Carnegie Mellon University
Tepper School of Business

DISSERTATION

Submitted in partial fulfillment of the requirements
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"Implementation of the Deferred Acceptance Algorithm in School Choice Application"

Presented by

Minyoung Rho

Accepted by

Robert Miller **4/12/2019**

Chair: Prof. Robert Miller **Date**

Approved by The Dean

Robert M. Dammon **4/15/2019**

Dean Robert M. Dammon **Date**

IMPLEMENTATION OF THE DEFERRED ACCEPTANCE ALGORITHM IN SCHOOL CHOICE APPLICATION

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I would not be here without the support from my family and friends.

Contents

[1] Any error in this draft is all mine.

Preface

Economists have extensively been studying and designing well-functioning algorithmic allocation (or matching) mechanisms where rank ordered lists are used in place of price (or willingness to pay). For example, public schools are "free" for students to attend, so many of the public school systems (Barcelona, Beijing, Boston, Denver, Ghana, New York City, and etc.) around the world adopted these mechanisms to allocate students to their public schools. In both theoretical and empirical literature, efforts to evaluate various implementations have been growing. This paper contributes to this literature and study the implementation of one of the most popular matching algorithm–the Deferred Acceptance algorithm.

The first chapter investigates the New York City high school matching market. As requested by the New York City Department of Education, Abdulkadiroglu et al. (2005) have chosen the Deferred Acceptance (DA, henceforth) algorithm to be the allocation mechanism for its promising theoretical properties. The allocation, when all the assumptions are satisfied, is *stable*, which means that there exist no student-school pair (or any coalition of students and schools) that can exchange their match outcome and end up with better allocation. Adding to this positive result, DA produces an allocation with optimal student welfare among all other stable allocations. Moreover, the mechanism induces *strategy-proofness*, where it is weakly dominant strategy for students to report their true preference when submitting their rank order list. Unfortunately, many of the assumptions required for such properties are not met in practice. Departures from the theoretic assumptions are: (1) schools do not strictly rank all of its applicants[2], (2) students do not rank all of their choices in the main round[3], and (3) DA is run twice[4]. Due to these violations, theoretical results mentioned above are no longer guaranteed, creating both instability and incentive to make strategic choices in the first round. I investigate the market data (including student choices and matching result) to

[2] **?** shows that the tie-breaking rule (to break weak ranking) currently used by schools in NYC creates inefficiency.

[3] Calsamiglia et al. (2010) provide evidence from the laboratory experiment that constraint on rank order list resulted in manipulation of their preferences.

[4] Dur et al. (2018)shows multi-stage DA in such setting (allowing students participation in the later rounds with changed rank order list) usually results in breaking of all positive properties. They also suggest a solution, a modified DA, and conditions required to achieve those properties in multi-stage setting.

build accurate picture. In particular, I find that there is a systematic relationship between student rank order choices (specifically, rank order list lengths) and matching outcomes (i.e. acceptance probabilities).

The second and the third chapter modifies the Deferred Acceptance algorithm to accommodate some of the market features introduced in the first chapter. In the second chapter, I develop a version of the Deferred Acceptance algorithm to allow schools to address multi-dimensional diversity concern (such as affirmative action based on race, test scores, and etc.) while respecting their preference rankings. I show that, in general, diversity considerations makes students non-substitutes, which makes the matching non-stable. If a school considers the diversity goals lexicographically, then I prove that such choice rule yields the best student-optimal matching and preserves strategy proofness.

In the third chapter, I develop another version of the Deferred Acceptance algorithm to combine two rounds. Inexplicably, most (including the New York City high school matching market) of such implementations feature an additional matching round (often called supplementary or scramble round) where students can re-participate in another matching round for leftover seats. This chapter first shows, using a simple example, that the current two-round implementation does not preserve any of the desirable characterizations of Deferred Acceptance algorithm. Then, the paper quantifies that 17-44% of the allocations have incentive to deviate (more precisely, justified envy) under the current two-round implementation in the NYC high school matching market. Finally, the paper designs a two (or multi)-round DA algorithm which preserves stability, strategy-proofness, and student optimality across two rounds.

In the fourth chapter, I recover student valuations for each school when it is assumed that parents game the system. To handle rank order list data with extremely large choice set, a computationally efficient, nonparametric estimation strategy is used for acceptance probabilities and valuations. The model produce closed form representation of student preferences where low acceptance probabilities increases valuations. Given that students with low socioeconomic status, on average, have lower acceptance probabilities to their top choice(s), those students have higher valuations for their top choice(s) compared the students with high socioeconomic status.

Literature Review

This dissertation contributes to both theoretical, behavioral and empirical literature on matching markets in the context of school choice. Seminal paper by Abdulkadiroglu and Sonmez (2003) have initiated taking mechanism design approach in a school choice setting, and Abdulkadiroglu et al. (2005) records the implementation of Deferred Acceptance algorithm to the New York City high school matching market. Implementation to the real-world school choice setting introduced many practical challenges. Many schools do not provide strict ranking of students; Erdil and Ergin (2008); Abdulkadiroglu et al. (2009); Kesten (2010); Abdulkadiroglu et al. (2011) have studied different methods to break ties. Many schools have to satisfy diversity rule (i.e., affirmative actions); Hafalir et al. (2013); Ehlers et al. (2014); Echenique and Yenmez (2015) studies such issues. The mechanism has centralized public schools only, and theoretical analysis of outside options (such as private schools) is limited; Feigenbaum et al. (2018) provides solutions to late attrition due to private or charter schools. The allocation algorithm is run twice; Pereyra (2013); Kennes et al. (2014) have studied a form of dynamic matching algorithms. Complementing such efforts, Chapters 2 and 3 of this dissertation provide solution to these challenges. Chapter 2 develops an algorithm and its desirable characterizations (similar to the format used in Echenique and Yenmez (2015)) when the diversity concern is multi-dimensional. Chapter 3 develops an algorithm and its desirable characterizations when the two (or more)-round Deferred Acceptance is run with changing preferences from student side. The dissertation also provides Julia code of these algorithms.

Despite the theoretical proof on the strategy-proofness of the Deferred Acceptance, there has been increasing number of papers questioning such behavior. Calsamiglia et al. (2010) has shown that constraining rank order list length can leads students to manipulate their choices in a laboratory setting. Fack et al. (ming) rejects that student choices are truth-telling using administrative data from Paris. Luflade (2018), using the sequential structure of the matching mechanism, provide evidence that students behave strategically when forming application lists using administrative data from Tunisia. Chapters 1 and 3 provide empirical and theoretical evidence for rejecting the assumption that students truthfully report. Chapter 1, comparing submissions from the two rounds, notices that rank order lists are not consistent across the rounds negating truthful reporting assumption, and suggests relationship between rank order list choices and matching outcome (i.e., acceptance probabilities). Chapter 3 provides cases and examples where violating truthful reporting is dominating strategy for students in case of two-round Deferred Acceptance.

Literature on estimating cardinal demand (or willingness to pay) from rank ordered list (also known as empirical matching literature) has been increasing. Many empirical methodologies have been used to overcome computational challenges posed by the rank ordered structure of discrete choice data. Since school choice markets are often large in number of participants, estimation procedure is often accompanied by large computational cost. In efforts to overcome computational burden, recent literature have come up with creative solutions: Calsamiglia et al. (2018) analogize the rank order list submission process as a dynamic discrete choice problem where each slot is considered as a time period; Abdulkadiroglu et al. (2017); Agarwal and Somaini (2018) uses Gibb's sampling when estimating parameter maximizing the likelihood function; Ajayi and Sidibe (2017) modeled the choice procedure as the stochastic portfolio choice problem of Chade and Smith (2006). Complementing this literature, I estimate the model using conditional choice probabilities (CCP) estimation, an arguments developed by Hotz and Miller (1993).

1 The New York City High School Matching Market

This chapter studies implementations of Deferred Acceptance in New York City high school matching market. The main finding is that there are many evidence against the claim that the submitted rank order lists are complete and truthful ordering of their preferences, contrary to the assumptions on the current empirical literature on this market. The chapter starts with describing participating students and schools, and records how Deferred Acceptance algorithm is implemented (Section 1.1). Then this chapter organizes theoretical (non-) characterizations of the modified Deferred Acceptance algorithm which are applicable for the described market (Section 1.2). Lastly, this chapter provide novel empirical evidence that there exist a relationship between length of student rank ordered lists are matching outcomes or acceptance probabilities (Section 1.3). These empirical and theoretical findings also motivates the rest of the chapters (Chapters 2 and 3 suggests improvements on the current implementation of the algorithm and the framework to analyze student choice in Chapter 4 is motivated by the findings from this chapter).

1.1 Why New York City High School Matching Market?

There are no shortage of centralized matching markets to study. The New York City high school matching market stands out due to its unique advantages, and among many are: size, diverse student population schools. New York City high school matching market is the largest market that has been studied in the matching market literature. Within the city, housing price (a proxy for socioeconomic status) ranges from lower than lower than 100,000 USD to a few millions; all races reside within the city with no one race consisting more than 50% of the population; and standardized test scores also varies across all possible points. High schools, mirroring the diverse student population, shows great differences in quality and selection rules. Having such large sample with various backgrounds makes this market a very attractive for both empirical and policy analysis.

1.1.1 Size

Every year there are more than 80,000 8th graders are waiting to be matched to one of more than 700 high school programs in New York City. In some cities (Boston and New York City), each school is divided further into program(s), and students apply to programs within the schools. For the purpose of understanding this paper, readers can think of schools and programs as same concept.

Compared to the other frequently studied markets such as Cambridge, Boston who has about 400 students every year, sheer number of participants itself makes this market enticing to study. Table 1 lists a few markets that has been empirically studied in the centralized school choice literature. Charlotte-Mecklenburg market seems large but it is a combination of 4 grades, which shows that for each grade, there are less than 10,000 students participating. For a single year matching market, the New York City high school is by far the largest system. Fortunately, the data includes all participants to the public high school matching market eliminating any concerns about sampling bias.

Table 1: School Choice Markets and Size

Market	No. Students	No. Schools/Programs
Cambridge, Boston (Elementary)	470	25
Beijing (Middle)	914	4
Barcelona	11,817	159
Charlotte-Mecklenburg (4th-8th)	36,887	Unknown
New York City	80,000+	700+

1.1.2 Students

8th grade student population in New York City are diverse in location of their residency, socioeconomic class, ethnicity and their academic achievements. For each student, zipcode[5],

[5] I match median housing price be found in the US Census Bureau to each students zipcode as a proxy for socioeconomic class. Zip code indicates a location measure, which is later used in relation to the chosen schools.

ethnicity, standardized test scores[6], and number of absent days are observed[7]. Descriptive statistics shown in Tables 3 and 2 provide distributions for the demographic information. New York City is largely Hispanic and Black population combined make 70% of the population (Table 2). Median housing price ranges from less than 33,400 USD to a few million USD (Table 3)[8]. With these extremes, the distribution looks normal with long tail on the right (few places with very high housing price) (Figure 1.1a). Average absent days are high, but largely driven by students with chronic absence (Figure 1.1b).

Table 2: Ethnic Composition of Student Population

School Year	Asian	Black	Hispanic	Other	White
2011-12	0.14	0.28	0.36	0.01	0.12
2012-13	0.14	0.27	0.37	0.01	0.12
2013-14	0.15	0.26	0.36	0.01	0.13
2014-15	0.14	0.26	0.37	0.01	0.13
2015-16	0.15	0.25	0.37	0.01	0.13

Table 3: Student Popoulation Information Descriptive Statistics

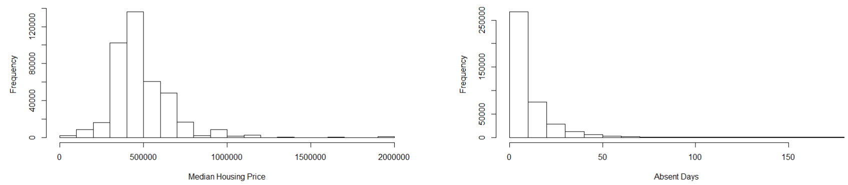
Statistic	N	Mean	St. Err.	Min	Max
HousingPrice	405,612	484,371	177,311	33,400	2,000,000
readingScore_percentile	352,203	0.512	0.290	0.0001	1
mathScore_percentile	356,505	0.509	0.288	0.012	1
daysAbsent	396,246	10	14	0	180

I look closer geographically and find that these variables are highly correlated. Each neighborhoods in the city has distinctive characterization of a representative students. The best way to illustrate this point is through maps. Before I introduce pockets of neighborhoods (and their distinct characteristics), I first divide the city into boroughs. Neighborhoods divided by their distinct characteristics doesn't perfectly coincide with the boroughs, but boroughs

[6]Standardized test scores data is transformed into percentiles every year, since difficulty of the test changes the distribution of the scores in some years. What matters in the matching market is relative scores to the student population each year, so the percentile measure is used to create comparable measure across the years.

[7]To give a rough idea on days absent measure, there are 180 school days in a year. 18 days of absence (either excused or unexcused) is considered chronic absence and can be in danger of not advancing to the next level.

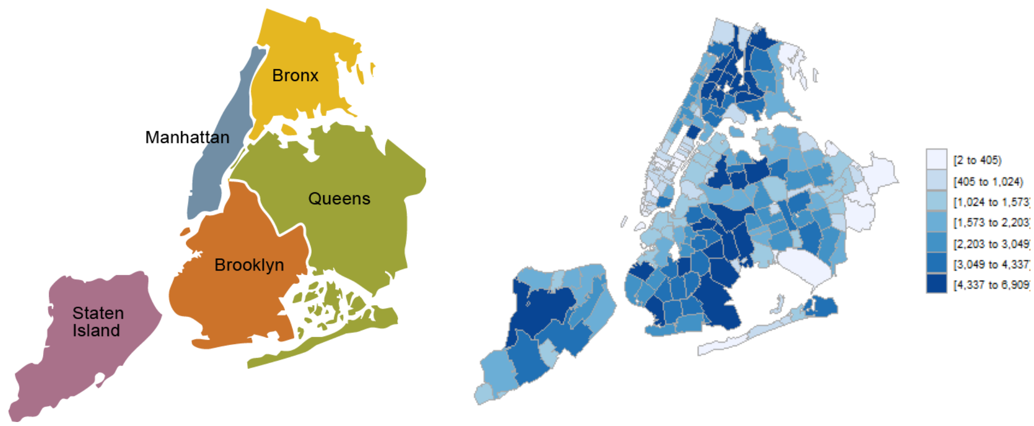
[8]I replace all values higher than 2 million to 2 million



(a) Distribution of Median Housing Price of Student Residency Zipcode

(b) Distribution of Absent Days

Figure 1.1: Student Information Distribution



(a) New York City by Borough

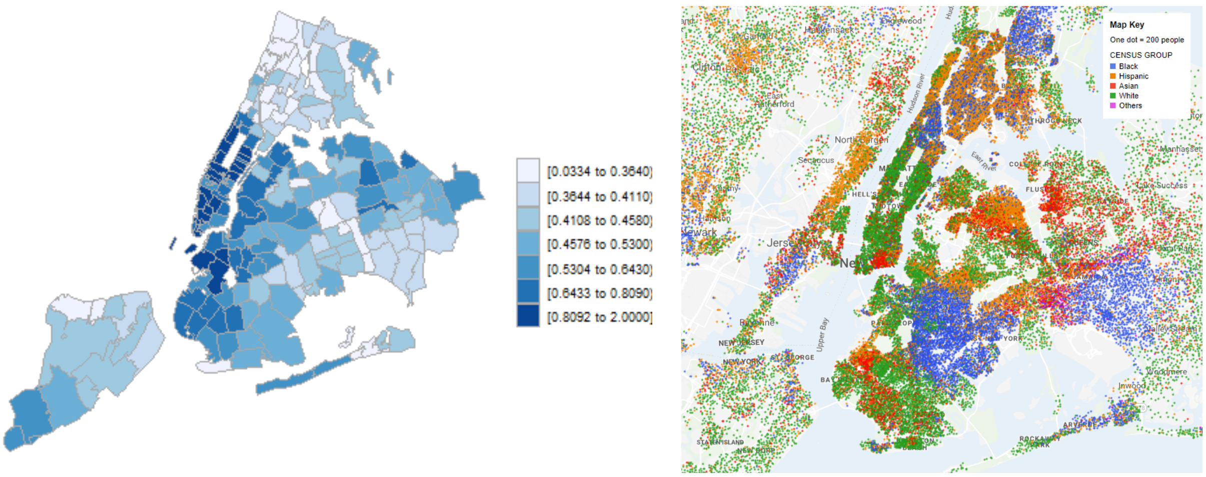
(b) Student Population Density Map

Figure 1.2: New York City Map and Student Population Density

are a good high level geographic segmentation for labeling purposes. There are 5 boroughs: Stanton Island, Brooklyn, Queens, Manhattan, and the Bronx (Figure 1.2a). Figure 1.2b shows density of the student residencies by zip code and show that majority of the student population reside in Bronx and Brooklyn bordering Queens.

Representing student demographic information using maps reveals that New York City is divided into neighborhoods with distinct and highly correlated characteristics. Figures 1.3 and 1.4 represent the student information geographically. Figure 1.3a looks at median housing price by zip code and Figure 1.3b racial composition for the corresponding region[9]. In Figure 1.4, each of the map corresponds to performance in standardized test score (the map shows

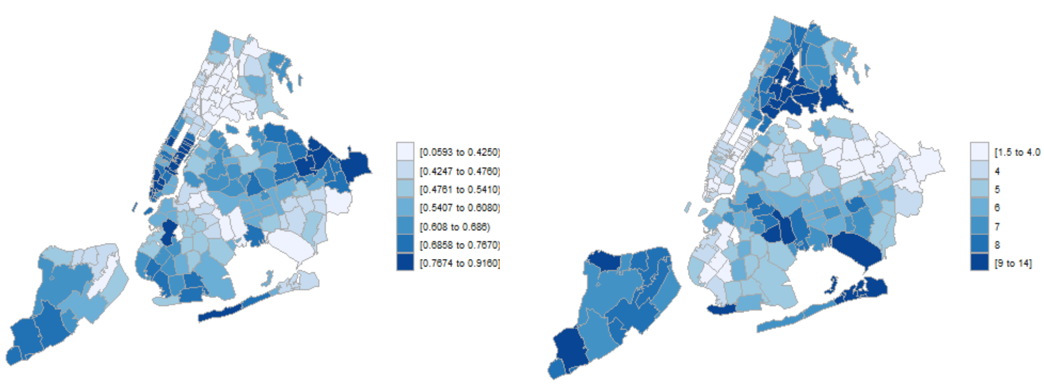
[9]Student data is only available upto zipcode level, so racial composition chart was taken from here: https://www.nytimes.com/interactive/2015/07/08/us/census-race-map.html. Each dot represents 6,000 p eople from Census data.



(a) Housing Price (in millions) by Zipcode

(b) Racial Composition of New York City Residents (not only Student Population) Neighborhoods

Figure 1.3: Neighborhood Characteristics: Racial Composition and Housing Price



(a) Standardized Test Score (English)

(b) Days Abent

Figure 1.4: Median Academic Performance of Student Population by Zipcode

English score but math score has similar shading) and absent days. Majority (70%) of the student population reside in neighborhood with the following characteristics: high density of Black and Hispanic residents, lower housing price, and lower academic performance. These neighborhoods (Orange and Blue regions in Figure 1.3), located in Bronx and (farther away from Manhattan) parts of Brooklyn and Queens, are characterized with lower housing price and lower academic performance. On the other hand, areas with lower student density, located in Manhattan coastal parts of Brooklyn nearing Manhattan and far east of Queens, are largely populated with White and Asian residents, has higher housing price, and is characterized with higher academic performance. One can profile a student based on each neighborhood. For example, a student residing in Manhattan is likely to be White, perform above average on standardized tests, is not absent from classes, and reside in expensive house. On the other hand, a student residing in Bronx, is likely to be either Black or Hispanic, have poor academic performance, and reside in relatively inexpensive house. Interesting neighborhood is the far east of Queens (Flushing and neighboring parts). A student from here is likely to be Asian, perform extremely well in standardized test, almost never misses classes, and reside in house with slightly above average price. Next section introduce schools to see if school qualities mirror the student characteristics of the neighborhood.

