$\text{conv}(S) = \bigcap_{i=1}^{t} \text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i)).$$

Furthermore, if v does not belong to any 4-cycle in $G(\bar{S})$, then $\text{conv}(S) = \text{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \bigcap_{i=1}^{t} \text{conv}(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$.

Proof. To ignore trivial cases, we assume $n \geq 3$ and $t \geq 2$. By Lemma 4.8, we know that $\text{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ can be described by star and edge inequalities together with $0 \leq x \leq 1$. Let $u \in \bar{S}_i \setminus N_{\bar{S}}[v]$ and $w \in \bar{S}_j \setminus N_{\bar{S}}[v]$ where $i \neq j$.

Claim 1. *No edge inequality of $\text{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ is active at both u and w.*

Proof of Claim. Consider an edge vr in the star $G(N_{\bar{S}}[v])$. Suppose for contradiction that the corresponding edge inequality is active at both u and w. Then each of u and w is adjacent in G_n to an endpoint of the edge. Since u and w cannot be adjacent to v by the definition of $N_{\bar{S}}[v]$, both are adjacent to r. Then (u, r, w) is a path contained in $G(\bar{S} \setminus \{v\})$, contradicting the assumption that u and w are disconnected in $G(\bar{S} \setminus \{v\})$. Hence, no edge inequality is active at both u and w, as required. ◇

Claim 2. *The skeleton graph of $\text{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ contains no edge connecting u and w.*

Proof of Claim. Suppose for contradiction that u and w are adjacent in the skeleton graph of $\mathrm{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$. Then we can find $n-1$ linearly independent inequalities in the description of $\mathrm{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$ that are active at both u and w.

It follows from Claim 1 that the only candidates are the star inequality and the bounds $0 \leq x \leq 1$. If the star inequality is active at both u and w, then each of u and w is adjacent to two vertices of $N_{\bar{S}}[v] \setminus \{v\}$ in G_n. As u and w belong to different components, no vertex is adjacent to u and w in G_n. Then there exist four distinct vertices in $N_{\bar{S}}[v] \setminus \{v\}$ two of which are adjacent to u and the other two which are adjacent to w. That means $n-4$ inequalities among $0 \leq x \leq 1$ are active at both u and w, so only $n-3$ linearly independent inequalities are active at both u and w. Thus, we may assume that the star inequality is not active at both u and w. Since u and w are at distance at least 2 in G_n, at most $n-2$ among $0 \leq x \leq 1$ are active at both, contradicting the supposition that u and w are adjacent on the skeleton graph of $\mathrm{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v])$, as required. ◇

Claim 2 implies that $S_i \setminus N_{\bar{S}}[v]$ and $\bar{S}_j \setminus N_{\bar{S}}[v]$ are disconnected. Then, by Theorem 4.3, we obtain

$$\mathrm{conv}(S) = \bigcap_{i=1}^{t} \mathrm{conv}\left(\{0,1\}^n \setminus \left(N_{\bar{S}}[v] \cup \bar{S}_i\right)\right).$$

For the second statement of Theorem 4.17, it is sufficient to prove the following claim:

Claim 3. *If v does not belong to any 4-cycle of $G(\bar{S})$, $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \cup \bar{S}_i)) = \mathrm{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \mathrm{conv}(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$.*

Proof of Claim. Let $W := N_{\bar{S}}[v] \setminus (\bar{S}_i \cup \{v\})$. It is sufficient to show that the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \setminus W))$ contains no edge connecting a vertex of $\bar{S}_i \setminus N_{\bar{S}}[v]$ to a vertex of W. Let $w \in W$ and $s \in \bar{S}_i \setminus N_{\bar{S}}[v]$. By the assumption that v does not belong to any square in $G(\bar{S})$, s is adjacent to at most one pendent vertex of $N_{\bar{S}}[v]$ in G_n. That means the star inequality is not active at s. We consider two cases. Consider first the case when s is adjacent to a vertex, denoted r, in $N_{\bar{S}}[v] \setminus W$. Then the edge inequality for vr is active at s, but no other edge inequality is active at s. Since w is adjacent to v, the edge inequality is also active at w. However, the distance in G_n between s and w is exactly 3 in this case. Thus at most $n-3$ bound inequalities are active at both s and w, for a total of at most $n-2$ linearly independent inequalities active at both. But we need $n-1$. So s and w are not connected by an edge of the skeleton in this case. Now consider the case where s is adjacent to no vertex of $N_{\bar{S}}[v] \setminus W$. Then no edge inequality is active at s. Since s and w are not adjacent in G_n, at most $n-2$ inequalities among $0 \leq x \leq 1$ are active at both u and w. Therefore, s and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v] \setminus W))$ in this case, either. Thus the assertion holds by Lemma 4.3, as required. ◇

This finishes the proof of Theorem 4.17. □

If $G(\bar{S})$ induces a forest, then $G(\bar{S})$ contains no square, and by Theorem 4.16, the Chvátal rank of Q_S is at most 3. We can directly prove this statement using Theorem 4.17.

Theorem 4.18 ([36])**.** *Let $S \subseteq \{0,1\}^n$ and $\bar{S} := \{0,1\}^n \setminus S$. If $G(\bar{S})$ is a forest, then the Chvátal rank of Q_S is at most 3.*

Proof. By Theorem 4.3, we may assume that $G(\bar{S})$ is connected, so $G(\bar{S})$ is a tree. We prove by induction on the size of the tree. The result holds if $|\bar{S}| \leq 3$. Let $G(\bar{S})$ induce a tree T and assume that the result holds for all trees with fewer vertices. The theorem holds if T is a star by Lemma 4.8, so we may assume that T is not a star. Let v be a non-pendant vertex of T and let $\bar{S}_1, \ldots, \bar{S}_t$ denote the connected components of $G(\bar{S} \setminus \{v\})$. Since v does not belong to any 4-cycle in $G(\bar{S})$, Theorem 4.17 implies that $\text{conv}(S) = \text{conv}(\{0,1\}^n \setminus N_{\bar{S}}[v]) \cap \bigcap_{i=1}^{t} \text{conv}(\{0,1\}^n \setminus (\{v\} \cup \bar{S}_i))$. Notice that the sets $N_{\bar{S}}[v]$ and $\{v\} \cup \bar{S}_i$ for $i = 1, \ldots, t$ have smaller cardinality than $\bar{S}$. Then the result follows from the induction hypothesis. □

Unlike Theorem 4.3, Theorem 4.17 cannot be extended to general polytopes, as shown by the following example.

Example. Let P be the polytope in $\mathbb{R}^2$ shown in Figure 4.4. Let $V := \{v_1, \ldots, v_8\}$ denote its vertex set and let $G = (V, E)$ be its skeleton graph. Let $S := \{v_5, v_6, v_7\}$ and $\bar{S} := V \setminus S$. In the figure the set of white vertices is S, while the set of black vertices is $\bar{S}$. Note that v_2 is a cut vertex of $G(\bar{S})$, and $N_{\bar{S}}[v_2] = \{v_1, v_2, v_3\}$. Therefore, $\bar{S}_1 := \{v_1, v_8\}$ and $\bar{S}_2 := \{v_3, v_4\}$ induce two distinct connected components of $G(\bar{S} \setminus \{v_2\})$. Note that $\text{conv}(S) \neq \text{conv}(V \setminus (N_{\bar{S}}[v_2] \cup \bar{S}_1)) \cap \text{conv}(V \setminus (N_{\bar{S}}[v_2] \cup \bar{S}_2))$ since $\text{conv}(S)$ is a triangle but the intersection of $\text{conv}(V \setminus \{v_1, v_2, v_3, v_4\})$ and $\text{conv}(V \setminus \{v_1, v_2, v_3, v_8\})$ is a parallelogram.

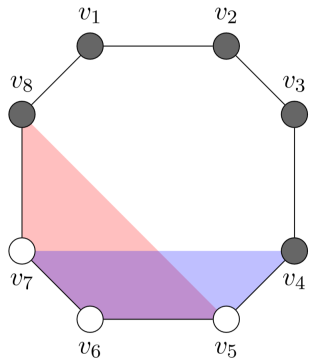


Figure 4.4: An example in $\mathbb{R}^2$

4.3.2 2-vertex cut

In this section, we prove Theorem 4.21 that is an extension of Theorem 4.17 to vertex cuts of cardinality 2. It will play a key role in proving the main result of Section 4.4. The proof of Theorem 4.21 entails analyzing the adjacency on the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ between two points in different connected components of the graph $G(\bar{S} \setminus \{v_1, v_2\})$. To do this, we need the following theorem that characterizes a linear description of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Lemma 4.19 ([36])**.** *Let $\bar{S} \subseteq \{0,1\}^n$ and $v_1, v_2 \in \bar{S}$. Then $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ is described by edge, star, square, cube, propeller inequalities and the bounds $0 \leq x \leq 1$.*

The following small lemma is also useful here and later in the next section:

Lemma 4.20 ([36]). *Let $S \subseteq \{0,1\}^n$ and $\bar{S} := \{0,1\}^n \setminus S$. Let $\bar{x}, \bar{y} \in S$ be two points at distance 2 in G_n, i.e., $\bar{y} = \bar{x}^{ij}$ for some $i, j \in N$. Then $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of* $\mathrm{conv}(S)$ *if, and only if, $\bar{x}^i$ or $\bar{x}^j$ is in $\bar{S}$.*

Proof. ($\Leftarrow$): Without loss of generality, we may assume that $\bar{x} = 0$ and $\bar{y} = e^1 + e^2$. If $e^1 \in \bar{S}$, then the corresponding vertex inequality $-x_1 + \sum_{i=2}^n x_i \geq 0$ is valid for $\mathrm{conv}(S)$ and active at both $\bar{x}$ and $\bar{y}$. We also know that $x_i \geq 0$ for $i \geq 3$ are all active at both $\bar{x}$ and $\bar{y}$. Since these $n-1$ inequalities are linearly independent, $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of $\mathrm{conv}(S)$. Likewise if $e^2 \in \bar{S}$. ($\Rightarrow$): If e^1, e^2 are in S, then $(0, e^1, e^2, e^1 + e^2)$ is a 2-dimensional face of $\mathrm{conv}(S)$. The center of the square can be obtained as a nontrivial convex combination of 4 distinct vertices of $\mathrm{conv}(S)$, and therefore it does not lie on any 1-dimensional face of $\mathrm{conv}(S)$. Thus the diagonal connecting 0 to $e^1 + e^2$ is not a face of $\mathrm{conv}(S)$. □

To prove Theorem 4.21, we first delete two star cutsets $N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ from $\{0,1\}^n$. If we can prove that no edge connects a vertex of $\bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ to a vertex of $\bar{S}_j \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ for $i \neq j$, the theorem follows by Theorem 4.3. Lemma 4.19 provides us with the linear description of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. Therefore, we only need to consider edge, star, square, propeller, cube inequalities and the bounds $0 \leq x \leq 1$ in order to analyze the adjacency of vertices on the polytope $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Theorem 4.21 ([36]). *Let $S \subseteq \{0,1\}^n$ and $\bar{S} := \{0,1\}^n \setminus S$. Let $\{v_1, v_2\}$ be a vertex cut of size 2 in $G(\bar{S})$. Let $\bar{S}_1, \ldots, \bar{S}_t$ denote the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. Then*

$$\mathrm{conv}(S) = \bigcap_{i=1}^{t} \mathrm{conv}\left(\{0,1\}^n \setminus \left(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] \cup \bar{S}_i\right)\right).$$

Proof. The assertion is trivially true if $n \leq 3$, so we may assume that $n \geq 4$. If $\bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ is nonempty for at most one i, then $\bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]) = \bar{S}$, and therefore, the theorem holds. Thus, we may assume that for some distinct i, j, $\bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ and $\bar{S}_j \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ are nonempty. Let $u \in \bar{S}_i \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ and $w \in \bar{S}_j \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. We will show that no edge in the skeleton graph of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ connects u and w.

Claim 1. *No edge inequality of* $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ *is active at both u and w.*

Proof of Claim. Suppose for contradiction that the edge inequality for an edge pq in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ is active at both u and w. Then each of u and w is adjacent to either p or q. If $p \in \{v_1, v_2\}$, then u and w cannot be adjacent to p since $u, w \notin N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$, implying in turn that $q \notin \{v_1, v_2\}$ and that u and w are adjacent to q. But (u, q, w) is a path contained in $G(\bar{S} \setminus \{v_1, v_2\})$, a contradiction as u and w are disconnected in $G(\bar{S} \setminus \{v_1, v_2\})$. Hence, we may assume that $p, q \notin S \setminus \{v_1, v_2\}$. This implies that there is a path in $G(\bar{S} \setminus \{v_1, v_2\})$ between u and w through the edge pq, which is again a contradiction. Therefore no edge inequality is active at both u and w, as required. ◇

Since u and w are disconnected in $G(\bar{S} \setminus \{v_1, v_2\})$, the distance in G_n between u and w is at least 2.

Claim 2. *We may assume that the distance between u and w in G_n is at least 3.*

Proof of Claim. $w = u^{k\ell}$ for some $k, \ell \in N$. Since u is adjacent to neither v_1 nor v_2, we get $u^k, u^\ell \notin \{v_1, v_2\}$. Besides, $u^k, u^\ell \notin \bar{S}$. Otherwise, u and w are connected in $G(\bar{S} \setminus \{v_1, v_2\})$, which contradicts the assumption. Then u and w are not adjacent in the skeleton graph by Lemma 4.20. Therefore, we may assume that the distance in G_n between u and w is at least 3, as required. $\diamond$

To prove Theorem 4.21, we consider different cases according to the distance between v_1 and v_2 in G_n. Without loss of generality, we may assume $v_1 = 0$. Recall that by Lemma 4.19, $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ is described by edge, star, square, and propeller (if it exists) inequalities.

Claim 3. *If the distance between v_1 and v_2 in G_n is 1, then u and w are not adjacent in the skeleton graph of* $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Proof of Claim. Without loss of generality, we may assume that $v_2 = e^1$ since $v_1 = 0$. Notice that each square in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains v_1v_2 as an edge and that $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains neither a tulip nor a propeller. If there is no square in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$, then no square inequality is active at u and w. If one exists, pick a square and consider the corresponding square inequality. Let p, q denote the other two vertices in the square. If the inequality is active at both u and w, then u and w are adjacent to a vertex in the square. Since u and w cannot be adjacent to any of v_1 and v_2, they are adjacent to either p or q. In this case, u and w are connected by the edge pq in $G(\bar{S} \setminus \{v_1, v_2\})$ which contradicts the assumption that u and w are disconnected. Hence no square inequality is active at both u and w.

Consider the star inequality for $N_{\bar{S}}[v_1]$. If it is active at both, then each of u and w is adjacent to two vertices in $N_{\bar{S}}[v_1] \setminus \{v_1\}$. Since u and w cannot have a common neighbor vertex in $N_{\bar{S}}[v_1] \setminus \{v_1\}$, there exist four distinct vertices $e^p, e^q, e^r, e^s \in N_{\bar{S}}[v_1] \setminus \{v_1\}$ such that $u = e^p + e^q$ and $w = e^r + e^s$. In addition, we know that $p, q, r, s > 1$, because u and w cannot be adjacent to v_2. That means that the star inequality for $N_{\bar{S}}[v_2]$ cannot be active at u and w. This implies that at most one star inequality is active at both u and w.

If a star inequality is active at both u and w, we observed that $n-4$ among $0 \le x \le 1$ are active at both and that the other star inequality is not active at both. Even if the propeller inequality is active at both u and w, we have only $n-2$ inequalities active at both u and w. In no star inequality is active at both, then we know that at most $n-3$ among $0 \le x \le 1$ are active at both by Claim 2. Regardless of whether a propeller inequality is active at u and w, we have at most $n-2$ inequalities active at both u and w. Therefore, u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$, as required. $\diamond$

Claim 4. *If the distance between v_1 and v_2 in G_n is 2, then u and w are not adjacent in the skeleton graph of* $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Proof of Claim. Without loss of generality, we may assume that $v_2 = e^1 + e^2$. Observe first that $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains at most one square, implying in turn that it contains none of cube, tulip, and propeller.

Consider the star inequality for $N_{\bar{S}}[v_1]$. If it is active at both u and w, we know that u and w can be written as $e^p + e^q$ and $e^r + e^s$, respectively, for some distinct p, q, r, s. and that $n-4$ inequalities among $0 \le x \le 1$ are active at both u and w. We need two more active inequalities. Then the other star inequality and the square inequality should be active at both u and w. Then we may assume that $p = 1$ and $r = 2$, so u and w can be written as $e^1 + e^q$ and $e^2 + e^s$, respectively. Without loss of generality, assume that

$q = 3$ and $s = 4$. Note that $\{e^1, e^2, e^3, e^4\} \subseteq N_{\bar{S}}[v_1]$ and $\{e^1, e^2, e^1 + e^2 + e^3, e^1 + e^2 + e^4\} \subseteq N_{\bar{S}}[v_2]$. In this case, the followings are $n-1$ inequalities that are active at both u and w.

$$x_i \geq 0 \text{ for } i \geq 5,\ \sum_{i=3}^{n} x_i \geq 1,\ \sum_{i=1}^{k_1} x_i + 2\sum_{j=k_1+1}^{n} x_j \geq 2,\ -x_1 - x_2 + \sum_{i=3}^{k_2} x_i + 2\sum_{j=k_2+1}^{n} x_j \geq 0$$

for some $k_1, k_2 \geq 4$. Note that $x_i = 0$ for $i \geq 5$ and $\sum_{i=1}^{k_1} x_i + 2\sum_{j=k_1+1}^{n} x_j = 2$ imply that $x_1+x_2+x_3+x_4 = 2$. Besides, $x_i = 0$ for $i \geq 5$ and $-x_1 - x_2 + \sum_{i=3}^{k_2} x_i + 2\sum_{j=k_2+1}^{n} x_j = 0$ imply that $-x_1 - x_2 + x_3 + x_4 = 0$. Then we get that $x_3 + x_4 = 1$ by adding the two equations. Since $x_3 + x_4 = 1$ and $x_i = 0$ for $i \geq 5$ imply $\sum_{i=3}^{n} x_i = 1$, it follows that at most $n-2$ linearly independent inequalities are active at both u and w in this case.

Therefore we may assume that no star inequality is active at both u and w. The only remaining candidates are at most $n-3$ inequalities among $0 \leq x \leq 1$ and the square inequality, so we have at most $n-2$ linearly independent inequalities active at both. Therefore, u and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. ◇

Claim 5. *If the distance between v_1 and v_2 in G_n is 3, then u and w are not adjacent in the skeleton graph of* $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Proof of Claim. Without loss of generality, we may assume that $v_2 = e^1 + e^2 + e^3$. Each square contains either v_1 or v_2 but not both. Suppose that a square inequality is active at both u and w. Without loss of generality, assume that the square is $(0, e^1, e^2, e^1 + e^2)$. Since u and w cannot be adjacent to $v_1 (= 0)$, they are adjacent to either e^1, e^2, or $e^1 + e^2$. However, this contradicts the assumption that u and w are disconnected. Hence, no square inequality is active at both u and w.

First, consider the case when a vertex in the cube $(0, e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1, e^1 + e^2 + e^3)$ is not in $\bar{S}$. Then it can be easily observed that $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$ contains none of tulip and propeller. By Lemma 4.19, $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$ is described by edge, star, and square inequalities together with $0 \leq x \leq 1$. Consider a star contained in $G(N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2])$. If the star is neither $N_{\bar{S}}[v_1]$ nor $N_{\bar{S}}[v_2]$, then it must be the case that the star is a subset of the cube. If the corresponding star inequality is active at u, then either u is in the cube or u is a vertex outside of the cube adjacent to the root r of the star. Note that a vertex in the cube is adjacent to either v_1 or v_2. This means that u cannot be in the cube, and u is adjacent to r. If the inequality is also active at w, then w is adjacent to r as well. Hence, we get that (u, r, w) is a path contained in $G(\bar{S} \setminus \{v_1, v_2\})$. Therefore, the star inequality is not active at both u and w. Thus, only the two star inequalities for $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$ can be active at both u and w. Consider the star inequality for $N_{\bar{S}}[v_1]$. If it is active at both u and w, then $n-4$ inequalities among $0 \leq x \leq 1$ are active at both u and w. But then at most $n-2$ inequalities are active at both u and w since no edge and square inequality is active at both u and w. If no star inequality is active at both u and w, then no inequality other than $0 \leq x \leq 1$ is active at both in fact. Since at most $n-3$ inequalities among $0 \leq x \leq 1$ are active at both u and w by Claim 2, we cannot find $n-1$ linearly independent inequalities active at both in this case, either.

Now consider the case when all the vertices in the cube are in $\bar{S}$. By Lemma 4.19, the cube inequality and the two star inequalities that correspond to $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$ together with $0 \leq x \leq 1$ describe $\mathrm{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. Suppose that the cube inequality is active at both u and w. Then u and w are adjacent to at least one vertex in the cube in G_n distinct from v_1 and v_2. That means u and

w are connected by six vertices $(e^1, e^2, e^3, e^1 + e^2, e^2 + e^3, e^3 + e^1)$ in $G(\bar{S} \setminus \{v_1, v_2\})$, contradicting the assumption that u and w are disconnected. Therefore, the cube inequality is not active at both u and w. If a star inequality is active at both u and w, then, as in the previous case, at most $n-2$ inequalities are active at both u and w, a contradiction. If no star inequality is active at both u and w, then no inequality other than $0 \leq x \leq 1$ is active at both u and w. Therefore, u and w are not adjacent in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$, as required. ◇

Claim 6. *If the distance between v_1 and v_2 in G_n is at least 4, then u and w are not adjacent in the skeleton graph of* $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$.

Proof of Claim. Notice that $N_{\bar{S}}[v_1]$ and $N_{\bar{S}}[v_2]$ are two separated stars. By Lemma 4.19, we know that edge and star inequalities together with $0 \leq x \leq 1$ describe $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. Consider the star inequality corresponding to $N_{\bar{S}}[v_1]$. If it is active at both u and w, then $n-4$ inequalities among $0 \leq x \leq 1$ are active at both u and w. Since no edge inequality is active at both u and w, we have at most $n-2$ inequalities that are active at both u and w since the only candidates are two star inequalities and the bounds. This contradicts to observation that there exist $n-1$ linearly independent inequalities that are active both u and w. ◇

Claims 3, 4, 5, 6 finish all the cases and show that u and w cannot be adjacent in the skeleton of $\text{conv}(\{0,1\}^n \setminus (N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]))$. This completes the proof of Theorem 4.21. □

It is natural to ask whether this theorem can be extended to vertex cuts of larger sizes. The 3-vertex cut case is open, but it turns out that Theorem 4.21 cannot be generalized to 4-vertex cutsets as shown by the following example.

Example ([36])**.** Consider $\bar{S} = ((\{0,1\}^4 \times \{0\}) \setminus \{e^1+e^2+e^3+e^4\}) \cup \{e^5\}$. Then $x_1+x_2+x_3+x_4+3x_5 \geq 4$ is a facet-defining inequality for $\text{conv}(S)$. Note that it cuts off all points in $\bar{S}$. In addition, $\bar{C} := \{e^1, e^2, e^3, e^4\}$ is a vertex cut of cardinality four in $\bar{S}$. Then $\bar{S}_1 := \{0, e^5\}$ and $\bar{S}_2 := \{e^1 + e^2 + e^3, e^1 + e^2 + e^4, e^1 + e^3 + e^4, e^2 + e^3 + e^4, e^1 + e^2, e^1 + e^3, e^1 + e^4, e^2 + e^3, e^2 + e^4, e^3 + e^4\}$ induce two connected components of $G(\bar{S} \setminus \bar{C})$. However,

$$\text{conv}(S) \neq \bigcap_{i=1}^{2} \text{conv}(\{0,1\}^5 \setminus (N_{\bar{S}}[e^1] \cup \ldots \cup N_{\bar{S}}[e^4] \cup \bar{S}_i))$$

since $x_1 + x_2 + x_3 + x_4 + 3x_5 \geq 4$ is not valid for $\text{conv}(\{0,1\}^5 \setminus (N_{\bar{S}}[e^1] \cup \ldots \cup N_{\bar{S}}[e^4] \cup \bar{S}_i))$ for $i = 1, 2$.

4.4 Graphs of tree width 2

Trees can be generalized using the notion of tree width. A connected graph has tree width one if, and only if, it is a tree. Next, we focus our attention on the case when $G(\bar{S})$ has tree width two. Instead of working directly with the definition of tree width, we will use the following characterization: A graph has tree width at most two if, and only if, it contains no K_4-minor; furthermore a graph with no K_4-minor and at least four vertices always has a vertex cut of size two. The main result of this section is that P has Chvátal rank at most 4 if $G(\bar{S})$, where $\bar{S} := \{0,1\}^n \setminus P$, has tree width 2.

The following considers a special case:

Lemma 4.22 ([36]). *Consider a star $\bar{N} = (\bar{x}, \bar{x}^{i_1}, \ldots, \bar{x}^{i_t})$ for some $\bar{x}$ and $t \geq 3$. Take a subset $\bar{T}$ of $\{\bar{x}^{i_j i_k} : 1 \leq j < k \leq t\}$ such that $\bar{x}^{i_j i_k}, \bar{x}^{i_k i_\ell} \in \bar{T}$ implies $\bar{x}^{i_j i_\ell} \notin \bar{T}$. Let $\bar{S}$ be the union of $\bar{N}$ and $\bar{T}$. Then* $\mathrm{conv}(S)$ *is described by the star inequality for $\bar{N}$, edge inequalities for the edges connecting $\bar{x}$ and pendant vertices of $G(\bar{S})$, square inequalities for all squares, propeller inequalities for all propellers and the bounds $0 \leq x \leq 1$.*

Let $v \in \bar{S}$. Let $M_{\bar{S}}[v]$ denote the set $N_{\bar{S}}[v] \cup \{v^{ij} \in \bar{S} : v^i, v^j \in N_{\bar{S}}[v]\}$. Then $M_{\bar{S}}[v]$ contains the closed neighborhood $N_{\bar{S}}[v]$ and the vertices in $\bar{S}$ at distance 2 from v that create a square when added to $N_{\bar{S}}[v]$. If $G(M_{\bar{S}}[v])$ is K_4-minor-free, $M_{\bar{S}}[v]$ is of the form $\bar{N} \cup \bar{T}$ in Lemma 4.22. Therefore Lemma 4.22 gives a description of $\mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v])$. Let v_1, v_2 be two vertices in $\bar{S}$ that are adjacent in G_n. The following lemma is similar to Lemma 4.19.

Lemma 4.23 ([36]). *Let $v_1, v_2 \in \bar{S}$ be adjacent vertices in G_n. If $G(\bar{S})$ has tree width 2, then* $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])) = \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_1]) \cap \mathrm{conv}(\{0,1\}^n \setminus M_{\bar{S}}[v_2])$.

The following lemma is similar to Lemma 4.20:

Lemma 4.24 ([36]). *Let $S \subseteq \{0,1\}^n$ and $\bar{S} = \{0,1\}^n \setminus S$. Let $\bar{x}, \bar{y} \in S$ be 2 points at distance 3 in G_n, i.e., $\bar{y} = \bar{x}^{ijk}$ for some i, j, k. Note that $(\bar{x}^i, \bar{x}^{ij}, \bar{x}^j, \bar{x}^{jk}, \bar{x}^k, \bar{x}^{ki})$ is a cycle of length 6 in G_n. Then $\bar{x}$ and $\bar{y}$ are adjacent in the skeleton of* $\mathrm{conv}(S)$ *if and only if there exist 3 consecutive vertices in the cycle that are contained in $\bar{S}$.*

We can now prove the last statement of Theorem 1.11.

> **Theorem 1.11 ([36]).** *Let $P \subseteq [0,1]^n$ be a rational polytope contained in the unit cube. Let $\bar{S} := \{0,1\}^n \setminus P$. Then the following statement holds:*
>
> *(4) If $G(\bar{S})$ has tree-width 2, then the Chvátal rank of P is at most 4.*

Proof. Let $S := \{0,1\}^n \cap P$. By Remark 1.12, it suffices to prove that the Chvátal rank of Q_S is at most 4. We argue by induction on $|\bar{S}|$. If $|\bar{S}| = 1$, then the Chvátal rank of Q_S is 1. Assume that the Chvátal rank of Q_S is at most 4 if $|\bar{S}| = t$ for some $t \geq 1$. Consider the case when $|\bar{S}| = t + 1$. We may assume that $G(\bar{S})$ is a connected graph.

Claim 1. *$G(\bar{S})$ contains neither tulip nor cube.*

Proof of Claim. Note that a tulip has three squares $(\bar{x}, \bar{x}^{i_1}, \bar{x}^{i_2}, \bar{x}^{i_1 i_2})$, $(\bar{x}, \bar{x}^{i_2}, \bar{x}^{i_3}, \bar{x}^{i_2 i_3})$, and $(\bar{x}, \bar{x}^{i_3}, \bar{x}^{i_1}, \bar{x}^{i_3 i_1})$ which are incident to a vertex $\bar{x}$. Hence, a tulip contains a K_4-minor. Likewise, a cube also contains a K_4-minor. Thus, $G(\bar{S})$ contains no tulip and cube, as required. ◇

If there is no propeller in $G(\bar{S})$, then the Chvátal rank of Q_S is at most 3 by Theorem 4.16. Thus, we may assume that $G(\bar{S})$ contains a propeller. Let v_1 and v_2 denote the two vertices in the axis of the propeller. The propeller contains at least three squares. Let (p, q, v_1, v_2) and (r, s, v_1, v_2) be two distinct squares contained in the propeller.

Claim 2. *$\{p, q\}$ is disconnected from $\{r, s\}$ in $G(\bar{S} \setminus \{v_1, v_2\})$, and $\{v_1, v_2\}$ is a vertex cut of $G(\bar{S})$.*

Proof of Claim. If there is a path connecting a vertex in $\{p, q\}$ and a vertex in $\{r, s\}$ in $G(\bar{S} \setminus \{v_1, v_2\})$, then those two squares and the path create a K_4-minor contained in $G(\bar{S})$, a contradiction. Hence, p, q are disconnected from r, s in $G(\bar{S} \setminus \{v_1, v_2\})$, implying in turn that $\{v_1, v_2\}$ is a vertex cut of $G(\bar{S})$. ◇

Let $\bar{S}_1, \ldots, \bar{S}_k$ be the connected components of $G(\bar{S} \setminus \{v_1, v_2\})$. We have shown that $k \geq 2$ by Claim 2. By Theorem 4.21, we get that

$$\operatorname{conv}(S) = \bigcap_{i=1}^{k} \operatorname{conv}\left(\{0,1\}^n \setminus \left(\bar{S}_i \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]\right)\right).$$

If $|\bar{S}_i \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]| < |\bar{S}|$ for every i, then the assertion follows directly from the induction hypothesis. Thus, we may assume that there exists j such that $\bar{S}_j \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2] = \bar{S}$. In this case, it can be readily checked that $\bar{S}_i \subset N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ for each $i \neq j$. Without loss of generality, we may assume that $\bar{S}_1$ and $\bar{S}_2$ denote two connected components that contain $\{p, q\}$ and $\{r, s\}$, respectively.

Claim 3. *One of $\bar{S}_1 \setminus \{p, q\}$ and $\bar{S}_2 \setminus \{r, s\}$ is empty.*

Proof of Claim. Suppose for contradiction that there exist u, w such that $u \in \bar{S}_1 \setminus \{p, q\}$ and $w \in \bar{S}_2 \setminus \{r, s\}$. Then we can find $u_0 \in \bar{S}_1 \setminus \{p, q\}$ and $w_0 \in \bar{S}_2 \setminus \{r, s\}$ such that u_0 is adjacent to one of p and q and w_0 is adjacent to one of r and s. Notice that $u_0, w_0 \notin N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$, implying in turn that $u_0 \notin \bar{S}_2 \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$ and that $w_0 \notin \bar{S}_1 \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]$. Then we obtain $|\bar{S}_1 \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]| < |\bar{S}|$ and $|\bar{S}_2 \cup N_{\bar{S}}[v_1] \cup N_{\bar{S}}[v_2]| < |\bar{S}|$, a contradiction to the assumption. ◇

Therefore, we may assume that $\bar{S}_1 \setminus \{p, q\}$ is empty. In other words, $\bar{S}_1 = \{p, q\}$, so the other vertices of $G(\bar{S} \setminus \{v_1, v_2\})$ are disconnected from p and q. Besides, p is adjacent to only v_1 and q, and q is adjacent to only v_2 and p in $G(\bar{S})$.

Let $w \in \bar{S} \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])$. Then w is not adjacent to p and q in G_n. We will show that w is adjacent to none of p and q in the skeleton graph of $\operatorname{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$.

Claim 4. *$\operatorname{conv}\left(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\})\right)$ is described by edge, star, square, propeller inequalities and the bounds $0 \leq x \leq 1$.*

Proof of Claim. Notice that $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$ can be written as $(M_{\bar{S}}[v_1] \setminus \{p, q\}) \cup (M_{\bar{S}}[v_2] \setminus \{p, q\})$. We know that $\{p, q\}$ is contained both $M_{\bar{S}}[v_1]$ and $M_{\bar{S}}[v_2]$. Since p and q are not adjacent to any vertices of $\bar{S} \setminus \{v_1, v_2\}$ other than themselves, we have $M_{\bar{S}}[v_\ell] \setminus \{p, q\} = M_{\bar{S} \setminus \{p,q\}}[v_\ell]$ for $\ell = 1, 2$. By Lemma 4.23, we get that

$$\operatorname{conv}(\{0,1\}^n \setminus (M_{\bar{S} \setminus \{p,q\}}[v_1] \cup M_{\bar{S} \setminus \{p,q\}}[v_2])) = \operatorname{conv}(\{0,1\}^n \setminus M_{\bar{S} \setminus \{p,q\}}[v_1]) \cap \operatorname{conv}(\{0,1\}^n \setminus M_{\bar{S} \setminus \{p,q\}}[v_2]).$$

Therefore,

$$\operatorname{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\})) = \operatorname{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \setminus \{p, q\})) \cap \operatorname{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_2] \setminus \{p, q\})).$$

By Lemma 4.22, this implies that the polytope $\operatorname{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$ is completely described by edge, star, square, propeller inequalities and the bounds $0 \leq x \leq 1$. ◇

Claim 5. *If w is not adjacent to a vertex in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$, then w is not adjacent to any of p and q in the skeleton graph of* $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$.

Proof of Claim. As w and p are not adjacent in G_n, the distance between them in G_n is at least 2. If w is not adjacent to any vertex in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$, then inequalities other than $0 \leq x \leq 1$ cannot be active at w. That means there exist at most $n-2$ linearly independent inequalities active at both w and p, so w and p are disconnected in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$. Likewise, w is also separated from q in the skeleton. Thus, we may assume that w is adjacent to a vertex of $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$. ◇

By Claim 5, we may assume that w is adjacent to a vertex in $M_{\bar{S}}[v_1] \setminus \{p, q\}$. We may further assume that $v_1 = 0$, $v_2 = e^1$, $p = e^2$, and $q = e^1 + e^2$. By the above assumption, w is adjacent to a vertex of either $N_{\bar{S}}[v_1] \setminus \{v_1\}$ or $M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$.

Claim 6. *If w is adjacent to a vertex in $N_{\bar{S}}[v_1] \setminus \{v_1\}$, then w is not adjacent to any of p and q in the skeleton graph of* $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$.

Proof of Claim. w can be written as $e^i + e^j$ for some i, j. If both e^i and e^j are in $\bar{S}$, then $e^i + e^j$ is contained in $M_{\bar{S}}[v_1]$. Thus, we may assume that $e^i \in \bar{S}$ and $e^j \notin \bar{S}$. Since w is not adjacent to v_2 and p, we get that $i, j > 2$. Consider the cube $(p, p^2, p^i, p^j, p^{2i}, p^{ij}, p^{j2}, p^{2ij})$. We know that $p^i = e^2 + e^i$, $p^j = e^2 + e^j$, $p^{2j} = e^j$ are not in $\bar{S}$. That is because p^i and p^j are both adjacent to p and $p^{2j} = e^j \notin \bar{S}$ by the assumption. Then those are not in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$, because $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\} \subseteq \bar{S}$. By Lemma 4.24, p and w are not adjacent in the skeleton.

It remains to show that q and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$. Note that any vertex of $N_{\bar{S}}[v_2] \setminus \{v_2, q\}$ is either 0 or $e^1 + e^\ell$ for some $\ell > 2$, indicating that $w = e^i + e^j$ is not adjacent to any vertex in $N_{\bar{S}}[v_2]$ in G_n. If w is adjacent to a vertex in $M_{\bar{S}}[v_2] \setminus N_{\bar{S}}[v_2]$, then the possible candidates are e^i and $e^1 + e^i + e^j$ since $e^j \notin \bar{S}$. We know that $e^i + e^j$ is adjacent to e^i. If $e^1 + e^i + e^j \in M_{\bar{S}}[v_2]$, then both $e^1 + e^i$ and $e^1 + e^j$ are in $N_{\bar{S}}[v_2]$. Then three squares $(0, e^1, e^i, e^1 + e^i)$, $(e^1, e^1 + e^i, e^1 + e^j, e^1 + e^i + e^j)$, and $(e^i, e^1 + e^i, e^i + e^j, e^1 + e^i + e^j)$ are contained in $G(\bar{S})$ in this case. However, these three squares form a K_4-minor, so $e^1 + e^i + e^j \notin M_{\bar{S}}[v_2]$. Therefore, e^i is the only vertex of $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$ adjacent to w in G_n.

The square inequalities for squares that have $0e^i$ as an edge are active at w, and the propeller inequality for the propeller that has $0e^i$ as its axis is active at w. We know that $p^i = e^2 + e^i$ is not in $\bar{S}$, so the square $(0, e^2, e^i, e^2 + e^i)$ of G_n is not contained in the propeller. Then $q (= e^1 + e^2)$ is adjacent to at most one square of the propeller, which is possibly $(0, e^1, e^i, e^1 + e^i)$. This means that at most one square inequality is active at both q and w, and the propeller inequality is not active at both. Since the distance in G_n between q and w is 4, at most $n-3$ linearly independent inequalities are active at both q and w. Therefore q and w are not adjacent in the skeleton, as required. ◇

By Claim 6, we may assume that w is not adjacent to $N_{\bar{S}}[v_1] \setminus \{v_1\}$ and that w is adjacent to $M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$.

Claim 7. *w is not adjacent to any of p and q in the skeleton graph of* $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$.

Proof of Claim. As w is adjacent to $M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$, w can be written as $e^i + e^j + e^k$ for some i, j, k where $e^i + e^j \in M_{\bar{S}}[v_1] \setminus N_{\bar{S}}[v_1]$. Then we know that both e^i and e^j are in $\bar{S}$. If i or j is 1, then w is adjacent to a vertex in $N_{\bar{S}}[v_2] \setminus \{v_2\}$. This reduces to the previous case. Thus, we may assume that $i, j > 1$. If i

or j is 2, then p is adjacent to $e^i + e^j \in \bar{S}$. This is impossible. Therefore, i and j are greater than 2. If $k = 1$, then w is $e^1 + e^i + e^j$. Since $0, e^1, e^i, e^j, e^i + e^j, e^1 + e^i + e^j$ are all in $\bar{S}$, both $e^1 + e^i$ and $e^1 + e^j$ are not in $\bar{S}$. Otherwise, $G(\bar{S})$ contains a K_4-minor. Therefore $w(= e^1 + e^i + e^j)$ is adjacent to nothing but $e^i + e^j$ among the vertices of $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$. Then only the square inequality for the square $(0, e^i, e^j, e^i + e^j)$ is active at w. Note that the distance in G_n between p and w is 4 and the distance in G_n between q and w is 3. Then there exist at most $n - 2$ linearly independent inequalities active at both w and each of p and q. Hence, neither p nor q is adjacent to w on the skeleton if $k = 1$. If $k = 2$, then $w = e^2 + e^i + e^j$. Since $p(= e^2)$ is not adjacent to any vertex other than 0 and $e^1 + e^2$, both $p^i(= e^2 + e^i)$ and $p^j(= e^2 + e^j)$ are not in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$. As the case when $k = 1$, q and w are not adjacent in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$. Besides, p and w are not adjacent in the skeleton by Lemma 4.20. Thus, we may assume that $k > 2$. If $e^i + e^k \in M_{\bar{S}}[v_1] \setminus \{p, q\}$, then we know that e^k also belongs to $\bar{S}$ by the definition of $M_{\bar{S}}[v_1]$. In this case, $(0, e^i, e^j, e^i + e^j)$, $(0, e^i, e^k, e^i + e^k)$, and $(e^i, e^i + e^j, e^i + e^k, e^i + e^j + e^k)$ create a K_4-minor in $G(\bar{S})$. Hence, we get that both $e^i + e^k$ and $e^j + e^k$ do not belong to $\bar{S}$. In fact, $e^i + e^j$ is the only vertex in $M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}$ which is adjacent to w in this case. Then only the square inequality for the square $(0, e^i, e^j, e^i + e^j)$ is active at w. Similarly, w is adjacent to neither p nor q in the skeleton in this case. ◇

To summarize, we have just shown that there is no edge connecting a vertex in $\{p, q\}$ and a vertex in $\bar{S} \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])$ in the skeleton of $\mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2] \setminus \{p, q\}))$. Then by Theorem 4.3, we get that

$$\mathrm{conv}(S) = \mathrm{conv}(\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])) \cap \mathrm{conv}(\{0,1\}^n \setminus (\bar{S} \setminus \{p, q\})).$$

Since $G(\bar{S} \setminus \{p, q\})$ is a subgraph of $G(\bar{S})$, it also has tree width 2. Besides, $|\bar{S} \setminus \{p, q\}| < |\bar{S}|$. Hence, the Chvátal rank of $Q_{\{0,1\}^n \setminus (\bar{S} \setminus \{p,q\})}$ is at most 4 by induction. By Lemma 4.23, we also know that the Chvátal rank of $Q_{\{0,1\}^n \setminus (M_{\bar{S}}[v_1] \cup M_{\bar{S}}[v_2])}$ is at most 4, implying in turn that the Chvátal rank of Q_S is also at most 4, as required. □

4.5 Proof of Theorem 1.14

Theorem 1.14 ([36]). *Let $P \subseteq [0,1]^n$ be a rational polytope, and let $S := P \cap \{0,1\}^n$. If the Chvátal rank of Q_S is at most k, then one can optimize a linear function over S in $O(n^k)$ time.*

Proof. The optimization problem is of the form $\min\{cx : x \in S\}$ where $c \in \mathbb{R}^n$. By complementing variables, we may assume $c \geq 0$. By hypothesis, $\mathrm{conv}(S) = Q_S^{(k)}$ for some constant k. We claim that an optimal solution can be found among the 0,1 vectors with at most $k + 1$ nonzero components. This will prove the theorem since there are only polynomially many such vectors. Indeed, if an optimal solution $\bar{x}$ has more than $k + 1$ nonzero components, any 0,1 vector $\bar{z}$ with $\mathrm{supp}(\bar{z}) \subset \mathrm{supp}(\bar{x})$ and $|\mathrm{supp}(\bar{z})| = k + 1$ satisfies $c\bar{z} \leq c\bar{x}$. Because $\mathrm{conv}(S) = Q_S^{(k)}$, Lemma 4.1 implies that the face of H_n of dimension $k + 1$

that contains 0 and $\bar{z}$ contains a feasible point $\bar{y} \in S$. Since $c\bar{y} \leq c\bar{z} \leq c\bar{x}$, the solution $\bar{y}$ is an optimal solution. □

For example, if $G(\bar{S})$ contains no 4-cycle, then the Chvátal rank of Q_S is at most 3 by Theorem 1.11, and therefore, Theorem 1.14 implies that optimizing a linear function over S can be done in $O(n^3)$ time.

Chapter 5

Generalized Chvátal closure

Let $S \subseteq \mathbb{Z}^n$, and let

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} \tag{5.1}$$

be a rational polyhedron where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. We denote by $P_I = P \cap \mathbb{Z}^n$ the integer hull of P. Let Π_P be defined as the set of all vectors that define valid inequalities for P with integral left-hand-side coefficients:

$$\Pi_P = \{(\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{R} \ : \ \exists \lambda \in \mathbb{R}^m_+ \text{ s.t. } \alpha = \lambda A,\ \beta \geq \lambda b\}. \tag{5.2}$$

and denote by Π^*_P the subset of Π_P that consists of the vectors defining supporting valid inequalities:

$$\Pi^*_P = \{(\alpha, \beta) \in \Pi_P \ : \ \beta = \max\{\alpha x : x \in P\}\}. \tag{5.3}$$

Given $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{R}$, recall that $\lfloor \beta \rfloor_{S,\alpha}$ is defined as follows:

$$\lfloor \beta \rfloor_{S,\alpha} = \begin{cases} \max\{\alpha z : \ z \in S,\ \alpha z \leq \beta\} & \text{if } \{z \in S : \ \alpha z \leq \beta\} \neq \emptyset \\ -\infty \quad \text{otherwise} \end{cases}$$

Given an inequality $\alpha x \leq \beta$ with $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{R}$ valid for P, we call $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$ the S-Chvátal-Gomory inequality for P obtained from $\alpha x \leq \beta$. Recall that the S-Chvátal closure of P, denoted P_S, is defined as the following:

$$P_S := \bigcap_{(\alpha,\beta) \in \Pi_P} \{x \in \mathbb{R}^n : \ \alpha x \leq \lfloor \beta \rfloor_{S,\alpha}\} = \bigcap_{(\alpha,\beta) \in \Pi^*_P} \{x \in \mathbb{R}^n : \ \alpha x \leq \lfloor \beta \rfloor_{S,\alpha}\}. \tag{5.4}$$

Hereinafter, we refer to a Chvátal-Gomory inequality (resp. cut) as a CG inequality (resp. cut) and refer to a S-Chvátal-Gomory inequality (resp. cut) as an S-CG inequality (resp. cut).

In this chapter, we study the following question:

Question. *Let $S = R \cap \mathbb{Z}^n$ for some rational polyhedron R, and let $P \subseteq \text{conv}(S)$ be a rational polyhedron. Is the S-Chvátal closure of P a rational polyhedron?*

In § 5.1, we prove some basic tools that are useful throughout this chapter and prove Proposition 1.17. In § 5.2, we study the question for the case when S is finite. In § 5.3, we consider the case when $S = T \times \mathbb{Z}^{n_2}$ for some finite $T \subseteq \mathbb{Z}^{n_1}$. In § 5.4, we study the case when

$$S = \left\{ \left(z^1, z^{2\ell}, z^{2u}, z^3\right) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_{2\ell}} \times \mathbb{Z}^{n_{2u}} \times \mathbb{Z}^{n_3} : \; z^1 \in T, \; z^{2\ell} \geq \ell^2, \; z^{2u} \leq u^2 \right\}$$

where $T \in \mathbb{R}^{n_1}$ is finite, $\ell^2 \in \mathbb{R}^{2\ell}$, $u^2 \in \mathbb{R}^{2u}$. We prove Theorem 1.16 in § 5.5. This chapter is based on [52].

5.1 Preliminaries

In this section, we prove some basic properties of the S-CG inequalities and the S-Chvátal clousre of a polyhedron.

Remark 5.1 (Pokutta [102]). *Let $S \subseteq \mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then the following statements hold:*

(1) $P_I \subseteq P_S \subseteq P$,

(2) if $S \subseteq S'$ for some $S' \subseteq \mathbb{Z}^n$, then $P_S \subseteq P_{S'}$,

(3) if Q is a rational polyhedron such that $P \subseteq Q$, then $P_S \subseteq Q_S$.

For $\Gamma \subseteq \Pi_P$, we define $P_{S,\Gamma}$ as follows:

$$P_{S,\Gamma} := \bigcap_{(\alpha,\beta)\in\Gamma} \left\{x \in P : \; \alpha x \leq \lfloor \beta \rfloor_{S,\alpha}\right\}.$$

Clearly, $P_S = P_{S,\Pi_P}$. We remark the following:

Remark 5.2 ([52]). *Let $S \subseteq \mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then the following statements hold:*

(1) if $\Gamma \subseteq \Pi_P$, then $P_S \subseteq P_{S,\Gamma}$,

(2) if $\Gamma \subseteq \Pi_P$ and $\Gamma = \bigcup_{i=1}^k \Gamma_i$, then $P_{S,\Gamma} = \bigcap_{i=1}^k P_{S,\Gamma_i}$.

In particular, to prove that $P_{S,\Gamma}$ is a rational polyhedron where $\Gamma = \bigcup_{i=1}^k \Gamma_i$, it suffices by Remark 5.2 to show that P_{S,Γ_i} is a rational polyhedron for $i \in [k]$. Remark 5.2 will be useful in § 5.3 and § 5.4.

Examples

We next present two simple examples to highlight the difference between regular CG cuts and S-CG cuts. The first example below highlights the strength of S-CG cuts.

Example 1 ([52])**.** Consider a rational polyhedron $P \subseteq \mathbb{R}^2$ such that the inequality $3x+5y \geq 3.4$ is valid. Clearly, the associated CG cut $3x + 5y \geq 4$ is valid for $P \cap \mathbb{Z}^n$. Notice that the CG cut is tight at point $(3, -1)$. Now, consider $S = \left\{x \in \mathbb{Z}^2 : 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 3\right\}$, and note that $(3, -1) \notin S$. In fact, the S-CG cut $3x + 5y \geq 5$ obtained from $3x + 5y \geq 3.4$ is valid for $P \cap S$. The S-CG cut is tight at point $(0, 1) \in S$. See Figure 5.1 for an illustration.

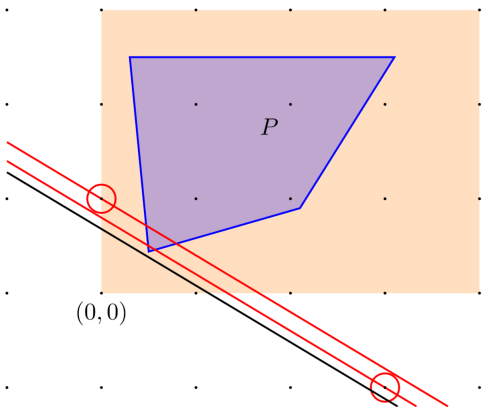


Figure 5.1: Illustration of an S-CG inequality

The next example highlights the fact that the S-Chvátal closure of a polyhedron can have facets that are not necessarily defined by S-CG cuts. In the following example, a sequence of S-CG cuts converge to an inequality that is not an S-CG cut itself.

Proposition 1.17 ([52])**.** *Let $S = \{0, 1\}^4$. There exists a polytope $P \subseteq [0, 1]^4$ whose S-Chvátal closure has a facet that cannot be induced by an S-CG inequality.*

Proof. Let $S = \{0, 1\}^4$, and let P be the convex hull of the following six points in $[0, 1]^4$:

$$P = \operatorname{conv}\left\{(1/2, 0, 0, 0)\,, (1, 0, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 1)\right\}.$$

Observe that $2x_1 + x_2 + x_3 + x_4 \geq 1$ is a valid inequality for P and is tight at the vertex $(1/2, 0, 0, 0)$. As the point $(0, 1, 0, 0) \in S$ satisfies $2x_1 + x_2 + x_3 + x_4 = 1$, one cannot obtain $2x_1 + x_2 + x_3 + x_4 \geq 2$ as an S-CG cut. However, we claim that $2x_1 + x_2 + x_3 + x_4 \geq 2$ is valid for the S-Chvátal closure of P. Note that for any $\delta > 0$, the inequality $2x_1 + (1 - \delta)x_2 + (1 - \delta)x_3 + (1 - \delta)x_4 \geq 1$ is valid for P as it is satisfied by all its vertices. Moreover, any point $x^* \in S$ that satisfies this inequality either has $x^*_1 = 1$ or $x^*_2 + x^*_3 + x^*_4 \geq 2$. Therefore, the smallest value of $2x_1 + (1 - \delta)x_2 + (1 - \delta)x_3 + (1 - \delta)x_4$ at such points in S is exactly $2 - 2\delta$. Therefore,

$$2x_1 + (1 - \delta)x_2 + (1 - \delta)x_3 + (1 - \delta)x_4 \geq 2 - 2\delta$$

is an S-CG cut. Taking the limit of this inequality as $\delta \to 0$, we can infer that $2x_1 + x_2 + x_3 + x_4 \geq 2$ is valid for P_S. As this inequality is facet-defining for P_I, it is also facet-defining for $P_S \supseteq P_I$. □

We next illustrate this fact in Figure 5.2, where $S \subseteq \mathbb{Z}^2$ is the set of black points, and $P \subseteq \mathbb{R}^2$ is the blue triangle. Observe that the closure has a facet that is not defined by an S-CG cut. The supporting hyperplane for P (which is parallel to this inequality – depicted with a thick line) also touches a point in S.

