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Diffusion Scaling of a Limit-Order Book: The Asymmetric Case

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Contents

Chapter 1

Introduction

1.1 Background

In the past 20 years, more and more traditional human traders have been replaced by automatic trading platforms. A recent report [27] estimates that 70 percent of global currencies trading volume was executed through electronic systems in 2013. Instead of gesturing and yelling to each other on the trading floor, traders use computers to accomplish trading through sending "orders" to the exchange, such as NYSE-ARCA, BATS or NASDAQ, and these orders wait to be executed in the "limit order book" (LOB).

Usually, orders are characterized by direction (buy or sell), price, amount (number of shares), and type (limit or market). Prices are multiples of the tick size, which is usually one cent. A limit order is an order to buy or sell a certain number of shares at a specified price, and it may not be executed if the given price cannot be met. Limit orders are accumulated in the LOB, which keeps a record of the quantities of limit orders at each price level. A market order is an order to buy or sell a certain number of shares immediately at the best available price in the LOB, and the LOB is updated once a market order is executed.

We can picture the LOB as a histogram where the horizontal axis indicates price ticks and each bar represents the number of limit order shares waiting at the corresponding price tick. We call a price the best bid price if it is the highest price at which there exists at least one limit buy order. Similarly, we call a price the best ask price if it is the lowest price at which there exists at least one limit sell order.

In real markets, besides market orders and limit orders, there are more complicated types of orders such as "iceberg" orders and "stop" orders. Iceberg orders are popular with investors who submit a large volume order. In order to avoid anticipatory action from other market participants, investors could conceal the full size of their orders by submitting iceberg orders which only publicly display a specified portion of the total order size. A stop order is an order to buy or sell an asset when its price surpasses a specified threshold, known as the stop price. In a liquid market, a stop order ensures that investors achieve a predetermined entry or exit price, limiting their loss or locking in profit. For example, an investor bought one share of stock A at $10 and now the stock is trading at $20. The investor can place a sell-stop order with stop price $15 to guarantee a profit of approximately $5 in case the price of stock A drops below $15.

In this thesis, we assume that the market only contains limit orders and market orders. For limit orders we only consider two-level arrivals: a Level I limit buy order is a limit buy order arriving at the price one tick below the best ask price, and a Level I limit sell order is a limit sell order arriving at the price one tick above the best bid price. Similarly, a Level II limit buy order is a limit buy order arriving at the price two ticks below the best ask price, and a Level II limit sell order is a limit sell order arriving at the price two ticks above the best bid price.

In modern financial markets, we usually see two different matching principles for order allocation: price-time priority and price/pro rata matching. In price-time priority, orders are automatically sorted according to price and time-of-entry criteria. Orders with the best possible prices always take precedence in the matching process over other orders with worse prices, and orders placed at the same price are executed according to the time of entry (i.e., first-in-first-out). In price/pro rata matching, the best priced orders in the book are still traded first, and when there are multiple orders at the best price, pro rata allocation allocates quantity of the incoming market order amongst all limit orders at the best price in the LOB. The allocation is proportional to the size of each limit order, and all limit orders at the best price are taken into account.

Given a certain amount of money, an investor might be interested in coming up with a strategy to maximize the expected value of his portfolio at the end of a pre-fixed time horizon, which is called an optimal execution problem. Obviously, the matching principle will be crucial since investors usually want their orders to get executed as soon as possible. We will not discuss the optimal execution problem in this document, and hence the priority of limit orders at individual prices does not matter now.

Another important feature of the LOB is that market participants are able without penalty to cancel their existing limit orders at any time before a match is made. According to Hautsch and Huang [13], more than 80% of all limit orders are cancelled before getting executed at NASDAQ. Because of its importance in real markets, cancellations occur in our limit order book model.

1.2 Our Contribution

The nature of the LOB's execution mechanism calls for a reasonable stochastic model which is consistent with statistical observations of the LOB in real markets. Since limit orders will wait in the queue at the specified price until they are executed against opposite market orders or are cancelled, we adapt queueing theory methodology to study the evolution of the LOB. We consider a “zero-intelligence Poisson” model, where “zero-intelligence” means there is no strategic play by the agents submitting orders, and “Poisson" refers to the fact that arrivals of market orders and limit orders are governed by Poisson processes. Moreover, we assume exponentially distributed waiting times before cancellations.

Our LOB model is a discrete-event system, and our goal is to approximate the discrete system by a system of Brownian Motions. Enlightened by heavy traffic theory, we can take a diffusion scaling of our zero-intelligence Poisson model, and take a limit to find the approximating system. In particular, at each price tick, we will define a process which refers to the number of orders at this price. Then we scale the process by accelerating time by a factor n, and dividing volume by $\sqrt{n}$, then pass to the limit as $n \to \infty$. We want to consider the simplest nontrivial model so that we can develop this methodology.

In our zero-intelligence Poisson model, we assume Poisson arrivals of limit orders and market orders with constant rates. All orders have the same size, which we set to be 1. Moreover, cancellations will occur two or more ticks away from the best available price. In particular, we assume that cancellations of limit buy orders come with Poisson arrival rate $\theta_b/\sqrt{n}$ per order in the n^{th} pre-limit model, where θ_b is a constant cancellation intensity. Similarly, cancellations of limit sell orders come with Poisson arrival rate $\theta_s/\sqrt{n}$ per order, where θ_s is a constant cancellation intensity.

Figure 1.1 is an example of a state of our LOB model in which positive queues represent limit buy orders and negative queues represents limit sell orders. Labels S to Y indicate the number of orders at corresponding prices. From the configuration in Figure 1.1, we see that the best bid price is at the $V - tick$ and the best ask price is at the $W - tick$. Therefore, according to the assumptions of our model, market buy orders will arrive at the $W - tick$ with a constant intensity λ_0; Level I limit buy orders will arrive at the $V - tick$ with a constant intensity λ_1; Level II limit buy orders will arrive at the $U - tick$ with a constant intensity λ_2. Similarly, market sell orders and two-Level limit orders will arrive at the $V - tick$, $W - tick$, and $X - tick$ with constant intensities μ_0, μ_1, and μ_2, respectively. Finally, cancellations of limit buy orders will happen at the $S - tick$ and $T - tick$, which are two or more ticks below the best bid price $V - tick$, and cancellations of limit sell orders will be at the $Y - tick$.

In order to obtain a diffusion-scaled limit, the arrival rates for market orders and limit orders must satisfy some appropriate technical conditions which will be explained later. Our study shows that the limiting LOB model has a two-tick spread at Lebesgue-almost-all time, i.e., there is always a price tick between the best bid price and the best ask price. Also, the queue at the best bid price and the queue at the best ask price are Brownian Motions. The convention in our model is that queues with buy orders have positive sign and queues with sell orders have negative sign, and with this sign convention, the queue at the best bid and queue at the best ask have positive correlation. The absolute values of these queues have negative correlation. Moreover, the queues which are adjacent to the best bid and the best ask are nailed at constant levels, and queues that are further away will be killed at zero level. See Figure 1.2.

The proof comes from ideas in queueing theory with a delicate analysis involving weak convergence in corresponding cádlág space with M_1 and sometimes J_1 topology.

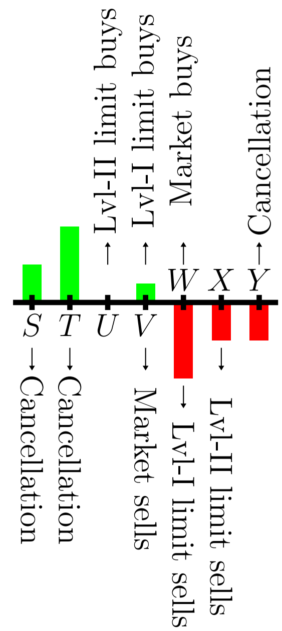

Figure 1.1: Limit-order book

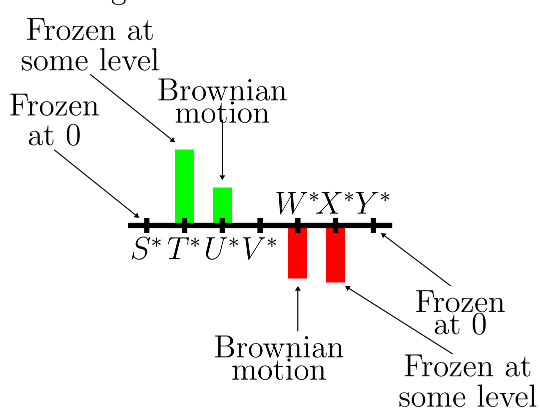

Figure 1.2: Limiting model 1

Also, the proof of the uniqueness of the limiting process uses a theorem that if a process and its absolute value are governed by two independent Brownian motions with a certain formula, then this process is unique in distribution. We will discuss this theorem and prove it later.

The limiting model is an approximation to the pre-limit model, so we should fit the LOB data to pre-limit models. To be precise, in Figure 1.2, the queue at the $V^* - tick$ having length zero means that in pre-limit models, the number of limit orders at this price is relatively small compared to the the number of orders at adjacent queues, but the number of limit orders at the $V^* - tick$ is not necessarily zero. For instance, suppose we want to use our limiting model to approximate the 100^{th} pre-limit model. Then $\sqrt{n} = 10$ and $V^* = 0$ means that the number of orders at $V^* - tick$ in the pre-limit model has a smaller order of magnitude than the number of orders at $U^* - tick$ and $W^* - tick$, i.e.,

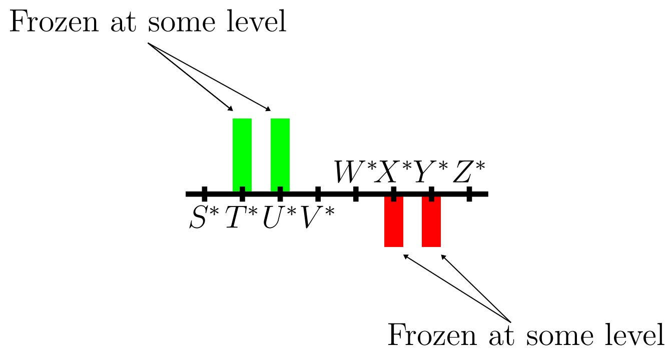

Figure 1.3: Renewal State

there might exist two or three limit buy orders at the $V^* - tick$ and 20 limit sell orders at the $W^* - tick$. Therefore, even though in the limiting model there is a two-tick spread almost all the time, the limit of the amount of time that there is a one-tick spread in the pre-limit models is not zero, which will be proved in (3.44).

Since both U^* and W^* are Brownian Motions, from the configuration shown in Figure 1.2, the limiting model will eventually reach a configuration in which either U^* or W^* hits zero. At this moment, the limiting model has a three-tick spread. The configuration in Figure 1.3 shows the this state when W^* vanishes before U^*. Following this moment, the process U^* will jump from its current position to a fixed level instantaneously, which will be proved later. We call this state a "renewal state" because this state will appear repeatedly in the limiting model. As in Figure 1.3, the system has a three-tick spread and the queues at the best bid price and the next best bid price are fixed at a constant level. Similarly, the queues at the best ask price and the next best ask price are fixed at another constant level, and all queues which are two or more ticks away from the best price are nailed at zero.

Starting from the renewal state in Figure 1.3, W^* might go on a negative excursion and U^* will behave like a Brownian motion positively correlated with W^*. If U^* reaches zero before W^* returns to zero, we will come to the state shown on the left of Figure 1.4. Similarly, starting from Figure 1.3, V^* has a chance to go on a positive excursion and X^* will behave like a Brownian motion positively correlated with V^*. If X^* reaches zero before V^* returns to zero, we will come to the state shown on the right of Figure 1.4. Hence, the limiting model will eventually reach one of the two adjacent renewal states. We say the process has a leftward renewal state transition if the process moves from Figure 1.3 to the left configuration in Figure 1.4, and it has a rightward renewal state transition if it moves to the right configuration in Figure 1.4. By applying Poisson random measure theory, we calculate the probability the limiting model makes a leftward transition and the probability of a rightward transition.

When the limiting model is not in a renewal state, the system will have a configuration like the one shown in Figure 1.2. Using Metzler [23], we are able to compute the joint density of the first passage times of U^* and W^* to zero given such a configuration. In particular, we can derive the density function of the waiting time from any intermediate

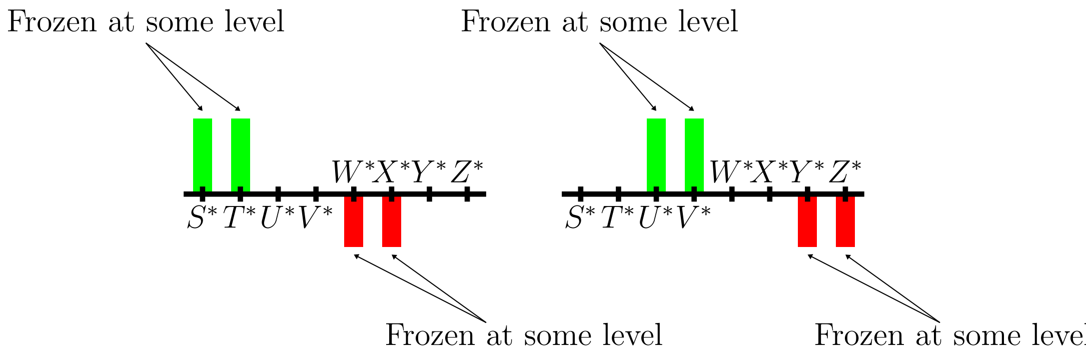

Figure 1.4: Two adjacent renewal states

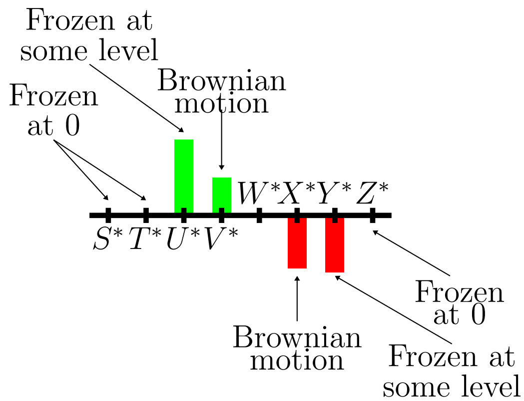

Figure 1.5: Limiting model 2

state to the next renewal state. Furthermore, by extending Metzler's result, we can also compute the characteristic function of the waiting time between two renewal states conditioned on the system making a leftward transition or rightward transition. This result shows that the process of renewal states is a semi-Markov process because the length of each queue in the renewal state is fixed (not dependent on the path of the process) and the waiting time between renewal states is not exponentially distributed.

Starting from the configuration in Figure 1.3, if we focus on the evolution of the process U^*, we can see that U^* behaves like a Brownian motion while W^* is nonzero, as shown in Figure 1.2, and it will jump from its current position to some fixed level whenever W^* reaches zero (renewal state). On the other hand, if V^* has positive orders, U^* will remain at that fixed level until V^* vanishes and W^* again becomes negative, see Figure 1.5. Eventually, we can show that the system will switch between these two configurations infinitely many times immediately following the moment the system reaches the renewal state shown in Figure 1.3. Therefore, we will see infinitely many jumps in U^*, and we shall see that those jumps are not absolutely summable. A consequence of this is that the process U^* is not a semi-martingale; the jumps cannot be embedded in a finite variation part nor a local-martingale part of a decomposition of U^*.

1.3 Related Literature

The need for modeling the LOB is recent since the earliest electronic trading platform was launched only twenty years ago. The literature on LOBs has grown rapidly as a have deeper understanding of the LOB trading process has developed. However, there is still no model that fully captures all significant features of electronic trading. Zero-intelligence models assume that order arrivals and cancellations are governed by stochastic processes whose parameters can be estimated directly from historical data. These are widely used because the falsifiable hypotheses can be tested through the comparison between the models' output and real data. One of the earliest work regard this is Garman [11], who uses a Poisson model to describe arrivals of buy and sell orders. In the second part of the paper, a LOB is constructed when there is no market maker. Under the assumption of Poisson arrivals of buy and sell orders, Luckock [22] constructs a continuous double auction model which yields a steady-state probability distribution for best bid and best ask. Smith, Farmer, Gillmemot, & Krishnamurthy [25] assumes exponential waiting time before cancellations and make testable predictions for basic properties of markets, such as price volatility, market depth, and bid-ask spread by using simulation. On a higher level, their work suggestes that a zero-intelligence model is useful to make strong predictions about the market. The idea of using a zero-intelligence model for LOB is also examined and checked against statistical analysis of historical data by Gould *et al* [12].

Two very closely related papers to ours are that of Cont, Stoikov, & Talreja [8] and Cont & Larrard [7]. Under the same setting of [25], [8] models the LOB as a finite-dimensional continuous time Markov chain and uses Laplace transforms to compute probabilities of basic events, such as the mid-price movement, and the execution of a market order before the best price moves. [7] proposes a simpler model in which there is always a one-tick spread and limit orders arrive only at the best available price. Their system has a price change once one of two queues at a best price is depleted, and the system shifts in the appropriate direction by one tick before the book is reinitialized. By applying the heavy traffic theory, they derive the diffusion-scaled limit of the LOB. We will use similar queueing theory techniques to develop a diffusion limit of the LOB. However, in contrast to [7], our model will work under a more general setting in which limit orders have 2-level depth and we do not have any assumption on the width of the spread. Moreover, our model does not reinitialize when the system reaches a renewal state.

Note that our LOB model assumes a discrete price grid, and we choose to scale the arrival rates and volumes of both standing and incoming orders. Alternatively, there are some literature which establishes the joint convergence of prices and volumes. Horst and Kreher [14] prove a scaling limit for a full LOB. Under their choice of scaling, the dynamics of volumes converge to two non-linear PDEs coupled with two non-linear ODEs which describe the limit of prices. The same technique is also used in papers by Horst with coauthors Kreher [15], Paulsen [16], and Xu [17]. Lakner, Reed, and Stoikov [21] and Lakner, Reed, and Simatos [20] assume limit orders are placed on the book according to a distribution which varies depending on the current best price, and they derive the limit of scaled measure-valued LOB process in the high frequency regime. Despite the fact that Poisson dynamics are widely used to model order flow in LOB, exceptions to this are papers by Abergel and Jedidi [1], where Hawkes-type dynamics are used, and Yang and Zhu [28], where Cox processes are used for arrivals and cancellations at the best bid and ask and the intensities depend on the order book imbalance. Another paper related to our work is Avellaneda, Reed, and Stoikov [3], where they use a modified model of [7] and assume that there is hidden liquidity at the best bid and ask because of iceberg orders or liquidity present at other exchanges. Through diffusion scaling technique, they derive closed-form solutions for the probability of a price uptick conditional on the sizes of best bid and ask.

Besides the zero-intelligence order book modeling, some have proposed a microeconomic approach, which models the evolution of the LOB as a result of interactions between rationally behaving agents. Instead of acting randomly, each agent makes decision strategically to maximize his own utility function. Follmer and Schweizer [10] first derives the diffusion approximation for stock prices by applying an invariance principle to a sequence of discrete-time-models resulting from market equilibrium. On top of a microstructure model, Bayraktar, Hoerst, & Sircar [4] adds inert investors to the market and shows that their presence creates long-range dependence in the macro model.

The work most closely related to our model is the thesis of Christopher Almost [2]. Following a simplified specification of the dynamics proposed in [8] where orders arrive according to Poisson processes whose intensities depend on the relative distance between price of arriving and opposite best price, [2] shows that the heavy-traffic scaled sequence of LOB models converges to a simple model where the scaled number of orders at each price level follows either a diffusion or a jump-diffusion process. In particular, by applying the "crush" argument of Peterson [24], [2] proves that the limiting model has a "two-tick" bid-ask spread, i.e., the distance between best bid and best ask is two price ticks, and the processes of best bid and best ask follow a pair of correlated Brownian motions. One assumption [2] makes during the derivation is that the model is symmetric where buy and sell orders of the same types arrive at the same rate. Moreover, the ratio of arrival rate of market order to that of limit order is assumed to be fixed in order to get the limit. This work is a generalization and extension of [2]. We first relax the "symmetric" assumption to the "asymmetric" case in which the model takes six parameters for arrival rates of market buys and market sells, as well as limit buys and limits sells on both Level-I and Level-II. We prove that the model allows three degrees of freedom among the six parameters in order to get a diffusion-scaled limit. We follow the same "crushing" technique used in [2] to derive the limit of both interior queues and bracketing queues. Different from [2] in which the interior queues of LOB converge to a *split Brownian motion*, we show that in the asymmetric LOB model, they converge to a *split two-variance Brownian motion* which is defined and closely studied in Chapter 4. We then extend [2] by discussing the waiting time between two different renewal states, which results in a price change in the limiting model. By applying P.Lévy's theory of Brownian local time, we calculate the probability of upward price movement and then derive the distribution of waiting time between two different renewal states.

Chapter 2

The zero-intelligence Poisson model

In this section we provide a detailed a description of our zero-intelligence Poisson model and state the the main results of its diffusion-scaled limit.

2.1 Arrivals of orders and their cancellations

In our model prices are multiples of the tick size, and we assume the model has a doubly infinite price tick grid. We also assume there are four types of orders: *market buys*, *market sells*, *limit buys* and *limit sells*, and each of these orders is of size 1. The state of the LOB is determined by the number of limit orders queued at each price tick. We use a set of histograms over price ticks to represent the state of the book: positive bars indicating the number of limit buys waiting to be executed at corresponding price ticks and negative bars indicating the number of limit sells waiting to be executed at corresponding price ticks. The limit buys are queued at strictly lower price ticks than the limit sells. The reason for this is that we assume investors are reasonable and they will not send a limit buy at a price higher than or equal to the price at which someone wants to sell. We call a price the best bid price if it is the highest price at which there exists at least one limit buy order. Similarly, we call a price the best ask price if it is the lowest price at which there exists at least one limit sell order.

We assume agents in the market do not engage in strategic play when submitting orders, and the arrivals of orders are Poisson processes. In particular, we assume the arrivals of market orders and limit orders are governed by the following rules until the LOB does not contain any limit buys or limit sells. The rate and the direction are indicated by parameters and arrows in Figure 2.1.

- Market buys: These orders arrive at the best ask price and arrivals occur at the jump times of a Poisson process with intensity $\lambda_0 > 0$. The arrival of a market buy order executes a limit sell order queued at the best ask price and thereby increases the queue length (reduces the number of limit sell orders in the queue, which is the absolute value of the queue length) at the best ask price by one unit.

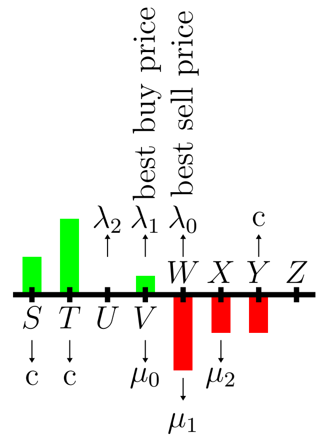

Figure 2.1: Limit-order book

- Level-I limit buys: These orders arrive at one tick below the best ask price and arrivals occur at the jump times of a Poisson process with intensity $\lambda_1 > 0$. These orders queue for later execution or cancellation. They increase the queue length at the price tick where they arrive.

- Level-II limit buys: These orders arrive at two ticks below the best ask price and arrivals occur at the jump times of a Poisson process with intensity $\lambda_2 > 0$. These orders queue for later execution or cancellation. They increase the queue length at the price tick where they arrive.

- Market sells: These orders arrive at the best bid price and arrivals occur at the jump times of a Poisson process with intensity $\mu_0 > 0$. The arrival of a market sell order executes a limit buy order queued at the best bid price and thereby decreases the queue length at the best bid price by one unit.

- Level-I limit sells: These orders arrive at one tick above the best bid price and arrivals occur at the jump times of a Poisson process with intensity $\mu_1 > 0$. These orders queue for later execution or cancellation. They decrease (make more negative) the queue length at the price tick where they arrive.

- Level-II limit sells: These orders arrive at two ticks above the best bid price and arrivals occur at the jump times of a Poisson process with intensity $\mu_2 > 0$. These orders queue for later execution or cancellation. They decrease (make more negative) the queue length at the price tick where they arrive.

- The six Poisson processes controlling the arrivals of market and limit buy and sell orders are independent.

In addition to arrivals of these four types of orders the state of the LOB might change by cancellations. When each limit order arrives, it is assigned an exponentially distributed *patience random variable*. Whenever a limit buy order is two or more ticks below the best bid price, its *cancellation clock* runs and if the cancellation clock reaches the value of the patience random variable for that order, the order is cancelled and therefore removed from the LOB. Similarly, whenever a limit sell order is two or more ticks above the the best ask price, its *cancellation clock* runs and if the cancellation clock reaches the value of the patience random variable for that order, the order is cancelled and therefore removed from the LOB. In particular, the cancellations are governed by the following rule, and they are indicated by the arrows labelled with "c" in Figure 2.1 in which the directions of the arrows show show the movement of the LOB.

- Cancellations of limit buys: The patience random variable associated to limit buys is exponentially distributed with mean $1/\theta_b$ where θ_b is a positive constant *cancellation rate* for limit buys.
- Cancellations of limit sells: The patience random variable associated to limit sells is exponentially distributed with mean $1/\theta_s$ where θ_s is a positive constant *cancellation rate* for limit sells.
- The patience random variables associated with different limit orders are independent of one another and also independent of six Poisson random processes controlling the arrivals of market orders and limit orders.

2.2 A sequence of pre-limit models

Our goal is to find a diffusion-scaled limit of the LOB. We consider a sequence of LOB models indexed by positive integers $n = 1, 2, \ldots$.. In the n^{th} model, arrivals of market orders and two-level limit orders are still governed by the six independent Poisson processes, and each of the orders is of size 1. However, the cancellation rates for limit buys and limit sells are scaled down by a factor of $1/\sqrt{n}$. In particular, the mean of the patience random variable for limit buys is $\sqrt{n}/\theta_b$, and the mean of the patience random variable for limit sells is $\sqrt{n}/\theta_s$. To perform the diffusion scaling, we scale the n^{th} pre-limit model by accelerating the time by a factor n, and dividing the volumes by $\sqrt{n}$, then pass to the limit as $n \to \infty$. In order to obtain a diffusion-scaled limit for the sequence of pre-limit models just described, we need the following assumption on the parameters of order arrivals.

Assumption 2.2.1 *There are two numbers $a > 1$ and $b > 1$ satisfying $a + b > ab$ such that*

$$\begin{aligned}
\lambda_1 &= (a-1)\lambda_0, \\
\lambda_2 &= (a+b-ab)\lambda_0, \\
\mu_1 &= (b-1)\mu_0, \\
\mu_2 &= (a+b-ab)\mu_0, \\
a\lambda_0 &= b\mu_0.
\end{aligned}$$

An immediate property of Assumption 2.2.1 that we shall use repeatedly is

$$c := \mu_0 - \lambda_1 = \lambda_0 - \mu_1 = (a+b-ab)\frac{\lambda_0}{b} > 0. \tag{2.1}$$

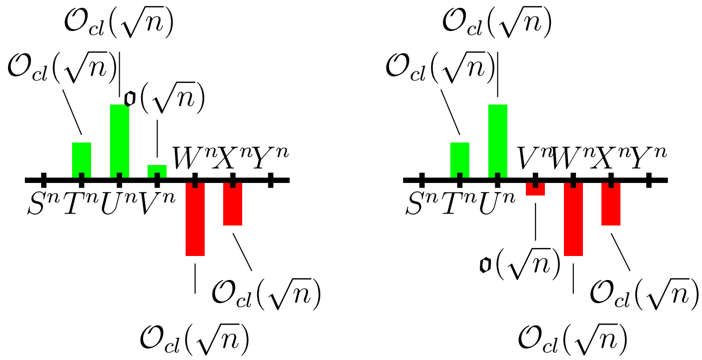

Figure 2.2: Initial configurations of our model

2.3 Interior queues and bracketing queues

Although the price ticks are doubly infinite, we focus on the price ticks centering at the best bid price and the best ask price. We assume that in the n^{th} pre-limit model, the LOB has initial configurations like those shown in Figure 2.2. Here, $\mathcal{O}_{cl}$ and $\mathfrak{o}$ indicate the usual big O and little o notations and we denote the number of orders queued at these ticks at time t by $S^n(t)$, $T^n(t)$, $U^n(t)$, $V^n(t)$, $W^n(t)$, $X^n(t)$, and $Y^n(t)$. In the left configuration, we see few limit buy orders sitting at the best bid price, and significantly more orders at the next best bid price and the best ask price. Similarly in the right configuration, there are few limit sell orders at the best ask price, and many more orders at the next best ask price and the best bid price. In both configurations we see an obvious difference between the number of orders at the best bid price and the number of orders at the best ask price, and we call this difference the *imbalance* of the LOB. Real markets frequently exhibit imbalances in the LOB. Therefore it is natural to assume that our model starts from one of these two configurations. We want to study the evolution of the LOB within a period when the best bid price is higher than or equal to the price tick of T^n and the best ask price is lower than or equal to the price tick of X^n. We observe that the dynamics of the LOB is determined by the locations of the best bid price and the best ask price. According to the rule of arrivals of orders we previously mentioned, the best bid price should be strictly lower than the best ask price. Hence, there are ten possible scenarios where the best bid price and the best ask price could be, and these are shown in Figure 2.3. The arrows indicate the directions of queues' movements while the parameters indicate the rates of arrivals.

Within each scenario shown in Figure 2.3, T^n, U^n, V^n, W^n, and X^n have the same dynamics. From the previous description of our model, we see that there are six independent, unit-intensity Poisson processes governing the arrivals of market orders and limit orders. We label these six Poisson processes by N_{MB}, N_{MS}, N_{LB1}, N_{LB2}, N_{LS1}, and N_{LS2}, where MB indicates "market buy", MS indicates "market sell", $LB1$ indicates "level-I limit buy", $LB2$ indicates "level-II limit buy", $LS1$ indicates "level-I limit sell", and $LS2$ indicates "level-II limit sell". Moreover, the cancellations of both limit buy orders and limit sell orders are governed by four independent Poisson processes, which are denoted by N_{CB2}, N_{CB3}, N_{CS2} and N_{CS3} where $CB2$ indicates cancellations of limit buy orders which

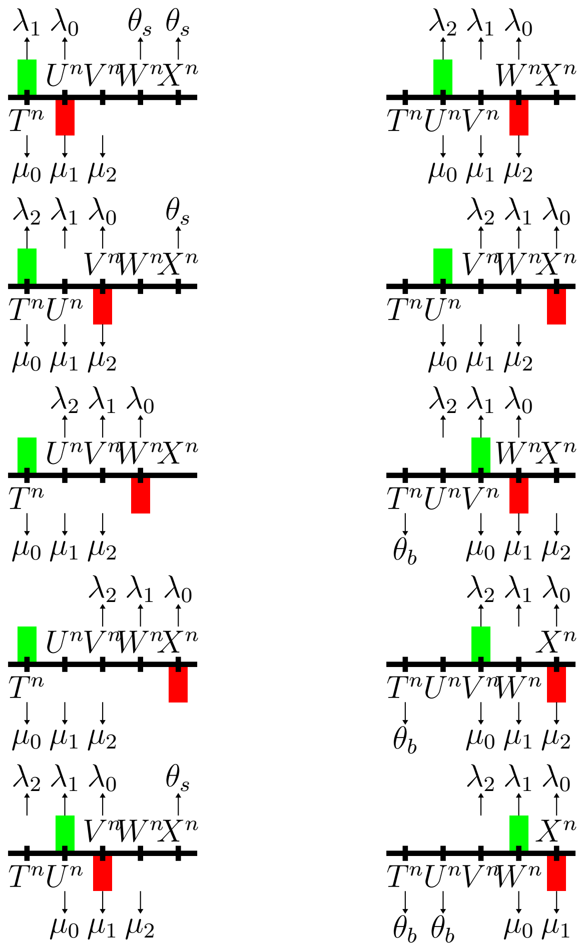

Figure 2.3: Ten possible dynamics for $(T^n, U^n, V^n, W^n, X^n)$

are two ticks away from the best bid price, $CB3$ indicates cancellations of limit buy orders which are three ticks away from the best bid price, $CS2$ indicates cancellations of limit sell orders which are two ticks away from the best ask price, and $CS3$ indicates cancellations of limit sell orders which are three ticks away from the best ask price. We also introduce two stochastic processes $p_b : [0, \infty) \to \{T, U, V, W, X\}$ and $p_a : [0, \infty) \to \{T, U, V, W, X\}$ where $p_b(t)$ tells where the best bid price is and $p_s(t)$ tells where the best ask price is at time $t \geq 0$. Then we can explicitly write down the dynamics for T^n, U^n, V^n, W^n, and X^n as follow,

$$
\begin{aligned}
dT^n(t) &= d\Bigg(\mathbb{1}_{\{p_b(t)=T, p_s(t)=U\}} N_{LB1} \circ \lambda_1 t + (\mathbb{1}_{\{p_b(t)=T, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=V\}}) N_{LB2} \circ \lambda_2 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=T, p_s(t)=U\}} + \mathbb{1}_{\{p_b(t)=T, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=T, p_s(t)=W\}} \\
&\quad + \mathbb{1}_{\{p_b(t)=T, p_s(t)=X\}}) N_{MS} \circ \mu_0 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=V, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V, p_s(t)=X\}}) N_{CB2} \circ \int_0^t \frac{\theta_b}{\sqrt{n}} \big(T^n(s)\big)^+ \, ds \\
&\quad - \mathbb{1}_{\{p_b(t)=W, p_s(t)=X\}} N_{CB3} \circ \int_0^t \frac{\theta_b}{\sqrt{n}} \big(T^n(s)\big)^+ \, ds \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
dU^n(t) &= d\Bigg(\mathbb{1}_{\{p_b(t)=T, p_s(t)=U\}} N_{MB} \circ \lambda_0 t + (\mathbb{1}_{\{p_b(t)=T, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=V\}}) N_{LB1} \circ \lambda_1 t \\
&\quad + (\mathbb{1}_{\{p_b(t)=T, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V, p_s(t)=W\}}) N_{LB2} \circ \lambda_2 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=U, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=X\}}) N_{MS} \circ \mu_0 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=T, p_s(t)=U\}} + \mathbb{1}_{\{p_b(t)=T, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=T, p_s(t)=W\}} \\
&\quad + \mathbb{1}_{\{p_b(t)=T, p_s(t)=X\}}) N_{LS1} \circ \mu_1 t \\
&\quad - \mathbb{1}_{\{p_b(t)=W, p_s(t)=X\}} N_{CB2} \circ \int_0^t \frac{\theta_b}{\sqrt{n}} \big(U^n(s)\big)^+ \, ds \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
dV^n(t) &= d\Bigg((\mathbb{1}_{\{p_b(t)=T, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=V\}}) N_{MB} \circ \lambda_0 t \\
&\quad + (\mathbb{1}_{\{p_b(t)=T, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V, p_s(t)=W\}}) N_{LB1} \circ \lambda_1 t \\
&\quad + (\mathbb{1}_{\{p_b(t)=T, p_s(t)=X\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=X\}} + \mathbb{1}_{\{p_b(t)=V, p_s(t)=X\}} \\
&\quad + \mathbb{1}_{\{p_b(t)=W, p_s(t)=X\}}) N_{LB2} \circ \lambda_2 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=V, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V, p_s(t)=X\}}) N_{MS} \circ \mu_0 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=U, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=U, p_s(t)=X\}}) N_{LS1} \circ \mu_1 t \\
&\quad - (\mathbb{1}_{\{p_b(t)=T, p_s(t)=U\}} + \mathbb{1}_{\{p_b(t)=T, p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=T, p_s(t)=W\}} \\
&\quad + \mathbb{1}_{\{p_b(t)=T, p_s(t)=X\}}) N_{LS2} \circ \mu_2 t \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
dW^n(t) \;=\; d\Big(& (\mathbb{1}_{\{p_b(t)=T,p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=U,p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V,p_s(t)=W\}}) N_{MB} \circ \lambda_0 t \\
& + (\mathbb{1}_{\{p_b(t)=T,p_s(t)=X\}} + \mathbb{1}_{\{p_b(t)=U,p_s(t)=X\}} + \mathbb{1}_{\{p_b(t)=V,p_s(t)=X\}} \\
& + \mathbb{1}_{\{p_b(t)=W,p_s(t)=X\}}) N_{LB1} \circ \lambda_1 t \\
& - \mathbb{1}_{\{p_b(t)=W,p_s(t)=X\}} N_{MS} \circ \mu_0 t - (\mathbb{1}_{\{p_b(t)=V,p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V,p_s(t)=X\}}) N_{LS1} \circ \mu_1 t \\
& - (\mathbb{1}_{\{p_b(t)=U,p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U,p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=U,p_s(t)=X\}}) N_{LS2} \circ \mu_2 t \\
& + \mathbb{1}_{\{p_b(t)=T,p_s(t)=U\}} N_{CS2} \circ \int_0^t \frac{\theta_s}{\sqrt{n}} \big(W^n(s)\big)^- \, ds \Big),
\end{aligned}
$$

$$
\begin{aligned}
dX^n(t) \;=\; d\Big(& (\mathbb{1}_{\{p_b(t)=T,p_s(t)=X\}} + \mathbb{1}_{\{p_b(t)=U,p_s(t)=X\}} + \mathbb{1}_{\{p_b(t)=V,p_s(t)=X\}} \\
& + \mathbb{1}_{\{p_b(t)=W,p_s(t)=X\}}) N_{MB} \circ \lambda_0 t - \mathbb{1}_{\{p_b(t)=W,p_s(t)=X\}} N_{LS1} \circ \mu_1 t \\
& - (\mathbb{1}_{\{p_b(t)=V,p_s(t)=W\}} + \mathbb{1}_{\{p_b(t)=V,p_s(t)=X\}}) N_{LS2} \circ \mu_2 t \\
& + \mathbb{1}_{\{p_b(t)=T,p_s(t)=U\}} N_{CS3} \circ \int_0^t \frac{\theta_s}{\sqrt{n}} \big(X^n(s)\big)^- \, ds \\
& + (\mathbb{1}_{\{p_b(t)=T,p_s(t)=V\}} + \mathbb{1}_{\{p_b(t)=U,p_s(t)=V\}}) N_{CS2} \circ \int_0^t \frac{\theta_s}{\sqrt{n}} \big(X^n(s)\big)^- \, ds \Big).
\end{aligned}
$$

In n^{th} pre-limit model, let us define the stopping time

$$\sigma^n := \min\{s \geq 0 : U^n(s) = 0 \text{ or } W^n(s) = 0\}, \tag{2.2}$$

which is the first time when either U^n or W^n hits zero. In chapter 3, we will study the evolution of the n^{th} pre-limit model before it reaches σ^n. In particular, we will derive the diffusion scaled limit of $(T^n, U^n, V^n, W^n, X^n)$ till σ^n. When this happens, we say the n^{th} pre-limit model reaches the second renewal state at σ^n.