1.1.3 Schools (and Programs)

School qualities also vary in terms of the academic achievement, selection criteria and etc. within commute providing favorable arguments for students having choices in high school selection. Table 4 describes variables used, and Table 5 presents summary statistics. I tried to use most variables from academic year 2013-2014, which is right in the middle of all available years of data, except for data on crimes and SAT scores. For crime data, years 2014-2015 had the least number of missing data. For SAT data, information from 2013-2014 was not available. Data on school quality (such as graduation rate and median SAT score) are available for more schools than data on crime rate.

Continuing with gaining insight geographically, I convert information from Table 5 into

Table 4: School Information Variables

Variable	Description
On Track Year1	Denotes the percentage of 9th grade students who were on track to graduate in four years by earning ten credits or more in core subjects at end of 2013 school year.
Graduation Rate	Denotes the percent of students who graduated ”on time” by earning a diploma four years after they entered 9th grade at end of 2013 school year.
College Career Rate	At the end of the 2012-13 school year, the percent of students who graduated from high school four years after they entered 9th grade and then enrolled in college, a vocational program, or a public service program within six months of graduation.
SAT Scores	Mean SAT score of graduating class of 2012.
(Type of) Crimes	Since 1998, the NYPD has been taked with the collection and maintenance of crime data for incidents occur in a building where NYC public schools are located during the school year of 2014-2015.
Total Student	Total number of students enrolled in the school as of the audited register in October 2013

Table 5: High School Summary Statistics

Statistic	N	Mean	St. Dev.	Min	Max
On Track Year 1	400	0.81	0.12	0.31	1.00
Graduation Rate	355	0.73	0.17	0.20	1.00
College Career Rate	339	0.55	0.20	0.06	1.00
Reading Score (SAT)	293	412.00	60.70	291	674
Math Score (SAT)	293	423.00	68.80	316	735
Writing Score (SAT)	293	406.00	62.20	285	678
Major Crimes	212	2.94	1.80	2	12
Other Crimes	212	9.79	9.51	2	27
Non Criminal Crimes	212	13.10	15.70	2	56
Property Crimes	212	7.23	6.68	2	22
Violent Crimes	212	4.42	3.92	2	17
Total Students	426	166.00	87.30	1	325

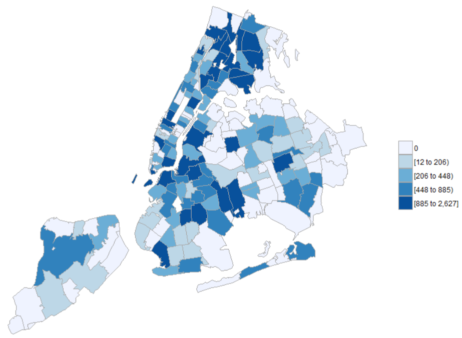


Figure 1.5: Total Number of Students Enrolled in Schools by Zipcode

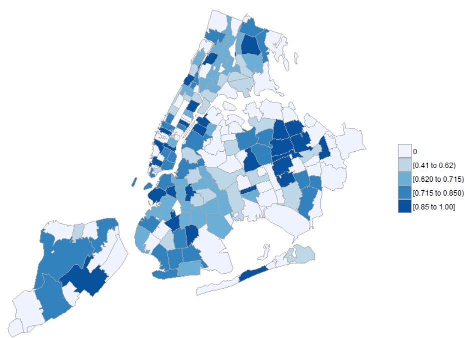


Figure 1.6: Median Graduation Rate of Schools by Zipcode

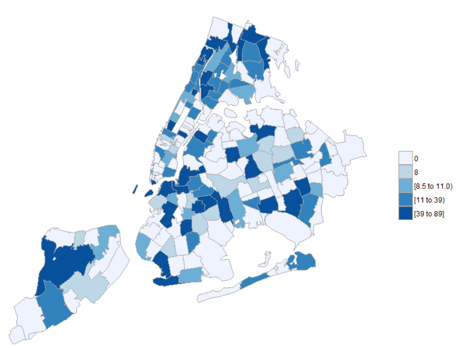


Figure 1.7: Total Number of Crimes in School Buildings

maps. First, I look at the the number of students enrolled per zip code in Figure 1.5. Next, I look at the quality of schools by areas in Figure 1.6.[10] Highest performing schools (Figure 1.4) are scattered throughout, although low in student enrollment count (Figure 1.5). Crime rates are related to the enrollment size rather than any other student demographic information (Figure 1.7).

Enrollment data supports the argument for school choice that students choose to move away from their neighborhood to attend high school. Comparing Figure 1.2b to Figure 1.5 shows that population dense areas aren't always the most enrolled area. Manhattan and coastal regions of Brooklyn and Queens near Manhattan has lower density in student population but enrolls rather larger proportion of students. Possibly because schools in these areas have higher academic qualities than other neighboring schools (Figure 1.6).

Programs. School choice often creates competition for the seats. To look at the competition, we have to look at application information, which is at the program level. So far we have focused on school level information, largely because the data available from "Open Data Law" is at the school level. However, in this market, students apply to a program and get matched to a program within a school. Any application information will be at a program level. For the purpose of this paper, readers can interchange the term program and the term school without any loss of meaning. Table 6 shows that there exist competition for the high school seats and the competition is quite steep (on average, programs have a little bit over 30% acceptance rate).

Table 6: Program Summary Statistics (Year 2013-2014)

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
applicant	679	300.4	485.1	10	62	325.5	4,046
admit	679	102.0	102.2	10	40	124	1,139

Given that there are some competitions, it is natural to ask how schools are admitting

[10] All other quality measure of schools are highly correlated with the graduation rate, so I only included one map. There are many zip codes without any information on public high schools, likely because there is no school in that zip code. In all maps, 0 represents zip codes without any information on public schools.

some students and not the others, and the admission method seem to be a good indicator for how competitive a seat at a particular seat is. Unlike other public high school markets where there is one selection rule for all schools [11], in New York City, each program uses different selection rules; there are 6 different admission methods across all programs. Descriptions of these different types of selection method schools are provided below:

1. Audition programs rank students based on student's performance in an audition.

2. Screened programs tank students based on their criteria including final report card, standardized test scores, and attendance.

3. Educational Option (Ed. Opt.) programs admit student based on distribution of student demographic (16% from the high reading level, 68% from middle reading level , and the rest 16% from the lower reading level). To further prevent bias selection, only half of the students matched to Ed. Opt. programs will be selected based on their rankings from the school, and the other half will be selected through randomized computer program. One thing to note is that if a student scored top 2% on ELA reading exam, and his/her first choice school is Ed. Opt. program, that student is guaranteed a position in the program.

4. Limited Unscreened programs give priority to students who showed interest in the school by attending information sessions or open house events.

5. Unscreened programs select randomly.

6. Zoned Priority schools prefer students who live in the zoned area of the high school, and Zoned Guarantee programs guarantee admissions to students live in the zoned area of the high school.

The above description already hints that programs which select student based on audition result or screen students based on academic records will likely to have more competition that those with zoned and unscreened selection rule. To see whether this hypothesis is true, I

[11] For example, Cambridge, Boston chooses students with siblings and residing in close proximity first.

Table 7: Selection Method Summary Statistics

method	SeatProp	AdmitProp
Audition	0.05	0.37
EdOpt	0.23	0.87
LimitedUnscreened	0.32	0.88
Screened	0.28	0.36
Screened For Language	0.02	0.51
Unscreened	0.01	0.97
Zoned	0.09	0.90

further investigate on average admission rates of programs by selection rules. Table 7 records (1) number of seat allocated for programs with each of the selection rules proportional to the total student population ("SeatProp") and (2) median admit proportion of each program calculated by dividing number of admitted students by number of applicants ("AdmitProp"). This proves that Audition and Screened programs tend to be more competitive and zoned and unscreened schools almost do not have any competition.

Tables 6 and 7 have shown that there exist a significant level of competition for large proportion of the seats. To allocate the students to schools in this situation, New York City implemented a version of Deferred Acceptance matching algorithm. Next section describes the allocation process through a centralized matching algorithm.

1.2 Allocation Procedure and Mechanism

This section records that many of the desired properties found in static, unconstrained (rank order list length) Deferred Acceptance algorithm do not hold in the current implementation (two-stage, constrained DA). I introduce the current implementation of Deferred Acceptance, list theoretical properties (and assumptions required) of Deferred Acceptance, identify required assumptions which are violated in this market, and revisit theoretical properties when these assumptions are relaxed as it is in the current implementation.

1.2.1 Allocation Procedure

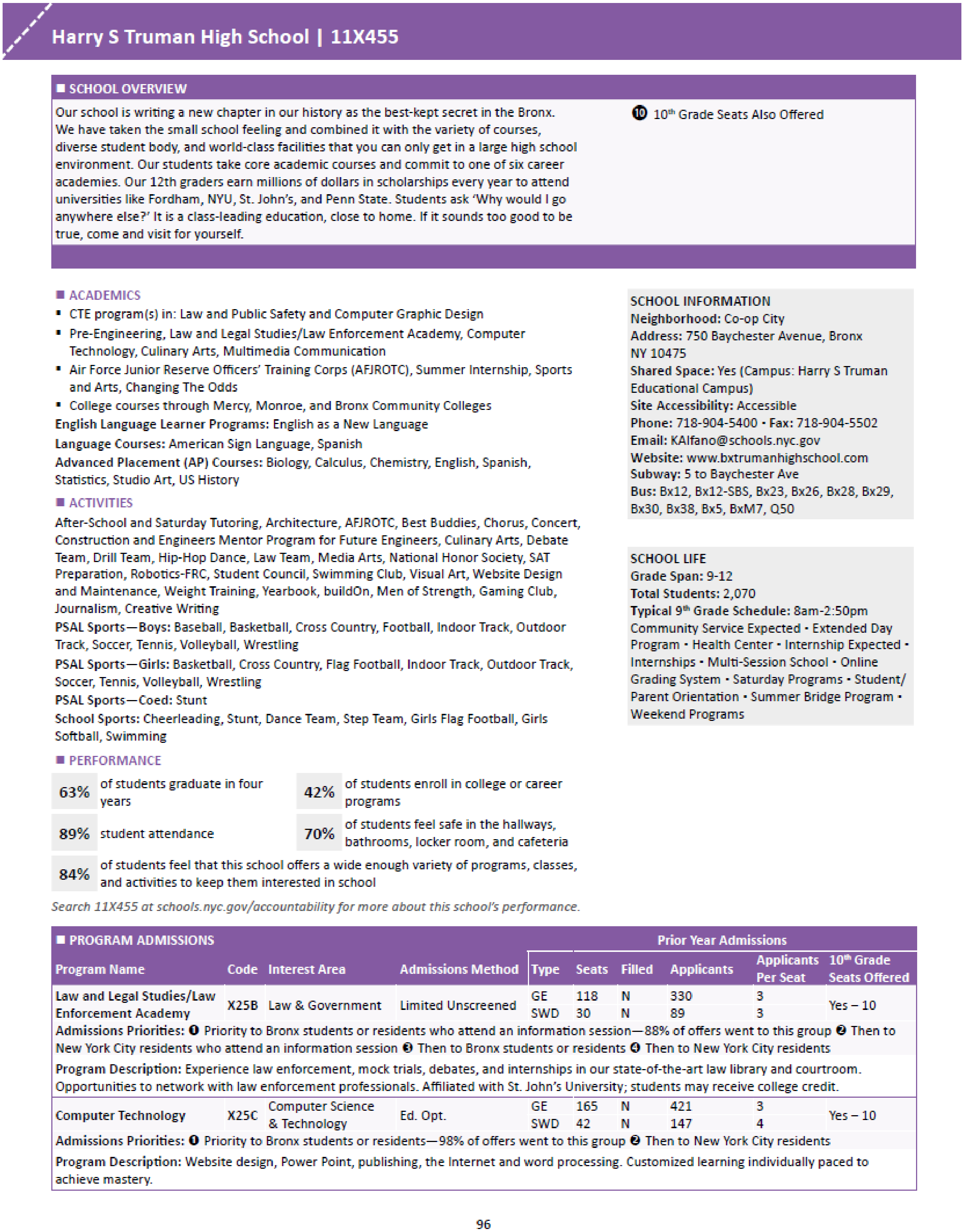
There has been different information on how the New York City implemented the Deferred Acceptance. This section gives the most recent information based on conversations with the New York City Department of Education. First thing to note is that the allocation process has 4 stages. The first and the last round is decentralized, and the middle two rounds run two separate Deferred Acceptance algorithms to allocate students to high schools. The following is timeline for the entire process, which takes almost a year:

1. *The Specialized Round*: A small subset of 8th graders who are applying to Specialized High Schools (including Stuyvesant, Bronx Science, and LaGuardia) collects information about the Specialized High School, take Specialized High School Admissions Test (SHSAT), and/or audition. Note that all students who are applying to this round also goes the rest of the process. This round is an extra step for all for those wish to apply. For the entire set of 8th graders, the centralized rounds proceeds in the following sequence:

2. *The Main Round:*

   (a) *Information Reveal for the Main Round*: At the beginning of the 8th grade school year, all students (including the students who will be applying to *the Specialized Round*) receive information about available high school programs. At this stage, NYCDOE releases 600+ page directory of all available public high schools in the NYC. There also exist high school fairs and information sessions. The booklet has detailed information about (1) how each of the program chooses their students and (2) applicant to admitted student ratio. Figure 1.8 is a page from the booklet which contains information about all available schools. The booklet is over 600 pages.

   (b) *Action for the Main Round*: By December, students submit their rank order list; students can rank upto 12 choices. High school programs receive a list of the Main round applicants and can create rank order among the Main round applicants.



Harry S Truman High School | 11X455

■ SCHOOL OVERVIEW

Our school is writing a new chapter in our history as the best-kept secret in the Bronx. We have taken the small school feeling and combined it with the variety of courses, diverse student body, and world-class facilities that you can only get in a large high school environment. Our students take core academic courses and commit to one of six career academies. Our 12th graders earn millions of dollars in scholarships every year to attend universities like Fordham, NYU, St. John's, and Penn State. Students ask 'Why would I go anywhere else?' It is a class-leading education, close to home. If it sounds too good to be true, come and visit for yourself.

❿ 10th Grade Seats Also Offered

■ ACADEMICS

- CTE program(s) in: Law and Public Safety and Computer Graphic Design
- Pre-Engineering, Law and Legal Studies/Law Enforcement Academy, Computer Technology, Culinary Arts, Multimedia Communication
- Air Force Junior Reserve Officers' Training Corps (AFJROTC), Summer Internship, Sports and Arts, Changing The Odds
- College courses through Mercy, Monroe, and Bronx Community Colleges

English Language Learner Programs: English as a New Language

Language Courses: American Sign Language, Spanish

Advanced Placement (AP) Courses: Biology, Calculus, Chemistry, English, Spanish, Statistics, Studio Art, US History

■ ACTIVITIES

After-School and Saturday Tutoring, Architecture, AFJROTC, Best Buddies, Chorus, Concert, Construction and Engineers Mentor Program for Future Engineers, Culinary Arts, Debate Team, Drill Team, Hip-Hop Dance, Law Team, Media Arts, National Honor Society, SAT Preparation, Robotics-FRC, Student Council, Swimming Club, Visual Art, Website Design and Maintenance, Weight Training, Yearbook, buildOn, Men of Strength, Gaming Club, Journalism, Creative Writing

PSAL Sports—Boys: Baseball, Basketball, Cross Country, Football, Indoor Track, Outdoor Track, Soccer, Tennis, Volleyball, Wrestling

PSAL Sports—Girls: Basketball, Cross Country, Flag Football, Indoor Track, Outdoor Track, Soccer, Tennis, Volleyball, Wrestling

PSAL Sports—Coed: Stunt

School Sports: Cheerleading, Stunt, Dance Team, Step Team, Girls Flag Football, Girls Softball, Swimming

■ PERFORMANCE

63%	of students graduate in four years	42%	of students enroll in college or career programs
89%	student attendance	70%	of students feel safe in the hallways, bathrooms, locker room, and cafeteria
84%	of students feel that this school offers a wide enough variety of programs, classes, and activities to keep them interested in school		

Search 11X455 at schools.nyc.gov/accountability for more about this school's performance.

SCHOOL INFORMATION

Neighborhood: Co-op City
Address: 750 Baychester Avenue, Bronx NY 10475
Shared Space: Yes (Campus: Harry S Truman Educational Campus)
Site Accessibility: Accessible
Phone: 718-904-5400 • **Fax:** 718-904-5502
Email: KAlfano@schools.nyc.gov
Website: www.bxtrumanhighschool.com
Subway: 5 to Baychester Ave
Bus: Bx12, Bx12-SBS, Bx23, Bx26, Bx28, Bx29, Bx30, Bx38, Bx5, BxM7, Q50

SCHOOL LIFE

Grade Span: 9-12
Total Students: 2,070
Typical 9th Grade Schedule: 8am-2:50pm
Community Service Expected • Extended Day Program • Health Center • Internship Expected • Internships • Multi-Session School • Online Grading System • Saturday Programs • Student/Parent Orientation • Summer Bridge Program • Weekend Programs

■ PROGRAM ADMISSIONS

Program Name	Code	Interest Area	Admissions Method	Prior Year Admissions: Type	Seats	Filled	Applicants	Applicants Per Seat	10th Grade Seats Offered
Law and Legal Studies/Law Enforcement Academy	X25B	Law & Government	Limited Unscreened	GE SWD	118 30	N N	330 89	3 3	Yes – 10
Computer Technology	X25C	Computer Science & Technology	Ed. Opt.	GE SWD	165 42	N N	421 147	3 4	Yes – 10

Law and Legal Studies/Law Enforcement Academy (X25B):

Admissions Priorities: ❶ Priority to Bronx students or residents who attend an information session—88% of offers went to this group ❷ Then to New York City residents who attend an information session ❸ Then to Bronx students or residents ❹ Then to New York City residents

Program Description: Experience law enforcement, mock trials, debates, and internships in our state-of-the-art law library and courtroom. Opportunities to network with law enforcement professionals. Affiliated with St. John's University; students may receive college credit.

Computer Technology (X25C):

Admissions Priorities: ❶ Priority to Bronx students or residents—98% of offers went to this group ❷ Then to New York City residents

Program Description: Website design, Power Point, publishing, the Internet and word processing. Customized learning individually paced to achieve mastery.

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Figure 1.8: A Page from 2017 NYC High School Directory

(c) *Result for the Main Round*: By March, match results from the main (and the Specialized High School) are revealed to students. For the subset of students who received both the Main and the Specialized High School round admission, they have to choose one.

3. *The Supplementary Round:*

   (a) *Information Reveal for the Supplementary Round*: Subset of high school program participates in the Supplementary round posts their seat availability for this round. There also exist high school fairs and information sessions for this subset of participating high school programs.

   (b) *Action for the Supplementary Round*: Students decide whether to participate in this round, and submits their rank order list for this round; students can rank up to 12 choices. High school programs receive a list of the Supplementary round applicants and can create rank order among the Supplementary round applicants.

   (c) *Result for the Supplementary Round*: Match results from the supplementary round are revealed to students. Note: If one gets matched to the Supplementary round, the Supplementary round results are the final match. If one did not get matched to the Supplementary round, one can keep their Main round match. If one does not get matched from either the Main or the Supplementary round, one is assigned to a program with vacancy.

4. *The Appeals Round*: There is no formal structure (like the last two rounds) to this round. This round exist for students who is qualified to appeal due to health or other reasons that the matched high school is not suited for his/her situation. The application process is decentralized.

Each year, there are more than 80,000 students, and Table 8 shows fraction of 8-th graders who are matched in each round:

Table 8: Proportion of Population and Their Match

	No Particip	Main (& Spec.)	Supplementary	Appeals	Total N.
2011-12	0.07	0.75	0.16	0.02	81,879
2012-13	0.07	0.75	0.16	0.02	80,833
2013-14	0.05	0.78	0.16	0.01	82,282
2014-15	0.04	0.80	0.14	0.02	80,609
2015-16	0.04	0.80	0.14	0.02	80,609

1.2.2 Allocation Algorithm

For this market, a student proposing Deferred Acceptance algorithm was chosen. Abdulkadiroglu et al. (2005) describes a rationale for the design choices that the resulting allocation properties for student proposing DA are stability, best welfare properties for students, and student strategy-proofness (it is a dominant strategy for students to state true preferences)[12]. In this subsection, I first describe how the student proposing DA works and formally define the aforementioned allocation properties associated with the algorithm.