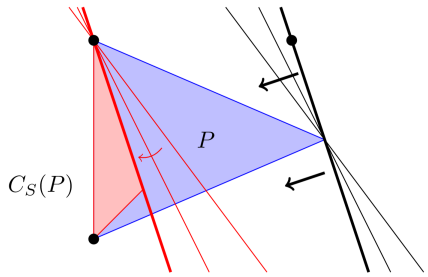


Figure 5.2: Some facets are not defined by S-CG inequalities

The polar lemma

We next show an important property of closures of polyhedra with respect to an infinite family of valid inequalities. The following lemma will be useful:

Lemma 5.3 (Polar lemma; see [55]). *Let $P \subseteq \mathbb{R}^n$ and $H \subseteq \mathbb{R}^{n+1}$ be rational polyhedra. Assume that $H \cap \mathbb{Z}^{n+1}$ is nonempty and is contained in the recession cone of H, denoted* $\mathrm{rec}(H)$*. Then*

$$\bigcap_{(\alpha,\beta)\in H\cap\mathbb{Z}^{n+1}} \{x \in P : \alpha x \leq \beta\} = \bigcap_{(\alpha,\beta)\in \mathrm{rec}(H)} \{x \in P : \alpha x \leq \beta\}. \tag{5.5}$$

Moreover, both sets are rational polyhedra.

Proof. By Meyer's Theorem [99], as $H \cap \mathbb{Z}^{n+1}$ is nonempty, $\mathrm{conv}(H \cap \mathbb{Z}^{n+1})$ is a rational polyhedron and has the same recession cone as H, namely $\mathrm{rec}(H)$. Let P_1 denote the set on the left-hand-side of equation (5.5), and let P_2 denote the right-hand-side set. As $H \cap \mathbb{Z}^{n+1} \subseteq \mathrm{rec}(H)$, P_2 is a subset of P_1. We will show, by contradiction, that for any $(\alpha, \beta) \in \mathrm{rec}(H)$, $\alpha x \leq \beta$ is valid for P_1, thereby proving that $P_1 \subseteq P_2$. Assume this is false. Then there exist $(\alpha, \beta) \in \mathrm{rec}(H)$ and $\bar{x} \in P_1$ such that $\alpha\bar{x} > \beta$. Consider an arbitrary $(\alpha^0, \beta^0) \in H \cap \mathbb{Z}^{n+1}$; then $\alpha^0\bar{x} \leq \beta^0$ as $\bar{x} \in P_1$. Therefore, we can choose a positive μ such that $\mu(\alpha\bar{x} - \beta) > \beta^0 - \alpha^0\bar{x}$. So, we have

$$(\alpha^0 + \mu\alpha)\bar{x} > \beta^0 + \mu\beta. \tag{5.6}$$

On the other hand, since $(\alpha^0, \beta^0) \in H \cap \mathbb{Z}^{n+1} \subseteq \text{conv}(H \cap \mathbb{Z}^{n+1})$ and $(\alpha, \beta) \in \text{rec}(H) = \text{rec}(\text{conv}(H \cap \mathbb{Z}^{n+1}))$, it follows that $(\alpha^0, \beta^0) + \mu(\alpha, \beta) \in \text{conv}(H \cap \mathbb{Z}^{n+1})$. Every vector of $H \cap \mathbb{Z}^{n+1}$ defines a valid inequality for P_1, and – by convexity – so does every vector of $\text{conv}(H \cap \mathbb{Z}^{n+1})$, implying in turn that $(\alpha^0 + \mu\alpha)\bar{x} \leq \beta^0 + \mu\beta$, a contradiction to (5.6). Therefore, $P_1 = P_2$.

To complete the proof, we show that P_2 is a rational polyhedron. As H is a rational polyhedron, $\text{rec}(H)$ is a rational polyhedral cone, and therefore, there exist $(\alpha^1, \beta^1), \ldots, (\alpha^r, \beta^r) \in \text{rec}(H) \cap \mathbb{Q}^{n+1}$ such that any $(\alpha, \beta) \in \text{rec}(H)$ can be written as a conic combination of these vectors. Therefore, P_2 is equal to $\{x \in P : \alpha^i x \leq \beta^i, i = 1, \ldots, r\}$, so P_2 is a rational polyhedron, as required. □

By Lemma 5.3, it suffices to argue the existence of a rational polyhedron $H \subseteq \mathbb{R}^{n+1}$ such that one can obtain the S-Chvátal closure of a rational polyhedron $P \subseteq \mathbb{R}^n$ after applying $\alpha x \leq \beta$ for $(\alpha, \beta) \in H \cap \mathbb{Z}^{n+1}$.

5.2 S-Chvátal closure for finite number of integer points

Recall that Theorem 1.15 by Dunkel and Schulz [55] states that the S-Chvátal closure of a rational polytope contained in the unit hypercube is polyhedral for $S = \{0, 1\}^n$. We extend this result to the case when S is any arbitrary finite subset of $\mathbb{Z}^n$.

Theorem 5.4 ([52]). *Let S be a finite subset of $\mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then the S-Chvátal closure P_S is a rational polyhedron.*

Proof. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a rational polyhedron, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Without loss of generality, we may assume that $P_S \neq \emptyset$ and that P_S is properly contained in P. Recall that P_S is described by the S-CG cuts obtained from vectors in Π_P^* (5.4). Take an $(\alpha, \beta) \in \Pi_P^*$. The S-CG cut $\alpha x \leq \lfloor \beta \rfloor_{S,\alpha}$ derived from $\alpha x \leq \beta$ partitions S into the following two sets:

$$L = \{z \in S : \alpha z \leq \lfloor \beta \rfloor_{S,\alpha}\} \quad \text{and} \quad G = \{z \in S : \alpha z > \beta\}.$$

Moreover, as S is finite, there is a finite number of such partitions. Therefore,

$$P_S = \bigcap_{(L,G) \in \Pi(S)} P_{(L,G)}$$

where $\Pi(S)$ is the family of all possible partitions of S and $P_{(L,G)}$ is the set obtained after applying all S-CG cuts that partition S into L and G. It is possible that, for some partition (L, G), there might not be any valid inequalities that partition S into L and G. In such a case, we let $P_{(L,G)} = P$.

Claim. *Let (L, G) be a partition of S. Then $P_{(L,G)} = \{x \in P : \alpha x \leq \gamma, \ \forall (\alpha, \gamma) \in H_{(L,G)}\}$, where*

$$H_{(L,G)} = \left\{ (\alpha, \gamma) \in \mathbb{Z}^{n+1} : \exists (\beta, \lambda) \in \mathbb{R} \times \mathbb{R}^m_+ \text{ s.t. } \begin{array}{rll} (\alpha, \beta) & = (\lambda A, \lambda b), & \\ \alpha z & \leq \gamma, & \forall z \in L \\ \alpha z & \geq \beta + \frac{1}{\Delta}, & \forall z \in G \\ \gamma & \leq \beta & \end{array} \right\}$$

and Δ is the product of all distinct sub-determinants of A.

Proof of Claim. Take an $(\alpha, \beta) \in \Pi_P^*$ such that $\alpha z \leq \lfloor \beta \rfloor_{S,\alpha}$ for all $z \in L$ and $\alpha z > \beta$ for all $z \in G$. We will argue that $(\alpha, \lfloor \beta \rfloor_{S,\alpha})$ is contained in the set $H_{(L,G)}$. As $(\alpha, \beta) \in \Pi_P^*$, we have $\alpha \in \mathbb{Z}^n$ and $\beta = \max\{\alpha x : x \in P\}$. Then there is an $\lambda \in \mathbb{R}_+^m$ such that $(\alpha, \beta) = (\lambda A, \lambda b)$. If G is empty, then $(\alpha, \lfloor \beta \rfloor_{S,\alpha}) \in H_{(L,G)}$. Alternatively, if $G \neq \emptyset$, then as $\beta = \max\{\alpha x : x \in P\}$ where α is integral, β has to be an integral multiple of $\frac{1}{\Delta}$. This implies that for any integral point z, if $\beta < \alpha z$ then $\beta \leq \alpha z - \frac{1}{\Delta}$. Hence, for all $z \in G$, we have $\beta \leq \alpha z - \frac{1}{\Delta}$, and therefore, $\alpha, \beta, \lfloor \beta \rfloor_{S,\alpha}, \lambda$ satisfy the constraints describing $H_{(L,G)}$. Consequently, $(\alpha, \lfloor \beta \rfloor_{S,\alpha}) \in H_{(L,G)}$, as desired.

In fact, for any $(\alpha, \gamma) \in H_{(L,G)}$, it can be proved that $\alpha x \leq \gamma$ is a valid S-CG cut. That is because $\alpha, \beta, \gamma, \lambda$ for some β, λ satisfy the constraints describing $H_{(L,G)}$, so it follows that $\alpha x \leq \beta$ is valid for P, and as there is no point $z \in S$ such that $\beta \geq \alpha z > \gamma$, $\alpha x \leq \gamma$ is a valid S-CG cut. This implies that $P_{(L,G)} = \{x \in P : \ \alpha x \leq \gamma, \ \forall (\alpha, \gamma) \in H_{(L,G)}\}$, as required. ◇

The recession cone of the linear programming relaxation of $H_{(L,G)}$ is

$$C_{(L,G)} = \left\{ (\alpha, \gamma) \in \mathbb{R}^{n+1} : \ \exists (\beta, \lambda) \in \mathbb{R} \times \mathbb{R}_+^m \text{ s.t. } \begin{array}{rll} (\alpha, \beta) & = (\lambda A, \lambda b), & \\ \alpha z & \leq \gamma, & \forall z \in L \\ \alpha z & \geq \beta, & \forall z \in G \\ \gamma & \leq \beta & \end{array} \right\},$$

and as $\frac{1}{\Delta} \geq 0$, we have $H_{(L,G)} \subseteq C_{(L,G)}$. Then Lemma 5.3 implies that $\{x \in P : \ \alpha x \leq \gamma, \ \forall (\alpha, \gamma) \in H_{(L,G)}\} = \{x \in P : \ \alpha x \leq \gamma, \ \forall (\alpha, \gamma) \in C_{(L,G)}\}$ and that $P_{(L,G)}$ is a rational polyhedron. Recall that P_S is the intersection of $P_{(L,G)}$ for all partitions (L, G) of S such that $H_{(L,G)} \neq \emptyset$. Since the number of such partitions (L, G) of S is finite, P_S is a rational polyhedron. □

As a direct corollary of Theorem 5.4, we obtain the following:

Corollary 5.5 ([52]). *Let $S = [\ell, u] \cap \mathbb{Z}^n$ for some $\ell, u \in \mathbb{Z}^n$ such that $\ell \leq u$, and let $P \subseteq [\ell, u]$ be a rational polyhedron. Then, P_S is a rational polyhedron.*

Notice that the set $C_{(L,G)}$ in the proof of Theorem 5.4 might strictly contain the set $H_{(L,G)}$. Therefore, for some $\alpha, \beta, \gamma, \lambda$ that satisfy the constraints describing $C_{(L,G)}$, we might have a point $z \in G$ that satisfies $\alpha z = \beta$. In this case, $\lfloor \beta \rfloor_{S,\alpha} = \beta > \gamma$ and therefore the inequality $\alpha x \leq \gamma$ cannot be obtained as an S-CG cut from $\alpha x \leq \beta$. In the example in the proof of Proposition 1.17, the limiting inequality that is facet-defining for the S-Chvátal closure but is not an S-CG cut precisely falls into this category.

5.3 Integer points in a cylinder

In § 5.2, we showed that P_S is a rational polyhedron if S is a finite subset of $\mathbb{Z}^n$ and P is a rational polyhedron. In this section, we consider the case where

$$S = T \times \mathbb{Z}^l \text{ for some finite } T \subseteq \mathbb{Z}^n, \tag{5.7}$$

$$P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^l : Ax + Cy \leq b\}, \tag{5.8}$$

and the matrices A, C, b have integral entries and m rows and $n, l, 1$ columns, respectively. For this case we will prove that P_S is a rational polyhedron.

As before, let Π_P be the set of all vectors that define (supporting) valid inequalities for P with integral left-hand-side coefficients:

$$\begin{aligned}\Pi_P = \{(\alpha,\gamma,\beta) \in \mathbb{Z}^n \times \mathbb{Z}^l \times \mathbb{R} \ : \ \exists \lambda \in \mathbb{R}^m_+ \ \text{ s.t. } \ (\alpha,\gamma,\beta) &= (\lambda A, \lambda C, \lambda b), \\ \beta &= \max\{\alpha x + \gamma y : (x,y) \in P\}\}. \quad (5.9)\end{aligned}$$

Clearly, Π_P can be partitioned into the sets Π_0 and $\Pi_P \setminus \Pi_0$ where

$$\Pi_0 = \{(\alpha,\gamma,\beta) \in \Pi_P : \gamma = \mathbf{0}\}. \quad (5.10)$$

In (5.10), $\mathbf{0}$ is the vector of all zeros of appropriate dimension. By Remark 5.2, $P_S = P_{S,\Pi_0} \cap P_{S,\Pi_P \setminus \Pi_0}$. To prove that P_S is a rational polyhedron, we will first argue that P_{S,Π_0} is a rational polyhedron. This result follows from the lemma below, which will also be used in § 5.4.

Lemma 5.6 (**Projection lemma [52]**). *Let T, S and P be defined as*

$$S = T \times \mathbb{Z}^l \text{ for some } T \subseteq \mathbb{Z}^n \quad \text{and} \quad P = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^l : Ax + Cy \le b\},$$

and the matrices A, C, b have integral components and m rows and $n, l, 1$ columns, respectively. Let $\Gamma \subseteq \Pi_0$, and let $\Omega = \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R} : (\alpha, \mathbf{0}, \beta) \in \Gamma\}$. If $Q = \text{proj}_x(P)$, then,

$$P_{S,\Gamma} = P \cap \left(Q_{T,\Omega} \times \mathbb{R}^l\right).$$

Proof. We first argue that $Q_{T,\Omega} = \text{proj}_x(P_{S,\Gamma})$. For any $(\alpha,\beta) \in \Omega$ (i.e., $(\alpha, \mathbf{0}, \beta) \in \Gamma$), we have

$$\lfloor \beta \rfloor_{T,\alpha} = \max\{\alpha x : x \in T, \alpha x \le \beta\} = \max\{\alpha x : (x,y) \in S, \alpha x \le \beta\} = \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}.$$

Let $(x,y) \in P_{S,\Gamma}$. Then for any $(\alpha,\beta) \in \Omega$, we have $\alpha x \le \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}$ and thus $\alpha x \le \lfloor \beta \rfloor_{T,\alpha}$, implying in turn that $x \in Q_{T,\Omega}$. Conversely, let $x \in Q_{T,\Omega}$. As $x \in Q$, there exists $y \in \mathbb{R}^l$ such that $(x,y) \in P$. Then for any $(\alpha, \mathbf{0}, \beta) \in \Gamma$, we have $\alpha x \le \lfloor \beta \rfloor_{T,\alpha}$ and thus $\alpha x \le \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}$, which in turn implies that $(x,y) \in P_{S,\Gamma}$. Therefore, $Q_{T,\Omega} = \text{proj}_x(P_{S,\Gamma})$, and it follows that

$$P_{S,\Gamma} \subseteq P \cap (Q_{T,\Omega} \times \mathbb{R}^l).$$

Suppose for a contradiction that $P_{S,\Gamma} \ne P \cap (Q_{T,\Omega} \times \mathbb{R}^l)$. Then there exists a point $(\bar{x},\bar{y}) \in P$ such that $\bar{x} \in Q_{T,\Omega}$ and $(\bar{x},\bar{y}) \notin P_{S,\Gamma}$. Since $(\bar{x},\bar{y}) \in P \setminus P_{S,\Gamma}$, there must exist some $(\alpha, \mathbf{0}, \beta) \in \Gamma$ such that $\alpha\bar{x} > \lfloor \beta \rfloor_{S,(\alpha,\mathbf{0})}$ and therefore $\alpha\bar{x} > \lfloor \beta \rfloor_{T,\alpha}$, a contradiction as $\bar{x} \in Q_{T,\Omega}$. Therefore, $P_{S,\Gamma} = P \cap (Q_{T,\Omega} \times \mathbb{R}^l)$, as required. □

Notice that $T \subseteq \mathbb{Z}^n$ in the statement in Lemma 5.6 does not need to be finite.

Lemma 5.7 ([52]). *Let S and P be defined as in (5.7)–(5.8), and let Π_0 be defined as in (5.10). Then P_{S,Π_0} is a rational polyhedron.*

Proof. Let $\Omega = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} : (\alpha, \mathbf{0}, \beta) \in \Pi_0\}$, and let $Q = \text{proj}_x(P)$. Then it follows that $\Omega = \Pi_Q$, and therefore, $Q_{T,\Omega} = Q_T$. So, Theorem 5.4 implies that $Q_{T,\Omega}$ is a rational polyhedron. Moreover, by Lemma 5.6, $P_{S,\Pi_0} = P \cap \left(Q_{T,\Omega} \times \mathbb{R}^l\right)$, implying in turn that P_{S,Π_0} is a rational polyhedron. □

The Chvátal closure of P is described by $\lambda Ax + \lambda Cy \leq \lfloor \lambda b \rfloor$ for $\lambda \in \mathbb{R}^m_+$ such that $(\lambda A, \lambda C) \in \mathbb{Z}^n$ and $\mathbf{0} \leq \lambda \leq \mathbf{1}$ [107]. So, a CG cut for a polyhedron is dominated by CG cuts obtained via bounded multipliers. For convention, we assume that an inequality dominates, or is dominated by, itself. The next result for S-CG cuts is analogous to this result. We define the following constant U that depends on P and T as follows:

$$U = \max \left\{ \mathbf{1}^\top |b - Ax| : x \in T \right\}. \tag{5.11}$$

where $|b - Ax|$ denotes the vector whose entries are the absolute values of the entries of $b - Ax$. Given a vector γ, let g.c.d.(γ) denote the greatest common divisor of the entries of γ.

Lemma 5.8 ([52]). *Let S, T, P and Π_P be defined as in (5.7)–(5.9). Then for any $(\alpha, \gamma, \beta) \in \Pi_P$, there exists $(\alpha', \gamma', \beta') \in \Pi_P$ that satisfies the following:*

(1) the S-CG cut derived from $(\alpha', \gamma', \beta')$ dominates the S-CG cut derived from (α, γ, β),

(2) either $\gamma' = \mathbf{0}$ or, letting $g' = $ g.c.d.(γ'), there exists $\mu \in \mathbb{R}^m$ with $\mathbf{0} \leq \mu < g'\mathbf{1}$ such that (a) $(\alpha', \gamma', \beta') = (\mu A, \mu C, \mu b)$ and (b) $|\beta' - \alpha' x| \leq g' U$ for all $x \in T$.

Proof. Let $(\alpha, \gamma, \beta) \in \Pi_P$. By the definition of Π_P, we have $(\alpha, \gamma, \beta) = (\lambda A, \lambda C, \lambda b)$ for some $\lambda \in \mathbb{R}^m_+$, and α, γ are integral vectors. If $\gamma = \mathbf{0}$, then the S-CG cut derived from $(\alpha, \gamma, \beta) = (\alpha, \mathbf{0}, \beta)$ dominates itself. Thus we assume that $\gamma \neq \mathbf{0}$. Let g denote g.c.d.(γ). If $\lambda_i < g$ for $i = 1, \ldots, m$, then $(\alpha', \beta', \gamma') = (\alpha, \beta, \gamma)$ is the desired vector as $|\beta - \alpha x| \leq g\mathbf{1}^\top |b - Ax|$, and therefore, we may assume that this is not the case. Let $\delta, \mu \in \mathbb{R}^m$ be defined by $\delta_i = g\lfloor \lambda_i / g \rfloor$ and $\mu = \lambda - \delta$. Clearly, $\delta_i \geq 0$ and $0 \leq \mu_i < g$ for each $i \in \{1, \ldots, m\}$ (here $\mu_i \equiv \lambda_i \pmod g$). Let $(\alpha', \gamma', \beta') = \mu(A, C, b)$. Then $\alpha' x + \gamma' y \leq \beta'$ is also a valid inequality.

Claim. $\lfloor \beta' \rfloor_{S,(\alpha',\gamma')} + \delta b \leq \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$.

Proof of Claim. Let $u = (\alpha, \gamma)$ and $v = (\alpha', \gamma')$. Suppose for a contradiction that $\lfloor \beta' \rfloor_{S,v} + \delta b > \lfloor \beta \rfloor_{S,u}$. Then

$$\lfloor \beta \rfloor_{S,u} < \delta b + \lfloor \beta' \rfloor_{S,v} = \beta - (\beta' - \lfloor \beta' \rfloor_{S,v}) \leq \beta. \tag{5.12}$$

As $\lfloor \beta' \rfloor_{S,v}$ is finite, there exists $(\bar{x}, \bar{y}) \in S$ such that $\alpha' \bar{x} + \gamma' \bar{y} = \lfloor \beta' \rfloor_{S,v}$, implying in turn that

$$\delta b + \lfloor \beta' \rfloor_{S,v} = \delta b + (\alpha - \delta A)\bar{x} + (\gamma - \delta C)\bar{y}.$$

Substituting this expression in (5.12) and rearranging terms, we get

$$\lfloor \beta \rfloor_{S,u} - \alpha \bar{x} < \delta b - \delta A \bar{x} + (\gamma - \delta C)\bar{y} \leq \beta - \alpha \bar{x}. \tag{5.13}$$

As all components of the vectors δ and γ are multiples of g, and $A, C, b, \bar{x}, \bar{y}$ are all integral, the expression

$$\frac{1}{g}(\delta b - \delta A \bar{x} + (\gamma - \delta C)\bar{y}) \tag{5.14}$$

is an integer. Since $\frac{1}{g}\gamma$ is an integral vector with g.c.d.$\left(\frac{1}{g}\gamma\right) = 1$, there exists $\hat{y} \in \mathbb{Z}^l$ such that $\frac{1}{g}\gamma\hat{y}$ is equal to the integer in (5.14), or equivalently

$$\gamma\hat{y} = \delta b - \delta A\bar{x} + (\gamma - \delta C)\bar{y}.$$

Substituting the right-hand-side of the above equation by $\gamma\hat{y}$ in (5.13), we obtain

$$\lfloor\beta\rfloor_{S,u} - \alpha\bar{x} < \gamma\hat{y} \leq \beta - \alpha\bar{x}$$

which implies that

$$\lfloor\beta\rfloor_{S,u} < \alpha\bar{x} + \gamma\hat{y} \leq \beta.$$

As $(\bar{x}, \hat{y}) \in S$, we get a contradiction. Therefore, it follows that $\lfloor\beta'\rfloor_{S,v} + \delta b \leq \lfloor\beta\rfloor_{S,u}$, as required. ◇

Adding $\delta(Ax + Cy) \leq \delta b$ to $\alpha' x + \gamma' y \leq \lfloor\beta'\rfloor_{S,(\alpha',\gamma')}$, we obtain $\alpha x + \gamma y \leq \lfloor\beta'\rfloor_{S,(\alpha',\gamma')} + \delta b$, implying in turn that $\alpha x + \gamma y \leq \lfloor\beta\rfloor_{S,(\alpha,\gamma)}$ is dominated by the inequality $\alpha' x + \gamma' y \leq \lfloor\beta'\rfloor_{S,(\alpha',\gamma')}$. If $\gamma' = \mathbf{0}$, the proof is complete. If $\gamma' \neq \mathbf{0}$, then we note that all components of γ' are multiples of g as $\gamma' = \gamma - \delta$ and all components of γ and δ are multiples of g. Therefore, g.c.d.$(\gamma') = g' = kg$ for some positive integer k and as $0 \leq \mu_i < g$, we have $0 \leq \mu_i < g'$, for all $i = 1, \ldots, m$ and (a) holds. To see that (b) also holds, note that $\beta' - \alpha' x = \mu b - \mu A x = \mu(b - Ax)$ for all $x \in T$. As A and b are fixed, and T is a finite set of integers, and $\mathbf{0} \leq \mu < g'\mathbf{1}$, the result follows with U defined in (5.11). □

Using Lemma 5.8, we can prove the following theorem:

Theorem 5.9 ([52]). *Let $S = T \times \mathbb{Z}^l$ for some finite $T \subseteq \mathbb{Z}^n$, and let $P \subseteq \mathbb{R}^{n+l}$ be a rational polyhedron. Then P_S is a rational polyhedron.*

Proof. If $P_S = \emptyset$, then P_S is trivially polyhedral. Thus, we may assume that $P_S \neq \emptyset$ and that $P \cap \text{conv}(S) \neq \emptyset$. Let P, Π_P and Π_0 be defined as in (5.7)–(5.10). Remark 5.2 implies that $P_S = P_{S,\Pi_0} \cap P_{S,\Pi_P \setminus \Pi_0}$, and Lemma 5.7 implies that P_{S,Π_0} is a rational polyhedron.

Let $\Theta = \mathbb{Z}^l \cap \{\delta C : \mathbf{0} \leq \delta \leq \mathbf{1}\}$, and let $T = \{x^1, \ldots, x^{|T|}\}$ and $I = \{1, \ldots, |T|\}$. Let U be defined as in (5.11). Given $\mu \in \Theta$ and $\ell \in [-U, U]^{|T|}$, we define $H_{(\mu,\ell)}$ as follows:

$$H_{(\mu,\ell)} := \left\{ \begin{array}{l} (\alpha,\gamma,\delta) \\ \quad \in \mathbb{Z}^{n+l+1} \end{array} : \exists \begin{pmatrix} \beta \in \mathbb{R} \\ \lambda \in \mathbb{R}^m_+ \\ g \in \mathbb{Z} \end{pmatrix} \text{ s.t. } \begin{array}{rcll} (\alpha,\gamma,\beta) & = & (\lambda A, \lambda C, \lambda b), & \\ \alpha x^i + g(\ell_i + 1) - \frac{1}{\Delta} & \geq & \beta, & \forall i \in I, \\ \alpha x^i + g\ell_i & \leq & \delta, & \forall i \in I, \\ \lambda & \leq & g\mathbf{1}, & \\ \delta & \leq & \beta, & \\ \gamma & = & g\mu, & \\ g & \geq & 1 & \end{array} \right\}$$

Claim 1. *Let $(\alpha,\gamma,\beta) \in \Pi^*_P \setminus \Pi_0$. Then $(\alpha,\gamma,\lfloor\beta\rfloor_{S,(\alpha,\gamma)}) \in H_{(\mu,\ell)}$ for some $\mu \in \Theta$ and $\ell \in [-U,U]^{|T|}$.*

Proof of Claim. Take a vector $(\alpha,\gamma,\beta) \in \Pi^*_P \setminus \Pi_0$ whose corresponding S-CG cut $\alpha x + \gamma y \leq \lfloor\beta\rfloor_{S,(\alpha,\gamma)}$ is dominated by no other S-CG cut. Then $\gamma \neq \mathbf{0}$ and $\beta = \max\{\alpha x + \gamma y : (x,y) \in P\}$. Moreover, by Lemma 5.8, we may assume that g.c.d.$(\gamma) = g$ for some $g \in \mathbb{Z}$, and $(\alpha,\gamma,\beta) = \lambda(A,C,b)$ for some

λ such that $\mathbf{0} \leq \lambda < g\mathbf{1}$. As $\gamma/g = (\lambda/g)C$ is an integral vector and $\mathbf{0} \leq \lambda/g < \mathbf{1}$, we see that $\gamma/g \in \Theta = \mathbb{Z}^l \cap \{\delta C : \mathbf{0} \leq \delta \leq \mathbf{1}\}$. Therefore, $\gamma = g\mu$ for some $\mu \in \Theta$.

By our choice of U in (5.11), for each $i \in I$, there exists an integer $\ell_i \in [-U, U]$ such that

$$g\ell_i \leq \beta - \alpha x^i < g(\ell_i + 1). \tag{5.15}$$

As $\beta = \max\{\alpha x + \gamma y : (x, y) \in P\}$ is finite, the maximum is achieved at a point in a minimal face of P. We may assume that this point is rational with all denominators of its components equal to a subdeterminant of (A, C). Therefore, β is an integer multiple of $\frac{1}{\Delta}$ for some $\Delta > 0$ that only depends on the data in (A, C). Hence, $\beta \leq \alpha x^i + g(\ell_i + 1) - \frac{1}{\Delta}$ for all $i \in I$. Let ℓ denote the vector whose entries are ℓ_i, $i \in I$. As the components of $\mu = \frac{1}{g}\gamma$ are relatively prime, we can find a vector $y^i \in \mathbb{Z}^l$ such that $\mu y^i = \ell_i$ for all $i \in I$. So, $\gamma y^i = g\ell_i$, and it follows from (5.15) that $\alpha x^i + \gamma y^i \leq \beta$. Since $(x^i, y^i) \in S$, we have that $\alpha x^i + g\ell_i \leq \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$. Therefore, $(\alpha, \gamma, \lfloor \beta \rfloor_{S,(\alpha,\gamma)}) \in H_{(\mu,\ell)}$, as required. ◇

Claim 2. *Let $\mu \in \Theta$ and $\ell \in [-U, U]^{|T|}$. Then $\alpha x + \gamma y \leq \delta$ for every $(\alpha, \gamma, \delta) \in H_{(\mu,\ell)}$ is valid for P_S.*

Proof of Claim. As $(\alpha, \gamma, \delta) \in H_{(\mu,\ell)}$, there exists some $\beta \geq \delta$ such that the inequality $\alpha x + \gamma y \leq \beta$ is valid for P. Moreover, $\delta \geq \max\{\alpha x^i + g\ell_i : i \in I\}$. Suppose for a contradiction that $\max\{\alpha x^i + g\ell_i : i \in I\} < \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$. Then $\alpha x^i + g\ell_i < \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$ for all $i \in I$. As δ is finite, so is $\lfloor \beta \rfloor_{S,(\alpha,\gamma)}$, and therefore $\lfloor \beta \rfloor_{S,(\alpha,\gamma)} = \alpha x^k + \gamma y^*$ for some $k \in I$ and $y^* \in \mathbb{Z}^l$. This implies that

$$\alpha x^k + g\ell_k < \lfloor \beta \rfloor_{S,(\alpha,\gamma)} = \alpha x^k + \gamma y^* \leq \beta < \alpha x^k + g(\ell_k + 1).$$

Subtracting αx^k throughout, we obtain

$$g\ell_k < \gamma y^* < g(\ell_k + 1),$$

a contradiction as $\gamma = g\mu$ and γy^* is a multiple of g. Hence, it follows that $\max\{\alpha x^i + g\ell_i : i \in I\} \geq \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$, implying in turn that $\delta \geq \lfloor \beta \rfloor_{S,(\alpha,\gamma)}$ and that $\alpha x + \gamma y \leq \delta$ is valid for P_S, as required. ◇

If $H_{(\mu,\ell)}$ is not empty, then the convex hull of $H_{(\mu,\ell)}$ is contained in its recession cone. For such $\mu \in \Theta$ and $\ell \in [-U, U]^{|T|}$, let $P_{(\mu,\ell)} := \{(x, y) \in P : \alpha x + \gamma y \leq \delta,\ \forall (\alpha, \gamma, \delta) \in H_{(\mu,\ell)}\}$. Then, by Lemma 5.3, $P_{(\mu,\ell)}$ is a rational polyhedron. By Claims 1 and 2, after letting $P_{(\mu,\ell)} := P$ when $H_{(\mu,\ell)} = \emptyset$, we have that

$$P_S = P_{S,\Pi_0} \cap \bigcap_{(\mu \in \Theta, \ell \in [-U,U]^{|T|})} P_{(\mu,\ell)},$$

implying in turn that P_S is a rational polyhedron. □

As a directly corollary of Theorem 5.9, we obtain the following result:

Corollary 5.10 ([52]). *Let $T = \{x \in \mathbb{R}^n : u \leq x \leq v\}$ for some $u \leq v \in \mathbb{Z}^n$, and let $S = (T \cap \mathbb{Z}^n) \times \mathbb{Z}^l$. Let $P \subseteq \mathbb{R}^{n+l}$ be a rational polyhedron. Then P_S is a rational polyhedron.*

5.4 Integer points with bounds on components

In this section, we consider the set

$$S_G = \left\{(z^1, z^{2l}, z^{2u}, z^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_{2l}} \times \mathbb{Z}^{n_{2u}} \times \mathbb{Z}^{n_3} : \ z^1 \in T_G, \ z^{2l} \geq \ell^2, \ z^{2u} \leq u^2\right\} \tag{5.16}$$

where $T_G \subseteq \mathbb{Z}^{n_1}$ is finite, $\ell^2 \in \mathbb{Z}^{n_{2l}}$, $u^2 \in \mathbb{Z}^{n_{2u}}$. We will show that the S_G-Chvátal closure of a rational polyhedron is again a rational polyhedron. To simplify the proof, we start with showing that if the result holds for the S_C-Chvátal closure of a rational polyhedron, where $n_2 = n_{2l} + n_{2u}$,

$$S_C = T_C \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}, \tag{5.17}$$

and $T_C \subseteq \mathbb{Z}_+^{n_1}$ is finite, then it also holds for the S_G-Chvátal closure. Throughout, we use N_1 to denote $\{1, \ldots, n_1\}$, and similarly, we use N_2 and N_3 to denote $\{1, \ldots, n_2\}$ and $\{1, \ldots, n_3\}$.

Remember that a unimodular transformation is a mapping $\tau : \mathbb{R}^n \to \mathbb{R}^n$ which maps $x \in \mathbb{R}^n$ to $Ux + v \in \mathbb{R}^n$ for some unimodular matrix $U \in \mathbb{R}^{n\times n}$ and some integral vector $v \in \mathbb{Z}^n$. Also note that the inverse mapping τ^{-1} is a unimodular transformation and that $\tau^{-1}(x) = U^{-1}x - U^{-1}v$.

Lemma 5.11 (Unimodular mapping lemma [52]). *Let $S \subseteq \mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a rational polyhedron contained in $\mathrm{conv}(S)$. Let τ be a unimodular transformation that maps $x \in \mathbb{R}^n$ to $Ux + v$ for some unimodular matrix $U \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{Z}^n$. Then $\tau(P) \subseteq \mathrm{conv}(\tau(S))$, and for any $\Pi \subseteq \Pi_P$,*

$$\tau(P_{S,\Pi}) = \tau(P)_{\tau(S),\tau(\Pi)}$$

where $\tau(\Pi) := \{(\pi U^{-1}, \pi_0 + \pi U^{-1}v) : (\pi, \pi_0) \in \Pi\} \subseteq \Pi_{\tau(P)}$. Moreover, $\tau(P_S) = \tau(P)_{\tau(S)}$.

Proof. It is clear that $\tau(\mathrm{conv}(S)) = \mathrm{conv}(\tau(S))$. As τ is a linear transformation and $P \subseteq \mathrm{conv}(S)$, it follows that $\tau(P) \subseteq \mathrm{conv}(\tau(S))$. For any $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{R}$, we have $\tau\left(\{x \in \mathbb{R}^n : \ \pi x \leq \pi_0\}\right) = \left\{y \in \mathbb{R}^n : \ \pi\tau^{-1}(y) \leq \pi_0\right\}$, which implies that $\pi x \leq \pi_0$ is valid for P if and only if $\pi U^{-1} y \leq \pi_0 + \pi U^{-1} v$ is valid for $\tau(P)$. Moreover,

$$\tau\left(\{x \in \mathbb{R}^n : \ \lfloor \pi_0 \rfloor_{S,\pi} < \pi x \leq \pi_0\}\right) = \left\{y \in \mathbb{R}^n : \ \lfloor \pi_0 \rfloor_{S,\pi} + \pi U^{-1} v < \pi U^{-1} y \leq \pi_0 + \pi U^{-1} v\right\}.$$

This implies that $\lfloor \pi_0 + \pi U^{-1} v \rfloor_{\tau(S),\pi U^{-1}} = \lfloor \pi_0 \rfloor_{S,\pi} + \pi U^{-1} v$. As a result,

$$\tau\left(\{x \in \mathbb{R}^n : \ \pi x \leq \lfloor \pi_0 \rfloor_{S,\pi}\}\right) = \left\{y \in \mathbb{R}^n : \ \pi U^{-1} y \leq \lfloor \pi_0 + \pi U^{-1} v \rfloor_{\tau(S),\pi U^{-1}}\right\}.$$

Therefore, we get $\tau(P_{S,\Pi}) = \tau(P)_{\tau(S),\tau(\Pi)}$. In particular, when $\Pi = \Pi_P$, we have $\tau(P_S) = \tau(P)_{\tau(S)}$. □

Using Lemma 5.11, we next show that we can simply work with S_C of the form (5.17) instead of S_G.

Lemma 5.12 ([52]). *If the S_C-Chvátal closure of a rational polyhedron is always a rational polyhedron for every S_C of the form (5.17), then so is the S_G-Chvátal closure of a rational polyhedron for every S_G of the form (5.16).*

Proof. As T_G is finite, $T_G \subseteq \left[\ell^1, u^1\right]$ for some $\ell^1, u^1 \in \mathbb{Z}^{n_1}$. Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ be the unimodular

transformation defined as follows:

$$\tau(x) = \tau\left((x^1, x^2, x^3, x^4)\right) := \left(x^1 - \ell^1,\ x^2 - \ell^2,\ -x^3 + u^3,\ x^4\right)$$

for $x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_{2l}} \times \mathbb{R}^{n_{2u}} \times \mathbb{R}^{n_3}$. Then $\tau(S_G) = T_C \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ where $T_C := \left\{z - \ell^1 : z \in T_G\right\}$ and $n_2 = n_{2l} + n_{2u}$. Notice that T_C is contained in $[\mathbf{0}, u]$ where $u := u^1 - \ell^1 \geq \mathbf{0}$. By Lemma 5.11, for any rational polyhedron P, we have $\tau(P_{S_G}) = \tau(P)_{S_C}$. Therefore, P_{S_G} is a rational polyhedron if and only if $\tau(P)_{S_C}$ is a rational polyhedron. □

By Lemma 5.12, we may simply work with S_C of the form (5.17), i.e. $S_C = T_C \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ for some finite $T_C \subseteq \mathbb{Z}_+^{n_1}$.

Lemma 5.13 ([52]). *Let $S_C = T_C \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ for some finite $T_C \subseteq \mathbb{Z}_+^{n_1}$, and let $P \subseteq \text{conv}(S_C)$ be a rational polyhedron. Then*

$$P_{S_C} = P_{S_0} \cap P_{S_C, \Pi_P^+} \cap P_{S_C, \Pi_P^-}$$

where $S_0 := T_C \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$, and

$$\Pi_P^+ := \left\{(\pi, \pi_0) \in \Pi_P :\ \pi = (\pi^1, \pi^2, \mathbf{0}),\ \pi^2 \geq \mathbf{0}\right\},$$

$$\Pi_P^- := \left\{(\pi, \pi_0) \in \Pi_P :\ \pi = (\pi^1, \pi^2, \mathbf{0}),\ \pi^2 \leq \mathbf{0}\right\}.$$

Proof. Notice that S_0 is obtained from S_C after relaxing the nonnegativity restriction on the second part of variables and that $S_C \subseteq S_0$, so $P_{S_C} \subseteq P_{S_0}$ by Remark 5.1. To prove the claim, we will argue that if $\pi x \leq \lfloor \pi_0 \rfloor_{S_C, \pi}$, the S-CG cut derived from $(\pi, \pi_0) \in \Pi_P$, is violated by a point in P_{S_0}, then it must be the case that $(\pi, \pi_0) \in \Pi_P^+ \cup \Pi_P^-$.

Let $(\pi, \pi_0) \in \Pi_P$ where $\pi = (\pi^1, \pi^2, \pi^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$. If $\lfloor \pi_0 \rfloor_{S_C, \pi} = \lfloor \pi_0 \rfloor_{S_0, \pi}$, then the associated S_C-CG cut $\pi x \leq \lfloor \pi_0 \rfloor_{S_C, \pi}$ is the same as the associated S_0-CG cut, implying that any S_C-CG cut that is violated by a point P_{S_0} must have $\lfloor \pi_0 \rfloor_{S_C, \pi} < \lfloor \pi_0 \rfloor_{S_0, \pi}$. This means that while S_0 contains a point $z = (z^1, z^2, z^3)$ such that $\pi z = \lfloor \pi_0 \rfloor_{S_0, \pi}$, there is no such point in S_C.

Suppose for a contradiction that $\pi^3 \neq \mathbf{0}$. Then $\pi_j^3 \neq \mathbf{0}$ for some $j \in N_3$. Let $r = (r^1, r^2, r^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$ where

$$r^1 = \mathbf{0}, \qquad r^2 = |\pi_j^3| \sum_{i \in N_2} e_2^i, \qquad r^3 = -\frac{|\pi_j^3|}{\pi_j^3} \left(\sum_{i \in N_2} \pi_i^2\right) e_3^j,$$

and e_2^i denotes the i^{th} unit vector in $\mathbb{R}^{n_2}$ and e_3^j denotes the j^{th} unit vector in $\mathbb{R}^{n_3}$. As $r^2 > \mathbf{0}$, there exists a sufficiently large integer N such that $\pi^2 z^2 + N r^2 \geq \mathbf{0}$, and therefore, $z + Nr \in S_C$. Moreover, it can be readily checked that $\pi r = 0$ and that $\pi(z + Nr) = \pi z$, implying in turn that $\lfloor \pi_0 \rfloor_{S_C, \pi} = \lfloor \pi_0 \rfloor_{S_0, \pi}$, a contradiction to our assumption that $\lfloor \pi_0 \rfloor_{S_C, \pi} < \lfloor \pi_0 \rfloor_{S_0, \pi}$. Therefore, it follows that $\pi^3 = \mathbf{0}$.

Next we argue that either $\pi^2 \geq \mathbf{0}$ or $\pi^2 \leq \mathbf{0}$ must hold. Suppose for a contradiction that there are distinct $i, j \in N_2$ such that $\pi_i^2 > 0$ and $\pi_j^2 < 0$. Let $J^+ = \{i \in N_2 : \pi_i^2 \geq 0\}$ and $J^- := \{j \in N_2 : \pi_j^2 < 0\}$.

As before, we construct a vector $r = (r^1, r^2, r^3) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{Z}^{n_3}$ where

$$r^1 = \mathbf{0}, \qquad r^2 = \left(\sum_{i \in J^+} \pi_i^2 \right) \sum_{j \in J^-} e_2^j + \left(- \sum_{j \in J^-} \pi_j^2 \right) \sum_{i \in J^+} e_2^i, \qquad r^3 = \mathbf{0}.$$

As $r^2 > \mathbf{0}$, there exists an integer N such that $\pi^2 z^2 + N r^2 \geq \mathbf{0}$ and therefore $z + Nr \in S_C$. Moreover, note that $\pi r = 0$, and therefore, $\pi(z + Nr) = \pi z$, which implies that $\lfloor \pi_0 \rfloor_{S_C, \pi} = \lfloor \pi_0 \rfloor_{S_0, \pi}$, a contradiction. Therefore, it follows that $\pi^2 \geq \mathbf{0}$ or $\pi^2 \leq \mathbf{0}$ holds, as desired. □

By Lemma 5.13, to show that P_{S_C} is a rational polyhedron, it is sufficient to show that both P_{S_C, Π_P^+} and P_{S_C, Π_P^-} are rational polyhedra. Next, we partition Π_P^+ and Π_P^- according to the sign pattern of the components of π^1. For $J \subseteq N_1$, we let

$$\Pi_P^{J+} = \left\{ (\pi, \pi_0) \in \Pi_P^+ : \ \pi_j^1 \geq 0 \ \forall j \in J, \ \pi_j^1 \leq 0 \ \ \forall j \in N_1 \setminus J \right\},$$

$$\Pi_P^{J-} = \left\{ (\pi, \pi_0) \in \Pi_P^- : \ \pi_j^1 \leq 0 \ \forall j \in J, \ \pi_j^1 \geq 0 \ \ \forall j \in N_1 \setminus J \right\}.$$

Then it follows from Lemma 5.13 that

$$P_{S_C} = P_{S_0} \cap \left(\cap_{J \subseteq N_1} P_{S_C, \Pi_P^{J+}} \right) \cap \left(\cap_{J \subseteq N_1} P_{S_C, \Pi_P^{J-}} \right). \tag{5.18}$$

Hence, we need to prove that $P_{S_C, \Pi_P^{J+}}$ and $P_{S_C, \Pi_P^{J-}}$ are rational polyhedra for all $J \subseteq N_1$. The following lemma will be useful:

Lemma 5.14 ([52]). *Let $S_C = T_C \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ for some finite $T_C \subseteq \mathbb{Z}_+^{n_1}$, and let $P \subseteq \operatorname{conv}(S_C)$ be a rational polyhedron. Then P_{S_C} is a rational polyhedron, provided that $Q_{L, \Pi_Q^{N_1+}}$ and $Q_{L, \Pi_Q^{N_1-}}$ are rational polyhedra for every $L = T' \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ where $T' \subseteq \mathbb{Z}_+^{n_1}$ is finite and every rational polyhedron $Q \subseteq \operatorname{conv}(L)$.*

Proof. Let $J \subseteq N_1$, and let $u \in \mathbb{Z}_+^{n_1}$ be such that $T_C \subseteq [\mathbf{0}, u]$. Consider the unimodular transformation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ that maps $x \in \mathbb{R}^n$ to $y = \tau(x) \in \mathbb{R}^n$ where

$$y_i := \begin{cases} -x_i + u_i, & \text{if } i \in N_1 \setminus J \\ x_i, & \text{otherwise.} \end{cases}$$

Let $Q := \tau(P)$ and $L := \tau(S_C)$. Then $L = T' \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ for some $T' \subseteq [\mathbf{0}, u] \cap \mathbb{Z}^{n_1}$. It follows from Lemma 5.11 that $Q \subseteq \operatorname{conv}(L)$. Moreover, Lemma 5.11 implies that $P_{S_C, \Pi_P^{J+}} = \tau^{-1}(Q_{L, \Pi_Q^{N_1+}})$ and that $P_{S_C, \Pi_P^{J-}} = \tau^{-1}(Q_{L, \Pi_Q^{N_1-}})$ where

$$\Pi_Q^{N_1+} = \left\{ (\pi, \pi_0) \in \Pi_Q : \ \pi = (\pi^1, \pi^2, \mathbf{0}) \geq \mathbf{0} \right\}, \tag{5.19}$$

$$\Pi_Q^{N_1-} = \left\{ (\pi, \pi_0) \in \Pi_Q : \ \pi = (\pi^1, \pi^2, \mathbf{0}) \leq \mathbf{0} \right\}. \tag{5.20}$$

Hence, if $Q_{L, \Pi_Q^{N_1+}}$ and $Q_{L, \Pi_Q^{N_1-}}$ are rational polyhedra for every rational polyhedron $Q \subseteq \mathbb{R}^n$ and $L = T' \times \mathbb{Z}_+^{n_2} \times \mathbb{Z}^{n_3}$ for $T' \subseteq \mathbb{Z}_+^{n_1}$ finite, then $P_{S_C, \Pi_P^{J+}}$ and $P_{S_C, \Pi_P^{J-}}$ are rational polyhedra for all $J \subseteq N_1$. So, by (5.18), P_{S_C} is a rational polyhedron. □

Finally, we observe that one only needs to study the following narrow case to prove the main result:

Proposition 5.15 ([52]). *Let $S_C = T_C \times \mathbb{Z}^{n_2}_+ \times \mathbb{Z}^{n_3}$ for some finite $T_C \subseteq \mathbb{Z}^{n_1}_+$, and let $P \subseteq \operatorname{conv}(S_C)$ be a rational polyhedron. Then P_{S_C} is a rational polyhedron, provided that W_{S,Π^U_W} and W_{S,Π^D_W}, where*

$$\Pi^U_W := \left\{(\pi, \pi_0) \in \Pi_W : \ \pi = (\pi^1, \pi^2) \geq \mathbf{0}\right\},$$
$$\Pi^D_W := \left\{(\pi, \pi_0) \in \Pi_W : \ \pi = (\pi^1, \pi^2) \leq \mathbf{0}\right\},$$

are rational polyhedra for every $S = T \times \mathbb{Z}^{n_2}_+$, $T \subseteq \mathbb{Z}^{n_1}_+$ finite, and every rational polyhedron $W \subseteq \operatorname{conv}(S)$.

Proof. Let $L = T \times \mathbb{Z}^{n_2}_+ \times \mathbb{Z}^{n_3}$ where $T \subseteq \mathbb{Z}^{n_1}_+$ is finite, and let $Q \subseteq \operatorname{conv}(L)$ be a rational polyhedron. Let $\Pi^{N_1+}_Q$ and $\Pi^{N_1-}_Q$ be defined as in (5.19)–(5.20). By Lemma 5.6, $Q_{L,\Pi^{N_1+}_Q}$ and $Q_{L,\Pi^{N_1-}_Q}$ are rational polyhedra, provided that W_{S,Π^U_W} and W_{S,Π^D_W} are rational polyhedra where $W = \operatorname{proj}_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}(Q)$ and $S = \operatorname{proj}_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}(L) = T \times \mathbb{Z}^{n_2}_+$. So, it follows from Lemma 5.14 that P_S is a rational polyhedron. □

5.4.1 Covering polyhedra

In this section, we consider covering polyhedra of the form

$$P^{\uparrow} = \left\{x \in \mathbb{R}^n : Ax \geq b\right\}, \tag{5.21}$$

where $A \in \mathbb{Z}^{m\times n}_+$ and $b \in \mathbb{Z}^m_+$. In this section, we will prove that if $P^{\uparrow} \subseteq \operatorname{conv}(S)$ where

$$S = T \times \mathbb{Z}^{n_2}_+, \quad T \subseteq \mathbb{Z}^{n_1}_+ \text{ finite}, \quad n = n_1 + n_2,$$

then $P^{\uparrow}_S$ is a rational polyhedron. Notice that every valid inequality for $P^{\uparrow}$ is of the form

$$\alpha x \geq \beta, \quad \alpha \geq \mathbf{0}, \ \beta \geq 0.$$

Given $(\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{R}$ such that $\alpha x \geq \beta$ is valid for $P^{\uparrow}$, the S-CG cut obtained from $\alpha x \geq \beta$ has the following form:

$$\alpha x \geq \lceil \beta \rceil_{S,\alpha}$$

where

$$\lceil \beta \rceil_{S,\alpha} := -\lfloor -\beta \rfloor_{S,-\alpha} = \begin{cases} \min\{\alpha z : \ z \in S, \ \alpha z \geq \beta\} & \text{if } \{z \in S : \ \alpha z \geq \beta\} \neq \emptyset \\ +\infty \quad \text{otherwise} \end{cases}$$

We assume for convention that $\{x \in \mathbb{R}^n : \alpha x \geq +\infty\} = \emptyset$. Hereinafter, we use notations $N = \{1, \ldots, n_1 + n_2\}$, $I_1 = \{1, \ldots, n_1\}$ and $I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$ for convenience.