In chapter 4, we will construct a stochastic process called *two-variance Brownian motion* through Poisson random measures, and introduce some results about excursions of Brownian motion, which will be applied in the proofs of following chapters.

Let us also define the stopping time

$$\tilde{\sigma}^n := \min\{s > \sigma : U^n(s) = 0 \text{ or } W^n(s) = 0\}. \tag{2.3}$$

Without loss of generality, we assume $W^n(\sigma^n) = 0$, and between σ^n and $\tilde{\sigma}^n$, we call (U^n, X^n) *bracketing processes*, and (V^n, W^n) *interior processes*. In chapter 5, we will first study the evolution of the interior processes (V^n, W^n) between the first renewal state and the second renewal state and prove the diffusion scaled interior processes weakly converge under the J_1 topology to a *split two-variance Brownian motion*. We then discuss the convergence of the bracketing processes (U^n, X^n) under the M_1 topology. In Chapter 6 we study the distribution of the time between the first renewal state and the second renewal state in the limiting model and compute the probability of leftward and rightward price movement.

Chapter 3

From initial state to the first renewal state

In this chapter we study the evolution of the LOB from the initial time until time σ^n, the first time U^n or W^n hits zero; see (2.2). Chapter 5 addresses the evolution of the LOB after σ^n.

Recall that in the n-th pre-limit model, we assume the LOB has one of the two initial configurations shown in Figure 2.2, i.e., $T^n(0)$, $U^n(0)$, $W^n(0)$ and $X^n(0)$ are of size $O(\sqrt{n})$ and $V^n(0) = o(\sqrt{n})$. More precisely, we assume the initial condition is nonrandom and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} T^n(0) = T^*(0) > 0, \quad \lim_{n\to\infty} \frac{1}{\sqrt{n}} U^n(0) = U^*(0) > 0, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} V^n(0) = V^*(0) = 0, \tag{3.2}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} W^n(0) = W^*(0) < 0, \quad \lim_{n\to\infty} \frac{1}{\sqrt{n}} X^n(0) = X^*(0) < 0. \tag{3.3}$$

Starting from this initial configuration, the LOB will evolve following the dynamics described in Figure 2.3. We study the evolution of $(T^n(t\wedge\sigma^n), U^n(t\wedge\sigma^n), V^n(t\wedge\sigma^n), W^n(t\wedge\sigma^n), X^n(t\wedge\sigma^n))$ and derive its diffusion scaled limit. We note that before time σ^n there are only three possible dynamics acting on $(T^n, U^n, V^n, W^n, X^n)$, depending on the sign of V^n, and these are shown in Figure 3.1.

After σ^n the dynamics acting on the LOB are more complicated because they are dependent on the locations of the bid and ask prices. To postpone consideration of these more complicated dynamics until Chapter 5, in this chapter we define the five-tuple of processes $(\mathcal{T}^n, \mathcal{U}^n, \mathcal{V}^n, \mathcal{W}^n, \mathcal{X}^n)$ that agrees with $(T^n, U^n, V^n, W^n, X^n)$ until time σ_1^n and thereafter continues following the dynamics shown in the left, right or middle configuration of Figure 3.1 depending on whether $\mathcal{V}^n$ is positive, negative, or zero, respectively. We

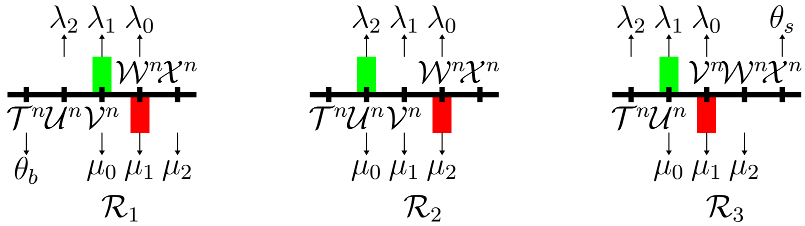

Figure 3.1: Three possible dynamics for $(T^n, U^n, V^n, W^n, X^n)$

apply the diffusion scaling to both five-tuples, defining

$$
\begin{aligned}
&\big(\widehat{T}^n(t), \widehat{U}^n(t), \widehat{V}^n(t), \widehat{W}^n(t), \widehat{X}^n(t)\big)\\
&\qquad = \left(\frac{1}{\sqrt{n}}T^n(nt), \frac{1}{\sqrt{n}}U^n(nt), \frac{1}{\sqrt{n}}V^n(nt)\frac{1}{\sqrt{n}}W^n(nt)\frac{1}{\sqrt{n}}X^n(nt)\right), \quad t \geq 0,\\
&\quad \widehat{\sigma}^n = \inf\{t \geq 0 : \widehat{U}^n(t) = 0 \text{ or } \widehat{W}^n(t) = 0\} = \frac{1}{n}\sigma^n,\\
&\big(\widehat{\mathcal{T}}^n(t), \widehat{\mathcal{U}}^n(t), \widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t), \widehat{\mathcal{X}}^n(t)\big)\\
&\qquad = \left(\frac{1}{\sqrt{n}}\widehat{\mathcal{T}}^n(nt), \frac{1}{\sqrt{n}}\widehat{\mathcal{U}}^n(nt), \frac{1}{\sqrt{n}}\widehat{\mathcal{V}}^n(nt), \frac{1}{\sqrt{n}}\widehat{\mathcal{W}}^n(nt), \frac{1}{\sqrt{n}}\widehat{\mathcal{X}}^n(nt)\right), \quad t \geq 0. \qquad (3.4)
\end{aligned}
$$

Because the stopping time σ_1^n plays no role in the evolution of the processes in (3.1), we are able to identify a limit $(\mathcal{T}^*, \mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*, \mathcal{X}^*)$ of $(\widehat{\mathcal{T}}^n, \widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n, \widehat{\mathcal{X}}^n)$. We then define

$$\sigma^* = \inf\{t \geq 0 : \mathcal{U}^*(t) = 0 \text{ or } \mathcal{W}^*(t) = 0\},$$

and show that the limit of

$$\big(\widehat{T}^n(t \wedge \widehat{\sigma}^n), \widehat{U}^n(t \wedge \widehat{\sigma}^n), \widehat{V}^n(t \wedge \widehat{\sigma}^n), \widehat{W}^n(t \wedge \widehat{\sigma}^n), \widehat{X}^n(t \wedge \widehat{\sigma}^n)\big), \quad t \geq 0,$$

is

$$\big(\mathcal{T}^*(t \wedge \sigma^*), \mathcal{U}^*(t \wedge \sigma^*), \mathcal{V}^*(t \wedge \sigma^*), \mathcal{W}^*(t \wedge \sigma^*), \mathcal{X}^*(t \wedge \sigma^*)\big), \quad t \geq 0;$$

see Theorem 3.7.5.

The behavior of the limit of the five-tuple of processes (3.4) as $n \to \infty$ is shown to be the following. Immediately after the initial time, $\mathcal{T}^*$ jumps to the value

$$\kappa_L := \frac{\lambda_2 \mu_1}{\theta_b \lambda_1} \qquad (3.5)$$

and remains there. Likewise, immediately after time zero $\mathcal{X}^*$ jumps to the value

$$\kappa_R := -\frac{\mu_2 \lambda_1}{\theta_s \mu_1} \qquad (3.6)$$

and stays there. The convergence of $\widehat{\mathcal{T}}^n$ to $\mathcal{T}^*$ and of $\widehat{\mathcal{X}}^n$ to $\mathcal{X}^*$ are in the M_1 topology on

$$D[0-, \infty) := \mathbb{R} \times D[0, \infty),$$

an extended version of $D[0,\infty)$ that allows for jumps at time zero; see Proposition 3.7.4. The process $\mathcal{V}^*$ is identically zero, and $(\mathcal{U}^*,\mathcal{W}^*)$ is a pair of Brownian motions with zero drift and covariance matrix given in Corollary 3.5.1. The convergence of $(\widehat{\mathcal{U}}^n,\widehat{\mathcal{V}}^n,\widehat{\mathcal{W}}^n)$ to $(\mathcal{U}^*,\mathcal{V}^*,\mathcal{W}^*)$ is in the J_1 topology. The simple structure of $(\mathcal{U}^*,\mathcal{W}^*)$ enables us to compute the joint distribution of the stopping times

$$\tau_{\mathcal{U}^*} = \inf\{t \geq 0 : \mathcal{U}^*(t) = 0\},$$
$$\tau_{\mathcal{W}^*} = \inf\{t \geq 0 : \mathcal{W}^*(t) = 0\},$$

and in particular, $\mathbb{P}\{\tau_{\mathcal{U}^*} < \tau_{\mathcal{W}^*}\}$ and $\mathbb{P}\{\tau_{\mathcal{U}^*} > \tau_{\mathcal{W}^*}\}$; see Corollary 3.6.1. Of course, $\sigma^* = \min(\tau_{\mathcal{U}^*,\mathcal{W}^*})$.

In Section 3.1 we prepare for the determination of the limit of $(\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n)$ by changing the variables from the triple $(\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n)$ to a new triple of processes (J^n,K^n,L^n). We scale these processes in Section 3.2 to obtain $(\widehat{J}^n,\widehat{K}^n,\widehat{L}^n)$. Section 3.3 is devoted to showing that $\widehat{K}^n \Longrightarrow 0$. In Section 3.4 we show that $(\widehat{J}^n,\widehat{L}^n)$ converges to a pair of correlated Brownian motions (J^*,L^*). Applying the inverse of the transformation in Section 3.1 to the limiting triple $(J^*,0,L^*)$, in Section 3.5 we obtain the triple $(\mathcal{U}^*,\mathcal{V}^*,\mathcal{W}^*)$ described above as the limit of $(\widehat{\mathcal{U}}^n,\widehat{\mathcal{V}}^n,\widehat{\mathcal{W}}^n)$. The computation of the joint distribution of $(\tau_{\mathcal{U}^*},\tau_{W^*})$ is in Section 3.6. Section 3.7 shows the convergence of $(\widehat{\mathcal{T}}^n,\widehat{\mathcal{X}}^n)$ to $(T^*(0)\mathbb{I}_{\{0\}}(\cdot)+\kappa_L\mathbb{I}_{(0,\infty)}(\cdot), X^*(0)\mathbb{I}_{\{0\}}(\cdot)+\kappa_R\mathbb{I}_{(0,\infty)}(\cdot))$.

3.1 Transformation of variables

Since there are three kinds of dynamics depending on the sign of $\mathcal{V}^n$, we can define three regions in $\mathbb{R}^5$,

$$\begin{aligned}
\mathcal{R}_1 &:= \big\{(t,u,v,w,x) : v > 0\big\},\\
\mathcal{R}_2 &:= \big\{(t,u,v,w,x) : v = 0\big\},\\
\mathcal{R}_3 &:= \big\{(t,u,v,w,x) : v < 0\big\},
\end{aligned}$$

and the dynamics acting on $(\mathcal{T}^n,\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n,\mathcal{X}^n)$ will be the same within each region. Although there are six independent Poisson processes governing the arrivals of market orders and limit orders, for convenience we will introduce twenty independent unit-intensity Poisson processes to describe the evolutions of $(\mathcal{T}^n,\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n,\mathcal{X}^n)$. This is possible because thinned Poisson processes are also Poisson processes. We denote these Poisson processes by $N_{i,\times,*}$, where $i = 1,2,3$ indicates the region where $(\mathcal{T}^n,\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n,\mathcal{X}^n)$ is, $\times \in \{\mathcal{T},\mathcal{U},\mathcal{V},\mathcal{W},\mathcal{X}\}$ indicates which of the processes among $(\mathcal{T}^n,\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n,\mathcal{X}^n)$ is affected by the Poisson process, and $* \in \{+,-\}$ indicates whether the Poisson process increases(+) or decreases(−) the affected process. For $i = 1,\ldots,3$, we define $P_i(t)$ to be the time $(\mathcal{T}^n,\mathcal{U}^n,\mathcal{V}^n,\mathcal{W}^n,\mathcal{X}^n)$ spends in region $\mathcal{R}_i$ up to time t. In particular, we have

$$P_i(t) = \int_0^t \mathbb{1}_{\{(\mathcal{T}^n(s),\mathcal{U}^n(s),\mathcal{V}^n(s),\mathcal{W}^n(s),\mathcal{X}^n(s))\in\mathcal{R}_i\}}\,ds, \quad i = 1,2,3.$$

Then, according to Figure 3.1, we have

$$\mathcal{T}^n(t) = \mathcal{T}^n(0) - N_{1,\mathcal{T},-}\Big(\int_0^t \frac{\theta_b}{\sqrt{n}}(\mathcal{T}^n(s))^+ dP_1(s)\Big) + N_{3,\mathcal{T},+}\big(\lambda_2 P_3(t)\big), \tag{3.7}$$

$$\begin{aligned}\mathcal{U}^n(t) &= \mathcal{U}^n(0) + N_{1,\mathcal{U},+}\big(\lambda_2 P_1(t)\big) + N_{2,\mathcal{U},+}\big(\lambda_2 P_2(t)\big) - N_{2,\mathcal{U},-}\big(\mu_0 P_2(t)\big) \\ &\quad + N_{3,\mathcal{U},+}\big(\lambda_1 P_3(t)\big) - N_{3,\mathcal{U},-}\big(\mu_0 P_3(t)\big),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{V}^n(t) &= \mathcal{V}^n(0) + N_{1,\mathcal{V},+}\big(\lambda_1 P_1(t)\big) - N_{1,\mathcal{V},-}\big(\mu_0 P_1(t)\big) + N_{2,\mathcal{V},+}\big(\lambda_1 P_2(t)\big) \\ &\quad - N_{2,\mathcal{V},-}\big(\mu_1 P_2(t)\big) + N_{3,\mathcal{V},+}\big(\lambda_0 P_3(t)\big) - N_{3,\mathcal{V},-}\big(\mu_1 P_3(t)\big),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{W}^n(t) &= \mathcal{W}^n(0) + N_{1,\mathcal{W},+}\big(\lambda_0 P_1(t)\big) - N_{1,\mathcal{W},-}\big(\mu_1 P_1(t)\big) + N_{2,\mathcal{W},+}\big(\lambda_0 P_2(t)\big) \\ &\quad - N_{2,\mathcal{W},-}\big(\mu_2 P_2(t)\big) - N_{3,\mathcal{W},-}\big(\mu_2 P_3(t)\big),\end{aligned} \tag{3.10}$$

$$\mathcal{X}^n(t) = \mathcal{X}^n(0) - N_{1,\mathcal{X},-}\big(\mu_2 P_1(t)\big) + N_{3,\mathcal{X},+}\Big(\int_0^t \frac{\theta_s}{\sqrt{n}}(\mathcal{X}^n(s))^- dP_3(s)\Big). \tag{3.11}$$

Recalling the positive constants a and b from Assumption 2.2.1, we define (J^n, K^n, L^n) as the continuous piecewise linear transformation of $(\mathcal{U}^n, \mathcal{V}^n, \mathcal{W}^n)$ given by

$$J^n(t) := \begin{cases} \mathcal{V}^n(t) + \mathcal{W}^n(t) & \text{if } \big(\mathcal{T}^n(t), \mathcal{U}^n(t), \mathcal{V}^n(t), \mathcal{W}^n(t), \mathcal{X}^n(t)\big) \in \mathcal{R}_1 \cup \mathcal{R}_2, \\ a\mathcal{V}^n(t) + \mathcal{W}^n(t) & \text{if } \big(\mathcal{T}^n(t), \mathcal{U}^n(t), \mathcal{V}^n(t), \mathcal{W}^n(t), \mathcal{X}^n(t)\big) \in \mathcal{R}_3. \end{cases} \tag{3.12}$$

$$K^n(t) := \mathcal{V}^n(t), \tag{3.13}$$

$$L^n(t) := \begin{cases} \mathcal{U}^n(t) + b\mathcal{V}^n(t) & \text{if } \big(\mathcal{T}^n(t), \mathcal{U}^n(t), \mathcal{V}^n(t), \mathcal{W}^n(t), \mathcal{X}^n(t)\big) \in \mathcal{R}_1, \\ \mathcal{U}^n(t) + \mathcal{V}^n(t) & \text{if } \big(\mathcal{T}^n(t), \mathcal{U}^n(t), \mathcal{V}^n(t), \mathcal{W}^n(t), \mathcal{X}^n(t)\big) \in \mathcal{R}_2 \cup \mathcal{R}_3. \end{cases} \tag{3.14}$$

Note that this transformation is invertible. Indeed, for $i = 1, 2, 3$, the image of $\mathcal{R}_i$ under this transformation is $\mathcal{R}'_i$, where the $\mathcal{R}'_i$ regions are defined by

$$\begin{aligned}\mathcal{R}'_1 &:= \big\{(j,k,l) : k > 0\big\}, \\ \mathcal{R}'_2 &:= \big\{(j,k,l) : k = 0\big\}, \\ \mathcal{R}'_3 &:= \big\{(j,k,l) : k < 0\big\},\end{aligned}$$

and the inverse map is

$$\mathcal{U}^n(t) = \begin{cases} L^n(t) - bK^n(t) & \text{if } \big(J^n(t), K^n(t), L^n(t)\big) \in \mathcal{R}'_1, \\ L^n(t) - K^n(t) & \text{if } \big(J^n(t), K^n(t), L^n(t)\big) \in \mathcal{R}'_2 \cup \mathcal{R}'_3, \end{cases} \tag{3.15}$$

$$\mathcal{V}^n(t) = K^n(t), \tag{3.16}$$

$$\mathcal{W}^n(t) = \begin{cases} J^n(t) - K^n(t) & \text{if } \big(J^n(t), K^n(t), L^n(t)\big) \in \mathcal{R}'_1 \cup \mathcal{R}'_2, \\ J^n(t) - aK^n(t) & \text{if } \big(J^n(t), K^n(t), L^n(t)\big) \in \mathcal{R}'_3. \end{cases} \tag{3.17}$$

It can be verified that the inverse transformation defined by (3.15), (3.16) and (3.17) is continuous on $\mathcal{R}' := \cup_{i=1}^3 \mathcal{R}'_i$.

An increase of $\mathcal{V}^n$ by one unit when $(\mathcal{T}^n, \mathcal{U}^n, \mathcal{V}^n, \mathcal{W}^n, \mathcal{X}^n)$ is in $\mathcal{R}_2$ increases L^n by b units. An increase or decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{T}^n, \mathcal{U}^n, \mathcal{V}^n, \mathcal{W}^n, \mathcal{X}^n)$ is in $\mathcal{R}_1$ increases or decreases, respectively, L^n by b units. Similarly, a decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{T}^n, \mathcal{U}^n, \mathcal{V}^n, \mathcal{W}^n, \mathcal{X}^n)$ is in $\mathcal{R}_2$ decrease J^n by a units. An increase or decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{T}^n, \mathcal{U}^n, \mathcal{V}^n, \mathcal{W}^n, \mathcal{X}^n)$ is in $\mathcal{R}_3$ increases or decreases, respectively, J^n by a units. Otherwise, all increases or decreases in $\mathcal{U}^n$, $\mathcal{V}^n$ or $\mathcal{W}^n$ by one unit increase or decrease J^n or L^n by one unit. It follows that

$$\begin{aligned} J^n(t) &= J^n(0) + N_{1,\mathcal{V},+} \circ \lambda_1 P_1(t) - N_{1,\mathcal{V},-} \circ \mu_0 P_1(t) + N_{1,\mathcal{W},+} \circ \lambda_0 P_1(t) \\ &\quad - N_{1,\mathcal{W},-} \circ \mu_1 P_1(t) + N_{2,\mathcal{V},+} \circ \lambda_1 P_2(t) - a N_{2,\mathcal{V},-} \circ \mu_1 P_2(t) \\ &\quad + N_{2,\mathcal{W},+} \circ \lambda_0 P_2(t) - N_{2,\mathcal{W},-} \circ \mu_2 P_2(t) + a N_{3,\mathcal{V},+} \circ \lambda_0 P_3(t) \\ &\quad - a N_{3,\mathcal{V},-} \circ \mu_1 P_3(t) - N_{3,\mathcal{W},-} \circ \mu_2 P_3(t), \end{aligned} \tag{3.18}$$

$$\begin{aligned} L^n(t) &= L^n(0) + N_{1,\mathcal{U},+} \circ \lambda_2 P_1(t) + b N_{1,\mathcal{V},+} \circ \lambda_1 P_1(t) - b N_{1,\mathcal{V},-} \circ \mu_0 P_1(t) \\ &\quad + N_{2,\mathcal{U},+} \circ \lambda_2 P_2(t) - N_{2,\mathcal{U},-} \circ \mu_0 P_2(t) + b N_{2,\mathcal{V},+} \circ \lambda_1 P_2(t) \\ &\quad - N_{2,\mathcal{V},-} \circ \mu_1 P_2(t) + N_{3,\mathcal{U},+} \circ \lambda_1 P_3(t) - N_{3,\mathcal{U},-} \circ \mu_0 P_3(t) \\ &\quad + N_{3,\mathcal{V},+} \circ \lambda_0 P_3(t) - N_{3,\mathcal{V},-} \circ \mu_1 P_3(t). \end{aligned} \tag{3.19}$$

Since $K^n = \mathcal{V}^n$, we have

$$\begin{aligned} K^n(t) &= K^n(0) + N_{1,\mathcal{V},+} \circ \lambda_1 P_1(t) - N_{1,\mathcal{V},-} \circ \mu_0 P_1(t) + N_{2,\mathcal{V},+} \circ \lambda_1 P_2(t) \\ &\quad - N_{2,\mathcal{V},-} \circ \mu_1 P_2(t) + N_{3,\mathcal{V},+} \circ \lambda_0 P_3(t) - N_{3,\mathcal{V},-} \circ \mu_1 P_3(t). \end{aligned} \tag{3.20}$$

In the region $\mathcal{R}_3$, K^n is negative and a change in K^n results in a change in $|K^n|$ of the same magnitude but the opposite direction. In the region $\mathcal{R}_2$, K^n is zero and a unit change in K^n results in a unit increase in $|K^n|$. Modifying (3.20) accordingly, we obtain

$$\begin{aligned} \big|K^n(t)\big| &= \big|K^n(0)\big| + N_{1,\mathcal{V},+} \circ \lambda_1 P_1(t) - N_{1,\mathcal{V},-} \circ \mu_0 P_1(t) + N_{2,\mathcal{V},+} \circ \lambda_1 P_2(t) \\ &\quad + N_{2,\mathcal{V},-} \circ \mu_1 P_2(t) - N_{3,\mathcal{V},+} \circ \lambda_0 P_3(t) + N_{3,\mathcal{V},-} \circ \mu_1 P_3(t). \end{aligned} \tag{3.21}$$

3.2 Diffusion scaling

Recall that the diffusion scaling of a sequence of processes Q^n is defined by,

$$\widehat{Q}^n(t) = \frac{1}{\sqrt{n}} Q^n(nt).$$

Because each of the regions $\mathcal{R}_i$, $i = 1, 2, 3$, is a cone, when we apply the diffusion scaling to the processes $\mathcal{T}^n$, $\mathcal{U}^n$, $\mathcal{V}^n$, $\mathcal{W}^n$, $\mathcal{X}^n$, J^n, K^n and L^n in the piecewise linear transformations (3.12), (3.13), and (3.14), we obtain the analogous formulas

$$\widehat{J}^n(t) := \begin{cases} \widehat{\mathcal{V}}^n(t) + \widehat{\mathcal{W}}^n(t) & \text{if } \big(\widehat{\mathcal{T}}^n(t), \widehat{\mathcal{U}}^n(t), \widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t), \widehat{\mathcal{X}}^n(t)\big) \in \mathcal{R}_1 \cup \mathcal{R}_2, \\ a\widehat{\mathcal{V}}^n(t) + \widehat{\mathcal{W}}^n(t) & \text{if } \big(\widehat{\mathcal{T}}^n(t), \widehat{\mathcal{U}}^n(t), \widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t), \widehat{\mathcal{X}}^n(t)\big) \in \mathcal{R}_3. \end{cases} \tag{3.22}$$

$$\widehat{K}^n(t) := \widehat{\mathcal{V}}^n(t), \tag{3.23}$$

$$\widehat{L}^n(t) := \begin{cases} \widehat{\mathcal{U}}^n(t) + b\widehat{\mathcal{V}}^n(t) & \text{if } \big(\widehat{\mathcal{T}}^n(t), \widehat{\mathcal{U}}^n(t), \widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t), \widehat{\mathcal{X}}^n(t)\big) \in \mathcal{R}_1, \\ \widehat{\mathcal{U}}^n(t) + \widehat{\mathcal{V}}^n(t) & \text{if } \big(\widehat{\mathcal{T}}^n(t), \widehat{\mathcal{U}}^n(t), \widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t), \widehat{\mathcal{X}}^n(t)\big) \in \mathcal{R}_2 \cup \mathcal{R}_3. \end{cases} \tag{3.24}$$

The inverse of this transformation is continuous. In fact, the inverse is given by replacing J^n by $\widehat{J}^n$, K^n by $\widehat{K}^n$, and L^n by $\widehat{L}^n$ in (3.15), (3.16) and (3.17),

$$\widehat{\mathcal{U}}^n(t) = \begin{cases} \widehat{L}^n(t) - b\widehat{K}^n(t) & \text{if } \big(\widehat{J}^n(t), \widehat{K}^n(t), \widehat{L}^n(t)\big) \in \mathcal{R}_1', \\ \widehat{L}^n(t) - \widehat{K}^n(t) & \text{if } \big(\widehat{J}^n(t), \widehat{K}^n(t), \widehat{L}^n(t)\big) \in \mathcal{R}_2' \cup \mathcal{R}_3', \end{cases} \tag{3.25}$$

$$\widehat{\mathcal{V}}^n(t) = \widehat{K}^n(t), \tag{3.26}$$

$$\widehat{\mathcal{W}}^n(t) = \begin{cases} \widehat{J}^n(t) - \widehat{K}^n(t) & \text{if } \big(\widehat{J}^n(t), \widehat{K}^n(t), \widehat{L}^n(t)\big) \in \mathcal{R}_1' \cup \mathcal{R}_2', \\ \widehat{J}^n(t) - a\widehat{K}^n(t) & \text{if } \big(\widehat{J}^n(t), \widehat{K}^n(t), \widehat{L}^n(t)\big) \in \mathcal{R}_3'. \end{cases} \tag{3.27}$$

The Continuous Mapping Theorem implies that we can determine the weak limit in the J_1 topology of $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)$ by determining the limit of $(\widehat{J}^n, \widehat{K}^n, \widehat{L}^n)$.

We next center the twenty independent unit-intensity Poisson processes appearing in (3.7) to (3.11), defining

$$M_{i,\times,*}(t) := N_{i,\times,*}(t) - t, \quad t \geq 0.$$

Each of these *compensated Poisson processes* is a martingale relative its own filtration, and these martingale are independent. For $n = 1, 2, \ldots$, their diffusion-scaled versions are

$$\widehat{M}^n_{i,\times,*}(t) := \frac{1}{\sqrt{n}}\big(M_{i,\times,*}(nt) - nt\big), \quad t \geq 0, \tag{3.28}$$

and each of these processes is likewise a martingale relative to its own filtration, and these processes are independent. For $i = 1, 2, 3$ and $n = 1, 2, \ldots$, we also define

$$\overline{P}^n_i(t) := \frac{1}{n} P_i(nt), \quad t \geq 0.$$

Replacing the Poisson processes in (3.18) and (3.19) by compensated Poisson processes and applying the diffusion scaling, we obtain

$$\begin{aligned} \widehat{J}^n(t) = {} & \widehat{J}^n(0) + \widehat{M}^n_{1,\mathcal{V},+} \circ \lambda_1 \overline{P}^n_1(t) - \widehat{M}^n_{1,\mathcal{V},-} \circ \mu_0 \overline{P}^n_1(t) + \widehat{M}^n_{1,\mathcal{W},+} \circ \lambda_0 \overline{P}^n_1(t) \\ & - \widehat{M}^n_{1,\mathcal{W},-} \circ \mu_1 \overline{P}^n_1(t) + \widehat{M}^n_{2,\mathcal{V},+} \circ \lambda_1 \overline{P}^n_2(t) - a\widehat{M}^n_{2,\mathcal{V},-} \circ \mu_1 \overline{P}^n_2(t) \\ & + \widehat{M}^n_{2,\mathcal{W},+} \circ \lambda_0 \overline{P}^n_2(t) - \widehat{M}^n_{2,\mathcal{W},-} \circ \mu_2 \overline{P}^n_2(t) + a\widehat{M}^n_{3,\mathcal{V},+} \circ \lambda_0 \overline{P}^n_3(t) \\ & - a\widehat{M}^n_{3,\mathcal{V},-} \circ \mu_1 \overline{P}^n_3(t) - \widehat{M}^n_{3,\mathcal{W},-} \circ \mu_2 \overline{P}^n_3(t), \end{aligned} \tag{3.29}$$

$$\begin{aligned} \widehat{L}^n(t) = {} & \widehat{L}^n(0) + \widehat{M}^n_{1,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_1(t) + b\widehat{M}^n_{1,\mathcal{V},+} \circ \lambda_1 \overline{P}^n_1(t) - b\widehat{M}^n_{1,\mathcal{V},-} \circ \mu_0 \overline{P}^n_1(t) \\ & + \widehat{M}^n_{2,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_2(t) - \widehat{M}^n_{2,\mathcal{U},-} \circ \mu_0 \overline{P}^n_2(t) + b\widehat{M}^n_{2,\mathcal{V},+} \circ \lambda_1 \overline{P}^n_2(t) \\ & - \widehat{M}^n_{2,\mathcal{V},-} \circ \mu_1 \overline{P}^n_2(t) + \widehat{M}^n_{3,\mathcal{U},+} \circ \lambda_1 \overline{P}^n_3(t) - \widehat{M}^n_{3,\mathcal{U},-} \circ \mu_0 \overline{P}^n_3(t) \\ & + \widehat{M}^n_{3,\mathcal{V},+} \circ \lambda_0 \overline{P}^n_3(t) - \widehat{M}^n_{3,\mathcal{V},-} \circ \mu_1 \overline{P}^n_3(t). \end{aligned} \tag{3.30}$$

The drift terms that arise from the centering of the Poisson processes vanish in (3.29) and (3.30) because, according to Assumption 2.2.1 and its consequence (2.1),

$$\begin{aligned}
(\lambda_1 - \mu_0 + \lambda_0 - \mu_1)\overline{P}_1^n &= 0, \\
(\lambda_1 - a\mu_1 + \lambda_0 - \mu_2)\overline{P}_2^n &= 0, \\
(a\lambda_0 - a\mu_1 - \mu_2)\overline{P}_3^n &= 0, \\
(\lambda_2 + b\lambda_1 - b\mu_0)\overline{P}_1^n &= 0, \\
(\lambda_2 - \mu_0 + b\lambda_1 - \mu_1)\overline{P}_2^n &= 0, \\
(\lambda_1 - \mu_0 + \lambda_0 - \mu_1)\overline{P}_3^n &= 0.
\end{aligned}$$

The filtration $\{\mathcal{F}^n(t)\}_{t\geq 0}$ we use for $\widehat{J}^n$ and $\widehat{L}^n$ is the one generated by the sixteen time-changed processes $\widehat{M}^n_{1,\mathcal{V},+} \circ \lambda_1\overline{P}_1^n, \ldots, \widehat{M}^n_{3,\mathcal{V},-} \circ \mu_1\overline{P}_3^n$ appearing in (3.29) and (3.30), and three occupation time processes $\overline{P}_1^n$, $\overline{P}_2^n$, and $\overline{P}_3^n$. These sixteen time-changed processes are not independent because of the coupling of the time changes. However, they are each martingales relative to the filtration $\{\mathcal{F}^n(t)\}_{t\geq 0}$, as are $\widehat{J}^n$ and $\widehat{L}^n$.