Student Proposing Deferred Acceptance Algorithm

Step 1. Each student applies to her more preferred school. Each school tentatively admits students according to their preference and total capacity and permanently rejects the rest.

$$\vdots$$

Step k. Similar to the first step, each student who was rejected at step $k-1$ applies to her next preferred school. Each school considers union of tentatively admitted students from step $k-1$ and applicants from step k. From this set, each school again tentatively admits a set of students according to their preference and total capacity, and permanently rejects the rest.

Step End. the algorithm stops when there are no rejections.

[12] The paper mentions that the properties hold in sufficiently simple environments, and that of course implementation faced some complications and introduced them.

At the end of the algorithm, if a student is unmatched, then one of the following happened:

1. A student never submitted rank order list.

2. A student was rejected from all schools in their list.

Allocation Properties (and required assumptions) of Student Proposing Deferred Acceptance Algorithm I formally define and state the properties of student-proposing DA algorithms studied from previous literature. The definition and theorems introduced in this paper are from the Roth and Sotomayor (1992). I didn't include the proof in this section since only the properties are of interest for this paper, but could easily be found in the book (Roth and Sotomayor (1992)).

Definition 1. A matching (or allocation) is pairwise stable if it is not blocked by any individual agent or any student-school (program) pair. A matching is group stable if it is not blocked by any coalition.

Theorem 2. *A matching is group stable if and only if it is pairwise stable.*

Definition 3. A matching is student-optimal if it is stable and there does not exist a stable matching that has better allocation for some students and no worse allocation for all students.

Theorem 4. *For any submitted lists of strict preferences, the matching of the student-proposing deferred acceptance algorithm yields the student-optimal stable matching.*

Definition 5. If a matching is weakly Pareto optimal, then there is no individually rational matching such that there exist a strictly better allocation for all students.

Theorem 6. *When the preferences are strict, the student-optimal stable matching is weakly Pareto optimal for the students.*

Definition 7. A mechanism is strategy-proof if truth-telling is a weakly dominant strategy equilibrium of the induced preference revelation game.

Theorem 8. *A stable matching procedure that yields the student-optimal stable matching makes it a weakly dominant strategy for all students to state their true preferences.*

In short, under the assumption that every agent is submitting a full list of strict preferences, a student-proposing static Deferred Acceptance algorithm is strategy-proof and results in student-optimal (or weak Pareto optimal) among stable allocation.

1.2.3 Relaxed Assumptions In Practice and Its Impact on Properties

This section revisits theoretical properties when some of the assumptions are relaxed. When theory is implemented in practice, often the assumptions required are not met. In the New York City high school matching, due to its large size, students and schools do not submit strict and complete preferences; students are restricted to list up to 12 schools (but most of the students submit even less choices); schools are not required to strictly rank its applicants (in fact about half of the schools either group rank students or do not rank). Because the ranking is not complete, there exists unmatched students at the end of the allocation; for those who are unmatched (and also open to those who are not satisfied with their choice), another round of allocation process is conducted.

Restriction on Student Rank Ordered List Length. Calsamiglia et al. (2010) has studied student behaviors when students are restricted to rank only few schools in an lab setting, and they find that restricting the rank order list has negative effect in truth-telling property. Suppose that students are allowed to rank up to M schools. Haeringer and Klijn (2009) studies student behavior once the schools to be listed are selected. Student cannot do better than submitting the true ordering among the selected M schools. Luflade (2018) claims that if there is a school that has garanteed admission among the top M schools, then students can do no better than reporting their top M schools.

Proposition 9. *(Calsamiglia et al. (2010)) Using the experiments designed, constraining rank order list reduces proportion of students preserving the original ranking, incurs less truncated truth telling strategy (submitting a choice list whose first M choices coincide with the true preference), and finally increases proportion of individuals exhibiting biases (Small School Bias: lowering the position of a more competitive school in the submitted list; District School Bias: raising the ranking of the district school in the submitted list).*

(Haeringer and Klijn (2009)) If a student finds at most M schools acceptable, then she can do no better than submitting her true preference. If a student finds more than M schools acceptable, then she can do no better than employing a strategy that selects M schools among the acceptable schools and ranking them according to her true preferences.

(Luflade (2018)) If a student has a perceived eligibility (or acceptance) probability 1 for (at least) one of her M most preferred programs, then students do not misreport their preferences over their choice set.

Such results shed some light into the submitted rank order lists, but do not paint a complete picture on how students choose to rank. Propositions suggest that once M schools to be ranked are selected, students will do no better than ranking them according to their preferences. However, it is not known how students are choosing to select the few schools to rank out of large choice set.

Schools' Non-Strict Rank Orderings. Some schools receive thousands of application and is only able to rank subset of the application, and others receive applications fewer than the capacity so do not rank students. There is no school that rank all of its applicants. For those who are not ranked, current practice breaks these unrecorded rank ties by randomly assigning a number to a student. If random tie-breaking ordering is not in accordance of students valuations (which is likely), there exist inefficiency.

Proposition 10. *Deferred Acceptance algorithm will result in student welfare maximizing (**among other stable allocations**) allocation if ties are broken using students' cardinal preference, or willingness to pay.*

Proof. Suppose schools without preference over individual applicants are ordered based on students' cardinal preference. The resulting allocation from DA is denoted μ. Further suppose there exists an ordering (tie-breaking), which generated stable matching with higher student welfare. I denote this matching μ'. Under μ', there must exist at least one school s which admitted at least one student with higher willingness to pay called i. This cannot be the case since i would have been admitted under μ, which is a contradiction. □

From the above proposition, it is shown that unless ties are broken according to applicants' cardinal valuations, the allocation can be improved. Because the matching market can only ask ordinal rankings of the choices, eliciting cardinal valuations from the ranking data requires an extra econometric step.

Multi-round. As described in the timeline, there are two rounds of DA. Even though a seat at the current school is guaranteed, not every student participates in the second round (Table 8). Decision to not participate could be the case that a more preferred school is not available in the second round or there could be another reason for not participating in the second round. If for some reason (possibly due to the cost of going through the allocation process) a part of the student body decides not to participate, then there is an incentive for some to not report his/her true preference.

Example 11. (Dur et al. (2018)) Let $C = \{c_1, c_2, c_3\}$ be schools and $S = \{s_1, s_2, s_3\}$ be students and seats available for each school is $q = (1, 1, 1)$. I further assume that the preferences doesn't change, and a student 1 (s_1) decides not to participate in the second round. The round invariant preferences are given as follows:

$$c_2 \succ_{s_1} s_1 \succ_{s_1} c_1 \qquad s_3 \succ_{c_1} s_2 \succ_{c_1} s_1$$

$$c_1 \succ_{s_2} c_2 \succ_{s_2} c_3 \qquad s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$$

$$c_2 \succ_{s_3} c_3 \succ_{s_3} c_1 \qquad s_3 \succ_{c_3} s_1 \succ_{c_3} s_2$$

If s_3 reports her true preference in the first round, then the allocation is $\mu^1 = \langle (s_1, c_2), (s_2, c_1), (s_3, c_3) \rangle$. There is no incentive for s_1 and s_2 to participate in the second round, so the matching stays. Now suppose s_3 submits non-truthful report in the first round: $c_2 \succ^1_{s_3} c_1 \succ^1_{s_3} c_3$. Then, $\mu^1 = \langle (s_1, s_1), (s_2, c_2), (s_3, c_1) \rangle$. $q^2_{c_3} = 1$, since no one was matched to c_3 and s_3 prefers c_3 to c_1, $s_3$applies. Since the second round is the last round, s_3 submits true preference. Suppose only s_2 and s_3 participates, since this is the last round, both participate with their true preference. $\mu^2 = \langle (s_2, c_1), (s_3, c_2) \rangle$, and $\mu^F = \langle (s_1, s_1), (s_2, c_1), (s_3, c_2) \rangle$.

Figure 1.9: Properties of Deferred Acceptance Allocation Revisited

	Stable	Strategy-Proof	Optimal (among stable all.)
All Assumptions Satisfied	O	O	O
Student: Restricting Length (Propositions 9)	X	X	X
School: Non-Strict Rank (Proposition 10)	O	O	X
Multi-round (Example 11)	X	X	X

Thus, the allocation is not only strategy-proof, and no longer stable or optimal (student s_1 and school c_2 can form a block). The scenario of a student calculating to strategize for a seat in the second round is unlikely in this large market. However, it is worth noting that current practice is not guaranteed strategy-proof for students, not stable, and also not efficient.

Table 1.9 summarizes violated assumptions and its consequences on the desired properties in Deferred Acceptance algorithm. Combining all three violations, the current allocation is no longer guaranteed to be strategy-proof, stable, nor optimal. There also has been growing evidence of possible strategic behaviors in a market using strategy-proof algorithms (Fack et al. (ming); Luflade (2018)). In the next section, I investigate the relationship between student choices and the matching outcome rather than taking the submitted list as true and complete ranking of the choice set.

1.3 Student Choices and Matching Outcome

First part of this section serves as an empirical counterpart to Section 1.2.3 and records all inconsistencies if student choices are assume to be complete and truthful. Then, it focuses on one aspects of New York City high school matching data is that majority of the students submit only few choices (Table 9). Literature (Abdulkadiroglu et al. (2017); Luflade (2018)) has assumed that shortening behavior indicates that only those listed are acceptable to students; students prefers outside option to all the unlisted schools. However, I provide evidence that suggests otherwise. Utilizing the two-round structure of this market, I suggest anomalies in the data if the submitted rank order lists are truthful and complete. I find answers to rank order list shortening behavior in matching outcome.

Table 9: Proportion of Student with ROL Length N

0	1	2	3	4	5	6	7	8	9	10	11	12
0.05	0.07	0.04	0.07	0.08	0.10	0.11	0.09	0.08	0.06	0.05	0.05	0.14

1.3.1 Inconsistencies In the Rank Order Lists Between the Two Rounds

Two round Deferred Acceptance set up allows for the comparison between the rank order choices of each student. I find several anomalies when when comparing evidences that contrasts the claim of truthful rank order list submission behavior in markets using Deferred Acceptance algorithm. First, students who has submitted short list do not prefer outside options to all of the other schools; second, rather than applying only to rejected schools in the supplementary round, majority (80%) apply to new schools; third, students do not fill their main round ROL, even though the student would have had higher chance of acceptance to the “newly added schools in the supplementary round” schools; fourth, majority of students do not apply to the rejected schools even though they are available in the second (or supplementary) round.

Most rejected students do not take the outside option, but rather re-submit the list in the supplementary round. If main round rank is complete, then all rejected students should take the outside option. However, the data tells us that it is not the case. When looking at the final allocation, majority of the rejected students do not take the outside option and participate again in the second (or supplementary) round. Table 10 records final allocation round for the rejected students (rejected students are 9% of the population). When looking at the “Supplementary R” column, the majority (67%) of the rejected students decide to participate in the supplementary round.

Most of the schools listed in the second (or supplementary) round are new schools. If students had submitted complete and truthful rank order list of the schools, then in the

Table 10: Final Allocation Round for Proportion of Rejected Students

Opt Out	Specialized HS	Supplementary R	Appeals R
.20	.05	.67	.07

Table 11: Examples of Strong and Weak Preference Change

Main Round	Supplemental Round	Supplemental Round
A	C	A
B	D	C
C		B
		D
	Weak Change	Strong Change

supplementary round, the data should show that students only participate to apply to rejected schools. Suppose that a student ranked A, B, and C on his supplementary rank order list. Further suppose that the same student had been rejected from A in the main round, and schools B and C were not recorded in his main round rank order lists. Then, for this student, 33.3% of the supplementary rank order list consists of rejected schools and 66.6% consists of the new schools. For the measure above, I took all individual percentages and looked at the average. The data reveals that only 10% of the supplementary rank order lists are rejected schools and 80% of the rank order lists are new schools.

I also check whether there has been a reverse in ordering from the first (or the main) round. I compare student rank order list from the main round and the supplementary round. If a student reversed ordering given in the main round, I define such change as strong change; if a student did not reverse the ordering in the main round, I define such change as weak change (Table 11). Looking at the data, reversing the rank order list from the first round almost never happens (strong change happens less than 5% of the supplementary participants). Since students mostly add new schools in the supplementary round, it is more likely scenario that students are listing new schools.

Even though students had leftover slots in the main round, students did not use it to list those new schools. If there is no cost associated with filling out additional schools, it is beneficial for students to record these new schools (recorded in the supplementary round) in the main round. However, 85% of the students who has slots left in the main round decided not to use those slots.

Moreover, had students applied to these schools in the main round, the likelihood of acceptance would have been higher. To estimate the acceptance probability, I look at the applicants' observed characteristics and their match outcome using the following equation for each school j:

$$\hat{q}\,(\text{accepted to } j|x) = \sum_{i \in A_j} \sum_{y} \left(\frac{k\left(\frac{x_{iy}-x}{h}\right)}{\sum_{i' \in A_j} \sum_{y'} k\left(\frac{x_{i'y'}-x}{h}\right)} \right) \cdot \mathbf{1}\,(\text{accepted to } j) \tag{1.1}$$

where A_j[13] is set of applicants and x is an index representation of arbitrary student characteristics. I used Gaussian kernel with Silverman's rule bandwidth. Figure 1.10 looks at the acceptance probability difference between the two rounds for the same school; for each choices recorded in the supplementary round, conditional acceptance probability from the main round minus acceptance probability in the supplementary round is calculated. Each bar in the Figure 1.10 represents a distribution of such measure in each slots. One can see that acceptance probabilities in the main round is higher (the distribution is largely above the 0 line) than the ones in the supplementary round.

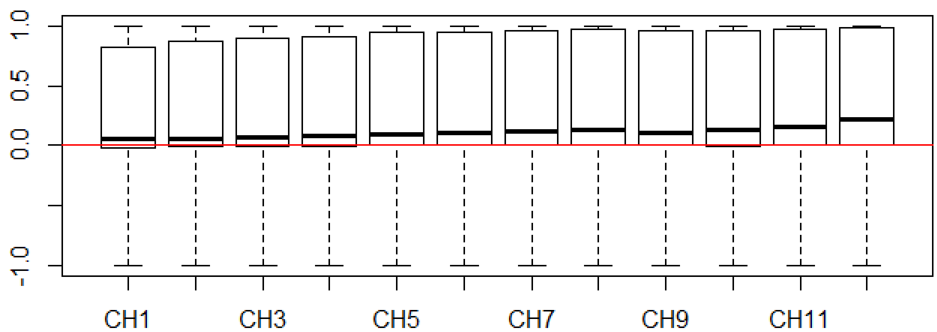
Students do not re-apply to the rejected schools even when they are available in the supplementary round. If one of the rejected schools are available in the supplementary round, then it is beneficial for students to participate in the supplementary round and re-apply to the rejected schools. However, majority (94%) of students do not re-apply. I calculated number of rejected students whose school is available (their estimated acceptance probability is higher than 0.1[14]) at the second round. Then, out of those students, I counted proportion of students who had re-applied to their rejected schools.

To the best of my knowledge, Narita (2018) is the only paper which focuses on comparing

[13]In student proposing Deferred Acceptance algorithm, a student is considered by the schools only applies upto the school that he or she applies. For example, suppose an individual i's rank order list looks like the following: $(1, 2, 3)$. Suppose that at the end of the algorithm i is matched to school 2. In this case, an individual $i \in A_1, A_2$ but $i \notin A_3$.

[14]I can change the number around from .2 to .01 and proportion stays robust to these changes

Figure 1.10: Acceptance Probability Difference to Choice N Schools

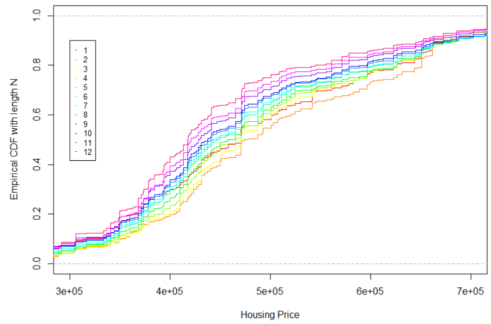


the rank order lists from the two rounds. In his paper, the self-reported reasons for changes in rank order lists are largely due to the new information or learning. Combined with the finding that majority choices in the supplementary list are new schools, it seems that the rank order lists submitted for the main round isn't complete. The next section investigates why students truncate their rank order list.

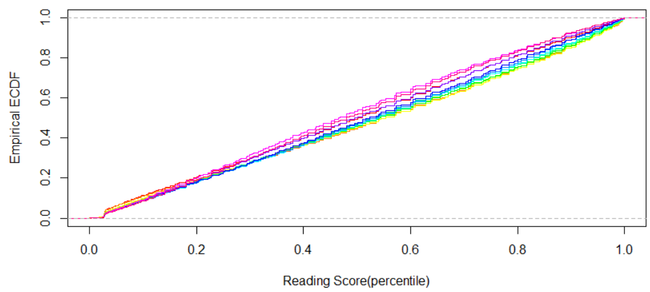
1.3.2 Rank Order List Stopping Choice and Acceptance Probabilities

Here, I argue that students decide to stop the ranking when one has high probability (or confidence) of being accepted to one of the previous choices. From Figure 1.8, one can find detailed information about the schools, programs, and its admission statistics. This ensures that students at least have a rough estimates on their probability of acceptance. Moreover, I find statistically significant relationship between the probability of acceptance, socioeconomic factors, and the list length.

Students with shorter lists (compared to those with longer list in the same quantile) are associated with higher socioeconomic factors; students with shorter lists reside in a higher housing price areas, have higher test scores, and are associated with being White. Figure 1.11 represents an empirical cumulative distributional functions for a group of student with



(a) Housing Price



(b) Reading Score

Figure 1.11: Empirical CDF of a Student Characteristic Grouped by Students with Length N ROL

different rank order list length. Rainbows in the graph shows that distributionally speaking, students with shorter lists reside in higher housing price zipcode and scores better on the standardized reading test.

Students with shorter list have higher probability of being accepted to their top choice(s). Table 12 records proportion of students matched to their n-th choice conditional on their rank order list length. Each row represents student groups with length N, and each of the column represents slot within the rank order list. For example, when looking at the first row, 90% of the students who has submitted one choice gets matched to that choice. When looking at the second row, 51% of the students who has submitted two choices gets matched to their first choice, 32% gets matched to their second choice, and 17% doesn't receive any match. The row continues down with the similar logic. The first few columns indicate that students with shorter rank order list is almost guaranteed to be matched up to their top choice(s). Final allocation data reveals that regardless of their ROL length, majority (more than 90%) of the students are matched with one of their submitted choices. Furthermore, using the expected acceptance probability calculated from Equation 1.1, I calculate the accumulated expected probabilities up to slot N. Denote q_{ij} as an expected acceptance probability for student i for school j. Suppose a student i has listed A, B, and C as her rank order list. Then, the accumulated acceptance probability up to B (slot 2) is $q_{iA} + (1 - q_{iA})\, q_{iB}$, and the accumulated acceptance probability up to C (slot 3) is $q_{iA} + (1 - q_{iA})\,(q_{iB} + (1 - q_{iB})\, q_{iC})$. Table 13 records mean of such measures by rank order list length. As expected from Table 12, students with longer list have smaller accumulated acceptance probabilities. The decision to stop the list happens once the accumulated acceptance probabilities are very high (90%).