We define the *support* of a vector $v \in \mathbb{R}^n$ to be the set $S \subseteq \{1, \ldots, n\}$ such that $v_i \neq 0$ if and only if $i \in S$, and we denote this by $\operatorname{support}(v)$. For any set $I \subseteq \{1, \ldots, n\}$, we let $\operatorname{support}(v, I) = \operatorname{support}(v) \cap I$ and we refer to this set as the *support of v on I*. Let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}$. For $j \in \operatorname{support}(\alpha)$, the intercept of the hyperplane $\{x \in \mathbb{R}^n : \alpha x = \beta\}$ on the nonnegative axis $\{x \in \mathbb{R}^n_+ : x_i = 0 \text{ for all } i \neq j\}$ equals β/α_j (and for convenience is referred to simply as an "intercept"). In the next result, we show that if all nondominated S-CG cuts for $P^{\uparrow}$ have bounded intercepts (in the components corresponding to the

support of the cut on I_2), then $P_S^\uparrow$ is a rational polyhedron. The following lemma will be useful in proving that $P_S^\uparrow$ is a rational polyhedron. Notice that, as $P^\uparrow = \{x \in \mathbb{R}^n : -Ax \leq -b\}$,

$$\Pi_{P^\uparrow} = \{(-\alpha, -\beta) \in \mathbb{Z}^n \times \mathbb{R} \ : \ \exists \lambda \in \mathbb{R}^m_+ \text{ s.t. } \alpha = \lambda A,\ \beta \leq \lambda b\} \tag{5.22}$$

and that

$$\Pi^*_{P^\uparrow} = \{(-\alpha, -\beta) \in \Pi_{P^\uparrow} \ : \ \beta = \min\{\alpha x : x \in P^\uparrow\}\}. \tag{5.23}$$

Lemma 5.16 ([52]). *Let M^* be a positive integer, and let*

$$\Pi = \{(\alpha, \beta) : \ (-\alpha, -\beta) \in \Pi^*_{P^\uparrow},\ \beta/\alpha_j \leq M^* \ \text{ for all } j \in \text{support}(\alpha, I_2)\}\,. \tag{5.24}$$

Then $P^\uparrow_{S,\Pi}$ is a rational polyhedron.

Proof. Let $S^* = T \times \{1, \ldots, M^*\}^{n_2}$. Then S^* is a finite subset of S, and by Remark 5.1, $P^\uparrow_{S^*,\Pi} \subseteq P^\uparrow_{S,\Pi}$. We claim the following:

Claim 1. $P^\uparrow_{S^*,\Pi} = P^\uparrow_{S,\Pi}$.

Proof of Claim. Let $(\alpha, \beta) \in \Pi$. Then $\alpha x \geq \beta$ is valid for $P^\uparrow$, $\alpha \geq \mathbf{0}$, $\beta \geq 0$, and $0 \leq \beta/\alpha_j \leq M^*$ for every $j \in I_2$ such that $\alpha_j > 0$. It is sufficient to show that $\lceil \beta \rceil_{S^*,\alpha} = \lceil \beta \rceil_{S,\alpha}$. Let $z^* = (z^1, z^2) \in S = T \times \mathbb{Z}^{n_2}_+$ be such that

$$\alpha z^* = \lceil \beta \rceil_{S,\alpha} = \min\{\alpha z : \ z \in S,\ \alpha z \geq \beta\} \tag{5.25}$$

If $z^* \in S^*$, then $\alpha z^* = \lceil \beta \rceil_{S^*,\alpha}$, so $\lceil \beta \rceil_{S^*,\alpha} = \lceil \beta \rceil_{S,\alpha}$ is satisfied. Thus, we may assume that $z^* \notin S^*$. Then for some $j \in [n_2]$, the j^{th} component of z^2 is larger than M^*. Let $\bar{z}^2$ be what is obtained from z^2 by reducing the component to M^*. Clearly $(z^1, \bar{z}^2) \in S$. If $\alpha_j > 0$, then $(z^1, \bar{z}^2)$ satisfies that $\alpha(z^1, \bar{z}^2) \geq \beta$ – as $\alpha_j M^* \geq \beta$ – and that $\alpha(z^1, \bar{z}^2) < \alpha z^*$, a contradiction to (5.25). This implies that $\alpha_j = 0$, so $\alpha z^* = \alpha(z^1, \bar{z}^2)$. Repeating this argument for each component of z^2 larger than M^*, we may assume that there exists $\bar{z} \in S^*$ such that $\alpha z^* = \alpha \bar{z}$, implying in turn that $\lceil \beta \rceil_{S^*,\alpha} = \lceil \beta \rceil_{S,\alpha}$. Therefore, $P^\uparrow_{S^*,\Pi} = P^\uparrow_{S,\Pi}$, as required. ◇

By Claim 1, it suffices to show that $P^\uparrow_{S^*,\Pi}$ is a rational polyhedron. The rest of the proof is similar to that of Theorem 5.4. We write $\Pi = \cup_{I \subseteq I_2} \Pi(I)$ where

$$\Pi(I) = \big\{(\alpha, \beta) \in \Pi : \alpha_j > 0 \ \text{ if and only if } j \in I\big\}$$

and $\Pi(I) = \cup_{G \subseteq S^*} \Pi(I, G)$ where

$$\Pi(I, G) = \Big\{(\alpha, \beta) \in \Pi(I) : \alpha z \geq \lceil \beta \rceil_{S^*,\alpha} \text{ if and only if } z \in G\Big\}.$$

Consequently,

$$P^\uparrow_{S^*,\Pi} = \cup_{I \subseteq I_2} \cup_{G \subseteq S^*} P^\uparrow_{S^*,\Pi(I,G)}.$$

We will show that $P^{\uparrow}_{S^*,\Pi(I,G)}$ is a rational polyhedron for every $I \subseteq I_2$ and $G \subseteq S^*$. Given $I \subseteq I_2$ and $G \subseteq S^*$, let $H_{(I,G)}$ be defined as

$$H_{(I,G)} = \left\{ (\alpha,\delta) \in \mathbb{Z}^{n+1} \ : \ \exists \begin{pmatrix} \beta \in \mathbb{R}, \\ \lambda \in \mathbb{R}^m_+ \end{pmatrix} \text{ s.t. } \begin{array}{rcll} \alpha & = & \lambda A, & \\ \beta & \leq & \lambda b, & \\ \alpha z & \geq & \delta, & \forall z \in G \\ \alpha z & \leq & \beta - \frac{1}{\Delta}, & \forall z \in S^* \setminus G \\ \delta & \geq & \beta & \\ \alpha_j & \geq & 1, & \forall j \in I \\ \alpha_j & = & 0, & \forall j \in N_2 \setminus I \\ M^* \alpha_j & \geq & \beta, & \forall j \in I \end{array} \right\}$$

where Δ is the product of all distinct nonzero sub-determinants of A.

Claim 2. *Let $(\alpha,\beta) \in \Pi(I,G)$. Then $(\alpha, \lceil \beta \rceil_{S^*,\alpha}) \in H_{(I,G)}$.*

Proof of Claim. As $(\alpha,\beta) \in \Pi$, it follows from (5.23) and (5.24) that $\beta = \min\{\alpha x : x \in P^{\uparrow}\}$. Therefore, β is an integer multiple of $1/\Delta$. Since $\alpha z < \beta$ for $z \in S^* \setminus G$ and αz is an integer, it follows that $\alpha z \leq \beta - \frac{1}{\Delta}$. It can also be checked that $\delta = \lceil \beta \rceil_{S^*,\alpha}$ together with α, β satisfies the constraints defining $H_{(I,G)}$. Therefore, $(\alpha, \lceil \beta \rceil_{S^*,\alpha}) \in H_{(I,G)}$, as required. ◇

Claim 3. *Let $(\alpha,\delta) \in H_{(I,G)}$. Then $\alpha x \geq \delta$ is valid for $P^{\uparrow}_{S^*,\Pi(I,G)}$.*

Proof of Claim. There exists β such that α, δ together with β satisfy the constraints in $H_{(I,G)}$. Notice that $\alpha x \geq \beta$ is valid for $P^{\uparrow}$ and that $\beta \leq \delta \leq \lceil \beta \rceil_{S^*,\alpha}$. Therefore, $\alpha x \geq \delta$ is implied by $\alpha x \geq \lceil \beta \rceil_{S^*,\alpha}$, so $\alpha x \geq \delta$ is valid for $P^{\uparrow}_{S^*,\Pi(I,G)}$, as required. ◇

By Claims 2 and 3,

$$P^{\uparrow}_{S^*,\Pi(I,G)} = \bigcap_{(\alpha,\delta) \in H_{(I,G)}} \left\{ x \in P^{\uparrow} : \ \alpha x \geq \delta \right\},$$

and by Lemma 5.3,

$$P^{\uparrow}_{S^*,\Pi(I,G)} = \bigcap_{(\alpha,\delta) \in C_{(I,G)}} \left\{ x \in P^{\uparrow} : \ \alpha x \geq \delta \right\}$$

where $C_{(I,G)}$ denotes the recession cone of the continuous relaxation of $H_{(I,G)}$. Moreover, $P^{\uparrow}_{S^*,\Pi(I,G)}$ is a rational polyhedron. □

We will next give a series of results which will show that all nondominated S-CG cuts for $P^{\uparrow}$ have "bounded" intercepts, in the sense that these inequalities belong to Π defined in (5.24). So, in the end, we will argue that $P^{\uparrow}_S = P^{\uparrow}_{S,\Pi}$.

Let $\lambda \in \mathbb{R}^m_+$. For $j \in [n]$, let $(\lambda A)_j$ denote the j^{th} component of λA, and consider the hyperplane $\{x : \lambda A x = \lambda b\}$. Notice that if each row a_i of A has the same support as λA, then the intercept on the positive x_j axis must lie between $\min_i\{b_i/a_{ij}\}$ and $\max_i\{b_i/a_{ij}\}$ for any j in support(λA). In other words, all intercepts are trivially bounded by a function of A and b. Therefore, the difficult case for us is when not all rows of A have the same support. In that case, $a_{ij} = 0$ for some i, and therefore, $\max_i\{b_i/a_{ij}\}$ is unbounded and the intercept on the positive x_j axis can be arbitrarily large.

Definition 5.17 ([52]). *Let $\lambda \in \mathbb{R}^m_+$, and let $\sigma : [m] \to [m]$ be a non-increasing order of the components in λ, i.e. $\lambda_{\sigma(1)} \geq \cdots \geq \lambda_{\sigma(m)}$.*

- *$t(\lambda, A)$ is defined as*

$$t(\lambda, A) = \min \left\{ j \in \{1, \ldots, m\} : \bigcup_{i=1}^{j} \text{support}(a_{\sigma(i)}, I_2) = \text{support}\,(\lambda A, I_2) \right\}. \tag{5.26}$$

  In words, $t(\lambda, A)$ denotes the smallest index $j \in \{1, \ldots, m\}$ such that the support of $\sum_{i=1}^{j} \lambda_{\sigma(i)} a_{\sigma(i)}$ on I_2 is the same as the support of $\lambda A = \sum_{i=1}^{\cdot} \lambda_{\sigma(i)} a_{\sigma(i)}$ on I_2.

- *The tilting ratio of λ with respect to A, denoted $r(\lambda, A)$, is defined as $\lambda_{\sigma(1)}/\lambda_{\sigma(t(\lambda,A))}$.*

In particular, $\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(t(\lambda,A))} > 0$ and $r(\lambda, A) > 0$.

We will later show (in Theorem 5.22) that for any $\lambda \in \mathbb{R}^m_+$, if $r(\lambda, A)$ is bounded above by a constant that depends only on A and b, then the intercepts of $\{x : \lambda A x = \lambda b\}$ corresponding to I_2 are also bounded above by a constant that depends only on A and b. We next focus on bounding $r(\lambda, A)$ for $\lambda \in \mathbb{R}^m_+$ defining a nondominated S-CG cut for $P^\uparrow$, with the bounding constants (that depend only on A and b, not on the cut) defined below.

Definition 5.18 ([52]). *Let $B = \max\{b_i : i \in [m]\}$, $C = \min\{a_{ij} : a_{ij} \neq 0, i \in [m], j \in [n]\}$, and $D = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$.*

$$M_1 = 2\,(mB + 2D)\,. \tag{5.27}$$

$$M_i = \left(2mB \prod_{j=1}^{i-1} M_j \right)^{i-1} M_1 \quad \text{for } i = 2, \ldots, m-1. \tag{5.28}$$

$$M = \prod_{i=1}^{m-1} M_i. \tag{5.29}$$

It can be readily observed that

Remark 5.19 ([52]). *Let $M_1, \ldots, M_{m-1}$ and M be defined as in Definition 5.18. Then we have $M_1 \geq 4$ as $m, B, D \geq 1$. and $B \geq 1$. Also $(M_i/M_1)^{1/(i-1)} \geq 4$ for all $i \geq 2$.*

We will show in the the following technical lemma that if $\lambda \in \mathbb{R}^m_+$ has tilting ratio $r(\lambda, A) > M$, then there exists a $\mu \in \mathbb{R}^m_+$ that defines an S-CG cut dominating the one defined by λ, but with $\|\mu\|_1 \leq \|\lambda\|_1 - 1$. We will need the following well-known result of Dirichlet:

Theorem 5.20 (**Simultaneous Diophantine Approximation theorem** [54]). *Let k be a positive integer. Given any real numbers $r_1, \ldots, r_k$ and $0 < \varepsilon < 1$, there exist integers $p_1, \ldots, p_k$ and q such that $\left| r_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}$ for $i = 1, \ldots, k$ and $1 \leq q \leq \left(\frac{1}{\varepsilon}\right)^k$.*

We are ready to prove the following technical lemma:

Lemma 5.21 ([52]). *Let $\lambda \in \mathbb{R}^m_+$ be such that $\lambda b = \min\{\lambda Ax : x \in P^\uparrow\}$. If $r(\lambda, A) > M$, then there exists $\mu \in \mathbb{R}^m_+$ that satisfies the following:*

(1) $\|\mu\|_1 \le \|\lambda\|_1 - 1$,

(2) $\mu b = \min\{\mu Ax : x \in P^\uparrow\}$,

(3) $\mu Ax \ge \lceil \mu b \rceil_{S,\mu A}$ *dominates* $\lambda Ax \ge \lceil \lambda b \rceil_{S,\lambda A}$.

Proof. After relabeling the rows of $Ax \ge b$, we may assume that $\lambda_1 \ge \cdots \ge \lambda_m$. If $t(\lambda, A) = 1$, we have $r(\lambda, A) = 1 \le M$, a contradiction to our assumption. So, $t(\lambda, A) \ge 2$. Let t stand for $t(\lambda, A)$. Let Δ be defined as

$$\Delta = \min\left\{(\lambda A)_j : j \in \text{support}(\lambda A, I_2)\right\}, \tag{5.30}$$

and let

$$k = \text{argmin}\left\{(\lambda A)_j : \ j \in \text{support}\,(\lambda A, I_2) \setminus \bigcup_{i=1}^{t-1} \text{support}(a_i, I_2)\right\}. \tag{5.31}$$

By the definition of t, it follows that $\text{support}\,(\lambda A, I_2) \setminus \bigcup_{i=1}^{t-1} \text{support}(a_i, I_2)$ is not empty, and therefore, k is a well-defined index. Moreover, by the definition of Δ in (5.30) and that of k in (5.31),

$$\Delta \le (\lambda A)_k = \sum_{i=t}^{m} \lambda_i a_{ik} \le \lambda_t \sum_{i=t}^{m} a_{ik} \le D\lambda_t. \tag{5.32}$$

Notice that

$$r(\lambda, A) = \frac{\lambda_1}{\lambda_t} = \frac{\lambda_1}{\lambda_2} \times \cdots \times \frac{\lambda_{t-1}}{\lambda_t} > M = M_1 \times \cdots \times M_{m-1},$$

so there exists some $\ell \in \{1, \ldots, t-1\}$ such that

$$\lambda_\ell / \lambda_{\ell+1} > M_\ell \quad \text{and} \quad \lambda_i / \lambda_{i+1} \le M_i \text{ for all } i = 1, \ldots, \ell - 1. \tag{5.33}$$

Claim 1. *If $\ell \ge 2$, there exist positive integers $p_1 \ge \cdots \ge p_\ell$ that satisfy the following:*

$$\left|\frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell}\right| < \frac{\varepsilon}{p_\ell}, \ i \in [\ell] \quad \textit{and} \quad p_\ell \le \left(\frac{1}{\varepsilon}\right)^{\ell-1}. \tag{5.34}$$

where $\varepsilon = (M_1/M_\ell)^{1/(\ell-1)}$.

Proof of Claim. We define $\varepsilon = (M_1/M_\ell)^{1/(\ell-1)}$. It follows from the Simultaneous Diophantine Approximation theorem (with $k = \ell - 1$ and $r_i = \lambda_i/\lambda_\ell$ for $i \in [\ell - 1]$) that there exist positive integers $p_1, \ldots, p_\ell$ satisfying (5.34). In fact, we may assume that $p_i \ge p_{i+1} \ge p_\ell$, because $\lambda_i \ge \lambda_{i+1}$ for $i = 1, \ldots, \ell - 1$. If $p_i < p_{i+1}$ for some $i \in \{1, \ldots, \ell - 1\}$, then increasing p_i till it becomes equal to p_{i+1} can only reduce the value of $|\lambda_i/\lambda_\ell - p_i/p_\ell|$. ◇

By Claim 1, if $\ell \ge 2$, there exist positive integers $p_1, \ldots, p_\ell$ that satisfy (5.34). Then we define $\mu_1, \ldots, \mu_m$ as follows:

$$\mu_i = \begin{cases} \lambda_i - p_i\Delta & \text{for } i = 1, \ldots, \ell, \\ \lambda_i & \text{otherwise} \end{cases} \tag{5.35}$$

In fact, even when $\ell = 1$, let μ be defined as in (5.34) with $p_1 = 1$.

Claim 2. *$\mu_1, \dots, \mu_\ell \geq \mu_{\ell+1} > 0$, and in particular, $\mu \in \mathbb{R}^m_+$.*

Proof of Claim. Let us consider the $\ell = 1$ case first. Notice that $\mu_1 = \lambda_1 - \Delta$ and $\mu_i = \lambda_i$ for $i \geq 2$. As $\lambda_1 > M_1\lambda_2$, it follows that $\mu_1 = \lambda_1 - \Delta > M_1\lambda_2 - \Delta$, so by (5.32), $\mu_1 > \lambda_2(M_1 - D)$. This in turn implies that $\mu_1 > \lambda_2$ as $M_1 - D \geq 1$ by Remark 5.19.

Now consider the case $\ell \geq 2$. Notice that

$$p_\ell \leq \frac{M_\ell}{M_1} \quad \text{and} \quad \lambda_i > \frac{p_i}{2p_\ell}\lambda_\ell, \ i \in [\ell] \tag{5.36}$$

where the first inequality follows from (5.34) and $\varepsilon = (M_1/M_\ell)^{1/(\ell-1)}$ and the second one follows from the fact that $\varepsilon \leq \frac{1}{2}$ & $|\lambda_i/\lambda_\ell - p_i/p_\ell| < \varepsilon/p_\ell \leq 1/(2p_\ell)$ and the fact that $p_i \geq p_\ell \geq 1$ for all $i \geq \ell$. We will first show that $\mu \in \mathbb{R}^m_+$. Clearly, we have $\mu_i \geq 0$ for $i \geq \ell + 1$, as $\mu_i = \lambda_i$ for these values of i. We next show that $\mu_1, \dots, \mu_\ell \geq \mu_{\ell+1}$. Let $i \in \{1, \dots, \ell\}$. By definition, we have

$$\lambda_\ell > M_\ell\lambda_{\ell+1} \geq M_1 p_\ell \lambda_{\ell+1} \quad \Rightarrow \quad \lambda_\ell/p_\ell > M_1\lambda_{\ell+1}.$$

As $\lambda_i > \frac{p_i}{2p_\ell}\lambda_\ell$, we can conclude that

$$\lambda_i > p_i M_1 \lambda_{\ell+1}/2 \quad \text{and} \quad \mu_i = \lambda_i - p_i\Delta > p_i(\frac{1}{2}M_1\lambda_{\ell+1} - \Delta)$$

But as $\Delta \leq D\lambda_t \leq D\lambda_{\ell+1}$, we can conclude that

$$\mu_i > p_i(\frac{1}{2}M_1 - D)\lambda_{\ell+1}.$$

Since $M_1/2 - D \geq 1$ by Remark 5.19 and $p_i \geq 1$, the inequality above implies that $\mu_i \geq \lambda_{\ell+1} = \mu_{\ell+1} > 0$ for all $i \leq \ell$, as required. ◇

Using Claim 2, we can prove the following:

Claim 3. *$\|\mu\|_1 \leq \|\lambda\|_1 - 1$, $\text{support}(\mu A, I_2) = \text{support}(\lambda A, I_2)$, and $t(\mu, A) = t(\lambda, A)$.*

Proof of Claim. Since we have $p_\ell \geq 1$ and $\Delta \geq 1$, it follows that $\|\mu\|_1 \leq \|\lambda\|_1 - 1$. We next prove that

$$\text{support}(\mu A, I_2) = \text{support}(\lambda A, I_2) \quad \text{and} \quad t(\mu, A) = t(\lambda, A).$$

In fact, Claim 2 implies that $\mu_i > 0$ if and only if $\lambda_i > 0$, for $i = 1, \dots, m$. Therefore, $\text{support}(\mu A) = \text{support}(\lambda A)$ and $t(\mu, A) = t(\lambda, A)$, as required. ◇

Putting Claims 2 and 3 together, it follows that $\mu \in \mathbb{R}^m_+$ and μ satisfies (1). Furthermore,

Claim 4. *$\mu b = \min\{\mu A x : x \in P^\uparrow\}$.*

Proof of Claim. We assumed that $\lambda b = \min\{\lambda A x : x \in P^\uparrow\}$. Recall that $P^\uparrow = \{x \in \mathbb{R}^n : Ax \geq b\}$. By the complementary slackness, $\lambda_i > 0$ if and only if $a_i x^* = b_i$ for all $x^* \in \text{argmin}\{\lambda A x : x \in P^\uparrow\}$. Notice that for all $x \in P^\uparrow$, $\mu A x \geq \mu b$. By Claim 2, we know that $\text{support}(\mu) = \text{support}(\lambda)$, implying in turn that for

all $x^* \in \operatorname{argmin}\{\lambda Ax : x \in P^\uparrow\}$, $\mu Ax^* = \sum_{i \in \operatorname{support}(\mu)} \mu_i b_i = \mu b$. Therefore, $\mu b = \min\{\mu Ax : x \in P^\uparrow\}$, as required. ◇

By Claim 4, μ satisfies (2). To complete the proof, we will show that $\mu Ax \geq \lceil \mu b \rceil_{S,\mu A}$ dominates $\lambda Ax \geq \lceil \lambda b \rceil_{S,\lambda A}$. Let $Q = \{x \in \mathbb{R}^n_+ : \mu b \leq \mu Ax \leq \mu b + \Delta\}$. We next prove the following claim, which requires a technical proof:

Claim 5. *There is no point $x \in Q$ that satisfies*

$$\sum_{i=1}^{\ell} p_i a_i x \geq 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{5.37}$$

Proof of Claim. Suppose for a contradiction that there exists $\tilde{x} \in Q$ satisfying (5.37). Recall that $k \in \operatorname{support}(\lambda A, I_2)$, so by Claim 3, $k \in \operatorname{support}(\mu A, I_2)$ and thus $(\mu A)_k > 0$. Let $v = \frac{\mu b}{(\mu A)_k} e^k$. Then

$$\mu Av = \mu b \quad \text{and} \quad \sum_{i=1}^{\ell} p_i a_i v = 0 \tag{5.38}$$

since $k \notin \bigcup_{i=1}^{t-1} \operatorname{support}(a_i, I_2)$ and $a_i e^k = 0$ for $i \leq t-1$. Since $\tilde{x}, v \in Q$, $\tilde{x}$ satisfies (5.37) and v satisfies (5.38), we can take a convex combination of these points to get a point $\bar{x} \in Q$ such that

$$\sum_{i=1}^{\ell} p_i a_i \bar{x} = 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{5.39}$$

As $\bar{x} \in Q$, we have $\mu A\bar{x} \leq \mu b + \Delta$, and we can rewrite this inequality as

$$\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) \leq -\sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + \Delta. \tag{5.40}$$

First, as $\left| \frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell} \right| < \frac{\varepsilon}{p_\ell}$ by (5.34), we have

$$\frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell} = \frac{\varepsilon_i}{p_\ell} \quad \Rightarrow \quad \lambda_i = \frac{\lambda_\ell}{p_\ell}(p_i + \varepsilon_i)$$

where $-\varepsilon < \varepsilon_i < \varepsilon$ for $i = 1, \ldots, \ell$ (ε_ℓ can be assumed to be zero). Recall that $\mu_i = \lambda_i - p_i \Delta$ for $i \leq \ell$ from (5.35) and that $\sum_{i=1}^{\ell} p_i a_i \bar{x} = 1 + \sum_{i=1}^{\ell} p_i b_i$ from (5.39), so we can rewrite the left hand side of (5.40) as the following:

$$\sum_{i=1}^{\ell} (\lambda_i - p_i\Delta)(a_i\bar{x} - b_i) = \sum_{i=1}^{\ell} [\frac{\lambda_\ell}{p_\ell}(p_i + \varepsilon_i) - p_i\Delta](a_i\bar{x} - b_i) = (\frac{\lambda_\ell}{p_\ell} - \Delta) + \frac{\lambda_\ell}{p_\ell} \sum_{i=1}^{\ell} \varepsilon_i (a_i\bar{x} - b_i). \tag{5.41}$$

Therefore, we obtain the following:

$$\begin{aligned}\frac{\lambda_\ell}{p_\ell}\left(1+\sum_{i=1}^{\ell}\varepsilon_i(a_i\bar{x}-b_i)\right) &\leq -\sum_{j=\ell+1}^{m}\mu_j(a_j\bar{x}-b_j)+2\Delta \\ \leq \sum_{j=\ell+1}^{m}\mu_j b_j+2\Delta &\leq \lambda_{\ell+1}(mB+2D)=\frac{1}{2}\lambda_{\ell+1}M_1 \end{aligned} \tag{5.42}$$

where the first inequality in (5.42) follows from (5.40) and (5.41), the second inequality follows from the assumption that $a_j \geq 0$ and $\bar{x} \geq \mathbf{0}$, the third inequality follows from the fact that $\mu_i = \lambda_i \leq \lambda_{\ell+1}$ for $i = \ell+1, \ldots, m$ by (5.35) and that $b_j \leq B$ by Definition 5.18, and the last equality follows from (5.27).

We will obtain a lower bound on the first term in (5.42). Note that

$$\sum_{i=1}^{\ell}\varepsilon_i\left(a_i\bar{x}-b_i\right)=\sum_{i=1}^{\ell}\varepsilon_i a_i\bar{x}-\sum_{i=1}^{\ell}\varepsilon_i b_i \geq -\varepsilon\sum_{i=1}^{\ell}(a_i\bar{x}+b_i) \tag{5.43}$$

where the inequality in (5.43) holds because $a_i\bar{x} \geq 0$, $b_i \geq 0$, and $-\varepsilon < \varepsilon_i < \varepsilon$. So, we need to lower bound $\sum_{i=1}^{\ell}(a_i\bar{x}+b_i)$. Notice that $\sum_{i=1}^{\ell} b_i \leq mB$ and that

$$\sum_{i=1}^{\ell}a_i\bar{x} \leq \sum_{i=1}^{\ell}\frac{p_i}{p_\ell}a_i\bar{x}=\frac{1}{p_\ell}\left(1+\sum_{i=1}^{\ell}p_i b_i\right) \leq \frac{1}{p_\ell}+B\sum_{i=1}^{\ell}\frac{p_i}{p_\ell} \tag{5.44}$$

where the first inequality follows from $p_i \geq p_\ell$ for $i \leq \ell$ by Claim 1 and the second inequality follows from, again, $p_i \geq p_\ell$ for $i \leq \ell$ and $b_i \leq B$. Moreover,

$$\sum_{i=1}^{\ell}\frac{p_i}{p_\ell} \leq 1+\sum_{i=1}^{\ell-1}\left(\frac{\lambda_i}{\lambda_\ell}+\frac{\varepsilon}{p_\ell}\right)=1+(\ell-1)\frac{\varepsilon}{p_\ell}+\sum_{i=1}^{\ell-1}\frac{\lambda_i}{\lambda_\ell} \leq 1+(\ell-1)\frac{\varepsilon}{p_\ell}+\sum_{i=1}^{\ell-1}\prod_{j=i}^{\ell-1}M_j \tag{5.45}$$

where the first inequality follows from $\frac{p_i}{p_\ell} \leq \frac{\lambda_i}{\lambda_\ell}+\frac{\varepsilon}{p_\ell}$ for $i \leq \ell-1$ by (5.34) and the second inequality follows from the fact that $\frac{\lambda_i}{\lambda_\ell}=\prod_{j=i}^{\ell-1}\frac{\lambda_j}{\lambda_{j+1}}$ and that $\frac{\lambda_j}{\lambda_{j+1}} \leq M_j$ for $j \leq \ell-1$. Putting (5.44), (5.45) and $\sum_{i=1}^{\ell} b_i \leq mB$ together, we obtain the following inequality:

$$\sum_{i=1}^{\ell}(a_i\bar{x}+b_i) \leq B\left(m+\frac{1}{Bp_\ell}+1+(\ell-1)\frac{\varepsilon}{p_\ell}+\sum_{i=1}^{\ell-1}\prod_{j=i}^{\ell-1}M_j\right)$$

The term $\sum_{i=1}^{\ell-1}\prod_{j=i}^{\ell-1}M_j$ can be bounded above by $(\ell-1)\prod_{j=1}^{\ell-1}M_j$. Moreover, it is not difficult to see that

$$m+\frac{1}{Bp_\ell}+1+(\ell-1)\frac{\varepsilon}{p_\ell} \leq \prod_{j=1}^{\ell-1}M_j.$$

Therefore,

$$\sum_{i=1}^{\ell}(a_i\bar{x}+b_i) \le Bm\prod_{j=1}^{\ell-1} M_j$$

It follows from (5.28) and (5.34) that $Bm\prod_{j=1}^{\ell-1} M_j = \frac{1}{2\varepsilon}$, implying in turn that

$$-\varepsilon\sum_{i=1}^{\ell}(a_i\bar{x}+b_i) \ge -\frac{1}{2}.$$

By (5.43), it follows that $\sum_{i=1}^{\ell}\varepsilon_i(a_i\bar{x}-b_i) \ge -\frac{1}{2}$. Then the left hand side of (5.42) is lower bounded by $\frac{\lambda_\ell}{2p_\ell}$, so we obtain $\lambda_\ell < p_\ell\lambda_{\ell+1}M_1$ from (5.42), implying in turn that $M_\ell < p_\ell M_1$ as we assumed that $\lambda_\ell > M_\ell\lambda_{\ell+1}$ (5.33). However, this contradicts the first inequality in (5.36). ◇

We now claim the following:

Claim 6. $\mu b \le \lceil \mu b\rceil_{S,\mu A} \le \mu b + \Delta$.

Proof of Claim. Let α, β denote $\mu A, \mu b$, respectively. By Claim 4, we have that $\beta = \min\{\alpha x : x \in P^{\uparrow}\}$. As $P^{\uparrow} \subseteq \text{conv}(S)$, it follows that $\beta \ge \min\{\alpha z : z \in S\}$. If $\beta = \min\{\alpha z : z \in S\}$, then $\beta = \lceil\beta\rceil_{S,\alpha}$. Thus we may assume that $\beta > \min\{\alpha z : z \in S\}$, so there exists $z \in S$ such that $\beta > \alpha z$. Let $j \in \text{support}(\alpha, I_2)$. Since $\frac{\beta-\alpha z}{\alpha_j} > 0$, it follows that $z + \lceil\frac{\beta-\alpha z}{\alpha_j}\rceil e^j \in S$. Observe that $\alpha\left(z + \lceil\frac{\beta-\alpha z}{\alpha_j}\rceil e^j\right) = \alpha z + \alpha_j\lceil\frac{\beta-\alpha z}{\alpha_j}\rceil$ and $\beta \le \alpha z + \alpha_j\lceil\frac{\beta-\alpha z}{\alpha_j}\rceil \le \beta + \alpha_j$. Therefore, we get $\lceil\beta\rceil_{S,\pi} \le \beta + \alpha_j$ for all $j \in \text{support}(\alpha, I_2)$, implying in turn by (5.30) that $\lceil\beta\rceil_{S,\pi} \le \beta + \Delta$, as required. ◇

Putting Claims 5 and 6 together, we are ready show the last piece of this lemma:

Claim 7. *$\mu Ax \ge \lceil\mu b\rceil_{S,\mu A}$ is implied by $\lambda Ax \ge \lceil\lambda b\rceil_{S,\lambda A}$ and the inequalities in $Ax \ge b$.*

Proof of Claim. There exists $z \in S$ such that $\mu Az = \lceil\mu b\rceil_{S,\mu A}$. Claim 6 implies that $\mu b \le \mu Az \le \mu b + \Delta$. Then, by Claim 5, it follows that

$$\sum_{i=1}^{\ell} p_i a_i z < 1 + \sum_{i=1}^{l} p_i b_i \quad \Rightarrow \quad \sum_{i=1}^{\ell} p_i a_i z = \sum_{i=1}^{\ell} p_i b_i - f$$

for some integer $f \in \left[0, \sum_{i=1}^{\ell} p_i b_i\right]$, as z is integral. Let $j = \text{argmin}\{(\lambda A)_j : j \in \text{support}(\lambda A, I_2)\}$. Then, by (5.30), $(\lambda A)_j = \Delta$. Consider $z + fe^j \in S$. Observe that

$$\lambda A\left(z + fe^j\right) = \lambda Az + f\left(\lambda A\right)_j = \left(\mu A + \Delta\sum_{i=1}^{\ell} p_i a_i\right) z + \Delta\sum_{i=1}^{\ell} p_i(b_i - a_i z) = \lceil\mu b\rceil_{S,\mu A} + \Delta\sum_{i=1}^{\ell} p_i b_i,$$

which implies that $\lambda A\left(z + fe^j\right) = \lceil\mu b\rceil_{S,\mu A} + \Delta\sum_{i=1}^{\ell} p_i b_i$. Since $\lceil\mu b\rceil_{S,\mu A} \ge \mu b$, we must have

$$\lceil\mu b\rceil_{S,\mu A} + \Delta\sum_{i=1}^{\ell} p_i b_i \ge \mu b + \Delta\sum_{i=1}^{\ell} p_i b_i = \lambda b.$$

Then $\lceil \mu b \rceil_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \geq \lceil \lambda b \rceil_{S,\lambda A}$. Hence, the inequality $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$ is dominated by $\mu A x \geq \lceil \mu b \rceil_{S,\mu A}$, as the former is implied by the latter and a nonnegative combination of the inequalities in $Ax \geq b$, as required. ◇

By Claim 7, μ satisfies (3), and this finishes the proof. □

Using Lemma 5.21, we can prove the following theorem. Recall that

$$\Pi = \{(\alpha, \beta) : \ (-\alpha, -\beta) \in \Pi^*_{P^\uparrow}, \ \beta/\alpha_j \leq M^* \ \text{ for all } j \in \text{support}(\alpha, I_2)\}.$$

Theorem 5.22 ([52]). *$P^\uparrow_S = P^\uparrow_{S,\Pi}$, and in particular, $P^\uparrow_S$ is a rational polyhedron.*

Proof. By (5.4), $P^\uparrow_S = P^\uparrow_{S,\Pi^*_{P^\uparrow}}$. As $\Pi \subseteq \Pi^*_{P^\uparrow}$, by Remark 5.2, $P^\uparrow_{S,\Pi^*_{P^\uparrow}} \subseteq P^\uparrow_{S,\Pi}$. We will show that $\alpha x \geq \beta$ for every $(-\alpha, -\beta) \in \Pi^*_{P^\uparrow}$ is valid for $P^\uparrow_{S,\Pi}$, thereby proving that $P^\uparrow_{S,\Pi^*_{P^\uparrow}} = P^\uparrow_{S,\Pi}$. To this end, take a vector (α, β) such that $(-\alpha, -\beta) \in \Pi^*_{P^\uparrow}$. It follows from (5.22) and (5.23) that $(\alpha, \beta) = (\lambda A, \lambda b)$ for some $\lambda \in \mathbb{R}^m_+$ and $\lambda b = \min\{\lambda A x : x \in P^\uparrow\}$. After relabeling the rows of $Ax \geq b$, we may assume that $\lambda_1 \geq \cdots \geq \lambda_m$.

We will first show that for each $j \in \text{support}(\lambda A, I_2)$,

$$0 \leq \frac{\lambda b}{(\lambda A)_j} \leq \frac{mB}{C} r(\lambda, A) \tag{5.46}$$

Let t stand for $t(\lambda, A)$. As A, b, λ are nonnegative and $(\lambda A)_j > 0$, we have $0 \leq \lambda b/(\lambda A)_j$. Furthermore,

$$\frac{\lambda b}{(\lambda A)_j} = \frac{\sum_{i=1}^m \lambda_i b_i}{\sum_{i=1}^m \lambda_i a_{ij}} \leq \frac{\lambda_1 \sum_{i=1}^m b_i}{\lambda_t \sum_{i=1}^t a_{ij}} = r(\lambda, A) \frac{\sum_{i=1}^m b_i}{\sum_{i=1}^t a_{ij}}.$$

As $\bigcup_{i=1}^t \text{support}\,(a_i, I_2) = \text{support}\,(\lambda A, I_2)$, we can infer that $0 \neq a_{kj} \geq C$ for some $1 \leq k \leq t$. Thus $\sum_{i=1}^t a_{ij} \geq C$. Besides, each $b_i \leq B$, and therefore $\sum_{i=1}^m b_i \leq mB$. We can conclude that $r(\lambda, A) \frac{\sum_{i=1}^m b_i}{\sum_{i=1}^t a_{ij}} \leq \frac{mB}{C} r(\lambda, A)$ and (5.46) follows.

Therefore, if $r(\lambda, A) \leq M$, then $\beta/\alpha_j = (\lambda b)/(\lambda A)_j \leq M^* = mBM/C$ for each $j \in \text{support}(\alpha, I_2)$, implying in turn that $(\alpha, \beta) \in \Pi$ and that $\alpha x \geq \beta$ is valid for $P^\uparrow_{S,\Pi}$. Thus, we may assume that $r(\lambda, A) > M$. Then, by Lemma 5.21, there exists a $\mu^1 \in \mathbb{R}^m_+$ such that

- $\|\mu^1\|_1 \leq \|\lambda\|_1 - 1$,
- $\mu^1 b = \min\{\mu^1 A x : x \in P^\uparrow\}$,
- $\mu^1 A x \geq \lceil \mu^1 b \rceil_{S,\mu^1 A}$ dominates $\lambda A x \geq \lceil \lambda b \rceil_{S,\lambda A}$.

We can repeat this argument and construct a sequence of vectors $\mu^1, \mu^2, \ldots$ such that each vector in the sequence defines an S-CG cut for $P^\uparrow$ that dominates the previous ones, and $\|\mu^i\|_1 \leq \|\mu^{i-1}\|_1 - 1$. Therefore, after at most $\|\lambda\|_1$ iterations, we must obtain a vector μ^j such that $r(\mu^j, A) \leq M$. Then $(\mu^j A, \mu^j b) \in \Pi$ and $\mu^j A x \geq \mu^j b$ is valid for $P^\uparrow_{S,\Pi}$. As $\lambda A x \geq \lambda b$ is implied by $\mu^j A x \geq \mu^j b$ and $Ax \geq b$, it follows that $\alpha x \geq \beta$ is valid for $P^\uparrow_{S,\Pi}$.

Therefore, $P_S^{\uparrow} = P_{S,\Pi}^{\uparrow}$. Since $P_{S,\Pi}^{\uparrow}$ is a rational polyhedron by Lemma 5.16, it follows that $P_S^{\uparrow}$ is a rational polyhedron, as required. □

5.4.2 Packing polyhedra

In this section, we consider packing polyhedra of the form

$$P^{\downarrow} = \Big\{ x \in \mathbb{R}^n : Ax \leq b \Big\}, \tag{5.47}$$

where $A \in \mathbb{Z}_+^{m \times n}$ and $b \in \mathbb{Z}_+^m$. In this section, we will prove that $P_S^{\downarrow}$ is a rational polyhedron where

$$S = T \times \mathbb{Z}_+^{n_2}, \quad T \subseteq \mathbb{Z}_+^{n_1} \text{ finite}, \quad n = n_1 + n_2.$$

Notice that every valid inequality for $P^{\downarrow}$ is of the form

$$\alpha x \leq \beta, \quad \alpha \geq \mathbf{0},\ \beta \geq 0.$$

Hereinafter, we use notations $N = \{1, \ldots, n_1 + n_2\}$, $I_1 = \{1, \ldots, n_1\}$ and $I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$ for convenience. The following lemma will be useful in proving that $P_S^{\downarrow}$ is a rational polyhedron. Recall that

$$\Pi_{P^{\downarrow}} = \{(\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{R} \ : \ \exists \lambda \in \mathbb{R}_+^m \text{ s.t. } \alpha = \lambda A,\ \beta \geq \lambda b\} \tag{5.48}$$

and that

$$\Pi_{P^{\downarrow}}^* = \{(\alpha, \beta) \in \Pi_{P^{\downarrow}} \ : \ \beta = \max\{\alpha x : x \in P^{\downarrow}\}\}. \tag{5.49}$$

Lemma 5.23 ([52]). *Let M^* be a positive integer, and let*

$$\Pi = \{(\alpha, \beta) \in \Pi_{P^{\downarrow}}^* : \ \beta/\alpha_j \leq M^* \ \text{ for all } j \in \operatorname{support}(\alpha, I_2)\}. \tag{5.50}$$

Then $P_{S,\Pi}^{\downarrow}$ is a rational polyhedron.

Proof. The proof is very similar to that of Lemma 5.16. Let $S^* = T \times \{1, \ldots, M^*\}^{n_2}$. Then S^* is a finite subset of S, and by Remark 5.1, $P_{S^*,\Pi}^{\downarrow} \subseteq P_{S,\Pi}^{\downarrow}$. We claim the following:

Claim 1. $P_{S^*,\Pi}^{\downarrow} = P_{S,\Pi}^{\downarrow}$.

Proof of Claim. Let $(\alpha, \beta) \in \Pi$. Then $\alpha x \leq \beta$ is valid for $P^{\downarrow}$, $\alpha \geq \mathbf{0}$, $\beta \geq 0$, and $0 \leq \beta/\alpha_j \leq M^*$ for every $j \in I_2$ such that $\alpha_j > 0$. Let $z^* = (z^1, z^2) \in S = T \times \mathbb{Z}_+^{n_2}$ be such that

$$\alpha z^* = \lfloor \beta \rfloor_{S,\alpha} = \max\{\alpha z : \ z \in S,\ \alpha z \leq \beta\} \tag{5.51}$$

Let $j \in I_2$. If $\alpha_j > 0$, then $\beta \leq M^* \alpha_j$, implying in turn that $z_j^* \leq M^*$. If $\alpha_j = 0$, then we may assume that $z_j^* = 0$. Therefore, we may assume that $z^* \in S^*$, so it follows that $\lfloor \beta \rfloor_{S^*,\alpha} = \lfloor \beta \rfloor_{S,\alpha}$. This implies that $P_{S^*,\Pi}^{\downarrow} = P_{S,\Pi}^{\downarrow}$, as required. ◇

By Claim 1, it suffices to show that $P^{\downarrow}_{S^*,\Pi}$ is a rational polyhedron. We write $\Pi = \cup_{I \subseteq I_2} \Pi(I)$ where

$$\Pi(I) = \big\{(\alpha, \beta) \in \Pi : \alpha_j > 0 \text{ if and only if } j \in I\big\}$$

and $\Pi(I) = \cup_{L \subseteq S^*} \Pi(I, L)$ where

$$\Pi(I, L) = \Big\{(\alpha, \beta) \in \Pi(I) : \alpha z \leq \lfloor \beta \rfloor_{S^*,\alpha} \text{ if and only if } z \in L\Big\}.$$

Consequently,

$$P^{\downarrow}_{S^*,\Pi} = \cup_{I \subseteq I_2} \cup_{L \subseteq S^*} P^{\downarrow}_{S^*,\Pi(I,L)}.$$

We will show that $P^{\downarrow}_{S^*,\Pi(I,L)}$ is a rational polyhedron for every $I \subseteq I_2$ and $L \subseteq S^*$. Given $I \subseteq I_2$ and $L \subseteq S^*$, let $H_{(I,L)}$ be defined as

$$H_{(I,L)} = \left\{ (\alpha, \delta) \in \mathbb{Z}^{n+1} \ : \ \exists \begin{pmatrix} \beta \in \mathbb{R}, \\ \lambda \in \mathbb{R}^m_+ \end{pmatrix} \text{ s.t. } \begin{array}{rcll} \alpha & = & \lambda A, & \\ \beta & \geq & \lambda b, & \\ \alpha z & \leq & \delta, & \forall z \in L \\ \alpha z & \geq & \beta + \frac{1}{\Delta}, & \forall z \in S^* \setminus L \\ \delta & \leq & \beta & \\ \alpha_j & \geq & 1, & \forall j \in I \\ \alpha_j & = & 0, & \forall j \in N_2 \setminus I \\ M^* \alpha_j & \geq & \beta, & \forall j \in I \end{array} \right\}$$

where Δ is the product of all distinct nonzero sub-determinants of A.

Claim 2. *Let $(\alpha, \beta) \in \Pi(I, L)$. Then $(\alpha, \lfloor \beta \rfloor_{S^*,\alpha}) \in H_{(I,L)}$.*

Proof of Claim. As $(\alpha, \beta) \in \Pi$, it follows from (5.49) and (5.50) that $\beta = \min\{\alpha x : x \in P^{\downarrow}\}$. Therefore, β is an integer multiple of $1/\Delta$. Since $\alpha z > \beta$ for $z \in S^* \setminus L$ and αz is an integer, it follows that $\alpha z \geq \beta + \frac{1}{\Delta}$. It can also be checked that $\delta = \lfloor \beta \rfloor_{S^*,\alpha}$ with α, β satisfies the constraints defining $H_{(I,L)}$. Therefore, $(\alpha, \lfloor \beta \rfloor_{S^*,\alpha}) \in H_{(I,L)}$, as required. ◇

Claim 3. *Let $(\alpha, \delta) \in H_{(I,L)}$. Then $\alpha x \leq \delta$ is valid for $P^{\downarrow}_{S^*,\Pi(I,L)}$.*

Proof of Claim. There exists β such that α, δ together with β satisfy the constraints in $H_{(I,L)}$. Notice that $\alpha x \leq \beta$ is valid for $P^{\downarrow}$ and that $\beta \geq \delta \geq \lfloor \beta \rfloor_{S^*,\alpha}$. Therefore, $\alpha x \leq \delta$ is implied by $\alpha x \leq \lfloor \beta \rfloor_{S^*,\alpha}$, so $\alpha x \leq \delta$ is valid for $P^{\downarrow}_{S^*,\Pi(I,L)}$, as required. ◇

By Claims 2 & 3 and Lemma 5.3,

$$P^{\downarrow}_{S^*,\Pi(I,L)} = \bigcap_{(\alpha,\delta) \in H_{(I,L)}} \big\{x \in P^{\downarrow} : \ \alpha x \leq \delta\big\} = \bigcap_{(\alpha,\delta) \in C_{(I,L)}} \big\{x \in P^{\downarrow} : \ \alpha x \leq \delta\big\}$$

where $C_{(I,L)}$ denotes the recession cone of the continuous relaxation of $H_{(I,L)}$. Moreover, by Lemma 5.3, $P^{\downarrow}_{S^*,\Pi(I,L)}$ is a rational polyhedron. □

As Lemma 5.21, we will prove Lemma 5.24. The proof of Lemma 5.24 is basically the same as that of Lemma 5.21. Given $\lambda \in \mathbb{R}^m_+$, as in Definition 5.17, we can define the tilting ratio of λ with respect to A, and we denote it by $r(\lambda, A)$. Let B, C, D, M_i for $i \in [m-1]$, and M be defined as in Definition 5.18.

Lemma 5.24 ([52]). *Let $\lambda \in \mathbb{R}^m_+$ be such that $\lambda b = \max\{\lambda Ax : x \in P^{\downarrow}\}$. If $r(\lambda, A) > M$, then there exists $\mu \in \mathbb{R}^m_+$ that satisfies the following:*

(1) $\|\mu\|_1 \le \|\lambda\|_1 - 1$,

(2) $\mu b = \max\{\mu Ax : x \in P^{\downarrow}\}$,

(3) $\mu Ax \le \lfloor \mu b \rfloor_{S,\mu A}$ *dominates* $\lambda Ax \le \lfloor \lambda b \rfloor_{S,\lambda A}$.

Proof. After relabeling the rows of $Ax \le b$, we may assume that $\lambda_1 \ge \cdots \ge \lambda_m$. Let $t(\lambda, A)$ be defined as in Definition 5.17. If $t(\lambda, A) = 1$, we have $r(\lambda, A) = 1 \le M$, a contradiction to our assumption. So, $t(\lambda, A) \ge 2$. Let t stand for $t(\lambda, A)$. Let Δ and k be defined as in (5.30) and (5.31). As $\text{support}(\lambda A, I_2) \setminus \bigcup_{i=1}^{t-1} \text{support}(a_i, I_2)$ is not empty, it follows that k is a well-defined index. Moreover, as $r(\lambda, A) > M_1 \times \cdots \times M_{m-1}$, there exists some $\ell \in \{1, \ldots, t-1\}$ such that

$$\lambda_\ell / \lambda_{\ell+1} > M_\ell \quad \text{and} \quad \lambda_i / \lambda_{i+1} \le M_i \text{ for all } i = 1, \ldots, \ell - 1. \tag{5.52}$$

Using the Simultaneous Diophantine Approximation theorem (with $k = \ell - 1$ and $r_i = \lambda_i / \lambda_\ell$ for $i \in [\ell - 1]$), as Claim 1 in the proof of Lemma 5.21, we can prove the following claim:

Claim 1. *If $\ell \ge 2$, there exist positive integers $p_1 \ge \cdots \ge p_\ell$ that satisfy the following:*

$$\left| \frac{\lambda_i}{\lambda_\ell} - \frac{p_i}{p_\ell} \right| < \frac{\varepsilon}{p_\ell},\ i \in [\ell] \quad \textit{and} \quad p_\ell \le \left(\frac{1}{\varepsilon} \right)^{\ell - 1}. \tag{5.53}$$

where $\varepsilon = (M_1 / M_\ell)^{1/(\ell-1)}$.

By Claim 1, if $\ell \ge 2$, there exist positive integers $p_1, \ldots, p_\ell$ that satisfy (5.53). As in the proof of Lemma 5.21, we define $\mu_1, \ldots, \mu_m$ as follows:

$$\mu_i = \begin{cases} \lambda_i - p_i \Delta & \text{for } i = 1, \ldots, \ell, \\ \lambda_i & \text{otherwise} \end{cases} \tag{5.54}$$

For the case $\ell = 1$, let μ be defined as in (5.53) with $p_1 = 1$. Notice that

$$p_\ell \le \frac{M_\ell}{M_1} \quad \text{and} \quad \lambda_i > \frac{p_i}{2 p_\ell} \lambda_\ell,\ i \in [\ell]. \tag{5.55}$$

As Claim 2 in Lemma 5.21, one can prove the following:

Claim 2. *$\mu_1, \ldots, \mu_\ell \ge \mu_{\ell+1} > 0$, and in particular, $\mu \in \mathbb{R}^m_+$.*

As a consequence of Claim 2, we obtain the following:

Claim 3. *$\|\mu\|_1 \le \|\lambda\|_1 - 1$, $\text{support}(\mu A, I_2) = \text{support}(\lambda A, I_2)$, and $t(\mu, A) = t(\lambda, A)$.*

Putting Claims 2 and 3 together, it follows that $\mu \in \mathbb{R}^m_+$ and μ satisfies (1). Furthermore,

Claim 4. $\mu b = \max\left\{\mu Ax : x \in P^{\downarrow}\right\}$.