Replacing the Poisson processes in (3.20) and (3.21) by compensated Poisson processes and applying the diffusion scaling, we obtain

$$\begin{aligned}
\widehat{K}^n(t) &= \widehat{K}^n(0) + \widehat{M}^n_{1,\mathcal{V},+} \circ \lambda_1\overline{P}_1^n(t) - \widehat{M}^n_{1,\mathcal{V},-} \circ \mu_0\overline{P}_1^n(t) + \widehat{M}^n_{2,\mathcal{V},+} \circ \lambda_1\overline{P}_2^n(t) \\
&\quad - \widehat{M}^n_{2,\mathcal{V},-} \circ \mu_1\overline{P}_2^n(t) + \widehat{M}^n_{3,\mathcal{V},+} \circ \lambda_0\overline{P}_3^n(t) - \widehat{M}^n_{3,\mathcal{V},-} \circ \mu_1\overline{P}_3^n(t) \\
&\quad + \sqrt{n}(\lambda_1 - \mu_0)\overline{P}_1^n(t) + \sqrt{n}(\lambda_1 - \mu_1)\overline{P}_2^n(t) \\
&\quad + \sqrt{n}(\lambda_0 - \mu_1)\overline{P}_3^n(t),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\left|\widehat{K}^n(t)\right| &= \left|\widehat{K}^n(0)\right| + \widehat{M}^n_{1,\mathcal{V},+} \circ \lambda_1\overline{P}_1^n(t) - \widehat{M}^n_{1,\mathcal{V},-} \circ \mu_0\overline{P}_1^n(t) + \widehat{M}^n_{2,\mathcal{V},+} \circ \lambda_1\overline{P}_2^n(t) \\
&\quad + \widehat{M}^n_{2,\mathcal{V},-} \circ \mu_1\overline{P}_2^n(t) - \widehat{M}^n_{3,\mathcal{V},+} \circ \lambda_0\overline{P}_3^n(t) + \widehat{M}^n_{3,\mathcal{V},-} \circ \mu_1\overline{P}_3^n(t) \\
&\quad + \sqrt{n}(\lambda_1 - \mu_0)\overline{P}_1^n(t) + \sqrt{n}(\lambda_1 + \mu_1)\overline{P}_2^n(t) \\
&\quad + \sqrt{n}(\mu_1 - \lambda_0)\overline{P}_3^n(t).
\end{aligned} \tag{3.32}$$

Since in n^{th} pre-limit model, we assume that

$$U^n(0)/\sqrt{n} \to u_0, \quad V^n(0)/\sqrt{n} \to 0, \quad W^n(0)/\sqrt{n} \to w_0,$$

where u_0 is a positive constant and w_0 is a negative constant, from (3.22), (3.23), and (3.24), we have

$$\widehat{J}^n(0) \to w_0, \quad \widehat{K}^n(0) \to 0, \quad \widehat{L}^n(0) \to u_0.$$

Replacing the Poisson processes in (3.7) and (3.11) by compensated Poisson processes, applying the diffusion scaling, we obtain

$$\begin{aligned}
\widehat{\mathcal{T}}^n(t) &= \widehat{\mathcal{T}}^n(0) - \widehat{M}^n_{1,\mathcal{T},-}\Big(\int_0^t \theta_b(\widehat{\mathcal{T}}^n(s))^+ d\overline{P}^n_1(s)\Big) + \widehat{M}^n_{3,\mathcal{T},+}\big(\lambda_2 \overline{P}^n_3(t)\big) \\
&\quad -\sqrt{n}\int_0^t \theta_b(\widehat{\mathcal{T}}^n(s))^+ d\overline{P}^n_1(s) + \sqrt{n}\lambda_2\overline{P}^n_3(t), \qquad (3.33)\\
\widehat{\mathcal{X}}^n(t) &= \widehat{\mathcal{X}}^n(0) - \widehat{M}^n_{1,\mathcal{X},-}\big(\mu_2\overline{P}^n_1(t)\big) + \widehat{M}^n_{3,\mathcal{X},+}\Big(\int_0^t \theta_s(\widehat{\mathcal{X}}^n(s))^- d\overline{P}^n_3(s)\Big) \\
&\quad -\sqrt{n}\mu_2\overline{P}^n_1(t) + \sqrt{n}\int_0^t \theta_s(\widehat{\mathcal{X}}^n(s))^- d\overline{P}^n_3(s). \qquad (3.34)
\end{aligned}$$

3.3 Crushing $\widehat{K}^n$

We denote by $\mathcal{D}([0,\infty),\mathbb{R}^d)$ the space of real-valued càdlàg functions from $[0,\infty)$ to $\mathbb{R}^d$. We shall use both the J_1 and M_1 topologies on this space; see Ethier and Kurtz [9] for the former and Whitt [26] for both topologies. A sequence of càdlàg processes is said to converge weakly-J_1 or weakly-M_1 if the measures induced on $\mathcal{D}([0,\infty),\mathbb{R}^d)$ converge weakly under the J_1 or M_1 topologies, respectively. We denote these convergences by $\overset{J_1}{\Longrightarrow}$ and $\overset{M_1}{\Longrightarrow}$. In this section we identify the weak-J_1 limit of the sequence $\{\widehat{K}^n\}_{n=1}^\infty$.

Definition 3.3.1 *We say that a sequence of càdlàg processes $\{X^n\}_{n=1}^\infty$ is* bounded above in probability *if for every $T>0$ and $\varepsilon>0$, there exists a K and a positive integer N such that*

$$\mathbb{P}\Big\{\sup_{0\le t\le T} X^n(t) > K\Big\} < \varepsilon \quad \forall n \ge N.$$

We say that $\{X^n\}_{n=1}^\infty$ is bounded below in probability *if $\{-X^n\}_{n=1}^\infty$ is bounded above in probability. We say that $\{X^n\}_{n=1}^\infty$ is* bounded in probability *and write $X^n = \mathcal{O}_{cl}(1)$ if $\{X^n\}_{n=1}^\infty$ is both bounded above and bounded below in probability, and for every subsequence $\{X^{n_k}\}_{k=1}^\infty$ of $\{X^n\}_{n=1}^\infty$, there exists a sub-subsequence $\{X^{n_{k_p}}\}_{p=1}^\infty$ such that*

$$X^{n_{k_p}} \overset{J_1}{\Longrightarrow} X^*,$$

where $X^*\in C([0,\infty),\mathbb{R}^d)$. We say that $X^n = \mathfrak{o}(1)$ if $X^n \overset{J_1}{\Longrightarrow} 0$, or equivalently, if for all $T>0$,

$$\sup_{0\le t\le T}\big|X^n(t)\big| \overset{\mathbb{P}}{\to} 0 \text{ as } n\to\infty.$$

Remark 3.3.2 *A classical result is that the diffusion-scaled compensated Poisson processes (3.28) converge weakly-J_1 to independent Brownian motions. See Billingsley [5], Section 17.3. This implies that these processes are $\mathcal{O}_{cl}(1)$. Since $\overline{P}^n_i(t)\le t$ and monotonic for $i=1,\dots,8$ and all $t\ge 0$, the time-changed diffusion-scaled compensated Poisson processes appearing in (3.29)–(3.32) are also $\mathcal{O}_{cl}(1)$.*

Theorem 3.3.3 $\widehat{K}^n \overset{J_1}{\Longrightarrow} 0$.

PROOF: We modify a proof due to Peterson [24]. For $t \geq 0$, we define

$$\tau^n(t) := \begin{cases} \sup\big\{s \in [0,t] : \widehat{K}^n(s) = 0\big\} & \text{if } \big\{s \in [0,t] : \widehat{K}^n(s) = 0\big\} \neq \emptyset, \\ 0 & \text{if } \big\{s \in [0,t] : \widehat{K}^n(s) = 0\big\} = \emptyset. \end{cases}$$

Because $\widehat{K}^n(s) \neq 0$ for $s \in (\tau^n(t), t]$, $\overline{P}_2^n$ is flat on this interval, and we have

$$\overline{P}_1^n(t) + \overline{P}_3^n(t) = \overline{P}_1^n\big(\tau^n(t)\big) + \overline{P}_3^n\big(\tau^n(t)\big) + t - \tau^n(t)$$

and

$$\overline{P}_2^n(t) = \overline{P}_2^n\big(\tau^n(t)\big).$$

Substituting this into (3.32), we obtain

$$\begin{aligned} 0 &\leq \big|\widehat{K}^n(t)\big| \\ &= \big|\widehat{K}^n\big(\tau^n(t)\big)\big| + \mathcal{O}_{cl}(1) - c\sqrt{n}\Big[\overline{P}_1^n(t) + \overline{P}_3^n(t) - \overline{P}_1^n\big(\tau^n(t)\big) - \overline{P}_3^n\big(\tau^n(t)\big)\Big] \\ &\leq \big|\widehat{K}^n\big(\tau^n(t)\big)\big| + \mathcal{O}_{cl}(1) - c\sqrt{n}\,\big(t - \tau^n(t)\big), \end{aligned} \tag{3.35}$$

where c is defined by (2.1). Since $\widehat{K}^n(\tau^n(t)) \to 0$ if $\tau^n(t) = 0$, and otherwise $|\widehat{K}^n(\tau^n(t))| = 1/\sqrt{n}$, (3.35) implies $\sqrt{n}(e - \tau^n)$ is bounded above and below, where e is the identify process $e(t) = t$ for all $t \geq 0$. This implies

$$\tau^n \overset{J_1}{\Longrightarrow} e, \tag{3.36}$$

and thus

$$0 \leq \overline{P}_i^n - \overline{P}_i^n \circ \tau^n \leq e - \tau^n = \mathfrak{o}(1). \tag{3.37}$$

Because the limits of the processes $\widehat{M}_{i,\times,*}^n$ are continuous, (3.37) implies that

$$\widehat{M}_{i,\times,*}^n \circ \alpha\overline{P}_i^n - \widehat{M}_{i,\times,*}^n \circ \alpha\overline{P}_i^n \circ \tau^n = \mathfrak{o}(1)$$

for any positive constant α. Therefore, we can upgrade the estimate in (3.35) to

$$0 \leq \big|\widehat{K}^n(t)\big| \leq \mathfrak{o}(1) + \mathfrak{o}(1) - c\sqrt{n}\big(t - \tau^n(t)\big),$$

which implies

$$\sqrt{n}\,(e - \tau^n) = \mathfrak{o}(1), \tag{3.38}$$

and we finish the proof. □

Remark 3.3.4 *From (3.31) and (3.32), we see that*

$$\begin{aligned} \sqrt{n}c(\overline{P}_3^n - \overline{P}_1^n) + \sqrt{n}(\lambda_1 - \mu_1)\overline{P}_2^n &= \mathcal{O}_{cl}(1), & (3.39) \\ -\sqrt{n}c(\overline{P}_3^n + \overline{P}_1^n) + \sqrt{n}(\lambda_1 + \mu_1)\overline{P}_2^n &= \mathcal{O}_{cl}(1), & (3.40) \end{aligned}$$

Dividing (3.39) and (3.40) by $\sqrt{n}$, we see that

$$c(\overline{P}_3^n - \overline{P}_1^n) + (\lambda_1 - \mu_1)\overline{P}_2^n \overset{J_1}{\Longrightarrow} 0, \tag{3.41}$$

$$-c(\overline{P}_3^n + \overline{P}_1^n) + (\lambda_1 + \mu_1)\overline{P}_2^n \overset{J_1}{\Longrightarrow} 0. \tag{3.42}$$

If we multiply (3.39) by $(\lambda_1 + \mu_1)$, multiply (3.40) by $-(\lambda_1 - \mu_1)$, and sum the two, we obtain

$$\sqrt{n}(\lambda_1 \overline{P}_3^n - \mu_1 \overline{P}_1^n) = \mathcal{O}_{cl}(1). \tag{3.43}$$

3.4 Convergence of $\widehat{J}^n$ and $\widehat{L}^n$

The proof of convergence of $\widehat{J}^n$ and identification of the limit proceeds through several steps. Along the way we identify the limits of the processes $\overline{P}_i^n$, $i = 1, \ldots, 8$.

Proposition 3.4.1

$$(\widehat{J}^n, \widehat{L}^n) \overset{J_1}{\Longrightarrow} (J^*, L^*),$$

where (J^*, L^*) is a two-dimensional correlated Brownian motion with the covariance matrix

$$\Sigma = \begin{bmatrix} c_J & c_{JL} \\ c_{JL} & c_L \end{bmatrix},$$

where

$$\begin{aligned}
c_J &= 2a\lambda_0 \frac{\lambda_1}{\lambda_0 + \lambda_1} + (\mu_2 + a^2\mu_1 + a\lambda_0)\frac{\lambda_0 - \mu_1}{\lambda_0 + \lambda_1} + (\mu_2 + a^2\mu_1 + a^2\lambda_0)\frac{\mu_1}{\lambda_0 + \lambda_1} \\
&= 2\lambda_0(a^2 - \frac{a^2}{b} + \frac{a}{b}), \\
c_L &= (b^2\lambda_1 + \lambda_2 + b^2\mu_0)\frac{\lambda_1}{\lambda_0 + \lambda_1} + (b^2\lambda_1 + \lambda_2 + a\lambda_0)\frac{\lambda_0 - \mu_1}{\lambda_0 + \lambda_1} + 2a\lambda_0 \frac{\mu_1}{\lambda_0 + \lambda_1} \\
&= 2\lambda_0(ab - b + 1), \\
c_{JL} &= (b\lambda_1 + b\mu_0)\frac{\lambda_1}{\lambda_0 + \lambda_1} + (b\lambda_1 + a\mu_1)\frac{\lambda_0 - \mu_1}{\lambda_0 + \lambda_1} + (a\lambda_0 + a\mu_1)\frac{\mu_1}{\lambda_0 + \lambda_1} \\
&= 2\lambda_0 \frac{2ab - a - b}{b}.
\end{aligned}$$

PROOF: Define

$$\begin{aligned}
\widehat{\Psi}_1^n &:= \widehat{M}_{1,\mathcal{V},+}^n \circ \lambda_1 e - \widehat{M}_{1,\mathcal{V},-}^n \circ \mu_0 e + \widehat{M}_{1,\mathcal{W},+}^n \circ \lambda_0 e - \widehat{M}_{1,\mathcal{W},-}^n \circ \mu_1 e, \\
\widehat{\Psi}_2^n &:= \widehat{M}_{2,\mathcal{V},+}^n \circ \lambda_1 e - a\widehat{M}_{2,\mathcal{V},-}^n \circ \mu_1 e + \widehat{M}_{2,\mathcal{W},+}^n \circ \lambda_0 e - \widehat{M}_{2,\mathcal{W},-}^n \circ \mu_2 e, \\
\widehat{\Psi}_3^n &:= a\widehat{M}_{3,\mathcal{V},+}^n \circ \lambda_0 e - a\widehat{M}_{3,\mathcal{V},-}^n \circ \mu_1 e - \widehat{M}_{3,\mathcal{W},-}^n \circ \mu_2 e, \\
\widehat{\Lambda}_1^n &:= \widehat{M}_{1,\mathcal{U},+}^n \circ \lambda_2 e + b\widehat{M}_{1,\mathcal{V},+}^n \circ \lambda_1 e - b\widehat{M}_{1,\mathcal{V},-}^n \circ \mu_0 e, \\
\widehat{\Lambda}_2^n &:= \widehat{M}_{2,\mathcal{U},+}^n \circ \lambda_2 e - \widehat{M}_{2,\mathcal{U},-}^n \circ \mu_0 e + b\widehat{M}_{2,\mathcal{V},+}^n \circ \lambda_1 e - \widehat{M}_{2,\mathcal{V},-}^n \circ \mu_1 e, \\
\widehat{\Lambda}_3^n &:= \widehat{M}_{3,\mathcal{U},+}^n \circ \lambda_1 e - \widehat{M}_{3,\mathcal{U},-}^n \circ \mu_0 e + \widehat{M}_{3,\mathcal{V},+}^n \circ \lambda_0 e - \widehat{M}_{3,\mathcal{V},-}^n \circ \mu_1 e,
\end{aligned}$$

so that $\widehat{J}^n = \widehat{J}^n(0)+\sum_{i=1}^3 \widehat{\Psi}_i^n\circ\overline{P}_i^n$, and $\widehat{L}^n = \widehat{L}^n(0)+\sum_{i=1}^3 \widehat{\Lambda}_i^n\circ\overline{P}_i^n$. Because $[\widehat{M}_{i,\times,*}^n, \widehat{M}_{i,\times,*}^n] \overset{J_1}{\Longrightarrow} e$ and these processes are independent, we have

$$\begin{aligned}
\left[\widehat{\Psi}_1^n, \widehat{\Psi}_1^n\right] &\overset{J_1}{\Longrightarrow} 2a\lambda_0 e =: A_1,\\
\left[\widehat{\Psi}_2^n, \widehat{\Psi}_2^n\right] &\overset{J_1}{\Longrightarrow} (\mu_2 + a^2\mu_1 + a\lambda_0)e =: A_2,\\
\left[\widehat{\Psi}_3^n, \widehat{\Psi}_3^n\right] &\overset{J_1}{\Longrightarrow} (\mu_2 + a^2\mu_1 + a^2\lambda_0)e =: A_3,\\
\left[\widehat{\Lambda}_1^n, \widehat{\Lambda}_1^n\right] &\overset{J_1}{\Longrightarrow} (b^2\lambda_1 + \lambda_2 + b^2\mu_0)e =: B_1,\\
\left[\widehat{\Lambda}_2^n, \widehat{\Lambda}_2^n\right] &\overset{J_1}{\Longrightarrow} (b^2\lambda_1 + \lambda_2 + a\lambda_0)e =: B_2,\\
\left[\widehat{\Lambda}_3^n, \widehat{\Lambda}_3^n\right] &\overset{J_1}{\Longrightarrow} 2a\lambda_0 e =: B_3,\\
\left[\widehat{\Psi}_1^n, \widehat{\Lambda}_1^n\right] &\overset{J_1}{\Longrightarrow} (b\lambda_1 + b\mu_0)e =: C_1,\\
\left[\widehat{\Psi}_2^n, \widehat{\Lambda}_2^n\right] &\overset{J_1}{\Longrightarrow} (b\lambda_1 + a\mu_1)e =: C_2,\\
\left[\widehat{\Psi}_3^n, \widehat{\Lambda}_3^n\right] &\overset{J_1}{\Longrightarrow} (a\lambda_0 + a\mu_1)e =: C_3,
\end{aligned}$$

The other cross variations are zero. Returning to the equations (3.41) and (3.42) and the obvious equation

$$\overline{P}_1^n + \overline{P}_2^n + \overline{P}_3^n = e,$$

we have

$$\overline{P}_1^n \overset{J_1}{\Longrightarrow} \frac{\lambda_1}{\lambda_0+\lambda_1}e, \quad \overline{P}_2^n \overset{J_1}{\Longrightarrow} \frac{\lambda_0-\mu_1}{\lambda_0+\lambda_1}e, \quad \overline{P}_3^n \overset{J_1}{\Longrightarrow} \frac{\mu_1}{\lambda_0+\lambda_1}e, \tag{3.44}$$

We have

$$\begin{aligned}
\left[\widehat{J}^n, \widehat{J}^n\right] &= \sum_{i=1}^3 \left[\widehat{\Psi}_i^n, \widehat{\Psi}_i^n\right]\circ\overline{P}_i^n\\
&\overset{J_1}{\Longrightarrow} \frac{\lambda_1}{\lambda_0+\lambda_1}A_1 + \frac{\lambda_0-\mu_1}{\lambda_0+\lambda_1}A_2 + \frac{\mu_1}{\lambda_0+\lambda_1}A_3\\
&= c_J e,
\end{aligned}$$

$$\begin{aligned}
\left[\widehat{L}^n, \widehat{L}^n\right] &= \sum_{i=1}^3 \left[\widehat{\Lambda}_i^n, \widehat{\Lambda}_i^n\right]\circ\overline{P}_i^n\\
&\overset{J_1}{\Longrightarrow} \frac{\lambda_1}{\lambda_0+\lambda_1}B_1 + \frac{\lambda_0-\mu_1}{\lambda_0+\lambda_1}B_2 + \frac{\mu_1}{\lambda_0+\lambda_1}B_3\\
&= c_L e,
\end{aligned}$$

$$\begin{aligned}
\left[\widehat{J}^n, \widehat{L}^n\right] &= \sum_{i=1}^3 \left[\widehat{\Psi}_i^n, \widehat{\Lambda}_i^n\right]\circ\overline{P}_i^n\\
&\overset{J_1}{\Longrightarrow} \frac{\lambda_1}{\lambda_0+\lambda_1}C_1 + \frac{\lambda_0-\mu_1}{\lambda_0+\lambda_1}C_2 + \frac{\mu_1}{\lambda_0+\lambda_1}C_3\\
&= c_{JL} e.
\end{aligned}$$

Since $\widehat{J}^n(0) \to w_0$ and $\widehat{L}^n(0) \to u_0$, we now apply [9], Theorem 1.4 of Section 7.1, to the sequence of martingales $\{\widehat{J}^n\}_{n=1}^{\infty}$ and $\{\widehat{L}^n\}_{n=1}^{\infty}$ relative to the filtrations $\{\mathcal{F}_s^n\}_{s\geq 0}$ to conclude that $(\widehat{J}^n, \widehat{L}^n)$ converges weakly-J_1 to (J^*, L^*), which is a two-dimensional correlated Brownian motion with the covariance matrix

$$\Sigma = \begin{bmatrix} c_J & c_{JL} \\ c_{JL} & c_L \end{bmatrix}.$$

□

3.5 Convergence of $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)$

Corollary 3.5.1

$$(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n) \stackrel{J_1}{\Longrightarrow} (\mathcal{U}^*, 0, \mathcal{W}^*),$$

where $(\mathcal{U}^*, \mathcal{W}^*)$ is a two-dimensional correlated Brownian motion with the covariance matrix

$$\Sigma = \begin{bmatrix} c_J & c_{JL} \\ c_{JL} & c_L \end{bmatrix}.$$

PROOF: The proof simply follows from (3.25) to (3.27), Theorem 3.3.3 and Proposition 3.4.1. □

Since $D[0,\infty)$ under the J_1 topology is separable, we can apply the Skorohod Representation Theorem to build a probability space $(\Omega, F, \mathbb{P})$ and random variables (for the convenience of the proof we do not relabel these), i.e., $(\mathcal{U}^*, 0, \mathcal{W}^*)$, $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{W}^n)_{n\geq 1}$, on this space such that we have pathwise convergence, i.e.,

$$(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{W}^n) \longrightarrow (\mathcal{U}^*, 0, \mathcal{W}^*)$$

almost surely. Let us define

$$\begin{aligned}
\tau_{\mathcal{U}}^n &= \inf\{t > 0 | \quad \widehat{\mathcal{U}}^n(t) \leq 0\}, \\
\tau_{\mathcal{W}}^n &= \inf\{t > 0 | \quad \widehat{\mathcal{W}}^n(t) \leq 0\}, \\
\tau_{\mathcal{U}} &= \inf\{t > 0 | \quad \mathcal{U}^*(t) \leq 0\}, \\
\tau_{\mathcal{W}} &= \inf\{t > 0 | \quad \mathcal{W}^*(t) \leq 0\}.
\end{aligned}$$

Proposition 3.5.2 *Under the probability space we mentioned above, we have*

$$\tau_{\mathcal{U}}^n \to \tau_{\mathcal{U}},$$
$$\tau_{\mathcal{W}}^n \to \tau_{\mathcal{W}}.$$

PROOF: The function $\Phi : C[0,\infty) \to [0,\infty]$ defined by

$$\Phi(x) = \min\{t \geq 0 : x(t) = 0\} \tag{3.45}$$

is almost surely continuous under Wiener measure. Because $\tau_{\mathcal{U}}^n = \Phi(\widehat{\mathcal{U}}^n)$ and $\widehat{\mathcal{U}}^n \to \mathcal{U}^*$ almost surely under the J_1 topology, the continuous mapping theorem implies

$$\tau_{\mathcal{U}}^n \to \Phi(\mathcal{U}^*) = \tau_{\mathcal{U}}$$

almost surely. We can prove the other convergence by the exact same argument. $\square$

Let us define,

$$\begin{aligned}\tilde{\sigma}^n &= \tau_{\mathcal{U}}^n \wedge \tau_{\mathcal{W}}^n,\\ \tilde{\sigma} &= \tau_{\mathcal{U}} \wedge \tau_{\mathcal{W}}.\end{aligned}$$

Corollary 3.5.3

$$(\widehat{\mathcal{U}}^n_{\cdot\wedge\tilde{\sigma}^n}, \widehat{\mathcal{V}}^n_{\cdot\wedge\tilde{\sigma}^n}, \widehat{\mathcal{W}}^n_{\cdot\wedge\tilde{\sigma}^n}) \xrightarrow{J_1} (\mathcal{U}^*_{\cdot\wedge\tilde{\sigma}}, 0, \mathcal{W}^*_{\cdot\wedge\tilde{\sigma}})$$

almost surely.

Proof: Because the mapping $\Theta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Theta(\alpha, \beta) = \alpha \wedge \beta \tag{3.46}$$

is continuous, from Proposition 3.5.2, we have

$$(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n, \tau_{\mathcal{U}}^n \wedge \tau_{\mathcal{W}}^n) \xrightarrow{J_1} (\mathcal{U}^*, 0, \mathcal{W}^*, \tau_{\mathcal{U}} \wedge \tau_{\mathcal{W}}) \tag{3.47}$$

almost surely. The mapping $(x, t) \mapsto x_{\cdot\wedge t}$ from $C[0,\infty) \times [0,\infty)$ to $C[0,\infty)$ is almost surely continuous under Wiener measure, and so from (3.47) and the continuous mapping theorem, we have

$$(\widehat{\mathcal{U}}^n_{\cdot\wedge\tilde{\sigma}^n}, \widehat{\mathcal{V}}^n_{\cdot\wedge\tilde{\sigma}^n}, \widehat{\mathcal{W}}^n_{\cdot\wedge\tilde{\sigma}^n}) \xrightarrow{J_1} (\mathcal{U}^*_{\cdot\wedge\tilde{\sigma}}, 0, \mathcal{W}^*_{\cdot\wedge\tilde{\sigma}})$$

almost surely. $\square$

3.6 Properties of the first passage times $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{W}}$

Since $(\mathcal{U}^*, \mathcal{W}^*)$ is a two-dimensional correlated Brownian motion starting at (w_0, u_0), and $(\tau_{\mathcal{U}}, \tau_{\mathcal{W}})$ are their first passage times, we can apply [23] to get following corollary. Since in [23], the two-dimensional Brownian motion (X_1, X_2) begins in the first quadrant (i.e., $X_1(0) > 0$, $X_2(0) > 0$) and has covariance matrix

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

and our correlated Brownian motion $(\mathcal{U}^*, \mathcal{W}^*)$ begins in the fourth quadrant (i.e., $u_0 > 0$, $w_0 < 0$) and has covariance matrix

$$\begin{bmatrix} c_J & c_{JL} \\ c_{JL} & c_L \end{bmatrix},$$

we identify $\mathcal{U}^*$ with X_1 and $-\mathcal{W}^*$ with X_2, so that, in the notation of [23],

$$\begin{aligned}
X_1(0) &:= u_0, \quad X_2(0) := -w_0, \\
\sigma_1 &:= \sqrt{c_J}, \quad \sigma_2 := \sqrt{c_G}, \\
\rho &:= -\frac{c_{GJ}}{\sqrt{c_J c_G}} < 0, \\
a_1 &:= \frac{u_0}{\sqrt{c_J}}, \quad a_2 := \frac{-w_0}{\sqrt{c_G}}.
\end{aligned}$$

From equations (2.4), (2.5), (3.2) and (3.3) in [23], we have

Corollary 3.6.1

$$\begin{aligned}
\mathbb{P}\{\tau_{\mathcal{U}} < \tau_{\mathcal{W}}\} &= \int_0^\infty \frac{1}{\alpha r_0} \frac{(r/r_0)^{(\pi/\alpha-1)} \sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha) + [(r/r_0)^{\pi/\alpha} + \cos(\pi\theta_0/\alpha)]^2} dr, \\
\mathbb{P}\{\tau_{\mathcal{U}} > \tau_{\mathcal{W}}\} &= \int_0^\infty \frac{1}{\alpha r_0} \frac{(r/r_0)^{(\pi/\alpha-1)} \sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha) + [(r/r_0)^{\pi/\alpha} - \cos(\pi\theta_0/\alpha)]^2} dr,
\end{aligned}$$

where

$$\begin{aligned}
\alpha &:= \tan^{-1}\Big(-\frac{\sqrt{1-\rho^2}}{\rho}\Big), \\
r_0 &:= \sqrt{\frac{a_1^2 + a_2^2 - 2\rho a_1 a_2}{1-\rho^2}}, \\
\theta_0 &:= \tan^{-1}\Big(\frac{a_2\sqrt{1-\rho^2}}{a_1 - \rho a_2}\Big).
\end{aligned}$$

Moreover, the joint density of $(\tau_{\mathcal{U}}, \tau_{\mathcal{W}})$ is given by

$$\mathbb{P}\{\tau_{\mathcal{U}} \in ds, \tau_{\mathcal{W}} \in dt\} = f(s,t) ds\, dt,$$

where for $s < t$ we have

$$\begin{aligned}
f(s,t) &= \frac{\pi \sin\alpha}{2\alpha^2\sqrt{s(t - s\cos^2\alpha)(t-s)}} \exp\Big(-\frac{r_0^2}{2s}\frac{t - s\cos(2\alpha)}{(t-s) + \big(t - s\cos(2\alpha)\big)}\Big) \\
&\quad \times \sum_{n=1}^\infty n \sin(\frac{n\pi(\alpha-\theta_0)}{\alpha}) I_{n\pi/2\alpha}\Big(\frac{r_0^2}{2s}\frac{t-s}{(t-s) + \big(t - s\cos(2\alpha)\big)}\Big),
\end{aligned}$$

and for $s > t$ we have

$$\begin{aligned}
f(s,t) &= \frac{\pi \sin\alpha}{2\alpha^2\sqrt{t(s - t\cos^2\alpha)(s-t)}} \exp\Big(-\frac{r_0^2}{2t}\frac{s - t\cos(2\alpha)}{(s-t) + \big(s - t\cos(2\alpha)\big)}\Big) \\
&\quad \times \sum_{n=1}^\infty n \sin(\frac{n\pi\theta_0}{\alpha}) I_{n\pi/2\alpha}\Big(\frac{r_0^2}{2t}\frac{s-t}{(s-t) + \big(s - t\cos(2\alpha)\big)}\Big),
\end{aligned}$$

where I_ν denotes the modified Bessel function of the first kind of order ν.

3.7 Convergence of $\widehat{\mathcal{T}}^n$ and $\widehat{\mathcal{X}}^n$

Theorem 3.7.1 *The sequence of càdlàg processes* $\{\widehat{\mathcal{T}}^n\}_{n=1}^\infty$ *and* $\{\widehat{\mathcal{X}}^n\}_{n=1}^\infty$ *are bounded in probability on compact time intervals.*

It suffices to prove $\{\widehat{\mathcal{T}}^n\}_{n=1}^\infty$ is bounded in probability on compact time intervals. The proof of this theorem is presented in Lemmas 3.7.2 and 3.7.3 below. For simplicity, we write $X^n = \widehat{\mathcal{O}_{cl}}(1)$ if $\{X^n\}_{n=1}^\infty$ is both bounded above and below in probability.

Lemma 3.7.2 *The sequence of processes* $\{\widehat{\mathcal{T}}^n\}_{n=1}^\infty$ *is bounded above in probability on compact time intervals.*

Proof: To simplify notation, we rewrite (3.33) as

$$\widehat{\mathcal{T}}^n = \widehat{\mathcal{T}}^n(0) + Y_1^n + Y_2^n + Y_3^n + Y_4^n, \tag{3.48}$$

where

$$\begin{aligned}
Y_1^n(t) &= -\widehat{M}^n_{1,\mathcal{T},-}\left(\int_0^t \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \, d\overline{P}_1^n(s)\right), && (3.49)\\
Y_2^n(t) &= \widehat{M}^n_{3,\mathcal{T},+}\big(\lambda_2 \overline{P}_3^n(t)\big), && (3.50)\\
Y_3^n(t) &= -\sqrt{n}\int_0^t \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \, d\overline{P}_1^n(s), && (3.51)\\
Y_4^n(t) &= \sqrt{n}\lambda_2 \overline{P}_3^n(t). && (3.52)
\end{aligned}$$

Then

$$Y_2^n \xRightarrow{J_1} Y_2^*, \tag{3.53}$$

where Y_2^* is a continuous process. We rewrite Y_4^n as

$$Y_4^n = \sqrt{n}\Big(\lambda_2 \overline{P}_3^n(t) - \frac{\lambda_2\mu_1}{\lambda_1}\overline{P}_1^n(t)\Big) + \sqrt{n}\frac{\lambda_2\mu_1}{\lambda_1}\overline{P}_1^n(t).$$

From (3.43) we see that

$$\sqrt{n}\Big(\lambda_2 \overline{P}_3^n(t) - \frac{\lambda_2\mu_1}{\lambda_1}\overline{P}_1^n(t)\Big) = \mathcal{O}_{cl}(1) = \widehat{\mathcal{O}_{cl}}(1),$$

so

$$Y_4^n = \frac{\lambda_2\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n + \widehat{\mathcal{O}_{cl}}(1). \tag{3.54}$$

If N is a unit-intensity Poisson process, then $-N(t) + \frac{1}{2}t$ is a supermartingale whose supremum S^* over $t \geq 0$ is finite almost surely. Therefore,

$$-\frac{1}{\sqrt{n}}\big(N(nt) - nt\big) - \sqrt{n}\,t = \frac{1}{\sqrt{n}}\left[-N(nt) + \frac{1}{2}nt\right] - \frac{1}{2}\sqrt{n}\,t \leq \frac{1}{\sqrt{n}}S^* - \frac{1}{2}\sqrt{n}\,t,$$

and hence by substituting N by $N_{1,\mathcal{T},-}$ and t by $\int_0^t \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \, d\overline{P}_1^n(s)$, we obtain

$$Y_1^n(t) + Y_3^n(t) \le \frac{1}{2} Y_3^n(t) + \frac{1}{\sqrt{n}} \widehat{\mathcal{O}_{cl}}(1). \tag{3.55}$$

Combining (3.53), (3.54), and (3.55) we obtain

$$\begin{aligned} \widehat{\mathcal{T}}^n(t) &\le \frac{1}{2} Y_3^n(t) + \frac{\lambda_2\mu_1}{\lambda_1} \sqrt{n} \overline{P}_1^n + Y_2^n(t) + \widehat{\mathcal{O}_{cl}}(1) \\ &= \sqrt{n} \int_0^t \left(\frac{\lambda_2\mu_1}{\lambda_1} - \frac{1}{2} \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \right) d\overline{P}_1^n(s) + \widehat{\mathcal{O}_{cl}}(1). \end{aligned} \tag{3.56}$$

Let us fix $T > 0$ and consider $t \in [0, T]$. Either

$$\int_0^t \left(\frac{\lambda_2\mu_1}{\lambda_1} - \frac{1}{2} \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \right) d\overline{P}_1^n(s) \le 0, \tag{3.57}$$

or else

$$\int_0^t \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \, d\overline{P}_1^n(s) \le 2 \frac{\lambda_2\mu_1}{\lambda_1} \overline{P}_1^n(t) \le 2 \frac{\lambda_2\mu_1}{\lambda_1} T. \tag{3.58}$$

We define

$$\tau^n(t) := \begin{cases} t & \text{if (3.57) holds,} \\ \sup\left\{ s \in [0,t] : \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \le 2\frac{\lambda_2\mu_1}{\lambda_1} \right\} & \text{if (3.58) holds.} \end{cases} \tag{3.59}$$

If (3.57) holds and $\tau^n(t) = t$, $\widehat{\mathcal{T}}^n(t)$ is bounded by the $\widehat{\mathcal{O}_{cl}}(1)$ term in (3.56). If (3.58) holds, then

$$\widehat{\mathcal{T}}^n(t) \le \widehat{\mathcal{T}}^n\big(\tau^n(t)\big) + \sum_{i=1}^4 \Big[Y_i^n(t) - Y_i^n\big(\tau^n(t)\big) \Big]. \tag{3.60}$$

We consider each of the four terms on the right-hand side of (3.60). Since the jumps in $\widehat{\mathcal{T}}^n$ are of size $\frac{1}{\sqrt{n}}$, we must have

$$\widehat{\mathcal{T}}^n\big(\tau^n(t)\big) \le 2 \frac{\lambda_2\mu_1}{\lambda_1} \frac{1}{\theta_b} + \frac{1}{\sqrt{n}}. \tag{3.61}$$