Table 12: Proportion of Students Matched to Slot N Conditional on ROL Length

	1	2	3	4	5	6	7	8	9	10	11	12	sum
1	0.90												0.90
2	0.51	0.32											0.83
3	0.49	0.17	0.18										0.84
4	0.47	0.17	0.10	0.12									0.85
5	0.45	0.17	0.10	0.07	0.08								0.87
6	0.44	0.18	0.10	0.07	0.05	0.05							0.89
7	0.43	0.17	0.11	0.07	0.05	0.04	0.04						0.91
8	0.42	0.17	0.11	0.07	0.05	0.04	0.03	0.03					0.92
9	0.40	0.18	0.11	0.07	0.05	0.04	0.03	0.02	0.02				0.93
10	0.40	0.17	0.11	0.07	0.05	0.04	0.03	0.03	0.02	0.02			0.94
11	0.40	0.17	0.11	0.08	0.05	0.04	0.03	0.02	0.02	0.01	0.01		0.95
12	0.41	0.18	0.11	0.07	0.05	0.04	0.03	0.02	0.02	0.01	0.01	0.01	0.96

Students' decision to stop the list is related to high accumulated acceptance probabilities from the previous choices (Tables 12 and 13). To separately measure impact of acceptance probabilities, I run a Probit regression of stopping decision (at all slots) on student characteristics and accumulated acceptance probabilities. Student characteristics, especially log housing price, are included as covariates to control for possibility of having an attractive outside option; the logic here is that wealthier students may have an attractive private school option, which may affect the decision to stop the list. In fact, Table 14 shows that high housing price

Table 13: Mean Estimated Accumulated Acceptance Probabilities Up to Slot N Conditional on ROL Length

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.77											
2	0.48	0.82										
3	0.46	0.67	0.86									
4	0.44	0.64	0.76	0.88								
5	0.42	0.62	0.75	0.83	0.90							
6	0.40	0.61	0.74	0.82	0.88	0.92						
7	0.39	0.60	0.72	0.81	0.86	0.90	0.94					
8	0.38	0.59	0.71	0.80	0.85	0.90	0.93	0.95				
9	0.37	0.57	0.70	0.79	0.85	0.89	0.92	0.94	0.96			
10	0.37	0.57	0.70	0.78	0.84	0.88	0.91	0.94	0.95	0.97		
11	0.37	0.58	0.70	0.79	0.85	0.89	0.92	0.94	0.95	0.96	0.98	
12	0.38	0.58	0.71	0.79	0.85	0.89	0.92	0.94	0.95	0.96	0.97	0.98

predicts higher probability of stopping. More importantly, controlling for the wealth, students decision to stop the list is positively correlated with accumulated acceptance probabilities. In summary, students associated with low socioeconomic status have higher uncertainty to be matched to their top choice(s), which could be a possible reason for elongated list.

Table 14: Probit Regression of Stopping Decision on Student Characteristics

	Dependent variable:
	Stop
Accum. Accept. Prob.	1.166*** (0.005)
log(HousingPrice)	0.071*** (0.003)
readingScore (percentile)	−0.026*** (0.006)
mathScore(percentile)	−0.024*** (0.007)
daysAbsent	−0.0005*** (0.0001)
ethnicityBLACK	−0.311*** (0.003)
ethnicityHISP	−0.207*** (0.003)
Constant	−2.825*** (0.042)
Observations	2,134,631
Log Likelihood	−757,160.000
Akaike Inf. Crit.	1,514,336.000
Note:	*p<0.1; **p<0.05; ***p<0.01

1.4 Conclusion

This chapter investigated the New York City high school matching market and concluded that the current implementation of Deferred Acceptance algorithm does not guarantee the desired characteristics: namely, stable matching, student strategy-proofness, and student-optimal matching among students. Motivated by the practical challenges seen in the current implementation, Chapters 2 and 3 provide modified versions of algorithm to improve the allocation outcome. This chapter concluded that the submitted rank ordered list cannot be considered as a complete and truthful list. The first question then is whether we could elicit true preferences from the submitted rank ordered list, and Chapter 4 attempts to recover the preferences from the submitted rank ordered list.

2 Lexicographic Choice Rule for Multi-dimensional Choice Rule with Welfare Maximizing Tie-Breaking

Note. Westcamp (2013) has similar content on development and characterization of lexicographic reserve choice rule (but not on tie-breaking with student preferences). In that paper, when the author uses a term "procedural", it is exchangeable for the term "lexicographic" in my paper. I have found the paper after I have completed this project.

One of the practical challenges in school choice application is that schools want to maintain certain distributions of students while admitting students according to their preferences. For example, in Cambridge, Botson, regulation encourages schools to reach 34% of the student population to be free/subsidized lunch eligible. Moreover, all schools want to reserve as many seats as they can for students with priority. Suppose a student is qualified for free/subsidized lunch and has highest priority to a school. Can this student be considered to fill seats for both distributional criteria? How should schools admit students (i.e., choice rule) in this case? This chapter studies different choice rules for schools to incorporate multi-dimensional distributional goals, and suggest a restrictive choice rule that results in stable, strategy-proof, and student optimal (among stable allocations) allocation.

Another practical challenge in school choice application is that a large subset of schools do not rank individual students, but rather group rank students. Currently students with same group ranking score are given randomly generated numbers (after the rank order list submission) to break ties. Tie-breaking rules are ex-ante stable, but can result in student non-optimal allocations. With increasing efforts in empirical matching literature (which aims to recover cardinal preferences from the observed rank ordered list), section 2.3 introduces that the cardinal preferences are available and can be used to break ties, resulting in student welfare maximizing tie-breaking rule. Using available data (or economy) from Agarwal and Somaini (2018), student welfare can be improved by 2% in Cambridge, Boston.

2.1 Deferred Acceptance in Controlled School Choice Setting: Axioms and Theorems

This section sets up a matching economy and introduces a choice rule where student can be identified with multiple type space (to be exact, T dimensions). Then it axiomizes conditions needed for choice rules to satisfy to achieve stable allocation after Deferred Acceptance algorithm is run.

First, I define the economy with notations. Notable departure from a traditional setting is introduction of choice rules of schools. Choice rule takes both ordinal preference ordering and distributional concern. It then outputs the admitted subset of students. With this definition of choice rule, I list axioms for a choice rule to satisfy in order to achieve stable, student-optimal, and strategy-proofness matching when used within DA algorithm.

2.1.1 Set Up

Notations in this paper follow closely to those of Echenique and Yenmez (2015).

Let $\mathcal{S}$ be a nonempty finite set of all students.

A priority, or a (strict) preference, on $\mathcal{S}$ is complete, transitive, and antisymmetric.

A **choice rule** is a function $C : \mathcal{S} \setminus \{\emptyset\} \to \mathcal{S}$ that maps nonempty set (of applicants) to a subset (of admitted students).

There exist **type spaces** $T_1, \ldots T_T$. Each of the type space T_t consists of a vector of different **attributes** within the type space $T_t = \{\ldots, k, \ldots\}$ and $|T_t| < \infty$. A type function $\tau : \mathcal{S} \to T_1 \times \cdots \times T_T$ maps a students to his or her attributes in each type spaces. Examples of type spaces are gender and race. Examples of attributes are male, female, Asian, and etc. Mathematically, $T_{\text{gender}} = \{\text{male}, \text{female}\}$, $T_{\text{race}} = \{\text{Asian}, \text{Black}, \text{Hispanic}, \text{White}\}$. A male Asian student s can be denoted with type function: $\tau(s) = (\text{male}, \text{Asian})$. Since attributes are not overlapping across type space, I use k for arbitrary attribute from all type space: $k \in T_1 \cup \cdots \cup T_T$.

Let $\mathcal{S}^k \equiv \{s \in \mathcal{S} : \tau(s) = \{\cdot, k, \cdot\}\}$ be a set of students with attribute k. For the simplicity of notation, Similarly, let $S^k \equiv \{s \in S : \tau(s) = \{\cdot, k, \cdot\}\}$ be a set of student in subset S with attribute. In other words, $S^k \equiv S \cap \mathcal{S}^k$.

A **matching market** is a tuple $\langle \mathcal{C}, \mathcal{S}, (\succ_s)_{s\in\mathcal{S}}, (C_c, q_c)_{c\in\mathcal{C}} \rangle$ where $\mathcal{C}$ is a finite set of schools; $\mathcal{S}$ is a finite set of students; $\succ_s$ is a strict preference ordering over $\mathcal{C} \cup \{s\}$ where $\{s\}$ is outside option for student s; and C_c is a choice rule over $\mathcal{S}$.

A **matching** (or an assignment) μ is a function on the set of agents such that

1. $\mu(c) \subseteq \mathcal{S}$ for all $c \in \mathcal{C}$ ad $\mu(s) \in \mathcal{C} \cup \{s\}$ for all $s \in \mathcal{S}$;

2. $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in \mathcal{C}$ and $s \in \mathcal{S}$;

3. $|\mu(c)| \leq q_c$ and $\mu(c) \subseteq S$ where q_c is the total capacity of school c.

A matching is **stable (or fair)** if every student prefers to be assigned to any school, rather than being unassigned and there exist no student-school pair, (s, c), such that s prefers c to her assignment and c prefers to give seat to s. Formally,

1. (individual rationality) $C_c(\mu(c)) = \mu(c)$ for all $c \in \mathcal{C}$, $\mu(s) \succeq_s \{s\}$ for all $s \in \mathcal{S}$; and

2. (no blocking, or no justifiable envy) $\nexists (c, s)$ such that $s \notin \mu(c)$, $c \succ_s \mu(s)$ and $s \in C_c(\mu(c) \cup \{s\})$.

A matching is **optimal among stable matching (or student-optimal)** if there is no other stable matching that results in higher allocation for some students and no worse allocation for all others.

A mechanism Φ is **group strategy-proof for students** if for any group of students $\tilde{S} \subseteq \mathcal{S}$ and for any student profile preference ordering $(\succ_s)_{s\in\mathcal{S}}$, there exist no other profile preference ordering $(\tilde{\succ}_s)_{s\in\tilde{S}}$ such that

$$\Phi\left((\tilde{\succ}_s)_{s\in\tilde{S}}, (\succ_s)_{s\in\mathcal{S}\setminus\tilde{S}}\right) \succ_s \Phi\left((\succ_s)_{s\in\mathcal{S}}\right)$$

Informally, there is no coalition of students where they can jointly manipulate their preferences and result in better matching. Sufficient condition for group strategy-proofness for students is also studied.

2.1.2 Axioms

The most crucial characteristic for a choice rule studied in controlled school choice literature is gross substitute (GS). Starting with Kelso and Crawford (1982) many scholars have prove that GS is sufficient condition for the existence of stable matching for Gale-Shapley's deferred acceptance algorithm. Theorem 1 in the next subsection will elaborate more. For now, I formally restate the property.

Gross Substitute (GS) Choice rule C satisfies gross substitute (GS) if $s \in S \subseteq S'$ and $s \in C(S)$ imply that $s \in C(S')$.

Interpretation of this axiom is that if a student survived competition among the bigger set, then that student should also be admitted for a smaller pool of applicants. It says that no student should be chosen or rejected because he or she complements another student. Unfortunately, in multi-dimensional case, GS is only satisfied to lexicographic reserve rule (proof is in the next subsection). Quota and different versions of reserve violates this condition, and thus stable matching is not guaranteed. No further characterization can be done, since existence of a stable matching cannot be guaranteed. Next property, unlike GS, is always satisfied as long as a choice function follows reserve rule.

Acceptance (A) Choice rule C satisfies acceptance if $C(S) = S$ when $|S| \leq q$ and$|C(S)| = q$ when $|S| > q$.

Acceptance condition ensures that a school is accepting students to its full capacity. If there is an empty seat, then a school is obligated to accept students until there is no empty seat. Now, the following two properties can follow from the two mentioned above. The proof is given in result section (Section 4.2) as lemma 2. Both of these two rules that I am about to mention plays crucial role in establishing important properties of the DA algorithm.

Law of Aggregate Demand (LAD) Choice rule C satisfies the law of aggregate demand if $S \subseteq S'$ implies $|C(S)| \leq |C(S')|$.

This axiom means that excluding some students should not increase the number of chosen students. The next property states that rejected students does not affect a set of already chosen students.

Irrelevance of Rejected Students (IRS) Choice rule C satisfies irrelevance of rejected students if $C(S') \subseteq S \subseteq S'$ then $C(S) = C(S')$.

The multiple dimension choice rule (lexicographic reserve) that we study in this paper (see Section 4 for details) satisfies the above four axioms. The importance of satisfying the four axioms are given in the next subsection; a choice rule that satisfies the four above guarantee stability and strategy proof-ness.

2.1.3 Theorems

Roth and Sotomayor (1992), in their book, identifies conditions for the existence of a student-optimal stable matching. The book introduces the matching problem between the firm and the workers, so I paraphrase the theorem and put into the school choice context.

Theorem 12. *(Roth and Sotomayor, 1992, Theorem 6.8) If a choice function of a school satisfies gross substitute (GS) and irrelevance of rejected students (IRS) with strict preference, then student-proposing deferred acceptance algorithm produces a student-optiomal stable matching.*

Hatfield and Kojima (2008) identifies conditions for group strategy-proofness of the mechanism.

Theorem 13. *(Hatfield and Kojima, 2008, Theorem 1) Suppose that a choice rule of a school satisfies gross substitute (GS) and law of aggregate demand (LAD). Then the student-optimal stable matching is group strategy-proof.*

This section summarized axiomatic conditions needed to achieve first stable resulting allocation when deferred acceptance is used. Sections 2.2 and 2.2.2 each describes an assumptions which has been violated in practice, and provide a solution.

2.2 Lexicographic Reserve Choice Rules

One of the key assumptions needed for an existence of a stable allocation is that schools' choice rules must satisfy gross substitute condition. The complication can arise because schools almost always have distributional goals for incoming admitted student sets, and moreover, often times the distributional goals are in multi-dimensional type space. In this section, I first discuss the distributional concern in multi-dimensional type space. I, then, provide conditions for a choice rule (also can be used as a guideline given to schools when implementing the algorithm) and finally define and prove the lexicographic choice rule for a DA algorithm resulting in a stable, student optimal, and strategy-proof allocation.

2.2.1 Distributional Concern: Violation of Substitute

When schools have distributional concern, a student can become complimentary to another admitted student. For example, suppose a school has the following preferences: $s_{W_1} \succ s_{W_2} \succ s_{B_1} \succ s_{B_2}$. W stands for a student who is White and B stands for a student who is Black. The school has distributional goal to admit 50% Black and 50% White students and can admit up to 2 students. If a school chooses s_{W_1} then s_{B_1} becomes a complimentary student to achieve. Student s_{W_2} is preferred, but will not be chosen.

This concept has first introduced by Kelso and Crawford (1982) and by Echenique and Yenmez (2015) in the school choice context that gross substitute is sufficient condition for the existence of stable matching from Gale-Shapley Deferred Acceptance algorithm. Echenique and Yenmez (2015) has already studied a few choice rules that satisfy gross substitute (and a few other axioms needed to satisfy the desirable properties of DA) when a student can be identified to fill a spot for at most one dimension of the distributional concern.

In practice, however, many schools have distributional goals such that a student can be qualified to fill more than one spot in the distributional goal. In New York City, "Educational Option" schools want to maintain 16% of high performing (in standardized test) student and also wants to admit as many students who reside in the same zipcode (priority 1) as the school as a school can. A student can be qualified for a high performance seat and also for a priority 1 seat. In Boston, all schools aim for 34% free lunch students all the while prioritizing students with enrolled sibling and residence in close proximity. Like these case, if a student is considered to fill more than one slots designated to satisfy distributional concern, does that choice rule satisfy gross substitute? Unfortunately, as Appendix B.1 shows, many of the intuitive choice rules fail to achieve gross substitute.

Fortunately, Section 2.2.2 provides positive result that if some restrictions are posed on how schools can define their distributional concern, there exist a choice rule that satisfies gross substitute (and other axioms).

2.2.2 Solution to Distributional Concern: Lexicographic Reserve Choice Rules

When students can belong to more than one diversity dimension (for example, a student who is considered in "Educational Option" school can be considered to fill both high reading score category but also belong to priority 1), substitute fails for most of the choice rules introduced (and studied) in Ehlers, Hafalir, Yenmez, and Yildirim (2014). Below, I develop a choice rule which satisfies substitute condition – and in turn results in stable matching – by considering characteristics of students lexicographically.

Before I formally introduce the choice rule in algorithmic form, I have to first mention conditions (or restrictions) that a schools have to abide. Firstly, a student can be identified with (or can fill) at most one slot reserved for school's distributional goals. For example, suppose a school wants 34% free lunch student and as many students with enrolled siblings as they can hold. The student who is both eligible for free lunch and have a enrolled sibling can only fill one spot, either a free lunch spot of a sibling spot. Secondly, schools, when indicating their distributional goals, have to order their of the distributional concern priorities. Continuing with the same example, a school has to indicate that it wants to first look to fill 34% free lunch slots and then look to fill remaining percentage with sibling students.

Formally, a choice rule is generated by lexicographic reserves if there exists a vector with length $\sum_{t=1}^{T} |T_t|$ where elements of the vector is number of seats reserved for all attributes at all type spaces such that for any $S \subseteq \mathcal{S}$,

1. there exist a strict priority $\succ$ over $\mathcal{S}$;

2. $|C\left(S^k\right)| \geq q^k \wedge |S^k|$[15] for all $k \in T_1 \cup \cdots \cup T_T$;

3. if $\tau(s) = \tau(s')$, $s \in C(S)$, and $s' \in S \backslash C(S)$, then it must be the case that $s \succ s'$; and

4. if $\emptyset \neq S \backslash C(S)$, then $|C(S)| = q$

Algorighmically,

Step 1: Check reserved seats for the students' attribute for the first type. *For example, suppose the school wants to consider racial diversity first, and the most preferred student is an Asian. Check if reserved seats for Asian are filled.* Then, follow the process below:

1. If the reserve seats are not filled, then admit the student and go to the next student (Step 2).

   (a) If the reserve seats are filled, go to the students' attribute for the next type space and repeat the process. *For example, suppose the school wants to consider gender diversity next, and the most prefereed student is a female. Check if reserved seats for female. Repeat the process.*

   (b) If reserved seats for all attributes of the student are filled, then do not accept the student, and go to the next student (Step 2).

Step 2: Repeat the same process (Step 1) for the second more preferred student.

$$\vdots$$

Step M: Repeat the same process until all the reserve requirements for all attributes for all the type space are filled, and then go to (Step End).

[15] $x \wedge y \equiv (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\})$

Step End: Excluding all the students already chosen, re-order the remaining students. Accept the students from the top until the capacity of the school is reached.

The restriction that a student can be identified with at most one type space is necessary to achieve substitute condition. Alternative rules which consider students with more than one type space violates substitute. Those alternative rules with counterexamples are introduced in Appendix B.1.

It may seem constraining, but the definition of type space generic enough that the type space can be constructed such that multiple characteristics of students can be considered. Again with the same example scenario, students who is eligible for free lunch and have siblings and have proximity to a school is considered type (or priority group) 1, and reserve 34% of the seat for priority 1. Students who has siblings but not eligible for free lunch (or priority group) 2, and so on. As long as schools clearly state what composition of student with certain characteristics (or combinations of characteristics) are wanted, this choice rule can be used in many settings.

Note that the reserve seats do not have to be point identified, that is a school doesn't always have to indicate percentage of desired distribution. A school can say "as many of this type of students as the remaining seat". In Ehlers, Hafalir, Yenmez, and Yildirim (2014), this concept is called soft floor.

The two theorems below states that the lexicogrphic reserve choice rule can result in stretegy-proof, student optimal, stable matching.

Theorem 14. *The lexicographic reserve choice rule satisfies GS, Acceptance (by definition), LAD and IRS.*

Proof. To show GS is satisfied, suppose there exist two sets of students: S, S' such that $S \subseteq S'$.

Lemma 15. *Let s_m denote m-th preferred student, where $m \in \{1, \ldots, M\}$, $M = |S'|$, and $\tau(s_m) = (k_1, \ldots, k_T)$. At all m stages, denote $r'^{k_t}(s_m)$ as remaining reserve seats to be filled for a set S'; similarly, $r^{k_t}(s_m)$ for a set S. Then, $r'^{k_t}(s_m) \leq r^{k_t}(s_m)$ for all $t \in \{1, \ldots, T\}$.*

Proof. At the beginning, since non of the reserves are filled, so $r'^{k_t}(s_m) \leq r^{k_t}(s_m)$ for all $t \in \{1,\ldots,T\}$ trivially holds. At step m:

1. Case where $s_m \in S'$ and $s_m \notin S$. Trivially, $r'^{k_t}(s_m) \leq r^{k_t}(s_m)$ for all $t \in \{1,\ldots,t\}$.