Proof of Claim. We assumed that $\lambda b = \max\left\{\lambda Ax : x \in P^{\downarrow}\right\}$. Let $X := \operatorname{argmax}\left\{\lambda Ax : x \in P^{\downarrow}\right\}$. By the complementary slackness, $\lambda_i > 0$ if and only if $a_i x^* = b_i$ for all $x^* \in X$. Notice that for all $x \in P^{\downarrow}$, $\mu Ax \leq \mu b$. By Claim 2, we know that $\operatorname{support}(\mu) = \operatorname{support}(\lambda)$, implying in turn that for all $x^* \in X$, $\mu Ax^* = \sum_{i \in \operatorname{support}(\mu)} \mu_i b_i = \mu b$. Therefore, $\mu b = \max\left\{\mu Ax : x \in P^{\downarrow}\right\}$, as required. ◇

To complete the proof, we will show that $\mu Ax \leq \lfloor \mu b \rfloor_{S,\mu A}$ dominates $\lambda Ax \leq \lfloor \lambda b \rfloor_{S,\lambda A}$. Let $Q = \{x \in \mathbb{R}^n_+ : \mu b - \Delta \leq \mu Ax \leq \mu b\}$. We next prove the following claim, which needs a technical proof:

Claim 5. *There is no point $x \in Q$ that satisfies*

$$\sum_{i=1}^{\ell} p_i a_i x \geq 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{5.56}$$

Proof of Claim. Suppose for a contradiction that there exists $\tilde{x} \in Q$ satisfying (5.56). Recall that $k \in \operatorname{support}(\lambda A, I_2)$, so by Claim 3, $k \in \operatorname{support}(\mu A, I_2)$ and thus $(\mu A)_k > 0$. Let $v = \frac{\mu b}{(\mu A)_k} e^k$. Then

$$\mu Av = \mu b \quad \text{and} \quad \sum_{i=1}^{\ell} p_i a_i v = 0 \tag{5.57}$$

since $k \notin \bigcup_{i=1}^{t-1} \operatorname{support}(a_i, I_2)$ and $a_i e^k = 0$ for $i \leq t-1$. Since $\tilde{x}, v \in Q$, $\tilde{x}$ satisfies (5.56) and v satisfies (5.57), we can take a convex combination of these points to get a point $\bar{x} \in Q$ such that

$$\sum_{i=1}^{\ell} p_i a_i \bar{x} = 1 + \sum_{i=1}^{\ell} p_i b_i. \tag{5.58}$$

Since $\bar{x} \in Q$, we have $\mu Ax \leq \mu b$, and this inequality can be rewritten as

$$\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) \leq - \sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j).$$

Since $\Delta > 0$, it follows that

$$\sum_{i=1}^{\ell} \mu_i (a_i \bar{x} - b_i) < - \sum_{j=\ell+1}^{m} \mu_j (a_j \bar{x} - b_j) + \Delta. \tag{5.59}$$

Note the (5.59) is the same as (5.40). The same argument used for proving Claim 5 in the proof of Lemma 5.21 can be repeated, and we obtain the desired contradiction. ◇

We now claim the following:

Claim 6. $\mu b - \Delta \leq \lfloor \mu b \rfloor_{S,\mu A} \leq \mu b$.

Proof of Claim. Let α, β denote $\mu A, \mu b$, respectively. There exists $z \in S$ such that $\alpha z = \lfloor \beta \rfloor_{S,\alpha}$. Let $j \in \text{support}(\alpha, I_2)$. Note that $z + e^j \in S$ and that $\alpha(z + e^j) = \alpha z + \alpha_j$. As $\alpha z = \lfloor \beta \rfloor_{S,\alpha}$, it follows that $\alpha(z + e^j) = \lfloor \beta \rfloor_{S,\alpha} + \alpha_j > \lfloor \beta \rfloor_{S,\alpha}$. That means $\alpha(z + e^j) > \beta$. So, we obtain $\lfloor \beta \rfloor_{S,\alpha} + \alpha_j > \beta$, which implies that $\lfloor \beta \rfloor_{S,\alpha} \geq \beta - \alpha_j$ for all $j \in \text{support}(\alpha, I_2)$. Thereofre, $\lfloor \beta \rfloor_{S,\alpha} \geq \beta - \Delta$ by (5.30), as required. ◇

Putting Claims 5 and 6 together, we can prove our last claim:

Claim 7. *$\mu Ax \leq \lfloor \mu b \rfloor_{S,\mu A}$ is implied by $\lambda Ax \leq \lfloor \lambda b \rfloor_{S,\lambda A}$ and the inequalities in $Ax \leq b$.*

Proof of Claim. There exists $z \in S$ such that $\mu Az = \lfloor \mu b \rfloor_{S,\mu A}$. Claim 6 implies that $\mu b - \Delta \leq \mu Az \leq \mu b$. Then, by Claim 5, it follows that

$$\sum_{i=1}^{\ell} p_i a_i z < 1 + \sum_{i=1}^{l} p_i b_i \quad \Rightarrow \quad \sum_{i=1}^{\ell} p_i a_i z = \sum_{i=1}^{\ell} p_i b_i - f$$

for some integer $f \in \left[0, \sum_{i=1}^{\ell} p_i b_i\right]$, as z is integral. Let $j = \text{argmin}\{(\lambda A)_j : j \in \text{support}(\lambda A, I_2)\}$. Then, by (5.30), $(\lambda A)_j = \Delta$. Consider $z + fe^j \in S$. Observe that

$$\lambda A\left(z + fe^j\right) = \left(\mu A + \Delta \sum_{i=1}^{\ell} p_i a_i\right) z + \Delta \sum_{i=1}^{\ell} p_i (b_i - a_i z) = \lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i,$$

which implies that $\lambda A\left(z + fe^j\right) = \lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i$. Since $\lfloor \mu b \rfloor_{S,\mu A} \leq \mu b$, we must have

$$\lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \leq \mu b + \Delta \sum_{i=1}^{\ell} p_i b_i = \lambda b.$$

Then $\lfloor \mu b \rfloor_{S,\mu A} + \Delta \sum_{i=1}^{\ell} p_i b_i \leq \lfloor \lambda b \rfloor_{S,\lambda A}$. So, the inequality $\lambda Ax \leq \lfloor \lambda b \rfloor_{S,\lambda A}$ is dominated by $\mu Ax \leq \lfloor \mu b \rfloor_{S,\mu A}$, as the former is implied by the latter and a nonnegative combination of the inequalities in $Ax \leq b$, as required. ◇

By Claim 7, μ satisfies (3), and this finishes the proof. □

Using Lemma 5.24, we can prove the following theorem. Recall that

$$\Pi = \{(\alpha, \beta) \in \Pi^*_{P^\downarrow} : \ \beta/\alpha_j \leq M^* \ \text{ for all } j \in \text{support}(\alpha, I_2)\}.$$

Theorem 5.25 ([52]). *$P^\downarrow_S = P^\downarrow_{S,\Pi}$, and in particular, $P^\downarrow_S$ is a rational polyhedron.*

Proof. By Remark 5.4, $P^\downarrow_S = P^\downarrow_{S,\Pi^*_{P^\downarrow}}$. As $\Pi \subseteq \Pi^*_{P^\downarrow}$, by Remark 5.2, $P^\downarrow_{S,\Pi^*_{P^\downarrow}} \subseteq P^\downarrow_{S,\Pi}$. We will show that, for every $(\alpha, \beta) \in \Pi^*_{P^\downarrow}$, $\alpha x \leq \beta$ is valid for $P^\downarrow_{S,\Pi}$, thereby proving that $P^\downarrow_{S,\Pi^*_{P^\downarrow}} = P^\downarrow_{S,\Pi}$. To this end, take a vector $(\alpha, \beta) \in \Pi^*_{P^\downarrow}$. It follows from (5.48) and (5.49) that $(\alpha, \beta) = (\lambda A, \lambda b)$ for some $\lambda \in \mathbb{R}^m_+$ and $\lambda b = \max\left\{\lambda Ax : x \in P^\downarrow\right\}$. After relabeling the rows of $Ax \geq b$, we may assume that $\lambda_1 \geq \cdots \geq \lambda_m$.

As in the proof of Lemma 5.22, we can show that, for each $j \in \text{support}(\lambda A, I_2)$,

$$0 \leq \frac{\lambda b}{(\lambda A)_j} \leq \frac{mB}{C} r(\lambda, A) \tag{5.60}$$

Therefore, if $r(\lambda, A) \leq M$, then $\beta/\alpha_j \leq M^* = mBM/C$ for each $j \in \text{support}(\alpha, I_2)$, implying in turn that $(\alpha, \beta) \in \Pi$ and that $\alpha x \leq \beta$ is valid for $P^{\downarrow}_{S,\Pi}$. Thus, we may assume that $r(\lambda, A) > M$. Then, by Lemma 5.24, there exists a $\mu^1 \in \mathbb{R}^m_+$ such that

- $\|\mu^1\|_1 \leq \|\lambda\|_1 - 1$,
- $\mu^1 b = \max\left\{\mu^1 Ax : x \in P^{\downarrow}\right\}$,
- $\mu^1 Ax \leq \lfloor \mu^1 b \rfloor_{S,\mu^1 A}$ dominates $\lambda Ax \leq \lfloor \lambda b \rfloor_{S,\lambda A}$.

After repeating this argument, we construct a sequence of vectors $\mu^1, \mu^2, \ldots$ such that each vector in the sequence defines an S-CG cut for $P^{\downarrow}$ that dominates the previous ones, and $\|\mu^i\|_1 \leq \|\mu^{i-1}\|_1 - 1$. Therefore, after at most $\|\lambda\|_1$ iterations, we get a vector μ^j with $r(\mu^j, A) \leq M$. Then $(\mu^j A, \mu^j b) \in \Pi$ and $\mu^j Ax \leq \mu^j b$ is valid for $P^{\downarrow}_{S,\Pi}$. As $\lambda Ax \leq \lambda b$ is implied by $\mu^j Ax \leq \mu^j b$ and $Ax \leq b$, it follows that $\alpha x \leq \beta$ is valid for $P^{\downarrow}_{S,\Pi}$. Consequently, this implies that $P^{\downarrow}_S = P^{\downarrow}_{S,\Pi}$. Since $P^{\downarrow}_{S,\Pi}$ is a rational polyhedron by Lemma 5.23, it follows that $P^{\downarrow}_S$ is a rational polyhedron, as required. □

5.5 Proof of Theorem 1.16

Now that we have proved Lemma 5.12, Proposition 5.15, Theorems 5.21 and 5.24, we are ready to prove the following theorem:

Theorem 5.26 ([52]). *Let*

$$S_G = \left\{\left(z^1, z^{2\ell}, z^{2u}, z^3\right) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_{2\ell}} \times \mathbb{Z}^{n_{2u}} \times \mathbb{Z}^{n_3} : \; z^1 \in T_G, \; z^{2\ell} \geq \ell^2, \; z^{2u} \leq u^2\right\}$$

where $T_G \in \mathbb{R}^{n_1}$ is finite, $\ell^2 \in \mathbb{R}^{2\ell}$, $u^2 \in \mathbb{R}^{2u}$. Let $P \subseteq \text{conv}(S_G)$ be a rational polyhedron. Then the S_G-Chvátal closure of P is a rational polyhedron.

Proof. By Lemma 5.12, we may assume that $S_G = S_C$ where $n_2 = n_{2l} + n_{2u}$,

$$S_C = T_C \times \mathbb{Z}^{n_2}_+ \times \mathbb{Z}^{n_3},$$

and $T_C \subseteq \mathbb{Z}^{n_1}_+$ is finite. Then, by Proposition 5.15, it is sufficient to show that W_{S,Π^U_W} and W_{S,Π^D_W}, where

$$\begin{aligned}\Pi^U_W &:= \left\{(\pi, \pi_0) \in \Pi_W : \; \pi = (\pi^1, \pi^2) \geq \mathbf{0}\right\}, \\ \Pi^D_W &:= \left\{(\pi, \pi_0) \in \Pi_W : \; \pi = (\pi^1, \pi^2) \leq \mathbf{0}\right\},\end{aligned}$$

are rational polyhedra for every $S = T \times \mathbb{Z}^{n_2}_+$, $T \subseteq \mathbb{Z}^{n_1}_+$ finite, and every rational polyhedron $W \subseteq \text{conv}(S)$. To this end, take a set $S = T \times \mathbb{Z}^{n_2}_+$ for some finite $T \subseteq \mathbb{Z}^{n_1}_+$ and a rational polyhedron $W \subseteq \text{conv}(S)$. Let

$P^\uparrow$ and $P^\downarrow$ be defined as follows:

$$P^\uparrow := W + \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \quad \text{and} \quad P^\downarrow := W - \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+.$$

Let $n = n_1 + n_2$. Since $W \subseteq \text{conv}(S)$ and $\text{conv}(S) \subseteq \mathbb{R}^n_+$, it can be easily proved that

$$P^\uparrow = \{x \in \mathbb{R}^n : Ax \geq b\} \quad \text{and} \quad P^\downarrow = \{x \in \mathbb{R}^n : Cx \leq d\}$$

for some matrices A, b, C, d of appropriate dimension whose entries are nonnegative integers.

Claim 1. $P^\uparrow_S \cap W = W_{S,\Pi^D_W}$.

Proof of Claim. We will show that $\Pi_{P^\uparrow} = \Pi^D_W$. Let $(-\alpha, -\beta) \in \Pi_{P^\uparrow}$. Then $\alpha x \geq \beta$ is a valid inequality for $P^\uparrow$. So, $\alpha x \geq \beta$ is valid for W, and there exists $\lambda \geq \mathbf{0}$ such that $\alpha = \lambda A$ and $\beta \leq \lambda b$. Since the entries of A are nonnegative, it follows that $\alpha \geq \mathbf{0}$, implying in turn that $(-\alpha, -\beta) \in \Pi^D_W$. Conversely, take $(-\alpha, -\beta) \in \Pi^D_W$. Then $\alpha x \geq \beta$ is valid for W and $\alpha \geq \mathbf{0}$, which implies that $\alpha x \geq \beta$ is valid for $P^\uparrow$ and that $(-\alpha, -\beta) \in \Pi_{P^\uparrow}$. Therefore, it follows that

$$W_{S,\Pi^D_W} = \{x \in W : \ \alpha x \geq \lceil \beta \rceil_{S,\alpha} \ \forall (-\alpha, -\beta) \in \Pi_{P^\uparrow}\} = W \cap P^\uparrow_S,$$

as required. ◇

Claim 2. $P^\downarrow_S \cap W = W_{S,\Pi^U_W}$

Proof of Claim. We will show that $\Pi_{P^\downarrow} = \Pi^U_W$. Let $(\alpha, \beta) \in \Pi_{P^\downarrow}$. Then $\alpha x \leq \beta$ is a valid inequality for $P^\downarrow$. So, $\alpha x \leq \beta$ is valid for W, and there exists $\lambda \geq \mathbf{0}$ such that $\alpha = \lambda C$ and $\beta \geq \lambda d$. Since the entries of C are nonnegative, it follows that $\alpha \geq \mathbf{0}$, implying in turn that $(\alpha, \beta) \in \Pi^U_W$. Conversely, take $(\alpha, \beta) \in \Pi^U_W$. Then $\alpha x \leq \beta$ is valid for W and $\alpha \geq \mathbf{0}$, which implies that $\alpha x \leq \beta$ is valid for $P^\downarrow$ and that $(\alpha, \beta) \in \Pi_{P^\downarrow}$. Therefore, it follows that

$$W_{S,\Pi^U_W} = \{x \in W : \ \alpha x \leq \lfloor \beta \rfloor_{S,\alpha} \ \forall (\alpha, \beta) \in \Pi_{P^\downarrow}\} = W \cap P^\downarrow_S,$$

as required. ◇

By Theorems 5.22 and 5.25, both $P^\uparrow_S$ and $P^\downarrow_S$ are rational polyhedra. In turn, by Claims 1 and 2, both W_{S,Π^D_W} and W_{S,Π^U_W} are rational polyhedra. Therefore, by Proposition 5.15, P_{S_C} is a rational polyhedron, implying in turn that the S_G-Chvátal closure of P is a rational polyhedron. □

Theorem 1.16 is a direct corollary of Theorem 5.26.

Theorem 1.16 ([52]). *Let*

$$S = \left\{\left(z^1, z^{2\ell}, z^{2u}, z^3\right) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_{2\ell}} \times \mathbb{Z}^{n_{2u}} \times \mathbb{Z}^{n_3} : \ \ell^1 \leq z^1 \leq u^1, \ z^{2\ell} \geq \ell^2, \ z^{2u} \leq u^2\right\}$$

where $\ell^1, u^1 \in \mathbb{R}^{n_1}$ such that $\ell^1 \leq u^1$, $\ell^2 \in \mathbb{R}^{2\ell}$, and $u^2 \in \mathbb{R}^{2u}$. Let $P \subseteq \text{conv}(S)$ be a rational polyhedron. Then the S-Chvátal closure of P is a rational polyhedron.

5.6 Further notes

We end this chapter with the following conjecture:

Conjecture 5.27. *Let $S = R \cap \mathbb{Z}^n$ for some rational polyhedron R, and let $P \subseteq \mathrm{conv}(S)$ be a rational polyhedron. Then the S-Chvátal closure of P is a rational polyhedron.*

Chapter 6

Intersecting restrictions in clutters

Take an integer $n \geq 3$. Recall that Δ_n, the delta of dimension n, is the clutter over ground set $[n]$ whose members are $\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}$ and that Δ_n is intersecting. Take an odd integer $n \geq 5$. Recall that an extended odd hole of dimension n is a clutter over ground set $[n]$ whose minimum cardinality members are $\{1,2\}$, $\{2,3\},\ldots,\{n-1,n\}$, $\{n,1\}$ and that the blocker of an extended odd hole is intersecting. We also saw that Q_6 and $\mathbb{L}_7$ are intersecting clutters.

Conjecture 1.26 *If a clutter $\mathcal{C}$ has no intersecting minor, then the following statements are equivalent:*

(i) $\mathcal{C}$ is ideal,

(ii) $\mathcal{C}$ has the packing property,

(iii) $\mathcal{C}$ has the max-flow min-cut property.

So, Conjecture 1.26 predicts that an ideal clutter has the max-flow min-cut property if, and only if, it has no intersecting minor.

Proposition 1.27 ([4]). *The $\tau = 2$ Conjecture and Conjecture 1.26 are equivalent.*

Proof. Assume that the $\tau = 2$ Conjecture is true. We show that Conjecture 1.26 is also true. To this end, take a clutter $\mathcal{C}$ containing no intersecting minor. If $\mathcal{C}$ has the max-flow min-cut property, then $\mathcal{C}$ has the packing property and $\mathcal{C}$ is ideal. Thus, it is sufficient to show that the idealness of $\mathcal{C}$ implies its max-flow min-cut property. To this end, we assume that $\mathcal{C}$ is ideal. Since the $\tau = 2$ Conjecture is true,

every ideal minimally non-packing clutter has covering number 2, and therefore, is intersecting. As $\mathcal{C}$ is ideal and has no intersecting minor, it follows that $\mathcal{C}$ has the packing property. By Proposition 1.23, $\mathcal{C}$ has the max-flow min-cut property if, and only if, $\mathcal{C}$ has the packing property. Therefore, $\mathcal{C}$ has the max-flow min-cut property, as required.

Assume that the $\tau = 2$ Conjecture is false. Then there is an ideal minimally non-packing clutter $\mathcal{C}$ with $\tau(\mathcal{C}) \geq 3$. Take an arbitrary element e. Notice that $\tau(\mathcal{C} \setminus \{e\}) \geq 2$. Since $\mathcal{C} \setminus \{e\}$ is a proper minor of $\mathcal{C}$, $\mathcal{C} \setminus \{e\}$ packs and thus $\nu(\mathcal{C} \setminus \{e\}) \geq 2$. This implies that $\nu(\mathcal{C}) \geq 2$, and therefore, $\mathcal{C}$ is not intersecting. Moreover, as every proper minor of $\mathcal{C}$ packs, $\mathcal{C}$ contains no intersecting clutter as a proper minor. Therefore, $\mathcal{C}$ has no intersecting minor. This in turn implies that $\mathcal{C}$ constitutes a counter-example to Conjecture 1.26, because $\mathcal{C}$ is ideal but does not pack. □

In §6.1, we prove Theorem 1.29 providing a characterization of clutters that have an intersecting restriction. In §6.2, we consider two classes of intersecting clutters, namely, the deltas and the blockers of extended odd holes, and we prove Theorem 1.31, stating that finding a delta or the blocker of an extended odd hole minor, or certifying that none exists can be done in polynomial time. This chapter is based on [4].

6.1 Finding an intersecting restriction

In this section, we prove Theorem 1.29. We begin by proving Remark 1.28.

Remark 1.28 ([4]). *A clutter $\mathcal{C}$ has an intersecting minor if, and only if, $\mathcal{C}$ has an intersecting restriction.*

Proof. ($\Leftarrow$): This direction is immediate, as a restriction is a minor. ($\Rightarrow$): Let $\mathcal{C} \setminus I/J$ be an intersecting minor for some disjoint subsets I, J of the ground set of $\mathcal{C}$. Let J_I be the set of elements that appear in a cover of $\mathcal{C} \setminus I$ of size one. Then $\mathcal{C} \setminus I/J_I$ is a restriction. As $\tau(\mathcal{C} \setminus I/J) \geq 2$, the elements in J_I must have been contracted. So, $J_I \subseteq J$ and $\mathcal{C} \setminus I/J$ is a contraction minor of $\mathcal{C} \setminus I/J_I$, implying that $\mathcal{C} \setminus I/J_I \neq \{\}, \{\emptyset\}$ and thus $\tau(\mathcal{C} \setminus I/J_I) \geq 2$. Thus, it is sufficient to argue that every two members of $\mathcal{C} \setminus I/J_I$ intersect. Suppose that $\mathcal{C} \setminus I/J_I$ has two disjoint members C_1, C_2. Since $\mathcal{C} \setminus I/J \neq \{\}$, none of $C_1 - (J \setminus J_I)$, $C_2 - (J \setminus J_I)$ is empty. Moreover, $C_1 - (J \setminus J_I)$ and $C_2 - (J \setminus J_I)$ are disjoint, implying in turn that $\mathcal{C} \setminus I/J$ has two disjoint members, a contradiction. Therefore, $\nu(\mathcal{C} \setminus I/J_I) = 1$, and $\mathcal{C} \setminus I/J_I$ is an intersecting restriction in $\mathcal{C}$, as required. □

We will need the following tool for recognizing a delta minor:

Theorem 6.1 (Abdi, Cornuéjols, Pashkovich [7]). *Let $\mathcal{C}$ be a clutter over ground set E. If $\mathcal{C}$ has three members $\{u, v\}, \{u, w\}, C$ for some distinct $u, v, w \in E$ such that $C \cap \{u, v, w\} = \{v, w\}$, then $\mathcal{C}$ has a delta as a minor.*

We say that a clutter is *strictly intersecting* if it is intersecting but no proper restriction is. Notice that if a clutter has an intersecting restriction, then it has one that is strictly intersecting. Then Remark 1.28 implies that if a clutter contains an intersecting minor, then it has a strictly intersecting restriction. Moreover, we remark the following:

Remark 6.2 ([4]). *Let $\mathcal{C}$ be a strictly intersecting clutter over ground set E. Then every intersecting minor of $\mathcal{C}$ is a contraction minor.*

Proof. Let $\mathcal{C} \setminus I/J$ be an intersecting minor of $\mathcal{C}$ for some disjoint $I, J \subseteq E$. Suppose for a contradiction that I is nonempty. Let $J_I := \{e \in E - I : \{e\} \text{ is a cover of } \mathcal{C} \setminus I\}$. Then $J_I \subseteq J$, since $\mathcal{C} \setminus I/J$ has no cover of size one. This implies that $\mathcal{C} \setminus I/J_I$ is a proper intersecting restriction in $\mathcal{C}$, contradicting the assumption that $\mathcal{C}$ is strictly intersecting. Therefore, $I = \emptyset$, as required. □

The following proposition is the key to proving Theorem 1.29:

Proposition 6.3 ([4]). *A strictly intersecting clutter has three members whose union is the ground set.*

Proof. Let $\mathcal{C}$ be a strictly intersecting clutter over ground set E.

Claim 1. *If $\mathcal{C}$ has a delta as a minor, then $\mathcal{C}$ has three members whose union is E.*

Proof of Claim. Suppose that $\mathcal{C} \setminus I/J = \Delta_n$ for some $n \geq 3$ and some disjoint $I, J \subseteq E$. Since Δ_n is intersecting, it follows from Remark 6.2 that $I = \emptyset$ and thus Δ_3 is a contraction minor of $\mathcal{C}$. So, $\mathcal{C}$ has three members C_1, C_2, C_3 such that $\{1,2\} \subseteq C_1 \subseteq \{1,2\} \cup J$, $\{1,3\} \subseteq C_2 \subseteq \{1,3\} \cup J$, and $\{2,3,\ldots,n\} \subseteq C_3 \subseteq \{2,3,\ldots,n\} \cup J$. Suppose for a contradiction that $C_1 \cup C_2 \cup C_3 \neq E$. Then there exists $e \in E - (E_1 \cup E_2 \cup E_3)$. Consider $\mathcal{C}' := \mathcal{C} \setminus \{e\}/(J - \{e\})$. Notice that $\{1,2\}$, $\{1,3\}$, $\{2,3,\ldots,n\}$ are still members of $\mathcal{C}'$, implying in turn that $\mathcal{C}'$ is intersecting. Since $\mathcal{C}'$ is not a contradiction minor, this contradicts Remark 6.2, and therefore, $C_1 \cup C_2 \cup C_3 = E$. ◇

By Claim 1, we may assume that $\mathcal{C}$ has no delta as a minor. We have $\tau(\mathcal{C}) \geq 2$, as $\mathcal{C}$ is intersecting. In fact, since $\mathcal{C}$ is strictly intersecting, we can prove the following claim:

Claim 2. *$\tau(\mathcal{C}) = 2$ and every element appears in a minimum cover.*

Proof of Claim. It is suffices to show that every element appears in a cover of size two. Suppose for a contradiction that there is an element $e \in E$ not contained in a cover of $\mathcal{C}$ of size two. Then every minimal cover of $\mathcal{C}$ containing e has cardinality at least three. Consider $\mathcal{C} \setminus \{e\}$. It follows from $b(\mathcal{C} \setminus \{e\}) = b(\mathcal{C})/\{e\}$ that every minimal cover of $\mathcal{C} \setminus \{e\}$ has cardinality at least two. Since the members of $\mathcal{C} \setminus \{e\}$ are members of $\mathcal{C}$, every two members of $\mathcal{C} \setminus \{e\}$ intersect. This implies that $\mathcal{C} \setminus \{e\}$ is a proper intersecting restriction of $\mathcal{C}$, a contradiction. Therefore, every element appears in a cover of size two, as required. ◇

Pick an element $u \in E$, and let U be defined as

$$U := \{v \in E : \{u, v\} \text{ is a cover of } \mathcal{C}\}.$$

By Claim 2, we know that $U \neq \emptyset$.

Claim 3. *U is not a cover of $\mathcal{C}$.*

Proof of Claim. Suppose for a contradiction that U is a cover of $\mathcal{C}$. Let B be a minimal cover contained in U. Then $|B| \geq 2$, and since $B \subseteq U$, there exist distinct $v, w \in B$ such that $\{u, v\}$, $\{u, w\}$ are covers of $\mathcal{C}$. So, $\{u, v\}$, $\{u, w\}$, B are minimal covers of $\mathcal{C}$. Since $B \cap \{u, v, w\} = \{v, w\}$, by Theorem 6.1, $b(\mathcal{C})$ has a delta as a minor. This implies that $\mathcal{C}$ has a delta as a minor, as $b(\Delta_n) = \Delta_n$ for $n \geq 3$, a contradiction to Claim 1. Therefore, U is not a cover, as required. ◇

By Claim 3, there is a member C_1 of $\mathcal{C}$ that is fully contained in $E - U$. Since $\{u\}$ is not a cover of $\mathcal{C}$, there is a member of $\mathcal{C}$ that does not contain u.

Claim 4. *Every member of $\mathcal{C}$ that excludes u properly contains U.*

Proof of Claim. Let C be a member of $\mathcal{C}$ not containing u. Since $\{u, v\}$ is a cover of $\mathcal{C}$, $v \in C$ for every $v \in U$, so it follows that $U \subseteq C$. Since $C_1 \cap U = \emptyset$ and $\mathcal{C}$ is intersecting, U is not a member of $\mathcal{C}$, implying in turn that $U \neq C$. Hence, U is a proper subset of C. ◇

In fact,

Claim 5. *$\mathcal{C}$ has two members C_2, C_3 such that $C_2 \cap C_3 = U$ and $C_2 \cup C_3 \subseteq E - \{u\}$.*

Proof of Claim. Notice that $\mathcal{C} \setminus \{u\}/U$ is a proper restriction of $\mathcal{C}$. By Claim 4, $\mathcal{C}$ has a member C such that $U \subset C \subseteq E - \{u\}$, which implies that $\mathcal{C} \setminus \{u\}/U \neq \{\}, \{\emptyset\}$. Since $\mathcal{C}$ is strictly intersecting, $\mathcal{C} \setminus \{u\}/U$ has two disjoint members C_2' and C_3'. This implies that $C_2 := C_2' \cup U$ and $C_3 := C_3' \cup U$ are members of $\mathcal{C}$, and therefore, $C_2 \cap C_3 = U$ and $C_2 \cup C_3 \subseteq E - \{u\}$. As a result, C_2 and C_3, are the desired members. ◇

Claim 6. $C_1 \cup C_2 \cup C_3 = E$.

Proof of Claim. Suppose for a contradiction that $C_1 \cup C_2 \cup C_3 \neq E$. Let $e \in E - (C_1 \cup C_2 \cup C_3)$. Then $e \notin U$. By Claim 2, $\{e, f\}$ for some $f \in E$ is a cover of $\mathcal{C}$. For $i \in [3]$, as $e \notin C_i$, it follows that $f \in C_i$. In particular, since $C_1 \subseteq E - U$, $f \notin U$. This implies that $f \in C_2 \setminus U$. By Claim 5, $(C_2 \setminus U) \cap C_3 = \emptyset$, implying in turn that $f \notin C_3$, a contradiction. Therefore, $C_1 \cup C_2 \cup C_3 = E$, as required. ◇

By Claim 6, C_1, C_2, C_3 are three members of $\mathcal{C}$ whose union is E, and this finishes the proof. □

Now we are ready to prove Theorem 1.29:

Theorem 1.29 ([4]). *Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent:*

(i) $\mathcal{C}$ contains an intersecting restriction,

(ii) There exist three distinct members C_1, C_2, C_3 such that the restriction of $\mathcal{C}$ obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$ is intersecting.

Proof. The direction **(i)**⇐**(ii)** is clear. **(i)**⇒**(ii)**: As $\mathcal{C}$ has an intersecting restriction, $\mathcal{C}$ contains a strictly intersecting clutter as a restriction. We may assume that for some $I \subseteq E$, the restriction of $\mathcal{C}$ obtained after restricting I is a strictly intersecting clutter. Let $J := \{e \in E - I : \{e\} \text{ is a cover of } \mathcal{C} \setminus I\}$. Then $\mathcal{C} \setminus I/J$ is a strictly intersecting restriction of $\mathcal{C}$. By Proposition 6.3, $\mathcal{C} \setminus I/J$ has three members C_1', C_2', C_3' whose union is $E - (I \cup J)$. Let $C_i := C_i' \cup J$ for $i \in [3]$. Then C_1, C_2, C_3 are members of $\mathcal{C} \setminus I$, because each element in J appears in every member of $\mathcal{C} \setminus I$. That means that C_1, C_2, C_3 are three members of $\mathcal{C}$ whose union is $E - I$. Restricting $E - (C_1 \cup C_2 \cup C_3) = I$ from $\mathcal{C}$, we obtain $\mathcal{C} \setminus I/J$, implying in turn that C_1, C_2, C_3 are the desired members of $\mathcal{C}$. □

We have shown in § 1.6 that Theorem 1.29 leads to a polynomial time algorithm finding an intersecting minor in a clutter or certifying that none exists (Theorem 1.30).

6.2 Finding a delta and the blocker of an extended odd hole minor

In this section, we prove Theorem 1.31, providing an algorithm that finds a delta or the blocker of an extended odd hole minor in a clutter or certifies that none exists. A main part of the algorithm is recognizing a *dense* restriction in a clutter. Let $\mathcal{C}$ be a clutter over ground set E such that $\tau(\mathcal{C}) \geq 2$. We say that $\mathcal{C}$ is *dense* if there exists $w \in \mathbb{R}_+^E$ such that

$$w(C) = \sum (w_e : e \in C) > \frac{\mathbf{1}^\top w}{2} \quad \forall C \in \mathcal{C}.$$

Remark 6.4. *Every dense clutter is non-ideal.*

Proof. Let $\mathcal{C}$ be a dense clutter over ground set E. Then $\tau(\mathcal{C}) \geq 2$ and there exists $w \in \mathbb{R}_+^E$ such that $w(C) > \frac{\mathbf{1}^\top w}{2}$ for all $C \in \mathcal{C}$. Let $\ell := \mathbf{1} \in \mathbb{R}_+^E$. Notice that $\min\{\ell(B) : B \in b(\mathcal{C})\} = \tau(\mathcal{C}) \geq 2$ and that $\min\{w(C) : C \in \mathcal{C}\} > \frac{w^\top \ell}{2}$, and therefore, we obtain

$$\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} > w^\top \ell.$$

By Theorem 1.20, $\mathcal{C}$ is non-ideal. □

The deltas and the blockers of extended odd holes are examples of dense clutters:

Remark 6.5. *The deltas and the blockers of extended odd holes are dense.*

Proof. Take an integer $n \geq 3$, and let $w := \left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)^\top \in \mathbb{R}_+^n$. Then

$$w(\{1,2\}) = w(\{1,3\}) = \cdots = w(\{1,n\}) = w(\{2,3,\ldots,n\}) = 1.$$

Since $\frac{\mathbf{1}^\top w}{2} = \frac{2n-3}{2n-2} < 1$, Δ_n is dense.

Take an odd integer $n \geq 5$, and let $w := (1, \ldots, 1)^\top \in \mathbb{R}^n_+$. Let $\mathcal{C}$ be the blocker of an extended odd hole of dimension n. Since every cover of an extended odd hole has cardinality at least $\frac{n+1}{2}$, $w(C) > \frac{n}{2} = \frac{\mathbf{1}^\top w}{2}$ for $C \in \mathcal{C}$, and therefore, $\mathcal{C}$ is dense, as required. □

It follows from Remarks 6.4 and 6.5 that the deltas and the blockers of extended odd holes are dense and thus non-ideal. In fact, it turns out that every dense clutter has a delta or the blocker of an extended odd hole as a minor [8].

Theorem 6.6 (Abdi and Lee [8]). *Let $\mathcal{C}$ be a clutter with m members over n elements. If $\mathcal{C}$ is dense, then $\mathcal{C}$ has a delta or the blocker of an extended odd hole as a minor, which can be found in $O(mn + n^4)$ time.*

Using this theorem, we obtain the following as a corollary:

Corollary 6.7 ([4]). *Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent:*

(i) $\mathcal{C}$ has a delta or the blocker of an extended odd hole as a minor,

(ii) $\mathcal{C}$ has a dense restriction.

Proof. **(i)⇐(ii)**: It follows from Theorem 6.6 that $\mathcal{C}$ has a delta or the blocker of an extended odd hole as a minor. **(i)⇒(ii)**: For some disjoint $I, J \subseteq E$, $\mathcal{C} \setminus I/J$ is a delta or the blocker of an extended odd hole. As $\mathcal{C} \setminus I/J$ is dense by Remark 6.5, for some $w \in \mathbb{R}_+^{E-(I \cup J)}$, $w(C') > \frac{1^\top w}{2}$ for all $C' \in \mathcal{C} \setminus I/J$. Now consider $J_I := \{e \in E - I : \{e\} \text{ is a cover of } \mathcal{C} \setminus I\}$. Then $J_I \subseteq J$, since $\tau(\mathcal{C} \setminus I/J) \geq 2$. That means that $\mathcal{C} \setminus I/J_I$ is a restriction of $\mathcal{C}$ and that $\mathcal{C} \setminus I/J$ is a contraction minor of $\mathcal{C} \setminus I/J_I$. Let $C \in \mathcal{C} \setminus I/J_I$. Then $C' \subseteq C \subseteq C' \cup (J \setminus J_I)$ for some $C' \in \mathcal{C} \setminus I/J$. Notice that we can extend w to a vector in $\mathbb{R}^{E-(I \cup J_I)}$ by setting $w_e := 0$ for all $e \in J \setminus J_I$. As $w_e = 0$ for $e \in J \setminus J_I$, we obtain $w(C) = w(C') > \frac{1^\top w}{2}$. Therefore, $\mathcal{C} \setminus I/J_I$ is dense, so $\mathcal{C}$ has a dense restriction. □

Therefore, to find a delta or the blocker of an extended odd hole minor in a clutter, it suffices to find a dense restriction.

In fact, given a clutter, one can test whether it is dense in polynomial time. Take integers $n, m \geq 1$ and a clutter $\mathcal{C}$ with m members over at most n elements. Denote by $T(n, m)$ the minimum time it takes to solve a linear program of the form

$$\begin{aligned} \text{maximize} \quad & z \\ \text{subject to} \quad & \sum (w_u : u \in C) \geq z \quad \forall C \in \mathcal{C} \\ & \mathbf{1}^\top w = 1 \\ & w \geq \mathbf{0} \end{aligned}$$

In particular, $T(n, m)$ is polynomial in n and m.

Remark 6.8 ([4]). *Let $\mathcal{C}$ be a clutter with m members over n elements. In time $T(n, m)$, one can determine whether $\mathcal{C}$ is dense.*

Proof. Notice that $\mathcal{C}$ is dense if, and only if, the optimum value of the above linear program is strictly greater than $\frac{1}{2}$. So, we can test whether $\mathcal{C}$ by solving the linear program for $\mathcal{C}$. □

Although testing whether a clutter is dense can be done in polynomial time, a clutter over ground set E has up to $2^{|E|}$ restrictions, because any subset of E can be restricted. Instead of enumerating all possible restrictions, we will use the following theorem, an analogue of Theorem 1.29 for clutters that have a dense restriction:

Theorem 6.9 ([4]). *Let $\mathcal{C}$ be a clutter over ground set E. Then the following statements are equivalent:*

(i) $\mathcal{C}$ contains a dense restriction,

(ii) There exist three distinct members C_1, C_2, C_3 such that the restriction of $\mathcal{C}$ obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$ is dense.

We will need the following tool:

Theorem 6.10 (Abdi and Lee [8]). *Let V be a set of cardinality at least 4. Let $\mathcal{C}$ be a clutter over ground set V where $\min\{|C| : C \in \mathcal{C}\} = 2$ and the minimum cardinality members correspond to the edges of a connected bipartite graph G over vertex set V with bipartition $R \cup B = V$. If R contains a member, then $\mathcal{C}$ has a delta or an extended odd hole as a minor.*

Using Theorem 6.10, we can prove the following:

Proposition 6.11 ([4]). *Take an odd integer $n \geq 5$, and let $\mathcal{C}$ be an extended odd hole over ground set $[n]$ whose minimum cardinality members are $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$. If $\mathcal{C}$ has no delta or extended odd hole as a proper minor, then for each $i \in [n]$,*

$$\left\{ i + 2k - 1 \pmod{n} : \; k = 1, 2, \ldots, \frac{n+1}{2} \right\}$$

is a minimal cover.

Proof. We will show that each set is a cover. As $\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}$ need to be covered, the minimality of each set follows. By symmetry, we may assume that $i = 1$. Suppose for a contradiction that $\{1,2,4,\ldots,n-1\}$ is not a cover. Then, for some $C \in \mathcal{C}$, $C \subseteq \{3,5,\ldots,n\}$. Consider $\mathcal{C}' := \{C\} \setminus \{1\}$. The minimum cardinality members of $\mathcal{C}'$ are $\{2,3\},\{3,4\},\{4,5\},\ldots,\{n-1,n\}$, and these members correspond to the edges of a connected bipartite graph with bipartition $\{2,4,\ldots,n-1\} \cup \{3,5,\ldots,n\}$. Since C is still a member of $\mathcal{C}'$ and contained in $\{3,5,\ldots,n\}$, it follows from Theorem 6.10 that $\mathcal{C}'$ has a delta or an extended odd hole as a minor, implying in turn that a delta or an extended odd hole as a minor is a proper minor of $\mathcal{C}$, a contradiction to our assumption. □

We say that a clutter is *strictly dense* if it is dense but no proper restriction is. Notice that if a clutter has a dense restriction, it has a strictly dense restriction. The following proposition is the key to proving Theorem 6.9:

Proposition 6.12 ([4]). *A strictly dense clutter has three members whose union is the ground set.*

Proof. Let $\mathcal{C}$ be a strictly dense clutter over ground set E.

Claim 1. *No proper deletion minor of $\mathcal{C}$ contains a delta or the blocker of an extended odd hole as a minor.*

Proof of Claim. If so, a proper deletion minor of $\mathcal{C}$ has a dense restriction by Corollary 6.7. Then it is a proper dense restriction of $\mathcal{C}$, contradicting our assumption that $\mathcal{C}$ is strictly dense. ◇

As $\mathcal{C}$ is dense, Theorem 6.6 implies that $\mathcal{C}$ has a delta or the blocker of an extended odd hole as a minor, and by Claim 1, it is a contraction minor. Pick a maximal $J \subseteq E$ such that $\mathcal{C}/J$ is a delta or the blocker of an extended odd hole. Then our maximal choice of J and Claim 1 imply that every proper minor of $\mathcal{C}/J$ is neither delta nor the blocker of an extended odd hole.

Claim 2. *$\mathcal{C}/J$ has three members C'_1, C'_2, C'_3 such that $C'_1 \cap C'_2 \cap C'_3 = \emptyset$ and $C'_1 \cup C'_2 \cup C'_3 = E - J$*

Proof of Claim. If $\mathcal{C}/J$ is a delta, we may assume that $\mathcal{C}/J = \Delta_n$ for some $n \geq 3$. Then $C'_1 := \{1, 2\}$, $C'_2 := \{1, 3\}$, $C'_3 := \{2, 3, \ldots, n\}$ are the desired members. Otherwise, we may assume that $\mathcal{C}/J$ is the blocker of an extended odd hole of dimension n, for some odd $n \geq 5$, whose minimum cardinality members are $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}$. As no proper minor of $\mathcal{C}/J$ is a delta or the blocker of an extended odd hole, it follows from Proposition 6.11 that $C'_1 := \{1, 2, 4, \ldots, n-1\}$, $C'_2 := \{2, 3, 5, \ldots, n\}$, $C'_3 := \{1, 3, 4, \ldots, n-1\}$ are members of $\mathcal{C}/J$. Notice that $C'_1 \cap C'_2 \cap C'_3 = \emptyset$ and $C'_1 \cup C'_2 \cup C'_3 = [n] - J$, implying in turn that C'_1, C'_2, C'_3 are the desired members. ◇

By Claim 2, $\mathcal{C}$ has three members C_1, C_2, C_3 such that $C'_i \subseteq C_i \subseteq C'_i \cup J$ for $i \in [3]$.

Claim 3. $C_1 \cup C_2 \cup C_3 = E$.

Proof of Claim. Suppose for a contradiction that $E - (C_1 \cup C_2 \cup C_3) \neq \emptyset$. Let $e \in E - (C_1 \cup C_2 \cup C_3)$. Consider $\mathcal{C}' := \mathcal{C} \backslash \{e\}/(J - \{e\})$. Notice that C'_1, C'_2, C'_3 are still members of $\mathcal{C}'$. By Claim 2, $C'_1 \cap C'_2 \cap C'_3 = \emptyset$, so $\tau(\mathcal{C}') \geq 2$. Since every member of $\mathcal{C}'$ contains a member of $\mathcal{C}/J$, $\mathcal{C}'$ must be dense too. Then Theorem 6.6 implies that $\mathcal{C}'$ has a delta or the blocker of an extended odd hole as a minor, and in particular, so does $\mathcal{C} \setminus \{e\}$. This is a contradiction to Claim 1. ◇

This finishes the proof. □

We are now ready to prove Theorem 6.9:

Proof of Theorem 6.9. The direction **(i)⇐(ii)** is immediate. **(i)⇒(ii)**: As $\mathcal{C}$ has a dense restriction, $\mathcal{C}$ contains a strictly dense clutter as a restriction. We may assume that for some $I \subseteq E$, the restriction of $\mathcal{C}$ obtained after restricting I is strictly dense. Let $J := \{e \in E - I : \{e\} \text{ is a cover of } \mathcal{C} \setminus I\}$. Then $\mathcal{C} \setminus I/J$ is a strictly dense restriction of $\mathcal{C}$. By Proposition 6.12, $\mathcal{C} \setminus I/J$ has three members C'_1, C'_2, C'_3 whose union is $E - (I \cup J)$. Let $C_i := C'_i \cup J$ for $i \in [3]$. Then C_1, C_2, C_3 are members of $\mathcal{C} \setminus I$, so C_1, C_2, C_3 are members of $\mathcal{C}$ whose union is $E - I$, implying in turn that C_1, C_2, C_3 are the desired members of $\mathcal{C}$. □

With the characterization given by Theorem 6.9 of when a clutter has a dense restriction, we can prove Theorem 1.31.

Theorem 1.31 ([4]). *Given a clutter $\mathcal{C}$ with m members over n elements where $m, n \geq 1$, one can find a delta or the blocker of an extended odd hole minor in $\mathcal{C}$ or certify that none exists in $O\left(n^4m^3(n+m)^{3.5}\log(n+m)\log\log(n+m)\right)$ time.*

Proof. Consider the following algorithm:

1. For all distinct $C_1, C_2, C_3 \in \mathcal{C}$,
   (a) take the restriction $\mathcal{C}'$ obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$,
   (b) test whether $\mathcal{C}'$ is dense, and
   (c) if $\mathcal{C}'$ is dense, find a delta or the blocker of an extended odd hole minor in $\mathcal{C}'$.
2. If the restriction obtained from every triple of distinct members is not dense, then conclude that $\mathcal{C}$ contains neither delta nor the blocker of an extended odd hole as a minor.

The correctness of this algorithm follows from Theorem 6.6 and Theorem 6.9. Notice that there are $O(m^3)$ triple of three distinct members of $\mathcal{C}$ and that, for every three distinct $C_1, C_2, C_3 \in \mathcal{C}$, it takes $O(mn)$ time to compute the restriction obtained after restricting $E - (C_1 \cup C_2 \cup C_3)$. For each restriction obtained, determining whether it is dense can be done in $T(n,m)$ time by Remark 6.8. If the restriction is dense, then it takes $O(mn+n^4)$ time to find a delta or the blocker of an extended odd hole minor by Theorem 6.6. As the algorithm checks at most one dense restriction of $\mathcal{C}$, the total running time is

$$O\left(m^3\left(T(n,m)+mn\right)\right) + O\left(mn+n^4\right).$$

We know from classic linear programming results that $T(n,m)$ is bounded above by a polynomial function in n, m. For instance, Renegar [103] gave a simple polynomial time algorithm for linear programming. After transforming the linear program into the standard form $\max\{c^\top x : Ax \geq b\}$, where A is an $m' \times n'$ matrix and L is the total number of bits needed to represent all entries of A, b, c, the linear program can be solved with $O((n'+m')^{1.5}n'^2L)$ arithmetic operations and $O((n'+m')^{1.5}n'^2L^2(\log L)(\log\log L))$ bit operations, the latter dominating the total running time. In our case, it can be readily checked that

$$m' \leq n+m+2 \quad \text{and} \quad n' \leq n+1 \quad \text{and} \quad L \leq (n+m+2)(n+1)+(n+m+2)+(n+1),$$

so

$$T(n,m) = O\left(n^4(n+m)^{3.5}\log(n+m)\log\log(n+m)\right).$$

Therefore, our algorithm terminates in

$$O\left(m^3n^4(n+m)^{3.5}\log(n+m)\log\log(n+m)\right)$$

time, as required. □

6.3 Further notes

We call a clutter *identically self-blocking* if it is equal to its blocker. Berge [19] gave the following characterization of identically self-blocking clutters.

Theorem 6.13 (Berge [19]). *A clutter $\mathcal{C}$ is identically self-blocking if, and only if, $\nu(\mathcal{C}) = \nu(b(\mathcal{C})) = 1$.*

Notice that an identically self-blocking clutter has a cover of cardinality one if and only if it has a member of cardinality one. In fact, $\{\{a\}\}$ is the only identically self-blocking clutter with a member of cardinality one. So, by Theorem 6.13, an identically self-blocking clutter other than $\{\{a\}\}$ is intersecting. Δ_n, $n \geq 3$ and $\mathbb{L}_7$ are examples of identically self-blocking clutters, and it was recently proved that

Theorem 6.14 (Abdi, Cornuéjols, Lee [3]). *An identically self-blocking clutter different from $\{\{a\}\}$ is non-ideal.*

Therefore, identically self-blocking clutters are intersecting and non-ideal. As Theorems 1.30 and 1.31, can we recognize an identically self-blocking minor in a clutter? We end this chapter with the following question.

Question 6.15. *Given a clutter $\mathcal{C}$ over ground set E, can we find an identically self-blocking minor in $\mathcal{C}$ or certify that none exists in time polynomial in $|E|, |\mathcal{C}|$?*

Chapter 7

Multipartite clutters

Take an integer $n \geq 1$. Recall that a multipartite clutter is a clutter whose ground set is partitioned into nonempty parts $E_1, \ldots, E_n$ where every member C satisfies

$$|C \cap E_i| = 1 \qquad \forall i \in [n].$$

The following is a consequence of Lehman's theorem [92]. We give an elementary proof of it.

Lemma 7.1 ([6]). *Let $\mathcal{C}$ be a minimally non-ideal clutter, and let E denote the ground set of $\mathcal{C}$. Then there is no subset F of E such that $|C \cap F| = 1$ for every member C of $\mathcal{C}$.*

Proof. Let $M(\mathcal{C})$ denote the incidence matrix of $\mathcal{C}$. Then $P := \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ has a fractional extreme point x^*, because $\mathcal{C}$ is non-ideal. Let $e \in E$. As $\mathcal{C}/\{e\}$ and $\mathcal{C} \setminus \{e\}$ are ideal, both $P \cap \{x : x_e = 0\}$ and $P \cap \{x : x_e = 1\}$ are integral polytopes. This implies that $0 < x^*_e < 1$ for each $e \in E$. Now, consider a nonsingular row submatrix A of $M(\mathcal{C})$ such that $Ax^* = \mathbf{1}$. Suppose that E has a subset F such that $|C \cap F| = 1$ for every member C of $\mathcal{C}$. Let χ_F denote the characteristic vector of F in $\{0,1\}^E$. Since $|C \cap F| = 1$ for every member C of $\mathcal{C}$, we have that $M(\mathcal{C})\chi_F = \mathbf{1}$ and thus $A\chi_F = \mathbf{1}$. As A is nonsingular, we obtain $x^* = \chi_F$, a contradiction. Therefore, there is no such subset F of E, as required. □

Recall that a minimally non-packing clutter is either ideal or minimally non-ideal. In fact, we obtain the following as an immediate consequence of Lemma 7.1:

Proposition 7.2 ([6]). *A minimally non-packing multipartite clutter is ideal.*

So, to refute the $\tau = 2$ Conjecture, it is sufficient to find a minimally non-packing multipartite clutter whose covering number is at least three. Recall that we call a clutter strictly polar if it has no intersecting restriction.