Because of the bound (3.58) on the argument of $\widehat{M}_{1,\mathcal{T},-}^n$, both $Y_1^n(t)$ and $Y_1^n\big(\tau^n(t)\big)$ are $\widehat{\mathcal{O}_{cl}}(1)$. Similarly, both $Y_2^n(t)$ and $Y_2^n\big(\tau^n(t)\big)$ are $\widehat{\mathcal{O}_{cl}}(1)$. It follows that

$$\begin{aligned} \widehat{\mathcal{T}}^n(t) &\le Y_3^n(t) - Y_3^n\big(\tau^n(t)\big) + \sqrt{n} \frac{\lambda_2\mu_1}{\lambda_1} \Big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\tau^n(t)\big) \Big) + \widehat{\mathcal{O}_{cl}}(1) \\ &= \sqrt{n} \int_{\tau^n(t)}^t \left(\frac{\lambda_2\mu_1}{\lambda_1} - \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \right) d\overline{P}_1^n(s) + \widehat{\mathcal{O}_{cl}}(1) \\ &\le -\sqrt{n} \frac{\lambda_2\mu_1}{\lambda_1} \Big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\tau^n(t)\big) \Big) + \widehat{\mathcal{O}_{cl}}(1), \end{aligned} \tag{3.62}$$

because $\theta(\widehat{\mathcal{T}}^n(s))^+ \ge 2\frac{\lambda_2\mu_1}{\lambda_1}$ for $s \in [\tau^n(t), t]$. Again we have an upper bound on $\widehat{\mathcal{T}}^n$. In conclusion, $\{\widehat{\mathcal{T}}^n\}_{n=1}^\infty$ is bounded above in probability on compact time intervals. $\square$

Lemma 3.7.3 *The sequence of processes $\{\widehat{\mathcal{T}}^n\}_{n=1}^\infty$ is bounded below in probability on compact time intervals.*

PROOF: We return to (3.33) and note that because $\widehat{\mathcal{T}}^n$ is bounded above in probability on compact time intervals and $d\overline{P}_1^n \leq dt$, the sequence of processes $\{\int_0^\cdot \theta_b(\widehat{\mathcal{T}}^n)^+ \, d\overline{P}_1^n\}_{n=1}^\infty$ is bounded in probability on compact time intervals. Consequently, the sequence of processes

$$\left\{\widehat{M}_{1,\mathcal{T},-}^n \circ \int_0^\cdot \theta_b(\widehat{\mathcal{T}}^n)^+ \, d\overline{P}_1^n\right\}_{n=1}^\infty$$

is bounded in probability on compact time intervals. In addition, the other process $\widehat{M}_{3,\mathcal{T},+}^n \circ \lambda_2 \overline{P}_3^n$ on the right-hand side of (3.33) involving a scaled, centered Poisson process is bounded in probability on compact time intervals. This permits us to write

$$\widehat{\mathcal{T}}^n(t) = \sqrt{n}\left[-\int_0^t \theta_b\big(\widehat{\mathcal{T}}^n(s)\big)^+ \, d\overline{P}_1^n(s) + \lambda_2 \overline{P}_3^n(t)\right] + \widehat{\mathcal{O}_{cl}}(1). \tag{3.63}$$

We define

$$\rho^n(t) := \sup\left\{s \in [0,t] : \widehat{\mathcal{T}}^n(s) \geq 0\right\}. \tag{3.64}$$

Then $\widehat{\mathcal{T}}^n(s) < 0$ for $\rho^n(t) < s \leq t$ and (3.64) implies

$$\begin{aligned}\widehat{\mathcal{T}}^n(t) &= \widehat{\mathcal{T}}^n\big(\rho^n(t)\big) + \sqrt{n}\lambda_2\big(\overline{P}_3^n(t) - \overline{P}_3^n\big(\rho^n(t)\big)\big) + \widehat{\mathcal{O}_{cl}}(1) \\ &\geq \widehat{\mathcal{T}}^n\big(\rho^n(t)\big) + \widehat{\mathcal{O}_{cl}}(1). \end{aligned}\tag{3.65}$$

Because $\widehat{\mathcal{T}}^n(\rho^n(t)) \geq -\frac{1}{\sqrt{n}}$, we conclude that $\widehat{\mathcal{T}}^n \geq \widehat{\mathcal{O}_{cl}}(1)$. $\square$

We can rewrite (3.33) and (3.34) as

$$\begin{aligned}\widehat{\mathcal{T}}^n(t) &= \widehat{\mathcal{T}}^n(0) - \widehat{M}_{1,\mathcal{T},-}^n\big(\int_0^t \theta_b(\widehat{\mathcal{T}}^n(s))^+ d\overline{P}_1^n(s)\big) + \widehat{M}_{3,\mathcal{T},+}^n\big(\lambda_2 \overline{P}_3^n(t)\big) \\ &\quad -\sqrt{n}\int_0^t \big(\theta_b(\widehat{\mathcal{T}}^n(s))^+ - \frac{\lambda_2\mu_1}{\lambda_1}\big) d\overline{P}_1^n(s) \\ &\quad +\sqrt{n}\big(\lambda_2 \overline{P}_3^n(t) - \frac{\lambda_2\mu_1}{\lambda_1}\overline{P}_1^n(t)\big), \end{aligned}\tag{3.66}$$

$$\begin{aligned}\widehat{\mathcal{X}}^n(t) &= \widehat{\mathcal{X}}^n(0) - \widehat{M}_{1,\mathcal{X},-}^n\big(\mu_2 \overline{P}_1^n(t)\big) + \widehat{M}_{3,\mathcal{X},+}^n\big(\int_0^t \theta_s(\widehat{\mathcal{X}}^n(s))^- d\overline{P}_3^n(s)\big) \\ &\quad +\sqrt{n}\int_0^t \big(\theta_s(\widehat{\mathcal{X}}^n(s))^- - \frac{\mu_2\lambda_1}{\mu_1}\big) d\overline{P}_3^n(s) \\ &\quad +\sqrt{n}\big(\frac{\mu_2\lambda_1}{\mu_1}\overline{P}_3^n(t) - \mu_2 \overline{P}_1^n(t)\big). \end{aligned}\tag{3.67}$$

Because of Theorem 3.7.1 and (3.43), we obtain

$$\widehat{\mathcal{T}}^n(t) = \widehat{\mathcal{T}}^n(0) - \sqrt{n}\int_0^t \big(\theta_b(\widehat{\mathcal{T}}^n(s))^+ - \frac{\lambda_2\mu_1}{\lambda_1}\big) d\overline{P}_1^n(s) + C_{\mathcal{T}}^n(t), \tag{3.68}$$

$$\widehat{\mathcal{X}}^n(t) = \widehat{\mathcal{X}}^n(0) + \sqrt{n}\int_0^t \big(\theta_s(\widehat{\mathcal{X}}^n(s))^- - \frac{\mu_2\lambda_1}{\mu_1}\big) d\overline{P}_3^n(s) + C_{\mathcal{X}}^n(t), \tag{3.69}$$

where $C^n_{\mathcal{T}} = \widehat{\mathcal{O}_{cl}}(1)$ and $C^n_{\mathcal{X}} = \widehat{\mathcal{O}_{cl}}(1)$. In fact, $C^n_{\mathcal{T}}$ and $C^n_{\mathcal{X}}$ have continuous limits along some subsequence.

Let $D[0-,T] = \mathbb{R} \times D[0,T]$ denote the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$ augmented by a value at $0-$. For $x \in D[0-,T]$, let the *augmented graph* of x be

$$\Gamma_x := \{(z,t) \in \mathbb{R} \times [0,T] \,:\, z \in [x(t-), x(t)]\}.$$

A *parametric representation* of x is a continuous nondecreasing function (u,r) mapping $[0,1]$ onto Γ_x. For the parametric representation, "nondecreasing" is with respect to the usual order on the domain $[0,1]$ and order on the graph defined above. Let $\Pi(x)$ be the set of parametric representations of x, and define

$$d(x_1,x_2) := \inf_{(u_1,r_1)\in\Pi(x_1),(u_2,r_2)\in\Pi(x_2)} \{\|u_1 - u_2\| \vee \|r_1 - r_2\|\},$$

where $\|\cdot\|$ is the supremum norm on $[0,1]$. Then the topology induced by d will be the M_1 topology on $D[0-,T]$. Once we have the topology on the compact domain, we can construct the topology on $D[0-,\infty)$ as usual.

For convenience, we denote $(D[0-,\infty), M_1)$ by $D_M[0-,\infty)$. In the following proposition, we will work under $D_M[0-,\infty)$. For each pre-limit processes $\widehat{\mathcal{T}}^n$ and $\widehat{\mathcal{X}}^n$, let

$$\widehat{\mathcal{T}}^n(0-) := \widehat{\mathcal{T}}^n(0), \quad \widehat{\mathcal{X}}^n(0-) := \widehat{\mathcal{X}}^n(0).$$

Proposition 3.7.4

$$\begin{aligned} \widehat{\mathcal{T}}^n &\Longrightarrow \mathcal{T}^*, \\ \widehat{\mathcal{X}}^n &\Longrightarrow \mathcal{X}^*, \end{aligned}$$

in $D_M[0-,\infty)$, where

$$\begin{aligned} \mathcal{T}^*(0-) &= t_0, \quad \mathcal{T}^*(t) = \frac{\lambda_2\mu_1}{\theta_b\lambda_1}, \text{ for } t \geq 0, \\ \mathcal{X}^*(0-) &= x_0, \quad \mathcal{X}^*(t) = -\frac{\mu_2\lambda_1}{\theta_s\mu_1}, \text{ for } t \geq 0. \end{aligned}$$

Proof: We appeal to Section 4.5 and 4.6 of [2]. Equation (4.51) in [2] considers the process

$$\widehat{\mathcal{V}}^n(t) = \widehat{\mathcal{V}}^n(0) + C^n(t) - \sqrt{n}\int_0^t \big(\theta(\widehat{\mathcal{V}}^n(s))^+ - 1\big) d\overline{P}^n_1(s),$$

where $(C^n, n \geq 0)$ converges in distribution to a continuous process. We have the analogous equation (3.68) for $\widehat{\mathcal{T}}^n$. However, in [2], $\widehat{\mathcal{V}}^n$ is modeulated by another process $\widehat{\mathcal{J}}^n$ that takes positive and negative excursions. When $\widehat{\mathcal{J}}^n$ is on a positive excursion, there is a positive queue $\widehat{\mathcal{W}}^n$ at the price tick adjacent to $\widehat{\mathcal{V}}^n$ to the right; when $\widehat{\mathcal{J}}^n$ is on a negative excursion, this price tick is empty and $\widehat{\mathcal{V}}^n$ is at the bid price. In our setting, prior to the stopping time σ^n, there is always a queue adjacent to $\widehat{\mathcal{T}}^n$ to the right, and we assume the dynamics of $\widehat{\mathcal{T}}^n$ are forever as they are prior to σ^n, According to Therorem 4.6.3 in [2], $\widehat{\mathcal{V}}^n$ converges in $D_M[0-,\infty)$ to a process $\mathcal{V}^*$ that is $1/\theta$ on the positive excursions of $\mathcal{J}^* = \lim_{n\to\infty} \widehat{J}^n$ and $\mathcal{V}^*(0-) = \lim_{n\to\infty} \widehat{\mathcal{V}}^n(0)$. This result in our setting establishes the claimed convergence of $\widehat{\mathcal{T}}^n$. The proof for $\widehat{\mathcal{X}}^n$ is similar. $\square$

Let $D_J[0,\infty)$ be $D[0,\infty)$ with the J_1 topology. We define

$$\mathbb{D} = D_M[0-,\infty) \times D_J[0,\infty) \times D_J[0,\infty) \times D_J[0,\infty) \times D_M[0-,\infty),$$

on which we use the product topology and the σ-algebra generated by this topology.

The processes

$$\begin{aligned} \widehat{\mathcal{S}}^n &:= (\widehat{\mathcal{T}}^n, \widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n, \widehat{\mathcal{X}}^n), \\ \mathcal{S}^* &:= (\mathcal{T}^*, \mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*, \mathcal{X}^*), \end{aligned}$$

take values in $\mathbb{D}$. From Corollary 3.5.3 and Proposition 3.7.4, we see that there exists a probability space on which we can define $\widehat{\mathcal{S}}^n$ and $\mathcal{S}^*$ so that

- $\widehat{\mathcal{S}}^n \to \mathcal{S}^*$ almost surely.
- Until the first time $\widehat{U}^n$ or $\widehat{V}^n$ vanishes, the distribution of $(\widehat{T}^n, \widehat{U}^n, \widehat{V}^n, \widehat{W}^n, \widehat{X}^n)$ on $\mathbb{D}$ agrees with the distribution of $\widehat{\mathcal{S}}^n$ on $\mathbb{D}$.

More specifically, let us define $\sigma : \mathbb{D} \to [0,\infty]$ by

$$\sigma(t,u,v,w,x) = \inf\{s \geq 0 \,:\, u(s) = 0 \text{ or } w(s) = 0\}.$$

We define

$$\widehat{S}^n = (\widehat{T}^n, \widehat{U}^n, \widehat{V}^n, \widehat{W}^n, \widehat{X}^n),$$

and

$$\begin{aligned} \widehat{S}^n_{stopped} &= (\widehat{T}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{U}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{V}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{W}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{X}^n_{\cdot\wedge\sigma(\widehat{S}^n)}), \\ \widehat{\mathcal{S}}^n_{stopped} &= (\widehat{\mathcal{T}}^n_{\cdot\wedge\sigma(\widehat{\mathcal{S}}^n)}, \widehat{\mathcal{U}}^n_{\cdot\wedge\sigma(\widehat{\mathcal{S}}^n)}, \widehat{\mathcal{V}}^n_{\cdot\wedge\sigma(\widehat{\mathcal{S}}^n)}, \widehat{\mathcal{W}}^n_{\cdot\wedge\sigma(\widehat{\mathcal{S}}^n)}, \widehat{\mathcal{X}}^n_{\cdot\wedge\sigma(\widehat{\mathcal{S}}^n)}), \\ \mathcal{S}^*_{stopped} &= (\mathcal{T}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{U}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{V}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{W}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{X}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}). \end{aligned}$$

$\widehat{S}^n_{stopped}$ is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\widehat{\mathcal{S}}^n_{stopped}$ and $\mathcal{S}^*_{stopped}$ are defined on another probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$.

Theorem 3.7.5 *The measure Q^n induced on $\mathbb{D}$ by $\widehat{S}^n_{stopped}$ is also the measure induced on $\mathbb{D}$ by $\widehat{\mathcal{S}}^n_{stopped}$. Moreover,*

$$\widehat{\mathcal{S}}^n_{stopped} \to \mathcal{S}^*_{stopped} \quad \mathbb{P}^1 \text{ almost surely.}$$

Let Q^* be the measure induced on $\mathbb{D}$ by $\mathcal{S}^*_{stopped}$. Then we have

$$Q^n \Rightarrow Q^*.$$

Chapter 4

Brownian motion preliminaries

In Chapter 3 we started the LOB with initial condition (3.1)–(3.3) and followed it until either U^n or W^n hit zero, or equivalently, until either $\widehat{\mathcal{U}}^n$ or $\widehat{\mathcal{W}}^n$ hit zero. In Chapter 5 we will assume without loss of generality that

$$\sigma^* = \tau_{\mathcal{W}^*} < \tau_{\mathcal{U}^*},$$

i.e., $\mathcal{W}^*$ reaches zero before $\mathcal{V}^*$. Under this assumption,

$$\lim_{n\to\infty} \widehat{U}^n(\sigma^n) = \mathcal{U}^*(\sigma^*) > 0 \qquad \lim_{n\to\infty} \widehat{V}^n(\sigma^n) = \mathcal{V}^*(\sigma^*) = 0,$$
$$\lim_{n\to\infty} \widehat{W}^n(\sigma^n) = \mathcal{W}^*(\sigma^*) = 0, \qquad \lim_{n\to\infty} \widehat{X}^n(\sigma^n) = \mathcal{X}^*(\sigma^*) = \kappa_R < 0.$$

These convergences are joint weak convergence in the J_1 topology, and by using the Skorohod Representation Theorem, we can put all processes on a common probability space so that the convergences are almost sure. In Chapter 5, we will reset the clock to zero at time σ^n for the pre-limit processes and at time σ^* for the limiting processes, and hence we will study the evolution of (U^n, V^n, W^n, X^n) beginning from the initial condition

$$U^*(0) := \lim_{n\to\infty} \frac{1}{\sqrt{n}} U^n(0) > 0, \quad V^*(0) := \lim_{n\to\infty} \frac{1}{\sqrt{n}} V^n(0) = 0, \tag{4.1}$$

$$W^*(0) := \lim_{n\to\infty} \frac{1}{\sqrt{n}} W^n(0) = 0, \quad X^*(0) := \lim_{n\to\infty} \frac{1}{\sqrt{n}} X^n(0) = \kappa_R < 0. \tag{4.2}$$

Similarly to the construction in Chapter 3, in Chapter 5 we will define processes $\widehat{\mathcal{U}}^n$, $\widehat{\mathcal{V}}^n$, $\widehat{\mathcal{W}}^n$ and $\widehat{\mathcal{X}}^n$ that agree with $\widehat{U}^n$, $\widehat{V}^n$, $\widehat{W}^n$ and $\widehat{X}^n$ for an initial period of time (in this case, until either $\widehat{U}^n$ or $\widehat{X}^n$ hits zero), but which continue to be governed by the same dynamics after this time as before it. We will discover that the limiting processes for $\widehat{\mathcal{V}}^n$ and $\widehat{\mathcal{W}}^n$ constitute a *split two-variance Brownian motion*, defined by the formula

$$(\mathcal{V}^*, \mathcal{W}^*) = \big(\max(G^*, 0), \min(G^*, 0)\big), \tag{4.3}$$

where G^* is a *two-variance Brownian motion* (see Definition 4.1.1 below) with variance c_+ per unit time on its positive excursions away from zero and variance c_- per unit time on its negative excursions away from zero.

In fact, what we are able to show initially is that every subsequence of $\{(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)\}_{n=1}^{\infty}$ has a sub-subsequence that converges weakly to a pair of processes $(\mathcal{V}^*, \mathcal{W}^*)$ satisfying (4.3). To see that the full sequence $\{(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)\}_{n=1}^{\infty}$ converges, we must show that a two-variance Brownian motion with given variances per unit time on positive and negative excursions is unique in law. It turns out that we also need to know how occupations times for the two variance Brownian motion are determined by Brownian motions used in its construction. This chapter provides these results.

In particular, Section 4.1 defines two-variance Brownian motion and derives its elementary properties. We choose to define it in such a way that the uniqueness in law of two-variance Brownian motion is immediate. We characterize it in the way that it will appear in Chapter 5. This characterization, which involves a detour through Brownian excursion theory, is developed in Section 4.2.

4.1 Two-variance Brownian motion

Definition 4.1.1 *Assume B is a standard Brownian motion starting at b_0, and c_+ and c_- are two positive real numbers. We call Z a* two-variance Brownian motion *if*

$$Z = B \circ (\frac{1}{c_+} P_B^+ + \frac{1}{c_-} P_B^-)^{-1},$$

where

$$P_B^{\pm}(t) = \int_0^t \mathbb{1}_{\{\pm B(s)>0\}} ds.$$

We say Z has variance c_+ per unit time on positive excursions and variance c_- per unit time on negative excursions.

Proposition 4.1.2 *Assume Z is a two-variance Brownian motion, i.e.*

$$Z = B \circ (\frac{1}{c_+} P_B^+ + \frac{1}{c_-} P_B^-)^{-1}.$$

Then

$$Z = B \circ (c_+ P_Z^+ + c_- P_Z^-), \tag{4.4}$$

where

$$\begin{aligned} P_Z^+(t) &= \int_0^t \mathbb{1}_{\{Z(s)>0\}} ds, \\ P_Z^-(t) &= \int_0^t \mathbb{1}_{\{Z(s)<0\}} ds. \end{aligned}$$

Moreover, for $t \geq 0$, we have

$$t = P_Z^+(t) + P_Z^-(t).$$

Proof: To prove (4.4), it suffices to show

$$(\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1} = (c_+P_Z^+ + c_-P_Z^-).$$

Note that

$$Z = B \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1}$$

implies

$$B = Z \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-).$$

Let

$$u = (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(s).$$

then

$$B(s) = Z \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(s) = Z(u).$$

Together with the definition of P_B^+ and P_B^-, we have

$$\begin{aligned} du &= d\big(\frac{1}{c_+}P_B^+(s) + \frac{1}{c_-}P_B^-(s)\big) \\ &= (\frac{1}{c_+}\mathbb{1}_{\{B(s)>0\}} + \frac{1}{c_-}\mathbb{1}_{\{B(s)<0\}})ds \\ &= (\frac{1}{c_+}\mathbb{1}_{\{Z(u)>0\}} + \frac{1}{c_-}\mathbb{1}_{\{Z(u)<0\}})ds, \end{aligned}$$

which also implies,

$$ds = (c_+\mathbb{1}_{\{Z(u)>0\}} + c_-\mathbb{1}_{\{Z(u)<0\}})du. \tag{4.5}$$

Since B is a standard Brownian motion, we know that Lebesgue measure of the time it spends at 0 till any time $t > 0$ is zero. Then through time change and (4.5), we have

$$\begin{aligned} t &= P_B^+(t) + P_B^-(t) \\ &= \int_0^t (\mathbb{1}_{\{B(s)>0\}} + \mathbb{1}_{\{B(s)<0\}})ds \\ &= \int_0^{(\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(t)} (c_+\mathbb{1}_{\{Z(u)>0\}} + c_-\mathbb{1}_{\{Z(u)<0\}})du \\ &= c_+P_Z^+ \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(t) + c_-P_Z^- \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(t) \\ &= (c_+P_Z^+ + c_-P_Z^-) \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(t), \end{aligned}$$

and this implies

$$(\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1} = (c_+P_Z^+ + c_-P_Z^-).$$

Since $\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-$ is absolutely continuous, and $B = Z \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)$, we see that $\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-$ maps the Lebesgue measure zero set $\{s : B(s) = 0\}$ to the Lebesgue zero set $\{u : Z(u) = 0\}$, which proves the last equation in the proposition. $\square$

Proposition 4.1.3 *Assume there exists a process Z satisfying*

$$Z = B \circ (c_+P_Z^+ + c_-P_Z^-),$$

where B is a standard Brownian motion starting at b_0, and

$$\begin{aligned} P_Z^+(t) &= \int_0^t \mathbb{1}_{\{Z(s)>0\}} ds, \\ P_Z^-(t) &= \int_0^t \mathbb{1}_{\{Z(s)<0\}} ds, \\ t &= P_Z^+(t) + P_Z^-(t). \end{aligned}$$

Then

$$Z = B \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1},$$

where

$$\begin{aligned} P_B^+(t) &= \int_0^t \mathbb{1}_{\{B(s)>0\}} ds, \\ P_B^-(t) &= \int_0^t \mathbb{1}_{\{B(s)<0\}} ds. \end{aligned}$$

In particular, Z is a two-variance Brownian motion.

Proof: The proof is similar to the proof of Proposition 4.1.2. It suffices to show

$$(\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1} = (c_+P_Z^+ + c_-P_Z^-).$$

Let

$$s = (c_+P_Z^+ + c_-P_Z^-)(u),$$

then

$$Z(u) = B \circ (c_+P_Z^+ + c_-P_Z^-)(u) = B(s).$$

Together with the definition of P_Z^+ and P_Z^-, we have

$$\begin{aligned} ds &= d\big(c_+P_Z^+(u) + c_-P_Z^-(u)\big) \\ &= (c_+\mathbb{1}_{\{Z(u)>0\}} + c_-\mathbb{1}_{\{Z(u)<0\}})du \\ &= (c_+\mathbb{1}_{\{B(s)>0\}} + c_-\mathbb{1}_{\{B(s)<0\}})du, \end{aligned}$$

which also implies

$$du = (\frac{1}{c_+}\mathbb{1}_{\{B(s)>0\}} + \frac{1}{c_-}\mathbb{1}_{\{B(s)<0\}})ds. \tag{4.6}$$

Since $t = P_Z^+(t) + P_Z^-(t)$, through the time change and (4.6), we have

$$\begin{aligned} t &= P_Z^+(t) + P_Z^-(t) \\ &= \int_0^t (\mathbb{1}_{\{Z(u)>0\}} + \mathbb{1}_{\{Z(u)<0\}}) du \\ &= \int_0^{(c_+P_Z^+ + c_-P_Z^-)(t)} (\frac{1}{c_+}\mathbb{1}_{\{B(s)>0\}} + \frac{1}{c_-}\mathbb{1}_{\{B(s)<0\}}) ds \\ &= \frac{1}{c_+} P_B^+ \circ (c_+P_Z^+ + c_-P_Z^-)(t) + \frac{1}{c_-} P_B^- \circ (c_+P_Z^+ + c_-P_Z^-)(t) \\ &= (\frac{1}{c_+} P_B^+ + \frac{1}{c_-} P_B^-) \circ (c_+P_Z^+ + c_-P_Z^-)(t), \end{aligned}$$

and this implies

$$(\frac{1}{c_+} P_B^+ + \frac{1}{c_-} P_B^-)^{-1} = (c_+P_Z^+ + c_-P_Z^-),$$

which completes the proof. □

From Proposition 4.1.2, we see that when Z is positive, Z behaves like an accelerated Brownian motion by factor c_+, and when Z is negative, it behaves like an accelerated Brownian motion by factor c_-. This is why we call Z a two-variance Brownian motion. In other words, each positive excursion of the standard Brownian motion B will be mapped to a positive excursion of Z with length stretched by $\frac{1}{c_+}$, and each negative excursion of the standard Brownian motion B will be mapped to a negative excursion of Z with length stretched by $\frac{1}{c_-}$. This leads us to construct such a process through a Poisson random measure. In particular, we want to build a Poisson random measure ν on $\mathcal{H} = ([0,\infty) \times C([0,\infty)))$ with intensity measure λ such that

$$\lambda(ds, de) = ds * n(de),$$

where e is the excursion starting at local time s, if there is such an excursion, and n is the excursion measure. The first entry in $\mathcal{H}$ indicates the local time when there is an excursion, and $n(de)$ describes the distribution of this excursion. We want to define a map from local time clock to chronological clock. Let $L_B^{-1} : [0,\infty) \to [0,\infty)$ be

$$L_B^{-1}(\theta) := \int_0^\theta \int_{C([0,\infty))} \sigma(e_s)\nu(ds\, de),$$

where $\sigma(e)$ represents the length of excursion e. Note that L_B^{-1} is right-continuous and is a strictly increasing, pure jump process. $L_B^{-1}(\ell)$ computes how much chronological time has passed when the local time of B reaches ℓ. Its inverse, which is continuous, is,

$$L_B(t) := \inf\{\theta \geq 0 : L_B^{-1}(\theta) > t\}.$$

From page 130 of [18], we can construct B as follows:

$$B(t) = \begin{cases} 0, & \text{if } L_B^{-1}(L_B(t)-) = L_B^{-1}(L_B(t)), \\ e_{L_B(t)}(t - L_B^{-1}(L_B(t)-)), & \text{if } L_B^{-1}(L_B(t)-) \neq L_B^{-1}(L_B(t)). \end{cases} \tag{4.7}$$

Note that $L_B^{-1}(L_B(t)-)$ is the starting time of the current excursion if B is on a excursion. Obviously, Z and B have their own clocks. Since

$$Z = B \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1},$$

the map between the two clocks, β : Clock of $Z \to$ Clock of B, is

$$\beta(t) := (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1}(t) =: s. \tag{4.8}$$

We also define

$$L_Z^{-1}(\ell) := \beta^{-1} \circ L_B^{-1}(\ell).$$

Since β is a strictly increasing continuous function, the right-continuous inverse of L_Z^{-1} is

$$L_Z(t) = L_B \circ \beta(t).$$

Whenever Z is on an excursion, B is also on an excursion, and the ratio of the length of excursion on Z to that of B is $\frac{1}{c_+}$ if the excursion is positive, and $\frac{1}{c_-}$ if the excursion is negative. So for any $t \geq 0$, let

$$\hat{e}_{L_Z(t)}(u) := \begin{cases} e_{L_B(s)}(c_+u), & \text{if } u \in [0, \frac{\sigma(e_{L_B(s)})}{c_+}] \text{ and } e_{L_B(s)}(c_+u) > 0, \\ e_{L_B(s)}(c_-u), & \text{if } u \in [0, \frac{\sigma(e_{L_B(s)})}{c_-}] \text{ and } e_{L_B(s)}(c_-u) < 0, \end{cases} \tag{4.9}$$

where $s = \beta(t)$. Assume Z is on an excursion at time t. Then the time when this excursion begins is

$$\begin{aligned} g_{start}(t) &:= \beta^{-1} \circ L_B^{-1} \circ (L_B(\beta(t))-) \\ &= L_Z^{-1}(L_Z(t)-). \end{aligned}$$

Now, we are ready to construct Z.

Proposition 4.1.4 *Let us define a process Z as follows:*

$$Z(t) = \begin{cases} 0, & \text{if } g_{start}(t) = L_Z^{-1}(L_Z(t)), \\ \hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)), & \text{if } g_{start}(t) \neq L_Z^{-1}(L_Z(t)), \end{cases} \tag{4.10}$$

where g_{start}, L_Z^{-1}, L_Z, and $\hat{e}_{L_Z}$ are defined previously. Then Z is a two-variance Brownian motion.

PROOF: From Definition 4.1.1, it suffices to show $B = Z \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)$, where B is a standard Brownian motion. From the definition of L_Z^{-1}, L_Z we have with $s = \beta(t)$,

$$\begin{aligned} L_Z^{-1}(L_Z(t)-) &= \beta^{-1} \circ L_B^{-1} \circ (L_B(\beta(t))-) = \beta^{-1} \circ L_B^{-1} \circ (L_B(s)-), \\ L_Z^{-1}(L_Z(t)) &= \beta^{-1} \circ L_B^{-1} \circ (L_B(\beta(t))) = \beta^{-1} \circ L_B^{-1} \circ (L_B(s)). \end{aligned}$$

Since β is continuous and strictly increasing, we have with $s = \beta(t)$

$$L_Z^{-1}(L_Z(t)-) = L_Z^{-1}(L_Z(t)) \Leftrightarrow L_B^{-1} \circ (L_B(s)-) = L_B^{-1} \circ (L_B(s)). \tag{4.11}$$

If $L_Z^{-1}(L_Z(t)-) \neq L_Z^{-1}(L_Z(t))$ and $\hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)) < 0$, from (4.9), we have

$$\begin{aligned}
&\hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)) = e_{L_B(s)}(c_-(t - L_Z^{-1}(L_Z(t)-))) \\
&\quad = e_{L_B(s)}(c_-(t - \beta^{-1} \circ L_B^{-1}(L_B \circ \beta(t)-))) \\
&\quad = e_{L_B(s)}(c_-(\beta^{-1}(s) - \beta^{-1} \circ L_B^{-1}(L_B(s)-))).
\end{aligned} \tag{4.12}$$

Note that $\beta^{-1} = \frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-$, and $L_B^{-1}(L_B(s)-))$ indicates the starting time (clock of B) of the current excursion which is negative. Therefore

$$\beta^{-1}(s) - \beta^{-1} \circ L_B^{-1}(L_B(s)-)) = \frac{1}{c_-}(s - L_B^{-1}(L_B(s)-)),$$

and (4.12) implies

$$\hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)) = e_{L_B(s)}(s - L_B^{-1}(L_B(s)-)). \tag{4.13}$$

Similarly, if $L_Z^{-1}(L_Z(t)-) \neq L_Z^{-1}(L_Z(t))$ and $\hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)) > 0$, we see

$$\begin{aligned}
&\hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)) = e_{L_B(s)}(c_+(t - L_Z^{-1}(L_Z(t)-))) \\
&\quad = e_{L_B(s)}(c_+(t - \beta^{-1} \circ L_B^{-1}(L_B \circ \beta(t)-))) \\
&\quad = e_{L_B(s)}(c_+(\beta^{-1}(s) - \beta^{-1} \circ L_B^{-1}(L_B(s)-))).
\end{aligned} \tag{4.14}$$

Since we are on a positive excursion of B, we have

$$\beta^{-1}(s) - \beta^{-1} \circ L_B^{-1}(L_B(s)-)) = \frac{1}{c_+}(s - L_B^{-1}(L_B(s)-)),$$

and (4.14) implies

$$\hat{e}_{L_Z(t)}(t - L_Z^{-1}(L_Z(t)-)) = e_{L_B(s)}(s - L_B^{-1}(L_B(s)-)). \tag{4.15}$$

Finally from (4.11), (4.13), and (4.15), we can rewrite (4.10) as

$$\begin{aligned}
&Z \circ (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(s) = Z \circ \beta^{-1}(s) = Z(t) \\
&= \begin{cases} 0, & \text{if } L_B^{-1}(L_B(s)-) = L_B^{-1}(L_B(s)), \\ e_{L_B(s)}(s - L_B^{-1}(L_B(s)-)), & \text{if } L_B^{-1}(L_B(s)-) \neq L_B^{-1}(L_B(s)). \end{cases} \\
&= B(s),
\end{aligned}$$

which completes the proof. $\square$

4.2 Brownian excursion theory

4.2.1 Construction of mappings

We denote by $C_r[0,\infty)$ the set of continuous functions $z\colon [0,\infty) \to \mathbb{R}$ such that $z(0) = r$. We introduce the metric d_r on $C_r[0,\infty)$ defined by

$$d_r(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(1 \wedge \sup_{0 \le t \le n} \big| x(t) - y(t) \big| \right).$$

Convergence under this metric is uniform convergence on compact sets. Let $\mathcal{B}^r$ be the Borel σ-algebra generated by this topology and $\mathcal{B}^{r_1,r_2}$ be the product σ-algebra on $C_{r_1}[0,\infty) \times C_{r_2}[0,\infty)$.

Definition 4.2.1 *We define the* Skorohod map $\Gamma\colon C[0,\infty) \to C[0,\infty)$ *by*

$$\Gamma(z)(t) = -\min_{0 \le s \le t} \big(z(s) \wedge 0 \big) \ \forall z \in C[0,\infty). \tag{4.16}$$

Remark 4.2.2 *For $z \in C[0,\infty)$, $\Gamma(z)$ is the unique nondecreasing function in $C[0,\infty)$ with the following three properties:*

(i) $\Gamma(z)(0) = \max\{-z(0), 0\}$,

(ii) $z(t) + \Gamma(z)(t) \ge 0$ *for all* $t \ge 0$, *and*

(iii) *on intervals where $z + \Gamma(z)$ is non-zero, $\Gamma(z)$ is constant.*

Given $r \in \mathbb{R}$, define

$$\mathcal{D}^r = \big\{ (z_+, z_-) \in C_{r^+}[0,\infty) \times C_{r^-}[0,\infty) : \liminf_{t\to\infty} z_+(t) = \liminf_{t\to\infty} z_-(t) = -\infty \big\}, \tag{4.17}$$

where $r^+ = \max\{0, r\}$ and $r^- = \max\{0, -r\}$. We show that $\mathcal{D}^r \in \mathcal{B}^{r^+,r^-}$ in Appendix A. Given $(z_+, z_-) \in \mathcal{D}^r$, we construct a function $z \in C_r[0,\infty)$ as follows. First set $\ell_\pm = \Gamma(z_\pm)$ so that $z_\pm + \ell_\pm \ge 0$. By the definition of $\mathcal{D}^r$, we have

$$\lim_{t\to\infty} \ell_+(t) = \lim_{t\to\infty} \ell_-(t) = \infty. \tag{4.18}$$

We define the mapping Φ_+ and Φ_- by

$$\begin{aligned}
\Phi_+(z_+, z_-)(t) &:= \sup \big\{ u \in [0,t] : \ell_+(u) = \ell_-(t-u) \big\}, \\
\Phi_-(z_+, z_-)(t) &:= \inf \big\{ u \in [0,t] : \ell_-(u) = \ell_+(t-u) \big\},
\end{aligned}$$

for $t \ge 0$ where $(z_+, z_-) \in C_{r^+}[0,\infty) \times C_{r^-}[0,\infty)$. Let

$$p_+(t) = \Phi_+(z_+, z_-)(t), \tag{4.19}$$
$$p_-(t) = \Phi_-(z_+, z_-)(t). \tag{4.20}$$

Then define

$$z = \Psi(z_+, z_-) := z_+ \circ p_+ - z_- \circ p_-. \tag{4.21}$$

We show that $\Psi \colon \mathcal{D}^r \to C_r[0,\infty)$ is measurable in the Appendix A.