2. Case where $s_m \in S \subseteq S'$. Suppose $r'^{k_t}(s_m) = r'^{k_t}(s_{m-1}) - 1$, which implies that $r'^{k_1} = \cdots = r'^{k_{t-1}} = 0$, then $r'^{k_1}(s_m) \leq r^{k_1}(s_m), \cdots, r'^{k_{t-1}}(s_m) \leq r^{k_{t-1}}(s_m)$. Assume by contradiction that $r^{k_{t+1}}(s_m) = r^{k_{t+1}}(s_{m-1}) - 1$, which implies that $r^{k_1}(s_m) = \cdots = r^{k_t}(s_m) = 0$, then, from the line abve ($r'^{k_1}(s_m) \leq r^{k_1}(s_m), \cdots, r'^{k_{t-1}}(s_m) \leq r^{k_{t-1}}(s_m)$), $r'^{k_1}(s_m) = \cdots = r'^{k_t}(s_m) = 0$. This is a contradiction.

□

Supose $s \in C(S')$ and $\tau(s) = (k_1,\ldots,k_T)$. I denote a set of students who are chosen from set S (or S') to fill one of the reserved seats $C(S)^{\text{reserve}}$ (or $C(S')^{\text{reserve}}$).

1. Case where $s \in C(S')^{\text{reserve}}$ (if s is selected to fill one of the reserve seats). $\sum_{t=1}^{T} r'^{k_t} > 0$ and Lemma 15 imply that $\sum_{t=1}^{T} r^{k_t} > 0$. Resulting in $s \in C(S)^{\text{reserve}}$. This also implies $C(S')^{\text{reserve}} \subseteq C(S)^{\text{reserve}}$.

2. Case where $s \in C(S') \backslash C(S')^{\text{reserve}}$ (if s is selected after the reserve seats are filled). I use contrapositive argument. Suppose $s \notin C(S) \backslash C(S')^{\text{reserve}}$, then there are at least $|C(S) \backslash C(S')^{\text{reserve}}|$ preferred students from the set $S \backslash C(S)^{\text{reserve}}$. Since we know $S \backslash C(S)^{\text{reserve}} \subseteq S \backslash C(S')^{\text{reserve}} \subseteq S' \backslash C(S')^{\text{reserve}}$, the same $|C(S) \backslash C(S')^{\text{reserve}}|$ preferred students are in this set $S' \backslash C(S')^{\text{reserve}}$. This results in $s \notin C(S') \backslash C(S')^{\text{reserve}}$.

From GS, $C(S') \subseteq C(S)$ and from Acceptance, $|C(S)| \leq |C(S')|$. Resulting in IRS: $C(S') = C(S)$. Combining GS, Acceptance and IRS implies LAD: $|C(S')| \leq |C(S)|$ □

Given the characterization of the choice rule generated by lexigographic reserve, we know the following:

Corollary 16. *The choice rule generated by lexigographic reserve produces the student-optimal stable matching.*

from Theorem 12

Corollary 17. *The choice rule generated by lexigographic reserve is group strategy proof for students.*

from Theorem 13.

This section introduced the choice rule that can produce an allocation with all the desirable characteristics consistent with choice rule without student characteristic composition concern under DA. One of the key assumptions needed to arrive at this results is strict preference from schools. Section 2.2.2 provides solutions when such assumption is violated.

2.3 Student Welfare Maximizing Tie-breaking

2.3.1 Violation of Strict Preference

Often schools do not indicate any preferences across individual students. Literature on solution methods for this problem is called "tie-breaking". Although somewhat extensively studied, current solution methods to break ties are rather arbitrary. These random tie-breaking can result in student welfare loss[16]. There have been efforts (Erdil and Ergin (2008), Ashlagi, Nikzad, and Romm (2016), and Abdulkadiroglu, Che, and Yasuda (2011)) to improve welfare using different tie-breaking rules, however, none guarantee maximum student welfare. Arbitrary tie-breaking rules are suggested because researchers only looked at at submitted ordinal rank order lists.

2.3.2 Solution to Non-Strict Preference: Student Welfare Maximizing Tie-breaking

In almost all economic settings, individual's actual preferences are not publicly observable. Researchers, hence, have to "elicit unobserved actual preferences of individual" as Mas-Colell, Whinston, and Green puts it. There has been increasing effort to map submitted rank order list (observed submission) into actual cardinal preferences, or willingness to pay, much like auction literature. In matching markets, where money isn't a mode of exchange, what

[16]When talking about welfare concern in school choice setting, welfare concern is in perspective of students.

agents are willing to "pay" requires some flexible interpretation. Agarwal and Somaini (2018) estimates the preference, or willingness to pay, as miles to which a student is willing to trevel.

When this limitation of not having cardinal information on preference is lifted, we can have simpler and cleaner welfare analysis.

In school choice literature, mechanisms are often designed and selected to maximize welfare of student[17]. When the cardinal preference of students are recovered and a school is indifferent among a subset of applicants, then it is clear that the tie should be broken using students' cardinal preferences. In other words, the school slots should be assigned to a student who values the school most. The following theorem formally argues this point.

Theorem 18. *Deferred Acceptance algorithm will result in student welfare maximizing (**among other stable allocations**) allocation if ties are broken using students' cardinal preference, or willingness to pay.*

Proof. Suppose schools without preference over individual applicants are ordered based on students' cardinal preference. The resulting allocation from DA is denoted μ. Further suppose there exists an ordering (tie-breaking), which generated stable matching with higher student welfare. I denote this matching μ'. Under μ', there must exist at least one school s which admitted at least one student with higher willingness to pay called i. This cannot be the case since i would have been admitted under μ, which is a contradiction. □

There has been increasing effort to estimate students' revealed preference (or willingness to pay) in the literature. In Section 2.4, I take Cambridge school choice market estimated by Agarwal and Somaini (2018) to quantify welfare gains from using student preference as tie-breaking rule.

2.4 Application: Cambridge, Boston High School Matching Market

Effort to recreate matching market economy has been increasing in the last decade since the data from matching algorithm implementation started to emerge. One of the first publications

[17]Knuth (1976) proved that the best outcome for one side of market is the worst for the other when all agents have strict preferences.

of this effort is by Agarwal and Somaini (2018), who replicate Cambridge Public School (CPS) economy where each year about 470 students to 13 schools are being allocated. Using this replicated market as an application, I quantify welfare gains from using student welfare maximizing tie-break introduced in Section 2.2.2. Welfare gain in this market is estimated around 2%.

2.4.1 Replicated Cambridge, Boston Economy

For this exercise, I use data provided by Agarwal and Somaini (2018) with their publication for a 2004 school year. In this economy, there are 430 students being allocated to 13 schools. Each student has a valuation to each of the schools denoted v_{ij} where i indexes students and j indexes schools. Student preferences are interpreted as normalized (to an option to be unmatched) miles willing to travel for each of the schools. All schools have the same distributional goals and no school provided strict preference to applications. The distributional goal for all school is:

1. Students with siblings and proximity priority[18] gets priorities.
2. Students with siblings only gets priorities
3. Students with proximity only gets priorities

In this economy, regulation encourages schools to have students who are eligible paid lunch to consist of 34% of the school's population. Agarwal and Somaini (2018) treats this regulation the following way. The provided data contains separate seat capacities for paid lunch students and students without lunch benefits. For example, in 2004, Graham & Perks school has 18 seats available for students without lunch benefits and 9 seats available for paid lunch students. I do not deviate from this set up.[19]

[18]Each school gives priority to students who resides within some pre-determined boundary. The proximity priority is known to student.

[19]If each school only wants to admit the fixed number of paid lunch students, possibly because of the limitation on funding available for paid lunch students, my algorithm can admit more than the designated number of seats.

	Welfare Mean (S.D.)	Welfare Mean in %
Student Preference Tie Breaking	884.82	100
Random Single Tie Breaking	870.53 (2.84)	98 (.32)
Random Multiple Tie Breaking	870.59 (2.75)	98 (.31)

Table 15: Performance of Lexicographic Choice Rule under Student Preference Tie Breaking Rule with 250 Simulations

2.4.2 Welfare Analysis

From Theorem 18, it is known that welfare using students' cardinal preference is going to be optimal. Here, I quantify the welfare gain. Before I dive into welfare analysis, I first define welfare:

$$\sum_i v_i^T Q_i$$

where $v_i \in \mathbb{R}^J$ is a vector of student preferences for all j schools and $Q_i \in \{0,1\}^J$ is a vector where an element Q_{ij} indicates 1 if a student i is matched to a school and 0 for all other schools ($\sum_j Q_{ij} \leq 1$).

To run this counterfactual exercise, I incorporated the lexicographic reserve choice rule describe in Section 2.2 within Deferred Acceptance algorithm.[20] From the distributional goal describe above, each student (or applicant) is mutually exclusively identified with a type 1, 2, or 3, respectively, and the seats for each types are available in the following manner: as many of type 1 students, then if available seats remain, as many of type 2, and so on. No school has preferences, so the ties can be broken in the following ways: (1) using student valuations to each school (described in Section 2.2.2), (2) single tie-breaking (each student is randomly assigned a number and that assigned number is used to break ties for all schools), or (3) multiple tie-breaking (each student is randomly assigned a number and the random number is newly drawn for each schools).

Table 15 presents the result that using student cardinal preferences, student valuation (willingness to travel to the assigned to school in miles) is 884.82 for the market. When using single or multiple tie breaking, the welfare values are 2% lower than the optimal measure.

[20] The code is available online, and Appendix **??** provides descriptions to use the code, if one wishes.

2.5 Conclusion

This chapter looks into two common complications arisen from schools during the implementation of Deferred Acceptance to achieve stable allocation. Section 2.2 provides conditions (or guidelines) for schools to define their distributional goals in a way that the lexicographic choice rule can guarantee stable, student-optimal, and strategy-proof matching. With the help of increasing effort to estimate student valuations from submitted rank order lists, Section 2.2.2 provides student welfare maximizing way to resolve the issue that schools do not provide strict preferences. As the literature on estimating student preferences develops on other markets using Deferred Acceptance algorithm, the algorithm provided in this chapter can be used for welfare analysis.

3 Two-Stage Deferred Acceptance with Passive Participants (with Umut Dur and Onur Kesten)[21]

When matching algorithms are implemented, many markets (including NYC and Denver public school choice market) add a separate round, often called a round 2 or a supplementary round. This paper investigates such phenomenon and asks the following questions: Why is the supplementary round needed? Is the current implementation stable and/or strategy-proof? If not, is there a way to fix it? We use the administrative data from the New York City high school matching market to find answers.

Every year in New York City, 15% of the student population in the New York City high school matching market submit the supplementary round (round 2) preference list. Majority (58%) of the supplementary round participants have matches from the main round (or round 1), yet re-submit rank order lists for the supplementary round. Moreover, when comparing the rank ordered lists from the main round to the supplementary round, majority (78%) of the supplementary rank ordered list consists of new (unlisted in the main round) schools. The supplementary round not only gives rejected students another chance of school choice, but also allows matched students to update their rank ordered list with new schools.

Currently, the main and the supplementary rounds are run separately, which causes the final allocation to be not stable and (theoretically) manipulable. Suppose there is a rejected student (student 1) from a school A in the main round. The rejected student decide not to participate in the supplementary round. In the supplementary round, another student (student 2), who happens to be less preferred to the rejected student (student 1) is by school A, becomes accepted. These students are called justified-envy in matching literature, and existence of justified envy students implies that the final allocations are not stable. In New York City, at least 3% of the student population is classified as justified envy students. Moreover, unlike the static Deferred Acceptance algorithm, the current implementation does not guarantee strategy-proofness to students. We provide a simple example to illustrate such point.

[21] Any error in this draft is all mine.

Example 19. Let $C = \{c_1, c_2, c_3\}$ be schools and $S = \{s_1, s_2, s_3\}$ be students and seats available for each school is $q = (1, 1, 1)$. I further assume that the preferences doesn't change. The round invariant preferences are given as follows:

$$c_2 \succ_{s_1} s_1 \succ_{s_1} c_1 \qquad s_3 \succ_{c_1} s_2 \succ_{c_1} s_1$$

$$c_1 \succ_{s_2} c_2 \succ_{s_2} c_3 \qquad s_2 \succ_{c_2} s_1 \succ_{c_2} s_3$$

$$c_2 \succ_{s_3} c_3 \succ_{s_3} c_1 \qquad s_3 \succ_{c_3} s_1 \succ_{c_3} s_2$$

If s_3 reports her true preference in the first round, then the allocation is $\mu^1 = \langle (s_1, c_2), (s_2, c_1), (s_3, c_3) \rangle$. There is no incentive for s_1 and s_2 to participate in the second round, so the matching stays.

Proof. Now suppose s_3 submits non-truthful report in the first round: $c_2 \succ^1_{s_3} c_1 \succ^1_{s_3} c_3$. Then, $\mu^1 = \langle (s_1, s_1), (s_2, c_2), (s_3, c_1) \rangle$. $q^2_{c_3} = 1$, since no one was matched to c_3 and s_3 prefers c_3 to c_1, $s_3$applies. Since the second round is the last round, s_3 submits true preference. Suppose only s_2 and s_3 participates, since this is the last round, both participate with their true preference. $\mu^2 = \langle (s_2, c_1), (s_3, c_2) \rangle$, and $\mu^F = \langle (s_1, s_1), (s_2, c_1), (s_3, c_2) \rangle$. □

Finally, the paper defines an economy where subset of students are allowed to update their rank order list and combines the two (but can be extended to multiple) rounds of Deferred Acceptance algorithm in such a way that preserves stability, student-optimality (among stable match), and strategy-proofness.

3.1 Two- Stage Deferred Acceptance in New York City

This section motivates the need for a study of an dynamic Deferred Acceptance algorithm allowing rank order lists for students to evolve. Another round of Deferred Acceptance algorithm is often added at the end for a subset of students and schools to re-match (e.g., NYC, Denver, and the Scramble round in Economics PhD job market organized by the American Economics Association). By investigating administrative data from the New York City high school matching market, we find that significant proportion of students (or proposing side) submit new set of schools in their supplementary round rank order lists as the application process evolves. One of the functions of supplementary round is to allow changes in rank order lists.

We then assume the following: for the supplementary round non-participants, the rank order lists from the main round are in alignment with their preferences; for the supplementary round participants, the rank order lists from the supplementary round are in alignment with their preferences; and school's preferences do not change from the main round submission. Although the strategy-proofness can no longer be guaranteed, we do not see an incentive to lie on the final submission of the rank order lists for students. Under such assumption, is the final allocation under the current implementation (two-separate Deferred Acceptance algorithm) stable? The answer is no. We quantify that 3% of the student population have been rejected from a school who admitted students less preferred. In matching literature, this is labeled as existence of justified envy students. In the rest of the paper, we define the economy that allows changes in student rank order lists, provide the modified Deferred Acceptance algorithm (called passive DA), and characterize the algorithm and the final allocation.

3.1.1 Changing Students' Rank Order Lists

As described in Section 1.2.1 New York City currently runs two separate rounds of Deferred Acceptance algorithm. In the main round, every student and every school participates. At the end of the main round, all participating students receives at most one match. Before the supplementary round starts, students keep collecting information about (1) available seats for the public schools at the supplementary round (2) private or charter school options, and (3) details (including other matching outcome of others for the main round) on public schools. Then, students decide to (not) participate in the supplementary round. Finally, the final match is revealed at the end of the supplementary round. The final outcome for those who are not matched in the supplementary round are assigned by the enrollment office. There are few (less than 5%) cases where students can appeal to be considered for a seat, but this is rather rare cases (such as medical reasons) and this paper does not focus on this step.

Possibly due to the information collected in between the main and the supplementary round, majority of the supplementary round participants re-submit their rank ordered lists with different set of schools even though they have a match from the main round. Table 16 records that 58% of the supplementary round participants have a match from the main round, in a school year 2015-2016. Moreover, the supplementary round rank order list mostly adds unlisted (in the main round) schools rather than re-applying to the rejected school. If the supplementary round rank order list switch the order of the main round rank order list, we label that the student has a strong change in rank ordered list; if the supplementary round rank order list do not switch the order of the main round rank order list, we label that the student has a weak change. As seen in Table 17, the second column is labeled as a weak change since the order of A prefer to B prefer to C has not been disrupted. However, in the last column, a student is labeled to have strong change because he/she prefers B to C in the main round, but he/she prefers C to B in the supplementary round. Only 6% of the supplementary round participants strongly change their preferences; 25% of the supplementary round participants applied again to one of their rejected schools; and 90% of the supplementary round participants include new (unlisted in the main round) schools. We conclude that one of the major functions for the supplementary round is to allow students to change (mostly to add new schools in) their rank ordered lists during the matching process. In Section 3.2, we define an economy where there are evolving (or changing) preferences of students.

Table 16: Allocation from the Main Round for the Supplementary Round Participants (2015-2016 School Year)

Main R Allocation	N	%	Total N
Rejected (from all)	4,291	33	12,965
Matched	7,503	58	12,965
Didn't Participate	1,171	9	12,965

Table 17: Examples of Strong and Weak Change in their Rank Ordered List

Main Round	Supplemental Round	Supplemental Round
A	C	A
B	D	C
C		B
		D
	Weak Change	Strong Change

3.1.2 Existence of Justified Envy Students

Because the matching process of the main round and the supplementary round is completely separate, for some schools, it is possible that the admitted student pool in the supplementary round can be less preferable than the main round. In such cases, there could be some rejected students from the main round who may be preferred to the admitted students in the supplementary round. We call such students *jusified-envy from the main round*, and it is defined as follows:

Definition (Justified-envy from the main round). If the student is rejected from a school in the Main round and there exist a student who is less preferred and is admitted from the school in the Supplementary round, then the rejected student from the Main round is called justified-envy from the main round.

Existence of *jusified-envy from the main round* students indicate that the final allocation is not stable. We find that there are many (at least 3% of the student population) *jusified-envy from the main round.* We make several assumptions to arrive at such conclusions. First, we assume that for the supplementary round non-participants, the rank order lists from the main round are in alignment with their preferences, and for the supplementary round participants, the rank order lists from the supplementary round are in alignment with their preferences. Second, school's preferences are in alignment with the main round submission. Under such assumption, we conservatively count *jusified-envy from the main round* if a student satisfies the following conditions:

1. there exist an admitted student from the supplementary round who has strictly lower (better) priority score; or

2. there exist a student who has (1) rejected from the Main round and (2) admitted from the Supplementary round, and the rejected-admitted student has either

   (a) strictly lower (better) priority score; or

   (b) same priority score and strictly lower (better) rank score

3. does not change preference in the second round

The second case leaves only 10% of the school programs (84 programs), since there aren't many rejected-admitted students. It is possible that there could be more *jusified-envy from the main round* students for schools who does not have such rejected-admitted students. Even though we count the second case for only 10% of the schools, we found that there are at least 2,647 students (3.2% of the population) who are justified envy from the main round.

By investigating administrative data from the New York City high school matching market, we find that there exist demand for an algorithm which (1) allows students (proposing side) to change rank order lists (from the proposing side) and (2) combines the two (or more) rounds so that the final allocation is stable (, student-optimal, and strategy-proof). The next sections define the economy that allows changes in student rank order lists, provide the modified Deferred Acceptance algorithm (called Two (or Multi)-Round Deferred Acceptance with Passive Participants), and characterize the algorithm and the final allocation.

3.2 The Dynamic Matching Market

Static matching model is denoted similarly to that in the previous section. The only different notation is that the schools' choice rules from the previous section have been replaced with a strict preference orderings, $\succ_c \forall c \in C$.

A set of students are denoted, $\{s_1, \ldots, s_N\}$ and a set of schools are denoted, $\{c_1, \cdots, c_J\}$. A maximum capacity of the schools are denoted $q_{c_j} \forall j$. $\succ_s$ is a preference relation of student i over all schools including unassignment, $\emptyset$. $\succ_c$ is the strict preference ordering of agents for school c. The economy (matching market) consists of tuple $\langle C, S, (\succ_s)_{s\in S}, (\succ_c, q_c)_{c\in C}\rangle$where C is a finite set of schools; S is a finite set of students; $\succ_s$ is a strict preference ordering over

$C \cup \{s\}$ where $\{s\}$ is outside option for student s; and $\succ_c$ is a strict preference ordering over S. Definitions of matching, mechanism, stability, Pareto efficiency, strategy proofness are similar to those in Section 2.1.1.