Remark 7.3 ([6]). *A minimally non-packing clutter with covering number at least three is strictly polar.*

Proof. Let $\mathcal{C}$ be a minimally non-packing clutter over ground set E with $\tau(\mathcal{C}) \geq 3$. We have $\nu(\mathcal{C} \setminus \{e\}) \geq 2$ for any $e \in E$, because $\tau(\mathcal{C} \setminus \{e\}) \geq 2$ and $\mathcal{C} \setminus \{e\}$ packs, implying in turn that $\mathcal{C}$ itself is not intersecting.

Hence, if $\mathcal{C}$ has an intersecting restriction, it must be a proper minor. Since every proper minor of $\mathcal{C}$ packs, $\mathcal{C}$ is strictly polar. □

Hence, by Proposition 7.2 and Remark 7.3,

Remark 7.4 ([6]). *A minimally non-packing multipartite clutter with covering number at least three is ideal and strictly polar.*

In this chapter, we study ideal strictly polar multipartite clutters as well as minimally non-packing multipartite clutters. In § 7.1, we show that the $\tau = 2$ Conjecture has many equivalent versions that are stated in terms of ideal strictly polar multipartite clutters, and we prove Theorems 1.32 and 1.34. In § 7.2, we study the induced clutters of multipartite clutters and provide their geometric interpretations, and we prove Theorem 1.35. In § 7.3, we study minimally non-packing multipartite clutters of bounded degree, and we prove Theorems 1.39, 1.41, 1.42 and Proposition 1.40. In § 7.4, we describe a pseudocode to generate strictly polar multipartite clutters that do not pack. This chapter is based on [6].

7.1 Multipartite clutters and the $\tau = 2$ Conjecture

The following tool will be useful throughout this chapter:

Proposition 7.5 ([6]). *Let $\mathcal{C}$ be a multipartite clutter containing no Δ_3 as a minor. If $\tau(\mathcal{C}) = 2$ and every element is in a minimum cover of $\mathcal{C}$, then $\mathcal{C}$ is a cuboid.*

Proof. Let the ground set of $\mathcal{C}$ be partitioned into $E_1, \ldots, E_n$. We may assume that E_i is a minimal cover for each $i \in [n]$. Since $\tau(\mathcal{C}) = 2$, $|E_i| \geq 2$ for $i \in [n]$. We claim that $|E_i| = 2$ for $i \in [n]$. Suppose for a contradiction that $|E_1| \geq 3$. Every element is contained in a member of $\mathcal{C}$, because it is in a minimum cover of $\mathcal{C}$. Let us pick 3 elements f_1, f_2, f_3 from E_1. By assumption, for $i \in \{1, 2, 3\}$, there is an element g_i such that $\{f_i, g_i\}$ is a minimum cover. Notice that $g_i \notin \{f_1, f_2, f_3\}$, because E_1 is a minimal cover. We claim that $\{g_1, g_2\}$, $\{g_2, g_3\}$, and $\{g_3, g_1\}$ are minimal covers of $\mathcal{C}$. By symmetry, it suffices to show that $\{g_1, g_2\}$ is a minimal cover of $\mathcal{C}$. Recall that g_1 is contained in the members of $\mathcal{C}$ not containing f_1 and g_2 is contained in the members of $\mathcal{C}$ not containing f_2. Since every member of $\mathcal{C}$ contains at most one of f_1 and f_2, it contains either g_1 or g_2. Therefore, $\{g_1, g_2\}$ is a cover of $\mathcal{C}$. This implies that $\{g_1, g_2\}$ is a minimal cover, because $\tau(\mathcal{C}) = 2$.

Now, consider the minor of $\mathcal{C}$, denoted by $\mathcal{C}'$, obtained after contracting all elements but g_1, g_2, g_3. Notice that $\{g_1, g_2\}$, $\{g_2, g_3\}$, and $\{g_3, g_1\}$ are still minimal covers of $\mathcal{C}'$. As $\{g_1, g_2\}$ is a cover of $\mathcal{C}'$, $\emptyset$ and $\{g_3\}$ are not members of $\mathcal{C}'$. Similarly, $\{g_2\}$ and $\{g_3\}$ are not members of $\mathcal{C}'$, either. Then $\{g_2, g_3\}$ is a member of $\mathcal{C}'$, because $\{g_1\}$ is not a cover. Likewise, $\{g_1, g_2\}$ and $\{g_3, g_1\}$ are also members of $\mathcal{C}'$. That means that $\mathcal{C}' = \Delta_3$, but this contradicts the assumption that $\mathcal{C}$ does not contain Δ_3 as a minor. Therefore, we get that $|E_1| = \cdots = |E_n| = 2$ and thus $\mathcal{C}$ is a cuboid, as required. □

Notice that

Remark 7.6 ([6]). *Let $\mathcal{C}$ be a clutter that does not pack but all of whose proper restrictions pack. Then every element appears in a minimum cover of $\mathcal{C}$.*

Proof. Let $e \in E$. Suppose for a contradiction that $e \in E$ does not appear in a minimum cover, then $\tau(\mathcal{C} \setminus \{e\}) = \tau(\mathcal{C})$, implying in turn that $\nu(\mathcal{C} \setminus \{e\}) = \tau(\mathcal{C})$ as $\mathcal{C} \setminus \{e\}$ packs. However, the members of $\mathcal{C} \setminus \{e\}$ are still members of $\mathcal{C}$, we have $\nu(\mathcal{C} \setminus \{e\}) \leq \nu(\mathcal{C}) < \nu(\mathcal{C}) < \tau(\mathcal{C})$, a contradiction. □

We are ready to prove Theorem 1.32.

Theorem 1.32 ([6]). *The $\tau = 2$ Conjecture, if true, implies that every minimally non-packing multipartite clutter is a cuboid.*

Proof. Let $\mathcal{C}$ be a minimally non-packing multipartite clutter. By Proposition 7.2, $\mathcal{C}$ is ideal. Then the $\tau = 2$ Conjecture, if true, implies that $\tau(\mathcal{C}) = 2$. Moreover, Remark 7.6 implies that every element appears in a minimum cover of $\mathcal{C}$. Then, by Proposition 7.5, $\mathcal{C}$ a cuboid, as required. □

Given a clutter $\mathcal{C}$ over ground set E and $w \in \mathbb{Z}_+^E$, the *replication of $\mathcal{C}$ with respect to w* is defined as the clutter obtained from $\mathcal{C}$ after replicating $w_e - 1$ times every element $e \in E$ with $w_e > 0$ and deleting from $\mathcal{C}$ every element $e \in E$ with $w_e = 0$. The following remark is a well-known fact about replication (See Remarks 2 and 3 [35]).

Remark 7.7 ([35]). *Let $\mathcal{C}$ be a clutter over ground set E. Given $w \in \mathbb{Z}_+^E$, let $\mathcal{D}$ denote the replication of $\mathcal{C}$ with respect to w. Then the following statements hold:*

(1) $\tau(D) = \tau(\mathcal{C}, w)$ and $\nu(D) = \nu(\mathcal{C}, w)$.

(2) If $\mathcal{C}$ is ideal, so is $\mathcal{D}$.

In fact, replication also preserves strict polarity.

Remark 7.8 ([6]). *Let $\mathcal{C}$ be a strictly polar clutter over ground set E. For every $w \in \mathbb{Z}_+^E$, the replication of $\mathcal{C}$ with respect to w is also strictly polar.*

Proof. Let e be an element of $\mathcal{C}$. $\mathcal{C} \setminus \{e\}$ is a minor of $\mathcal{C}$, so every restriction of $\mathcal{C} \setminus \{e\}$ is a restriction of $\mathcal{C}$. Therefore, $\mathcal{C} \setminus \{e\}$ is strictly polar. To complete the proof, it suffices to argue that $\mathcal{D}$, the clutter obtained from $\mathcal{C}$ after replicating e, is strictly polar. Denote by e' the element obtained by replicating e. Then

$$\mathcal{D} = \mathcal{C} \cup \{C - e + e' : e \in C \in \mathcal{C}\}.$$

Notice that $\mathcal{D} \setminus \{e'\} = \mathcal{C}$ and $\mathcal{D} \setminus \{e\} \cong \mathcal{C}$. Moreover, any restriction of $\mathcal{D} \setminus \{e'\}$ or $\mathcal{D} \setminus \{e\}$ two disjoint members. Let $\mathcal{D} \setminus I/J$ be a nontrivial restriction of $\mathcal{D}$. We may assume that $I \cap \{e, e'\} = \emptyset$. Then $J \cap \{e, e'\} = \emptyset$ as well, implying in turn that $\mathcal{C} \setminus I/J$ is also a nontrivial restriction of $\mathcal{C}$. So, $\mathcal{C} \setminus I/J$ has two disjoint members. Since $I \cap \{e, e'\} = \emptyset$ and $J \cap \{e, e'\} = \emptyset$, the members of $\mathcal{C} \setminus I/J$ are still members of $\mathcal{D} \setminus I/J$. That means $\mathcal{D} \setminus I/J$ contains the two disjoint members in $\mathcal{C} \setminus I/J$. Therefore, $\mathcal{D}$ is strictly polar, as required. □

Moreover, a replication of a multipartite clutter is also multipartite with the same number of parts.

Remark 7.9 ([6]). *Let $\mathcal{C}$ be a multipartite clutter over ground set E that is partitioned into n parts so that every member of $\mathcal{C}$ intersects each part exactly once. Let $w \in \mathbb{Z}_+^E$. Then the replication of $\mathcal{C}$ with respect to w is also multipartite and its ground set is also partitioned into n parts.*

Proof. Let $E_1, \ldots, E_n$ partition E so that for every member $C \in \mathcal{C}$, $|C \cap E_i| = 1$ for $i = 1, \ldots, n$. Let $e \in E$. It suffices to prove that $\mathcal{C}'$, both the clutter obtained from $\mathcal{C}$ after replicating e once, denoted $\mathcal{C}'$, and $\mathcal{C} \setminus \{e\}$ are multipartite clutters with n parts. We may assume that $e \in E_1$. Notice that

$$\mathcal{C} \setminus \{e\} = \{C \in \mathcal{C} : \ C \subseteq (E_1 - \{e\}) \cup E_2 \cup \cdots \cup E_n\}.$$

As $|C \cap (E_1 - \{e\})| = |C \cap E_2| = \cdots = |C \cap E_n| = 1$ for every $C \in \mathcal{C} \setminus \{e\}$, $\mathcal{C} \setminus \{e\}$ is a multipartite clutter with n parts. Denote by e' the element obtained by replicating e. Then

$$\mathcal{C}' = \mathcal{C} \cup \{C - e + e' : \ e \in C \in \mathcal{C}\}.$$

Notice that the ground set of $\mathcal{C}'$ is partitioned into $E_1 \cup \{e'\}, E_2, \ldots, E_n$ and that for every $C \in \mathcal{C}'$, $|C \cap (E_1 \cup \{e'\})| = |C \cap E_2| = \cdots = |C \cap E_n| = 1$. Therefore, $\mathcal{C}'$ is also a multipartite clutter, as required. □

Using Remarks 7.7, 7.8, and 7.9, we are ready to prove the following theorem:

Theorem 7.10 ([6]). *The following statements are equivalent:*

(i) **(The polarity Conjecture [2])** *Every ideal strictly polar cuboid has the packing property.*

(ii) *Every ideal strictly polar cuboid has the max-flow min-cut property.*

(iii) **(Conjecture 1.33)** *Every ideal strictly polar multipartite clutter packs.*

(iv) *Every ideal strictly polar multipartite clutter has the packing property.*

(v) *Every ideal strictly polar multipartite clutter has the max-flow min-cut property.*

(vi) **(The $\tau = 2$ Conjecture)** *Every ideal minimally non-packing clutter has covering number two.*

Proof. **(iii) ⇒ (ii)**: Suppose that there exists $S \subseteq \{0,1\}^n$ for some $n \geq 1$ such that $\text{mult}(S)$ is ideal and strictly polar but does not have the max-flow min-cut property. Choose $w \in \mathbb{Z}_+^{2n}$ so that $\tau(\text{mult}(S), w) > \nu(\text{mult}(S), w)$. Let $\mathcal{C}$ denote the replication of $\text{mult}(S)$ with respect to w. Then, by Remarks 7.7, 7.8, and 7.9, $\mathcal{C}$ is an ideal strictly polar multipartite clutter with $\tau(\mathcal{C}) = \tau(\text{mult}(S), w)$ and $\nu(\mathcal{C}) = \nu(\text{mult}(S), w)$, implying that $\mathcal{C}$ does not pack, a contradiction as we assumed (iii) holds. Therefore, we get that (iii) implies (ii), as required.

(ii) ⇒ (i) is straightforward, because the max-flow min-cut property implies the packing property.

(i) ⇔ (vi) follows from Theorem 1.14 in [2].

(vi) ⇒ (v): Suppose for a contradiction that there exists an ideal strictly polar multipartite clutter $\mathcal{C}$ that does not have the max-flow min-cut property. Let E be the ground set of $\mathcal{C}$. Then choose $w \in \mathbb{Z}_+^E$

such that $\tau(\mathcal{C}, w) > \nu(\mathcal{C}, w)$. Let $\mathcal{D}$ denote the replication of $\mathcal{C}$ with respect to w. By Remarks 7.7, 7.8, and 7.9, $\mathcal{D}$ is an ideal strictly polar multipartite clutter with $\tau(\mathcal{D}) = \tau(\mathcal{C}, w)$ and $\nu(\mathcal{D}) = \nu(\mathcal{C}, w)$. Then $\mathcal{D}$ does not pack, so $\mathcal{D}$ contains an ideal minimally non-packing minor $\mathcal{D}'$. (vi) implies that $\tau(\mathcal{D}') = 2$, and therefore, $\mathcal{D}'$ is an intersecting minor of $\mathcal{D}$. However, it follows from Remark 1.28 that $\mathcal{D}$ has an intersecting restriction, a contradiction as $\mathcal{D}$ is strictly polar. Therefore, we can conclude that every ideal strictly polar multipartite clutter has the max-flow min-cut property if (vi) is true, as required.

(v) $\Rightarrow$ **(iv)** $\Rightarrow$ **(iii)** follows immediately from the definition of the max-flow min-cut property and that of the packing property. □

In particular, the equivalence of **(iii)** and **(vi)** in Theorem 7.10 implies

Theorem 1.34 ([6]). *The $\tau = 2$ Conjecture and Conjecture 1.33 are equivalent.*

7.2 Induced clutters

Using Lemma 7.1, we can prove Theorem 1.35:

Theorem 1.35 ([6]). *A multipartite clutter is ideal if, and only if, all of its induced clutters are ideal.*

Proof. Let $\mathcal{C}$ be a multipartite clutter whose ground set is partitioned into nonempty parts $E_1, \ldots, E_n$. **(⇒)**: If $\mathcal{C}$ is ideal, then all of its induced clutters are ideal, as every minor of $\mathcal{C}$ is ideal. **(⇐)**: Assume that $\mathcal{C}$ is non-ideal. Then it has a minimally non-ideal minor $\mathcal{C}' := \mathcal{C} \setminus I/J$ obtained after deleting I and contracting J for some disjoint subsets $I, J \subseteq E_1 \cup \cdots \cup E_n$. Observe that $\mathcal{C} \setminus I$ is another multipartite clutter whose ground set is partitioned into nonempty parts $F_1, \ldots, F_n$ where $F_i := E_i \setminus I$ for $i \in [n]$. By Lemma 7.1, the ground set of $\mathcal{C}'$ does not have any of $F_1, \ldots, F_n$ as a subset. This implies that for each $i \in [n]$, $J \cap F_i \neq \emptyset$, so we have that $J \cap E_i \neq \emptyset$. Then, $\mathcal{C}'$ is a minor of an induced clutter. Therefore, one of $\mathcal{C}$'s induced clutters is non-ideal, as required. □

Recall that there is a way to represent multipartite clutters geometrically. More precisely, Remarks 1.37 and 1.38 show that there is a one-to-one correspondence between a multipartite clutter whose ground set is partitioned into $E_1, \ldots, E_n$ with $|E_i| = \omega_i \geq 1$ for $i \in [n]$ and a subset of $V(H_{\omega_1,\ldots,\omega_n})$. Given a subset $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$, recall that $\text{mult}(S)$ is the clutter over ground set $[\sum_{i=1}^{n} \omega_i]$ whose members are

$$C_v := \left\{ v_i + \sum_{j=1}^{i-1} \omega_j : \ i \in [n] \right\}, \quad v \in S.$$

Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$, where $n \geq 1$ and $\omega_i \geq 1$ for $i \in [n]$. The set obtained from $S' := S \cap \{x : x_i \notin J_i \text{ for } i \in [n]\}$, for some $J_i \subseteq [\omega_i]$ for $i \in [n]$, after dropping the coordinates where the points in S' agree on is called a *set-restriction* of S. We say that S has $R \subseteq V(H_{\delta_1,\ldots,\delta_\ell})$, where $\ell \geq 1$ and $\delta_i \geq 1$ for $i \in [\ell]$, as a set-restriction if a set-restriction of S is isomorphic to R. For example, $R_{1,1}$ is a set-restriction of

$$S = \left\{ \begin{array}{l} (1,3,1),(2,3,1),(3,1,1),(3,2,1), \\ (1,1,2),(1,2,2),(2,1,2),(2,2,2),(3,3,2), \\ (1,1,3),(1,2,3),(2,1,3),(2,2,3),(3,3,3) \end{array} \right\}$$

since $S \cap \{x : x_1 \neq 1, x_2 \neq 1, x_3 \neq 3\} = \{(2,3,1),(3,2,1),(2,2,2),(3,3,2)\}$ is isomorphic to $R_{1,1}$ (see Figure 7.1).

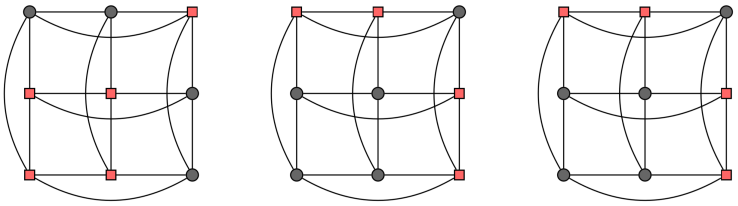


Figure 7.1: A set in $V(H_{3,3,3})$ that has $R_{1,1}$ as a set-restriction

It can be easily shown that

Remark 7.11 ([6]). *Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ where $n \geq 1$ and $\omega_i \geq 1$ for $i \in [n]$, and let $R \subseteq V(H_{\delta_1,\ldots,\delta_\ell})$ where $\ell \geq 1$ and $\delta_i \geq 1$ for $i \in [\ell]$. If S has R as a set-restriction, then $\mathrm{mult}(R)$ is a restriction of $\mathrm{mult}(S)$.*

Conversely,

Remark 7.12 ([6]). *Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ where $n \geq 1$ and $\omega_i \geq 1$ for $i \in [n]$, and let $\mathcal{C}$ be a restriction of $\mathrm{mult}(S)$. Then there exists a set-restriction R of S such that $\mathcal{C} \cong \mathrm{mult}(R)$.*

For $a, b \in [\omega_1] \times \cdots \times [\omega_n]$, denote by $d(a,b)$ the number of coordinates a and b differ on, i.e. $d(a,b)$ is the Hamming distance between a and b. Moreover, for $a, b \in [\omega_1] \times \cdots \times [\omega_n]$, define the *distance between* a, b, denoted $\mathrm{dist}(a,b)$, as the length of a shortest ab-path in $H_{\omega_1,\ldots,\omega_n}$.

Remark 7.13 ([6]). *Take integers $n \geq 1$ and $\omega_1, \ldots, \omega_n \geq 1$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be vertices in $V(H_{\omega_1,\ldots,\omega_n})$. Then the following statements hold:*

(1) The distance between x and y in $H_{\omega_1,\ldots,\omega_n}$ is exactly $d(x,y)$.

(2) The distance between x and y is at most n.

(3) Let $k \in \{0,1,\ldots,n\}$ be the distance between x and y, and let $H[x,y]$ be the vertex-induced subgraph consisting of all the vertices that lie on a shortest xy-path. Then the smallest set-restriction of $V(H_{\omega_1,\ldots,\omega_n})$ containing $V(H[x,y])$ is a hypercube of dimension k.

Proof. **(1)**: We argue by induction on the distance between x and y in $H_{\omega_1,\ldots,\omega_n}$. The distance between two vertices is 1 if, and only, if they differ in exactly 1 coordinate and the hamming distance between them is also 1 in this case. Assume that for any pair of two vertices at distance k for some $k \geq 1$, the hamming distance between them is also k. Consider the case when the distance between x and y is $k+1$. Take a shortest path from x to y, and let y' denote the vertex sitting right before y on the path. Then the distance between x and y' is k and $d(x,y') = k$ by the induction hypothesis. As y' and y differ in just one coordinate, it is clear that $d(x,y) \leq d(x,y') + 1$. So, $d(x,y) \leq k+1$. On the other hand, we can construct a path from x to y of distance $d(x,y)$ in $H_{\omega_1,\ldots,\omega_n}$ by changing one of the coordinates where x and y are different at a time. That means the distance between x and y is at most $d(x,y)$, so we get $k+1 \leq d(x,y)$. Therefore, the distance between x and y is exactly $d(x,y)$, as required.

(2): x and y have n coordinates, and in particular, $d(x,y) \leq n$. So, the distance between x and y is at most n by (1).

(3): We would like to show that the vertex set of $H[x,y]$ is exactly $\{z \in V(H_{\omega_1,\ldots,\omega_n}) : \ z_i = x_i \text{ or } z_i = y_i \text{ for } i \in [n]\}$. Let $z \in V(H[x,y])$. We claim that for each $i \in [n]$, either $z_i = x_i$ or $z_i = y_i$. Suppose not. Then $z_i \neq x_i, y_i$ for some $i \in [n]$. In this case, we consider z' obtained after replacing the i^{th} component of z by x_i. Notice that $\text{dist}(x,z') = \text{dist}(x,z) - 1$, while $\text{dist}(z',y) \leq \text{dist}(z,y)$. As the Hamming distance satisfies the triangle inequality, $\text{dist}(x,y) \leq \text{dist}(x,z') + \text{dist}(z',y)$. So, $\text{dist}(x,y)$ is strictly less than $\text{dist}(x,z) + \text{dist}(z,y)$, implying that z does not lie on a minimum xy-path, a contradiction. Thus, $z \in H[x,y]$ satisfies $z_i = x_i$ or $z_i = y_i$ for each $i \in [n]$. Conversely, we claim that $z = (z_1, \ldots, z_n)$ with $z_i = x_i$ or $z_i = y_i$ for $i \in [n]$ is contained in $H[x,y]$. Let I and J are defined as follows:

$$I := \{i \in [n] : \ z_i \neq x_i\} \quad \text{and} \quad J := \{i \in [n] : \ z_i \neq y_i\}.$$

Then, $\text{dist}(x,z) = |I|$ and $\text{dist}(z,y) = |J|$. As $z_i = x_i$ or $z_i = y_i$ for each $i \in [n]$, $I \cup J = \{i \in [n] : \ x_i \neq y_i\}$ and $\text{dist}(x,y) = |I \cup J|$. Moreover, I and J are disjoint, because $z_i = x_i$ or $z_i = y_i$ for $i \in [n]$. As a result, we obtain $\text{dist}(x,y) = |I| + |J| = \text{dist}(x,z) + \text{dist}(z,y)$. This implies that z is on a minimum xy-path, so z is a vertex in $H[x,y]$. Therefore, we obtain

$$V(H[x,y]) = \{z \in V(H_{\omega_1,\ldots,\omega_n}) : \ z_i = x_i \text{ or } z_i = y_i \text{ for } i \in [n]\}.$$

As x and y have $n-k$ common coordinates, the vertices in $H[x,y]$ agree on exactly those $n-k$ coordinates. Hence, the smallest set-restriction of $H_{\omega_1,\ldots,\omega_n}$ containing $H[x,y]$ is obtained from $H[x,y]$ after dropping the common coordinates, implying that the set-restriction is a hypercube of dimension k, as required. □

For $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ and $x \in V(H_{\omega_1,\ldots,\omega_n})$, let $\text{ind}(S,x)$ be defined as the minor of $\text{mult}(S)$ obtained after contracting the elements in C_x. In other words,

$$\text{ind}(S,x) := \text{mult}(S)/C_x = \text{the minimal sets of } \{C_v - C_x : \ v \in S\}.$$

Notice that $\text{ind}(S,x)$ is an induced clutter of $\text{mult}(S)$. We call $\text{ind}(S,x)$ the *induced clutter of* $\text{mult}(S)$ *with respect to* x. Observe that $\text{ind}(S,x) = \{\emptyset\}$ if x is a feasible vertex.

Remark 7.14 ([6]). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Each induced clutter of* $\text{mult}(S)$ *is* $\text{ind}(S,x)$ *for some $x \in V(H_{\omega_1,\ldots,\omega_n})$.*

For two vertices $x, y \in V(H_{\omega_1,\ldots,\omega_n})$, we say that x *sees* y if y is the only feasible vertex in $H[x,y]$ (see

Remark 7.13(3) for the definition of $H[x, y]$). The following proposition provides a geometric interpretation of the members of an induced clutter.

Proposition 7.15 ([6]). *Let $\mathcal{C}$ be a multipartite clutter, and let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ with $\omega_i \geq 1$ for $i \in [n]$ be its Hamming representation. For $x \in V(H_{\omega_1,\ldots,\omega_n})$, there is a bijection between the following two sets:*

- *the members of induced clutter* $\operatorname{ind}(S, x)$ *of* $\mathcal{C}$,
- *the vertices that x sees.*

More precisely, $v \in S$ is a vertex that x sees if, and only if, $C_v - C_x$ is a member of $\operatorname{ind}(S, x)$.

Proof. If $x \in S$, then $\operatorname{ind}(S, x) = \{\emptyset\}$ and x itself is the only vertex that x sees. The assertion trivially holds in this case. Thus, we may assume that $x \notin S$.

Let C be a member of $\operatorname{ind}(S, x)$. Then $C = C_v - C_x$ for some $v \in S$. We claim that x sees v. Suppose that there exists another feasible vertex u in $H[x, v]$. Then $\operatorname{ind}(S, x)$ has a member contained in $C_u - C_x$. However, $C_u - C_x$ is strictly contained in $C_v - C_x$, because $\{u_i : u_i \neq x_i,\ i \in [n]\}$ is a proper subset of $\{v_i : v_i \neq x_i,\ i \in [n]\}$. This implies that $\operatorname{ind}(S, x)$ is not a clutter, a contradiction. Therefore, v is the only feasible vertex in $H[x, v]$, so x sees v, as required.

Let v be a vertex that x sees. We claim that $C_v - C_x$ is a member of $\operatorname{ind}(S, x)$. Suppose not. Then we can find $u \in S$ such that $C_u - C_x$ is strictly contained in $C_v - C_x$. This implies that $\{u_i : u_i \neq x_i,\ i \in [n]\}$ is strictly contained in $\{v_i : v_i \neq x_i,\ i \in [n]\}$, thereby indicating that u is contained in $H[x, v]$, a contradiction. Hence, $C_v - C_x$ is a member of $\operatorname{ind}(S, x)$, as required. □

7.3 Multipartite clutters of bounded degree

Recall that the degree of $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ is defined as the maximum number of vertices in $\overline{S} := V(H_{\omega_1,\ldots,\omega_n}) - S$ that a vertex in $\overline{S}$ is adjacent to.

> **Theorem 1.39** ([6]). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree at most k. Then every minimally non-ideal minor of* $\operatorname{mult}(S)$, *if any, has at most k elements.*

Proof. Let $\mathcal{C}^*$ be a minimally non-ideal minor of $\operatorname{mult}(S)$, if any. Theorem 1.35 implies that $\mathcal{C}^*$ is a minor of $\operatorname{ind}(S, x)$ for some $x \in V(H_{\omega_1,\ldots,\omega_n})$. Observe that $x \notin S$, as $\operatorname{ind}(S, x) = \{\emptyset\}$ otherwise. Since S is of degree at most k, x has at least $\sum_{i=1}^{n}(\omega_i - 1) - k$ feasible neighbors by Remark 1.36. Recall that for each neighbor y of x, C_x and C_y have $n-1$ common elements. Then $C_y - C_x$ has exactly 1 element, so $\operatorname{ind}(S, x)$ has at least $\sum_{i=1}^{n}(\omega_i - 1) - k$ members of cardinality 1. Since a minimally non-ideal clutter does not contain a member of cardinality 1, $\sum_{i=1}^{n}(\omega_i - 1) - k$ elements of $\operatorname{ind}(S, x)$ that belong to members of cardinality 1 in $\operatorname{ind}(S, x)$ are contracted to obtain $\mathcal{C}^*$. Therefore, $\mathcal{C}^*$ has at most k elements, as required. □

One can easily verify the following remark:

Remark 7.16. *A clutter whose members are pairwise disjoint has the max-flow min-cut property.*

We obtain the following remark as an application of Kőnig's theorem on bipartite matching and Remark 7.16:

> **Proposition 1.40 ([6]).** *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Then the following statements hold:*
>
> *(1) if $n \leq 2$, then $\text{mult}(S)$ has the max-flow min-cut property, and*
>
> *(2) if $\text{mult}(S)$ does not pack, then $n \geq 3$ and $\omega_n \geq 2$.*

Proof. **(1)**: If $n = 1$, C_x has size 1 for each $x \in S$, and therefore, the members of $\text{mult}(S)$ are pairwise disjoint. Then $\text{mult}(S)$ has the max-flow min-cut property by Remark 7.16. Consider the $n = 2$ case. $\text{mult}(S)$ has the max-flow min-cut property if, and only if, the replication of $\text{mult}(S)$ with respect to w for every $w \in \mathbb{Z}_+^{\omega_1+\omega_2}$ packs. By Remark 7.9, replications of $\text{mult}(S)$ are also multipartite clutters with 2 parts. By Remark 1.38, the replication of $\text{mult}(S)$ with respect to w for each $w \in \mathbb{Z}_+^{\omega_1+\omega_2}$ is isomorphic to $\text{mult}(S')$ where $S' \subseteq V(H_{\omega_1',\omega_2'})$ for some $\omega_1', \omega_2' \geq 1$. Therefore, it is sufficient to show that every multipartite clutter whose ground set is partitioned into 2 parts packs.

Given $S \subseteq V(H_{\omega_1,\omega_2}) = [\omega_1] \times [\omega_2]$, we construct a bipartite graph G as follows:

$$V(G) = [\omega_1] \cup [\omega_2] \quad \text{and} \quad E(G) = \{uv : (u, v) \in S \subseteq [\omega_1] \times [\omega_2]\}.$$

Notice that $\tau(\text{mult}(S))$ is exactly the minimum cardinality of a vertex cover in G, whereas $\nu(\text{mult}(S))$ is exactly the maximum cardinality of a matching in G. Then Kőnig's theorem implies that $\tau(\text{mult}(S)) = \nu(\text{mult}(S))$, so $\text{mult}(S)$ packs.

(2): If $\omega_n = 1$, then $\tau(\text{mult}(S)) = 1$ and thus $\text{mult}(S)$ packs. If $n \leq 2$, $\text{mult}(S)$ packs by part (1). Therefore, if $\text{mult}(S)$ does not pack, $n \geq 3$ and $\omega_n \geq 2$, as required. □

We will need the following remark to prove Theorem 1.41:

Remark 7.17 ([6]). *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. If $\text{mult}(S)$ does not pack but all of its proper restrictions pack, then $\nu(\text{mult}(S) \setminus C_v) = \tau(\text{mult}(S) \setminus C_v) \leq \omega_n - 2$ for every $v \in S$.*

Proof. Let $v \in S$, and consider C_v of $\text{mult}(S)$. Notice that there are at most $\tau(\text{mult}(S))-2$ pairwise disjoint members of $\text{mult}(S)$ that are disjoint from C_v. Otherwise, $\text{mult}(S)$ contains at least $\tau(\text{mult}(S))$ pairwise disjoint members, a contradiction to the assumption that $\text{mult}(S)$ does not pack. So, $\nu(\text{mult}(S) \setminus C_v) \leq \tau(\text{mult}(S)) - 2$. Observe that $\nu(\text{mult}(S) \setminus C_v) \leq \omega_n - 2$, because $\tau(\text{mult}(S)) \leq \omega_n$. As every proper restriction of $\text{mult}(S)$ packs, $\text{mult}(S) \setminus C_v$ packs and thus $\tau(\text{mult}(S) \setminus C_v) = \nu(\text{mult}(S) \setminus C_v) \leq \omega_n - 2$. □

Now we are ready to prove Theorem 1.41.

Theorem 1.41 ([6]). *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree at most k. Then the following statements hold:*

(1) if $\mathrm{mult}(S)$ contains no Δ_3 as a minor and does not pack but all of its proper restrictions pack, then $k \geq \sum_{i=1}^{n-1}(\omega_i - 2)$,

(2) if $\mathrm{mult}(S)$ contains no Δ_3 as a minor and does not pack, every proper restriction of $\mathrm{mult}(S)$ packs, and $k = \sum_{i=1}^{n-1}(\omega_i - 2)$, then $\mathrm{mult}(S) \cong Q_6$.

Proof. For simplicity G denote $H_{\omega_1,\ldots,\omega_n}$. Let the ground set of $\mathrm{mult}(S)$ be partitioned into $E_1 \cup \cdots \cup E_n$ with $|E_i| = \omega_i$ for $i \in [n]$. We claim that

Claim 1. *If $\mathrm{mult}(S)$ does not pack but all of its proper restrictions pack, then there exist $L_i \subseteq E_i$ for $i \in [n]$ that satisfy the following:*

(a) $1 \leq |L_i| \leq \omega_i - 1$ for $i \in [n]$ and $\sum_{i=1}^{n}(|L_i| - 1) \geq \sum_{i=1}^{n-1}(\omega_i - 2)$.

(b) $R_1 \cup \cdots \cup R_n$ where $R_i := E_i - L_i$ for $i \in [n]$ is a cover of $\mathrm{mult}(S)$.

Proof of Claim. Let $v \in S$. Let B be a minimum cover of $\mathrm{mult}(S) \setminus C_v$. Then $\mathrm{mult}(S) \setminus (C_v \cup B)$ has no members. Let L_i for $i \in [n]$ be defined as $E_i - (C_v \cup B)$. We know that $|E_i \cap C_v| = 1$ for $i \in [n]$ and that $|B| \leq \omega_n - 2$ by Remark 7.17. So, $1 \leq |L_i| \leq \omega_i - 1$ for $i \in [n]$. In addition,

$$\sum_{i=1}^{n} |L_i| = \sum_{i=1}^{n} |E_i - C_v| - |B| = \sum_{i=1}^{n} (|E_i| - 1) - |B|.$$

As $|B| \leq \omega_n - 2$ and $|E_i| = \omega_i$ for $i \in [n]$, it is clear that $\sum_{i=1}^{n}(|L_i| - 1) \geq \sum_{i=1}^{n-1}(\omega_i - 2)$. Notice that $R_i = E_i \cap (C_v \cup B)$ for $i \in [n]$ and $R_1 \cup \cdots \cup R_n = C_v \cup B$ is a cover of $\mathrm{mult}(S)$. ◇

(1): By Claim 1(b), no member of $\mathrm{mult}(S)$ is fully contained in $L_1 \cup \cdots \cup L_n$. In other words, $v \in V(G)$ such that $C_v \subseteq L_1 \cup \cdots \cup L_n$ is infeasible. In turn, G has a subgraph $H \cong H_{|L_1|,\ldots,|L_n|}$ such that $S \cap V(H) = \emptyset$. Let u be a vertex in H. Then the number of u's neighbors in H is $\sum_{i=1}^{n}(|L_i| - 1)$ by Remark 1.36(2). By Claim 1(a), u has at least $\sum_{i=1}^{n-1}(\omega_i - 2)$ infeasible neighbors. Therefore, we get that $k \geq \sum_{i=1}^{n-1}(\omega_i - 2)$, as H is a subgraph of G.

(2): We further assume that $k = \sum_{i=1}^{n-1}(\omega_i - 2)$, and we want to show that $\mathrm{mult}(S)$ is isomorphic to Q_6. Any vertex in H has $\sum_{i=1}^{n}(|L_i| - 1)$ infeasible neighbors that are in H. As $\sum_{i=1}^{n-1}(\omega_i - 2)$ is the maximum number of infeasible neighbors of a vertex, $\sum_{i=1}^{n-1}(\omega_i - 2) \geq \sum_{i=1}^{n}(|L_i| - 1)$ and thus we have $\sum_{i=1}^{n-1}(\omega_i - 2) = \sum_{i=1}^{n}(|L_i| - 1)$. Moreover, we know that any vertex outside H that is adjacent to a vertex

in H is feasible. Since a vertex u such that C_u is fully contained in $L_1 \cup \cdots \cup L_n$ is infeasible, every vertex $v \in V(H)$ such that C_v is fully contained in one of the following sets is feasible:

$$\begin{aligned} N_1 &:= R_1 \cup L_2 \cup \cdots \cup L_n, \\ N_2 &:= L_1 \cup R_2 \cup \cdots \cup L_n, \\ &\vdots \\ N_n &:= L_1 \cup L_2 \cup \cdots \cup R_n. \end{aligned}$$

We first show that $\omega_n = 2$. Suppose for contradiction that $\omega_n \geq 3$. We claim the following:

Claim 2. *$|\{i \in [n] : \ |L_i| = 1\}| \leq 1$ and $|\{i \in [n] : \ |L_i| = \omega_i - 1\}| \leq n - 1$.*

Proof of Claim. If there exist distinct $p, q \in [n]$ such that $|L_p| = |L_q| = 1$, then

$$\sum_{i=1}^{n}(|L_i| - 1) = \sum_{i \neq p,q}(|L_i| - 1) \leq \sum_{i=1}^{n-2}(\omega_i - 2) < \sum_{i=1}^{n-1}(\omega_i - 2)$$

where the last inequality is from $\omega_{n-1} \geq \omega_n \geq 3$. So, we have $|\{i \in [n] : \ |L_i| = 1\}| \leq 1$. If $|L_i| = \omega_i - 1$ for all $i \in [n]$, then

$$\sum_{i=1}^{n}(|L_i| - 1) = \sum_{i=1}^{n}(\omega_i - 2) > \sum_{i=1}^{n-1}(\omega_i - 2)$$

where the last inequality is implied by $\omega_n \geq 3$. Therefore, $|\{i \in [n] : \ |L_i| = \omega_i - 1\}| \leq n - 1$. ◇

Let i^* be the index in $[n]$ defined as follows:

1. If there is $i \in [n]$ such that $|L_i| = 1$, then choose this i for i^*.
2. If not, there is $i \in [n]$ such that $|L_i| \leq \omega_i - 2$ by Claim 2. Choose such i for i^*.

Pick a vertex w such that $C_w \subseteq N_{i^*}$, and remember that $w \in S$. We will argue that $\tau(\text{mult}(S) \setminus C_w) \geq \omega_n - 1$, a contradiction to Remark 7.17, thereby showing that $\omega_n = 2$. Any member of $\text{mult}(S)$ that is fully contained in

$$\begin{aligned} N'_1 &:= R'_1 \cup L'_2 \cup \cdots \cup L'_n, \\ N'_2 &:= L'_1 \cup R'_2 \cup \cdots \cup L'_n, \\ &\vdots \\ N'_n &:= L'_1 \cup L'_2 \cup \cdots \cup R'_n \end{aligned}$$

where $L'_i := L_i - C_w$ and $R'_i := R_i - C_w$ is still a member of $\text{mult}(S) \setminus C_w$. In fact, we will show that we need at least $\omega_n - 1$ elements to cover all the members contained in $N'_1 \cup \cdots \cup N'_n$. Let B be a cover of $\text{mult}(S) \setminus C_w$.

Claim 3. *B satisfies one of the following statements:*

(i) $L'_i \cup R'_i \subseteq B$ for some $i \in [n]$.

(ii) $L'_i \cup L'_j \subseteq B$ *for some distinct* $i, j \in [n]$.

(iii) $R'_1 \cup \cdots \cup R'_n \subseteq B$.

Proof of Claim. If there is $j \in [n]$ such that $L'_j \subseteq B$, then the members of $\mathrm{mult}(S) \setminus C_w$ contained in N'_i for $i \neq j$ are covered by B. To cover N'_j, B contains either R'_j or L'_i for some $i \neq j$. In this case, B satisfies either (i) or (ii). If not, there exists $e_i \in L_i$ such that $e_i \notin B$ for each $i \in [n]$. To cover the members contained in

$$\begin{gathered} R'_1 \cup \{e_2\} \cup \cdots \cup \{e_n\}, \\ \{e_1\} \cup R'_2 \cup \cdots \cup \{e_n\}, \\ \vdots \\ \{e_1\} \cup \{e_2\} \cup \cdots \cup R'_n, \end{gathered}$$

B must contain $R'_1 \cup \cdots \cup R'_n$. So, B satisfies (iii) in this case. ◇

By Claim 3, it is sufficient to claim the following to show that $|B| \geq \omega_n - 1$.

Claim 4. *The following statements hold:*

(i) $|L'_i \cup R'_i| \geq \omega_n - 1$ *for every* $i \in [n]$.

(ii) $|L'_i \cup L'_j| \geq \omega_n - 1$ *for every distinct* $i, j \in [n]$.

(iii) $|R'_1 \cup \cdots \cup R'_n| \geq \omega_n - 1$.

Proof of Claim. As $C_w \subseteq N_{i^*}$, we have

$$|R'_i| = \begin{cases} |R_i|, & \text{for } i \neq i^* \\ |R_i| - 1, & \text{for } i = i^* \end{cases} \quad \text{and} \quad |L'_i| = \begin{cases} |L_i| - 1, & \text{for } i \neq i^* \\ |L_i|, & \text{for } i = i^* \end{cases}.$$

Then $|L'_i \cup R'_i| = |L'_i| + |R'_i| = |L_i| + |R_i| - 1 = \omega_i - 1$ for each $i \in [n]$, so (i) holds.

Recall how we chose i^*. In fact, due to the choice of i^*, we can easily check that

$$1 \leq |L'_i|, |R'_i| \leq \omega_i - 2, \quad i \in [n].$$

Moreover,

$$\sum_{i=1}^{n} |L'_i| = -(n-1) + \sum_{i=1}^{n} |L_i| = 1 + \sum_{i=1}^{n-1} (\omega_i - 2).$$

Now, we are ready to show (ii) and (iii). Suppose that $|L'_p \cup L'_q| \leq \omega_n - 2$ for some distinct $p, q \in [n]$. Then we get

$$\sum_{i=1}^{n} |L'_i| = (|L'_p| + |L'_q|) + \sum_{i \neq p,q} |L'_i| \leq (\omega_n - 2) + \sum_{i \neq p,q} (\omega_i - 2) \leq \sum_{i=1}^{n-1} (\omega_i - 2).$$

This implies $\sum_{i=1}^{n} |L_i'| \leq \sum_{i=1}^{n-1} (\omega_i - 2)$, a contradiction. Thus, (ii) holds. In addition, observe that

$$|R_1' \cup \cdots \cup R_n'| = \sum_{i=1}^{n} |R_i'| = \sum_{i=1}^{n} (\omega_i - 1 - |L_i'|) = \sum_{i=1}^{n} (\omega_i - 1) - 1 - \sum_{i=1}^{n-1} (\omega_i - 2) = (\omega_n - 1) + (n-1).$$

Since $n \geq 3$, $|R_1' \cup \cdots \cup R_n'| \geq \omega_n - 1$ and thus (iii) holds. ◇

By Claim 3 and Claim 4, we know that $\tau(\text{mult}(S) \setminus C_w) \geq \omega_n - 1$, but this contradicts Remark 7.17. Therefore, $\omega_n = 2$.

Since $\omega_n = 2$ and $\text{mult}(S)$ does not pack, we have $\tau(\text{mult}(S)) = \omega_n = 2$. Since every proper restriction of $\text{mult}(S)$ packs, every element is in a minimum cover of $\text{mult}(S)$ by Remark 7.6. Then, by Proposition 7.5, $\text{mult}(S)$ is a cuboid. Since $\text{mult}(S)$ is a cuboid, we may assume that S is a subset of $\{0,1\}^n$. $\omega_1 = \cdots = \omega_n = 2$ implies $k = 0$ so that all the neighbors of an infeasible vertex in S are feasible. Let $v \in S$. Since $\text{mult}(S)$ does not pack, $\mathbf{1} - v \in \overline{S}$. Then the neighbors of $\mathbf{1} - v$ are all feasible. Thus, every neighbor of v are all infeasible because it is the antipodal vertex of a neighbor of $\mathbf{1} - v$. Therefore, S always contains $R_{1,1}$ as a set-restriction since $n \geq 3$. Then by Remark 7.11, $\text{mult}(S)$ has $\text{mult}(R_{1,1}) = Q_6$ as a restriction. Since every proper minor of $\text{mult}(S)$ packs, it must be isomorphic to Q_6. □

Consider $P_3 := \{(2,2,1),(2,1,2),(1,2,2)\}$ and $S_3 := \{(2,2,1),(2,1,2),(1,2,2),(2,2,2)\}$.

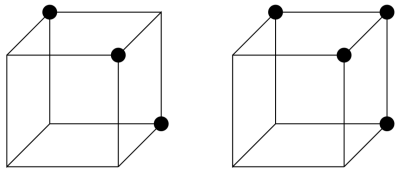


Figure 7.2: P_3, S_3

Lemma 7.18 ([2], Lemma 6.2)**.** *Take integers $n \geq 3$, $k \geq 0$. Let $S \subseteq V(H(n,2))$ be of degree at most k. If $\text{mult}(S)$ does not pack and S has none of $P_3, S_3, R_{1,1}$ as a set-restriction, then $n \leq 2k+1$.*

We will need the following lemma, an extension of Lemma 7.18:

Lemma 7.19 ([6])**.** *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ be of degree at most k. If $\nu(\text{mult}(S)) < \omega_n$ and S has none of $P_3, S_3, R_{1,1}$ as a set-restriction, then $n \leq 2k+1$.*

Proof. We argue by induction on ω_n. First, consider the case $\omega_n = 2$. Then $H_{\omega_1,\ldots,\omega_n}$ contains $H \cong H(n,2)$ as a subgraph. As H is a subgraph, $S' := S \cap V(H)$ is of degree at most k in H. Moreover, $\tau(\text{mult}(S')) \leq 2$. If $\tau(\text{mult}(S')) = 1$, then the vertices in S' agree on a coordinate, and therefore, there is an infeasible vertex of degree is at least $n-1$. That means that $n - 1 \leq k$, so $n \leq 2k+1$ clearly holds in this case. Thus we may assume that $\tau(\text{mult}(S')) = 2$. As we assumed that $\nu(\text{mult}(S)) < 2$, $\nu(\text{mult}(S')) < 2$ and $\text{mult}(S')$ does not pack. Then, by Lemma 7.18, we get that $n \leq 2k+1$, as required.

For the induction step, consider the case when $\omega_n \geq 3$. If $S = \emptyset$, the degree of S is $\sum_{i=1}^{n}(\omega_i - 1)$ by Remark 1.36, and therefore, the degree of S is at least $2n$. This implies that $k \geq 2n$, so $n \leq 2k+1$ holds if $S = \emptyset$. Thus we may assume that S is not empty. Let $v \in S$. Consider the subgraph H of $H_{\omega_1,\dots,\omega_n}$ induced by $\{x \in H_{\omega_1,\dots,\omega_n} : x_i \neq v_i,\ i \in [n]\}$. Then $H \cong H_{\omega_1-1,\dots,\omega_n-1}$, and $S' := S \cap V(H)$ is of degree at most k as H is a subgraph. Notice that $\mathrm{mult}(S') = \mathrm{mult}(S) \setminus C_v$ and that $\nu(\mathrm{mult}(S')) \leq \omega_n - 2$. It is also true that S' has none of $P_3, S_3, R_{1,1}$ as a set-restriction, because S' itself is a set-restriction of S. Then, by the induction hypothesis applied to $S' \subseteq V(H_{\omega_1-1,\dots,\omega_n-1})$, we obtain $n \leq 2k+1$, as required. □

Now we are ready to prove Theorem 1.42.

Theorem 1.42 ([6]). *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and $k \geq 0$. Let $S \subseteq V(H_{\omega_1,\dots,\omega_n})$ be of degree at most k. If $\mathrm{mult}(S)$ has a restriction that does not pack, then it has one with at most $\max\left\{\frac{11}{2}k + \frac{1}{2},\ 6\right\}$ elements.*

Proof. It can be readily checked that $\mathrm{mult}(R_{1,1})$, $\mathrm{mult}(P_3)$, $\mathrm{mult}(S_3)$ are clutters over 6 elements that do not pack. Thus we may assume that $\mathrm{mult}(S)$ contains none of $\mathrm{mult}(R_{1,1})$, $\mathrm{mult}(P_3)$, $\mathrm{mult}(S_3)$ as a restriction. In particular, $\mathrm{mult}(S)$ has no $Q_6 (= \mathrm{mult}(R_{1,1}))$ as a restriction, and by Remark 7.11, S contains none of $P_3, S_3, R_{1,1}$ as a set-restriction. Let $\mathcal{C}$ be a restriction of $\mathrm{mult}(S)$ that does not pack but all of its restrictions pack. By Remark 7.12, there exists a set-restriction $R \subseteq V(H_{\delta_1,\dots,\delta_\ell})$, for some $\ell \geq 1$ and $\delta_i \geq 1$ for $i \in [\ell]$, of S such that $\mathcal{C} \cong \mathrm{mult}(R)$. As we assumed that $\mathrm{mult}(S)$ has no Q_6 as a restriction, $\mathrm{mult}(R) \ncong Q_6$. Moreover, as $\mathrm{mult}(R)$ does not pack, $\ell \geq 3$ by Proposition 1.40 (2). Then, by Lemma 7.19, we have $\ell \leq 2k+1$. By Theorem 1.41, $1 + \sum_{i=1}^{\ell-1}(\delta_i - 2) \leq k$. Notice that

$$\sum_{i=1}^{\ell} \delta_i \leq \frac{\ell}{\ell-1}\sum_{i=1}^{\ell-1}\delta_i \leq \frac{\ell}{\ell-1}(2\ell - 3 + k) = 2\ell + \frac{\ell}{\ell-1}(k-1) \leq (4k+2) + \frac{3}{2}(k-1) = \frac{11}{2}k + \frac{1}{2}.$$

As $\mathrm{mult}(R)$ has $\sum_{i=1}^{\ell}\delta_i$ elements, it has at most $\frac{11}{2}k + \frac{1}{2}$ elements. □

7.4 A pseudocode to generate strictly polar multipartite clutters that do not pack

In this section, we will describe a pseudocode for generating strictly polar multipartite clutters that do not pack. Take integers $n \geq 1$, $\omega_1, \dots, \omega_n \geq 1$. Let $\omega := \min\{\omega_i : i \in [n]\}$. We say that $\{v^1, \dots, v^\omega\} \subseteq V(H_{\omega_1,\dots,\omega_n})$ is a *general diagonal* of $H_{\omega_1,\dots,\omega_n}$ if $v^1, \dots, v^\omega$ are ω vertices at pairwise distance n. Given $S \subseteq V(H_{\omega_1,\dots,\omega_n})$, a general diagonal consists of some feasible vertices and some infeasible ones as in the figure below (black and red vertices represent feasible and infeasible vertices, respectively). Figure 7.3 shows a general diagonal of $H_{8,7}$. Note that the picture in Figure 7.3 is not a grid, and in fact, each row represents K_7 and each column represents K_8.