We first develop properties of $p_\pm$ and then properties of z.

Lemma 4.2.3 *The functions $p_\pm$ defined by (4.19) and (4.20) have zero initial condition and are nondecreasing and continuous. In addition,*

$$\ell_+ \circ p_+ = \ell_- \circ p_-, \tag{4.22}$$

$$p_+ + p_- = e, \tag{4.23}$$

where e is the identify function $e(t) = t$ for all $t \geq 0$.

Proof: Since $z_+(0) \geq 0$ and $z_-(0) \geq 0$, we have $\ell_+(0) = \ell_-(0) = 0$. It is obvious that $p_\pm$ satisfy $p_\pm(0) = 0$. Because $\ell_\pm$ is nondecreasing and continuous and $\ell_\pm(0) = 0$, for each $t \geq 0$, there exists u_1 such that $\ell_+(u_1) = \ell_-(t - u_1)$ and there exists u_2 (in fact, we can take $u_2 = t - u_1$), such that $\ell_-(u_2) = \ell_+(t - u_2)$. Therefore, $p_\pm$ takes values in $[0, t]$. The maximum and minimum in (4.19) and (4.20) are obtained because both ℓ_+ and ℓ_- are continuous. It is apparent that the maximum u_1 for which $\ell_+(u_1) = \ell_-(t - u_1)$ corresponds to the minimum $u_2 = t - u_1$ for which $\ell_-(u_2) = \ell_+(t - u_2)$, and hence (4.23) holds. By construction, $\ell_+(p_+(t)) = \ell_-(t - p_+(t))$, and (4.23) implies (4.22).

To see that p_+ is nondecreasing, let $0 \leq t_1 < t_2$ be given. Then

$$\ell_+\big(p_+(t_1)\big) = \ell_-\big(t_1 - p_+(t_1)\big).$$

If, in addition,

$$\ell_+\big(p_+(t_1)\big) = \ell_-\big(t_2 - p_+(t_1)\big),$$

then because $p_+(t_2)$ is the maximum of all numbers satisfying $\ell_+(u) = \ell_-(t_2 - u)$, we have $p_+(t_2) \geq p_+(t_1)$. If instead

$$\ell_+\big(p_+(t_1)\big) < \ell_-\big(t_2 - p_+(t_1)\big),$$

then $\ell_+(p_+(t_2)) = \ell_-(t_2 - p_+(t_2))$ implies $p_+(t_2) > p_+(t_1)$.

Suppose now that $t_n \downarrow t$. Then

$$\ell_+\big(p_+(t_n)\big) = \ell_-\big(t_n - p_+(t_n)\big), \quad n = 1, 2, \ldots.$$

Letting $n \to \infty$ and using the continuity of $\ell_\pm$, we obtain

$$\ell_+\big(\lim_{n\to\infty} p_+(t_n)\big) = \ell_-\big(t - \lim_{n\to\infty} p_+(t_n)\big).$$

According to the definition of p_+, this implies that $p_+(t) \geq \lim_{n\to\infty} p(t_n)$. Because p_+ is nondecreasing, we also have $p_+(t) \leq \lim_{n\to\infty} p_+(t_n)$, and hence p_+ is right continuous. Equation (4.23) implies that p_- is right continuous as well.

To see that p_- is nondecreasing, let $0 \le t_1 < t_2$ be given. Then

$$\ell_-\big(p_-(t_2)\big) = \ell_+\big(t_2 - p_-(t_2)\big).$$

If, in addition,

$$\ell_-\big(p_-(t_2)\big) = \ell_+\big(t_1 - p_-(t_2)\big),$$

then because $p_-(t_1)$ is the minimum of all numbers satisfying $\ell_-(u) = \ell_+(t_1 - u)$, we have $p_-(t_2) \ge p_-(t_1)$. If instead

$$\ell_-\big(p_-(t_2)\big) > \ell_+\big(t_1 - p_-(t_2)\big),$$

then $\ell_-(p_-(t_1)) = \ell_+(t_1 - p_-(t_1))$ implies $p_-(t_2) > p_-(t_1)$.

Suppose now that $t_n \uparrow t$. Then

$$\ell_-\big(p_-(t_n)\big) = \ell_+\big(t_n - p_-(t_n)\big), \quad n = 1, 2, \ldots.$$

Letting $n \to \infty$ and using continuity of $\ell_\pm$, we obtain

$$\ell_-\big(\lim_{n\to\infty} p_-(t_n)\big) = \ell_-\big(t - \lim_{n\to\infty} p_-(t_n)\big).$$

According to the definition of p_-, this implies that $p_-(t) \le \lim_{n\to\infty} p_-(t_n)$. Because p_- is nondecreasing, we also have $p_-(t) \ge \lim_{n\to\infty} p_-(t_n)$, and hence p_- is left continuous. Equation (4.23) implies that p_+ is left continuous as well. $\square$

Lemma 4.2.4 *For z defined by (4.21), we have*

$$|z| = z_+ \circ p_+ + z_- \circ p_- + \Gamma(z_+ \circ p_+ + z_- \circ p_-), \tag{4.24}$$

$$\Gamma(z_+ \circ p_+ + z_- \circ p_-) = \ell_+ \circ p_+ + \ell_- \circ p_- = 2\ell_\pm \circ p_\pm. \tag{4.25}$$

In addition,

$$\int_0^t \mathbb{1}_{\{\pm z(s) > 0\}} ds \le p_\pm(t) \le \int_0^t \mathbb{1}_{\{\pm z(s) \ge 0\}} ds, \quad t \ge 0. \tag{4.26}$$

PROOF: We can write

$$z = (z_+ + \ell_+) \circ p_+ - (z_- + \ell_-) \circ p_- \tag{4.27}$$

as the difference of the nonnegative functions $(z_+ + \ell_+) \circ p_+$ and $(z_- + \ell_-) \circ p_-$. We first show that at each $t \ge 0$, we cannot have both $(z_+ + \ell_+) \circ p_+(t)$ and $(z_- + \ell_-) \circ p_-(t)$ positive. Without loss of generality, let us assume $(z_+, z_-) \in \mathcal{D}^r$ where $r \ge 0$. Then let

$$c = \inf\big\{s \ge 0 : z_+(s) = 0\big\}.$$

From (4.17), we know $c < \infty$ is well defined. Because $\ell_+ = \Gamma(z_+)$, we have $\ell_+(s) = 0$ for all $s \in [0, c]$. Hence, by (4.19) and (4.20), we know that $p_+(s) = s$ and $p_-(s) = 0$ for all $s \in [0, c]$, which implies

$$(z_+ + \ell_+) \circ p_+(s) \ge 0,$$
$$(z_- + \ell_-) \circ p_-(s) = 0,$$

for all $s \in [0, c]$. Moreover, $(z_+ + \ell_+) \circ p_+(c) = 0$. Therefore, we showed that $(z_+ + \ell_+) \circ p_+(t)$ and $(z_- + \ell_-) \circ p_-(t)$ cannot be both positive for $t \le c$. Next, assume for some $t > c$ that

$$(z_+ + \ell_+) \circ p_+(t) > 0 \tag{4.28}$$

and define

$$a = \sup \big\{ s \in [c, t] : (z_+ + \ell_+) \circ p_+(s) = 0 \big\}, \tag{4.29}$$

$$b = \inf \big\{ s \in [t, \infty) : (z_+ + \ell_+) \circ p_+(s) = 0 \big\}. \tag{4.30}$$

Note that $(z_+ + \ell_+) \circ p_+(c) = 0$, we have $a \in [c, t)$ and $(z_+ + \ell_+) \circ p_+(a) = 0$. We have $b \in (t, \infty]$, and because p_+ is nondecreasing, $p_+(b)$ is defined in $[0, \infty]$. On the interval or half-line (a, b), $(z_+ + \ell_+) \circ p_+$ is strictly positive, and Remark 4.2.2(iii) implies that $\ell_+ \circ p_+$ is constant and equal to $\ell_+(p_+(a))$. Note that $p_+(a) < p_+(t)$.

For $\delta \in (0, p_+(t) - p_+(a))$, we have

$$\ell_+\big(p_+(a) + \delta)\big) = \ell_+\big(p_+(a)\big) = \ell_-\big(a - p_+(a)\big), \tag{4.31}$$

and we must also have

$$\ell_-\big(a - p_+(a)\big) > \ell_-\big(a - p_+(a) - \delta\big), \tag{4.32}$$

or else $u = p_+(a) + \delta$ would satisfy the equation

$$\ell_+(u) = \ell_-(a - u),$$

a contradiction to the definition of $p_+(a)$. We conclude that

$$\ell_+\big(p_+(a)\big) = \ell_-\big(a - p_+(a)\big) > \ell_-(a - u) \quad \forall u \in \big(p_+(a), a\big]. \tag{4.33}$$

Now consider $s \in [a, b)$. Because $p_+(a) \le p_+(s) \le p_+(b)$ and ℓ_+ is constant on $[p_+(a), p_+(b))$, we have

$$\begin{aligned} p_+(s) &= \max \big\{ u \in [0, s] : \ell_+(u) = \ell_-(s - u) \big\} \\ &= \sup \big\{ u \in \big[p_+(a), s \wedge p_+(b)\big] : \ell_+(u) = \ell_-(s - u) \big\} \\ &= \sup \big\{ u \in \big[p_+(a), s \wedge p_+(b)\big] : \ell_+\big(p_+(a)\big) = \ell_-(s - u) \big\}. \end{aligned} \tag{4.34}$$

Relation (4.33) shows that if u were not constrained from above, the supremum in (4.34) would be attained when $s - u = a - p_+(a)$, i.e., at $u = p_+(a) + s - a$. However, because of the constraint, the supremum is attained instead at $u = (p_+(a) + s - a) \wedge p_+(b)$, i.e.,

$$p_+(s) = \big(p_+(a) + s - a\big) \wedge p_+(b) \quad \forall s \in [a, b). \tag{4.35}$$

We wish to remove the term $\wedge p_+(b)$ in (4.35). If $p_+(b) = \infty$, this is trivial. If $p_+(b) < \infty$ and $b = \infty$, then whenever $b_n \to \infty$ we have $p_+(b_n) \to p_+(b) < \infty$. But (4.19) implies

$$\ell_+\big(p_+(b_n)\big) = \ell_-\big(b_n - p_+(b_n)\big),$$

and the left-hand side converges to $\ell_+\big(p_+(b)\big) < \infty$, whereas (4.18) implies that the right-hand side converges to ∞. Because of this contradiction, we conclude that whenever $p_+(b) < \infty$ then also $b < \infty$, in which case $(z_+ + \ell_+) \circ p_+(b) = 0$ and $(z_+ + \ell_+) \circ p_+(s) > 0$ for $s \in [t, b)$. But (4.35) implies that $p_+(s) = p_+(b)$ for $s \in [p_+(b) - p_+(a) + a, b)$, and hence $(z_+ + \ell_+) \circ p_+(s) = (z_+ + \ell_+) \circ p_+(b) = 0$ for s in this interval. It follows that $b \le p_+(b) - p_+(a) + a$, and consequently

$$p_+(b) \ge p_+(a) + b - a. \tag{4.36}$$

If $p_+(b) - p_+(a) + a \ge b$, so the interval $[p_+(b) - p_+(a) + a, b)$ is empty, we again have (4.36). Inequality (4.36) permits us to remove the term $\wedge p_+(b)$ in (4.30), and we conclude that

$$p_+(s) = p_+(a) + s - a \quad \forall s \in [a, b). \tag{4.37}$$

For $\delta \in (0, t - a)$, we have from (4.37) that

$$p_+(t - \delta) = p_+(t) - \delta, \tag{4.38}$$

and from the definition of p_+ that

$$\ell_+\big(p_+(t)\big) = \ell_-\big(t - p_+(t)\big).$$

We must also have

$$\ell_-\big(t - p_+(t)\big) > \ell_-\big(t - \delta - p_+(t)\big), \tag{4.39}$$

or else $u = p_+(t)$ would satisfy the equation $\ell_+(u) = \ell_-(t - \delta - u)$, implying $p_+(t - \delta) \ge p_+(t)$, a contradiction to (4.38). We rewrite (4.39) as

$$\ell_-\big(p_-(t)\big) > \ell_-\big(p_-(t) - \delta\big)$$

and conclude that ℓ_- is not constant in an open interval containing $p_-(t)$. According to Remark 4.2.2(iii), $(z_- + \ell_-) \circ p_-(t) = 0$.

Because not both $(z_+ + \ell_+) \circ p_+(t)$ and $(z_- + \ell_-) \circ p_-(t)$ can be positive, from the representation (4.27) we have

$$|z| = (z_+ + \ell_+) \circ p_+ + (z_- + \ell_-) \circ p_- = z_+ \circ p_+ + z_- \circ p_- + \ell_+ \circ p_+ + \ell_- \circ p_-.$$

Equation (4.22) implies that

$$\ell_+ \circ p_+ + \ell_- \circ p_- = 2\ell_\pm \circ p_\pm.$$

Because $\ell_+ \circ p_+$ is constant on intervals where z is strictly positive and $\ell_- \circ p_-$ is constant on intervals where z is strictly negative, the nondecreasing process $2\ell_\pm \circ p_\pm$ is constant on intervals where $|z|$ is nonzero. According to Remark 4.2.2(iii),

$$2\ell_\pm \circ p_\pm = \Gamma(z_+ \circ p_+ + z_- \circ p_-).$$

Being open, the set

$$\{t > 0 : (z_+ + \ell_+) \circ p_+(t) > 0\} = \bigcup_{i \in I} (a_i, b_i)$$

is the union of disjoint nonempty open intervals, where the index set I if finite or countably infinite and one of these intervals may be an open half-line. Equation (4.37) implies that

$$p_+(t) - p_+(a_i) = \int_{a_i}^{t} \mathbb{1}_{\{z(s)>0\}} ds \quad \forall t \in [a_i, b_i). \tag{4.40}$$

Since p_+ is nondecreasing and $p_+(s) = s\ \forall s \in [0, c]$, we have

$$p_+(t) \geq \int_0^t \mathbb{1}_{\{z(s)>0\}} ds, \quad t \geq 0. \tag{4.41}$$

A symmetric argument shows that

$$p_-(t) \geq \int_0^t \mathbb{1}_{\{z(s)<0\}} ds, \quad t \geq 0.$$

Therefore,

$$p_+(t) = t - p_-(t) \leq \int_0^t \mathbb{1}_{\{z(s)\geq 0\}} ds \ \text{ and } \ p_-(t) = t - p_+(t) \leq \int_0^t \mathbb{1}_{\{z(s)\leq 0\}} ds. \qquad \square$$

4.2.2 Disintegration of two-variance Brownian motion

Let Z be a two-variance Brownian motion defined in Definition 4.1.1, i.e.,

$$Z(t) := B \circ (\frac{1}{c_+} P_B^+ + \frac{1}{c_-} P_B^-)^{-1}(t),$$

where B is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$\begin{aligned} A(t) &:= (\frac{1}{c_+} P_B^+ + \frac{1}{c_-} P_B^-)(t), \\ A^{-1}(s) &:= \inf\{t \geq 0 : A(t) > s\}. \end{aligned}$$

Then A is adapted to the filtration $\{\mathcal{F}_t\}$ generated by B. Because

$$\{A^{-1}(s) \leq t\} = \{A(t) \geq s\} \in \mathcal{F}_t,$$

A^{-1} is a stopping time of $\mathcal{F}$. Let $\mathcal{G}_s = \mathcal{F}_{A^{-1}(s)}$. By the Optional Sampling Theorem, Z is a martingale with respect to the filtration $\{\mathcal{G}_s\}$. We define

$$M_\pm(s) = \pm \int_0^s \mathbb{1}_{\{\pm Z(u)>0\}} dZ(u), \tag{4.42}$$

$$P_Z^\pm(s) = \int_0^s \mathbb{1}_{\{\pm Z(u)>0\}} ds, \tag{4.43}$$

$$(P_Z^\pm)^{-1}(t) = \inf\left\{s \geq 0 : P_Z^\pm(s) > t\right\}, \tag{4.44}$$

$$Z_\pm = B^\pm(0) + M_\pm \circ (P_Z^\pm)^{-1}, \tag{4.45}$$

$$L_\pm = \Gamma(Z_\pm), \tag{4.46}$$

where $B^+(0) = \max\{0, B(0)\}$ and $B^-(0) = \max\{0, -B(0)\}$. Note that M_+ and M_- are martingales relative to $\{\mathcal{G}_s\}_{s\geq 0}$ and $\langle M_+, M_-\rangle = 0$.

Lemma 4.2.5

$$Z = B(0) + M_+ - M_- = Z_+ \circ P_Z^+ - Z_- \circ P_Z^-. \tag{4.47}$$

PROOF: Since $Z(t) = Z(0) + \int_0^t dZ(u)$, and Proposition 4.1.2 implies

$$\mathbb{1}_{\{Z(u)>0\}} + \mathbb{1}_{\{Z(u)<0\}} = 1 \tag{4.48}$$

a.s. for all $u \geq 0$, we have

$$\begin{aligned} Z(t) &= Z(0) + \int_0^t \mathbb{1}_{\{Z(u)>0\}} dZ(u) + \int_0^t \mathbb{1}_{\{Z(u)<0\}} dZ(u) \\ &= B(0) + M_+ - M_- = Z_+ \circ P_Z^+ - Z_- \circ P_Z^-, \end{aligned}$$

which completes the proof.

Lemma 4.2.6 *The processes Z_+ and Z_- are independent Brownian motions (relative to their own filtrations) with variances c_+ and c_- per unit time, i.e., there exists two independent standard Brownian motions B_+ and B_- such that*

$$\begin{aligned} Z_+ &= B_+ \circ c_+ e, \\ Z_- &= B_- \circ c_- e, \end{aligned}$$

where $e(t) = t$ for $t \geq 0$.

PROOF: For each $t \geq 0$, we define

$$B_\pm(t) = Z_\pm(t/c_\pm) = B^\pm(0) + M_\pm \circ (P_Z^\pm)^{-1}(t/c_\pm).$$

Let

$$T^\pm(t) := \inf\{u \geq 0 : \langle M_\pm\rangle(u) > t\}. \tag{4.49}$$

We first show

$$T^\pm(t) = (P_Z^\pm)^{-1}(t/c_\pm). \tag{4.50}$$

By definition of $M_\pm$ and the time change $u = (\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)(s)$, we have

$$M_\pm(t) = \pm \int_0^t \mathbb{1}_{\{\pm Z(u)>0\}}\, dZ(u) = \pm \int_0^{(\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1}(t)} \mathbb{1}_{\{\pm B(s)>0\}}\, dB(s). \tag{4.51}$$

Note that because $du = (\frac{1}{c_+}\mathbb{1}_{\{B(s)>0\}} + \frac{1}{c_-}\mathbb{1}_{\{B(s)<0\}})ds$ and $Z(u) = B(s)$, we have

$$ds = (c_+\mathbb{1}_{\{Z(u)>0\}} + c_-\mathbb{1}_{\{Z(u)<0\}})du,$$

so (4.51) implies

$$\begin{aligned}\langle M_\pm\rangle(t) &= \int_0^{(\frac{1}{c_+}P_B^+ + \frac{1}{c_-}P_B^-)^{-1}(t)} \mathbb{1}_{\{\pm B(s)>0\}}\,ds \\ &= c_\pm \int_0^t \mathbb{1}_{\{\pm Z(u)>0\}}\,du = c_\pm P_Z^\pm(t),\end{aligned} \tag{4.52}$$

Hence from (4.52), we obtain

$$\begin{aligned}T^\pm(t) &= \inf\{u \geq 0 : \langle M_\pm\rangle(u) > t\} \\ &= \inf\{u \geq 0 : c_\pm P_Z^\pm(u) > t\} \\ &= \inf\{u \geq 0 : P_Z^\pm(u) > t/c_\pm\} = (P_Z^\pm)^{-1}(t/c_\pm),\end{aligned} \tag{4.53}$$

which proves (4.50). According to (4.52), we have $\lim_{t\to\infty}\langle M_\pm\rangle(t) = \infty$ almost surely. Therefore, the result of the lemma follows from Knight's Theorem, Theorem 3.4.13, p. 179 of [19]. □

To relate these processes to the construction in Subsection A.1, we need the following result.

Lemma 4.2.7 *The processes $P_Z^\pm$ satisfy*

$$P_Z^+(t) = \Phi_+(Z_+, Z_-), \tag{4.54}$$

$$P_Z^-(t) = \Phi_-(Z_+, Z_-), \tag{4.55}$$

almost surely.

Proof: We first want to show that

$$L_+ \circ P_Z^+ = L_- \circ P_Z^-. \tag{4.56}$$

Without loss of generality, let us assume $Z(0) \geq 0$, and let

$$T_1^+ := \inf\{t \geq 0 : Z(t) \leq 0\}, \quad T_1^- := \inf\{t \geq 0 : Z(t) \geq 0\} = 0.$$

Obviously, $\forall t \leq T_1^+$, from Lemma 4.2.5 we have

$$0 \leq Z(t) = Z_+ \circ P_Z^+(t) = Z_+(t).$$

Therefore $L_+(t) = 0$ and $P_Z^-(t) = 0$, which implies

$$0 = L_+ \circ P_Z^+(t) = L_- \circ P_Z^-(t) = 0, \forall t \leq T_1^+.$$

We observe that $Z_\pm + L_\pm$ is a nonnegative process. For $s \geq T_1^\pm$, we define

$$U_\pm(s) := \max\{v \in [T_1^\pm, s] : -Z_\pm(v) = L_\pm(s)\}$$

so that $U_\pm(T_1^+) = T_1^+$ and

$$- Z_\pm \circ U_\pm = L_\pm, \tag{4.57}$$

on $[T_1^+, \infty)$. Define also

$$\tau_\pm(t) := \max\{u \in [T_1^\pm, t] : P_Z^\pm(u) = U_\pm \circ P_Z^\pm(t)\}, \quad t \geq T_1^\pm,$$

so that $P_Z^\pm \circ \tau_\pm = U_\pm \circ P_Z^\pm$, and hence, in light of (4.57),

$$Z_\pm \circ P_Z^\pm \circ \tau_\pm = Z_\pm \circ U_\pm \circ P_Z^\pm = -L_\pm \circ P_Z^\pm,$$

on $[T_1^\pm, \infty)$. Let $t \geq T_1^\pm$ be given. At time $\tau_+(t)$, either Z is on a negative excursion that began at some time $\ell(t) < \tau_+(t)$, or else $Z(\tau_+(t)) \geq 0$. In the latter case,

$$\begin{aligned}
0 &\leq Z\big(\tau_+(t)\big) \\
&= Z_+ \circ P_Z^+ \circ \tau_+(t) - Z_- \circ P_Z^- \circ \tau_+(t) \\
&= -L_+ \circ P_Z^+(t) - (Z_- + L_-) \circ P_Z^- \circ \tau_+(t) + L_- \circ P_Z^- \circ \tau_+(t) \\
&\leq -L_+ \circ P_Z^+(t) + L_- \circ P_Z^- \circ \tau_+(t) \\
&\leq -L_+ \circ P_Z^+(t) + L_- \circ P_Z^-(t).
\end{aligned}$$

In the event that Z is on a negative excursion at time $\tau_+(t)$ that began at time $\ell(t) < \tau_+(t)$, we have $P_Z^+(\ell(t)) = P_Z^+(\tau_+(t))$, and hence

$$\begin{aligned}
0 &= Z\big(\ell(t)\big) \\
&= Z_+ \circ P_Z^+ \circ \ell(t) - Z_- \circ P_Z^- \circ \ell(t) \\
&= Z_+ \circ P_Z^+ \circ \tau_+(t) - Z_- \circ P_Z^- \circ \ell(t) \\
&= -L_+ \circ P_Z^+(t) - (Z_- + L_-) \circ P_Z^- \circ \ell(t) + L_- \circ P_Z^- \circ \ell(t) \\
&\leq -L_+ \circ P_Z^+(t) + L_- \circ P_Z^- \circ \ell(t) \\
&\leq -L_+ \circ P_Z^+(t) + L_- \circ P_Z^-(t).
\end{aligned}$$

We conclude that $L_+ \circ P_Z^+(t) \leq L_- \circ P_Z^-(t)$ for all $t \geq T_1^+$. For the opposite inequality, we note first that $P_Z^-(T_1^+) = 0$, so $\tau_-(T_1^+) = T_1^+$. Therefore, for $t \geq T_1^+$, at $\tau_-(t)$, either Z is on a positive excursion that began at some time $\ell(t) \in [T_1^+, \tau_-(t))$, or else $Z(\tau_-(t)) \leq 0$. In the latter case

$$\begin{aligned}
0 &\geq Z\big(\tau_-(t)\big) \\
&= Z_+ \circ P_Z^+ \circ \tau_-(t) - Z_- \circ P_Z^- \circ \tau_-(t) \\
&= (Z_+ + L_+) \circ P_Z^+ \circ \tau_-(t) - L_+ \circ P_Z^+ \circ \tau_-(t) + L_- \circ P_Z^-(t) \\
&\geq -L_+ \circ P_Z^+ \circ \tau_-(t) + L_- \circ P_Z^-(t) \\
&\geq -L_+ \circ P_Z^+(t) + L_- \circ P_Z^-(t).
\end{aligned}$$

In the event that Z is on a positive excursion at time $\tau_-(t)$ that began at $\ell(t) \in [T_1^+, \tau_-(t)\big)$, we have $P_Z^+(\ell(t)) = P_Z^-(\tau_-(t))$, and hence

$$\begin{aligned} 0 &= Z\big(\ell(t)\big) \\ &= Z_+ \circ P_Z^+ \circ \ell(t) - Z_- \circ P_Z^- \circ \ell(t) \\ &= Z_+ \circ P_Z^+ \circ \ell(t) - Z_- \circ P_Z^- \circ \tau_-(t) \\ &= (Z_+ + L_+) \circ P_Z^+ \circ \ell(t) - L_+ \circ P_Z^+ \circ \ell(t) + L_- \circ P_Z^-(t) \\ &\geq -L_+ \circ P_Z^+ \circ \ell(t) + L_- \circ P_Z^-(t) \\ &\geq -L_+ \circ P_Z^+(t) + L_- \circ P_Z^-(t). \end{aligned}$$

We conclude that $L_+ \circ P_Z^+(t) \geq L_- \circ P_Z^-(t)$ for all $t \geq T_1^+$. This completes the proof of (4.56).

We denote $Q_+(t) = \Phi_+(Z_+, Z_-)(t)$ and $Q_-(t) = \Phi_+(Z_-, Z_+)$ so that

$$L_+\big(Q_+(t)\big) = L_-\big(t - Q_+(t)\big), \quad L_-\big(Q_-(t)\big) = L_+\big(t - Q_-(t)\big), \quad t \geq 0. \tag{4.58}$$

Because $P_Z^+(t) + P_Z^-(t) = t$ for all $t \geq 0$, (4.56) implies

$$L_+\big(P_Z^+(t)\big) = L_-\big(t - P_Z^+(t)\big), \quad L_-\big(P_Z^-(t)\big) = L_+\big(t - P_Z^-(t)\big), \quad t \geq 0.$$

The definition of Φ_+ then implies that $P_Z^\pm(t) \leq Q_\pm(t)$ and

$$t = P_Z^+(t) + P_Z^-(t) \leq Q_+(t) + Q_-(t). \quad t \geq 0.$$

To show that $P_Z^\pm = Q_\pm$, it suffices to show that

$$Q_+(t) + Q_-(t) \leq t, \quad t \geq 0, \ a.s. \tag{4.59}$$

The Brownian motions $Z_\pm$ have variances $c_\pm$ per unit time that may differ from one. The Brownian motions $Z_\pm/\sqrt{c_\pm}$ are standard. For these standard Brownian motions, we define stopping times

$$S_\pm^b := \inf\{t \geq 0 : -Z_\pm/\sqrt{c_\pm} > b\}, \quad T_\pm^b := \inf\{t \geq 0 : -Z_\pm/\sqrt{c_\pm} = b\}, \quad b > 0.$$

Then

$$S_\pm^b = \inf\{t \geq 0 : L_\pm/\sqrt{c_\pm} > b\}, \quad T_\pm^b = \inf\{t \geq 0 : L_\pm/\sqrt{c_\pm} = b\}, \quad b > 0.$$

Then $L_\pm$ is constant and equal to $b\sqrt{c_\pm}$ on the interval $[T_\pm^b, S_\pm^b]$ if and only if $T_\pm^b < S_\pm^b$. In other words, $L_\pm$ has a "flat spot" at level $b\sqrt{c_\pm}$ if and only $T_\pm^b < S_\pm^b$. According to [19], Section 6.2D, there are Poisson random measures $\nu_\pm$ on $(0, \infty)$, both with Lévy measure $d\ell/\sqrt{2\pi\ell^3}$, $\ell > 0$, such that

$$S_\pm^b = \int_{(0,\infty)} \ell \nu_\pm\big((0, b] \times d\ell\big), \quad b > 0$$

Furthermore, it is apparent that

$$T_\pm^b = \int_{(0,\infty)} \ell \nu_\pm\big((0, b) \times d\ell\big) \quad b > 0.$$

Because Z_+ and Z_- are independent, so are the Poisson random measures ν_+ and ν_-. A Poisson random measure charges only countably many time points, i.e., there are only countably many points $b > 0$ for which $\nu_+(\{b\} \times (0, \infty)) > 0$. Because ν_- is independent of ν_+, the probability that ν_- charges $\sqrt{c_+}/\sqrt{c_-}$ times one of the countably many points charged by ν_+ is zero. Therefore,

$$\begin{aligned}
&\mathbb{P}\big\{\text{There exists } c > 0 \text{ such that } L_+ \text{ and } L_- \text{ both have a flat spot at level } c > 0.\big\} \\
&\quad = \mathbb{P}\big\{\text{There exists } c > 0 \text{ such that } S_+^{c/\sqrt{c_+}} > T_+^{c/\sqrt{c_+}} \text{ and } S_-^{c/\sqrt{c_-}} > T_-^{c/\sqrt{c_-}}.\big\} \\
&\quad = \mathbb{P}\big\{\text{There exists } c > 0 \text{ s. t. } \nu_+\big(\{c/\sqrt{c_+}\} \times (0,\infty)\big) > 0 \text{ and } \nu_-\big(\{c/\sqrt{c_-}\} \times (0,\infty)\big) > 0.\big\} \\
&\quad = 0. \qquad (4.60)
\end{aligned}$$

In other words, the probability that L_+ and L_- both have a "flat spot" at the same level is zero.

Suppose $Q_+(t) + Q_-(t) > t$ for some $t \geq 0$. Then $Q_-(t) > t - Q_+(t) \geq 0$ and $Q_+(t) > t - Q_-(t) \geq 0$. Using these inequalities and (4.58), we obtain

$$\begin{aligned}
L_+\big(Q_+(t)\big) &= L_-\big(t - Q_+(t)\big) \leq L_-\big(Q_-(t)\big), \\
L_-\big(Q_-(t)\big) &= L_+\big(t - Q_-(t)\big) \leq L_+\big(Q_+(t)\big).
\end{aligned}$$

These two equations show that

$$c := L_+\big(Q_+(t)\big) = L_+\big(t - Q_-(t)\big) = L_-\big(Q_-(t)\big) = L_-\big(t - Q_+(t)\big).$$

In other words, both L_+ and L_- have a "flat spot" at level c. According to (4.60), the probability of this is zero. Hence, (4.59) holds almost surely for every rational $t \geq 0$. But $Q_+(t) + Q_-(t)$ is continuous, so (4.59) holds almost surely for every $t \geq 0$. This implies (4.54). Because $P_Z^+(t) + P_Z^-(t) = t$, we also have (4.55). ☐

Theorem 4.2.8 *Let Z be a two-variance Brownian motion and let $Z_\pm$ be defined by (4.45). Then $Z = \Psi(Z_+, Z_-)$, where Ψ is defined by (4.21).*

Proof: Combine Lemmas 4.2.5 and 4.2.6. ☐

4.2.3 Reconstruction of two-variance Brownian motion

Given $r \in \mathbb{R}$, let $\mathcal{B}_{\mathcal{D}^r}^{r^+,r^-}$ denote the trace σ-field of $\mathcal{B}^{r^+,r^-}$ on $\mathcal{D}^r$. Given any measure $\mathbb{Q}$ on $(\mathcal{D}^r, \mathcal{B}_{\mathcal{D}^r}^{r^+,r^-})$, Ψ induces a measure $\mathbb{Q} \circ \Psi^{-1}$ on $(C_r[0,\infty), \mathcal{B}^r)$ defined by

$$\mathbb{Q} \circ \Psi^{-1}(A) = \mathbb{Q}\big\{(z_+, z_-) \in \mathcal{D}^r : \Psi(z_+, z_-) \in A\big\} = \mathbb{Q}\big(\Psi^{-1}(A)\big), \quad A \in \mathcal{B}^r.$$

Let $\mathbb{W}_r^c$ denote *one-variance Wiener measure* on $(C_r[0,\infty), \mathcal{B})$, under which the coordinate mapping process is a Brownian motion with variance c per unit time starting at r, and $\mathbb{W}_{r_1}^{c_1} \otimes \mathbb{W}_{r_2}^{c_2}$ denote the product of two one-variance Wiener measures on $(C_{r_1}[0,\infty) \times C_{r_2}[0,\infty), \mathcal{B}^{r_1,r_2})$ where $r_1 \geq 0$ and $r_2 \geq 0$. Note that $(\mathbb{W}_{r^+}^{c_1} \otimes \mathbb{W}_{r^-}^{c_2})(\mathcal{D}^r) = 1$. We let $\mathbb{W}_{r^+}^{c_1} \otimes \mathbb{W}_{r^-}^{c_2}|_{\mathcal{D}^r}$ denote $\mathbb{W}_{r^+}^{c_1} \otimes \mathbb{W}_{r^-}^{c_2}$ restricted to $\mathcal{D}^r$. Let Z be a two-variance Brownian motion starting at r, we let $\mathbb{W}_r^{c_+,c_-}$ denote the measure induced by Z on $\big(C_r[0,\infty), \mathcal{B}\big)$, and call this measure *two-variance Wiener measure*.

Theorem 4.2.9 $(\mathbb{W}^{c+}_{r^+} \otimes \mathbb{W}^{c-}_{r^-}|_{\mathcal{D}^r}) \circ \Psi^{-1} = \mathbb{W}^{c_+,c_-}_r$.