Since there are two rounds, components of each matching round is denoted with superscript $t = \{1, 2\}$, and final matching result super scripted with F. Participating students in the later rounds are subsets of the student population, $S^t \subseteq S$, and they can change their preferences: $(\succ_s^1)_{s \in S^2} \neq (\succ_s^2)_{s \in S^2}$ or $(\succ_s^1)_{s \in S^2} = (\succ_s^2)_{s \in S^2}$, but all schools do not change their preferences: $(\succ_c^1)_{c \in C} = (\succ_c^2)_{c \in C}$.

Notable feature of this market is that preference of students can vary across rounds. We additionally assuming that students do not expect anyone's preference to vary across rounds, supported by Narita (2018) using learning argument.

A **final matching** (or an assignment) μ^F is a function on the set of agents such that

1. $(\mu^1(c) \setminus S^2) \cup \mu^2(c) \subseteq \mathcal{S}$ for all $c \in C$ and $\left(\left((\mu^1(s))_{s \in S} \setminus (\mu^1(s))_{s \in S^2}\right) \cup (\mu^2(s))_{s \in S^2}\right) \in C \cup \{s\}$ for all $s \in \mathcal{S}$;

2. $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in \mathcal{C}$ and $s \in \mathcal{S}$;

The final matching is first round match for those who did not participate in the second round and second round match for those who participated in the second round. Mathematically, $(\mu^1(c) \setminus S^2) \cup \mu^2(c)$ and $\left(\left((\mu^1(s))_{s \in S} \setminus (\mu^1(s))_{s \in S^2}\right) \cup (\mu^2(s))_{s \in S^2}\right)$.

A **final matching is stable (or fair)** if every stuent prefers to be assigned to any school, rather than being unassigned and there exist no student-school pair, (s, c), such that s prefers c to her assignment and c prefers to give seat to s. Formally,

1. (individual rationality) $C_c\left(\mu^F(c)\right) = \mu^F(c)$ for all $c \in \mathcal{C}$, $\mu^F(s) \succ_s^1 \{s\}$ for all $s \in S \setminus S^2$, $\mu^F(s) \succ_s^2 \{s\}$ for all $s \in S^2$; and

2. (non-wasteful)$|\mu^F(c)| \leq q_c$ and $\mu^F(c) \subseteq S$ where q_c is the total capacity of school c.

3. (no blocking, or no justifiable envy) $\nexists (c, s)$ such that $s \notin \mu^F(c)$, $c \succ_s^2 \left(\mu^F(s) \cup \{s\}\right)$ for all $s \in S^2$, $c \succ_s^1 \left(\mu^F(s) \cup \{s\}\right)$ for all $s \in S \setminus S^2$, and $s \succ_c \mu(c)$.

A mechanism Φ is **group strategy-proof for all stages** if for any group of students $\tilde{S} \subseteq \mathcal{S}$ at any stage and for the following preference ordering $\succ_s \equiv \left\langle (\succ_s^1)_{s \in S \setminus S^2}, (\succ_s^1)_{s \in S^2} \right\rangle$, there exist no other profile preference ordering $(\tilde{\succ}_s)_{s \in \tilde{S}}$ at any stage such that

$$\Phi\left((\tilde{\succ}_s)_{s \in \tilde{S}}, (\succ_s)_{s \in \mathcal{S} \setminus \tilde{S}}, q\right) \succ_s \Phi\left((\succ_s)_{s \in \mathcal{S}}, q\right)$$

Informally, there is no coalition of students where they can jointly manipulate their preferences and result in better matching. Sufficient condition for group strategy-proofness for students is also studied.

3.3 Multi Round DA with Passive Participation Algorithm

In this section, we introduce the Multi Round Deferred Acceptance with Passive Participation algorithm. This algorithm combines multiple rounds of Deferred Acceptance. By allowing the non-participants to passively participate and ensuring their match from the previous round, the resulting final allocation is stable, student-optimal, and strategy-proof. We first introduce the algorithm.

Algorithm: The Main Round:

Step 1. Each student applies to her most preferred school. Each school tentatively admits students according to their preference and total capacity and permanently rejects the rest.

$$\vdots$$

Step k. Similar to the first step, each student who was rejected at step $k-1$ applies to her next preferred school. Each school considers union of tentatively admitted students from step $k-1$ and applicants from step k. From this set, each school again tentatively admits a set of students according to their preference and total capacity, and permanently rejects the rest.

Step End. the algorithm stops when there are no rejections.

The Supplementary Round:

Step 1. (Schools' Preference Update) The schools give the highest priorities to the students with the main round match to guarantee the main round allocation seat.

Step 2. (Students' Preference Update) For the participating students, add the main round matched school at the end of their rank order list. Then, participating student applies to new most preferred school. Each school tentatively admits students according to their preference and total capacity and permanently rejects the rest.

$$\vdots$$

Step k. Similar to the first step, each student who was rejected at step $k-1$ applies to her next preferred school. Each school considers union of tentatively admitted students from step $k-1$ and applicants from step k. From this set, each school again tentatively admits a set of students according to their preference and total capacity, and permanently rejects the rest.

Step End. the algorithm stops when there are no rejections.

3.3.1 Characterization

In the supplementary round, **active participants** are students who submitted new preference list and **passive participants** are students who did not submit the preference list. Following the multi round DA with passive participants, the following theorems ensure that the final allocation is stable and student-optimal; moreover, the algorithm is strategy-proof.

Theorem 20. *Let* $\Phi = \left(\phi^{Main}, \phi^{Supplementary}\right)$ *be a system where all students participate (both actively and passively), and schools guarantee passive participants' first round match seats in* *the second, or Supplementary, round. If the two round uses DA-DA, then, Φ is the unique roundwise stable system satisfying immune both within round and across round manipulation.*

Proof. We know that static DA mechanism is strategy-proof, so Ψ is immune to manipulation at each round. Hence, we only need to prove Ψ is immune to across round manipulation.

Suppose there exist a student s_i who can be better off by manipulating in the main round. Denote$\mu^F(s_i) = c$ as a match outcome from the main round under truth-telling in the first round and $\check{\mu}^F(s_i) = c'$as a match outcome from the main round under manipulative report in the first round, and $c' \succ_{s_i} c$. We only prove the case where $(\succ_s^2)_{s\in S} = (\succ_s^1)_{s\in S}$, since the assumption of this setting is that at the initial stage (ex-ante) students do not know that the future preference will change.

For this proof, we use McVitie-Wilson version of the DA algorithm when there is at most N iteration, since the algorithm doesn't move to $i+1$ student unless all the students before i has been assigned. Consider a problem where $\phi\left((\succ_s^1)_{s\in S\backslash s_i}, (\succ_c |S\backslash s_i)_{c\in C}, q\right)$ where $\succ_c |S\backslash s_i$ is school preference of all c excluding s_i from the list and remaining everything else equal. Denote the outcome of this mechanism as v. Consider a problem where $\phi\left((\succ_s^1)_{s\in S\backslash s_i}, (\succ_c |\check{\mu}^1)_{c\in C}, q\right)$ where $\succ_c |\check{\mu}^1$ is school preference of all c with priority given to matched students from $\check{\mu}^1$ and $\check{\mu}^1$ is a first round match result using manipulated preference from student s_i and true preference from all other students. The resulting match outcome is called $\check{v}$.

Claim 21. $v(s_j) = \check{v}(s_j)$ for all $s_j \in S\backslash s_i$. In words, if s_i, the manipulator, is excluded from the algorithm, the resulting algorithm for both truthful reporting algorithm and manipulated reporting algorithm are the same for all $s_j \neq s_i$.

Proof. If a is rejected from a school c in the following problem: $\phi\left((\succ_s^1)_{s\in S\backslash s_i}, (\succ_c |S\backslash s_i)_{c\in C}, q\right)$ if and only if she is also rejected from school c in the following problem: $\phi\left((\succ_s^1)_{s\in S\backslash s_i}, (\succ_c |\check{\mu}^1)_{c\in C}, q\right)$ at any step.

($\Longrightarrow$)If a student was rejected from a school c in the following problem: $\phi\left((\succ_s^1)_{s\in S\backslash s_i}, (\succ_c |S\backslash s_i)_{c\in C}, q\right)$ then the student doesn't have any priority from $\phi^1\left((\succ_s^1)_{s\in S\backslash s_i}, (\succ_c |\check{\mu}^1)_{c\in C}, q\right)$ which means that the student has to compete for the seat that are left from the main round. However, there are no leftover seats, since there exists q_c students that c prefers in the system already.

($\Longleftarrow$)If a student is rejected from school c from the following problem: $\phi\left(\left(\succ_s^1\right)_{s\in S\backslash s_i}, \left(\succ_c |\breve{\mu}^1\right)_{c\in C}, q\right)$, then automatically he or she is rejected from this problem: $\phi\left(\left(\succ_s^1\right)_{s\in S\backslash s_i}, \left(\succ_c |S\backslash s_i\right)_{c\in C}, q\right)$.

$\square$

Case I: $s_j \succ_{c'} s_i$ for all $s_j \in v^{-1}(c')$.

There are at least $q_{c'}$ preferred students already taking all the seats at c'. Hence, s_i will not be admitted to c', even if s_i deviates to $\check{\succ}_{s_i}^1$.

Case II: $\exists s_j \in v^{-1}(c')$ such that $s_i \succ_{c'} s_j$

From the above claim, we know that relative school preferences remain preserved from the truth telling and manipulative rounds excluding s_i. This means that once the rejection cycle starts because s_i took over the least preferred s_j at c'. The rejection cycle will be the same as the one in the truth-telling one. In other words, s_i cannot be assigned to c' after a manipulation in the first round, because the same rejection cycle from the truth-telling will reject s_i. $\square$

3.4 Conclusion

This chapter motivates the need for a combined algorithm when the students are allowed to change their rank order list, and hence the market adds another round of Deferred Acceptance algorithm for students with changed rank order list. Because the two rounds are fun separately in New York City, the final matching is not stable, and at the first round submission is not guaranteed to be strategy proof. To remedy this, we provide provide a modified algorithm called, the Multi Round Deferred Acceptance with Passive Participant. The final allocation for this algorithm is stable, student-optimal, and not manipulable in any round.

4 Student Rank Order List Choice Model: A Dynamic Discrete Choice

This section models how students fill his/her rank ordered list considering not only his/her valuation for schools but also acceptance probabilities. As shown in Section 1.3, accumulated acceptance probabilities across the rank order list sums up to be close to 1 for students; regardless of whether a student listed 1 or 12 schools, a student is almost likely to be matched to one of her choices. By adopting dynamic discrete choice framework and using the rejection probability as a discount factor in this framework, I build a empirically tractable model for discrete rank order list behavior incorporating acceptance probabilities. Building such model is not a simple task, especially when the choice set is large. In New York, students can choose rank order list of any length out of 700+ high school programs. The number of possible choices becomes quickly uncountable. In fact, 80% of the submitted rank order list choice is a unique list. In the literature, Hastings et al. (2009) have used a method of exploded-mixed-logit model and Abdulkadiroglu et al. (2017) and Agarwal and Somaini (2018) have used Gibb's sampling method. However, both methods are computationally expensive in this market. The exploded-mixed-logit method uses maximum likelihood method and Gibb's Sampling with large choice set requires extremely large number of samples. For computational efficiency and tractability, the rank order list structure is re-imagined as a discrete dynamic choice problem (Calsamiglia et al. (2018)) where each slot in the rank order list is analogous to a time period in the literature.

From the model, the student valuations for schools (or the parameters of interest) can be represented as a simple and intuitive function of conditional acceptance and choice probabilities (McFadden (1978); Rust (1987); Hotz and Miller (1993)). Exploiting such representation, I apply this model to the New York City data. The estimation is done in two steps. In the first step, the conditional acceptance and choice probabilities are calculated using choice and outcome data. The second step uses the estimates from the first stage to calculate valuations. This estimation method not only reduces computation time, but also provides tractability in terms of how each of the first step estimates interacts with the valuation.

Finally, the estimates are used to predict student choices without the influence of school selections. This empirical exercise yields the following result. In order to demonstrate the role of acceptance probabilities in student choices, acceptance probabilities are set to 1 for all schools (when students choose based on their valuations only). I find that majority (65%) of the students change their first slot choices; only about 35% student choices are robust to this change in acceptance probabilities.

4.1 Model

In this economy, there exist I students and J schools are to be allocated. An arbitrary student is indexed by $i \in \{1, \ldots, I\} \subseteq \mathcal{I}$ and an arbitrary school is indexed by $j \in \mathcal{J} \equiv \{1, \ldots, J\} \cup \{0\}$ where 0 is choosing default option. Each student i is endowed with a vector of characteristics, X_i. Each school j is endowed with a seat capacity $\omega_j \in [0, 1]$ which represents the maximum fraction of the population that school j can admit. The total capacity of all schools has a lower bound that is equal to the total number of students in the market (i.e. $\sum_j I\omega_j \geq I$).

Each student submits a rank order list of schools. Each school has a selection rule, which creates a ranking among the applicants. Given the three elements: (1) rankings from the students, (2) school selection rules, and (3) the capacity at each school, the centralized Deferred Acceptance algorithm determines how students and schools are matched. At the end of the process, every student has to be matched to at least and at most one school. Additionally, there exist a default school for each student that the student is guaranteed to be matched if the student is not matched to a school on his or her rank order list.

4.1.1 Endowments

Each student i is endowed with a vector of characteristics, $X_i \in \mathcal{X}$. A student's vector of characteristics includes their academic record (standardized test scores and attendance), ethnicity, and location of their residential zip code.

Student i's valuation for a school j is conditional on the student's characteristics, and is denoted $\mu_j(X_i)$, or simply μ_{ij}. One thing to mention here is that two students with same characteristics and location will have the same valuation for each school j. In other words, $\mu_{ij} = \mu_{i'j}\ \forall j$, if $X_i = X_{i'}$. Additionally, for each student i, his or her valuation for the default school is denoted μ_{i0}, and normalized to 0[22]. There also exists an unobserved component of utility which is denoted ϵ_{ijt}.[23].

4.1.2 Acceptance Probability

Each school receives applicants which are only a subset of the population. I denote applicants to a program j as $A_j \subseteq \mathcal{I}$. Under Deferred Acceptance algorithm, a student isn't necessarily an applicant to every program in their rank order list. For example, suppose an individual i's rank order list has length 3 and looks like the following: $(1, 2, J)$. Suppose that at the end of the algorithm i is matched to school 2. In this case, an individual $i \in A_1, A_2$ but $i \notin A_J$. Given applicants, each school j has a selection rule, denoted $S_j : \mathcal{X}^{|A_j|} \times \mathbb{R} \to \{0,1\}^{|A_j|}$. Each selection rule S_j inputs characteristics of all applicants (including the distance between the student and a school j) and the seat capacity, and it outputs admission decision.

Students do not have full information on other student individual choices, but rather know the distribution of applicant characteristics for each program j. The distribution of applicant characteristics for program j is denoted as $F_{X_{A_j}}$. I assume that $F_{X_{A_j}}$ is stationary over slot t (and also over all market years), since DA defers its admission decision until all the applicants (hence slots) are considered.

Beliefs over competitor characteristics distribution and selection rule allow a student to estimate acceptance probability for the school, denoted as follows:

$$q_{ij} = \int S_j \left(X_i, \xi_{ij} | F_{X_{A_j}}, I\omega_j \right) dG\left(\xi_{ij}\right) \tag{4.1}$$

where $S_j\left(X_i, \xi_{ij}|A_j, I\omega_j\right)$ is selection rule for student i with characteristic x_i for program j

[22]Please refer to the identification section.

[23]Students face an unobserved utility shock for choosing a school j at a slot t. Allowing ϵ to vary not only across schools, but also across slots serves a practical purpose. A student can choose any permutation of school with any length. In the case of NYC, there are $J = 800+$ and $T \in \{1, \ldots, 12\}$, which makes the cardinality of the choice set uncountable. To solve this problem, I represent the rank order list choice process as dynamic discrete choice model (Calsamiglia et al. (2018)), and ϵ_{ijt} allows the dynamic representation.

given A_j as the applicants and $I\omega_j$ as seat capacity. ξ_{ij} is an unknown to students but the distribution of ξ is known.[24]

The assumption here is that an an atomless student's choice of school has little affect on the outcome of the matching. This seems like a plausible assumption given the number of players (students) in this market. Therefore, I am assuming for the rest of the paper that students know their acceptance probability (from knowing applicant distribution, seat capacity, and selection rule for each school) and take it as given when they make there rank order lists.

4.1.3 Student Problem

Given the above endowments, students fill out rank order lists sequentially. Student choice procedures consistently follow the assignment mechanism. Student Proposing Deferred Acceptance algorithm sequentially goes through student rank order lists and only moves down the lists only if students are rejected from the previous choices. At every slot, if a student i is matched to her choice, d_t, out of a choice set at time t, C_t. The student's per period utility is $\mu_{i,d_t} + \epsilon_{i,d_t,t}$; if a student i is rejected, her per period utility is $\epsilon_{i,d_t,t} + V_{t+1}$ where V_{t+1} is a continuation value. I recursively represent this procedure for a student (index i omitted):

$$V_t(X, \epsilon_t) = \max_{d_t} \left\{ \mathbf{1}\left(\text{accepted to } d_t\right) \mu_{d_t \in C_t} + \mathbf{1}\left(\text{rejected to } d_t\right) V_{t+1} + \epsilon_{d_t,t} \right\}$$

A student is guaranteed to be accepted to their default school, so there is no continuation value once the default school is chosen. In other words, if a student i fills his or her list up to T, then $\mu_{i,d_{iT+1}} = \mu_{i,0}$. The optimal decision is denoted as:

$$\delta_t = \arg\max V_t$$

When the choice is being made, however, students do not know whether they will be either

[24] To justify the existence of this shock, the result section shows that conditional on the student characteristics, acceptance probabilities are mostly not close to 0 or 1. This shock could come from random selection criteria for some of the schools or there exist a more structured criteria but unobserved by students.

accepted or rejected, because ξ_{ij} introduced at Equation 4.1 is unknown. Thus, indicator function for acceptance is replaced by acceptance probabilities, q_{ij} from Equation 4.1. Given that students take acceptance probabilities as given, they face the following recursive value function:

$$V_t\left(X, \epsilon_t | \varrho_t\right) = \max_{d_t}\left\{q_{d_t}\mu_{d_t} + \epsilon_{d_t,t} + \left(1 - q_{d_t}\right)V_{t+1}\right\} \tag{4.2}$$

where

$$\varrho_t = \Pi_{\tau=1}^{t-1}\left(1 - q_{\delta_\tau}\right) \tag{4.3}$$

At each period, students come into the period with their endowed characteristics and probability of being rejected from all choices before (state space). At period 1, everyone has same probability of being rejected from previous choice (which is none). Then, given this state and realized idiosyncratic preference shock, ϵ_t, a student chooses d_t which maximizes Equation 4.2. Moreover, every student has a default school with acceptance probability of 1 at the end of the list.

The value function of a rank order list with arbitrary length T can be written in a matrix form below:

$$\begin{bmatrix} q_{d_1} \\ \varrho_2 q_{d_2} \\ \vdots \\ \varrho_{T+1}\cdot 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_{d_1} \\ \mu_{d_2} \\ \vdots \\ \mu_0 \end{bmatrix} + \begin{bmatrix} \epsilon_{d_1,1} \\ \epsilon_{d_2,2} \\ \vdots \\ \epsilon_{d_{T+1},T+1} \end{bmatrix} \tag{4.4}$$

where $\otimes$ and $+$ indicates element-wise operations. A vector on the left indicates overall expected probability of being matched to a school at choice t, d_t, a vector in the middle is valuation for those choices, and the last vector is additively separable (by assumption) idiosyncratic parts of the utilities. Such formation forces students tho choose the rank order list given that the sum of all expected allocation to all the choices are 1; sum of the elements in the vector is 1 for all students.