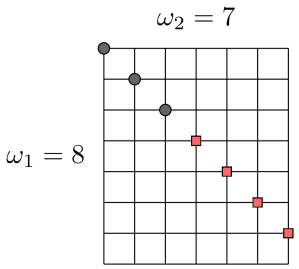


Figure 7.3: A general diagonal of $H_{8,7}$

Remark 7.20 ([6]). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Then*

$$\nu(\text{mult}(S)) = \max\{|L \cap S| : L \text{ is a general diagonal of } H_{\omega_1,\ldots,\omega_n}\}.$$

In particular, if there is a general diagonal all of whose vertices are feasible, then mult(S) *packs.*

Proof. By Remark 7.13, $\nu(\text{mult}(S))$ is equal to the maximum number of vertices in S that are at pairwise distance n, so $\nu(\text{mult}(S))$ is the maximum number of feasible vertices that a general diagonal has. Moreover, if a general diagonal has all of its vertices feasible, then it has ω_n feasible vertices at pairwise distance n and thus $\nu(\text{mult}(S)) = \omega_n$, implying in turn that mult(S) packs. □

By Remark 7.20, one can test whether a multipartite clutter packs by checking the general diagonals in its Hamming representation. How do we check if a multipartite clutter is strictly polar? We know that Theorem 1.29 provides a characterization of when a clutter is strictly polar, but this characterization is stated in terms of its members. In fact, for a multipartite clutter, the characterization can be rewritten with respect to the vertices in its Hamming representation.

Remark 7.21 ([6]). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Then the following statements are equivalent:*

(i) mult(S) *is strictly polar,*

(ii) For every three distinct vertices $u, v, w \in S$, the smallest set-restriction of S containing u, v, w has a general diagonal with at least two feasible vertices.

Proof. **(i)⇒(ii)**: Assume that mult(S) is strictly polar. Let u, v, w be three distinct vertices in S, and let R denote the smallest set-restriction of S containing u, v, w. Then R is obtained from $S \cap \{x : x_i \in \{u_i, v_i, w_i\} \text{ for } i \in [n]\}$ after dropping every coordinate $i \in [n]$ with $u_i = v_i = w_i$. Notice that mult(R) is isomorphic to the restriction of mult(S) obtained after restricting $E - (C_u \cup C_v \cup C_w)$ where E denotes the ground set of mult(S). In particular, $\tau(\text{mult}(R)) \geq 2$. As mult$(S)$ has no intersecting restriction, mult(R) is not intersecting, and therefore, $\nu(\text{mult}(R)) \geq 2$. So, by Remark 7.20, it follows that R has a general diagonal with at least two feasible vertices. **(ii)⇒(i)**: We will show the contrapositive of this direction. Assume that mult(S) is not strictly polar. Then mult(S) has an intersecting restriction. By

Theorem 1.29, there exist three vertices $u, v, w \in S$ such that the restriction of $\text{mult}(S)$ obtained after restricting $E - (C_u \cup C_v \cup C_w)$ is intersecting. Then the restriction is isomorphic to $\text{mult}(R)$ where R is the smallest set-restriction of S containing u, v, w. As $\text{mult}(R)$ is intersecting, $\nu(\text{mult}(R)) = 1$, and therefore, every general diagonal of R has at most one feasible vertex, as required. □

The following remark provides the last ingredient for our pseudocode:

Remark 7.22 ([6]). *Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. If $\text{mult}(S)$ is strictly polar and does not pack, then $n \geq 3$ and $\omega_n \geq 3$.*

Proof. Assume that $\text{mult}(S)$ is strictly polar and does not pack. By Proposition 1.40 (2), if $n \leq 2$ or $\omega_n = 1$, then $\text{mult}(S)$ packs. As $\text{mult}(S)$ does not pack, $n \geq 3$ and $\omega_n \geq 2$. Suppose for a contradiction that $\omega_n = 2$. Then we have $\tau(\text{mult}(S)) \leq 2$. If $\tau(\text{mult}(S)) = 1$, then $\text{mult}(S)$ packs. If $\tau(\text{mult}(S)) = 2$, as $\text{mult}(S)$ is not intersecting, it follows that $\nu(\text{mult}(S)) = 2$ and thus $\text{mult}(S)$ packs. This implies that $\text{mult}(S)$ packs, a contradiction. Therefore, it follows that $\omega_n \geq 3$, as required. □

Now we are ready to describe our algorithm for generating strictly polar multipartite clutters that do not pack. The correctness of our algorithm follows from Theorem 1.39, Remarks 7.20, 7.21, 7.22. A *partial set* is a triple $P = (F, I, U)$ where F, I, U partitions $V(H_{\omega_1,\ldots,\omega_n}) = [\omega_1] \times \cdots \times [\omega_n]$. We refer to F, I and U as the *feasible* points, *infeasible* points and *undecided* points of P, respectively. If $U = \emptyset$, F is the *corresponding set* of P. Now we are ready to describe our algorithm.

Input:

- dimension n & rook dimensions $\omega_1, \ldots, \omega_n$,
- degree $k \in \left\{1 + \sum_{i=1}^{n-1}(\omega_i - 2),\ \ldots,\ \sum_{i=1}^{n}(\omega_i - 1)\right\}$.

Output:

- all non-isomorphic sets of degree k in $V(H_{\omega_1,\ldots,\omega_n})$ whose multipartite clutters are strictly polar and do not pack.

Algorithm

0. Check if $n \geq 3$ and $\omega_i \geq 3$ for all $i \in [n]$. If not, there is no subset of $V(H_{\omega_1,\ldots,\omega_n})$ whose multipartite clutter is strictly polar and does not pack.
1. Enumerate all non-isomorphic sets of degree k in $V(H_{\omega_1-1,\ldots,\omega_n-1})$ whose multipartite clutters are ideal and strictly polar. Call these sets *configurations*.
2. Let $\mathcal{P}$ be the family of all partial sets originating from a configuration, i.e. initialize

$$\mathcal{P} := \{(S,\ V_0 - S,\ V - V_0) :\ S \text{ is a configuration}\}$$

where $V_0 := V(H_{\omega_1-1,\ldots,\omega_n-1})$ and $V := V(H_{\omega_1,\ldots,\omega_n})$.

3. While $\mathcal{P}$ has a partial set $P = (F, I, U)$ with $U \neq \emptyset$

   (a) If there is a general diagonal all but one of whose vertices are feasible and whose remaining vertex is undecided, update P by making the undecided point infeasible.

   (b) If P has an infeasible point with at least $k+1$ infeasible neighbors, remove P from $\mathcal{P}$ and restart Step 3.

   (c) If P has an infeasible point with k infeasible neighbors, update P by making the undecided neighbors feasible.

   (d) If P has an undecided point with at least $k+1$ infeasible neighbors, update P by making the undecided point feasible.

   (e) If there is a general diagonal all whose vertices are feasible, remove P from $\mathcal{P}$ and restart Step 3.

   (f) If there exist three distinct feasible points u, v, w such that the smallest set-restriction R of $V(H_{\omega_1,\ldots,\omega_n})$ containing u, v, w has no undecided point and has no general diagonal with at least two feasible points, remove P from $\mathcal{P}$ and restart Step 3.

   (g) Otherwise, take an undecided point q. Let P_1 and P_2 be the partial sets obtained from P after making q feasible and infeasible, respectively. Set $\mathcal{P} := \mathcal{P} \triangle \{P, P_1, P_2\}$.

Step (a) makes sure that the corresponding multipartite clutter does not pack. While adding an infeasible point in Step (a), an infeasible point with degree greater than k may have been created, and if so, Step (b) prunes the partial set. Steps (c) and (d) make sure that there is no infeasible point of degree greater than k. Adding feasible points in Steps (c) and (d) may have made the multipartite clutter pack, and if so, Step (e) prunes the partial set. Step (f) checks whether the multipartite clutter contains an intersecting restriction. Step (g) makes sure that the multipartite clutter is strictly polar.

At this point, the partial sets in $\mathcal{P}$ have no undecided point. Let $\mathcal{S}$ be the family of sets corresponding to the partial sets in $\mathcal{P}$.

4. From every isomorphic class in $\mathcal{S}$, keep only one set and filter out the other ones.

5. Output the sets in $\mathcal{S}$ whose multipartite clutters do not pack.

End of Algorithm

Our computational experiment showed the following result:

Theorem 7.23 ([6]). *Up to isomorphism, there are precisely 60 subsets of $V(H_{3,3,3})$ whose multipartite clutters are strictly polar and do not pack.*

By Theorem 1.34, if the $\tau = 2$ Conjecture is true, then every strictly polar multipartite clutter that does not pack is non-ideal. By Theorem 1.39, if the degree of a set is k, then every minimally non-ideal minor of its multipartite clutter, if any, has at most k element. In particular, a set $S \subseteq V(H_{3,3,3})$ has degree at most 6, so if $\text{mult}(S)$ is non-ideal, every minimally non-ideal minor of it is one of Δ_3, Δ_4, Δ_5, Δ_6, C_5^2, $b(C_5^2)$. In fact, as every member of $\text{mult}(S)$ has size 3 and its ground set is partitioned into three parts, none of Δ_4, Δ_5, Δ_6 is a minimally non-ideal minor of $\text{mult}(S)$. Moreover, Δ_3 and $b(C_5^2)$ are intersecting clutters, so any strictly polar clutter has none of Δ_3, $b(C_5^2)$ as a minor. Therefore, it is sufficient to check C_5^2. Using this fact, we came to the following conclusion.

Theorem 7.24 ([6]). *The multipartite clutters of the 60 subsets of $V(H_{3,3,3})$ have C_5^2 as a minor, and thus, are non-ideal.*

Theorems 7.23 and 7.24 have the following consequence:

> **Theorem 1.43 ([6]).** *Let $\mathcal{C}$ be a multipartite clutter over at most 9 elements. If $\mathcal{C}$ is ideal and strictly polar, then $\mathcal{C}$ packs.*

Proof. Assume that $\mathcal{C}$ is ideal and strictly polar. By Remark 1.38, there exists a set $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$ for some $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$ such that $\mathcal{C} = \text{mult}(S)$. By Remark 7.22, if $n \leq 2$ or $\omega_n \leq 2$, then $\mathcal{C}$ packs. Thus we may assume that $n \geq 3$ and $\omega_n \geq 3$. So, it follows that $n = 3$ and that $\omega_1 = \omega_2 = \omega_3 = 3$. Then Theorems 7.23 and 7.24 imply that $\mathcal{C}$ packs. □

7.5 Further notes

So far, we checked that there is no counter-example to Conjecture 1.33 among multipartite clutters over at most 9 elements. Our next step is to generate and check multipartite clutters over 10 to 12 elements that are ideal and strictly polar. To do so, we need to go through subsets of $V(H_{4,3,3})$, $V(H_{4,4,3})$, $v(H_{4,4,4})$, and $V(H_{3,3,3,3})$. We end this chapter with the following question.

Question 7.25. *Does any of $V(H_{4,3,3})$, $V(H_{4,4,3})$, $V(H_{4,4,4})$, $V(H_{3,3,3,3})$ have a subset whose multipartite clutter is ideal and strictly polar but does not pack?*

Chapter 8

The reflective product

Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Recall that

$$S_1 \times S_2 = \{(x, y) \in V(G_1) \times V(G_2) : \ x \in S_1 \text{ and } y \in S_2\}$$
$$S_1 * S_2 = (S_1 \times S_2) \cup \left(\overline{S_1} \times \overline{S_2}\right)$$

Let $\mathcal{C}_1, \mathcal{C}_2$ be clutters over disjoint ground sets E_1, E_2, respectively. Define the *product* of $\mathcal{C}_1$ and $\mathcal{C}_2$ as the clutter over ground set $E_1 \cup E_2$ whose members are

$$\mathcal{C}_1 \times \mathcal{C}_2 := \{C_1 \cup C_2 : C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2\}$$

and the *coproduct* of $\mathcal{C}_1$ and $\mathcal{C}_2$ as the clutter over ground set $E_1 \cup E_2$ whose members are

$$\mathcal{C}_1 \oplus \mathcal{C}_2 := \text{ the minimal sets of } \mathcal{C}_1 \cup \mathcal{C}_2.$$

Remark 8.1 ([2, 6])**.** *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Then the following statements hold:*

(1) $\text{mult}(S_1 \times S_2) = \text{mult}(S_1) \times \text{mult}(S_2)$,

(2) $\text{mult}(S_1 * S_2) = \text{mult}(S_1 \times S_2) \oplus \text{mult}(\overline{S_1} \times \overline{S_2}) = \text{mult}(S_1 \times S_2) \cup \text{mult}(\overline{S_1} \times \overline{S_2})$.

Proof. **(1)**: $\text{mult}(S_1 \times S_2) = \left\{C_{(x,y)} : \ (x, y) \in S_1 \times S_2\right\} = \{C_x \cup C_y : \ x \in S_1, \ y \in S_2\} = \text{mult}(S_1) \times \text{mult}(S_2)$. **(2)**: As $S_1 * S_2 = (S_1 \times S_2) \cup \left(\overline{S_1} \times \overline{S_2}\right)$, it follows that $\text{mult}(S_1 * S_2)$ is the clutter of the minimal sets in $\text{mult}(S_1 \times S_2) \cup \text{mult}(\overline{S_1} \times \overline{S_2})$. As the members of $\text{mult}(S_1 \times S_2)$ and $\text{mult}(\overline{S_1} \times \overline{S_2})$ have the same cardinality $n_1 + n_2$, they are the minimal sets of $\text{mult}(S_1 \times S_2) \cup \text{mult}(\overline{S_1} \times \overline{S_2})$, implying in turn that $\text{mult}(S_1 * S_2) = \text{mult}(S_1 \times S_2) \cup \text{mult}(\overline{S_1} \times \overline{S_2})$. □

In § 8.1, we show some basic facts on the products and coproducts of clutters. In § 8.2, we study the products and reflective products of sets and their multipartite clutters, and we prove Theorem 1.44. In

§ 8.3, we prove Theorem 1.45 implying that an ideal minimally non-packing multipartite clutter obtained by a reflective product must be a cuboid, and we prove Theorem 1.46 giving a characterization of an ideal minimally non-packing cuboid obtained by a reflective product. The material in this chapter is based on a submitted paper [2] and a working paper [6].

8.1 Products and coproducts of clutters

In this section, we prove Proposition 8.3 on the products and coproducts and clutters.

Remark 8.2 ([2]). *For clutters $\mathcal{C}_1, \mathcal{C}_2$ over disjoint ground sets, the following statements hold:*

(1) $b(\mathcal{C}_1 \times \mathcal{C}_2) = b(\mathcal{C}_1) \oplus b(\mathcal{C}_2)$ *and* $b(\mathcal{C}_1 \oplus \mathcal{C}_2) = b(\mathcal{C}_1) \times b(\mathcal{C}_2)$,

(2) for an element e *of* $\mathcal{C}_1$, $(\mathcal{C}_1 \times \mathcal{C}_2) \setminus \{e\} = (\mathcal{C}_1 \setminus \{e\}) \times \mathcal{C}_2$ *and* $(\mathcal{C}_1 \times \mathcal{C}_2)/\{e\} = (\mathcal{C}_1/\{e\}) \times \mathcal{C}_2$,

(3) for an element e *of* $\mathcal{C}_1$, $(\mathcal{C}_1 \oplus \mathcal{C}_2) \setminus \{e\} = (\mathcal{C}_1 \setminus \{e\}) \oplus \mathcal{C}_2$ *and* $(\mathcal{C}_1 \oplus \mathcal{C}_2)/\{e\} = (\mathcal{C}_1/\{e\}) \oplus \mathcal{C}_2$.

Proof. **(1)**: It suffices to show that $b(\mathcal{C}_1 \times \mathcal{C}_2) = b(\mathcal{C}_1) \oplus b(\mathcal{C}_2)$. Let B be a member of $b(\mathcal{C}_1) \oplus b(\mathcal{C}_2)$. Then B is a minimal cover of $\mathcal{C}_1$ or $\mathcal{C}_2$, so B is a cover of $\mathcal{C}_1 \times \mathcal{C}_2$. Hence, B contains a member of $b(\mathcal{C}_1 \times \mathcal{C}_2)$. Conversely, take a member B of $b(\mathcal{C}_1 \times \mathcal{C}_2)$. Then B is a minimal cover of $\mathcal{C}_1 \times \mathcal{C}_2$. Suppose for a contradiction that B is neither a cover of $\mathcal{C}_1$ nor a cover of $\mathcal{C}_2$. Then there exist $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $B \cap C_1 = B \cap C_2 = \emptyset$, a contradiction as $C_1 \cup C_2$ is a member of $\mathcal{C}_1 \times \mathcal{C}_2$. Therefore, B is a cover of $\mathcal{C}_1$ or $\mathcal{C}_2$, implying in turn that B contains a member of $b(\mathcal{C}_1) \oplus b(\mathcal{C}_2)$. Hence, $b(\mathcal{C}_1 \times \mathcal{C}_2) = b(\mathcal{C}_1) \oplus b(\mathcal{C}_2)$. **(2)** and **(3)** are immediate. □

Using this remark, we can easily prove the following:

Proposition 8.3 ([2]). *Let $\mathcal{C}_1, \mathcal{C}_2$ be clutters over disjoint ground sets. Then the following statements hold:*

(1) if $\mathcal{C}_1, \mathcal{C}_2$ are ideal, then so are $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$,

(2) if $\mathcal{C}_1, \mathcal{C}_2$ pack, then so do $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$,

(3) if $\mathcal{C}_1, \mathcal{C}_2$ are strictly polar, then so are $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$,

(4) if $\mathcal{C}_1, \mathcal{C}_2$ have the packing property, then so do $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$,

(5) if $\mathcal{C}_1, \mathcal{C}_2$ have the max-flow min-cut property, then so do $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$.

Proof. If $\mathcal{C}_1 = \{\emptyset\}$, then $\mathcal{C}_1 \times \mathcal{C}_2 = \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{\emptyset\}$. If $\mathcal{C}_1 = \{\}$, then $\mathcal{C}_1 \times \mathcal{C}_2 = \{\}$ and $\mathcal{C}_1 \oplus \mathcal{C}_2 = C_2$. In both cases, the assertions trivially hold. Therefore, we we may assume that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are nontrivial.

(1): We first show that $\mathcal{C}_1 \oplus \mathcal{C}_2$ is ideal. Notice that $Q(\mathcal{C}_1 \oplus \mathcal{C}_2) = Q(\mathcal{C}_1) \times Q(\mathcal{C}_2)$. As $Q(\mathcal{C}_1)$ and $Q(\mathcal{C}_2)$ are integral polyhedra, so is $Q(\mathcal{C}_1 \oplus \mathcal{C}_2)$ and thus $\mathcal{C}_1 \oplus \mathcal{C}_2$ is ideal. By Theorem 1.20, $b(\mathcal{C}_1)$ and $b(\mathcal{C}_2)$ are ideal, implying in turn that $b(\mathcal{C}_1) \oplus b(\mathcal{C}_2)$ is ideal. So, by Remark 8.2 (1), $b(\mathcal{C}_1 \times \mathcal{C}_2)$ is ideal, and it follows from Theorem 1.20 that $\mathcal{C}_1 \times \mathcal{C}_2$ is ideal, as required.

(2): By Remark 8.2 (2), $\tau(\mathcal{C}_1 \times \mathcal{C}_2) = \min\{\tau(\mathcal{C}_1), \tau(\mathcal{C}_2)\}$. Since both $\mathcal{C}_1, \mathcal{C}_2$ pack, each of them has $\tau(\mathcal{C}_1 \times \mathcal{C}_2)$ disjoint members, thereby leading to $\tau(\mathcal{C}_1 \times \mathcal{C}_2)$ disjoint members in $\mathcal{C}_1 \times \mathcal{C}_2$. Thus, $\mathcal{C}_1 \times \mathcal{C}_2$ packs. Moreover, by Remark 8.2 (3), $\tau(\mathcal{C}_1 \oplus \mathcal{C}_2) = \tau(\mathcal{C}_1) + \tau(\mathcal{C}_2)$. Since $\mathcal{C}_1$ has $\tau(\mathcal{C}_1)$ disjoint members and $\mathcal{C}_2$ has $\tau(\mathcal{C}_2)$ disjoint members, it follows that $\mathcal{C}_1 \oplus \mathcal{C}_2$ has $\tau(\mathcal{C}_1 \oplus \mathcal{C}_2)$ disjoint members. Thus, $\mathcal{C}_1 \oplus \mathcal{C}_2$ packs.

(3): To prove that $\mathcal{C}_1 \times \mathcal{C}_2$ is strictly polar, we will show that every restriction of $\mathcal{C}_1 \times \mathcal{C}_2$ is not intersecting. Let $\mathcal{C}$ be a restriction of $\mathcal{C}_1 \times \mathcal{C}_2$. Then, by Remark 8.2 (2), $\mathcal{C} = \mathcal{C}_1' \times \mathcal{C}_2'$ for some restriction $\mathcal{C}_1'$ of $\mathcal{C}_1$ and some restriction $\mathcal{C}_2'$ of $\mathcal{C}_2$. It can be readily checked that $\mathcal{C}$ is not intersecting if $\mathcal{C}_1'$ or $\mathcal{C}_2'$ is trivial. Thus, we may assume that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ are nontrivial. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are strictly polar, each of $\mathcal{C}_1'$ and $\mathcal{C}_2'$ has two disjoint members, thereby leading to two disjoint members of $\mathcal{C}$. This implies that $\mathcal{C}$ is not intersecting. Therefore, $\mathcal{C}_1 \times \mathcal{C}_2$ is strictly polar. To prove that $\mathcal{C}_1 \oplus \mathcal{C}_2$ is strictly polar, we will show that every restriction of $\mathcal{C}_1 \oplus \mathcal{C}_2$ is not intersecting. Let $\mathcal{C}$ be a restriction of $\mathcal{C}_1 \oplus \mathcal{C}_2$. By Remark 8.2 (3), $\mathcal{C} = \mathcal{C}_1' \oplus \mathcal{C}_2'$ for some minor $\mathcal{C}_1'$ of $\mathcal{C}_1$ and some minor $\mathcal{C}_2'$ of $\mathcal{C}_2$. If $\mathcal{C}_1' = \{\emptyset\}$, then $\mathcal{C} = \{\emptyset\}$, so $\mathcal{C}$ is not intersecting. If $\mathcal{C}_1' = \{\}$, then $\mathcal{C} = \mathcal{C}_2'$ and thus $\mathcal{C}_2'$ is a restriction of $\mathcal{C}_2$, implying in turn that $\mathcal{C}$ is not intersecting as $\mathcal{C}_2$ is strictly polar. Thus, we may assume that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ are nontrivial. This implies that $\mathcal{C}$ has two disjoint members, so it is not intersecting. Therefore, $\mathcal{C}_1 \oplus \mathcal{C}_2$ is strictly polar.

(4): By Remark 8.2 (2), every minor of $\mathcal{C}_1 \times \mathcal{C}_2$ is the product of a minor of $\mathcal{C}_1$ and a minor of $\mathcal{C}_2$, so it follows from part (2) that every minor of $\mathcal{C}_1 \times \mathcal{C}_2$ packs. Hence, $\mathcal{C}_1 \times \mathcal{C}_2$ has the packing property. Similarly, it follows from Remark 8.2 (3) and part (3) that $\mathcal{C}_1 \oplus \mathcal{C}_2$ has the packing property.

(5): Let E_1 and E_2 be the ground sets of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let $w_1 \in \mathbb{Z}_+^{E_1}, w_2 \in \mathbb{Z}_+^{E_2}$. For $i \in \{1, 2\}$, let $\mathcal{D}_i$ denote the replication of $\mathcal{C}_i$ with respect to w_i. As $\mathcal{C}_1$ and $\mathcal{C}_2$ have the max-flow min-cut property, it follows from Remark 7.7 that $\mathcal{D}_1$ and $\mathcal{D}_2$ pack. In fact, the replication of $\mathcal{C}_1 \times \mathcal{C}_2$ with respect to (w_1, w_2) is $\mathcal{D}_1 \times \mathcal{D}_2$ and that of $\mathcal{C}_1 \oplus \mathcal{C}_2$ is $\mathcal{D}_1 \oplus \mathcal{D}_2$. So, by part (3), both $\mathcal{D}_1 \times \mathcal{D}_2$ and $\mathcal{D}_1 \oplus \mathcal{D}_2$ pack, implying in turn that $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$ have the max-flow min-cut property. □

8.2 Products and reflective products of sets

Using Remark 8.1 (1), we can show that the set product preserves the properties we considered so far:

Proposition 8.4 ([2, 6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1, \ldots, \omega_{n_1}}$ and $G_2 = H_{\delta_1, \ldots, \delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Then the following statements hold:*

(1) if $\mathrm{mult}(S_1), \mathrm{mult}(S_2)$ *are ideal, then so is* $\mathrm{mult}(S_1 \times S_2)$,

(2) if $\mathrm{mult}(S_1), \mathrm{mult}(S_2)$ *pack, then so does* $\mathrm{mult}(S_1 \times S_2)$,

(3) if $\mathrm{mult}(S_1), \mathrm{mult}(S_2)$ *are strictly polar, then so is* $\mathrm{mult}(S_1 \times S_2)$,

(4) if $\mathrm{mult}(S_1), \mathrm{mult}(S_2)$ *have the packing property, then so does* $\mathrm{mult}(S_1 \times S_2)$,

(5) if $\mathrm{mult}(S_1), \mathrm{mult}(S_2)$ *have the max-flow min-cut property, then so does* $\mathrm{mult}(S_1 \times S_2)$.

Proof. By Remark 8.1 (1), we have $\mathrm{mult}(S_1 \times S_2) = \mathrm{mult}(S_1) \times \mathrm{mult}(S_2)$. Therefore, the assertions follow from Proposition 8.3. □

Recall that for $i \in [2]$, $\mathrm{ind}(S_i, x_i) = \mathrm{mult}(S_i)/C_{x_i}$ is the induced clutter of $\mathrm{mult}(S_i)$ with respect to $x_i \in V(G_i)$ (Section 7.2).

Remark 8.5 ([2, 6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Then, viewing $\mathrm{mult}(S_1)$ and $\mathrm{mult}(S_2)$ as clutters over disjoint ground sets, we have that*

$$\mathrm{ind}\,(S_1 \times S_2, (x_1, x_2)) = \mathrm{ind}(S_1, x_1) \times \mathrm{ind}(S_2, x_2)$$

for every $(x_1, x_2) \in V(G_1) \times V(G_2)$.

Moreover,

Proposition 8.6 ([6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Then, viewing $\mathrm{mult}(S_1)$ and $\mathrm{mult}(S_2)$ as clutters over disjoint ground sets, we have*

$$\mathrm{ind}(S_1 * S_2, (x_1, x_2)) = \begin{cases} \{\emptyset\} & \text{if } x_1 \in S_1 \text{ and } x_2 \in S_2 \\ \{\emptyset\} & \text{if } x_1 \in \overline{S_1} \text{ and } x_2 \in \overline{S_2} \\ \mathrm{ind}(S_1, x_1) \oplus \mathrm{ind}(\overline{S_2}, x_2) & \text{if } x_1 \in \overline{S_1} \text{ and } x_2 \in S_2 \\ \mathrm{ind}(\overline{S_1}, x_1) \oplus \mathrm{ind}(S_2, x_2) & \text{if } x_1 \in S_1 \text{ and } x_2 \in \overline{S_2}. \end{cases}$$

for every $(x_1, x_2) \in V(G_1) \times V(G_2)$.

Proof. By Remark 8.1 (2), $\mathrm{ind}(S_1 * S_2, (x_1, x_2)) = \mathrm{ind}(S_1 \times S_2, (x_1, x_2)) \oplus \mathrm{ind}(\overline{S_1} \times \overline{S_2}, (x_1, x_2))$. If $x_1 \in S_1, x_2 \in S_2$ or $x_1 \in \overline{S_1}, x_2 \in \overline{S_2}$, then $\mathrm{ind}(S_1 \times S_2, (x_1, x_2)) = \emptyset$ or $\mathrm{ind}(\overline{S_1} \times \overline{S_2}, (x_1, x_2))$, and therefore, $\mathrm{ind}(S_1 * S_2, (x_1, x_2)) = \{\emptyset\}$. If $x_1 \in \overline{S_1}$ and $x_2 \in S_2$, then $\mathrm{ind}(\overline{S_1}, x_1) = \{\emptyset\}$ and $\mathrm{ind}(S_2, x_2) = \{\emptyset\}$. So, by Remark 8.5, $\mathrm{ind}(S_1 \times S_2, (x_1, x_2)) = \mathrm{ind}(S_1, x_1)$ and $\mathrm{ind}(\overline{S_1} \times \overline{S_2}, (x_1, x_2)) = \mathrm{ind}(\overline{S_2}, x_2)$, so $\mathrm{ind}(S_1 * S_2, (x_1, x_2)) = \mathrm{ind}(S_1, x_1) \oplus \mathrm{ind}(\overline{S_2}, x_2)$. Similarly, if $x_1 \in S_1$ and $x_2 \in \overline{S_2}$, $\mathrm{ind}(S_1 * S_2, (x_1, x_2)) = \mathrm{ind}(\overline{S_1}, x_1) \oplus \mathrm{ind}(S_2, x_2)$. □

We are ready to prove Theorem 1.44.

Theorem 1.44 ([6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. If $\mathrm{mult}(S_1)$, $\mathrm{mult}(\overline{S_1})$, $\mathrm{mult}(S_2)$, $\mathrm{mult}(\overline{S_2})$ are ideal, then so are $\mathrm{mult}(S_1 * S_2)$, $\mathrm{mult}\left(\overline{S_1 * S_2}\right)$.*

Proof. Since $\overline{S_1 * S_2} = \overline{S_1} * S_2$, it is sufficient to consider $\mathrm{mult}(S_1 * S_2)$. To prove that $\mathrm{mult}(S_1 * S_2)$ is ideal, it suffices by Theorem 1.35 to prove that the induced clutters of $\mathrm{mult}(S_1 * S_2)$ are ideal. To this end, take $(x_1, x_2) \in V(G_1) \times V(G_2)$. Since $\mathrm{mult}(S_1)$, $\mathrm{mult}(\overline{S_1})$, $\mathrm{mult}(S_2)$, $\mathrm{mult}(\overline{S_2})$ are ideal, all of $\mathrm{ind}(S_1, x_1)$, $\mathrm{ind}(S_2, x_2)$, $\mathrm{ind}(\overline{S_1}, x_1)$, $\mathrm{ind}(\overline{S_2}, x_2)$ are ideal, implying in turn that $\mathrm{ind}(S_1 * S_2, (x_1, x_2))$ is ideal by Proposition 8.6. □

8.3 Minimally non-packing multipartite clutters obtained by the reflective product

We will need the following remark:

Remark 8.7 ([6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Then*

$$\tau(\text{mult}(S_1 * S_2)) \leq \min\left\{\tau(\text{mult}(S_1)) + \tau(\text{mult}(\overline{S_2})), \tau(\text{mult}(\overline{S_1})) + \tau(\text{mult}(S_2))\right\}.$$

Proof. By Remark 8.1 (1), a cover of $\text{mult}(S_1)$ is a cover of $\text{mult}(S_1 \times S_2)$ and a cover of $\text{mult}(\overline{S_2})$ is a cover of $\text{mult}(\overline{S_1} \times \overline{S_2})$. That means the union of a cover of $\text{mult}(S_1)$ and a cover of $\text{mult}(\overline{S_2})$ is a cover of $\text{mult}(S_1 * S_2)$ by Remark 8.1 (2). Therefore, $\tau(\text{mult}(S_1 * S_2)) \leq \tau(\text{mult}(S_1)) + \tau(\text{mult}(\overline{S_2}))$. Similarly, we obtain $\tau(\text{mult}(S_1 * S_2)) \leq \tau(\text{mult}(\overline{S_1})) + \tau(\text{mult}(S_2))$, as required. □

Take integers $n \geq 1$, $\omega_1 \geq \cdots \geq \omega_n \geq 1$. Recall that $\left\{v^1, \ldots, v^{\omega_n}\right\} \subseteq V(H_{\omega_1,\ldots,\omega_n})$ is a general diagonal of $H_{\omega_1,\ldots,\omega_n}$ if $v^1, \ldots, v^{\omega_n}$ are ω_n vertices at pairwise distance n. The following remark can be readily proved:

Remark 8.8 ([6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Let $G = G_1 \square G_2$. If L is a general diagonal of G, then there exist general diagonals of G_1 and G_2, L_1 and L_2, respectively, such that $L \subseteq L_1 \times L_2$. Conversely, if L_1 and L_2 are general diagonals of G_1 and G_2, respectively, then there is a general diagonal L of G such that $L \subseteq L_1 \times L_2$.*

We will need the following claim:

Remark 8.9 ([6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. Let L_1 and L_2 be general diagonals of G_1 and G_2, respectively. Then*

$$\nu(\text{mult}(S_1 * S_2)) \geq \min\{|L_1 \cap S_1|, |L_2 \cap S_2|\} + \min\left\{|L_1 \cap \overline{S_1}|, |L_2 \cap \overline{S_2}|\right\}.$$

Proof. If two vertices $(u_1, v_1), (u_2, v_2) \in (L_1 \cap S_1) \times (L_2 \cap S_2)$ are at distance $n_1 + n_2$, then the distance between u_1, u_2 is n_1 and the distance between v_1, v_2 is n_2. For $i = 1, 2$, $(L_i \cap S_i)$ has $|L_i \cap S_i|$ vertices at distance n_i, so $(L_1 \cap S_1) \times (L_2 \cap S_2)$ contains exactly $\min\{|L_1 \cap S_1|, |L_2 \cap S_2|\}$ vertices at pairwise distance $n_1 + n_2$. Similarly, $(L_1 \cap \overline{S_1}) \times (L_2 \cap \overline{S_2})$ contains exactly $\min\left\{|L_1 \cap \overline{S_1}|, |L_2 \cap \overline{S_2}|\right\}$ vertices at pairwise distance $n_1 + n_2$.

Moreover, $(u, v) \in (L_1 \cap S_1) \times (L_2 \cap S_2)$ and $(u', v') \in (L_1 \cap \overline{S_1}) \times (L_2 \cap \overline{S_2})$ are at distance $n_1 + n_2$, because the distance between u and u' is n_1 and the distance between v and v' is n_2. As a result, $(L_1 \cap S_1) \times (L_2 \cap S_2) \cup (L_1 \cap \overline{S_1}) \times (L_2 \cap \overline{S_2})$ contains $\min\{|L_1 \cap S_1|, |L_2 \cap S_2|\} + \min\left\{|L_1 \cap \overline{S_1}|, |L_2 \cap \overline{S_2}|\right\}$ vertices at pairwise distance $n_1 + n_2$. As $(L_1 \cap S_1) \times (L_2 \cap S_2) \cup (L_1 \cap \overline{S_1}) \times (L_2 \cap \overline{S_2})$ is a subset of $S_1 * S_2$, we get that $\nu(\text{mult}(S_1 * S_2)) \geq \min\{|L_1 \cap S_1|, |L_2 \cap S_2|\} + \min\left\{|L_1 \cap \overline{S_1}|, |L_2 \cap \overline{S_2}|\right\}$, as required. □

It is easy to observe the following:

Remark 8.10 ([6]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, where $G_1 = H_{\omega_1,\ldots,\omega_{n_1}}$ and $G_2 = H_{\delta_1,\ldots,\delta_{n_2}}$ for some $\omega_1, \ldots, \omega_{n_1}, \delta_1, \ldots, \delta_{n_2} \geq 1$. If $S_1 * S_2$ does not pack but all of its proper restrictions pack, then S_1, $\overline{S_1}$, S_2, $\overline{S_2}$ are nonempty.*

Proof. Suppose for a contradiction that $\overline{S_2} = \emptyset$. Then $S_2 = V(G_2)$ and $S = S_1 \times V(G_2)$, which implies that $\mathrm{mult}(S_1)$ and $\mathrm{mult}(V(G_2))$ are proper restrictions of $\mathrm{mult}(S)$. By Remark 8.1 (1), $\mathrm{mult}(S) = \mathrm{mult}(S_1) \times \mathrm{mult}(V(G_2))$. So, Proposition 8.4 (2) implies that $\mathrm{mult}(S)$ packs, a contradiction. Therefore, $\overline{S_2}$ is nonempty. Similarly, we can argue that $S_1, \overline{S_1}, S_2$ are all nonempty. □

Recall that $H(n, \omega)$ denotes $H_{\omega,\ldots,\omega}$, so $H(n, 2)$ is the skeleton graph of the n-dimensional hypercube. The following remark will be useful to prove Theorem 1.45.

Remark 8.11 ([6]). *Take an integer $n \geq 1$ and an antipodally symmetric set $S \subseteq V(H(n, 2))$. If both S and $\overline{S}$ are nonempty, then $\mathrm{mult}(S * \{1\})$ does not pack.*

Proof. Take $u \in S$ and $v \in \overline{S}$. Let $\overline{u}$ and $\overline{v}$ denote the antipodal of u and that of v in $H(n, 2)$, respectively. Notice that $(u, 1)$, $(\overline{u}, 1)$, $(v, 2)$, $(\overline{v}, 2) \in S * \{1\}$ and that they do not agree on a coordinate, implying in turn that $\tau(\mathrm{mult}(S * \{1\})) = 2$. To show that $\mathrm{mult}(S * \{1\})$ does not pack, it suffices to argue that $S * \{1\}$ does not have antipodal vertices in $H(n + 1, 2)$, thereby showing that $\nu(\mathrm{mult}(S * \{1\})) = 1$. Let $(w, 1) \in S * \{1\}$. Then $w \in S$. As S is antipodally symmetric, the antipodal of w in $H(n, 2)$, denoted $\overline{w}$, is also contained in S. That means $(\overline{w}, 2)$, the antipodal of $(w, 1)$ in $H(n + 1, 2)$, is not in $S * \{1\}$. Similarly, for every $(w, 2) \in S * \{1\}$, the antipodal of $(w, 2)$ in $H(n + 1, 2)$ is not in $S * \{1\}$. Therefore, $\tau(\mathrm{mult}(S * \{1\})) = 2$ and $\nu(\mathrm{mult}(S * \{1\})) = 1$, so $\mathrm{mult}(S * \{1\})$ does not pack. □

We are ready to prove Theorem 1.45.

Theorem 1.45 ([6]). *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$ and a set $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. Assume that $\mathrm{mult}(S)$ contains no Δ_3 as a minor and does not pack but all of its proper restrictions pack. If S is obtained by a reflective product, then $\omega_1 = \cdots = \omega_n = 2$, and therefore, $\mathrm{mult}(S)$ is a cuboid.*

Proof. By Proposition 7.5 and Remark 7.6, it is sufficient to show that $\omega_n = 2$. Suppose that $\omega_n \geq 3$ for the sake of contradiction. As S is obtained by a reflective product, there exist $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ such that $S = S_1 * S_2$, where $G = G_1 \square G_2$. Then $G_1 \cong H_{\delta_1,\ldots,\delta_{n_1}}$ and $G_2 \cong H_{\gamma_1,\ldots,\gamma_{n_2}}$ for some $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$, $\delta_1 \geq \cdots \geq \delta_{n_1} \geq 3$, and $\gamma_1 \geq \cdots, \gamma_{n_2} \geq 3$ such that $\{\delta_1, \ldots, \delta_{n_1}, \gamma_1, \ldots, \gamma_{n_2}\} = \{\omega_1, \ldots, \omega_n\}$ as multisets. It follows from Remark 8.10 that $S_1, \overline{S_1}, S_2, \overline{S_2}$ are all nonempty.

Let τ denote $\tau(\mathrm{mult}(S))$. Then $\tau \leq \omega_n$ and $\nu(\mathrm{mult}(S)) \leq \tau - 1$ as $\mathrm{mult}(S)$ does not pack. In fact, $\nu(\mathrm{mult}(S)) = \tau - 1$. Suppose otherwise. Then $\nu(\mathrm{mult}(S)) \leq \tau - 2$ and what is obtained after deleting one element from $\mathrm{mult}(S)$ does not pack, since deleting one element from a clutter reduces its covering number

by at most one and does not increase its packing number. As every proper restriction of $\text{mult}(S)$ packs, we get $\nu(\text{mult}(S)) = \tau - 1$. As $\nu(\text{mult}(S)) = \tau - 1$, there is a general diagonal L of G with $|L \cap S| = \tau - 1$.

By Remark 8.8, there exist general diagonals of G_1 and G_2, L_1 and L_2, respectively, such that $L \subseteq L_1 \times L_2$. Let $p_i := |L_i \cap S_i|$, $q_i := |L_i \cap \overline{S_i}|$ for $i = 1, 2$. Then $p_1 + q_1 = \delta_{n_1} \geq \omega_n \geq 3$ and $p_2 + q_2 = \gamma_{n_2} \geq \omega_n \geq 3$.

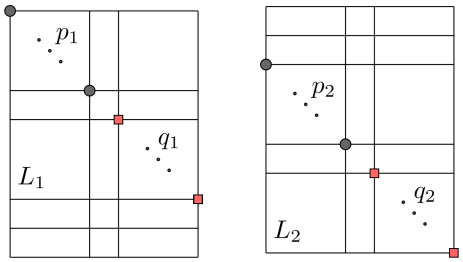


Figure 8.1: L_1 in G_1 and L_2 in G_2

Claim 1. *Either*

- $p_1 < p_2$, $q_2 < q_1$, *and* $p_1 + q_2 = \tau - 1$, *or*
- $p_2 < p_1$, $q_1 < q_2$, *and* $p_2 + q_1 = \tau - 1$.

Proof of Claim. Without loss of generality, we may assume that $\delta_{n_1} \geq \gamma_{n_2}$, so $p_1 + q_1 \geq p_2 + q_2$. Let us consider the case when $p_1 \geq p_2$ and $q_1 \geq q_2$. By Remark 8.9, $\nu(\text{mult}(S)) \geq p_2 + q_2$. However, we have $p_2 + q_2 = \gamma_{n_2} \geq \omega_n > \tau - 1$, a contradiction as we already argued that $\nu(\text{mult}(S)) = \tau - 1$. Therefore, either $p_1 < p_2$ or $q_1 < q_2$.

If $p_1 < p_2$, then we get that $q_1 > q_2$ because $\delta_{n_1} \geq \gamma_{n_2}$. In this case, $(L_1 \cap S_1) \times (L_2 \cap S_2)$ and $\left(L_1 \cap \overline{S_1}\right) \times \left(L_2 \cap \overline{S_2}\right)$ contain p_1 and q_2 vertices at pairwise distance $n_1 + n_2$, respectively. Note also that $(L_1 \cap S_1) \times (L_2 \cap S_2) \cup \left(L_1 \cap \overline{S_1}\right) \times \left(L_2 \cap \overline{S_2}\right)$ contains $L \cap S$, so $p_1 + q_2 \geq |L \cap S| = \tau - 1$. We know from Remark 8.9 that $p_1 + q_2 \leq \nu(\text{mult}(S)) = \tau - 1$. So, $p_1 + q_2 = \tau - 1$. Similarly, if $q_1 < q_2$, then we can show that $p_1 > p_2$ and $p_2 + q_1 = \tau - 1$. ◇

By Claim 1, we may assume that $p_1 < p_2$, $q_2 < q_1$, and $p_1 + q_2 = \tau - 1$. We will argue that a general diagonal of G_1 has at most p_1 feasible vertices and a general diagonal of G_2 has at least p_2 feasible vertices. We need the following two claims to argue that.

Claim 2. *Let D be a general diagonal of G_1, and let $p := |D \cap S_1|$, $q := |D \cap \overline{S_1}|$. Let F be the vertex-induced subgraph of G_1 that consists of the vertices at distance n_1 from each vertex in $D \cap S_1$, and let H be the vertex-induced subgraph of G_1 that consists of the vertices at distance n_1 from each vertex in $D \cap \overline{S_1}$. Then the following statements hold:*

(1) if G_1 does not contain a general diagonal with exactly $p + 1$ feasible vertices, then $V(F) \cap S_1 = \emptyset$,

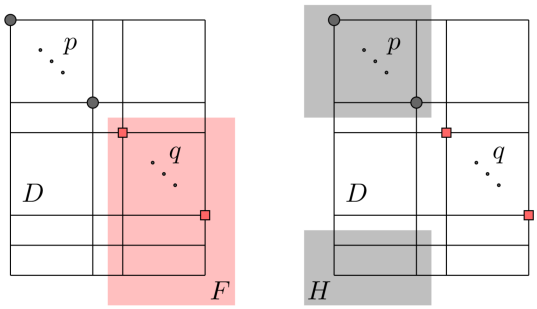


Figure 8.2: F and H in G_1

(2) if G_1 does not contain a general diagonal with exactly $p-1$ feasible vertices, then $V(H) \subseteq S_1$.

Proof of Claim. By the symmetry between S_1 and $\overline{S_1}$, it is enough to show **(1)**.

Assume that G_1 does not contain a general diagonal with exactly $p+1$ feasible vertices. Notice that $F \cong H_{(\delta_1-p),\ldots,(\delta_{n_1}-p)}$. If $q=0$, then F has no vertex and we automatically get $V(F) \cap S_1 = \emptyset$. Consider the case $q=1$. Then D has $p = \delta_{n_1} - q = \delta_{n_1} - 1$ feasible vertices. Suppose for a contradiction that there is a vertex $v \in V(F) \cap S_1$. Then v is a feasible vertex that is at distance n_1 from any vertex in $D \cap S_1$. That means $\{v\} \cup (D \cap S_1)$ is a general diagonal of G_1 with exactly $\delta_{n_1} = p+1$ feasible vertices, a contradiction to our assumption. Therefore, $V(F) \cap S_1 = \emptyset$. Thus, we may assume that $q \geq 2$.

Notice that $D \cap \overline{S_1}$ is contained in F. That means $D \cap \overline{S_1}$ is a general diagonal of F that contains no vertex in S_1. To prove $V(F) \cap S_1 = \emptyset$, we will argue that a general diagonal D_F of F such that $D_F \cap S_1 = \emptyset$ satisfies the following:

> ($\star$) Any vertex that is a neighbor of a vertex in D_F is not feasible, and any vertex that is a neighbor of a vertex in D_F is on a general diagonal of F without a feasible vertex.

Note that D_F has q vertices $u_1, \ldots, u_q$. Let $u \in V(F)$ be a vertex that is adjacent to D_F, and we may assume that u is adjacent to u_1. If u is at distance n_1 from all of $u_2, \ldots, u_q$, then $\{u, u_2, \ldots, u_q\}$ is a general diagonal of F. Consider $(D \cap S_1) \cup \{u, u_2, \ldots, u_q\}$. This is a general diagonal of G_1. So if $u \in S_1$, then $(D \cap S_1) \cup \{u, u_2, \ldots, u_q\}$ contains exactly $p+1$ feasible vertices, a contradiction to our assumption. Thus, u is not contained in S_1, and also, $\{u, u_2, \ldots, u_q\}$ is a general diagonal of F with no vertex in S_1.

Otherwise, at least one of $u_2, \ldots, u_q$ is at distance less than n_1 from u. In fact, since u and u_1 are adjacent, u is at distance $n_1 - 1$ from one point and at distance n_1 from all the other points in $u_2, \ldots, u_q$. Thus, we may assume that u has exactly one common coordinate with u_2 but is at distance n_1 from the other vertices. Let Q denote the smallest hypercube containing u_1 and u_2. Then u belongs to Q. We will show that $V(Q) \cap S_1$ is antipodally symmetric. Let $w \in V(Q)$, and let $\overline{w}$ be the antipodal of w in Q. Consider a general diagonal $(D \cap S_1) \cup \{w, \overline{w}, u_3, \ldots, u_q\}$. If $|\{w, \overline{w}\} \cap S_1| = 1$, then the general diagonal has exactly $p+1$ vertices, contrary to our assumption. Therefore, $w \in S_1$ if and only if $\overline{w} \in S_1$, thereby implying that $V(Q) \cap S_1$ is antipodally symmetric. Since both S_2 and $\overline{S_2}$ are nonempty, we can find $v \in S_2$ and $v' \in \overline{S_2}$ such that v, v' are adjacent in G_2. Notice that S has $(V(Q) \cap S_1) * \{v\}$ as a restriction. After

dropping the coordinates where the vertices in $(V(Q) \cap S_1) * \{v\}$ agree, we obtain a set isomorphic to $(V(Q) \cap S_1) * \{1\}$.

Suppose for a contradiction that $u \in S_1$. Then $V(Q)$ contains both a feasible vertex (u) and an infeasible vertex (u_1), which means both $V(Q) \cap S_1$ and $V(Q) \cap \overline{S_1}$ are nonempty.

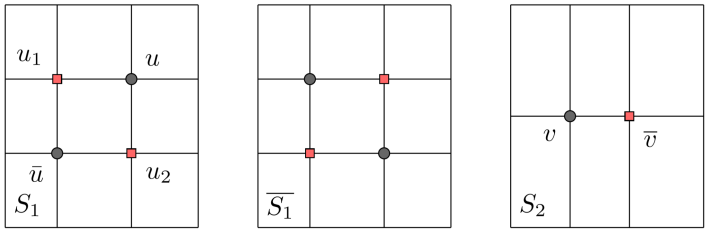


Figure 8.3: S_1, $\overline{S_1}$, and S_2

By Remark 8.11, mult $((V(Q) \cap S_1) * \{0\})$ does not pack. However, this contradicts the assumption that mult(S) has no proper restriction that does not pack. That means that $u \notin S_1$ and that u is contained in a general diagonal $\{u, \bar{u}, u_3, \ldots, u_q\}$ of F containing no vertex in S_1. Therefore a general diagonal D_F of F with no feasible vertex satisfies $(\star)$.

Now we are ready to prove that $V(F) \cap S_1 = \emptyset$. Let $u \in V(F)$. We argue that u is contained in a general diagonal of F with no vertex in S_1, by induction on the distance between u and $D \cap \overline{S_1} (= D \cap \overline{S_1} \cap V(F))$. If the distance is 0, then u is contained in $D \cap \overline{S_1}$ and thus the assertion holds as $D \cap \overline{S_1}$ is a general diagonal of F. Assume that the assertion holds for any vertex with distance from $D \cap \overline{S_1}$ at most k for some $k \geq 0$. Let the distance from u to $D \cap \overline{S_1}$ be $k+1$. Then it is adjacent to a vertex that is at distance k from $D \cap \overline{S_1}$, so it is contained in a general diagonal with no vertex in S_1 by the induction hypothesis. By $(\star)$, u is infeasible and belongs to a general diagonal of F with no vertex in S_1. This complete the induction step. So, $V(F) \cap S_1 = \emptyset$, as required. $\diamond$

Claim 3. *Let D be a general diagonal of G_2, and let $p := |D \cap S_2|$, $q := |D \cap \overline{S_2}|$. Let F be the vertex-induced subgraph of G_2 that consists of the vertices at distance n_2 from each vertex in $D \cap S_2$, and let H be the vertex-induced subgraph of G_2 that consists of the vertices at distance n_2 from each vertex in $D \cap \overline{S_2}$. Then the following statements hold:*

(1) if G_2 does not contain a general diagonal with exactly $p+1$ feasible vertices, then $V(F) \cap S_2 = \emptyset$,

(2) if G_2 does not contain a general diagonal with exactly $p-1$ feasible vertices, then $V(H) \subseteq S_2$.

Proof of Claim. Claim 3 follows from Claim 2 by the symmetry between G_1 and G_2. $\diamond$

Let F_1 be the subgraph of G_1 that consists of the vertices in G_1 that are at distance n_1 from each vertex in $L_1 \cap S_1$, and let H_2 be the subgraph of G_2 that consists of the vertices in G_2 that are at distance n_2 from each vertex in $L_2 \cap \overline{S_2}$ (see Figure 8.4). Then we obtain $V(F_1) \cap S_1 = \emptyset$ by Claim 2(1) and Claim 3(1) and $V(H_2) \subseteq S_2$ by Claim 2(2) and Claim 4(2).