Proof: Let Z be a two-variance Brownian motion on some probability space $(\Omega, , \mathbb{P})$. Then Z induces a two-variance Wiener measure on $C_r[0,\infty)$, i.e., $\mathbb{P} \circ Z^{-1} = \mathbb{W}^{c_+,c_-}_r$. Let $Z_\pm$ be defined by (4.45), so that (Z_+, Z_-) induces the product of one-variance Wiener measure $\mathbb{W}^{c+}_{r^+} \otimes \mathbb{W}^{c-}_{r_2}|_{\mathcal{D}^r}$ on $(\mathcal{D}^r, \mathcal{B}^{r^+,r^-}_{\mathcal{D}^r})$, i.e., $\mathbb{P} \circ (Z_+, Z_-)^{-1} = \mathbb{W}^{c+}_{r^+} \otimes \mathbb{W}^{c-}_{r_2}|_{\mathcal{D}^r}$. From Theorem 4.2.8 we have

$$\mathbb{W}^{c_+,c_-}_r = \mathbb{P} \circ Z^{-1} = \mathbb{P} \circ (Z_+, Z_-)^{-1} \circ \Psi^{-1} = (\mathbb{W}^{c+}_{r^+} \otimes \mathbb{W}^{c-}_{r_2}|_{\mathcal{D}^r}) \circ \Psi^{-1}.$$

□

Corollary 4.2.10 *Given $r \in \mathbb{R}$, let Z_+ and Z_- be independent Brownian motions with variances c_+ and c_- per unit time starting at r^+ and r^-. Then $Z = \Psi(Z_+, Z_-)$ is a two-variance Brownian motion (relative to its own filtration). Moreover*

$$Z = Z_+ \circ P^+_Z - Z_- \circ P^-_Z.$$

In particular, from Lemma 4.2.4 and Lemma 4.2.7, we have

$$|Z| = Z_+ \circ P^+_Z + Z_- \circ P^-_Z + \Gamma(Z_+ \circ P^+_Z + Z_- \circ P^-_Z), \tag{4.61}$$

$$\Gamma(Z_+ \circ P^+_Z + Z_- \circ P^-_Z) = L_+ \circ P^+_Z + L_- \circ P^-_Z = 2L_\pm \circ P^\pm_Z. \tag{4.62}$$

Proof: Because (Z_+, Z_-) induces the product of one-variance Wiener measure, $\mathbb{W}^{c+}_{r^+} \otimes \mathbb{W}^{c-}_{r^-}|_{\mathcal{D}^r}$, on $(\mathcal{D}^r, \mathcal{B}^{r^+,r^-}_{\mathcal{D}^r})$, and $\Psi(Z_+, Z_-)$ induces a two-variance Wiener measure $\mathbb{W}^{c_+,c_-}_r$ on $(C_r[0,\infty), \mathcal{B}^r)$, from Theorem 4.2.9, we have Z is a two-variance Brownian motion. According to Proposition 4.1.2, we have $t = P^+_Z(t) + P^-_Z(t)$, which implies

$$\int_0^t \mathbb{1}_{\{\pm Z(s) > 0\}} ds = \int_0^t \mathbb{1}_{\{\pm Z(s) \geq 0\}} ds.$$

Therefore, from Lemma 4.2.4, we obtain

$$Z = \Psi(Z_+, Z_-) = Z_+ \circ P^+_Z - Z_- \circ P^-_Z.$$

□

Chapter 5

From renewal state to the next renewal state

From Theorem 3.7.5 in Chapter 3, we see that

$$(\widehat{T}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{U}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{V}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{W}^n_{\cdot\wedge\sigma(\widehat{S}^n)}, \widehat{X}^n_{\cdot\wedge\sigma(\widehat{S}^n)}) \Longrightarrow (\mathcal{T}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{U}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{V}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{W}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}, \mathcal{X}^*_{\cdot\wedge\sigma(\mathcal{S}^*)}),$$

under the topology on the space $\mathbb{D}$, and the limiting model reaches the first renewal state at time $\sigma(\mathcal{S}^*)$. Note that $\sigma(\mathcal{S}^*)$ is the first time when either $\mathcal{U}^*$ or $\mathcal{W}^*$ reaches zero. Without loss of generality, in this chapter, we assume that $\mathcal{W}^*$ reaches zero before $\mathcal{U}^*$. We are now interested in the evolution of (U^n, V^n, W^n, X^n) after $\sigma^n(\widehat{S}^n)$. According to Corollary 3.5.3 and Proposition 3.7.4, we have

$$(\widehat{U}^n_{\sigma(\widehat{S}^n)}, \widehat{V}^n_{\sigma(\widehat{S}^n)}, \widehat{W}^n_{\sigma(\widehat{S}^n)}, \widehat{X}^n_{\sigma(\widehat{S}^n)}) \Longrightarrow (\mathcal{U}^*_{\sigma(\mathcal{S}^*)}, 0, 0, \mathcal{X}^*_{\sigma(\mathcal{S}^*)}),$$

where $\mathcal{U}^*_{\sigma(\mathcal{S}^*)} > 0$ and $\mathcal{X}^*_{\sigma(\mathcal{S}^*)} < 0$. For convenience of our discussion, we reset the clock of the n^{th} pre-limit model at σ^n. In particular, the reset LOB has the following initial condition,

$$\begin{aligned} U^n(0)/\sqrt{n} &\to u_0 > 0, \\ V^n(0)/\sqrt{n} &\to 0, \\ W^n(0)/\sqrt{n} &\to 0, \\ X^n(0)/\sqrt{n} &\to x_0 < 0, \end{aligned}$$

where Proposition 3.7.4 imples $x_0 = -\frac{\mu_2\lambda_1}{\theta_s\mu_1}$ and $u_0 = \frac{\lambda_2\mu_1}{\theta_b\lambda_1}$. Let us define

$$\eta^n := \inf\{t \geq 0 \,|\, U^n(t) \leq 0 \text{ or } X^n(t) \geq 0\}.$$

We call (U^n, X^n) *bracketing processes*, and (V^n, W^n) *interior processes* until the LOB reaches η^n. We are going to study the evolution of the stopped process $(U^n_{\cdot\wedge\eta^n}, V^n_{\cdot\wedge\eta^n}, W^n_{\cdot\wedge\eta^n}, X^n_{\cdot\wedge\eta^n})$ and its diffusion scaled limit.

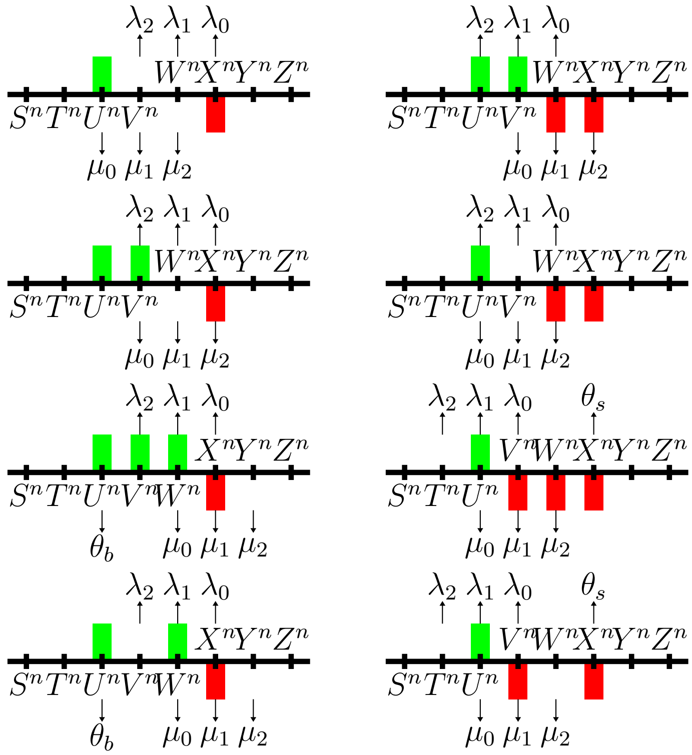

Figure 5.1: Bracketing queues U^n, X^n and interior queues V^n, W^n

5.1 The interior processes

To study the interior processes V^n and W^n, in Section 5.1.1 we change the variables similarly to what we did in Section 3.1 to obtain new processes G^n and H^n. In Section 5.1.2 we then apply the diffusion scaling in order to obtain a process $\widehat{G}^n$ that is a martingale and a second process $\widehat{H}^n$ that is shown in Section 5.1.3 to converge to zero. The limit of $\widehat{G}^n$ is shown in Section 5.1.4 to be a two-variance Brownian motion. In terms of the limits of $\widehat{H}^n$ and $\widehat{G}^n$, in Section 5.1.5 we provide the limits of $\widehat{\mathcal{V}}^n$ and $\widehat{\mathcal{W}}^n$. The convergences in this section are weak convergence of probability measures in the J_1 topology on $D[0,\infty)$.

In principle, V^n can be either positive, zero, or negative and W^n can be either positive, zero, or negative. However, V^n cannot be negative when W^n is positive because that would mean a limit sell order at a price below the price of a limit buy order, which is impossible under the rule of order arrivals. Therefore, there are eight possibilities for the pair (V^n, W^n) when U^n and X^n are bracketing queues, and these are illustrated in Figure 5.1. In each of these eight configuration, the arrows indicate the directions of queues' movement and the parameters show the locations and rates of arrivals of market and limit orders or cancellations. For convenience, we only show cancellations on (U^n, V^n, W^n, X^n). We see from this figure that during the time when U^n and X^n are bracketing processes, (V^n, X^n) is a two-dimensional Markov process on the two-dimensional integer lattice $\mathbb{Z}^2$

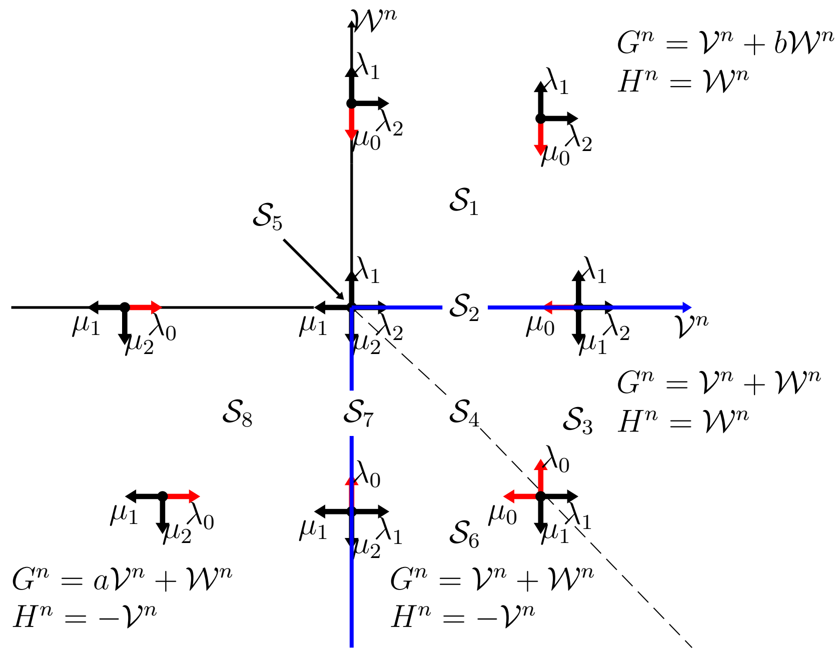

Figure 5.2: Bracketing queues U^n, X^n and interior queues V^n, W^n

intersected with

$$\mathcal{S} := \big\{(v,w) : v \geq 0\big\} \cup \big\{(v,w) : w \leq 0\big\}.$$

In order to avoid here a discussion of the possibility that the bracketing queues U^n and X^n are no longer valid till certain time, i.e., either U^n or X^n hits zero, we consider a pair of processes $(\mathcal{V}^n, \mathcal{W}^n)$ that has the same dynamics as (V^n, W^n) but is defined by these dynamics for all time, not just during the period of time that U^n and X^n are bracketing processes. Figure 5.2 shows the transitions of this two-dimensional process, where the rates and directions of transitions are indicated by arrows.

In order to make the dynamics of $(\mathcal{V}^n, \mathcal{W}^n)$ more precise, we divide $\mathcal{S}$ into eight regions

$$\begin{aligned}
\mathcal{S}_1 &:= \big\{(v,w) : v \geq 0, w > 0\big\},\\
\mathcal{S}_2 &:= \big\{(v,w) : v > 0, w = 0\big\},\\
\mathcal{S}_3 &:= \big\{(v,w) : v > 0, -v < w < 0\big\},\\
\mathcal{S}_4 &:= \big\{(v,w) : v > 0, -v = w\big\},\\
\mathcal{S}_5 &= \big\{(v,w) : v = 0, w = 0\big\},\\
\mathcal{S}_6 &:= \big\{(v,w) : w < 0, 0 < v < -w\big\},\\
\mathcal{S}_7 &:= \big\{(v,w) : w < 0, v = 0\big\},\\
\mathcal{S}_8 &:= \big\{(v,w) : w \leq 0, v < 0\big\}.
\end{aligned}$$

From Figure 5.2 we see that the dynamics of $(\mathcal{V}^n, \mathcal{W}^n)$ is the same within each of these eight regions. On the other hand, the types of orders which affect $\mathcal{V}^n$ or $\mathcal{W}^n$ might be different in different regions. Although there are six independent Poisson processes governing the arrivals of market orders and limit orders, for convenience we will thin these Poisson processes according to the regions in which $(\mathcal{V}^n, \mathcal{W}^n)$ is located in order to obtain thirty independent unit-intensity Poisson processes to describe the evolutions of $(\mathcal{V}^n, \mathcal{W}^n)$. In particular, we denote these Poisson processes by $N_{i,\times,*}$, where $i = 1, \dots, 8$ indicates the region in which the Poisson process acts, $\times \in \{\mathcal{V}, \mathcal{W}\}$ indicates which of the processes $\mathcal{V}^n$ or $\mathcal{W}^n$ is affected by the Poisson process, and $* \in \{+, -\}$ indicates whether the Poisson process increases$(+)$ or decreases$(-)$ the affected process. For $i = 1, \dots, 8$, we define $P_i(t)$ to be the time $(\mathcal{V}^n, \mathcal{W}^n)$ spends in region $\mathcal{S}_i$ up to time t. In particular, we have

$$P_i(t) = \int_0^t \mathbb{1}_{\{(\mathcal{V}(s),\mathcal{W}(s))\in\mathcal{S}_i\}}\, ds, \quad i = 1, \dots, 8.$$

Then

$$\begin{aligned}
\mathcal{V}^n(t) \;=\;& \mathcal{V}^n(0) + N_{1,\mathcal{V},+} \circ \lambda_2 P_1(t) - N_{2,\mathcal{V},-} \circ \mu_0 P_2(t) + N_{2,\mathcal{V},+} \circ \lambda_2 P_2(t) \\
& - N_{3,\mathcal{V},-} \circ \mu_0 P_3(t) + N_{3,\mathcal{V},+} \circ \lambda_1 P_3(t) - N_{4,\mathcal{V},-} \circ \mu_0 P_4(t) + N_{4,\mathcal{V},+} \circ \lambda_1 P_4(t) \\
& - N_{5,\mathcal{V},-} \circ \mu_1 P_5(t) + N_{5,\mathcal{V},+} \circ \lambda_2 P_5(t) - N_{6,\mathcal{V},-} \circ \mu_0 P_6(t) + N_{6,\mathcal{V},+} \circ \lambda_1 P_6(t) \\
& - N_{7,\mathcal{V},-} \circ \mu_1 P_7(t) + N_{7,\mathcal{V},+} \circ \lambda_1 P_7(t) - N_{8,\mathcal{V},-} \circ \mu_1 P_8(t) + N_{8,\mathcal{V},+} \circ \lambda_0 P_8(t), \quad (5.1) \\
\mathcal{W}^n(t) \;=\;& \mathcal{W}^n(0) - N_{1,\mathcal{W},-} \circ \mu_0 P_1(t) + N_{1,\mathcal{W},+} \circ \lambda_1 P_1(t) - N_{8,\mathcal{W},-} \circ \mu_2 P_8(t) \\
& - N_{2,\mathcal{W},-} \circ \mu_1 P_2(t) + N_{2,\mathcal{W},+} \circ \lambda_1 P_2(t) - N_{3,\mathcal{W},-} \circ \mu_1 P_3(t) + N_{3,\mathcal{W},+} \circ \lambda_0 P_3(t) \\
& - N_{4,\mathcal{W},-} \circ \mu_1 P_4(t) + N_{4,\mathcal{W},+} \circ \lambda_0 P_4(t) - N_{5,\mathcal{W},-} \circ \mu_2 P_5(t) + N_{5,\mathcal{W},+} \circ \lambda_1 P_5(t) \\
& - N_{6,\mathcal{W},-} \circ \mu_1 P_6(t) + N_{6,\mathcal{W},+} \circ \lambda_0 P_6(t) - N_{7,\mathcal{W},-} \circ \mu_2 P_7(t) + N_{7,\mathcal{W},+} \circ \lambda_0 P_7(t). \quad (5.2)
\end{aligned}$$

5.1.1 Transformation of variables

Recalling the positive constants a and b from Assumption 2.2.1, we define (G^n, H^n) to be the continuous piecewise linear transformation of $(\mathcal{V}^n, \mathcal{W}^n)$ given by

$$G^n(t) := \begin{cases} \mathcal{V}^n(t) + b\mathcal{W}^n(t) & \text{if } \big(\mathcal{V}^n(t), \mathcal{W}^n(t)\big) \in \mathcal{S}_1 \cup \mathcal{S}_2, \\ \mathcal{V}^n(t) + \mathcal{W}^n(t) & \text{if } \big(\mathcal{V}^n(t), \mathcal{W}^n(t)\big) \in \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6, \\ a\mathcal{V}^n(t) + \mathcal{W}^n(t) & \text{if } \big(\mathcal{V}^n(t), \mathcal{W}^n(t)\big) \in \mathcal{S}_7 \cup \mathcal{S}_8. \end{cases} \quad (5.3)$$

$$H^n(t) := \begin{cases} \mathcal{W}^n(t) & \text{if } \big(\mathcal{V}^n(t), \mathcal{W}^n(t)\big) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3, \\ \mathcal{W}^n(t) = -\mathcal{V}^n(t) & \text{if } \big(\mathcal{W}^n(t), \mathcal{V}(t)\big) \in \mathcal{S}_4 \cup \mathcal{S}_5, \\ -\mathcal{V}^n(t) & \text{if } \big(\mathcal{V}^n(t), \mathcal{W}^n(t)\big) \in \mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8. \end{cases} \quad (5.4)$$

Note that this transformation is invertible. Indeed, for $i = 1, \ldots, 8$, the image of $\mathcal{S}_i'$ under this transformation is $\mathcal{S}_i$, where the $\mathcal{S}_i'$ regions are defined by

$$\begin{aligned}
\mathcal{S}_1' &:= \big\{(g,h) : h > 0, g \geq bh\big\},\\
\mathcal{S}_2' &:= \big\{(g,h) : h = 0, g > 0\big\},\\
\mathcal{S}_3' &:= \big\{(g,h) : h < 0, g > 0\big\},\\
\mathcal{S}_4' &:= \big\{(g,h) : h < 0, g = 0\big\},\\
\mathcal{S}_5' &:= \big\{(g,h) : g = h = 0\big\},\\
\mathcal{S}_6' &:= \big\{(g,h) : h < 0, g < 0\big\},\\
\mathcal{S}_7' &:= \big\{(g,h) : h = 0, g < 0\big\},\\
\mathcal{S}_8' &:= \big\{(g,h) : h > 0, g \leq -ah\big\},
\end{aligned}$$

and the inverse map is

$$\mathcal{V}^n(t) = \begin{cases} G^n(t) - bH^n(t) & \text{if } \big(G^n(t), H^n(t)\big) \in \mathcal{S}_1' \cup \mathcal{S}_2', \\ G^n(t) - H^n(t) & \text{if } \big(G^n(t), H^n(t)\big) \in \mathcal{S}_3', \\ -H^n(t) & \text{if } \big(G^n(t), H^n(t)\big) \in \mathcal{S}_4' \cup \mathcal{S}_5' \cup \mathcal{S}_6' \cup \mathcal{S}_7' \cup \mathcal{S}_8', \end{cases} \tag{5.5}$$

$$\mathcal{W}^n(t) = \begin{cases} H^n(t) & \text{if } \big(G^n(t), H^n(t)\big) \in \mathcal{S}_1' \cup \mathcal{S}_2' \cup \mathcal{S}_3' \cup \mathcal{S}_4' \cup \mathcal{S}_5', \\ G^n(t) + H^n(t) & \text{if } \big(G^n(t), H^n(t)\big) \in \mathcal{S}_6', \\ G^n(t) + aH^n(t) & \text{if } \big(G^n(t), H^n(t)\big) \in \mathcal{S}_7' \cup \mathcal{S}_8'. \end{cases} \tag{5.6}$$

It can be verified that the inverse transformation defined by (5.5) and (5.6) is continuous on $\mathcal{S} := \cup_{i=1}^8 \mathcal{S}_i$.

A decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_5 \cup \mathcal{S}_7$ decrease G^n by a units. An increase or decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_8$ increases or decreases, respectively, G^n by a units. Similarly, an increase of $\mathcal{W}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_2 \cup \mathcal{S}_5$ increases G^n by b units. An increase or decrease of $\mathcal{W}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_1$ increases or decreases, respectively, G^n by b units. Otherwise, all increases or decreases in $\mathcal{V}^n$ or $\mathcal{W}^n$ by one unit increase or decrease G^n by one unit. It follows that

$$\begin{aligned}
G^n = \; & G^n(0) + N_{1,\mathcal{V},+} \circ \lambda_2 P_1 - bN_{1,\mathcal{W},-} \circ \mu_0 P_1 + bN_{1,\mathcal{W},+} \circ \lambda_1 P_1 \\
& + N_{2,\mathcal{V},+} \circ \lambda_2 P_2 - N_{2,\mathcal{V},-} \circ \mu_0 P_2 - N_{2,\mathcal{W},-} \circ \mu_1 P_2 + bN_{2,\mathcal{W},+} \circ \lambda_1 P_2 \\
& + N_{3,\mathcal{V},+} \circ \lambda_1 P_3 - N_{3,\mathcal{V},-} \circ \mu_0 P_3 - N_{3,\mathcal{W},-} \circ \mu_1 P_3 + N_{3,\mathcal{W},+} \circ \lambda_0 P_3 \\
& + N_{4,\mathcal{V},+} \circ \lambda_1 P_4 - N_{4,\mathcal{V},-} \circ \mu_0 P_4 - N_{4,\mathcal{W},-} \circ \mu_1 P_4 + N_{4,\mathcal{W},+} \circ \lambda_0 P_4 \\
& + N_{5,\mathcal{V},+} \circ \lambda_2 P_5 - aN_{5,\mathcal{V},-} \circ \mu_1 P_5 - N_{5,\mathcal{W},-} \circ \mu_2 P_5 + bN_{5,\mathcal{W},+} \circ \lambda_1 P_5 \\
& + N_{6,\mathcal{V},+} \circ \lambda_1 P_6 - N_{6,\mathcal{V},-} \circ \mu_0 P_6 - N_{6,\mathcal{W},-} \circ \mu_1 P_6 + N_{6,\mathcal{W},+} \circ \lambda_0 P_6 \\
& + N_{7,\mathcal{V},+} \circ \lambda_1 P_7 - aN_{7,\mathcal{V},-} \circ \mu_1 P_7 - N_{7,\mathcal{W},-} \circ \mu_2 P_7 + N_{7,\mathcal{W},+} \circ \lambda_0 P_7 \\
& + aN_{8,\mathcal{V},+} \circ \lambda_0 P_8 - aN_{8,\mathcal{V},-} \circ \mu_1 P_8 - N_{8,\mathcal{W},-} \circ \mu_2 P_8.
\end{aligned} \tag{5.7}$$

On the other hand, in the region $\mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8$, G^n is negative and a change in G^n results in a change in $|G^n|$ of the same magnitude but the opposite direction. In the region $\mathcal{S}_4 \cup \mathcal{S}_5$,

G^n is zero and a change in G^n results in an increase in $|G^n|$ of the same magnitude. Modifying (5.7) accordingly, we obtain

$$\begin{aligned} |G^n| \;=\; & |G^n(0)| + N_{1,\mathcal{V},+}\circ\lambda_2 P_1 - bN_{1,\mathcal{W},-}\circ\mu_0 P_1 + bN_{1,\mathcal{W},+}\circ\lambda_1 P_1 \\ & +N_{2,\mathcal{V},+}\circ\lambda_2 P_2 - N_{2,\mathcal{V},-}\circ\mu_0 P_2 - N_{2,\mathcal{W},-}\circ\mu_1 P_2 + bN_{2,\mathcal{W},+}\circ\lambda_1 P_2 \\ & +N_{3,\mathcal{V},+}\circ\lambda_1 P_3 - N_{3,\mathcal{V},-}\circ\mu_0 P_3 - N_{3,\mathcal{W},-}\circ\mu_1 P_3 + N_{3,\mathcal{W},+}\circ\lambda_0 P_3 \\ & +N_{4,\mathcal{V},+}\circ\lambda_1 P_4 + N_{4,\mathcal{V},-}\circ\mu_0 P_4 + N_{4,\mathcal{W},-}\circ\mu_1 P_4 + N_{4,\mathcal{W},+}\circ\lambda_0 P_4 \\ & +N_{5,\mathcal{V},+}\circ\lambda_2 P_5 + aN_{5,\mathcal{V},-}\circ\mu_1 P_5 + N_{5,\mathcal{W},-}\circ\mu_2 P_5 + bN_{5,\mathcal{W},-}\circ\lambda_1 P_5 \\ & -N_{6,\mathcal{V},+}\circ\lambda_1 P_6 + N_{6,\mathcal{V},-}\circ\mu_0 P_6 + N_{6,\mathcal{W},-}\circ\mu_1 P_6 - N_{6,\mathcal{W},+}\circ\lambda_0 P_6 \\ & -N_{7,\mathcal{V},+}\circ\lambda_1 P_7 + aN_{7,\mathcal{V},-}\circ\mu_1 P_7 + N_{7,\mathcal{W},-}\circ\mu_2 P_7 - N_{7,\mathcal{W},+}\circ\lambda_0 P_7 \\ & -aN_{8,\mathcal{V},+}\circ\lambda_0 P_8 + aN_{8,\mathcal{V},-}\circ\mu_1 P_8 + N_{8,\mathcal{W},-}\circ\mu_2 P_8. \end{aligned} \tag{5.8}$$

An increase of $\mathcal{W}^n$ by one unit or a decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_4 \cup \mathcal{S}_5$ increases H^n by one unit. An increase or decrease of $\mathcal{W}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ increases or decreases H^n by one unit, respectively, and an increase or decrease of $\mathcal{V}^n$ by one unit when $(\mathcal{V}^n, \mathcal{W}^n)$ is in $\mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8$ decreases or increases H^n by one unit, respectively. Other changes in $\mathcal{V}^n$ and $\mathcal{W}^n$ do not affect H. It follows that

$$\begin{aligned} H^n \;=\; & H^n(0) + N_{1,\mathcal{W},+}\circ\lambda_1 P_1 - N_{1,\mathcal{W},-}\circ\mu_0 P_1 + N_{2,\mathcal{W},+}\circ\lambda_1 P_2 - N_{2,\mathcal{W},-}\circ\mu_1 P_2 \\ & +N_{3,\mathcal{W},+}\circ\lambda_0 P_3 - N_{3,\mathcal{W},-}\circ\mu_1 P_3 + N_{4,\mathcal{W},+}\circ\lambda_0 P_4 + N_{4,\mathcal{V},-}\circ\mu_0 P_4 \\ & +N_{5,\mathcal{W},+}\circ\lambda_1 P_5 + N_{5,\mathcal{V},-}\circ\mu_1 P_5 + N_{6,\mathcal{V},-}\circ\mu_0 P_6 - N_{6,\mathcal{V},+}\circ\lambda_1 P_6 \\ & +N_{7,\mathcal{V},-}\circ\mu_1 P_7 - N_{7,\mathcal{V},+}\circ\lambda_1 P_7 + N_{8,\mathcal{V},-}\circ\mu_1 P_8 - N_{8,\mathcal{V},+}\circ\lambda_0 P_8. \end{aligned} \tag{5.9}$$

In the region $\mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_6$, H^n is negative and a change in H^n results in a change in $|H^n|$ of the same magnitude but the opposite direction. In the region $\mathcal{S}_2 \cup \mathcal{S}_5 \cup \mathcal{S}_7$, H^n is zero and a unit change in H^n results in a unit increase in $|H^n|$. Modifying (5.9) accordingly, we obtain

$$\begin{aligned} |H^n| \;=\; & |H^n(0)| + N_{1,\mathcal{W},+}\circ\lambda_1 P_1 - N_{1,\mathcal{W},-}\circ\mu_0 P_1 + N_{2,\mathcal{W},+}\circ\lambda_1 P_2 + N_{2,\mathcal{W},-}\circ\mu_1 P_2 \\ & -N_{3,\mathcal{W},+}\circ\lambda_0 P_3 + N_{3,\mathcal{W},-}\circ\mu_1 P_3 - N_{4,\mathcal{W},+}\circ\lambda_0 P_4 - N_{4,\mathcal{V},-}\circ\mu_0 P_4 \\ & +N_{5,\mathcal{W},+}\circ\lambda_1 P_5 + N_{5,\mathcal{V},-}\circ\mu_1 P_5 - N_{6,\mathcal{V},-}\circ\mu_0 P_6 + N_{6,\mathcal{V},+}\circ\lambda_1 P_6 \\ & +N_{7,\mathcal{V},-}\circ\mu_1 P_7 + N_{7,\mathcal{V},+}\circ\lambda_1 P_7 + N_{8,\mathcal{V},-}\circ\mu_1 P_8 - N_{8,\mathcal{V},+}\circ\lambda_0 P_8. \end{aligned} \tag{5.10}$$

5.1.2 Diffusion scaling

Recall that the diffusion scaling of a sequence of processes Q^n is defined by,

$$\widehat{Q}^n(t) = \frac{1}{\sqrt{n}} Q^n(nt).$$

Because each of the regions $\mathcal{S}_i$, $i = 1, \ldots, 8$, is a cone, when we apply the diffusion scaling to the processes $\mathcal{V}^n$, $\mathcal{W}^n$, G^n and H^n in the piecewise linear transformations (5.3) and

(5.4) we obtain the analogous formulas

$$\widehat{G}^n(t) \;:=\; \begin{cases} \widehat{\mathcal{V}}^n(t) + b\widehat{\mathcal{W}}^n(t) & \text{if } \big(\widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t)\big) \in \mathcal{S}_1 \cup \mathcal{S}_2, \\ \widehat{\mathcal{V}}^n(t) + \widehat{\mathcal{W}}^n(t) & \text{if } \big(\widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t)\big) \in \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6, \\ a\widehat{\mathcal{V}}^n(t) + \widehat{\mathcal{W}}^n(t) & \text{if } \big(\widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t)\big) \in \mathcal{S}_7 \cup \mathcal{S}_8. \end{cases} \tag{5.11}$$

$$\widehat{H}^n(t) \;:=\; \begin{cases} \widehat{\mathcal{W}}^n(t) & \text{if } \big(\widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t)\big) \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3, \\ \widehat{\mathcal{W}}^n(t) = -\widehat{\mathcal{V}}^n(t) & \text{if } \big(\widehat{\mathcal{W}}^n(t), \widehat{\mathcal{V}}^n(t)\big) \in \mathcal{S}_4 \cup \mathcal{S}_5, \\ -\widehat{\mathcal{V}}^n(t) & \text{if } \big(\widehat{\mathcal{V}}^n(t), \widehat{\mathcal{W}}^n(t)\big) \in \mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8. \end{cases} \tag{5.12}$$

The inverse of this transformation is continuous. In fact, the inverse is given by replacing G by $\widehat{G}^n$ and H by $\widehat{H}^n$ in (5.5) and (5.6),

$$\widehat{\mathcal{V}}^n(t) \;=\; \begin{cases} \widehat{G}^n(t) - b\widehat{H}^n(t) & \text{if } \big(\widehat{G}^n(t), \widehat{H}^n(t)\big) \in \mathcal{R}_1 \cup \mathcal{R}_2, \\ \widehat{G}^n(t) - \widehat{H}^n(t) & \text{if } \big(\widehat{G}^n(t), \widehat{H}^n(t)\big) \in \mathcal{R}_3, \\ -\widehat{H}^n(t) & \text{if } \big(\widehat{G}^n(t), \widehat{H}^n(t)\big) \in \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_6 \cup \mathcal{R}_7 \cup \mathcal{R}_8, \end{cases} \tag{5.13}$$

$$\widehat{\mathcal{W}}^n(t) \;=\; \begin{cases} \widehat{H}^n(t) & \text{if } \big(\widehat{G}^n(t), \widehat{H}^n(t)\big) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \cup \mathcal{R}_5, \\ \widehat{G}^n(t) + \widehat{H}^n(t) & \text{if } \big(\widehat{G}^n(t), \widehat{H}^n(t)\big) \in \mathcal{R}_6, \\ \widehat{G}^n(t) + a\widehat{H}^n(t) & \text{if } \big(\widehat{G}^n(t), \widehat{H}^n(t)\big) \in \mathcal{R}_7 \cup \mathcal{R}_8. \end{cases} \tag{5.14}$$

The Continuous Mapping Theorem implies that we can determine the weak limit in the J_1 topology of $(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)$ by determining the limit of $(\widehat{G}^n, \widehat{H}^n)$.

We next center the thirty independent unit-intensity Poisson processes appearing in (5.1), (5.2), defining

$$M_{i,\times,*}(t) := N_{i,\times,*}(t) - t, \quad t \geq 0.$$

Each of these *compensated Poisson processes* is a martingale relative its own filtration, and these martingale are independent. For $n = 1, 2, \ldots$, their diffusion-scaled versions are

$$\widehat{M}^n_{i,\times,*}(t) := \frac{1}{\sqrt{n}}\big(M_{i,\times,*}(nt) - nt\big), \quad t \geq 0, \tag{5.15}$$

and each of these processes is likewise a martingale relative to its own filtration, and these processes are independent. For $i = 1, \ldots, 8$ and $n = 1, 2, \ldots$, we also define

$$\overline{P}^n_i(t) := \frac{1}{n} P_i(nt), \quad t \geq 0.$$

Replacing the Poisson processes in (5.7) by compensated Poisson processes and applying the diffusion scaling, we obtain ScGP

$$
\begin{aligned}
\widehat{G}^n \;=\;& \widehat{G}^n(0)+\widehat{M}^n_{1,\mathcal{V},+}\circ\lambda_2\overline{P}^n_1-b\widehat{M}^n_{1,\mathcal{W},-}\circ\mu_0\overline{P}^n_1+b\widehat{M}^n_{1,\mathcal{W},+}\circ\lambda_1\overline{P}^n_1\\
&+\widehat{M}^n_{2,\mathcal{V},+}\circ\lambda_2\overline{P}^n_2-\widehat{M}^n_{2,\mathcal{V},-}\circ\mu_0\overline{P}^n_2-\widehat{M}^n_{2,\mathcal{W},-}\circ\mu_1\overline{P}^n_2+b\widehat{M}^n_{2,\mathcal{W},+}\circ\lambda_1\overline{P}^n_2\\
&+\widehat{M}^n_{3,\mathcal{V},+}\circ\lambda_1\overline{P}^n_3-\widehat{M}^n_{3,\mathcal{V},-}\circ\mu_0\overline{P}^n_3-\widehat{M}^n_{3,\mathcal{W},-}\circ\mu_1\overline{P}^n_3+\widehat{M}^n_{3,\mathcal{W},+}\circ\lambda_0\overline{P}^n_3\\
&+\widehat{M}^n_{4,\mathcal{V},+}\circ\lambda_1\overline{P}^n_4-\widehat{M}^n_{4,\mathcal{V},-}\circ\mu_0\overline{P}^n_4-\widehat{M}^n_{4,\mathcal{W},-}\circ\mu_1\overline{P}^n_4+\widehat{M}^n_{4,\mathcal{W},+}\circ\lambda_0\overline{P}^n_4\\
&+\widehat{M}^n_{5,\mathcal{V},+}\circ\lambda_2\overline{P}^n_5-a\widehat{M}^n_{5,\mathcal{V},-}\circ\mu_1\overline{P}^n_5-\widehat{M}^n_{5,\mathcal{W},+}\circ\mu_2\overline{P}^n_5+b\widehat{M}^n_{5,\mathcal{W},-}\circ\lambda_1\overline{P}^n_5\\
&+\widehat{M}^n_{6,\mathcal{V},+}\circ\lambda_1\overline{P}^n_6-\widehat{M}^n_{6,\mathcal{V},-}\circ\mu_0\overline{P}^n_6-\widehat{M}^n_{6,\mathcal{W},-}\circ\mu_1\overline{P}^n_6+\widehat{M}^n_{6,\mathcal{W},+}\circ\lambda_0\overline{P}^n_6\\
&+\widehat{M}^n_{7,\mathcal{V},+}\circ\lambda_1\overline{P}^n_7-a\widehat{M}^n_{7,\mathcal{V},-}\circ\mu_1\overline{P}^n_7-\widehat{M}^n_{7,\mathcal{W},-}\circ\mu_2\overline{P}^n_7+\widehat{M}^n_{7,\mathcal{W},+}\circ\lambda_0\overline{P}^n_7\\
&+a\widehat{M}^n_{8,\mathcal{V},+}\circ\lambda_0\overline{P}^n_8-a\widehat{M}^n_{8,\mathcal{V},-}\circ\mu_1\overline{P}^n_8-\widehat{M}^n_{8,\mathcal{W},-}\circ\mu_2\overline{P}^n_8. \qquad (5.16)
\end{aligned}
$$

The drift terms that arise from the centering of the Poisson processes vanish in (5.16) because, according to Assumption 2.2.1 and its consequence (2.1),

$$
\begin{aligned}
(\lambda_2-b\mu_0+b\lambda_1)\overline{P}^n_1 &= 0,\\
(\lambda_2-\mu_0-\mu_1+b\lambda_1)\overline{P}^n_2 &= 0,\\
(\lambda_1-\mu_0-\mu_1+\lambda_0)\overline{P}^n_3 &= 0,\\
(\lambda_1-\mu_0-\mu_1+\lambda_0)\overline{P}^n_4 &= 0,\\
(\lambda_2-a\mu_1-\mu_2+b\lambda_1)\overline{P}^n_5 &= 0,\\
(\lambda_1-\mu_0-\mu_1+\lambda_0)\overline{P}^n_6 &= 0,\\
(\lambda_1-a\mu_1-\mu_2+\lambda_0)\overline{P}^n_7 &= 0,\\
(a\lambda_0-a\mu_1-\mu_2)\overline{P}^n_8 &= 0.
\end{aligned}
$$

The filtration $\{\mathcal{F}^n(t)\}_{t\geq 0}$ we use for $\widehat{G}^n$ is the one generated by the thirty time-changed processes $\widehat{M}^n_{1,\mathcal{V},I}\circ\lambda_2\overline{P}^n_1,\ldots,\widehat{M}^n_{8,\mathcal{W},-}\circ\mu_2\overline{P}^n_8$ appearing in (5.16). These are not independent because of the coupling of the time changes. However, they are each martingales relative to the filtration $\{\mathcal{F}^n(t)\}_{t\geq 0}$, as is $\widehat{G}^n$.