4.2 Identification

In this section, I provide closed form representation of student valuations for schools, denoted $\boldsymbol{\mu}$. At each slot, the value function consists of the expected utility of being matched to the current choice and the expected utility of continuing to fill out more schools. The expected utility of being matched to the current choice is acceptance probability of the current choice multiplied by valuation of the current choice plus idiosyncratic preference shock. Expected utility of continuing to fill out more school(s) is expected rejection probability of the current choice (1- acceptance probability of the current choice) multiplied by expected future valuation, also known as continuation value. In other words,

$$V_t\left(d_t, \epsilon_{d_t} | X, \varrho_t\right) = q_{d_t}\mu_{d_t} + \epsilon_{d_t} + \left(1 - q_{d_t}\right)\bar{V}_{t+1}\left(X, \varrho_{t+1}\right)$$

To arrive at the closed form representation of $\boldsymbol{\mu} \equiv [\mu_1, \ldots, \mu_J]'$, results from Hotz and Miller (1993) are used. Some notations need to be introduced first.[25] Ex-ante (before ϵ_t is realized) value function, $\bar{V}_t\left(X, \varrho_t\right)$, is denoted as

$$\bar{V}_t\left(X, \varrho_t\right) \equiv \int V_t\left(X, \varrho_t\right) dF\left(\epsilon_t\right) \tag{4.5}$$

An ex-post conditional value function, $v_t\left(d_t | X, \varrho_t\right)$, is denoted as:

$$v_t\left(d_t | X, \varrho_t\right) \equiv q_{d_t}\mu_{d_t} + \left(1 - q_{d_t}\right)\bar{V}_{t+1}\left(X, \varrho_{t+1}\right) \tag{4.6}$$

With an assumption on distribution on the error term,$\epsilon \sim GEV$, the ex-ante value function and the conditional value function can further be represented as:

$$\bar{V}_t\left(X, \varrho_t\right) = -\ln\left[p\left(d_t^* | X, \varrho_t\right)\right] + v_t\left(d_t^* | X, \varrho_t\right) + \gamma \tag{4.7}$$

[25] Notation used here are similar to those used in Arcidiacono and Ellickson (2011), and derivations can be found in the paper.

where d_t^* can be any arbitrary choice. The right hand side can be interpreted as: a value function of choosing d_t^* , $v_t\left(d_t^*|X,\varrho_t\right)$, a nonnegative adjustment term for choosing d_t^*, $-\ln\left[p\left(d_t^*|X,\varrho_t\right)\right]$, and the mean of the type 1 extreme value distribution, γ.

Normalizing $v_t\left(0|X,\varrho_t\right)=0$ for all t[26] and selecting d_{t+1}^* as choice of selecting the default school, the ex-post conditional value function for choosing d_t is:

$$
\begin{aligned}
v_t\left(d_t|X,\varrho_t\right) &= q_{d_t}\mu_{d_t} + \left(1-q_{d_t}\right)\left[-\ln\left[p\left(d_{t+1}^*|X,\varrho_{t+1}\right)\right] + v_{t+1}\left(d_{t+1}^*|X,\varrho_{t+1}\right) + \gamma\right] \\
&= q_{d_t}\mu_{d_t} + \left(1-q_{d_t}\right)\left[-\ln\left[p\left(0|X,\varrho_{t+1}\right)\right] + \gamma\right]
\end{aligned}
$$

This structure is analogous to an optimal stopping problems. By choosing d_{t+1}^* to be the default (or “exit”) choice which does not have any future value term, $v_t\left(d_t|X,\varrho_t\right)$ can be expressed in terms of static utility and one period ahead default choice probability.

With the assumption that ϵ's are drawn from GEV distribution, closed-form expression for the conditional choice probabilities are:

$$
p_t\left(d_t|X,\varrho_t\right) = \frac{\exp\left(v_t\left(d_t|X,\varrho_t\right)\right)}{\sum_{d_t'}\exp\left(v_t\left(d_t'|X,\varrho_t\right)\right)}
$$

and the log difference in the conditional choice probabilities simplifies to:

$$
\begin{aligned}
\ln\left[p_t\left(d_t|X,\varrho_t\right)\right] - \ln\left[p_t\left(0|X,\varrho_t\right)\right] &= v_t\left(d_t|X,\varrho_t\right) - v_t\left(0|X,\varrho_t\right) && (4.8)\\
&= q_{d_t}\mu_{d_t} + \left(1-q_{d_t}\right)\left[-\ln\left[p_{t+1}\left(0|X,\varrho_{t+1}\right)\right] + \gamma\right] - 0 && (4.9)
\end{aligned}
$$

Finally, we arrive at the closed form representation of student valuation:

$$
\begin{aligned}
\mu_{d_t} &= \mu_{d_t}\left(q_{d_t}, p_t\left(d_t|X,\varrho_t\right), p_t\left(0|X,\varrho_t\right), p_{t+1}\left(0|X,\varrho_{t+1}\right)\right) \\
&= \frac{\ln\left[p_t\left(d_t|X,\varrho_t\right)\right] - \ln\left[p_t\left(0|X,\varrho_t\right)\right] - \left(1-q_{d_t}\right)\left(-\ln\left[p_{t+1}\left(0|X,\varrho_{t+1}\right)\right] + \gamma\right)}{q_{d_t}} && (4.10)
\end{aligned}
$$

[26]Note that in our set up, normalizing $v_t\left(C_t, X, 0\right)$ for all t is equivalent to $\mu_0=0$ since conditional value function for the terminal choice is student valuation for the default school: $v_t\left(C_t, X, 0\right) = q_0\cdot\mu_0 = 1\cdot\mu_0$.

For intuition, Equation 4.10 can be broken down into three parts:

$$\mu_{d_t} = \frac{(1) - (1 - (3))\,(2)}{(3)}$$

where

$$\begin{aligned}
(1) &= \ln\left[p_t\left(d_t|X, \varrho_t\right)\right] - \ln\left[p_t\left(0|X, \varrho_t\right)\right] \\
(2) &= -\ln\left[p_{t+1}\left(0|X, \varrho_{t+1}\right)\right] + \gamma \\
(3) &= q_{d_t}
\end{aligned} \tag{4.11}$$

(1) in Equation 4.11 is the normalized value function for choice d_t, (2) is the continuation value, and (3) is the acceptance probability to d_t. This equation intuitively shows how valuations change depending on different values of (1), (2), and (3). Notice that such valuation units (in utils) doesn't have an interpretation, and this paper doesn't further parameterize the valuations in order to remain fully nonparametric.

4.3 Estimation

As shown in Section 4.2, the optimized choice (probability) can be represented as a function of the acceptance and choice probabilities, which can easily be retrieved from the data. Hence, the estimation is done in two-steps: in the first step, all choice probabilities are calculated using choice and outcome data; in the second step, the probabilities in in Equation 4.10 is replaced with the estimates from the first step to solve for valuations. This strategy is known as the two-step estimator, and it not only reduces computational burden of value function iteration but also provide tractability of each estimates.

For the rest of this section, I describe the process of how the two-step estimator is used for this model, using the following data:

$$data = \{X_{i,y}, d_{i,j,t,y}, a_{i,j,y}\}_{i \in I, j \in J, t \in \{1,\ldots,13\}, y \in \{2011,\ldots,2016\}}$$

where X_i is student characteristics (ethnicity, latitude and longitude of residential zip code, standardized test score percentiles, and number of absent days), $d_{i,j,t} \in \{0, 1\}$ is student i's choice of school j at slot t, and finally $a_{i,j} \in \{0, 1, \mathrm{NA}\}$ is whether or not a student has been admitted to school j (NA when i has not applied to school j).

4.3.1 First Stage: Conditional Probabilities

In the first stage, conditional on student characteristics (and additional state variable), (1) probabilities of being accepted to a school j for all j and (2) probabilities of choosing a school j at slot t are estimated. For conditional acceptance probabilities, given observed student characteristics, the probability of acceptance is calculated using the sample of applicants for the school j. For conditional choice probabilities, given observed student characteristics and the rejection probabilities up to t (or probability of reaching slot t), the probabilities of choosing a school j for all j and the probability of stopping the list are calculated using the sample of students who has rank order list length of at least $t - 1$.

Given such large sample of 300,000+ students, nonparametric estimation method is used to calculate conditional probabilities. For the consistency of the estimates, Nadaraya-Watson estimator with Gaussian Kernel is used. Rather than incurring computational burden of choosing bandwidth using data, I use a Silverman's rule of thumb.

Conditional Acceptance Probabilities In the student proposing Deferred Acceptance algorithm, a student is not considered by all of the schools in the rank order list. The student applies to schools in the rank order list only up to the school to which (s)he is matched. For example, suppose an individual i's rank order list looks like the following: $(1, 2, 3)$. Suppose that at the end of the algorithm i is matched to school 2. In this case, an individual has only applied to schools 1 and 2, or $i \in A_1, A_2$ but $i \notin A_3$. The estimation, thus, uses applicants in the sample.

Selection rules and acceptance probabilities are calculated using the following equation:

$$\hat{q}\left(j|X\right)=\frac{\sum_{i\in A_j}\sum_y K\left(\frac{X_{iy}-X}{h}\right)\cdot a_{ijy}}{\sum_{i\in A_j}\sum_y K\left(\frac{X_{iy}-X}{h}\right)}$$

where A_j is set of applicants and h is a bandwidth. For student characteristics that are continuous, I smooth over those states using Gaussian kernel and Silverman's rule of thumb bandwidth. For a student characteristic that is not continuous, in this case ethnicity, I use simple bin estimators. Notice that acceptance probabilities, q_{ij} for all i and j, are stationary, and hence not indexed by t.

Conditional Choice Probabilities Conditional choice probabilities are calculated for each slot. At different slots, two components of input change: one is additional conditioning variable, the rejection probabilities (ϱ_t), and the other is the sample of students used at each slot. As one goes down the slots, both the rejection probabilities, ϱ_t, and number of students who has rank order length of at least $t-1$ in the data diminish.

Conditional choice probabilities are calculated similarly using the following equation:

$$\hat{p_t}\left(j|X,\varrho_t\right)=\frac{\sum_{i\in I\setminus\left(\cup_i\{d_{i0\tau y}=1\}_{\tau=1}^{t-1}\right)}\sum_y K\left(\frac{X_{iy}-X}{h}\right)\cdot\mathbf{1}\left(d_{ijty}=0\right)}{\sum_{i\in I\setminus\left(\cup_i\{d_{i0\tau y}=1\}_{\tau=1}^{t-1}\right)}\sum_y K\left(\frac{X_{iy}-X}{h}\right)} \tag{4.12}$$

Here, I denote all conditional variables as X to simplify the notation. At the first slot, the conditional choice probabilities are calculated conditional on student characteristics only. At the second (and on) slot(s), the conditional choice probabilities are additionally conditioned on rejection probabilities, ϱ_t and only the students with their rank order list length at least as long as $t-1$, are included.

4.3.2 Second Stage: Valuations

Valuations are calculated using Equation 4.10. Exploiting the structure of optimal stopping problem, the valuations are calculated using the following equationusing only $\hat{q}\left(j|X_i\right)=\hat{q_{ij}}$,

$\hat{p_1}(j|X, \varrho_t)$, and $\hat{p_2}(j|X, \varrho_t)$:

$$\mu_{ij} = \frac{\ln[\hat{p_1}(j|X_1)] - \ln[\hat{p_1}(0|X_1)] - (1-\hat{q_{ij}})(-\ln[\hat{p_2}(0|X_i,(1-\hat{q_{ij}}))] + \gamma)}{\hat{q_{ij}}} \forall i, j$$

The formula follows from Equation 4.10 and insights follows from Equation 4.11.

4.4 Application to NYC

In this section, I take the data from the New York City high school matching market to the model with a specific goal of disentangling the role of acceptance probabilities in student choices. To do so, I need estimates for acceptance probabilities and student valuations. From the first stage estimates, acceptance probabilities are introduced not only to provide information about student choices, but also to build intuition for the valuations, the second stage estimate. Then, the second stage estimates are used to conduct an empirical exercise to see how student choices change in the absence of acceptance probabilities. Rest of the section describes the results at each step in detail (including model fit at each stage).

4.4.1 First Stage Estimates

Equation 4.11 provides intuitions for how valuations are constructed from the model. A student values a school (normalized to taking an outside option) higher if conditional choice probabilities are higher, conditional acceptance probabilities are lower, and conditional choice probability of choosing default school next period is lower. Each of the estimates (probabilities) play an important role in building valuations for the school, and are described in detail in this section.

Conditional acceptance probabilities not only explain large part of valuation[27], as expected from Equation 4.11, but also construct the weight that each choice carries in the value function (Equation 4.4). In the model specified in Section 4, the conditional acceptance probabilities measure the percentage by which each choice in the rank order list occupies utility. On

[27]Since acceptance probability is the denominator for the valuation function, valuation exponentially increase with low probability.

average, the first choice carries more than 50% of the weight and the weights for later choices decrease exponentially, indicating that the first choice is taken seriously by students.

From conditional choice probabilities, I select frequently chosen schools and call this set of schools a consideration set. If the conditional choice probability for a school is higher than choosing an outside option (if a school is chosen less frequently by students with similar characteristics than the default school option), then I select the school to the consideration set. Since valuations for the schools in this set are positive, in an event of changing acceptance probabilities (in the next section), these schools are considered as alternative choices. On average, there are about 25 schools in this set.

Acceptance Probability There are $I \times J$ number of such measure, so I present the result by different student groups. Continuing with the partitioning students by ROL length (similarly to Section **??**), Table 18 represents the mean acceptance probabilities for schools chosen by each group at each slot. Rows indicate groups with different rank order list length and columns indicate the schools chosen at each slot. Recall that students with longer list is, distributionally speaking, associated with lower housing price and lower academic performance, and etc. Students with longer list lengths tend to apply to schools that are more out of reach than those with shorter lists. Although school choice has opened up opportunities to apply to any schools in the city, 18 shows that admission (and thus final allocation) opportunities differ significantly across different student groups.[28]

Conditional acceptance probabilities are used to determine how each student values each schools. The conditional acceptance probabilities, which ranges from 0 to 1, is inversely related to (the denominator of the valuation function) valuation. In other words, if a student has chosen to use up the slot even if the acceptance probabilities are small, then it must mean that the student values the school highly. Following the logic, students with longer lists, also associated with lower socioeconomic status, tend to value their first choice schools highly compared to students with shorter lists, also associated with higher socioeconomic status.

[28] Media has paid close attention to such allocation result and Harris and Fessenden (2017) (in New York Times) is one of the most notable articles written.

Notice that valuations are in relation to the default option. A student with a default school which is not desirable will have many schools with comparatively high valuation. For those schools, even if the acceptance probabilities are low, a student will choose to apply.

Table 18: Estimated Acceptance Probabilities (Mean)

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.77											
2	0.48	0.66										
3	0.46	0.45	0.60									
4	0.44	0.42	0.44	0.55								
5	0.42	0.41	0.43	0.44	0.53							
6	0.40	0.40	0.41	0.43	0.44	0.51						
7	0.39	0.39	0.40	0.42	0.43	0.44	0.50					
8	0.38	0.38	0.39	0.41	0.42	0.43	0.45	0.49				
9	0.37	0.37	0.39	0.39	0.41	0.42	0.43	0.45	0.49			
10	0.37	0.37	0.39	0.39	0.40	0.42	0.42	0.44	0.45	0.49		
11	0.37	0.38	0.39	0.40	0.41	0.43	0.43	0.44	0.45	0.46	0.51	
12	0.38	0.38	0.39	0.40	0.41	0.43	0.43	0.44	0.45	0.46	0.47	0.51

In the model specified in Section 4, acceptance probabilities play one other important function: to determine how important each slot choice is in a student's overall utility. Suppose a student has submitted rank order list length of 4 and the student's acceptance probability for the first choice is .80. The later 3 choices only consist of at most 20% of the overall utility for a student, and hence will play much smaller role. Table 19 calculates such measures, mean acceptance probabilities (Equation 4.3) of receiving slot t choice. This measure can be interpreted as a utility weight for which a choice at slot t is responsible. As one goes down the slots at any length, probabilities of reaching to that point and being admitted is quickly converging to 0. Of course this is expected since the rejection rates from the previous slots are multiplied. However, it is worth noting that the rate at which the later choices loses its weight in one's utility functions is exponential. It seems that only the first few choices will have significant weight on a student's utility. [29]

I focus on the measures from the first slot only for a practical reason for the rest of the

[29] This provides additional support for using optimal stopping estimation methods using conditional choice probability estimates from the first two choices.

Table 19: Estimated Expected Probability of Receiving Slot N Allocation (Mean)

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.77											
2	0.48	0.34										
3	0.46	0.21	0.19									
4	0.44	0.20	0.12	0.11								
5	0.42	0.21	0.13	0.08	0.07							
6	0.40	0.21	0.13	0.08	0.06	0.05						
7	0.39	0.21	0.13	0.08	0.06	0.04	0.03					
8	0.38	0.20	0.13	0.08	0.06	0.04	0.03	0.02				
9	0.37	0.20	0.13	0.09	0.06	0.04	0.03	0.02	0.02			
10	0.37	0.20	0.13	0.08	0.06	0.04	0.03	0.02	0.02	0.01		
11	0.37	0.20	0.13	0.09	0.06	0.04	0.03	0.02	0.02	0.01	0.01	
12	0.38	0.20	0.13	0.08	0.06	0.04	0.03	0.02	0.02	0.01	0.01	0.01

section.[30] This paper aims at a particular exercise: to evaluate how much of student choices are driven by school side (i.e. acceptance probabilities). To see if a student choice changes when school side effect on decision making process is eliminated, acceptance probabilities for all schools are set to 1. Since a student only moves down the list when the student is rejected from the previous choices, when acceptance probabilities for the first slot is set to 1, the future value zeroes out. I only need estimates for the first slot.

Conditional Choice Probabilities: Construction of Consideration Sets When predicting whether a student will make a different choice given different acceptance probabilities, it is unlikely that students consider all 700+ available schools. I select a subset of schools with positive valuation, called a consideration set. Interpretation for this set is the following. For the first slot, a student looks around the neighborhood and see where students apply. If enough of the "neighbors" (students with similar characteristics) apply, I put the school in my consideration set.[31] In model terms, if the numerator of the Equation 4.10, then the school is in the consideration set. A student considers about 25 schools with standard deviation of about 25. Table 20 records the median and standard errors of the size of consideration set. This set is used in the next section.

[30] The model can estimate the rest of the slots.

[31] As more "neighbors" apply to a particular school, valuation for that school also increases.

Table 20: Median (Standard Deviation) Size of Consideration Set at Slot 1

Group by ROL Length	$\lvert$Consid. Set at $t = 1\rvert$
1	22 (25)
2	24 (22.7)
3	24 (22.9)
4	26 (24.1)
5	27 (25)
6	28 (25.9)
7	28 (26.1)
8	29 (26.7)
9	29 (26.4)
10	29 (27)
11	28 (27)
12	28 (27.6)

Note that in the model, slot 2 (and on) conditional choice probabilities, conditional acceptance probabilities, and hence consideration set resets. At slot t, depending on rejection probabilities from the previous choices,ϱ_t, all of the estimates differ. It is possible for a student to choose one school from a consideration set of 22 at $t = 1$ and choose to stop the list at $t = 2$.

4.4.2 Second Stage Estimates

Intuition for how the valuations look like can be deduced from Equation 4.11. For each student, the valuation are higher for the schools with higher conditional choice probabilities and lower acceptance probabilities. Valuations are in units of abstract utilities, called a util or utils, and is in relation to the default school of each student. For interpretability, economist often further parameterize this value. However, for the purpose of this exercise (with a specific goal of disentangling the role of acceptance probabilities in student choices), rather than interpreting individual student valuation, I simply use these valuations to arrive at the result for Section 4.4.4.

4.4.3 Model Fit

For acceptance probabilities, the model fit is calculated using Deferred Acceptance algorithm allocation outcome. In the model, since acceptance probabilities indicate selection rule for each schools, I consider acceptance probabilities as an estimate for how schools rank students. As inputs for Deferred Acceptance algorithm, student rank order list data and acceptance probabilities are used. The resulting allocation matched 83% of the data. For conditional choice probabilities, I select the school with the highest estimated conditional choice probabilities for the first choice, and it matches the first slot choice data with 81%. Finally, using the valuation and future values (conditional choice probability of choosing default school in the next period), the model's first slot choice matches the first slot data with 79%.