Now we are ready to prove the following:

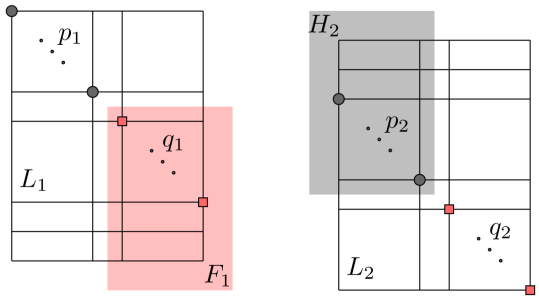


Figure 8.4: F_1 in G_1 and H_2 in G_2

Claim 4. *The following statements hold.*

(1) G_1 does not have a general diagonal with at least $p_1 + 1$ vertices in S_1,

(2) G_2 does not have a general diagonal with at most $p_2 - 1$ vertices in S_2.

Proof of Claim. **(1)**: We show that G_1 does not have a general diagonal with exactly k vertices in S_1 for any $k \geq p_1 + 1$. We argue by induction on k. Let us consider the base case $k = p_1 + 1$. Suppose that G_1 has a general diagonal L'_1 with $|L'_1 \cap S_1| = p_1 + 1$. In this case, $|L'_1 \cap \overline{S_1}| = q_1 - 1$. Since $p_1 < p_2$ and $q_2 < q_1$ by Claim 1, we have that $p_1 + 1 \leq p_2$ and $q_2 \leq q_1 - 1$. That means that $(L'_1 \cap S_1) \times (L_2 \cap S_2)$ contains $p_1 + 1$ vertices at pairwise distance $n_1 + n_2$, while $\left(L'_1 \cap \overline{S_1}\right) \times \left(L_2 \cap \overline{S_2}\right)$ contains q_2 vertices that are at pairwise distance $n_1 + n_2$. This implies that G has a general diagonal with at least $p_1 + q_2 + 1$ vertices in S. By Claim 1, $p_1 + q_2 + 1 = \tau$, a contradiction as $\nu(\text{mult}(S)) = \tau - 1$. Therefore, G_1 contains no general diagonal with exactly $p_1 + 1$ vertices in S_1.

Now assume that G_1 does not have a general diagonal with k vertices in S_1 for some $k \geq p_1 + 1$. We would like to show that G_1 does not have a general diagonal with $k+1$ vertices in S_1 either. Suppose for a contradiction that there is a general diagonal L'_1 of G_1 with $|L'_1 \cap S_1| = k+1$. Then $|L'_1 \cap \overline{S_1}| = \delta_{n_1} - k - 1$. Let H'_1 be the vertex-induced subgraph of G_1 that consists of the vertices at distance n_1 from each vertex in $L'_1 \cap \overline{S_1}$. The induction hypothesis and Claim 3(2) imply that $V(H'_1) \subseteq S_1$. Notice that $H'_1 \cong H_{(\delta_1 - \delta_{n_1} + k + 1), \ldots, (\delta_{n_1} - \delta_{n_1} + k + 1)}$ and $F_1 \cong H_{(\delta_1 - p_1), \ldots, (\delta_{n_1} - p_1)}$. Observe that for each j,

$$(\delta_j - \delta_{n_1} + k + 1) + (\delta_j - p_1) = 2\delta_j - \delta_n + k + 1 - p_1 \geq \delta_j + 2.$$

This means that $V(F_1)$ and $V(H'_1)$ overlap. However, we observed that $V(F_1) \cap S_1 = \emptyset$, a contradiction as $V(H'_1) \subseteq S_1$. Therefore, G_1 does not have a general diagonal with more than $p_1 + 1$ vertices in S_1, as required.

(2): By Claim 2(2), we know that G_2 does not have a general diagonal with exactly $q_2 + 1$ vertices in $\overline{S_2}$. By the symmetry, we can similarly show that G_2 does not have a general diagonal with k vertices in $\overline{S_2}$ for any $k > q_1 + 1$. Therefore, G_2 does not have a general diagonal with less than $p_2 - 1$ vertices in S_2, as required. ◇

By Claim 4, the maximum number of feasible vertices that a general diagonal of G_1 has is p_1 and the maximum number of infeasible vertices that a general diagonal of G_2 has is q_2. That means that

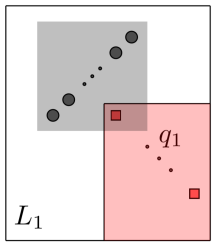


Figure 8.5: L_1 in G_1: a contradiction

$\nu(\text{mult}(S_1)) = p_1$ and $\nu(\text{mult}(\overline{S_2})) = q_2$. Since $\text{mult}(S_1)$ and $\text{mult}(\overline{S_2})$ are proper restrictions of $\text{mult}(S)$, they pack, so we get $\tau(\text{mult}(S_1)) = p_1$ and $\tau(\text{mult}(\overline{S_2})) = q_2$. By Remark 8.7, $\tau(\text{mult}(S_1 * S_2)) \leq p_1 + q_2$, but this is a contradiction as $p_1 + q_2 = \tau - 1$. Therefore, we have that $\omega_n = 2$, and by Proposition 7.5 and Remark 7.6, $\mathcal{C}$ is a cuboid. □

By Theorem 1.45, an ideal minimally non-packing multipartite clutter obtained by a reflective product must be a cuboid. Lastly, we prove Theorem 1.46.

Theorem 1.46 ([2]). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{1,2\}^{n_1}$ and $S_2 \subseteq \{1,2\}^{n_2}$, where $\text{mult}(S_1 * S_2)$ does not pack but all of its proper restrictions pack. Then one of the following statements holds:*

(i) $S_1 * S_2 \cong R_{k,1}$ for some $k \geq 1$,

(ii) $n_1 = 1$ and $S_2, \overline{S_2}$ are antipodally symmetric and strictly connected, or

(iii) $n_2 = 1$ and $S_1, \overline{S_1}$ are antipodally symmetric and strictly connected.

Moreover, $S_1 * S_2 \cong \overline{S_1 * S_2}$.

Proof. Let us start with the following claim:

Claim 1. *Either $n_1 = 1$ and S_2 is antipodally symmetric, or $n_2 = 1$ and S_1 is antipodally symmetric.*

Proof of Claim. We first argue that one of S_1 and S_2 is antipodally symmetric. Suppose not. Then for $i \in [2]$, there exists $u_i \in S_i$ such that the antipodal of u_i, denoted $\overline{u_i}$, is in $\overline{S_i}$. So, $(u_1, u_2), (\overline{u_1}, \overline{u_2}) \in S_1 * S_2$. Notice that $(\overline{u_1}, \overline{u_2})$ is the antipodal of (u_1, u_2), implying in turn that $C_{(u_1,u_2)}$ and $C_{(\overline{u_1},\overline{u_2})}$ are disjoint members of $\text{mult}(S_1 * S_2)$ and that $\text{mult}(S_1 * S_2)$ packs, a contradiction. We may therefore assume that S_2 is antipodally symmetric. Suppose for a contradiction that $n_1 \neq 1$. Since both S_1 and $\overline{S_1}$ are nonempty by Remark 8.10, S_1 has $\{1\} \subseteq \{1,2\}$ as a proper set-restriction as we assumed that $n_1 \geq 2$. So, $S_1 * S_2$ contains $\{1\} * S_2$ as a proper set-restriction, so $\text{mult}(\{1\} * S_2)$ is a proper restriction of $\text{mult}(S_1 * S_2)$ by Remark 7.11. It follows from Remark 8.11 that $\text{mult}(\{1\} * S_2)$ does not pack, a contradiction to our assumption that every proper restriction of $\text{mult}(S_1 * S_2)$ packs. Therefore, $n_1 = 1$, as required. ◇

By Claim 1, we may assume that $n_2 = 1$ and S_1 is antipodally symmetric. Since $n_2 = 1$ and $S_2, \overline{S_2}$ are nonempty, we may assume that $S_2 = \{1\}$. In turn, $S_1 * S_2 = S_1 * \{1\}$.

Claim 2. $S_1 * S_2 \cong \overline{S_1 * S_2}$.

Proof of Claim. Recall that $\overline{S_1 * S_2} = S_1 * \overline{S_2} = S_1 * \{2\}$. Since $\{1\} \cong \{2\}$, $S_1 * S_2 \cong \overline{S_1 * S_2}$, as required. ◇

As S_1 is antipodally symmetric, $\overline{S_1}$ is also antipodally symmetric. So, if $S_1, \overline{S_1}$ are strictly connected, then (iii) holds. Thus, we may assume that either S_1 or $\overline{S_1}$ is not strictly connected. We will show that (i) holds in this case. By Claim 2, it is sufficient to show that $S_1 * S_2 = R_{k,1}$ or $\overline{S_1} * S_2 = R_{k,1}$ for some $k \geq 1$. So, without loss of generality, we may assume that S_1 is not strictly connected.

Claim 3. $S_1 * S_2 = R_{k,1}$ *for some* $k \geq 1$.

Proof of Claim. Since S_1 is not strictly connected, one of its set-restrictions is not connected. Let $R \subseteq \{1,2\}^n$ be a set-restriction of S_1 that is not connected. Then there exist vertices $a, b \in R$ such that there is no path between a and b in the subgraph of $H(n,2)$ induced by R; among all possible such pairs of vertices, we take a, b so that $d(a,b)$, the number of coordinates a and b differ on, is minimized. By Remark 7.13 (1), the distance between a and b in $H(n,2)$ is exactly $d(a,b)$. As a and b are disconnected, $d(a,b) \geq 2$, so $d(a,b) = k+1$ for some $k \geq 1$. Let $H[a,b]$ be the vertex-induced subgraph of $H(n,2)$ consisting of all the vertices that lie on a shortest ab-path. By Remark 7.13 (3), the smallest set-restriction of $V(H(n,2))$ containing $V(H[a,b])$ is a hypercube of dimension $k+1$. Moreover, by our choice of a, b, $R \cap V(H[a,b]) = \{a,b\}$, implying in turn that the smallest set-restriction of R containing a, b is isomorphic to $\{\mathbf{1}^{k+1}, \mathbf{2}^{k+1}\}$. Therefore, S_1 contains a set-restriction isomorphic to $\{\mathbf{1}^{k+1}, \mathbf{2}^{k+1}\}$. Since $S_2 = \{1\}$. $S_1 * S_2$ has a set-restriction isomorphic to $\{\mathbf{1}^{k+1}, \mathbf{2}^{k+1}\} * \{1\} = R_{k,1}$. Our assumption that $\text{mult}(S_1 * S_2)$ does not pack indeed indicates that $S_1 * S_2 = R_{k,1}$, as required. ◇

This finishes the proof. □

8.4 Further notes

We have seen that $R_{k,1}$, $k \geq 1$ and R_5 (see § 1.9 for their definitions) are sets obtained by a reflective product and that their multipartite clutters are ideal and minimally non-packing. By Theorems 1.45 and 1.46, we know that an ideal minimally non-packing multipartite clutter obtained by a reflective product must be the cuboid of $R_{k,1}$ for some $k \geq 1$ or $S * \{1\}$ where S is antipodally symmetric and strictly connected. Is there an antipodally symmetric and strictly connected set S different from R_5 such that $\text{mult}(S * \{1\})$ is ideal and minimally non-packing?

Recall that $R_5 = C_4 * \{1\}$ where

$$C_4 = \{1111, 2111, 2211, 2221, 2222, 1222, 1122, 1112\}$$

and that C_4 is antipodally symmetric and strictly connected. There is a generalization of C_4 & R_5, namely,

$$C_{k-1} := \left\{ \mathbf{1}^{k-1} + \sum_{i=1}^{d} e^i,\ \mathbf{2}^{k-1} - \sum_{i=1}^{d} e^i :\ d \in [k-1] \right\} \subseteq \{1,2\}^{k-1}$$
$$R_k := C_{k-1} * \{1\}.$$

It is easy to show that C_{k-1} is antipodally symmetric and strictly connected. It turns out that $\text{mult}(R_k)$ is non-ideal for $k \geq 6$. So, we end this chapter by introducing the following conjecture:

Conjecture 8.12. *$\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are the only sets whose cuboids are ideal minimally non-packing and obtained by a reflective product.*

Chapter 9

Ideal vector spaces

Take an integer $n \geq 1$ and a prime power q. Let $GF(q)$ denote the finite field of order q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. As the element set of $GF(q)$ is isomorphic to $[q]$, there is a bijection $f : GF(q) \to [q]$. Then we can define $\text{mult}(S)$ as the clutter over ground set $[qn]$ whose members are C_v for $v = (v_1, \ldots, v_n) \in GF(q)^n$, where

$$C_v := \{f(v_i) + (i-1)q : \ i \in [n]\} \subseteq [qn]\,.$$

So, $\text{mult}(S)$ is a multipartite clutter whose ground set is partitioned into n sets of size q. Recall that we asked the following two questions: When is $\text{mult}(S)$ ideal?

- (Question 1.47) When does $\text{mult}(S)$ have the max-flow min-cut property?
- (Question 1.48) When is $\text{mult}(S)$ ideal?

Recall that Theorem 1.49 provides answers to these questions for when $q = 2$. In this chapter, we study these questions for prime powers other than 2.

Question 1.47 is answered in § 9.1 by Theorem 1.51 for the $q = 4$ case and in § 9.1 – 9.2 by Theorem 1.50 for the prime powers other than 4. We split the proof of Theorem 1.50 into two parts: one for the case when q is not a power of 2 in § 9.1 and the other for the case when q is a power of 2 in § 9.2. Theorem 1.50 also answers Question 1.48 when q is not 4. Question 1.48 when $q = 4$ is answered in § 9.3 by Theorem 1.52. This chapter is based on [5].

9.1 Theorem 1.50 for when the characteristic of $GF(q)$ is not 2

In this section, we prove Theorems 1.50 and 1.51.

Lemma 9.1 ([5]). *Take integers $n \geq 3$, $\omega_1 \geq \cdots \geq \omega_n \geq 2$. Let $S \subseteq V(H_{\omega_1,\ldots,\omega_n})$. If $\text{mult}(S)$ contains no Δ_3 as a minor, then for any distinct $a, b, c \in S$ and distinct $i, j, k \in [n]$ such that*

$$a_i = b_i \neq c_i, \quad b_j = c_j \neq a_j, \quad c_k = a_k \neq b_k,$$

there exists $d \in S - \{a, b, c\}$ that satisfies the following:

(1) $d_\ell \in \{a_\ell, b_\ell, c_\ell\}$ for all $\ell \in [n]$, and

(2) at least two of $d_i = c_i$, $d_j = a_j$, $d_k = b_k$ hold.

Proof. Let V denote the ground set of $\mathrm{mult}(S)$, and let $I := V - (C_a \cup C_b \cup C_c)$. Let J denote the set of elements in $C_a \cup C_b \cup C_c$ that correspond to $\{a_\ell, b_\ell, c_\ell : \ell \in [n] - \{i, j, k\}\}$. Then the members of $\mathrm{mult}(S) \setminus I/J$ correspond to

$$R := \{(v_i, v_j, v_k) : \ v \in S, \ v_\ell \in \{a_\ell, b_\ell, c_\ell\} \text{ for } \ell \in [n]\}.$$

The incidence matrix of $\mathrm{mult}(S) \setminus I/J$ looks like the following:

$$\begin{array}{c} \\ a \\ b \\ c \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} a_i & \overbrace{\mathbf{c_i}} & \overbrace{\mathbf{a_j}} & b_j & c_k & \overbrace{\mathbf{b_k}} \end{array} \\ \begin{pmatrix} 1 & \mathbf{0} & \mathbf{1} & 0 & 1 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} & 1 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{0} & 1 & 1 & \mathbf{0} \\ & & & \vdots & & \end{pmatrix} \end{array}$$

Note that R contains (a_i, a_j, a_k), (b_i, b_j, b_k), and (c_i, c_j, c_k). Suppose that there is no $d \in S - \{a, b, c\}$ that satisfies (1) and (2). Let $d \in S$ with $d_\ell \in \{a_\ell, b_\ell, c_\ell\}$ for $\ell \in [n]$. Since d satisfies (1), d does not satisfy (2). Then (d_i, d_j, d_k) can be (c_i, b_j, c_k), (a_i, a_j, c_k), (a_i, b_j, b_k), or (a_i, b_j, c_k). That means

$$R \subseteq \{(a_i, a_j, a_k), (b_i, b_j, b_k), (c_i, c_j, c_k), (c_i, b_j, c_k), (a_i, a_j, c_k), (a_i, b_j, b_k), (a_i, b_j, c_k)\}.$$

Observe that a row of $M(\mathrm{mult}(S) \setminus I/J)$ other than the ones for a, b, c, if any, has at least two nonzero entries in the columns for a_i, b_j, c_k. After contracting the columns for c_i, a_j, b_k, the resulting incidence matrix is one of the following two.

$$\begin{array}{ccc} a_i & b_j & c_k \\ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, & \quad & \begin{array}{ccc} a_i & b_j & c_k \end{array} \\ & & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \end{array}$$

This implies that we obtain Δ_3 after contracting c_i, a_j, b_k from $\mathrm{mult}(S) \setminus I/J$, a contradiction to the assumption that $\mathrm{mult}(S)$ has no Δ_3 minor. □

Recall that the characteristic of $GF(q)$ is the smallest integer ℓ such that $\underbrace{a + \cdots + a}_{\ell} = 0$ for all $a \in GF(q)$.

Lemma 9.2 ([5])**.** *Let q be a prime power. Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. If S does not admit a basis with vectors of pairwise disjoint supports, then the following statements holds:*

(1) $\mathrm{mult}(S)$ *contains Δ_3 or Q_6 as a minor.*

(2) If q is not a power of 2, then $\mathrm{mult}(S)$ *contains* Δ_3 *as a minor.*

(3) If $q = 2^k$, $k \geq 3$, then $\mathrm{mult}(S)$ *has* C_5^2 *as a minor.*

Proof of (1)&(2). Assume that S does not admit a basis with vectors of pairwise disjoint supports. We will show that if $\mathrm{mult}(S)$ does not contain Δ_3 as a minor, then q is a power of 2 and $\mathrm{mult}(S)$ contains Q_6 as a minor.

Assume that S contains no Δ_3 as a minor. Let $v^1, \ldots, v^r \in GF(q)^n$ be a basis of S. After elementary arithmetic operations over $GF(q)$, we may assume that for each $i = 1, \ldots, r$,

$$v^i_i = 1 \quad \text{and} \quad v^i_j = 0 \quad \forall j \in [r] - \{i\}$$

Since there is no basis of S with vectors of pairwise disjoint supports, we may assume that $v^1_{r+1}, v^2_{r+1} \neq 0$. Let x and y be the multiplicative inverse of v^1_{r+1} and that of v^2_{r+1} in $GF(q)$, respectively. Let $a := \mathbf{0} \in GF(q)^n$, $b := xv^1$, and $c := yv^2$. Notice that $a, b, c \in S$ and that a, b, c satisfy

$$(a_1, a_2, a_{r+1}) = (0, 0, 0), \quad (b_1, b_2, b_{r+1}) = (x, 0, 1), \quad (c_1, c_2, c_{r+1}) = (0, y, 1).$$

Let R be the restriction $\{d \in S : d_j \in \{a_j, b_j, c_j\} \text{ for } j \in [n]\}$ of S.

Claim 1. $R \subseteq \left\{\lambda_1 v^1 + \lambda_2 v^2 : \ \lambda_1 \in \{0, x\}, \ \lambda_2 \in \{0, y\}\right\}$.

Proof of Claim. Let $u \in R$. Then $u = \sum_{j=1}^r \lambda_j v^j$ for some $\lambda_1, \ldots, \lambda_r \in GF(q)$. Since $\{a_j, b_j, c_j\} = \{0\}$ for $j = 3, \ldots, r$, it follows that $u_3 = \cdots = u_r = 0$, which implies that $\lambda_3 = \cdots = \lambda_r = 0$ and so $u = \lambda_1 v^1 + \lambda_2 v^2$. Notice that $\lambda_1 \in \{0, x\}$ and $\lambda_2 \in \{0, y\}$, because $\{a_1, b_1, c_1\} = \{0, x\}$ and $\{a_2, b_2, c_2\} = \{0, y\}$. ◇

Claim 2. *q is a power of 2 and* $R = \left\{\lambda_1 v^1 + \lambda_2 v^2 : \ \lambda_1 \in \{0, x\}, \ \lambda_2 \in \{0, y\}\right\}$.

Proof of Claim. By Lemma 9.1, R contains $d \notin \{a, b, c\}$ such that

$$(d_1, d_2, d_{r+1}) = (0, y, 0), \ (x, 0, 0), \ (x, y, 1), \text{ or } (x, y, 0). \tag{9.1}$$

By Claim 1, $d \in \left\{\lambda_1 v^1 + \lambda_2 v^2 : \ \lambda_1 \in \{0, x\}, \ \lambda_2 \in \{0, y\}\right\}$. As $d \neq a, b, c$, it must be the case that $xv^1 + yv^2 = d$, so $xv^1 + yv^2 \in R$. In particular, $R = \left\{\lambda_1 v^1 + \lambda_2 v^2 : \ \lambda_1 \in \{0, x\}, \ \lambda_2 \in \{0, y\}\right\}$. Since $d = xv^1 + yv^2$, we obtain $(xv^1 + yv^2)_{r+1} = 1 + 1 = d_{r+1} \in \{0, 1\}$. Since $1 \neq 0$, we have $1 + 1 = 0$, so q is a power of 2, as required. ◇

By Claim 2, $R = \left\{\lambda_1 v^1 + \lambda_2 v^2 : \ \lambda_1 \in \{0, x\}, \ \lambda_2 \in \{0, y\}\right\}$, so the projection of R onto $1, 2, r+1$ is $R_{1,1}$, so $\mathrm{mult}(S)$ has a Q_6 minor. So, we have shown that if $\mathrm{mult}(S)$ has no Δ_3 as a minor, then q is a power of 2 and $\mathrm{mult}(S)$ contains Q_6 as a minor, as required. □

We will prove Lemma 9.2 (3) in Section 9.2. Lemma 9.2 (1) implies the following theorem:

Theorem 9.3 ([5]). *Let q be a prime power. Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then the following statements are equivalent:*

(i) $\mathrm{mult}(S)$ *contains no* Δ_3, Q_6 *as a minor.*

(ii) $S = \langle v^1, \ldots, v^r \rangle$ *where* $v^1, \ldots, v^r \in GF(q)^n$ *have pairwise disjoint supports.*

(iii) $\mathrm{mult}(S)$ *has the max-flow min-cut property.*

Proof. Direction **(iii)⇒(i)** is straightforward, and direction **(i)⇒(ii)** follows from Lemma 9.2 (1). Thus, what remains is to show direction **(ii)⇒(iii)**.

Assume that $S = \langle v^1, \ldots, v^r \rangle$ where $v^1, \ldots, v^r \in GF(q)^n$ have pairwise disjoint supports. For $i \in [r]$, let u^i denote the subvector of v^i that consists of v^i's nonzero coordinates. Then, after a possible rearrangement of the coordinates, we may assume that S is expressed as the Cartesian product of $r+1$ sets as follows:

$$S = \langle u^1 \rangle \times \langle u^2 \rangle \times \cdots \times \langle u^r \rangle \times \{\mathbf{0}\}$$

where the length of $\mathbf{0}$ is the number of coordinates not covered by the supports of $v^1, \ldots, v^r$. By Remark 8.1 (1), we have that

$$\mathrm{mult}(S) = \mathrm{mult}\left(\langle u^1 \rangle\right) \times \mathrm{mult}\left(\langle u^2 \rangle\right) \times \cdots \times \mathrm{mult}\left(\langle u^r \rangle\right) \times \mathrm{mult}\left(\{\mathbf{0}\}\right).$$

Notice that for any distinct $x, y \in GF(q)$, xu^i and yu^i do not have common coordinates. Thus C_{xu^i} and C_{yu^i}, the members of $\mathrm{mult}\left(\langle u^i \rangle\right)$ corresponding to xu^i and yu^i, are disjoint. That means that the members of $\mathrm{mult}\left(\langle u^i \rangle\right)$ are pairwise disjoint. By Remark 7.16, $\mathrm{mult}\left(\langle u^i \rangle\right)$ has the max-flow min-cut property. $\mathrm{mult}(\{\mathbf{0}\})$ has only one member, so it also has the max-flow min-cut property. So Remark 8.4 (5) implies that S has the max-flow min-cut property. □

Theorem 1.51 is an immediate corollary of Theorem 9.3.

Theorem 1.51 ([5]). *Let* $n \geq 3$*, and let* $S \subseteq GF(4)^n$ *be a vector space over* $GF(4)$*. Then the following statements are equivalent:*

(i) $\mathrm{mult}(S)$ *contains no* Δ_3, Q_6 *as a minor,*

(ii) $S = \langle v^1, \ldots, v^r \rangle$ *where* $v^1, \ldots, v^r \in GF(4)^n$ *have pairwise disjoint supports,*

(iii) $\mathrm{mult}(S)$ *has the max-flow min-cut property.*

Using Lemma 9.2 (2) and Theorem 9.3, we can prove Theorem 1.50 for the case when q is not a power of 2.

Theorem 1.50 ([5]). *Let* q *be a prime power other than* $2, 4$*. Let* $n \geq 3$*, and let* $S \subseteq GF(q)^n$ *be a vector space over* $GF(q)$*. Then the following statements are equivalent:*

(i) $\mathrm{mult}(S)$ *contains no* Δ_3, Q_6, C_5^2 *as a minor,*

(ii) $S = \langle v^1, \ldots, v^r \rangle$ *where* $v^1, \ldots, v^r \in GF(q)^n$ *have pairwise disjoint supports,*

(iii) mult(S) *has the max-flow min-cut property,*

(iv) mult(S) *is ideal.*

Proof of Theorem 1.50 when q is not a power of 2. Take a prime power q other than $2, 4$ and an integer $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. By Theorem 9.3, **(i)** – **(iii)** are equivalent. Clearly, if mult(S) has the max-flow min-cut property, then mult(S) is ideal. It remains to be shown that **(iv)**⇒**(ii)**. If mult(S) is ideal, then it has no Δ_3 as a minor. By Lemma 9.2 (2), S has a basis with vectors of pairwise disjoint supports, so **(ii)** holds. □

9.2 Theorem 1.50 when q is a power of 2

To finish the proof of Theorem 1.50, we will show that **(ii)** and **(iv)** are equivalent if $q = 2^k$ for $k \geq 3$, but more techniques are involved in this case. Recall that for a vector space over $GF(2)$, there is a binary matroid associated with it. In fact, for any prime power q and any vector space over $GF(q)$, there is an associated matroid:

> For $S = \{x \in GF(q)^n : Ax = \mathbf{0}\}$, denote by $\mathcal{M}(S)$ the linear matroid over $GF(q)$ represented by A.

Remark 9.4 ([5]). *Let q be a prime power. Let $n \geq 1$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Let $\mathcal{S}$ be defined as the clutter of the minimal supports of the points in $S - \{\mathbf{0}\}$. Then $\mathcal{S}$ is the clutter of circuits of $\mathcal{M}(S)$.*

Proof. First, we show that $\mathcal{S}$ is the clutter of circuits of a matroid. It suffices to check that $\mathcal{S}$ satisfies the Circuit Elimination Axiom. Let C_1, C_2 be distinct sets in $\mathcal{S}$ such that $i \in C_1 \cap C_2$ for some $i \in [n]$. We need to show that $(C_1 \cup C_2) - \{i\}$ contains a set in $\mathcal{S}$. Then there exist two points u, v such that support$(u) = C_1$ and support$(v) = C_2$. Let $x = u_i^{-1}$ and $y = -v_i^{-1}$. Consider $xu + yv \in S - \{\mathbf{0}\}$. Since $x, y \neq 0$, support$(xu) = C_1$ and support$(yv) = C_2$, so support$(xu + yv) \subseteq C_1 \cup C_2$. Moreover, $(xu + yv)_i = xu_i + yv_i = 1 - 1 = 0$. This means that $i \notin$ support$(xu + yv)$ and that support$(xu + yv) \subseteq (C_1 \cup C_2) - \{i\}$. Therefore, $(C_1 \cup C_2) - \{i\}$ contains a set in $\mathcal{S}$, as required.

To complete the proof, it is sufficient to show that the clutter of circuits of $\mathcal{M}(S)$ is precisely $\mathcal{S}$. As S is a vector space over $GF(q)$, $S = \{x \in GF(q)^n : Ax = \mathbf{0}\}$ for some matrix A whose entries are in $GF(q)$. Let $A_{(\cdot,1)}, \ldots, A_{(\cdot,n)}$ denote the columns of A. Let $C \subseteq [n]$. Then the columns in $\{A_{(\cdot,j)} : j \in C\}$ are linearly dependent if, and only if, there exists $x \in GF(q)^n$ such that $Ax = \mathbf{0}$ and support$(x) \subseteq C$. Therefore, C is a circuit of $\mathcal{M}(S)$ if, and only if, C is a minimal support of the points in $S - \{\mathbf{0}\}$. That means that the clutter of circuits of $\mathcal{M}(S)$ is precisely the clutter of the minimal supports of the points in $S - \{\mathbf{0}\}$, as required. □

Next, we remark that matroid deletions and contractions in $\mathcal{M}(S)$ correspond to 0-restrictions and projections in S. For a matroid M and disjoint subsets I, J of the ground set of M, we denote by $M \setminus I/J$ the matroid minor of M obtained after deleting I and contracting J. Let $\mathcal{C}(M)$ denote the clutter of the circuits of M. We leave the following as an easy exercise for the reader.

Remark 9.5. *Let $\mathcal{C}_1, \mathcal{C}_2$ be clutters over the same ground set. If every member of $\mathcal{C}_1$ contains a member of $\mathcal{C}_2$ and every member of $\mathcal{C}_2$ contains a member of $\mathcal{C}_1$, then $\mathcal{C}_1 = \mathcal{C}_2$.*

With Remark 9.5, we can prove the following lemma:

Lemma 9.6 ([5])**.** *Let q be a prime power. Let $n \geq 1$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then $\mathcal{M}(S) \setminus I/J$, for some disjoint $I, J \subseteq [n]$, is precisely $\mathcal{M}(S')$ where $S' \subseteq GF(q)^{n-|I|-|J|}$ is the vector space over $GF(q)$ obtained from*

$$S \cap \{x \in GF(q)^n : \ x_i = 0 \ \forall i \in I\}$$

after dropping coordinates in $I \cup J$.

Proof. It is clear that S' is a vector space over $GF(q)$, so $\mathcal{M}(S')$ is well-defined. To prove that $\mathcal{M}(S)\setminus I/J = \mathcal{M}(S')$, we will show that $\mathcal{C}(\mathcal{M}(S) \setminus I/J) = \mathcal{C}(\mathcal{M}(S'))$. Let $C_1 \in \mathcal{C}(\mathcal{M}(S) \setminus I/J)$. Then there exists $C \in \mathcal{C}(\mathcal{M}(S))$ such that $C \cap I = \emptyset$ and $C_1 = C - J$. By Remark 9.4, $C_1 = \text{support}(x)$ for some $x \in S$. As $C \cap I = \emptyset$, it follows that $x_i = 0$ for $i \in I$, which implies that there exists $x' \in S' - \{\mathbf{0}\}$ such that $\text{support}(x') = \text{support}(x) - J$. So, by Remark 9.4, there exists $C_2 \in \mathcal{C}(\mathcal{M}(S'))$ such that $C_2 \subseteq C_1$. Therefore, every member of $\mathcal{C}(\mathcal{M}(S) \setminus I/J)$ contains a member of $\mathcal{C}(\mathcal{M}(S'))$. Let $C_2 \in \mathcal{C}(\mathcal{M}(S'))$. By Remark 9.4, $C_2 = \text{support}(x')$ for some $x' \in S'$. This implies that there is some $x \in S$ such that $x_i = 0$ for $i \in I$ and $\text{support}(x) - J = \text{support}(x')$. Since $\text{support}(x)$ contains a circuit of $\mathcal{M}(S)$ by Remark 9.4 and $\text{support}(x) \cap I = \emptyset$, $C_2 = \text{support}(x')$ contains a circuit of $\mathcal{M}(S) \setminus I/J$. Therefore, by Remark 9.5, $\mathcal{C}(\mathcal{M}(S) \setminus I/J) = \mathcal{C}(\mathcal{M}(S'))$, as required. □

Note that $\text{mult}(S')$, where S' is defined as in Lemma 9.6, is a minor of $\text{mult}(S)$. So, if $\text{mult}(S)$ is ideal, then $\text{mult}(S')$ is also ideal. For $t \geq 3$, let A_t denote the graph that consists of two vertices and t parallel edges connecting them (see Figure 9.1). Hereinafter, for a graph G, we denote by $M(G)$ the cycle matorid

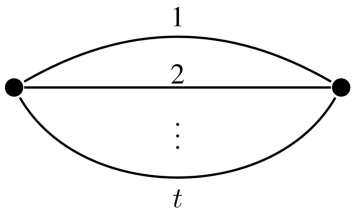


Figure 9.1: Graph on two verties and parallel edges

of G. We will show in Proposition 9.12 that if $q = 2^k$ for some $k \geq 3$, and $M(A_t)$ is the matroid associated with S', a vector space over $GF(q)$, then $\text{mult}(S')$ must be non-ideal. This in turn implies that if the multipartite clutter of a vector space S over $GF(q)$ is ideal, then $\mathcal{M}(S)$ does not contain $M(A_t)$, $t \geq 3$ as

a minor, and this fact will be the key for finishing the proof of Theorem 1.50. The following remark will be useful.

Remark 9.7 ([5]). *Let q be a prime power. Let $n \geq 1$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Let $f_i : GF(q) \to GF(q)$ be a bijection for $i \in [n]$, and $g : GF(q)^n \to GF(q)^n$ be the bijection defined as*

$$g(x) := (f_1(x_1), \ldots, f_n(x_n)), \quad x \in GF(q)^n.$$

Then $\text{mult}(S) \cong \text{mult}(g(S))$.

Proof. Let the ground set of $\text{mult}(S)$ be partitioned into $E_1, \ldots, E_n$ where E_i corresponds to the i^{th} coordinate of the points in S for $i \in [n]$. After relabeling the elements in E_i with respect to f_i for $i \in [n]$, we obtain $\text{mult}(g(S))$, thereby showing that $\text{mult}(S) \cong \text{mult}(g(S))$, as required. □

With Remark 9.7, we can prove the following:

Remark 9.8 ([5]). *Let q be a prime power. Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then $\mathcal{M}(S) = M(A_n)$ if, and only if,* $\text{mult}(S) \cong \text{mult}(\{x \in GF(q)^n : x_1 + \cdots + x_n = 0\})$.

Proof. Let $\{1, 2, 3 \ldots, n\}$ denote the edge set of A_n. Then the cycle space of A_n is generated by $\{1, 2\}$, $\{1, 3\}, \ldots, \{1, n\}$. (⇐): Let $\mathcal{S}$ be the clutter of the minimal supports of the points in $S - \{\mathbf{0}\}$. Then $\mathcal{S} = \{\{i, j\} : i \neq j\}$, so $\mathcal{M}(S) = M(A_n)$ by Remark 9.4. (⇒): Since $\mathcal{M}(S) = M(A_n)$, S contains $n - 1$ points $u^1, \ldots, u^{n-1}$ whose supports are $\{1, 2\}$, $\{1, 3\}, \ldots, \{1, n\}$, respectively. Notice that $u^1, \ldots, u^{n-1}$ are linearly independent over $GF(q)$, so the rank of S is at least $n-1$. On the other hand, the rank is less than n, because $S \neq GF(q)^n$. Thus, $S = \langle u^1, \ldots, u^{n-1} \rangle$. After scaling the u^is, if necessary, we may assume that the first coordinate of each u^i is 1. Hence, $u^1, \ldots, u^{n-1}$ are of the following form:

$$\begin{array}{c} u^1 \\ u^2 \\ \vdots \\ u^{n-1} \end{array} \begin{bmatrix} 1 & \lambda_1 & 0 & \cdots & 0 \\ 1 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \lambda_{n-1} \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_{n-1} \in GF(q) - \{0\}$. Notice that $\{x \in GF(q)^n : x_1 + \cdots + x_n = 0\} = \langle v^1, \ldots, v^{n-1} \rangle$ where

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^{n-1} \end{array} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

implying in turn that

$$\{x \in GF(q)^n : x_1 + \cdots + x_n = 0\} = \left\{(x_1, \lambda_1^{-1} x_2, \lambda_2^{-1} x_3, \ldots, \lambda_{n-1}^{-1} x_n) : x \in S\right\}.$$

So, by Remark 9.7, $\text{mult}(S) \cong \text{mult}(\{x \in GF(q)^n : x_1 + \cdots + x_n = 0\})$. □

Let $S = \{x \in GF(q)^n : x_1 + \cdots + x_n = 0\}$. We will show that $\text{mult}(S)$ is non-ideal if $q = 2^k$ for some $k \geq 3$. One way to argue that a multipartite clutter is non-ideal is to find an induced clutter that is non-ideal by Lemma 1.35. To see how an induced clutter of $\text{mult}(S)$ looks, we define an n-partite n-uniform hypergraph H_n as follows:

- The vertex set of H_n has n parts, where each part is a distinct copy $GF(q)$.
- $E_n = \big\{\{x_1, \ldots, x_n\} : (x_1, \ldots, x_n) \in S,\ x_i \text{ belongs to the } i^{\text{th}} \text{ part}, i \in [n]\big\}$ is the set of edges in H_n.

Then there is a one-to-one correspondence between E_n and S. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \notin S$. Let $H_{n,\alpha}$ be what is obtained from H_n after "contracting" vertices $\alpha_1, \ldots, \alpha_n$. More precisely, $H_{n,\alpha}$ be defined as follows:

- The vertex set of $H_{n,\alpha}$ has n parts $V_1 \cup \cdots \cup V_n$ where $V_i = GF(q) - \{\alpha_i\}$ for $i \in [n]$.
- $E_{n,\alpha} = \{e - \{\alpha_1, \ldots, \alpha_n\} : e \in E_n\}$ is the set of edges in $H_{n,\alpha}$.

Notice that $V_1, \ldots, V_n$ are still symmetric.

Remark 9.9 ([5]). *Let $\alpha \notin S$. Then there is a one-to-one correspondence between the vertices & the minimal edges of $H_{n,\alpha}$ and the elements & the members of $\text{ind}(S, \alpha)$, the induced clutter of $\text{mult}(S)$ with respect to α.*

The following lemma provides a characterization of the edges of $H_{n,\alpha}$.

Lemma 9.10 ([5]). *Let q be a power of 2. Let $n \geq 3$, and let $\alpha \in GF(q)^n$ with $\sigma := \alpha_1 + \cdots + \alpha_n \neq 0$. Let $e \subseteq V_1 \cup \cdots \cup V_n$. Then the following statements are equivalent:*

(i) e is an edge in $H_{n,\alpha}$.

(ii) e contains at most one vertex in V_i for each $i \in [n]$ and $\sum(v : v \in e) = \sigma + \sum(\alpha_i : e \cap V_i \neq \emptyset)$.

Proof. **(i)⇒(ii)** There exists $x = (x_1, \ldots, x_n) \in S$ such that $e = \{x_1, \ldots, x_n\} - \{\alpha_1, \ldots, \alpha_n\}$. Then $e \cap V_i = \{x_i\} - \{\alpha_i\}$, implying that $e \cap V_i$ has at most one vertex. Without loss of generality, we may assume that $x = (x_1, \ldots, x_k, \alpha_{k+1}, \ldots, \alpha_n)$ for some $1 \leq k \leq n$. Then $e = \{x_1, \ldots, x_k\}$. Since $x \in S$, we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{k} x_i + \sum_{j=k+1}^{n} \alpha_j = 0.$$

As the characteristic of $GF(q)$ is 2, $\sum_{i=1}^{k} x_i = -\sum_{i=1}^{k} x_i$, implying in turn that $\sum_{i=1}^{k} x_i = \sum_{j=k+1}^{n} \alpha_j$. As $\sum_{i=1}^{n} \alpha_i = \sigma$, we also get $\sum_{j=k+1}^{n} \alpha_j = \sigma + \sum_{i=1}^{k} \alpha_i$, and therefore, we obtain $\sum_{i=1}^{k} x_i = \sigma + \sum_{i=1}^{k} \alpha_i$, as required.

(i)⇐(ii) Without loss of generality, we may assume that $e = \{x_1, \ldots, x_k\}$ where $x_i \in V_i$ for $i \in [k]$. Since $\sum_{i=1}^{k} x_i = \sigma + \sum_{i=1}^{k} \alpha_i$, we have $\sum_{i=1}^{k} x_i + \sum_{j=k+1}^{n} \alpha_j = \sigma + \sum_{i=1}^{n} \alpha_i$, implying in turn that $(x_1, \ldots, x_k, \alpha_{k+1}, \ldots, \alpha_n) \in S$. As $e = \{x_1, \ldots, x_k, \alpha_{k+1}, \ldots, \alpha_n\} - \{\alpha_1, \ldots, \alpha_n\}$, e is an edge of $H_{n,\alpha}$, as required. □

Using Lemma 9.10, we can show the following proposition providing a characterization of the edges of size 1 and 2.

Proposition 9.11 ([5]). *Let q be a power of 2. Let $n \geq 3$, and let $\alpha \in GF(q)^n$ with $\sigma := \alpha_1 + \cdots + \alpha_n \neq 0$. Then the following statements hold:*

(1) The edges of size one in $E_{n,\alpha}$ are $\{\alpha_1 + \sigma\}, \ldots, \{\alpha_n + \sigma\}$.

(2) The edges of cardinality 2 in $E_{n,\alpha}$ form a graph that consists of $\frac{q}{2}-1$ connected components $G_1, \ldots, G_{\frac{q}{2}-1}$ satisfying the following: for each $j = 1, \ldots, \frac{q}{2} - 1$,

- *G_j's vertex set is $\left\{\beta_1^j, \beta_1^j + \sigma\right\} \cup \cdots \cup \left\{\beta_n^j, \beta_n^j + \sigma\right\}$ where $\beta_i^j, \beta_i^j + \sigma \in V_i - \{\alpha_i + \sigma\}$ for $i \in [n]$,*
- *G_j is a bipartite graph with bipartition $\left\{\beta_1^j, \ldots, \beta_n^j\right\} \cup \left\{\beta_1^j + \sigma, \ldots, \beta_n^j + \sigma\right\}$,*
- *$\beta_i^j = \beta_1^j + \alpha_1 + \alpha_i$ for $i \in [n]$, and*
- *G_j's edge set is $\left\{\left\{\beta_i^j, \beta_k^j + \sigma\right\} : i \neq k\right\}$, i.e. G_j is obtained from a complete bipartite graph after removing the edges of perfect matching $\left\{\left\{\beta_i^j, \beta_i^j + \sigma\right\} : i \in [n]\right\}$.*

(3) The edges of cardinality 1 or 2 in $E_{n,\alpha}$ are minimal.

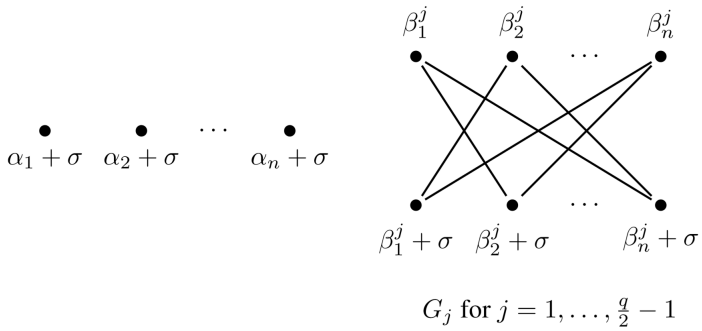


Figure 9.2: The edges of size 1&2 of $H_{n,\alpha}$

Proof. **(1)** By Lemma 9.10, e is an edge of size 1 if, and only if, $e = \{\sigma + \alpha_i\}$ for some $i \in [n]$. Therefore, $\{\alpha_1 + \sigma\}, \ldots, \{\alpha_n + \sigma\}$ are the edges of size 1 in $H_{n,\alpha}$, as required.

(2) First, we will argue that an edge of cardinality 2 contains none of $\alpha_1 + \sigma, \ldots, \alpha_n + \sigma$. Let $\{u, v\}$ be an edge of size 2 where $u \in V_i$ and $v \in V_j$ for some $i \neq j$. Then we get $u + v = \sigma + \alpha_i + \alpha_j$ by Lemma 9.10. If $u = \alpha_i + \sigma$, then $v = \alpha_j$, contradicting the assumption that $v \in V_j$. Therefore, the edges of cardinality 2 are contained in $V' := (V_1 - \{\alpha_1 + \sigma\}) \cup \cdots \cup (V_n - \{\alpha_n + \sigma\})$. Notice that we have preserved the symmetry between $V_1 - \{\alpha_1 + \sigma\}, \ldots, V_n - \{\alpha_n + \sigma\}$ and that $V_1 - \{\alpha_1 + \sigma\}$ is not different from the other $V_i - \{\alpha_i + \sigma\}$'s.

Observe that $V_1-\{\alpha_1+\sigma\}$ has $q-2$ vertices and that $V_1-\{\alpha_1+\sigma\}$ can be partitioned as $V_1-\{\alpha_1+\sigma\} = \{\beta_1^1, \beta_1^1+\sigma\} \cup \cdots \cup \left\{\beta_1^{\frac{q}{2}-1}, \beta_1^{\frac{q}{2}-1}+\sigma\right\}$, with $\frac{q}{2}-1$ sets of cardinality 2, where $\beta_1^1, \ldots, \beta_1^{\frac{q}{2}-1}$ are distinct vertices. For $i = 2, \ldots, n$ and $j = 1, \ldots, \frac{q}{2}-1$, we denote by $\beta_i^j \in V_i$ the vertex satisfying $\beta_i^j = \beta_1^j + \alpha_1 + \alpha_i$.

Claim 1. *$V_i - \{\alpha_i+\sigma\} = \{\beta_i^1, \beta_i^1+\sigma\} \cup \cdots \cup \left\{\beta_i^{\frac{q}{2}-1}, \beta_i^{\frac{q}{2}-1}+\sigma\right\}$ for $i = 1, \ldots, n$.*

Proof of Claim. We may assume that $i \geq 2$. Let j, ℓ be distinct indices in $\left[\frac{q}{2}-1\right]$. As $\beta_1^j \neq \beta_1^\ell$, we get $\beta_i^j \neq \beta_i^\ell$. Similarly, $\beta_1^j \neq \beta_1^\ell + \sigma$ implies $\beta_i^j \neq \beta_i^\ell + \sigma$. Therefore, $\beta_i^1, \beta_i^1+\sigma, \ldots, \beta_i^{\frac{q}{2}-1}, \beta_i^{\frac{q}{2}-1}+\sigma$ are distinct vertices, so $\{\beta_i^1, \beta_i^1+\sigma\}, \cdots, \left\{\beta_i^{\frac{q}{2}-1}, \beta_i^{\frac{q}{2}-1}+\sigma\right\}$ partition $V_i - \{\alpha_i+\sigma\}$, as required. ◇

By Claim 1, each vertex in V' is β_i^j or $\beta_i^j + \sigma$ for some $i \in [n]$ and $j \in \left[\frac{q}{2}-1\right]$. Now we are ready to characterize what the edges of size 2 are.

Claim 2. *Let u, v be distinct verties in V'. Then $\{u, v\}$ is an edge in $E_{n,\alpha}$ if, and only if, $u = \beta_i^j$ and $v = \beta_k^j + \sigma$ or $u = \beta_i^j + \sigma$ and $v = \beta_k^j$ for some $j \in \left[\frac{q}{2}-1\right]$ and distinct $i, k \in [n]$.*

Proof of Claim. ($\Leftarrow$) Without loss of generality, we may assume that $j = 1$, $i = 1$, and $k = 2$. As $\beta_2^1 = \beta_1^1 + \alpha_1 + \alpha_2$, we have $\beta_1^1 + \beta_2^1 + \sigma = \alpha_1 + \alpha_2 + \sigma$. So, by Lemma 9.10, $\{u, v\}$ is an edge.

($\Rightarrow$) Without loss of generality, we may assume that $u \in V_1$, $v \in V_2$. Then $u = \beta_1^j$ or $u = \beta_1^j + \sigma$ for some $j \in \left[\frac{q}{2}-1\right]$. If $u = \beta_1^j$, then by Lemma 9.10, $v = \beta_1^j + \alpha_1 + \alpha_2 + \sigma = \beta_2^j + \sigma$. Similarly, if $u = \beta_1^j + \sigma$, we can argue that $v = \beta_2^j$, as required. ◇

For $j \in \left[\frac{q}{2}-1\right]$, let G_j denote the graph induced by $\left\{\beta_1^j, \ldots, \beta_n^j\right\} \cup \left\{\beta_1^j + \sigma, \ldots, \beta_n^j + \sigma\right\}$. By Claim 2, the edge set of G_j is precisely $\left\{\left\{\beta_i^j, \beta_k^j + \sigma\right\} : i \neq k\right\}$. Moreover, Claim 2 also implies that there is no edge between G_j and G_ℓ if $j \neq \ell$, as required.

(3) Since $\alpha \notin S$, $\emptyset \notin E$, implying in turn that all the edges of size 1 are minimal. From part (2), we know that no edge of size 2 contains an edge of size 1, and therefore, every edge of size 2 is also minimal, as required. □

Proposition 9.12 ([5]). *Let $q = 2^k$ for some $k \geq 3$, and let $S \subseteq GF(q)^3$ be a vector space over $GF(q)$ such that $\mathcal{M}(S)$ is isomorphic to $M(A_3)$. Then* mult(S) *has C_5^2 as a minor.*

Proof. By Remark 9.8, we may assume that $S = \{x \in GF(q)^n : x_1+x_2+x_3 = 0\}$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \notin S$. We will show that the induced clutter of mult(S) with respect to α, denoted ind(S, α), has C_5^2 as a minor. By Remark 9.9, the members of ind(S, α) are the minimal edges of $H_{3,\alpha}$. Let $\sigma = \alpha_1 + \alpha_2 + \alpha_3$, and we choose $a, b \in GF(q)$ such that $a \in GF(q) - \{\alpha_1, \alpha_1 + \sigma\}$ and $b \in GF(q) - \{\alpha_1, \alpha_1 + \sigma, a, a + \sigma\}$.

Claim 1. *$a + b + \alpha_1 \in GF(q) - \{\alpha_1, \alpha_1 + \sigma, a, a + \sigma, b, b + \sigma\}$.*

Proof of Claim. If $a + b + \alpha_1 = \alpha_1$ or $\alpha_1 + \sigma$, then $b = a$ or $b = a + \sigma$, contradicting the choice of b. If $a+b+\alpha_1 = a$ or $a+\sigma$, then $b = \alpha_1$ or $b = \alpha_1+\sigma$, contradicting the choice of b. If $a+b+\alpha_1 = b$ or $b+\sigma$, then $a = \alpha_1$ or $a = \alpha_1+\sigma$, a contradiction as $a \notin \{\alpha_1, \alpha_1+\sigma\}$. Therefore, $a+b+\alpha_1 \notin \{\alpha_1, \alpha_1+\sigma, a, a+\sigma, b, b+\sigma\}$, as required. ◇

Consider $H_{3,\alpha}$. By Proposition 9.11 (2), the edges of cardinality 2 in $H_{3,\alpha}$ form a graph with $\frac{q}{2}-1$ connected components $G_1,\dots,G_{\frac{q}{2}-1}$ where the vertex set of G_j is

$$\left\{\beta_1^j,\beta_1^j+\sigma\right\}\cup\left\{\beta_2^j,\beta_2^j+\sigma\right\}\cup\left\{\beta_3^j,\beta_3^j+\sigma\right\}$$

where $\beta_i^j,\beta_i^j+\sigma\in V_i-\{\alpha_i+\sigma\}$ for $i\in[3]$. Furthermore, $G_1,\dots,G_{\frac{q}{2}-1}$ are 6-cycles by Proposition 9.11 (2). As $\frac{q}{2}-1\geq 3$, without loss of generality, we may assume that $\beta_1^1=a$, $\beta_1^2=b$, and $\beta_1^3=a+b+\alpha_1$, i.e. G_1,G_2,G_3 contain $a,b,a+b+\alpha_1\in V_1-\{\alpha_1+\sigma\}$, respectively.