Replacing the Poisson processes in (5.8), (5.9) and (5.10) by compensated Poisson processes, applying the diffusion scaling, and using Assumption 2.2.1 to simplify, we obtain

$$
\begin{aligned}
\left|\widehat{G}^n\right| \;=\;& \left|\widehat{G}^n(0)\right|+\widehat{M}^n_{1,\mathcal{V},+}\circ\lambda_2\overline{P}^n_1-b\widehat{M}^n_{1,\mathcal{W},-}\circ\mu_0\overline{P}^n_1+b\widehat{M}^n_{1,\mathcal{W},+}\circ\lambda_1\overline{P}^n_1\\
&+\widehat{M}^n_{2,\mathcal{V},+}\circ\lambda_2\overline{P}^n_2-\widehat{M}^n_{2,\mathcal{V},-}\circ\mu_0\overline{P}^n_2-\widehat{M}^n_{2,\mathcal{W},-}\circ\mu_1\overline{P}^n_2+b\widehat{M}^n_{2,\mathcal{W},+}\circ\lambda_1\overline{P}^n_2\\
&+\widehat{M}^n_{3,\mathcal{V},+}\circ\lambda_1\overline{P}^n_3-\widehat{M}^n_{3,\mathcal{V},-}\circ\mu_0\overline{P}^n_3-\widehat{M}^n_{3,\mathcal{W},-}\circ\mu_1\overline{P}^n_3+\widehat{M}^n_{3,\mathcal{W},+}\circ\lambda_0\overline{P}^n_3\\
&+\widehat{M}^n_{4,\mathcal{V},+}\circ\lambda_1\overline{P}^n_4+\widehat{M}^n_{4,\mathcal{V},-}\circ\mu_0\overline{P}^n_4+\widehat{M}^n_{4,\mathcal{W},-}\circ\mu_1\overline{P}^n_4+\widehat{M}^n_{4,\mathcal{W},+}\circ\lambda_0\overline{P}^n_4\\
&+\widehat{M}^n_{5,\mathcal{V},+}\circ\lambda_2\overline{P}^n_5+a\widehat{M}^n_{5,\mathcal{V},-}\circ\mu_1\overline{P}^n_5+\widehat{M}^n_{5,\mathcal{W},-}\circ\mu_2\overline{P}^n_5+b\widehat{M}^n_{5,\mathcal{W},-}\circ\lambda_1\overline{P}^n_5\\
&-\widehat{M}^n_{6,\mathcal{V},+}\circ\lambda_1\overline{P}^n_6+\widehat{M}^n_{6,\mathcal{V},-}\circ\mu_0\overline{P}^n_6+\widehat{M}^n_{6,\mathcal{W},-}\circ\mu_1\overline{P}^n_6-\widehat{M}^n_{6,\mathcal{W},+}\circ\lambda_0\overline{P}^n_6\\
&-\widehat{M}^n_{7,\mathcal{V},+}\circ\lambda_1\overline{P}^n_7+a\widehat{M}^n_{7,\mathcal{V},-}\circ\mu_1\overline{P}^n_7+\widehat{M}^n_{7,\mathcal{W},-}\circ\mu_2\overline{P}^n_7-\widehat{M}^n_{7,\mathcal{W},+}\circ\lambda_0\overline{P}^n_7\\
&-a\widehat{M}^n_{8,\mathcal{V},+}\circ\lambda_0\overline{P}^n_8+a\widehat{M}^n_{8,\mathcal{V},-}\circ\mu_1\overline{P}^n_8+\widehat{M}^n_{8,\mathcal{W},-}\circ\mu_2\overline{P}^n_8\\
&+2a\lambda_0\sqrt{n}(\overline{P}^n_4+\overline{P}^n_5), \qquad (5.17)
\end{aligned}
$$

$$\begin{aligned}
\widehat{H}^n &= \widehat{H}^n(0) + \widehat{M}^n_{1,\mathcal{W},+}\circ\lambda_1\overline{P}^n_1 - \widehat{M}^n_{1,\mathcal{W},-}\circ\mu_0\overline{P}^n_1 + \widehat{M}^n_{2,\mathcal{W},+}\circ\lambda_1\overline{P}^n_2 - \widehat{M}^n_{2,\mathcal{W},-}\circ\mu_1\overline{P}^n_2\\
&\quad + \widehat{M}^n_{3,\mathcal{W},+}\circ\lambda_0\overline{P}^n_3 - \widehat{M}^n_{3,\mathcal{W},-}\circ\mu_1\overline{P}^n_3 + \widehat{M}^n_{4,\mathcal{W},+}\circ\lambda_0\overline{P}^n_4 + \widehat{M}^n_{4,\mathcal{V},-}\circ\mu_0\overline{P}^n_4\\
&\quad + \widehat{M}^n_{5,\mathcal{W},+}\circ\lambda_1\overline{P}^n_5 + \widehat{M}^n_{5,\mathcal{V},-}\circ\mu_1\overline{P}^n_5 + \widehat{M}^n_{6,\mathcal{V},-}\circ\mu_0\overline{P}^n_6 - \widehat{M}^n_{6,\mathcal{V},+}\circ\lambda_1\overline{P}^n_6\\
&\quad + \widehat{M}^n_{7,\mathcal{V},-}\circ\mu_1\overline{P}^n_7 - \widehat{M}^n_{7,\mathcal{V},+}\circ\lambda_1\overline{P}^n_7 + \widehat{M}^n_{8,\mathcal{V},-}\circ\mu_1\overline{P}^n_8 - \widehat{M}^n_{8,\mathcal{V},+}\circ\lambda_0\overline{P}^n_8\\
&\quad + (\lambda_1+\mu_1)\sqrt{n}\,\overline{P}^n_5 + (\lambda_0+\mu_0)\sqrt{n}\,\overline{P}^n_4 + c\sqrt{n}\,(\overline{P}^n_3 + \overline{P}^n_6 - \overline{P}^n_1 - \overline{P}^n_8)\\
&\quad + \sqrt{n}\,(\mu_1-\lambda_1)(\overline{P}^n_7 - \overline{P}^n_2),
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\big|\widehat{H}^n\big| &= \big|\widehat{H}^n(0)\big| + \widehat{M}^n_{1,\mathcal{W},+}\circ\lambda_1\overline{P}^n_1 - \widehat{M}^n_{1,\mathcal{W},-}\circ\mu_0\overline{P}^n_1 + \widehat{M}^n_{2,\mathcal{W},+}\circ\lambda_1\overline{P}^n_2 + \widehat{M}^n_{2,\mathcal{W},-}\circ\mu_1\overline{P}^n_2\\
&\quad - \widehat{M}^n_{3,\mathcal{W},+}\circ\lambda_0\overline{P}^n_3 + \widehat{M}^n_{3,\mathcal{W},-}\circ\mu_1\overline{P}^n_3 - \widehat{M}^n_{4,\mathcal{W},+}\circ\lambda_0\overline{P}^n_4 - \widehat{M}^n_{4,\mathcal{V},-}\circ\mu_0\overline{P}^n_4\\
&\quad + \widehat{M}^n_{5,\mathcal{W},+}\circ\lambda_1\overline{P}^n_5 + \widehat{M}^n_{5,\mathcal{V},-}\circ\mu_1\overline{P}^n_5 - \widehat{M}^n_{6,\mathcal{V},-}\circ\mu_0\overline{P}^n_6 + \widehat{M}^n_{6,\mathcal{V},+}\circ\lambda_1\overline{P}^n_6\\
&\quad + \widehat{M}^n_{7,\mathcal{V},-}\circ\mu_1\overline{P}^n_7 + \widehat{M}^n_{7,\mathcal{V},+}\circ\lambda_1\overline{P}^n_7 + \widehat{M}^n_{8,\mathcal{V},-}\circ\mu_1\overline{P}^n_8 - \widehat{M}^n_{8,\mathcal{V},+}\circ\lambda_0\overline{P}^n_8\\
&\quad + (\lambda_1+\mu_1)\sqrt{n}\,(\overline{P}^n_5 + \overline{P}^n_2 + \overline{P}^n_7) - c\sqrt{n}\,(\overline{P}^n_1 + \overline{P}^n_3 + \overline{P}^n_6 + \overline{P}^n_8)\\
&\quad - (\lambda_0+\mu_0)\sqrt{n}\,\overline{P}^n_4,
\end{aligned} \tag{5.19}$$

where $c > 0$ is defined by (2.1). Because $(\widehat{G}^n, \widehat{H}^n)$ is adapted to $\{\mathcal{F}^n(t)\}_{t\geq 0}$ and for $i = 1, \ldots, 8$,

$$\overline{P}^n_i(t) = \int_0^t \mathbb{1}_{\{(\widehat{G}^n(u), \widehat{H}^n(u))\in\mathcal{R}_i\}}du, \quad t \geq 0, \tag{5.20}$$

the time-change processes $\overline{P}^n_i$ are also adapted to $\{\mathcal{F}^n(t)\}_{t\geq 0}$. Consequently, $|\widehat{G}^n|$ and $|\widehat{H}^n|$ are adapted to this filtration. According to the initial condition, we have

$$\widehat{\mathcal{V}}^n(0) \to 0, \quad \widehat{\mathcal{W}}^n(0) \to 0.$$

From (5.11) and (5.12), we have

$$\widehat{G}^n(0) \to 0, \quad \widehat{H}^n(0) \to 0.$$

5.1.3 Crushing $\widehat{H}^n$

Theorem 5.1.1 $\widehat{H}^n \overset{J_1}{\Longrightarrow} 0.$

Proof: This proof follows the same logic as the proof of Theorem 3.3.3. Because it is notationally different, we give the details. For $t \geq 0$, we define

$$\tau^n(t) := \begin{cases} \sup\big\{s \in [0,t] : \widehat{H}^n(s) = 0\big\} & \text{if } \big\{s \in [0,t] : \widehat{H}^n(s) = 0\big\} \neq \emptyset,\\ 0 & \text{if } \big\{s \in [0,t] : \widehat{H}^n(s) = 0\big\} = \emptyset. \end{cases}$$

Because $\widehat{H}^n(s) \neq 0$ for $s \in (\tau^n(t), t]$, $\overline{P}^n_2$, $\overline{P}^n_5$, and $\overline{P}^n_7$ are flat on this interval, and we have

$$\begin{aligned}
&\overline{P}^n_1(t) + \overline{P}^n_3(t) + \overline{P}^n_4(t) + \overline{P}^n_6(t) + \overline{P}^n_8(t)\\
&\quad = \overline{P}^n_1\big(\tau^n(t)\big) + \overline{P}^n_3\big(\tau^n(t)\big) + \overline{P}^n_4\big(\tau^n(t)\big) + \overline{P}^n_6\big(\tau^n(t)\big) + \overline{P}^n_8\big(\tau^n(t)\big) + t - \tau^n(t)
\end{aligned}$$

and

$$\overline{P}_2^n(t) + \overline{P}_5^n(t) + \overline{P}_7^n(t) = \overline{P}_2^n\big(\tau^n(t)\big) + \overline{P}_5^n\big(\tau^n(t)\big) + \overline{P}_7^n\big(\tau^n(t)\big).$$

Substituting this into (5.19), we obtain

$$\begin{aligned} 0 &\le \big|\widehat{H}^n(t)\big| \\ &= \big|\widehat{H}^n\big(\tau^n(t)\big)\big| + \mathcal{O}_{cl}(1) - c\sqrt{n}\Big[\overline{P}_1^n(t) + \overline{P}_3^n(t) + \overline{P}_6^n(t) + \overline{P}_8^n(t) \\ &\qquad -\overline{P}_1^n\big(\tau^n(t)\big) - \overline{P}_3^n\big(\tau^n(t)\big) - \overline{P}_6^n\big(\tau^n(t)\big) - \overline{P}_8^n\big(\tau^n(t)\big)\Big] \\ &\qquad -(\lambda_0+\mu_0)\sqrt{n}\Big[\overline{P}_4^n(t) - \overline{P}^n\big(\tau^n(t)\big)\Big] \\ &\le \big|\widehat{H}^n\big(\tau^n(t)\big)\big| + \mathcal{O}_{cl}(1) - \min(c, \lambda_0+\mu_0)\sqrt{n}\big(t - \tau^n(t)\big). \end{aligned} \tag{5.21}$$

But $\widehat{H}^n(\tau^n(t)) \to 0$ if $\tau^n(t) = 0$, and otherwise $|\widehat{H}^n(\tau^n(t))| = 1/\sqrt{n}$, so (5.21) implies $\sqrt{n}(e - \tau^n) = \mathcal{O}_{cl}(1)$, where e is the identify process $e(t) = t$ for all $t \ge 0$. This implies

$$\tau^n \stackrel{J_1}{\Longrightarrow} e, \tag{5.22}$$

and thus

$$0 \le \overline{P}_i^n - \overline{P}_i^n \circ \tau^n \le e - \tau^n = \mathfrak{o}(1). \tag{5.23}$$

Because the limits of the processes $\widehat{M}_{i,\times,*}^n$ are continuous, (5.23) implies that

$$\widehat{M}_{i,\times,*}^n \circ \alpha\overline{P}_i^n - \widehat{M}_{i,\times,*}^n \circ \alpha\overline{P}_i^n \circ \tau^n = \mathfrak{o}(1)$$

for any positive constant α. Therefore, we can upgrade the estimate in (5.21) to

$$0 \le \big|\widehat{H}^n(t)\big| \le \mathfrak{o}(1) + \mathfrak{o}(1) - \min(c, \lambda_0+\mu_0)\sqrt{n}\big(t - \tau^n(t)\big),$$

which implies

$$\sqrt{n}\,(e - \tau^n) = \mathfrak{o}(1). \tag{5.24}$$

In particular, $|\widehat{H}^n| = \mathfrak{o}(1)$. $\square$

Remark 5.1.2 *Dividing (5.18) and (5.19) by $\sqrt{n}$ and passing to the limit, we see that*

$$(\lambda_1+\mu_1)\overline{P}_5^n + (\lambda_0+\mu_0)\overline{P}_4^n + c(\overline{P}_3^n + \overline{P}_6^n - \overline{P}_1^n - \overline{P}_8^n) + (\mu_1 - \lambda_1)(\overline{P}_7^n - \overline{P}_2^n) \stackrel{J_1}{\Longrightarrow} 0, \tag{5.25}$$

$$(\lambda_1+\mu_1)(\overline{P}_5^n + \overline{P}_2^n + \overline{P}_7^n) - c(\overline{P}_1^n + \overline{P}_3^n + \overline{P}_6^n + \overline{P}_8^n) - (\lambda_0+\mu_0)\overline{P}_4^n \stackrel{J_1}{\Longrightarrow} 0. \tag{5.26}$$

5.1.4 Convergence of $\widehat{G}^n$

The proof of convergence of $\widehat{G}^n$ and identification of the limit proceeds through several steps. Along the way we identify the limits of the processes $\overline{P}_i^n$, $i = 1, \ldots, 8$.

Lemma 5.1.3 *Let $\varphi_k\colon \mathbb{R} \to [0,\infty)$ be defined for $k = 1, 2, \ldots$ by*

$$\varphi_k(\xi) = \begin{cases} 0 & \text{if } \xi \le -\frac{1}{k}, \\ k\xi + 1 & \text{if } -\frac{1}{k} \le \xi \le 0, \\ -k\xi + 1 & \text{if } 0 \le \xi \le \frac{1}{k}, \\ 0 & \text{if } \xi \ge \frac{1}{k}. \end{cases}$$

Define F_k, $k = 0, 1, \ldots,$ mapping $D[0,\infty)$ to $[0,\infty)$ by

$$\begin{aligned} F_0(x) &= \int_0^\infty e^{-s} \mathbb{1}_{\{0\}}\big(x(s)\big)\,ds, \\ F_k(x) &= \int_0^\infty e^{-s} \varphi_k\big(x(s)\big)\,ds, \quad k = 1, 2, \ldots. \end{aligned}$$

For $k = 1, 2, \ldots,$ F_k is continuous in the Skorohod topology, and $F_0 = \inf_{k\ge1} F_k$ is upper semi-continuous.

Proof: Obviously, $F_0 = \inf_{k\ge1} F_k$, so it suffices to show that F_k is continuous for $k = 1, 2, \ldots.$

Let $k \ge 1$ be given. We recall from [9], Section 3.5, that a metric for the Skorohod topology on $D[0,\infty)$ is

$$d(x,y) := \inf_{\lambda\in\Lambda} \left[\gamma(\lambda) \vee \int_0^\infty e^{-u} d(x,y,\lambda,u)\,du\right],$$

where

$$\begin{aligned} d(x,y,\lambda,u) &:= 1 \wedge \sup_{t\ge0} \big|x(t\wedge u) - y\big(\lambda(t)\wedge u\big)\big|, \\ \gamma(\lambda) &:= \operatorname*{ess\,sup}_{t\ge0} \big|\log \lambda'(t)\big| \\ &= \sup_{s>t\ge0} \left|\log \frac{\lambda(s)-\lambda(t)}{s-t}\right|, \end{aligned}$$

and Λ is the set of all strictly increasing Lipschitz continuous functions λ mapping $[0,\infty)$ onto $[0,\infty)$ with $\gamma(\lambda) < \infty$. Let $x_n \to x$ in the Skorohod topology on $D[0,\infty)$. Then there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ in Λ such that

$$\lim_{n\to0} \gamma(\lambda_n) \vee \int_0^\infty e^{-u} d(x,x_n,\lambda_n,u)\,du = 0. \tag{5.27}$$

We compute

$$
\begin{aligned}
\big|F_k(x_n) - F_k(x)\big| &= \left|\int_0^\infty e^{-s}\big[\varphi_k\big(x_n(s)\big) - \varphi_k\big(x(s)\big)\big]ds\right| \\
&\le \left|\int_0^\infty e^{-s}\Big[\varphi_k\Big(x_n(s)\Big) - \varphi_k\Big(x\big(\lambda_n(s)\big)\Big)\Big]\,ds\right| \\
&\qquad + \left|\int_0^\infty e^{-s}\Big[\varphi_k\Big(x\big(\lambda_n(s)\big)\Big) - \varphi_k\Big(x(s)\Big)\Big]ds\right|.
\end{aligned}
$$

We consider the last two terms separately. Note that φ_k is bounded by 1 and is Lipschitz with constant k. Moreover, from definition of γ and (5.27), for sufficiently large n, $\lambda_n(s) \le s+1$ for all $s \ge 0$. Therefore, we have

$$
\begin{aligned}
&\left|\int_0^\infty e^{-s}\Big[\varphi_k\Big(x_n(s)\Big) - \varphi_k\Big(x\big(\lambda_n(s)\big)\Big)\Big]\,ds\right| \\
&\le \int_0^\infty e^{-s}\Big(2 \wedge k\big|x_n(s) - x\big(\lambda_n(s)\big)\big|\Big)ds \\
&\le 2k\int_0^\infty e^{-s}\Big(1 \wedge \sup_{t\ge 0}\big|x_n\big(t\wedge(s+1)\big) - x\big(\lambda_n(t)\wedge(s+1)\big)\big|\Big)ds \\
&\le 2ke\int_1^\infty e^{-u} d(x_n, x, \lambda_n, u)du,
\end{aligned}
$$

which has limit zero as $n \to 0$. Being Lipschitz, each λ_n is absolutely continuous, $\lambda_n(0) = 0$, λ_n' is defined almost everywhere, and $|\lambda_n' - 1|$ is uniformly bounded by a constant that goes to zero as $n \to \infty$. Therefore,

$$
\begin{aligned}
&\left|\int_0^\infty e^{-s}\Big[\varphi_k\Big(x\big(\lambda_n(s)\big)\Big) - \varphi_k\Big(x(s)\Big)\Big]ds\right| \\
&= \left|\int_0^\infty e^{-s}\varphi_k\Big(x\big(\lambda_n(s)\big)\Big)ds - \int_0^\infty e^{-\lambda_n(t)}\varphi_k\Big(x\big(\lambda_n(t)\big)\Big)\lambda_n'(t)dt\right| \\
&\le \int_0^\infty e^{-s}\big|1 - e^{s-\lambda_n(s)}\lambda_n'(s)\big|ds,
\end{aligned}
$$

and this has limit zero as $n \to \infty$ because $|1 - e^{s-\lambda_n(s)}\lambda_n'(s)|$ converges pointwise to zero and is bounded by $|1 - e^{s/2}|$ uniformly in s for sufficiently large n. This concludes the proof that F_k is continuous for $k = 1, 2, \ldots$. $\square$

Proposition 5.1.4 *The sequence of processes $\{\widehat{G}^n\}_{n=1}^\infty$ is tight in the J_1-topology, every convergent subsequence of this sequence has a continuous limit, and the limit spends zero Lebesgue time at the origin.*

Proof: Define

$$\begin{aligned}
\widehat{\Psi}_1^n &:= \widehat{M}_{1,\mathcal{V},+}^n \circ \lambda_2 e - b\widehat{M}_{1,\mathcal{W},-}^n \circ \mu_0 e + b\widehat{M}_{1,\mathcal{W},+}^n \circ \lambda_1 e,\\
\widehat{\Psi}_2^n &:= \widehat{M}_{2,\mathcal{V},+}^n \circ \lambda_2 e - \widehat{M}_{2,\mathcal{V},-}^n \circ \mu_0 e - \widehat{M}_{2,\mathcal{W},-}^n \circ \mu_1 e + b\widehat{M}_{2,\mathcal{W},+}^n \circ \lambda_1 e,\\
\widehat{\Psi}_3^n &:= \widehat{M}_{3,\mathcal{V},+}^n \circ \lambda_1 e - \widehat{M}_{3,\mathcal{V},-}^n \circ \mu_0 e - \widehat{M}_{3,\mathcal{W},-}^n \circ \mu_1 e + \widehat{M}_{3,\mathcal{W},+}^n \circ \lambda_0 e,\\
\widehat{\Psi}_4^n &:= \widehat{M}_{4,\mathcal{V},+}^n \circ \lambda_1 e - \widehat{M}_{4,\mathcal{V},-}^n \circ \mu_0 e - \widehat{M}_{4,\mathcal{W},-}^n \circ \mu_1 e + \widehat{M}_{4,\mathcal{W},+}^n \circ \lambda_0 e,\\
\widehat{\Psi}_5^n &:= \widehat{M}_{5,\mathcal{V},+}^n \circ \lambda_2 e - a\widehat{M}_{5,\mathcal{V},-}^n \circ \mu_1 e - \widehat{M}_{5,\mathcal{W},-}^n \circ \mu_2 e + b\widehat{M}_{5,\mathcal{W},+}^n \circ \lambda_1 e,\\
\widehat{\Psi}_6^n &:= \widehat{M}_{6,\mathcal{V},+}^n \circ \lambda_1 e - \widehat{M}_{6,\mathcal{V},-}^n \circ \mu_0 e - \widehat{M}_{6,\mathcal{W},-}^n \circ \mu_1 e + \widehat{M}_{6,\mathcal{W},+}^n \circ \lambda_0 e,\\
\widehat{\Psi}_7^n &:= \widehat{M}_{7,\mathcal{V},+}^n \circ \lambda_1 e - a\widehat{M}_{7,\mathcal{V},-}^n \circ \mu_1 e - \widehat{M}_{7,\mathcal{W},-}^n \circ \mu_2 e + \widehat{M}_{7,\mathcal{W},+}^n \circ \lambda_0 e,\\
\widehat{\Psi}_8^n &:= a\widehat{M}_{8,\mathcal{V},+}^n \circ \lambda_0 e - a\widehat{M}_{8,\mathcal{V},-}^n \circ \mu_1 e - \widehat{M}_{8,\mathcal{W},-}^n \circ \mu_2 e,
\end{aligned}$$

so that $\widehat{G}^n = \widehat{G}^n(0) + \sum_{i=1}^8 \widehat{\Psi}_i^n \circ \overline{P}_i^n$. Because $[\widehat{M}_{i,\times,*}^n, \widehat{M}_{i,\times,*}^n] \overset{J_1}{\Longrightarrow} e$ and these processes are independent, we have

$$\begin{aligned}
\big[\widehat{\Psi}_1^n, \widehat{\Psi}_1^n\big] &\overset{J_1}{\Longrightarrow} (\lambda_2 + b^2\lambda_1 + b^2\mu_0)e =: A_1, &(5.28)\\
\big[\widehat{\Psi}_2^n, \widehat{\Psi}_2^n\big] &\overset{J_1}{\Longrightarrow} (\lambda_2 + b^2\lambda_1 + b\mu_0)e =: A_2, &(5.29)\\
\big[\widehat{\Psi}_3^n, \widehat{\Psi}_3^n\big] &\overset{J_1}{\Longrightarrow} 2a\lambda_0 e =: A_3, &(5.30)\\
\big[\widehat{\Psi}_4^n, \widehat{\Psi}_4^n\big] &\overset{J_1}{\Longrightarrow} (a\lambda_0 + b\mu_0)e =: A_4, &(5.31)\\
\big[\widehat{\Psi}_5^n, \widehat{\Psi}_5^n\big] &\overset{J_1}{\Longrightarrow} (\lambda_2 + a^2\mu_1 + \mu_2 + b^2\lambda_1)e =: A_5, &(5.32)\\
\big[\widehat{\Psi}_6^n, \widehat{\Psi}_6^n\big] &\overset{J_1}{\Longrightarrow} 2b\mu_0 e =: A_6, &(5.33)\\
\big[\widehat{\Psi}_7^n, \widehat{\Psi}_7^n\big] &\overset{J_1}{\Longrightarrow} (\mu_2 + a^2\mu_1 + a\lambda_0)e =: A_7, &(5.34)\\
\big[\widehat{\Psi}_8^n, \widehat{\Psi}_8^n\big] &\overset{J_1}{\Longrightarrow} (\mu_2 + a^2\mu_1 + a^2\lambda_0)e =: A_8. &(5.35)
\end{aligned}$$

We next define

$$A^n = \sum_{i=1}^8 A_i \circ \overline{P}_i^n, \tag{5.36}$$

which is a strictly increasing, piecewise linear process with slope bounded between $m := \min\{A_i' : i = 1, \dots, 8\}$ and $M := \max\{A_i' : i = 1, \dots, 8\}$. Let I^n be the inverse of A^n, a strictly increasing, piecewise linear process whose slope is bounded between $1/M$ and $1/m$. We observed using (5.20) that each $\overline{P}_i^n$ is $\{\mathcal{F}^n(t)\}_{t\geq 0}$-adapted. Therefore, for each $s \geq 0$, $I^n(s)$ is a stopping time for the filtration $\{\mathcal{F}^n(t)\}_{t\geq 0}$. We have

$$\begin{aligned}
\big[\widehat{G}^n \circ I^n, \widehat{G}^n \circ I^n\big] &= \sum_{i=1}^8 \big[\widehat{\Psi}_i^n, \widehat{\Psi}_i^n\big] \circ \overline{P}_i^n \circ I^n\\
&= \sum_{i=1}^8 \big(\big[\widehat{\Psi}_i^n, \widehat{\Psi}_i^n\big] - A_i\big) \circ \overline{P}_i^n \circ I^n + \sum_{i=1}^8 A_i \circ \overline{P}_i^n \circ I^n\\
&= \sum_{i=1}^8 \big(\big[\widehat{\Psi}_i^n, \widehat{\Psi}_i^n\big] - A_i\big) \circ \overline{P}_i^n \circ I^n + e.
\end{aligned}$$

Because $\overline{P}_i^n \circ I^n \leq \frac{1}{m}e$, it follows from (5.28)–(5.35) that

$$\sum_{i=1}^{8} \left(\left[\widehat{\Psi}_i^n, \widehat{\Psi}_i^n \right] - A_i \right) \circ \overline{P}_i^n \circ I^n \xRightarrow{J_1} 0,$$

or equivalently,

$$\left[\widehat{G}^n \circ I^n, \widehat{G}^n \circ I^n \right] \xRightarrow{J_1} e. \tag{5.37}$$

Since $\widehat{G}^n(0) \to 0$, we now apply [9], Theorem 1.4 of Section 7.1, to the sequence of martingales $\{\widehat{G}^n \circ I^n\}_{n=1}^{\infty}$ relative to the filtrations $\{\mathcal{F}_{I^n(s)}^n\}_{s \geq 0}$ to conclude that $\widehat{G}^n \circ I^n$ converges weakly-J_1 to a standard Brownian motion starting at zero, i.e.,

$$\widehat{G}^n \circ I^n \xRightarrow{J_1} B^*, \tag{5.38}$$

where B^* is a standard Brownian motion starting at zero. Since the time changes A^n satisfy the uniform bound $A^n \leq Me$, we conclude that the sequence $\{\widehat{G}^n\}_{n=1}^{\infty} = \{\widehat{G}^n \circ I^n \circ A^n\}_{n=1}^{\infty}$ is tight.

We now show that the limit of every convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$ is continuous. We recall from Theorem 10.2, Section 3.10 of [9] that if $X^n \xRightarrow{J_1} X$, then X is continuous if and only if $J(X^n) \xRightarrow{J_1} 0$, where for $t \geq 0$,

$$J(x)(t) := \sup_{0 \leq u \leq t} \left| x(u) - x(u-) \right|, \quad x \in D[0, \infty).$$

Given a convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$, we have $\widehat{G}^n \circ I^n \xRightarrow{J_1} B^*$, where B^* a standard Brownian motion, hence a continuous process. We have

$$J(\widehat{G}^n)(t) = J(\widehat{G}^n \circ I^n \circ A^n)(t) \leq J(\widehat{G}^n \circ I^n)(Mt),$$

and hence $J(\widehat{G}^n) \xRightarrow{J_1} 0$ along the convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$. Consequently, the limit of this convergent subsequence is continuous.

Finally, we assume that G^* is the limit of a convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$ and show that G^* spends zero Lebesgue time at the origin. We have already shown that along this subsequence, $\widehat{G}^n \circ I^n$ converges to a Brownian motion B^*, which spends zero Lebesgue time at the origin. Therefore, given $\varepsilon > 0$, there exists K such that $F_k(B^*) < \varepsilon$ for $k \geq K$, where we are using the notation of Lemma 5.1.3. Consequently, there exists N such that $F_k(\widehat{G}^n \circ I^n) < \varepsilon$ for $k \geq K$ and $n \geq N$. Making the change of variable $s = A^n(u)$, we see that

$$m \int_0^{\infty} e^{-Mu} \varphi_k\big(\widehat{G}^n(u)\big) du \leq \int_0^{\infty} e^{-s} \varphi_k\big(\widehat{G}^n \circ I^n(s)\big) ds = F_k\big(\widehat{G}^n \circ I^n\big) < \varepsilon.$$

Taking the limit as $n \to \infty$ along the convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$ and using the continuity of F_k proved in Lemma 5.1.3, we see further that

$$\int_0^{\infty} e^{-Mu} \mathbb{1}_{\{G^*(u)=0\}} du \leq \int_0^{\infty} e^{-Mu} \varphi_k\big(G^*(u)\big) du \leq \varepsilon / m.$$

Because $\varepsilon > 0$ is arbitrary, we conclude that G^* spends zero Lebesgue time at the origin. $\square$

Corollary 5.1.5 *We have* $\overline{P}_4^n \overset{J_1}{\Longrightarrow} 0$ *and* $\overline{P}_5^n \overset{J_1}{\Longrightarrow} 0$.

PROOF: Again we use the notation of Lemma 5.1.3. Suppose $\{\mathbb{P}^n\}_{n=1}^{\infty}$ is a sequence of measures on $D[0,\infty)$ converging weakly to a measure $\mathbb{P}$. We have

$$\limsup_{n\to\infty} \int_{D[0,\infty)} F_0\, d\mathbb{P}^n \le \lim_{n\to\infty} \int_{D[0,\infty)} F_k\, d\mathbb{P}^n = \int_{D[0,\infty)} F_k\, d\mathbb{P}.$$

Letting $k \to \infty$, we conclude that

$$\limsup_{n\to\infty} \int_{D[0,\infty)} F_0 d\mathbb{P}^n \le \int_{D[0,\infty)} F_0 d\mathbb{P}. \tag{5.39}$$

Let $\mathbb{P}^n$ be the probability measure induced on $D[0,\infty)$ by the process $\widehat{G}^n \circ I^n$ in (5.38). In the proof of Proposition 5.1.4, we showed that this sequence converges weakly to Wiener measure. We have

$$\begin{aligned}
m\mathbb{E}\int_0^\infty e^{-Mu}\big(d\overline{P}_4^n(u) + d\overline{P}_5^n(u)\big) &\le \mathbb{E}\int_0^\infty e^{-A^n(u)}\big(A_4'(u)d\overline{P}_4^n(u) + A_5'(u)d\overline{P}_5^n(u)\big)\\
&= \mathbb{E}\int_0^\infty e^{-A^n(u)}\mathbb{1}_{\{\widehat{G}^n(u)=0\}}dA^n(u)\\
&= \mathbb{E}\int_0^\infty e^{-s}\mathbb{1}_{\{\widehat{G}^n\circ I^n(s)=0\}}ds\\
&= \int_{D[0,\infty)} F_0 d\mathbb{P}^n.
\end{aligned}$$

By (5.39), the limit of this last expression is zero because Brownian motion spends zero Lebesgue time at the origin. We conclude that $\overline{P}_4^n + \overline{P}_5^n$ converges to zero uniformly on compact time intervals in probability, which is equivalent to the convergence stated in the corollary. $\square$

We have shown in (5.38) that $\widehat{G}^n \circ I^n$ converges to Brownian motion, and we want to identify the limit of $\widehat{G}^n = \widehat{G}^n \circ I^n \circ A^n$. Thus we need to determine the limit of A^n given by (5.36). To do this, we must determine the limits of the processes $\overline{P}_i^n$, $i = 1, \ldots, 8$. We have just done that for $\overline{P}_4^n$ and $\overline{P}_5^n$. For the other processes, we have the following result.