4.4.4 Student Choice to Changed Acceptance Probabilities

Finally, I ask the following question: how many students will change their choices if schools are not allowed to rank students? To answer the question, for each student, acceptance probabilities for all school are set equal (to 1).[32] I, then, compare to the model outcome (student choices) before manipulation of acceptance probabilities to the student choice after the manipulation.

I predict that most of the students (65%) change their first choices. In other words, about 35% of the population applied to their highest valued schools, or reported truthfully. Table 21 shows the result by partitioning students by their rank order list length from the data. Student groups with shorter list, especially students who only submits one school, respond much more sensitively (81% of the students changed their choice with ROL length 1) than student group with longer list respond (62% the students changed their choice with ROL length 12). Students who have submitted only one choice (in the data) seem to have already considered acceptance probabilities and reported their best response at the first slot. For other

[32]Exercise of setting all acceptance probabilities to 1 may sound naive and perhaps Utopian. However, this paper is not aimed at conducting aggregate welfare analysis. Instead, this paper focuses on student demand under a partial equilibrium setting. To make a claim on welfare, a full general equilibrium model considering both student demand and school demand has to be evaluated and aggregate welfare analysis is out of scope of this paper. Although out of scope of this paper, the model in Section 4 can be extended and the equilibrium is defined in Appendix.

Table 21: Proportion of Students Changing the Choice when Acceptance Probabilities

Group by ROL Length	% Change (Mean)	% Change (Std)
1	.85	.295
2	.735	.412
3	.719	.426
4	.706	.431
5	.704	.433
6	.701	.439
7	.698	.440
8	.701	.439
9	.704	.439
10	.697	.444
11	.684	.453
12	.682	.454

groups with longer lists, although larger proportion about 35% of the population is predicted to not change their choices even when they are allowed to be admitted to any schools that they want, indicating that this is their true preference, majority for these groups will still likely to change their choices when acceptance probabilities are not a concern.

4.5 Conclusion

This paper provides thorough description of the New York City high school matching market, documents systematic relationship between student rank order list choices and the matching outcomes, develops a model that captures this relationship, and finally applies model to the NYC to separate student choice driven by student demand from school's selection.

To receive a seat at the desired high school, students are allowed choose (take a shot at receiving an admission) up to 12 schools of their choice, but most rank fewer schools. Moreover, when not matched to one of the listed choices, students neither take an outside option nor are allocated randomly, and instead show clear preferences for the leftover schools. This paper attributes the reason for short rank order lists to acceptance probabilities based on the empirical finding that regardless of the rank order list, most (90%) students are matched to one of their choices. Since failing to account for the effect of acceptance probabilities on student choices can lead to under or over estimation of student valuation, I develop a computationally tractable model that captures this relationship.

Applying such model to the data, at the first slot, (1) a student considers about 25 schools on average (a result from the choice probabilities); (2) valuations of the schools that a student considers are inversely related to his or her acceptance probabilities (a result from solving the model); and (3) finally, in a scenario where schools are not allowed to rank, 65% of the population would respond differently (a result from a counterfactual analysis). Moreover, the student group with rank order list of length 1 responds the most sensitively to the acceptance probability change.

This exercise provides the intuition that a student who has chosen to apply to one school with a high acceptance probability, who is associated with high socioeconomic characteristics, are the most sensitive to acceptance probability for their first-choice school. It is important to understand student choice sensitivity in response to the change in school admission policy, especially since such policy change is discussed by the current Mayor of New York City, Bill de Blasio. Findings from the paper show that the response (in student rank order list choices, which ultimately result in the final allocation) to such policy change can differ greatly across various student groups.

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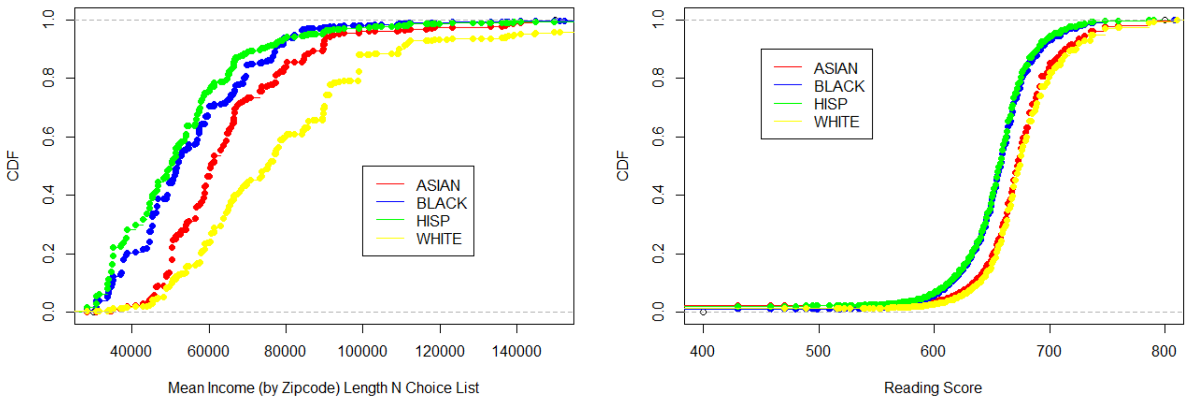

Figure A.1: Student Information Conditioned by Race

A Appendix for Chapter 1

A.1 Data

Data consists of students' zipcode (and corresponding median housing price), standardized test scores (math and English), and their attendance rate (number of absent days), and ethnicity.

Using the zipcode information, I match each student with median housing price (or median income) for his or her zipcode from the United States Census.

Travel time between students' zipcode to school was calculated using Google Maps at 7:30am using public transportation method.

For ease of interpretation, I have converted the standardized test scores of students into percentiles. For days absent, there are 180 school days in each school year. One of the requirements for passing to the next grade is 90% attdance (days absent 18 days or less). Median days absent for the market is 2, and mean is 10.

A.2 Figures

Below are list of figures not included in the main section of the paper:

Harry S Truman High School | 11X455

SCHOOL OVERVIEW

Our school is writing a new chapter in our history as the best-kept secret in the Bronx. We have taken the small school feeling and combined it with the variety of courses, diverse student body, and world-class facilities that you can only get in a large high school environment. Our students take core academic courses and commit to one of six career academies. Our 12th graders earn millions of dollars in scholarships every year to attend universities like Fordham, NYU, St. John's, and Penn State. Students ask 'Why would I go anywhere else?' It is a class-leading education, close to home. If it sounds too good to be true, come and visit for yourself.

❿ 10th Grade Seats Also Offered

ACADEMICS

- CTE program(s) in: Law and Public Safety and Computer Graphic Design
- Pre-Engineering, Law and Legal Studies/Law Enforcement Academy, Computer Technology, Culinary Arts, Multimedia Communication
- Air Force Junior Reserve Officers' Training Corps (AFJROTC), Summer Internship, Sports and Arts, Changing The Odds
- College courses through Mercy, Monroe, and Bronx Community Colleges

English Language Learner Programs: English as a New Language

Language Courses: American Sign Language, Spanish

Advanced Placement (AP) Courses: Biology, Calculus, Chemistry, English, Spanish, Statistics, Studio Art, US History

ACTIVITIES

After-School and Saturday Tutoring, Architecture, AFJROTC, Best Buddies, Chorus, Concert, Construction and Engineers Mentor Program for Future Engineers, Culinary Arts, Debate Team, Drill Team, Hip-Hop Dance, Law Team, Media Arts, National Honor Society, SAT Preparation, Robotics-FRC, Student Council, Swimming Club, Visual Art, Website Design and Maintenance, Weight Training, Yearbook, buildOn, Men of Strength, Gaming Club, Journalism, Creative Writing

PSAL Sports—Boys: Baseball, Basketball, Cross Country, Football, Indoor Track, Outdoor Track, Soccer, Tennis, Volleyball, Wrestling

PSAL Sports—Girls: Basketball, Cross Country, Flag Football, Indoor Track, Outdoor Track, Soccer, Tennis, Volleyball, Wrestling

PSAL Sports—Coed: Stunt

School Sports: Cheerleading, Stunt, Dance Team, Step Team, Girls Flag Football, Girls Softball, Swimming

SCHOOL INFORMATION

Neighborhood: Co-op City
Address: 750 Baychester Avenue, Bronx NY 10475
Shared Space: Yes (Campus: Harry S Truman Educational Campus)
Site Accessibility: Accessible
Phone: 718-904-5400 • **Fax:** 718-904-5502
Email: KAlfano@schools.nyc.gov
Website: www.bxtrumanhighschool.com
Subway: 5 to Baychester Ave
Bus: Bx12, Bx12-SBS, Bx23, Bx26, Bx28, Bx29, Bx30, Bx38, Bx5, BxM7, Q50

SCHOOL LIFE

Grade Span: 9-12
Total Students: 2,070
Typical 9th Grade Schedule: 8am-2:50pm
Community Service Expected • Extended Day Program • Health Center • Internship Expected • Internships • Multi-Session School • Online Grading System • Saturday Programs • Student/Parent Orientation • Summer Bridge Program • Weekend Programs

PERFORMANCE

63%	of students graduate in four years	42%	of students enroll in college or career programs
89%	student attendance	70%	of students feel safe in the hallways, bathrooms, locker room, and cafeteria
84%	of students feel that this school offers a wide enough variety of programs, classes, and activities to keep them interested in school		

Search 11X455 at schools.nyc.gov/accountability for more about this school's performance.

PROGRAM ADMISSIONS

Program Name	Code	Interest Area	Admissions Method	Prior Year Admissions: Type	Seats	Filled	Applicants	Applicants Per Seat	10th Grade Seats Offered
Law and Legal Studies/Law Enforcement Academy	X25B	Law & Government	Limited Unscreened	GE	118	N	330	3	Yes – 10
				SWD	30	N	89	3	
Computer Technology	X25C	Computer Science & Technology	Ed. Opt.	GE	165	N	421	3	Yes – 10
				SWD	42	N	147	4	

Law and Legal Studies/Law Enforcement Academy (X25B):

Admissions Priorities: ❶ Priority to Bronx students or residents who attend an information session—88% of offers went to this group ❷ Then to New York City residents who attend an information session ❸ Then to Bronx students or residents ❹ Then to New York City residents

Program Description: Experience law enforcement, mock trials, debates, and internships in our state-of-the-art law library and courtroom. Opportunities to network with law enforcement professionals. Affiliated with St. John's University; students may receive college credit.

Computer Technology (X25C):

Admissions Priorities: ❶ Priority to Bronx students or residents—98% of offers went to this group ❷ Then to New York City residents

Program Description: Website design, Power Point, publishing, the Internet and word processing. Customized learning individually paced to achieve mastery.

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Figure A.2: A Page from NYC High School Directory Booklet

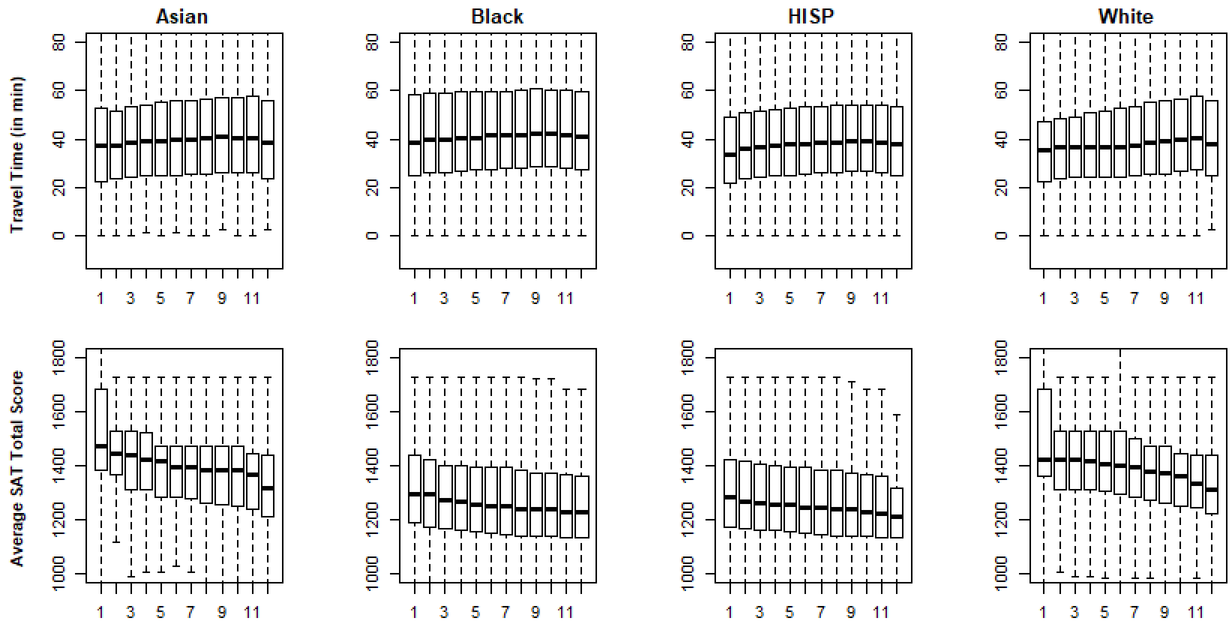

Figure A.3: Characteristics of Programs Chosen at Slot N.

Characteristics of Slot N Choices. I look further into the rank order lists by investigating characteristics of programs applied at slot N. Figure A.3 presents distribution of program characteristics chosen at slot N (I additionally divided by race). Application behavior divided by slots show that all students prefer to go to quality schools (earlier slots have higher college ready rate and decreases for later slots), but school quality of later choices for Whites and Asians doesn't decrease as sharply. Predicted acceptance probability to the top choices are slightly higher for White than the rest of the population. Lastly, all choices have around 40 minutes of travel time, on average.

B Appendix for Chapter 2

B.1 Alternative Choice Rules (Quota & Reserves)

This section examines more intuitive choice rules for quota and reserves in multi-dimensional case. However, it is found that those choice functions fail to satisfy GS, which is required for a stable matching. I will introduce those choice rules, and counter-examples to GS.

B.1.1 Quota

Each school chooses the highest-ranked students conditional on not exceeding any of the quotas. In this case, q^k is upper bounds on the number of students of type k.

Formally, choice function, C, is generated by quotas if there exists a vector with length $\sum_{t=1}^{T} |T_t|$ where elements of the vector is number of quota seats for all attributes at all type spaces such that for any $S \subseteq \mathcal{S}$,

1. there exist a strict priority $\succ$ over $\mathcal{S}$;
2. $|C\left(S^k\right)| \leq q^k$ for all $k \in T_1 \cup \cdots \cup T_T$;
3. if $s \in C(S)$, $s' \in S \backslash C(S)$, and s'$\succ$s, then it must be the case that $\tau(s) \neq \tau(s')$; and $|C\left(S^{\tau(s')}\right)| = q^{\tau(s')}$ for all attributes in $\tau(s')$ in each type spaces; and
4. if $s \in S \backslash C(S)$, then either $|C(S)| = q$ or $|C(S)^{\tau(s)}| = q^{\tau(s)}$for at least one attribute in $\tau(s)$.

If a choice function, C, is generated by quotas defined above, then I can find counterexamples where GS fails.

Counterexample. Suppose there are three students: $\{s_1, s_2, s_3\}$, and their types are:

$$\tau(s_1) = (\text{female}, \text{White})$$

$$\tau(s_2) = (\text{male}, \text{White})$$

$$\tau(s_3) = (\text{male}, \text{Black})$$

and a school's preference is $s_1 \succ s_2 \succ s_3$. Capacity of school is 2; $q = 2$. Quota for each types are given as follows: $q^{\text{male}} = q^{\text{female}} = q^{\text{White}} = q^{\text{Black}} = 1$.

Let $S = \{s_2, s_3\}$ and $S' = \{s_1, s_2, s_3\}$.

Then,$C(S) = \{s_2\}$ and $C(S') = \{s_1, s_3\}$. $s_3 \in C(S')$ but $s_3 \notin C(S)$, which proves that GS is not satisfied.$\square$

B.1.2 Diversity-Focused Reserve

A choice function, C, is generated by diversity-focused reserve if there exists a vector with length $\sum_{t=1}^{T} |T_t|$ where elements of the vector is number of seats reserved for all attributes at all type spaces such that for any $S \subseteq \mathcal{S}$,

1. there exist a strict priority $\succ$ over $\mathcal{S}$;
2. $|C\left(S^k\right)| \geq q^k \wedge |S^k|$[33] for all $k \in T_1 \cup \cdots \cup T_T$;
3. if $s \in C(S)$, $s' \in S \backslash C(S)$, and s'$\succ$s, then it must be the case that $\tau(s) \neq \tau(s')$ and $\sum_{k \in \tau(s)} I(k) > \sum_{k \in \tau(s')} I(k)$ where

$$I_k(k) = \begin{cases} 1 & \text{if } |C\left(S^k\right)| \leq q^k \\ 0 & \text{otherwise} \end{cases}$$

; and

4. if $\emptyset \neq S \backslash C(S)$, then $|C(S)| = q$

[33] $x \wedge y \equiv (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\})$

If a choice function, C, is generated by (diversity-focused version of) reserves defined above, then GS fails.

Counterexample. Suppose there are four students: $\{s_1, s_2, s_3, s_4\}$, and their types are:

$$\tau(s_1) = (\text{male}, \text{White})$$

$$\tau(s_2) = (\text{female}, \text{White})$$

$$\tau(s_3) = (\text{male}, \text{Black})$$

$$\tau(s_4) = (\text{female}, \text{Black})$$

and a school's preference is $s_1 \succ s_2 \succ s_3 \succ s_4$. Capacity of school is 2; $q = 2$. Reserve for each types are given as follows: $q^{\text{male}} = q^{\text{female}} = q^{\text{White}} = q^{\text{Black}} = 1$.

Let $S = \{s_2, s_3, s_4\}$ and $S' = \{s_1, s_2, s_3 s_4\}$.

Then,$C(S) = \{s_2, s_3\}$ and $C(S') = \{s_1, s_4\}$. $s_4 \in C(S')$ but $s_4 \notin C(S)$, which proves that GS is not satisfied.□

B.1.3 Preference-focused Reserve

A choice function, C, is generated by preference-focused reserve if there exists a vector with length $\sum_{t=1}^{T} |T_t|$ where elements of the vector is number of seats reserved for all attributes at all type spaces such that for any $S \subseteq \mathcal{S}$,

1. there exist a strict priority $\succ$ over $\mathcal{S}$;
2. $|C\left(S^k\right)| \geq q^k \wedge |S^k|$[34] for all $k \in T_1 \cup \cdots \cup T_T$;
3. if $s \in C(S)$, $s' \in S \backslash C(S)$, and s'$\succ$s, then it must be the case that $\tau(s) \neq \tau(s')$ and $|C\left(S^{\tau(s)}\right)| \leq q^{\tau(s)}$ for at least one $\tau(s)$; and
4. if $\emptyset \neq S \backslash C(S)$, then $|C(S)| = q$

[34] $x \wedge y \equiv (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\})$

If a choice function, C, is generated by (preference-focused version of) reserves defined above, then GS fails.

Counterexample. Suppose there are five students: $\{s_1, s_2, s_3, s_4, s_5\}$, and their types are:

$$\tau(s_1) = (\text{male}, \text{Black}, \text{low})$$

$$\tau(s_2) = (\text{male}, \text{White}, \text{high})$$

$$\tau(s_3) = (\text{female}, \text{White}, \text{high})$$

$$\tau(s_4) = (\text{female}, \text{Black}, \text{high})$$

$$\tau(s_4) = (\text{female}, \text{White}, \text{low})$$

and a school's preference is $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5$. Capacity of school is 3; $q = 3$. Reserve for each types are given as follows: $q^{\text{male}} = q^{\text{Black}} = q^{\text{low}} = 1$.

Let $S = \{s_2, s_3, s_4, s_5\}$ and $S' = \{s_1, s_2, s_3 s_4, s_5\}$.

Then,$C(S) = \{s_2, s_4, s_5\}$ and $C(S') = \{s_1, s_2, s_3\}$. $s_3 \in C(S')$ but $s_1 \notin C(S)$, which proves that GS is not satisfied.$\square$