Claim 2. *The following statements hold:*

(1) $\beta_1^1+\sigma=a+\sigma$, $\beta_2^1+\sigma=a+\alpha_1+\alpha_2+\sigma$, *and* $\beta_3^1=a+\alpha_1+\alpha_3$.

(2) $\beta_2^2=b+\alpha_1+\alpha_2$ *and* $\beta_2^2+\sigma=b+\alpha_1+\alpha_2+\sigma$.

(3) $\beta_3^3+\sigma=a+b+\alpha_3+\sigma$.

Proof of Claim. The claim follows from Proposition 9.11 (2). ◇

Now keep vertices $\beta_1^1,\beta_1^1+\sigma,\beta_2^1+\sigma,\beta_3^1$ in G_1, $\beta_2^2,\beta_2^2+\sigma$ in G_2, and $\beta_3^3+\sigma$ in G_3 and delete the other vertices from $H_{n,\alpha}$ (see Figure 9.3 for an illustration). Let H denote the resulting subgraph of $H_{n,\alpha}$.

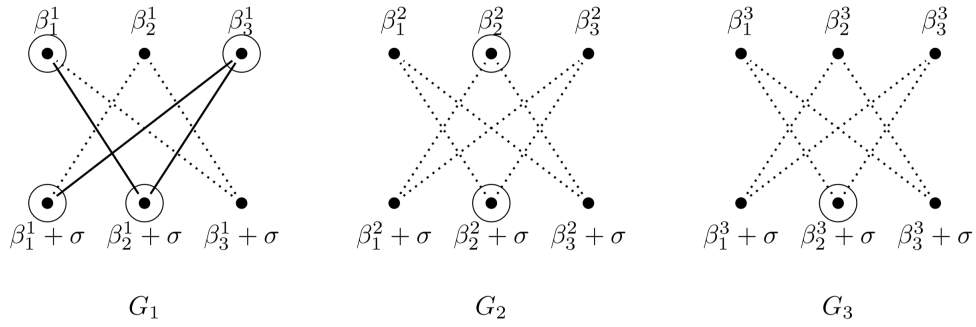


Figure 9.3: The subgraph of $H_{n,\alpha}$ after deleting the vertices

Then the minimal edges of H represent the members of the minor of $\mathrm{ind}(S,\alpha)$ obtained after deleting the elements corresponding to the vertices deleted from $H_{n,\alpha}$, and let $\mathcal{C}$ denote the minor.

As $\alpha_i+\sigma$ for $i\in[n]$ are deleted, we know from Proposition 9.11 (1) that H contains no edge of size 1. By Proposition 9.11 (2), H has 3 edges of size 2: $\left\{\beta_1^1,\beta_2^1+\sigma\right\}$, $\left\{\beta_3^1,\beta_1^1+\sigma\right\}$, $\left\{\beta_3^1,\beta_2^1+\sigma\right\}$, and these are the only ones.

Claim 3. $\left\{\beta_1^1,\beta_2^2,\beta_3^3+\sigma\right\}$ *and* $\left\{\beta_1^1+\sigma,\beta_2^2+\sigma,\beta_3^3+\sigma\right\}$ *are the only edges of size greater than 2 in* H.

Proof of Claim. H contains no vertex in V_i for $i>3$, so H has no edge of size greater than 3 by Lemma 9.10. We also know from Lemma 9.10 that an edge of size 3 contains one vertex from each V_1,V_2,V_3. We claim that no edge of size 3 in H contains an edge of size 2. Suppose that an edge $\{u,v,w\}$ in H contains

an edge $\{u, v\}$ in H for a contradiction. Then $u \in V_i$ and $v \in V_j$ for some distinct $i, j \in \{1, 2, 3\}$ and $u + v = \sigma + \alpha_i + \alpha_j$ by Lemma 9.10. However, $w \in V_k$ for $k \in \{1, 2, 3\} - \{i, j\}$, and $w = \alpha_k$ by Lemma 9.10, a contradiction as $\alpha_k \notin V_k$. The subsets of $V(H)$ not containing an edge of size 2 but one vertex from each of V_1, V_2, V_3 are the following:

$$\left\{\beta_1^1, \beta_2^2, \beta_3^1\right\},\ \left\{\beta_1^1, \beta_2^2 + \sigma, \beta_3^1\right\},\ \left\{\beta_1^1, \beta_2^2, \beta_3^3 + \sigma\right\},\ \left\{\beta_1^1, \beta_2^2 + \sigma, \beta_3^3 + \sigma\right\},$$

$$\left\{\beta_1^1 + \sigma, \beta_2^1 + \sigma, \beta_3^3 + \sigma\right\},\ \left\{\beta_1^1 + \sigma, \beta_2^2, \beta_3^3 + \sigma\right\},\ \left\{\beta_1^1 + \sigma, \beta_2^2 + \sigma, \beta_3^3 + \sigma\right\}.$$

By Lemma 9.10, a subset $\{x_1, x_2, x_3\}$ where $x_i \in V_i$ for $i = 1, 2, 3$ is an edge if, and only if, $x_1 + x_2 + x_3 = \sigma + \alpha_1 + \alpha_2 + \alpha_3$. Notice that $\beta_1^1 + \beta_2^2 + \beta_3^1 = b + \alpha_2 + \alpha_3$ cannot be $\sigma + \alpha_1 + \alpha_2 + \alpha_3$, because b is not $\alpha_1 + \sigma$ by our choice of b. This implies that $\left\{\beta_1^1, \beta_2^2, \beta_3^1\right\}$ is not an edge. Similarly, $\left\{\beta_1^1, \beta_2^2 + \sigma, \beta_3^1\right\}$ is not an edge, because $b \neq \alpha_1$. Notice also that $\{\beta_1^1 + \sigma, \beta_2^1 + \sigma, \beta_3^3 + \sigma\}$ is not an edge, because $\beta_1^1 + \sigma + \beta_2^1 + \sigma + \beta_3^3 + \sigma = a + b + \alpha_1 + \alpha_2 + \alpha_3 + \sigma$ cannot be $\sigma + \alpha_1 + \alpha_2 + \alpha_3$ by our assumption that $a \neq b$. Observe that $\beta_1^1 + \beta_2^2 + \beta_3^3 + \sigma = \sigma + \alpha_1 + \alpha_2 + \alpha_3$, implying in turn that $\left\{\beta_1^1, \beta_2^2, \beta_3^3 + \sigma\right\}$ and $\left\{\beta_1^1 + \sigma, \beta_2^2 + \sigma, \beta_3^3 + \sigma\right\}$ are edges, whereas $\left\{\beta_1^1, \beta_2^2 + \sigma, \beta_3^3 + \sigma\right\}$ and $\left\{\beta_1^1 + \sigma, \beta_2^2, \beta_3^3 + \sigma\right\}$ are not. Therefore, $\left\{\beta_1^1, \beta_2^2, \beta_3^3 + \sigma\right\}$ and $\left\{\beta_1^1 + \sigma, \beta_2^2 + \sigma, \beta_3^3 + \sigma\right\}$ are the only edges of size at least 3 in H, as required. ◇

Now that we have characterized all the edges of H, we know that the incidence matrix of $\mathcal{C}$ is isomorphic to the following 0,1 matrix:

$$\begin{array}{c} \begin{array}{ccccccc} \beta_1^1 & \beta_2^1 + \sigma & \beta_3^1 & \beta_1^1 + \sigma & \beta_3^3 & \beta_2^2 & \beta_2^2 + \sigma \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Contracting the elements corresponding to $\beta_2^2, \beta_2^2 + \sigma$ from $\mathcal{C}$, we obtain C_5^2, and thus, $\mathcal{C}$ contains C_5^2 as a minor. Since $\mathcal{C}$ is a minor of $\mathrm{ind}(S, \alpha)$, $\mathrm{ind}(S, \alpha)$ also has C_5^2 as a minor, as required. □

In particular, if $\mathrm{mult}(S)$ is ideal, then $\mathcal{M}(S)$ does not contain $M(A_3)$ as a minor. Brylawski [24] proved the following theorem, which will be used to complete the proof of Theorem 1.50 and will be also useful later to prove Theorem 1.52. Let $G = (V, E)$ be a connected graph. A *block* or *2-vertex-connected component* of G is a maximal vertex-induced subgraph of G that is 2-vertex-connected. We call a graph a *series-parallel network* if each of its blocks is a series-parallel graph, a loop, or a bridge.

Theorem 9.13 ([24]). *Let M be a matroid. Then the following statements are equivalent:*

(i) M contains none of $U_{2,4}$ and $M(K_4)$ as a minor.

(ii) M is the cycle matroid of a series-parallel network.

A graph H is a *minor* of a graph G if H can be obtained from G after a series of edge deletions, edge contractions, and deletions of isolated vertices. If G is connected, then H is a minor of G if and only if $G = H \setminus E_1/E_2$ for some disjoint subsets E_1, E_2 of $E(G)$.

Remark 9.14 (see Chapter 3.2 in [101]). *Let G, H be graphs. If H is a minor of G, then $M(H)$ is a minor of $M(G)$.*

Now we are ready to prove Lemma 9.2 (3).

Proof of Lemma 9.2 (3). Take integers $n, k \geq 3$. Let $q = 2^k$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. We will prove that if $\text{mult}(S)$ has no C_5^2 as a minor, then S has a basis with vectors of pairwise disjoint supports.

Assume that S admits a basis with vectors of pairwise disjoint supports. Then, by Proposition 9.12, $\mathcal{M}(S)$ does not contain $M(A_3)$ as a minor.

Claim 1. *$U_{2,4}$ contains $M(A_3)$ as a minor.*

Proof of Claim. Let $\{1,2,3,4\}$ be the ground set of $U_{2,4}$. Then $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ is the set of circuits of $U_{2,4}$. Contracting 4 from $U_{2,4}$, we obtain the minor of $U_{2,4}$ whose ground set is $\{1,2,3\}$ and whose circuits are $\{1,2\},\{1,3\},\{2,3\}$, that is $M(A_3)$, as required. ◇

Claim 2. *$\mathcal{M}(S)$ is the cycle matroid of a graph whose circuits are pairwise edge-disjoint.*

Proof of Claim. Observe that K_4 has A_3 as a minor, and therefore, $M(K_4)$ contains $M(A_3)$ as a minor. Then $\mathcal{M}(S)$ does not contain $M(K_4)$ as a minor, while we know from Claim 1 that $\mathcal{M}(S)$ contains no $U_{2,4}$ as a minor. Then Theorem 9.13 implies that $\mathcal{M}(S)$ is a graphic matroid, so let G denote its underlying graph. We also know that G does not contain A_3 as a minor, implying in turn that two distinct circuits of G are edge-disjoint. ◇

Suppose for a contradiction that S does not admit a basis with vectors of pairwise disjoint supports. Let $S = \langle v^1, \ldots, v^r \rangle$ for some vectors $v^1, \ldots, v^r \in GF(q)$. After elementary arithmetic operations over $GF(q)$, we may assume that for each $i = 1, \ldots, r$,

$$v_i^i = 1 \quad \text{and} \quad v_j^i = 0 \quad \forall j \in [r] - \{i\}.$$

Claim 3. *For each $i \in [r]$, the support of v^i is minimal among the points in $S - \{\mathbf{0}\}$.*

Proof of Claim. Suppose for a contradiction that there is a vector $u \in S - \{\mathbf{0}\}$ whose support is properly contained in the support of v^i. As $u \in S$, u can be expressed as $\sum_{i=1}^r \lambda_i v^i$ for some $\lambda \in GF(q)^r$. By our supposition, we have $\lambda_j = 0$ for every $j \neq i$. As u is nonzero, λ_i is nonzero, implying in turn that the support of u is the same as that of v^i, a contradiction. ◇

As the generators of S are not pairwise disjoint, v^i and v^j for some distinct i, j have their supports intersect. By Claim 3, the supports of v^i and v^j are both minimal, which means $\mathcal{M}(S)$ has two circuits that are not edge-disjoint. However, this contradicts Claim 2. Therefore, there is a basis of S with vectors of pairwise disjoint support, as required. □

Now we are ready to finish the proof of Theorem 1.50.

Proof of Theorem 1.50 when $q = 2^k$, $k \geq 3$. Take an integer $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. By Theorem 9.3, three statements **(i)**, **(ii)**, **(iii)** are equivalent. As the max-flow min-cut property implies idealness and **(ii)**, **(iii)** are equivalent, we have **(iii)⇒(iv)**. It remains to be shown that **(iv)⇒(ii)**. We need to prove that if mult(S) is ideal, then S has a basis with vectors of pairwise disjoint supports. If mult(S) is ideal, then it has no C_5^2 as a minor. Then it follows from Lemma 9.2 (3) that S has a basis with vectors of pairwise disjoint supports, as required. □

9.3 Theorem 1.52

Take an integer $n \geq 3$, and let $S \subseteq GF(4)^n$ be a vector space over $GF(4)$. If mult(S) is ideal, then it has no Δ_3 minor. In fact, Theorem 1.52 states that the converse is also true, namely, if mult(S) has no Δ_3 minor, then it is ideal. We will prove this in this section. We start this section by proving the following two propositions.

Proposition 9.15 ([5])**.** *Let q be a power of 2, and let $S \subseteq GF(q)^4$ be a vector space over $GF(q)$. If $\mathcal{M}(S)$ is isomorphic to $U_{2,4}$, then* mult(S) *has Δ_3 as a minor.*

Proof. Suppose for a contradiction that mult(S) has no Δ_3 as a minor. Since the rank of $U_{2,4}$ is 2, the rank of S is $4 - 2 = 2$. Let $v^1, v^2 \in GF(q)^4$ be two generators of S. By elementary row operations, we may assume that $(v_1^1, v_2^1) = (1, 0)$ and $(v_1^2, v_2^2) = (0, 1)$. Then

$$\begin{array}{c} \\ v^1 \\ v^2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array}\right]$$

where $x, y, z, w \in GF(q)$. Each circuit of $U_{2,4}$ has size 3, so $x, y, z, w \neq 0$. Then $a := (-x^{-1}z)v^1$, $b := v^2$, $c := a + b$ are vectors in S. Observe that

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \left[\begin{array}{c|c|c|c} -x^{-1}z & 0 & -z & -x^{-1}yz \\ 0 & 1 & z & w \\ -x^{-1}z & 1 & 0 & -x^{-1}yz + w \end{array}\right]$$

and that $a_1 = c_1 \neq b_1$, $b_2 = c_2 \neq a_2$. We also have that $a_3 = b_3 \neq c_3$, because q being a power of 2 implies $z + z = 0$ and $z = -z$. By Lemma 9.1, there is a vector $d \in GF(q)^4$ that satisfies at least two of $d_1 = b_1 = 0$, $d_2 = a_2 = 0$, $d_3 = c_3 = 0$ and satisfies $d_4 \in \{-x^{-1}yz, w, -x^{-1}yz + w\}$. But then the support of d has size at most 2. Since every circuit of $U_{2,4}$ has size 3, $d = \mathbf{0}$, and therefore, $d_4 = -x^{-1}yz + w = 0$. This implies the support of c has size 2, a contradiction. □

K_4 is the complete graph on 4 vertices, and we denote by K_4/e what is obtained from K_4 after contracting an edge from it (see Figure 9.4).

Proposition 9.16 ([5])**.** *Let $q = 2^k$ for some $k \geq 2$, and let $S \subseteq GF(q)^5$ be a vector space over $GF(q)$. If $\mathcal{M}(S)$ is isomorphic to $M(K_4/e)$, then* mult(S) *has Δ_3 as a minor.*

Proof. In Figure 9.4, we can see that the fundamental circuits $\{1, 4, 5\}$, $\{2, 4\}$, $\{3, 5\}$ with respect to spanning tree $\{4, 5\}$ generate the cycle space of K_4/e. Pick vectors $v^1, v^2, v^3 \in S$ whose supports are the

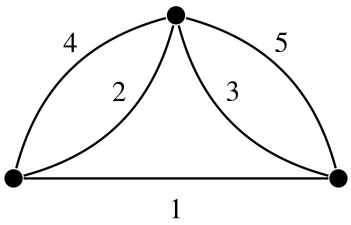


Figure 9.4: K_4/e

three circuits. Notice that these vectors are linearly independent. Since the rank of S is $5-2=3$, v^1, v^2, v^3 generate S. After elementary row operations, S is generated by the 3 vectors v^1, v^2, v^3 of the following forms:

$$\begin{array}{c} v^1 \\ v^2 \\ v^3 \end{array} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & z & 0 \\ 0 & 0 & t & 0 & w \end{array}\right]$$

where $t, x, y, z, w \neq 0$. Since $q > 2$, we may assume that z and w are distinct nonzero elements in $GF(q)$. Now consider the restriction S' of S:

$$S' := S \cap \left\{ x \in GF(q)^5 : \ x_1 \in \{0, z, w\},\ x_2 \in \{0, x\},\ x_3 \in \{0, ty\} \right\}.$$

We will show that mult(S') has Δ_3 as a minor, implying in turn that mult(S) also has Δ_3 as a minor. Notice that

$$S' = \left\{ \sum_{i=1}^{3} \lambda_i v^i : \ \lambda_1 \in \{0, z, w\},\ \lambda_2 \in \{0, x\},\ \lambda_3 \in \{0, y\} \right\}.$$

Consider the three distinct points $a := zv^1$, $b := wv^1$, $c := xv^2 + yv^3$ in S':

$$\begin{array}{c} a \\ b \\ c \end{array} \left[\begin{array}{ccc|cc} z & 0 & 0 & zx & zy \\ w & 0 & 0 & wx & wy \\ 0 & x & ty & zx & wy \end{array}\right]$$

As $z \neq w$, we have that $c_4 = a_4 \neq b_4$ and $b_5 = c_5 \neq a_5$. We also have $a_3 = b_3 \neq c_3$, because $ty \neq 0$. Suppose for a contradiction that mult(S') has no Δ_3 as a minor. By Lemma 9.1, there is $d \in S' - \{a, b, c\}$ that satisfies

(1) $d_1 \in \{0, z, w\}$, $d_2 \in \{0, x\}$, $d_3 \in \{0, ty\}$, $d_4 \in \{zx, wx\}$, $d_5 \in \{zy, wy\}$, and

(2) at least two of $d_3 = ty$, $d_4 = wx$, $d_5 = zy$ hold.

The points of $S' - \{a, b, c\}$ are the following:

$$S' - \{a, b, c\} = \left\{ \begin{array}{c|c|c} (0,0,0,0,0) & (0,x,0,zx,0) & (0,0,ty,0,wy) \\ (z,x,0,0,zy) & (z,0,ty,zx,(z+w)y) & (w,x,0,(z+w)x,wy) \\ (w,0,ty,wx,0) & (z,x,ty,0,(z+w)y) & (w,x,ty,(z+w)x,0) \end{array} \right\}.$$

Since $z, w \neq 0$ and $z \neq w$, $(z+w)x \notin \{zx, wx\}$ and $(z+w)y \notin \{zy, wy\}$. Since $z, w, x, y \neq 0$, $0 \notin \{zx, wx\}$ and $0 \notin \{zy, wy\}$. This indicates that no point in $S' - \{a, b, c\}$ satisfies condition (1), a contradiction. Therefore, $\mathrm{mult}(S')$ has Δ_3 as a minor, and so does $\mathrm{mult}(S)$, as required. □

By Propositions 9.15 and 9.16, if $\mathrm{mult}(S)$ for has no Δ_3 minor, then $\mathcal{M}(S)$ has none of $U_{2,4}$, $M(K_4/e)$ as a minor. In that case, as $M(K_4/e)$ is a minor of $M(K_4)$, it follows by Theorem 9.13 that $\mathcal{M}(S)$ is the cycle matroid of a graph not containing K_4/e as a minor. How does a graph with no K_4/e minor look like? We will prove Proposition 9.18 that characterizes graphs with no K_4/e minor.

Recall that a block of G is a maximal vertex-induced subgraph of G that is 2-vertex-connected. The following is well-known in graph theory:

Proposition 9.17 (see Proposition 5.3 [20]). *Let $G = (V, E)$ be a connected graph. Then the following statements hold:*

(1) any two blocks of G have at most one vertex in common, and if they have one, it is a cut-vertex of G,

(2) each cycle of G is contained in a block of G, and

(3) the blocks of G decomposes G.

So, we may associate a connected graph $G = (V, E)$ with a bipartite graph $\mathcal{B}(G)$ where

- the cut-vertices of G form one color class of $\mathcal{B}(G)$,
- the blocks of G form the other color class of $\mathcal{B}(G)$, and
- cut-vertex u and block B are adjacent in $\mathcal{B}(G)$ if $u \in V(B)$.

It is well-known that $\mathcal{B}(G)$ is a tree all of whose leaves are blocks of G (see [20]).

Proposition 9.18 ([5]). *Let $G = (V, E)$ be a connected graph. If G contains no K_4/e as a minor, then each block of G is a subdivision of A_t for some $t \geq 2$, a bridge, or a loop.*

Proof. Assume that G contains no K_4/e minor. We will prove by induction on the number of edges that each block of G is a subdivision of A_t for some $t \geq 2$, a bridge, or a loop. The base case is trivial. For the induction step, we may assume that G has at least 3 edges. If G has more than one block, a block of G has less edges than G has, so by the induction hypothesis, each block of G is a subdivision of A_t for some $t \geq 2$, a bridge, or a loop. Thus we may assume that G is 2-vertex-connected. In particular, G has no loop.

Let e be an edge of G. By the induction hypothesis, each block of $G - \{e\}$ is a subdivision of A_t fo some $t \geq 2$ or a bridge. We first prove the following claim:

Claim 1. *$\mathcal{B}(G - \{e\})$ is a path, and e is incident to internal vertices of the end blocks.*

Proof of Claim. If $\mathcal{B}(G-\{e\})$ is a single vertex, the assertion follows. Thus we may assume that $G-\{e\}$ has at least two blocks. Since G is 2-vertex-connected, e connects two distinct blocks B_1, B_2 of $G-\{e\}$. Then, after putting e back, the blocks of $G-\{e\}$ on the fundamental cycle C of $\mathcal{B}(G-\{e\})$ obtained after adding edge B_1B_2 become one block in G. In fact, since G is 2-vertex-connected, G has no other block. This implies that $G-\{e\}$ has no block other than the ones on C. So, $\mathcal{B}(G-\{e\})$ contains no vertex outside C, and therefore, $\mathcal{B}(G-\{e\})$ is a path where B_1, B_2 are its two ends. If e is not incident to an internal vertex of B_1, then e is incident to the cut-vertex of B_1, implying that B_1 is separated from B_2 in G, a contradiction. Thus e is incident to an internal vertex of B_1. Similarly, e is incident to an internal vertex of B_2, as required. ◇

Next, we claim the following:

Claim 2. *All but at most one block of $G-\{e\}$ are bridges.*

Proof of Claim. We may assume that $G-\{e\}$ has at least two blocks. Then, by Claim 1, $\mathcal{B}(G-\{e\})$ is a path $B_1, u_1, B_2, \ldots, u_{k-1}, B_k$ for some $k \geq 2$, where $B_1, \ldots, B_k$ are the blocks of $G-\{e\}$ and u_ℓ is the cut-vertex separating B_ℓ and $B_{\ell+1}$ for $\ell \in [k-1]$. Moreover, by Claim 1, $e = u_0u_k$, where u_0 is an internal vertex of B_1 and u_k is an internal vertex of B_k.

Suppose for a contradiction that $G-\{e\}$ has two blocks that are not bridges. Then B_i, B_j for some distinct $i, j \in [k]$ are not bridges. In particular, B_i has a cycle C_i and B_j has a cycle C_j. After contracting the edges of B_ℓ for $\ell \in [k]-\{i,j\}$ from $G-\{e\}$, the vertices in $B_1, \ldots, B_{i-1}$ are identified with u_{i-1}, the vertices in $B_{i+1}, \ldots, B_{j-1}$ are identified with u_{j-1}, and the vertices in $B_{j+1}, \ldots, B_k$ are identified with u_j. Therefore, the resulting graph is $u_{i-1}, B_i, u_{j-1}, B_j, u_j$, where u_{i-1} and u_j are internal vertices of B_i and B_j, respectively, and u_{j-1} is the cut-vertex separating B_i, B_j. Notice that e connects u_{i-1} and u_j after the contraction, because u_0, u_k were identified with u_{i-1}, u_j, respectively. We then delete the edges outside of the cycles C_i, C_j. After adding e back, we obtain a subdivision of K_4/e, a contradiction as G has no K_4/e minor. Therefore, at most one block of $G-\{e\}$ is a bridge. ◇

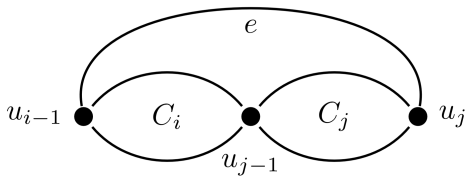


Figure 9.5: $e = u_{i-1}u_j$

If every block of $G-\{e\}$ is a bridge, then it follows from Claim 1 that G is a cycle, so G is a subdivision of A_2. Thus we may assume that a block B of $G-\{e\}$ is a subdivision of A_t for some $t \geq 2$. Then, by Claim 2, the other blocks of $G-\{e\}$ are bridges.

Claim 3. *G is the union of B and a path P whose ends are two vertices in B and whose interior vertices are disjoint from $V(B)$.*

Proof of Claim. It follows from Claim 1 that e and the bridges of $G - \{e\}$ form a path P connecting two vertices of B. An interior vertex of P, if exists, is in a block of $G - \{e\}$ other than B, so it is not contained in $V(B)$, as required. $\diamond$

As B is a subdivision of A_t for some $t \geq 2$, B is a disjoint union of internally vertex-disjoint uv-paths for some distinct $u, v \in V(B)$. Let $P_1, \ldots, P_t$ be the uv-paths.

Claim 4. *If $t = 2$, G is a subdivision of A_3.*

Proof of Claim. If $t = 2$, B is a cycle and P connects two vertices on the cycle, and by Claim 3, G is the union of three internally vertex-disjoint paths connecting the two vertices. So, G is a subdivision of A_3. $\diamond$

By Claim 4, we may assume that $t \geq 3$. We will show that P is an uv-path, thereby proving that G is a subdivision of A_{t+1}.

Claim 5. *P is an uv-path.*

Proof of Claim. Suppose for a contradiction that P is not a uv-path. Then one of P's two ends is not in $\{u, v\}$.

First, consider the case when one end of P is in $\{u, v\}$. Without loss of generality, we may assume that one end of P is u and the other end is $w \in V - \{u, v\}$. Without loss of generality, assume that w is on P_1. Then the subgraph of G obtained after deleting the edges $E - E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3)$ (see Figure 9.6 for an illustration) is a subdivision of K_4/e, contradicting the assumption that G has no K_4/e minor.

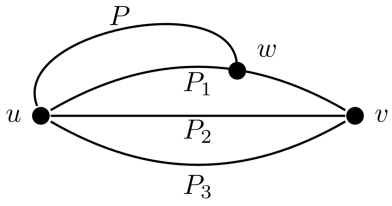


Figure 9.6: $w \notin \{u, v\}$

Now consider the case when both ends of P are not in $\{u, v\}$. Let the ends of P be $w_1, w_2 \in V - \{u, v\}$. There are two cases to consider: w_1, w_2 are on the same uv-path of B, or w_1, w_2 are on different uv-paths. If w_1, w_2 are on the same uv-path, we may assume that they are on P_1 without loss of generality. In this case, deleting the edges $E - E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3)$ and contracting the edges of the uw_1-path on P_1 (see Figure 9.7 for an illustration), we obtain a subdivision of K_4/e, a contradiction.

If w_1, w_2 are on different uv-paths, we may assume that w_1 is on P_1 and w_2 is on P_2 without loss of generality. Deleting the edges $E - E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3)$ and contracting the edges of P (see Figure 9.7 for an illustration), we obtain a subdivision of K_4/e, a contradiction as G has no K_4/e minor. $\diamond$

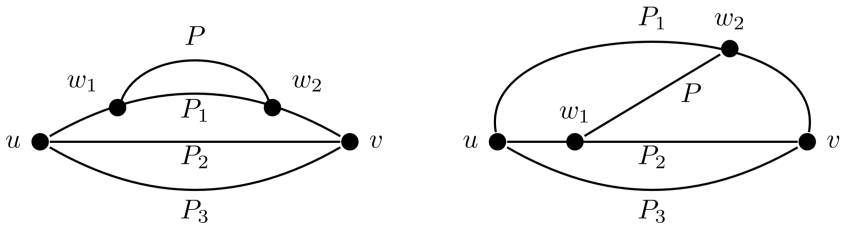


Figure 9.7: $w_1, w_2 \notin \{u, v\}$

By Claims 3 and 5, P is an uv-path that is internally vertex-disjoint from $P_1, \ldots, P_t$, implying in turn that G is a subdivision of A_{t+1}. This finishes the proof. □

The *direct sum* of ℓ matroids $M_1, \ldots, M_\ell$ with pairwise disjoint ground sets is defined as

$$M_1 \oplus \cdots \oplus M_\ell = (E_1 \cup \cdots \cup E_\ell,\ \{I_1 \cup \cdots \cup I_\ell :\ I_i \in \mathcal{I}_i,\ i \in [\ell]\})$$

where E_i and $\mathcal{I}_i$ are the ground set and family of independent sets of M_i for $i = 1, \ldots, \ell$.

Remark 9.19 (see Chapter 4.1 in [101])**.** *Let M be the cycle matroid of a graph G, and let $G_1, \ldots, G_k$ be the blocks of G. Then $M = M(G_1) \oplus \cdots \oplus M(G_k)$.*

Putting Proposition 9.18 and Remark 9.19 together, we can prove the following lemma:

Lemma 9.20 ([5])**.** *Let $q = 2^k$ for some $k \geq 2$, and let S be a vector space over $GF(q)$. If* mult(S) *has no Δ_3 as a minor, then for some $k \geq 1$,*

$$\mathcal{M}(S) = M_1 \oplus \cdots \oplus M_k$$

where M_i is the cycle matroid of a subdivision of A_t for some $t \geq 2$, a bridge, or a loop for each $i \in [k]$.

Proof. If mult(S) has no Δ_3 as a minor, then $\mathcal{M}(S)$ contains none of $U_{2,4}$ and $M(K_4/e)$ as a minor by Lemma 9.6, Propositions 9.15 and 9.16. As $M(K_4/e)$ is a minor of $M(K_4)$, by Theorem 9.13, $\mathcal{M}(S)$ is the cycle matroid of a series-parallel network not containing K_4/e as a minor, denoted as G. Then by Proposition 9.18, each block of G is a subdivision of A_t for some $t \geq 2$, a bridge, or a loop. So, the assertion follows from Remark 9.19, as required. □

The following remark shows how to represent the direct sum of two matroids with their representations:

Remark 9.21 (see Chapter 4.2 in [101])**.** *Let A_1 and A_2 be $GF(q)$-representations of matroids M_1, M_2 with disjoint ground sets, respectively. Then $M_1 \oplus M_2$ can be represented by*

$$\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$$

Using Remark 9.21, we can prove the following lemma:

Lemma 9.22 ([5]). *Let q be a power of 2. Let $n \geq 3$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. If $\mathcal{M}(S) = M_1 \oplus M_2$ for some $GF(q)$-representable matroids M_1, M_2, then there exist vector spaces S_1 and S_2 over $GF(q)$ satisfying the following:*

(1) $S = S_1 \times S_2$.

(2) $\mathcal{M}(S_i) = M_i$ *for* $i = 1, 2$.

Proof. For $i \in [2]$, let A_i be a $GF(q)$-representation of M_i. By Remark 9.21, $\mathcal{M}(S)$ can be represented by

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$$

By Remark 9.4, $S = \{x \in GF(q)^n : \ Ax = \mathbf{0}\}$, and therefore,

$$S = \{(x_1, x_2) \in GF(q)^{n_1} \times GF(q)^{n_2} : \ A_1x_1 = \mathbf{0}, \ A_2x_2 = \mathbf{0}\}$$

where n_i denotes the number of columns in A_i for $i = 1, 2$, and thus S can be written as $S_1 \times S_2$ where

$$S_i = \{x_i \in GF(q)^{n_i} : \ A_ix_i = \mathbf{0}\} \quad \text{for } i = 1, 2.$$

Then, by Remark 9.4, $\mathcal{M}(S_i) = M_i$ for $i = 1, 2$, as required. □

We know by Proposition 8.4 (1) that the product operations preserves idealness, so it suffices by Lemmas 9.20 and 9.22 to show that $\text{mult}(S)$ is ideal for any vector space S over $GF(4)$ whose associate matroid $\mathcal{M}(S)$ is the cycle matroid of a subdivision of A_t for some $t \geq 2$, a bridge, or a loop.

Let $\mathcal{C}$ be a clutter over ground set E. For an element $e \in E$, the clutter obtained from $\mathcal{C}$ after *duplicating* e is

$$\{C : \ e \notin C \in \mathcal{C}\} \cup \{C \cup \{e'\} : \ e \in C \in \mathcal{C}\}$$

where $e' \notin E$. A *duplication* of $\mathcal{C}$ is what is obtained from $\mathcal{C}$ after a series of duplicating elements. It is a well-known fact that duplication preserves the idealness of a clutter.

Remark 9.23 ([5]). *Let $\mathcal{C}$ be a clutter over ground set is E, and let $\mathcal{C}'$ be a duplication of $\mathcal{C}$. Then $\mathcal{C}$ is ideal if, and only if, $\mathcal{C}'$ is ideal.*

Remark 9.24 is the last ingredient to prove Theorem 1.52.

Remark 9.24 ([5]). *Let q be a prime power. Let $n \geq 1$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. If $\mathcal{M}(S)$ has elements $i, j \in [n]$ in series, then there exists a vector space $S' \subseteq GF(q)^{n-1}$ over $GF(q)$ such that*

- $\mathcal{M}(S') = \mathcal{M}(S)/\{j\}$,
- $\text{mult}(S)$ *is isomorphic to a duplication of* $\text{mult}(S')$.

Proof. Without loss of generality, we may assume that $i = n-1$ and $j = n$. Let $S' \subseteq GF(q)^{n-1}$ be what is obtained from S after dropping the n^{th} coordinate of the points in S. Then S' is a vector space, and by Lemma 9.6, $\mathcal{M}(S') = \mathcal{M}(S)/\{n\}$.

Let $x \in S$. Then support(x) is the union of some circuits of $\mathcal{M}(S)$ by Remark 9.4. As $n-1, n$ are series elements, a circuit of $\mathcal{M}(S)$ contains $n-1$ if and only if it contains n, implying in turn that $n-1 \in$ support(x) if and only if $n \in$ support(x). Let $v^1, \ldots, v^r$ be a basis of S. If $n \in$ support(x) for some $x \in S$, then $n \in$ support(v^ℓ) for some $\ell \in [r]$, and thus, we may assume that $n \in$ support(v^1) and that $v^1_n \neq 0$. After scaling the v^ℓ's, if necessary, we may assume that $v^\ell_n = 0$ for $\ell \in [r] - \{1\}$. Since $n-1 \in$ support(x) if and only if $n \in$ support(x) for $x \in S$, we have that $v^1_{n-1} \neq 0$ and $v^\ell_{n-1} = 0$ for $\ell \in [r] - \{1\}$. Then for some $y, z \in GF(q) - \{0\}$,

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^r \end{array} \left[\begin{array}{c|cc} \cdots & y & z \\ \cdots & 0 & 0 \\ \vdots & \vdots & \\ \cdots & 0 & 0 \end{array}\right].$$

By Remark 9.7, we may assume that $y = z$. Moreover, S' is generated by $u^1, \ldots, u^r$, where

$$\begin{array}{c} u^1 \\ u^2 \\ \vdots \\ u^r \end{array} \left[\begin{array}{c|c} \cdots & y \\ \cdots & 0 \\ \vdots & \vdots \\ \cdots & 0 \end{array}\right].$$

Therefore, mult(S) is isomorphic to the clutter obtained from mult(S') after duplicating the q elements in the part of mult(S')'s ground set corresponding to n. □

Now we are ready to prove Theorem 1.52.

Theorem 1.52 ([5]). *Let $n \geq 3$, and let $S \subseteq GF(4)^n$ be a vector space over $GF(4)$. Then the following statements are equivalent:*

(i) mult(S) *contains no Δ_3 as a minor,*

(ii) $S = S_1 \times \cdots \times S_k$ *where for each $i \in [k]$,*

- $S_i = \{0\}$,
- $S_i = GF(4)$, *or*
- $S_i = \langle v^1, \ldots, v^r \rangle$ *where $r \geq 1$ and $v^1, \ldots, v^r$ are vectors of the following form, after permuting the coordinates:*

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^r \end{array} \left[\begin{array}{c|c|c|c|c} u^0 & u^1 & \mathbf{0} & \cdots & \mathbf{0} \\ u^0 & \mathbf{0} & u^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u^0 & \mathbf{0} & \mathbf{0} & \cdots & u^r \end{array}\right] \qquad (\star)$$

for some vectors $u^0, u^1 \ldots, u^r$ *of nonzero entries,*

(iii) $\mathrm{mult}(S)$ *is ideal.*

Proof. **(iii)⇒(i)**: Since Δ_3 is non-ideal, $\mathrm{mult}(S)$ contains no Δ_3 as a minor, as required. **(i)⇒(ii)**: By Lemma 9.20, $\mathcal{M}(S) = M_1 \oplus \cdots \oplus M_k$ for some $k \geq 1$ where for each $i \in [k]$, M_i is the cycle matroid of a subdivision A_t for some $t \geq 2$, a bridge, or a loop. Lemma 9.22 implies that there exist vector spaces $S_1, \ldots, S_k$ such that $S = S_1 \times \cdots \times S_k$ and $\mathcal{M}(S_i) = M_i$ for $i \in [k]$. One can easily prove the following claim:

Claim 1. *Let* T *be a vector space over* $GF(4)$*. Then the following statements hold:*

(1) if $\mathcal{M}(T)$ *is the cycle matroid of a bridge, then* $T = \{0\}$*,*

(2) if $\mathcal{M}(T)$ *is the cycle matroid of a loop, then* $T = GF(4)$*.*

We also need the following claim:

Claim 2. *Let* T *be a vector space over* $GF(4)$*. If* $\mathcal{M}(T)$ *is the cycle matroid of a subdivision of* A_t *for some* $t \geq 2$*, then* $T = \langle v^1, \ldots, v^{t-1} \rangle$ *where* $v^1, \ldots, v^{t-1}$ *are vectors of the form* $(\star)$ *for some vectors* $u^0, u^1 \ldots, u^{t-1}$ *of nonzero entries.*

Proof of Claim. Assume that $\mathcal{M}(T) = M(G)$ where G is a subdivision of A_t for some $t \geq 2$. Notice that G consists of two vertices and t internally vertex-disjoint paths connecting them. Let $P_0, \ldots, P_{t-1}$ denote the paths, and let $E(P_0), \ldots, E(P_{t-1})$ denote their edge sets. Then it follows from Remark 9.4 that that $P_0 \cup P_i$ is a circuit of G for each $i \in [t-1]$, so T contains a point whose support is $E(P_0) \cup E(P_i)$. Therefore, T contains $t-1$ points $v^1, \ldots, v^{t-1}$ of the following form:

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^{t-1} \end{array} \left[\begin{array}{c|c|c|c|c} u_1^0 & u^1 & \mathbf{0} & \cdots & \mathbf{0} \\ u_2^0 & \mathbf{0} & u^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{t-1}^0 & \mathbf{0} & \mathbf{0} & \cdots & u^{t-1} \end{array}\right]$$

where $u_1^0, \ldots, u_{t-1}^0 \in GF(4)^{|E(P_0)|}$ and $u^i \in GF(4)^{|E(P_i)|}$ for $i \in [n]$ are vectors of nonzero entries. As the cycle space of G is generated by $P_0 \cup P_1, \ldots, P_0 \cup P_t$, the rank of T is $t-1$, implying in turn that $T = \langle v^1, \ldots, v^{t-1} \rangle$. We will show that for each $i \in [t-1]$, $u_i^0 = \lambda_i u_1^0$ for some $\lambda_i \in GF(4) - \{0\}$. As $P_1 \cup P_2$ is a circuit of G, by Remark 9.4, there is a point $v \in T$ whose support is $E(P_1) \cup E(P_2)$. Then v can be written as $v = \mu_1 v^1 + \mu_2 v^2$ for some $\mu_1, \mu_2 \in GF(4) - \{0\}$. As the support of v is $E(P_1) \cup E(P_2)$, we have that $\mu_1 u_1^0 + \mu_2 u_2^0 = 0$, which implies that $u_2^0 = \lambda_2 u_1^0$ for some nonzero λ_2. Similarly, we obtain $u_i^0 = \lambda_i u_1^0$ for some nonzero λ_i for $i \in [t-1]$, as required. Therefore, after scaling v^i's if necessary, we may assume that $u_1^0 = \cdots = u_{t-1}^0$, as required. ◇

By Claims 1 and 2, for each $i \in [k]$, either $S_i = \{0\}$, $S_i = GF(4)$, or $S_i = \langle v^1, \ldots, v^{t-1} \rangle$ where $t \geq 2$ and $v^1, \ldots, v^{t-1} \in GF(4)^n$ are of the form $(\star)$ for some vectors $u^0, u^1 \ldots, u^{t-1}$ of nonzero entries, as required.

(ii)⇒(iii): It suffices by Proposition 8.4 (1) to show that $\text{mult}(S_i)$ is ideal for every $i \in [k]$. Let $i \in [k]$. If $S_i = \{0\}$ or $GF(4)$, then the members of $\text{mult}(S_i)$ are pairwise disjoint, and therefore, $\text{mult}(S_i)$ is ideal by Remark 7.16. Thus we may assume that $S_i = \langle v^1, \ldots, v^r \rangle$ where $r \geq 1$ and $v^1, \ldots, v^r$ are vectors of the form $(\star)$ for some vectors $u^0, u^1 \ldots, u^r$ of nonzero entries. We will need the following claim:

Claim 3. *Let $n \geq 1$, and let $T = \{x \in GF(4)^n : x_1 + \cdots + x_n = 0\}$. Then $\text{mult}(T)$ is ideal.*

Proof of Claim. By Proposition 1.40 (1), $\text{mult}(T)$ is ideal if $n \leq 2$. Thus we may assume that $n \geq 3$. By Theorem 1.35, it suffices to argue that all induced clutters of $\text{mult}(T)$ are ideal. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \notin T$. We will show that the induced clutter of $\text{mult}(T)$ with respect to α, denoted $\text{ind}(T, \alpha)$, is ideal. By Remark 9.9, the members of $\text{ind}(T, \alpha)$ are the minimal edges of $H_{n,\alpha}$. Let $\sigma = \alpha_1 + \cdots + \alpha_n$. $H_{n,\alpha}$ has n edges of cardinality 1, $\{\alpha_1 + \sigma\}, \ldots, \{\alpha_n + \sigma\}$ by Proposition 9.11 (1). By Proposition 9.11 (2), the edges of cardinality 2 form a connected bipartite graph G where

- G is bipartite on $\{\beta_1, \ldots, \beta_n\} \cup \{\beta_1 + \sigma, \ldots, \beta_n + \sigma\}$ where $\{\beta_i, \beta_i + \sigma\} = GF(4) - \{\alpha_i, \alpha_i + \sigma\}$ for $i \in [n]$,
- $\beta_i = \beta_1 + \alpha_1 + \alpha_i$ for $i \in [n]$, and
- the edge set of G is $\{\{\beta_i, \beta_k + \sigma\} : i \neq k\}$.

We will show that there is no minimal edge of cardinality at least 3 in $H_{n,\alpha}$. Suppose for a contradiction that $H_{n,\alpha}$ contains a minimal edge e whose cardinality is at least 3. As e is minimal, e does not contain any of the edges in $H_{n,\alpha}$ of cardinality 1 or 2, and therefore, $e \subseteq \{\beta_1, \ldots, \beta_n\}$ or $e \subseteq \{\beta_1 + \sigma, \ldots, \beta_n + \sigma\}$. Without loss of generality, we may assume that $e = \{\beta_1, \ldots, \beta_k\}$ for some $k \geq 3$. Then, by Lemma 9.10, we have $\sum_{i=1}^k \beta_i = \sigma + \sum_{i=1}^k \alpha_i$. Substituting $\beta_i = \beta_1 + \alpha_1 + \alpha_i$ for $i = 2, \ldots, k$, we obtain $\sum_{i=1}^k (\beta_1 + \alpha_1) = \sigma$. Since σ is nonzero and $\sum_{i=1}^k (\beta_1 + \alpha_1)$ is either $\beta_1 + \alpha_1$ or 0, we get $\beta_1 + \alpha_1 = \sigma$. However, $\beta_1 + \alpha_1 = \sigma$ in turn implies that $\beta_i = \beta_1 + \alpha_1 + \alpha_i = \alpha_i + \sigma$, contradicting the assumption that $\beta_i \in GF(4) - \{\alpha_i, \alpha_i + \sigma\}$. Therefore, $H_{n,\alpha}$ does not have a minimal edge of cardinality at least 3, as required.

Thus the members of $\text{ind}(T, \alpha)$ have size either 1 or 2. Let $\mathcal{C}$ be what is obtained from $\text{ind}(T, \alpha)$ after deleting every element that appears in a member of cardinality 1. As no minimally non-ideal clutter has a member of cardinality 1, $\text{ind}(T, \alpha)$ is ideal if, and only if, $\mathcal{C}$ is ideal. Notice that $M(\mathcal{C})$, the incidence matrix of $\mathcal{C}$, is the edge - vertex incidence matrix of a bipartite graph. It follows from Kőnig's theorem for bipartite matching that $\mathcal{C}$ is ideal. Therefore, $\text{ind}(T, \alpha)$ is ideal, and $\text{mult}(T)$ is ideal, as required. ◇

Let $T = \langle w^1, \ldots, w^r \rangle$ where

$$\begin{array}{c} \\ w^1 \\ w^2 \\ \vdots \\ w^r \end{array} \begin{array}{c} \begin{array}{ccccc} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \end{array} \\ \left[\begin{array}{c|c|c|c|c} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array}\right] \end{array}.$$

Then $T = \left\{x \in GF(4)^{r+1} : x_1 + \cdots + x_{r+1} = 0\right\}$, so by Claim 3, $\text{mult}(T)$ is ideal. Let d_ℓ denote the number of entries in u^ℓ for $\ell = 0, 1, \ldots, r$, and let T' be defined as

$$T' := \left\{ (\underbrace{x_1, \ldots, x_1}_{d_0}, \underbrace{x_2, \ldots, x_2}_{d_1}, \ldots, \underbrace{x_{r+1}, \ldots, x_{r+1}}_{d_r}) : (x_1, x_2, \ldots, x_{r+1}) \in T \right\}.$$

Then T' is generated by $y^1, \ldots, y^r$ where

$$\begin{array}{c} \\ y^1 \\ y^2 \\ \vdots \\ y^r \end{array} \begin{bmatrix} \overbrace{\mathbf{1}}^{d_0} & \overbrace{\mathbf{1}}^{d_1} & \overbrace{\mathbf{0}}^{d_2} & \cdots & \overbrace{\mathbf{0}}^{d_r} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{bmatrix}.$$

Notice that $\mathrm{mult}(T')$ is a duplication of $\mathrm{mult}(T)$. As $\mathrm{mult}(T)$ is ideal by Claim 3, it follows from Remark 9.23 that $\mathrm{mult}(T')$ is ideal. Since $u^0, u^1 \ldots, u^r$ have nonzero entries and $S_i = \langle v^1, \ldots, v^r \rangle$ where $v^1, \ldots, v^r$ are of the form $(\star)$, S_i can be obtained from T' by taking coordinate-wise bijections. So, Remark 9.7 implies that $\mathrm{mult}(S_i) \cong \mathrm{mult}(T')$, thereby showing that $\mathrm{mult}(S_i)$ is ideal.

Since $S = S_1 \times \cdots \times S_k$ and $\mathrm{mult}(S_i)$ is ideal for $i \in [k]$, $\mathrm{mult}(S)$ is ideal by Proposition 8.4 (1), as required. □

Chapter 10

Conclusion

In this thesis, we have discussed polyhedral and combinatorial aspects of integer linear programming. In the first part, we studied the following questions about the Chvátal-Gomory cuts and the split cuts for integer linear programming:

(1) when is the Chvátal rank (or split rank) of a rational polyhedron equal to one?

(2) when the Chvátal rank (or split rank) of a rational polyhedron is one, can we optimize a linear function over the integer points in the polyhedron in polynomial time?

(3) when is the Chvátal rank of a polytope in the 0,1 hypercube small?

(4) when S is a proper subset of the integer lattice, is the S-Chvátal closure of a rational polyhedron also a rational polyhedron?

Theorems 1.3 and 1.4 imply that answering (1) is hard in general, but it is still interesting to study the question for some special cases. For example, finding a polynomial time algorithm for recognizing t-perfect graphs and finding a structural characterization of t-perfect graphs are important open questions not only in integer linear programming but also in combinatorial optimization. It follows from Propositions 1.7 and 1.8 that the problem in (2) is in complexity class NP $\cap$ co-NP, but we saw that finding a polynomial time algorithm might be difficult because it seems hard to exploit the condition on the Chvátal rank (or split rank). For (3), we proved Theorem 1.11 providing some sufficient conditions under which a polytope in the 0,1 hypercube has Chvátal rank at most 4, and the conditions are stated in terms of the infeasible 0,1 points. By Theorem 1.16, the answer to (4) when S is the set of integer points in Q where Q is a polyhedron defined by bound constraints on some variables is in the affirmative. Let us revisit the following conjecture for (4):

Conjecture 5.27. *Let $S = R \cap \mathbb{Z}^n$ for some rational polyhedron R, and let $P \subseteq \text{conv}(S)$ be a rational polyhedron. Then the S-Chvátal closure of P is a rational polyhedron.*

In the second part, we studied ideal clutters and clutters with the max-flow min-cut property. The $\tau = 2$ Conjecture, due to Cornuéjols, Guenin, and Margot, is the main topic of the second part.

> **The $\tau = 2$ Conjecture ([35]).** *If a clutter is ideal and minimally non-packing, then its covering number is two.*

In an attempt to prove the $\tau = 2$ Conjecture, we introduced multipartite clutters, and Theorem 1.34 shows that Conjecture 1.33 is equivalent to the $\tau = 2$ Conjecture.

> **Conjecture 1.33.** *If a multipartite clutter is ideal and strictly polar, then it packs.*

Theorem 1.29 provides a characterization of when a clutter is strictly polar, which in turn leads to a polynomial time algorithm for recognizing strictly polar clutters (Theorem 1.30). Theorem 1.39 provides a way of testing whether a multipartite clutter is ideal. Based on Theorems 1.29 and 1.39, we wrote a computer program to check multipartite clutters over at most 9 elements, and Theorem 1.43 confirms Conjecture 1.33 for the multipartite clutters over at most 9 elements. The next imminent task is to go over multipartite clutters over 10 to 12 elements.

> **Question 7.25.** *Does any of $V(H_{4,3,3})$, $V(H_{4,4,3})$, $V(H_{4,4,4})$, $V(H_{3,3,3,3})$ have a subset whose multipartite clutter is ideal and strictly polar but does not pack?*

We also considered two special classes of multipartite clutters. Theorem 1.45 implies that if the multipartite clutter of a set obtained by a reflective product is minimally non-packing, then its covering number must be two. Theorems 1.49 – 1.52 characterize when the multipartite clutter of a vector space is ideal and when the multipartite clutter of a vector space has the max-flow min-cut property, and Theorems 1.49 – 1.52 imply that Q_6 is the only ideal minimally non-packing clutter that is the multipartite clutter of a vector space. Therefore, the $\tau = 2$ Conjecture holds for these two classes of multipartite clutters. The question as to validate the $\tau = 2$ Conjecture or come up with a counter-example is certainly an exciting open question in the field of integer programming and combinatorial optimization.

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