Proposition 5.1.6 *Consider a convergent subsequence of* $\{\widehat{G}^n\}_{n=1}^{\infty}$ *with limit* G^*. *Define*

$$\overline{P}_{G^*}^+(t) = \int_0^t \mathbb{1}_{\{G^*(s)>0\}}ds, \quad \overline{P}_{G^*}^-(t) = \int_0^t \mathbb{1}_{\{G^*(s)<0\}}ds, \quad t \ge 0, \tag{5.40}$$

Then along the same subsequence of indices for which $\widehat{G}^n \Longrightarrow G^*$, *we have*

$$(\widehat{G}^n, \widehat{H}^n, \overline{P}_1^n, \ldots, \overline{P}_8^n) \overset{J_1}{\Longrightarrow} (G^*, 0, \overline{P}_1, \ldots, \overline{P}_8), \tag{5.41}$$

where $\overline{P}_4 = \overline{P}_5 = 0$, *and*

$$\overline{P}_1 = \frac{\lambda_1}{\lambda_0 + \lambda_1}\overline{P}^{+}_{G^*}, \quad \overline{P}_2 = \frac{\lambda_0 - \mu_1}{\lambda_0 + \lambda_1}\overline{P}^{+}_{G^*}, \quad \overline{P}_3 = \frac{\mu_1}{\lambda_0 + \lambda_1}\overline{P}^{+}_{G^*}, \tag{5.42}$$

$$\overline{P}_6 = \frac{\lambda_1}{\mu_0 + \mu_1}\overline{P}^{-}_{G^*}, \quad \overline{P}_7 = \frac{\mu_0 - \lambda_1}{\mu_0 + \mu_1}\overline{P}^{-}_{G^*}, \quad \overline{P}_8 = \frac{\mu_1}{\mu_0 + \mu_1}\overline{P}^{-}_{G^*}. \tag{5.43}$$

PROOF: For notational simplicity, we assume that $\widehat{G}^n \Longrightarrow G^*$ along the full sequence. Each $\overline{P}^n_i$ is nondecreasing and Lipschitz with Lipschitz constant 1. Moreover, $\overline{P}^n_i(0) = 0$. This together with Theorem 5.1.1 and Proposition 5.1.4 implies that the sequence $\{(\widehat{G}^n, \widehat{H}^n, \overline{P}^n_1, \ldots, \overline{P}^n_9)\}_{n=1}^{\infty}$ is tight. Given any subsequence of this sequence, there is a sub-subsequence that converges weakly-J_1 to a limit $(G^*, H^*, \overline{P}_1, \ldots, \overline{P}_8)$. We know from Theorem 5.1.1, Proposition 5.1.4 and Corollary 5.1.5 that $H^* \equiv 0$, G^* is continuous, and $\overline{P}_4 \equiv 0$, $\overline{P}_5 \equiv 0$. We also know that $\overline{P}_1, \overline{P}_2, \overline{P}_3, \overline{P}_6, \overline{P}_7$ and $\overline{P}_8$ are Lipschitz continuous with Lipschitz constant 1. We show that these last six processes satisfy (5.42) and (5.43), and hence they do not depend on the sub-subsequence. It will then follow that these are the limits of the full sequence.

Substituting $\overline{P}_4 = \overline{P}_5 = 0$ into (5.25) and (5.26), we obtain

$$c\big(\overline{P}_3 + \overline{P}_6 - \overline{P}_1 - \overline{P}_8) + (\mu_1 - \lambda_1)(\overline{P}_7 - \overline{P}_2) = 0, \tag{5.44}$$

$$(\lambda_1 + \mu_1)(\overline{P}_2 + \overline{P}_7) - c(\overline{P}_1 + \overline{P}_3 + \overline{P}_6 + \overline{P}_8) = 0. \tag{5.45}$$

We also have $\sum_{i=1}^{8} \overline{P}^n_i = e$, which implies

$$\overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \overline{P}_6 + \overline{P}_7 + \overline{P}_8 = e. \tag{5.46}$$

From (2.1), (5.45) and (5.46), we obtain

$$\overline{P}_1 + \overline{P}_3 + \overline{P}_6 + \overline{P}_8 = \frac{\lambda_1 + \mu_1}{\lambda_0 + \lambda_1}e, \tag{5.47}$$

$$\overline{P}_2 + \overline{P}_7 = \frac{\lambda_0 - \mu_1}{\lambda_0 + \lambda_1}e. \tag{5.48}$$

Adding (5.44) and (5.45), we see that

$$\overline{P}_1 + \overline{P}_8 = \frac{\lambda_1}{c}\overline{P}_2 + \frac{\mu_1}{c}\overline{P}_7. \tag{5.49}$$

Subtracting (5.45) from (5.44) yields

$$\overline{P}_3 + \overline{P}_6 = \frac{\mu_1}{c}\overline{P}_2 + \frac{\lambda_1}{c}\overline{P}_7. \tag{5.50}$$

We use the Skorohod representation theorem to put the pre-limit and the limit processes on a common probability space so that the convergence of the subsequence of $(\widehat{G}^n, \widehat{H}^n, \overline{P}^n_1, \ldots, \overline{P}^n_8)$ to the continuous process $(G^*, 0, \overline{P}_1, \ldots, \overline{P}_8)$ is uniform on compact time intervals almost surely. Because each $\overline{P}_i$ is Lipschitz, to identify $\overline{P}_i$ it suffices to identify $\overline{P}_i'$ for Lebesgue-almost every $t \geq 0$. We identify $\overline{P}_i'(t)$ for all t such that $G^*(t) \neq 0$, a set of full Lebesgue measure by Proposition 5.1.4.

Assume first that $G^*(t) > 0$. Then for sufficiently large n, $\widehat{G}^n$ is strictly positive in a neighborhood of t. We see from (5.11) that in this neighborhood, $(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)$ is in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, which implies that $\overline{P}_6^n$, $\overline{P}_7^n$ and $\overline{P}_8^n$ are constant for sufficiently large n, and consequently their limits $\overline{P}_6$, $\overline{P}_7$ and $\overline{P}_8$ are constant in this neighborhood. In particular,

$$\overline{P}_6'(t) = \overline{P}_7'(t) = \overline{P}_8'(t) = 0 \text{ if } G^*(t) > 0. \tag{5.51}$$

Equation (5.48) implies

$$\overline{P}_2'(t) = \frac{\lambda_0 - \mu_1}{\lambda_0 + \lambda_1} \text{ if } G^*(t) > 0. \tag{5.52}$$

Substitution of this into (5.49) yields

$$\overline{P}_1'(t) = \frac{\lambda_1}{\lambda_0 + \lambda_1} \text{ if } G^*(t) > 0. \tag{5.53}$$

Substitution of this into (5.47) results in

$$\overline{P}_3'(t) = \frac{\mu_1}{\lambda_0 + \lambda_1} \text{ if } G^*(t) > 0 \tag{5.54}$$

An analogous argument for t such that $G^*(t) < 0$ yields

$$\overline{P}_1'(t) = \overline{P}_2'(t) = \overline{P}_3'(t) = 0 \text{ if } G^*(t) < 0, \tag{5.55}$$

and

$$\overline{P}_6'(t) = \frac{\lambda_1}{\mu_0 + \mu_1}, \quad \overline{P}_7'(t) = \frac{\mu_0 - \lambda_1}{\mu_0 + \mu_1}, \quad \overline{P}_8'(t) = \frac{\mu_1}{\mu_0 + \mu_1} \text{ if } G^*(t) < 0. \tag{5.56}$$

Integrating (5.51)–(5.56), we obtain (5.42) and (5.43). ☐

Corollary 5.1.7 *Under the assumptions of Proposition 5.1.6, along the same subsequence of indices for which $\widehat{G}^n \Longrightarrow G^*$, the sequence of processes $\{A^n\}_{n=1}^{\infty}$ defined by (5.36) satisfies*

$$A^n \overset{J_1}{\Longrightarrow} c_+ \overline{P}_{G^*}^+ + c_- \overline{P}_{G^*}^- \tag{5.57}$$

and

$$G^* = B^* \circ (c_+ \overline{P}_{G^*}^+ + c_- \overline{P}_{G^*}^-). \tag{5.58}$$

where

$$\begin{aligned} c_+ &= 2\lambda_0(1 + ab - b) = c_L, \\ c_- &= 2\lambda_0(\frac{a}{b} + a^2 - \frac{a^2}{b}) = c_J, \end{aligned}$$

and c_L and c_J are defined in Proposition 3.4.1.

PROOF: For (5.57), it suffices to verify that $A_1 \circ \overline{P}_1 + A_2 \circ \overline{P}_2 + A_3 \circ \overline{P}_3 = c_+ \overline{P}^+_{G^*}$ and $A_6 \circ \overline{P}_6 + A_7 \circ \overline{P}_7 + A_8 \circ \overline{P}_8 = c_- \overline{P}^-_{G^*}$. This is a lengthy but direct computation using Assumption 2.2.1, (5.28)–(5.35), (5.42) and (5.43).

Because $(\widehat{G}^n \circ I^n, A^n) \overset{J_1}{\Longrightarrow} (B^*, c_+ \overline{P}^+_{G^*} + c_- \overline{P}^-_{G^*})$ (see (5.38)) and the Brownian motion B^* is continuous, we can invoke the time-change lemma in Section 14 of [5] to obtain (5.58). That lemma is stated for $D[0,1]$, but the modification of the proof to obtain the result for $D[0,\infty)$ is straightforward. □

Theorem 5.1.8 *Every weakly convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$ converges to the same limit, i.e., all limits induce the same probability measure on $C[0,\infty)$. In particular, the limit is a two-variance Brownian motion defined in Definition 4.1.1.*

PROOF: From Corollary 5.1.7 and Proposition 4.1.3, we see that the limit of every convergent subsequence of $\{\widehat{G}^n\}_{n=1}^{\infty}$ is a two-variance Brownian motion. We want to point out that in (5.58), B^* is a standard Brownian motion whose measure is unique among all convergent subsequences. From Proposition 4.1.3, we can rewrite (5.58) as

$$G^* = B^* \circ \left(\frac{1}{c_+} P^+_{B^*} + \frac{1}{c_-} P^-_{B^*}\right)^{-1},$$

where

$$\begin{aligned} P^+_{B^*}(t) &= \int_0^t \mathbb{1}_{\{B^*(s)>0\}} ds, \\ P^-_{B^*}(t) &= \int_0^t \mathbb{1}_{\{B^*(s)<0\}} ds. \end{aligned}$$

Since the probability measure induced by B^* is the same among all weakly convergent subsequences of $\{\widehat{G}^n\}_{n=1}^{\infty}$, we complete the proof. □

5.1.5 Convergence of $(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n)$

Because of the inverse map defined in (5.13) and (5.14), applying Theorem 5.1.1 and Theorem 5.1.8, we obtain the following result.

Corollary 5.1.9

$$(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n) \overset{J_1}{\Longrightarrow} (\max\{G^*, 0\}, \min\{G^*, 0\}). \tag{5.59}$$

We refer to the process on the right hand side of (5.59) as *split two-variance Brownian motion*.

5.2 The bracketing processes

The dynamics of the bracketing processes $\widehat{\mathcal{U}}^n$ and $\widehat{\mathcal{X}}^n$ depend on the state of the interior processes $\widehat{\mathcal{V}}^n$ and $\widehat{\mathcal{W}}^n$. We shall see that the diffusion-scaled bracketing processes converge in the M_1 topology on

$$D[0-,\infty) := \mathbb{R}\times D[0,\infty)$$

to *snapped Brownian motions.* More specifically, when G^* is on a negative excursion, $\mathcal{U}^*$, the limit of $\widehat{\mathcal{U}}^n$, is a Brownian motion, but when G^* is on a positive excursion, $\mathcal{U}^*$ is frozen at κ_L. Analogously, when G^* is on a positive excursion, $\mathcal{X}^*$ is a Brownian motion, but when G^* is on a negative excursion, $\mathcal{X}^*$ is frozen at κ_R. To determine the dependence between $\mathcal{U}^*$ and the negative excursions of G^*, we decompose $\mathcal{U}^*$ into two processes, one of which is the excursion of G^* itself and the other of which is independent of G^*. A similar decomposition applies to $\mathcal{X}^*$ and the positive excursions of G^*.

To establish limits for $\widehat{\mathcal{U}}^n$ and $\widehat{\mathcal{X}}^n$, in Section 5.2.1 we first establish stochastic boundedness of the sequences $\{\widehat{\mathcal{U}}^n\}_{n=1}^\infty$ and $\{\widehat{\mathcal{X}}^n\}_{n=1}^\infty$. The decomposition of $\mathcal{U}^*$ when G^* is on a negative excursion into the negative excursion of G^* and an independent process requires a lengthy technical analysis, which is contained in Section 5.2.2. The analogous decomposition of $\mathcal{X}^*$ when G^* is on a positive excursion into the positive excursion of G^* and an independent process is then stated without proof. After that the convergence of $(\widehat{\mathcal{U}}^n,\widehat{\mathcal{X}}^n)$ to $(\mathcal{U}^*,\mathcal{X}^*)$ is straightforward, and the proof is given in Section 5.2.3. To conclude we need to establish that the stopping times when $\widehat{\mathcal{U}}^n$ or $\widehat{\mathcal{X}}^n$ hits zero converge to the stopping time when $\mathcal{U}^*$ or $\mathcal{X}^*$ hits zero. This is the main content of Section 5.2.4. This section completes the determination of the limit of the sequence of diffusion-scaled limit-order books.

In order to avoid a discussion of the possibility that the bracketing processes are no longer valid till a certain time, we consider a pair of processes $(\mathcal{U}^n,\mathcal{X}^n)$ that has the same dynamics as (U^n,X^n) but is defined by these dynamics for all time, not just during the period of time when U^n and X^n are bracketing processes. Note that cancellations might happen on $\mathcal{U}^n$ or $\mathcal{X}^n$. Therefore the dynamics of the n^{th} pre-limit model does depend on n. The LOB has eight possible configurations depending on the locations of the best bid price and the best ask price, as shown in Figure 5.1. Within each configuration, the dynamics of $\mathcal{U}^n$ and $\mathcal{X}^n$ are the same, so we can use the same way to write down the dynamics of $\mathcal{U}^n$ and $\mathcal{X}^n$ as what we did in previous subsection. In particular, we can introduce eighteen independent unit-intensity Poisson processes $N_{i,\times,*}$, where $i=1,\dots,8$ indicates the region where $\mathcal{V},\mathcal{W}$ is, $\times\in\{\mathcal{U},\mathcal{X}\}$ indicates which of the processes $\mathcal{U}^n$ or $\mathcal{X}^n$ is affected by the Poisson process, and $*\in\{+,-\}$ indicates whether the Poisson process increases($+$) or decreases($-$) the affected process. Hence, we have

$$\begin{aligned} d\mathcal{U}^n(t) \;=\; & d\Big(-N_{1,\mathcal{U},-}\big(\int_0^t \frac{\theta_b}{\sqrt{n}}(\mathcal{U}^n(s))^+\,dP_1(s)\big)+N_{3,\mathcal{U},+}\circ\lambda_2P_3(t) \\ & +N_{4,\mathcal{U},+}\circ\lambda_2P_4(t)-N_{5,\mathcal{U},-}\circ\mu_0P_5(t)+N_{6,\mathcal{U},+}\circ\lambda_2P_6(t)+N_{7,\mathcal{U},+}\circ\lambda_2P_7(t) \\ & -N_{7,\mathcal{U},-}\circ\mu_0P_7(t)+N_{8,\mathcal{U},+}\circ\lambda_1P_8(t)-N_{8,\mathcal{U},-}\circ\mu_0P_8(t)\Big), \end{aligned} \tag{5.60}$$

$$\begin{aligned}
d\mathcal{X}^n(t) &= d\Big(N_{8,\mathcal{X},+}\big(\int_0^t \frac{\theta_s}{\sqrt{n}}(\mathcal{X}^n(s))^-\,dP_8(s)\big) - N_{6,\mathcal{X},-}\circ\mu_2 P_6(t)\\
&\quad + N_{5,\mathcal{X},+}\circ\lambda_0 P_5(t) - N_{4,\mathcal{X},-}\circ\mu_2 P_4(t) - N_{3,\mathcal{X},-}\circ\mu_2 P_3(t) + N_{2,\mathcal{X},+}\circ\lambda_0 P_2(t)\\
&\quad - N_{2,\mathcal{X},-}\circ\mu_2 P_2(t) + N_{1,\mathcal{X},+}\circ\lambda_0 P_1(t) - N_{1,\mathcal{X},-}\circ\mu_1 P_1(t)\Big). \qquad (5.61)
\end{aligned}$$

We next center the eighteen independent unit-intensity Poisson processes appearing in (5.60), (5.61), defining

$$M_{i,\times,*}(t) := N_{i,\times,*}(t) - t, \quad t \geq 0.$$

Each of these *compensated Poisson processes* is a martingale relative its own filtration, and these martingale are independent. For $n = 1, 2, \ldots$, their diffusion-scaled versions are

$$\widehat{M}^n_{i,\times,*}(t) := \frac{1}{\sqrt{n}}\big(M_{i,\times,*}(nt) - nt\big), \quad t \geq 0,$$

and each of these processes is likewise a martingale relative to its own filtration, and these processes are independent. Replacing the Poisson processes in (5.60) and (5.61) by the centered Poisson processes and applying the diffusion scaling, we obtain

$$\begin{aligned}
\widehat{\mathcal{U}}^n(t) &= \widehat{\mathcal{U}}^n(0) - \widehat{M}^n_{1,\mathcal{U},-}\big(\int_0^t \theta_b(\widehat{\mathcal{U}}^n(s))^+\,d\overline{P}^n_1(s)\big) + \widehat{M}^n_{3,\mathcal{U},+}\circ\lambda_2\overline{P}^n_3(t)\\
&\quad + \widehat{M}^n_{4,\mathcal{U},+}\circ\lambda_2\overline{P}^n_4(t) - \widehat{M}^n_{5,\mathcal{U},-}\circ\mu_0\overline{P}^n_5(t) + \widehat{M}^n_{6,\mathcal{U},+}\circ\lambda_2\overline{P}^n_6(t) + \widehat{M}^n_{7,\mathcal{U},+}\circ\lambda_2\overline{P}^n_7(t)\\
&\quad - \widehat{M}^n_{7,\mathcal{U},-}\circ\mu_0\overline{P}^n_7(t) + \widehat{M}^n_{8,\mathcal{U},+}\circ\lambda_1\overline{P}^n_8(t) - \widehat{M}^n_{8,\mathcal{U},-}\circ\mu_0\overline{P}^n_8(t)\\
&\quad + \sqrt{n}\big(-\int_0^t \theta_b(\widehat{\mathcal{U}}^n(s))^+\,d\overline{P}^n_1(s) + \lambda_2\overline{P}^n_3(t) + \lambda_2\overline{P}^n_4(t) - \mu_0\overline{P}^n_5(t)\\
&\quad + \lambda_2\overline{P}^n_6(t) - (\mu_0-\lambda_2)\overline{P}^n_7(t) - (\mu_0-\lambda_1)\overline{P}^n_8(t)\big), \qquad (5.62)
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathcal{X}}^n(t) &= \widehat{\mathcal{X}}^n(0) + \widehat{M}^n_{8,\mathcal{X},+}\big(\int_0^t \theta_s(\widehat{\mathcal{X}}^n(s))^-\,d\overline{P}^n_8(s)\big) - \widehat{M}^n_{6,\mathcal{X},-}\circ\mu_2\overline{P}^n_6(t)\\
&\quad + \widehat{M}^n_{5,\mathcal{X},+}\circ\lambda_0\overline{P}^n_5(t) - \widehat{M}^n_{4,\mathcal{X},-}\circ\mu_2\overline{P}^n_4(t) - \widehat{M}^n_{3,\mathcal{X},-}\circ\mu_2\overline{P}^n_3(t) + \widehat{M}^n_{2,\mathcal{X},+}\circ\lambda_0\overline{P}^n_2(t)\\
&\quad - \widehat{M}^n_{2,\mathcal{X},-}\circ\mu_2\overline{P}^n_2(t) + \widehat{M}^n_{1,\mathcal{X},+}\circ\lambda_0\overline{P}^n_1(t) - \widehat{M}^n_{1,\mathcal{X},-}\circ\mu_1\overline{P}^n_1(t)\\
&\quad + \sqrt{n}\big(\int_0^t \theta_s(\mathcal{X}^n(s))^-\,d\overline{P}^n_8(s) - \mu_2\overline{P}^n_6(t) + \lambda_0\overline{P}^n_5(t) - \mu_2\overline{P}^n_4(t)\\
&\quad - \mu_2\overline{P}^n_3(t) + (\lambda_0-\mu_2)\overline{P}^n_2(t) + (\lambda_0-\mu_1)\overline{P}^n_1(t)\big). \qquad (5.63)
\end{aligned}$$

5.2.1 Stochastic boundedness of $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{X}}^n)$

Proposition 5.2.1 *We define the following processes*

$$\begin{aligned}
\mathfrak{A}^n_1 &= N_{1,\mathcal{W},+}\circ\lambda_1 P^n_1 - N_{1,\mathcal{W},-}\circ\mu_0 P^n_1 + N_{2,\mathcal{W},+}\circ\lambda_1 P^n_2 + N_{5,\mathcal{W},+}\circ\lambda_1 P^n_5,\\
\mathfrak{A}^n_3 &= N_{2,\mathcal{W},-}\circ\mu_1 P^n_2 - N_{3,\mathcal{W},+}\circ\lambda_0 P^n_3 + N_{3,\mathcal{W},-}\circ\mu_1 P^n_3 - N_{4,\mathcal{W},+}\circ\lambda_0 P^n_4,\\
\mathfrak{A}^n_6 &= N_{7,\mathcal{V},+}\circ\lambda_1 P^n_7 - N_{6,\mathcal{V},-}\circ\mu_0 P^n_6 + N_{6,\mathcal{V},+}\circ\lambda_1 P^n_6 - N_{4,\mathcal{V},-}\circ\mu_0 P^n_4,\\
\mathfrak{A}^n_8 &= N_{8,\mathcal{V},-}\circ\mu_1 P^n_8 - N_{8,\mathcal{V},+}\circ\lambda_0 P^n_8 + N_{7,\mathcal{V},-}\circ\mu_1 P^n_7 + N_{5,\mathcal{V},-}\circ\mu_1 P^n_5.
\end{aligned}$$

Obviously, from (5.9) and (5.10), we have

$$\begin{aligned} H^n &= H^n(0) + \mathfrak{A}_1^n - \mathfrak{A}_3^n + \mathfrak{A}_8^n - \mathfrak{A}_6^n, \\ |H^n| &= |H^n(0)| + \mathfrak{A}_1^n + \mathfrak{A}_3^n + \mathfrak{A}_8^n + \mathfrak{A}_6^n. \end{aligned}$$

The scaled versions of $\mathfrak{A}_1^n$, $\mathfrak{A}_3^n$, $\mathfrak{A}_6^n$, *and* $\mathfrak{A}_8^n$ *are denoted by* $\widehat{\mathfrak{A}}_1^n$, $\widehat{\mathfrak{A}}_3^n$, $\widehat{\mathfrak{A}}_6^n$, *and* $\widehat{\mathfrak{A}}_8^n$. *Then*

$$\widehat{\mathfrak{A}}_1^n \xrightarrow{J_1} 0, \quad \widehat{\mathfrak{A}}_8^n \xrightarrow{J_1} 0, \quad \widehat{\mathfrak{A}}_3^n + \widehat{\mathfrak{A}}_6^n \xrightarrow{J_1} 0.$$

PROOF: According to (5.18) and (5.19), we have

$$\begin{aligned} \widehat{H}^n &= \widehat{H}^n(0) + \widehat{\mathfrak{A}}_1^n - \widehat{\mathfrak{A}}_3^n + \widehat{\mathfrak{A}}_8^n - \widehat{\mathfrak{A}}_6^n, \\ |\widehat{H}^n| &= |\widehat{H}^n(0)| + \widehat{\mathfrak{A}}_1^n + \widehat{\mathfrak{A}}_3^n + \widehat{\mathfrak{A}}_8^n + \widehat{\mathfrak{A}}_6^n. \end{aligned}$$

Since $\widehat{H}^n(0) \to 0$, from Theorem 5.1.1, we have

$$\widehat{\mathfrak{A}}_1^n - \widehat{\mathfrak{A}}_3^n + \widehat{\mathfrak{A}}_8^n - \widehat{\mathfrak{A}}_6^n = \widehat{H}^n - \widehat{H}^n(0) \xrightarrow{J_1} 0, \tag{5.64}$$

$$\widehat{\mathfrak{A}}_1^n + \widehat{\mathfrak{A}}_3^n + \widehat{\mathfrak{A}}_8^n + \widehat{\mathfrak{A}}_6^n = |\widehat{H}^n| - |\widehat{H}^n(0)| \xrightarrow{J_1} 0. \tag{5.65}$$

Adding and subtracting (5.64) and (5.65), we obtain

$$\widehat{\mathfrak{A}}_1^n + \widehat{\mathfrak{A}}_8^n \xrightarrow{J_1} 0, \quad \widehat{\mathfrak{A}}_3^n + \widehat{\mathfrak{A}}_6^n \xrightarrow{J_1} 0. \tag{5.66}$$

We now show that we can separate the first convergence in (5.66) to obtain

$$\widehat{\mathfrak{A}}_1^n \xrightarrow{J_1} 0, \quad \widehat{\mathfrak{A}}_8^n \xrightarrow{J_1} 0. \tag{5.67}$$

We consider an interval of time $[0, \tau_0]$ where

$$\tau_0 := \inf\{t \geq 0 : \big(G^n(t), H^n(t)\big) \notin \mathcal{S}_1'\}.$$

If $\tau_0 = 0$, $\mathfrak{A}_1^n(\tau_0) = \mathfrak{A}_1^n(0) = 0$. If $\tau_0 > 0$, $\big(G^n(\tau_0), H^n(\tau_0)\big) \in \partial\mathcal{S}_1'$ so $H^n(\tau_0) = 0$. Since the jumps in H^n agree with the jumps in $\mathfrak{A}_1^n$ on $[0, \tau_0)$ and $\mathfrak{A}_1^n(0) = 0$, we have

$$\mathfrak{A}_1^n(\tau_0) = \mathfrak{A}_1^n(\tau_0) - \mathfrak{A}_1^n(0) = H^n(\tau_0) - H^n(0) = -H^n(0). \tag{5.68}$$

Then let us consider a sequence of subinterval $[\sigma_i, \tau_i]_{i=1}^{\infty}$ on $[\tau_0, \infty)$ such that $[\sigma_i, \tau_i]$ is the i^{th} positive excursion of G^n starting from τ_0. Specifically, if $\tau_0 > 0$, and $\big(G^n(\tau_0), H^n(\tau_0)\big) \in \mathcal{S}_2'$ then $\sigma_1 = \tau_0$ and

$$G^n(\sigma_1) > 0 = G^n(\tau_1) = G^n(\sigma_i-) = G^n(\tau_i),$$

for $i \geq 2$. Otherwise, $\sigma_1 > \tau_0$ and

$$G^n(\sigma_i-) = 0 = G^n(\tau_i),$$

for $i \geq 1$. For j between 1 and some finite number (which may be 0), let $[\sigma_{ij}, \tau_{ij})$ be the j^{th} interval contained in $[\sigma_i, \tau_i)$ on which G^n is in $\mathcal{S}_1'$. Specifically, σ_{ij} is the j^{th} time inside

$[\sigma_i, \tau_i)$ when (G^n, H^n) moves from $\mathcal{S}'_5 \cup \mathcal{S}'_2$ to $\mathcal{S}'_1$, and τ_{ij} is the first time after σ_{ij} when (G^n, H^n) moves from $\mathcal{S}'_1$ back to $\mathcal{S}'_5 \cup \mathcal{S}'_2$. For $i \geq 1$ and $j \geq 1$, there is a jump in either $N_{2,\mathcal{W},+} \circ \lambda_1 P_2$ or $N_{5,\mathcal{W},+} \circ \lambda_1 P_5$ at time σ_{ij} in order to move (G^n, H^n) from $\mathcal{S}'_5 \cup \mathcal{S}'_2$ into $\mathcal{S}'_1$, so that $\Delta \mathfrak{A}^n_1(\sigma_{ij}) = \Delta H^n(\sigma_{ij}) = 1$. Note that the jumps in H^n agree with the jumps in $\mathfrak{A}^n_1$ on $[\sigma_{ij}, \tau_{ij})$. Because $H^n(\sigma_{ij}-) = H^n(\tau_{ij}) = 0$, we have

$$\mathfrak{A}^n_1(\tau_{ij}) - \mathfrak{A}^n_1(\sigma_{ij}-) = 0.$$

We observe that on $[\sigma_i, \tau_i)$, $\mathfrak{A}^n_1$ only jumps on $[\sigma_{ij}, \tau_{ij})$ for some j. Hence if $\tau_0 > 0$ and $(G^n(\tau_0), H^n(\tau_0)) \in \mathcal{R}_2$ we have

$$\mathfrak{A}^n_1(\tau_1) - \mathfrak{A}^n_1(\sigma_1) = 0, \quad \mathfrak{A}^n_1(\tau_i) - \mathfrak{A}^n_1(\sigma_i-) = 0,$$

for $i \geq 2$. Because $\mathfrak{A}^n_1$ is constant on the complement of the union of the intervals $[0, \tau_0)$ and $[\sigma_i, \tau_i)$, from (5.68), we have

$$\mathfrak{A}^n_1(\tau_0) = \mathfrak{A}^n_1(\sigma_1) = \mathfrak{A}^n_1(\tau_1) = \mathfrak{A}^n_1(\sigma_i-) = \mathfrak{A}^n(\tau_i), \tag{5.69}$$

for $i \geq 2$. Otherwise, if $\tau_0 = 0$ or $(G^n(\tau_0), H^n(\tau_0)) \in \mathcal{R}_5$, we have

$$\mathfrak{A}^n_1(\tau_i) - \mathfrak{A}^n_1(\sigma_i-) = 0,$$

and

$$\mathfrak{A}^n_1(\tau_0) = \mathfrak{A}^n_1(\sigma_i-) = \mathfrak{A}^n_1(\tau_i), \tag{5.70}$$

for $i \geq 1$.

Since P_8, P_7 and $N_{5,\mathcal{V},-} \circ \mu_1 P_5$ are constant on $[\sigma_i, \tau_i]$ and have no jump at σ_i, we also have $\mathfrak{A}^n_8$ is constant on $[\sigma_i, \tau_i]$, which implies

$$\mathfrak{A}^n_8(\tau_i) - \mathfrak{A}^n_8(\sigma_i-) = 0. \tag{5.71}$$

Obviously if $\tau_0 > 0$, $\mathfrak{A}^n_8$ is zero on $[0, \tau_0]$.

If $\tau_0 > 0$, and $\big(G^n(\tau_0), H^n(\tau_0)\big) \in \mathcal{S}'_2$, consider the process

$$\tilde{\mathfrak{A}}^n(t) := \begin{cases} \mathfrak{A}^n_1(\tau_0) + \mathfrak{A}^n_8(\tau_0), & \text{if } 0 \leq t < \tau_0, \\ \mathfrak{A}^n_1(\sigma_1) + \mathfrak{A}^n_8(\sigma_1), & t \in [\sigma_1, \tau_1), \\ \mathfrak{A}^n_1(\sigma_i-) + \mathfrak{A}^n_8(\sigma_i-), & t \in [\sigma_i, \tau_i), \text{for } i \geq 2, \\ \mathfrak{A}^n_1(t) + \mathfrak{A}^n_8(t), & \text{otherwise.} \end{cases}$$

According to (5.69), and (5.71), $\tilde{\mathfrak{A}}^n$ is continuous at τ_0, σ_i and τ_i, and in fact, $\tilde{\mathfrak{A}}^n = \mathfrak{A}^n_8 - H^n(0)$.

If $\tau_0 > 0$, and $\big(G^n(\tau_0), H^n(\tau_0)\big) \in \mathcal{S}'_5$, consider the process

$$\tilde{\mathfrak{A}}^n(t) := \begin{cases} \mathfrak{A}^n_1(\tau_0) + \mathfrak{A}^n_8(\tau_0), & \text{if } 0 \leq t < \tau_0, \\ \mathfrak{A}^n_1(\sigma_i-) + \mathfrak{A}^n_8(\sigma_i-), & t \in [\sigma_i, \tau_i), \text{for } i \geq 1, \\ \mathfrak{A}^n_1(t) + \mathfrak{A}^n_8(t), & \text{otherwise.} \end{cases}$$

According to (5.70), and (5.71), $\tilde{\mathfrak{A}}^n$ is continuous at τ_0, σ_i and τ_i, and in fact, $\tilde{\mathfrak{A}}^n = \mathfrak{A}_8^n - H^n(0)$.

If $\tau_0 = 0$, then consider the process

$$\tilde{\mathfrak{A}}^n(t) := \begin{cases} \mathfrak{A}_1^n(\sigma_i-) + \mathfrak{A}_8^n(\sigma_i-), & t \in [\sigma_i, \tau_i), \text{for } i \geq 1, \\ \mathfrak{A}_1^n(t) + \mathfrak{A}_8^n(t), & \text{otherwise.} \end{cases}$$

According to (5.70), and (5.71), $\tilde{\mathfrak{A}}^n$ is continuous at σ_i and τ_i, and in fact, $\tilde{\mathfrak{A}}^n = \mathfrak{A}_8^n$.

This shows that for any $T > 0$, we have

$$\max_{0 \leq t \leq nT} \left|\mathfrak{A}_8^n(t)\right| \leq |H^n(0)| + \max_{0 \leq t \leq nT} \left|\tilde{\mathfrak{A}}(t)\right| \leq |H^n(0)| + \max_{0 \leq t \leq nT} \left|\mathfrak{A}_1^n(t) + \mathfrak{A}_8^n(t)\right|,$$

and since $H^n(0)/\sqrt{n} \to 0$, we have

$$\limsup_{n\to\infty} \max_{0 \leq t \leq T} \left|\widehat{\mathfrak{A}}_8^n(t)\right| \leq \limsup_{n\to\infty} \max_{0 \leq t \leq T} \left|\widehat{\mathfrak{A}}_1^n(t) + \widehat{\mathfrak{A}}_8^n(t)\right|.$$

Because $\widehat{\mathfrak{A}}_1^n + \widehat{\mathfrak{A}}_8^n \overset{J_1}{\Longrightarrow} 0$, we have $\widehat{\mathfrak{A}}_8^n \overset{J_1}{\Longrightarrow} 0$, and therefore $\widehat{\mathfrak{A}}_1^n \overset{J_1}{\Longrightarrow} 0$. □

We observe that

$$\begin{aligned} \widehat{\mathfrak{A}}_1^n &= \widehat{M}_{1,\mathcal{W},+}^n \circ \lambda_1 \overline{P}_1^n - \widehat{M}_{1,\mathcal{W},-}^n \circ \mu_0 \overline{P}_1^n + \widehat{M}_{2,\mathcal{W},+}^n \circ \lambda_1 \overline{P}_2^n + \widehat{M}_{5,\mathcal{W},+}^n \circ \lambda_1 \overline{P}_5^n \\ &\quad + \sqrt{n}\big[(\lambda_1 - \mu_0)\overline{P}_1^n + \lambda_1 \overline{P}_2^n + \lambda_1 \overline{P}_5^n\big], \qquad (5.72) \\ \widehat{\mathfrak{A}}_8^n &= \widehat{M}_{8,\mathcal{V},-}^n \circ \mu_1 \overline{P}_8^n - \widehat{M}_{8,\mathcal{V},+}^n \circ \lambda_0 \overline{P}_8^n + \widehat{M}_{7,\mathcal{V},-}^n \circ \mu_1 \overline{P}_7^n + \widehat{M}_{5,\mathcal{V},-}^n \circ \mu_1 \overline{P}_5^n \\ &\quad + \sqrt{n}\big[(\mu_1 - \lambda_0)\overline{P}_8^n + \mu_1 \overline{P}_7^n + \mu_1 \overline{P}_5^n\big]. \qquad (5.73) \end{aligned}$$

Since

$$\widehat{M}_{i,\times,*}^n \overset{J_1}{\Longrightarrow} B_{i,\times,*},$$

where $B_{i,\times,*}$ is a standard Brownian motion, also from Proposition 5.1.6 and Theorem 5.1.8, we have

$$\begin{aligned} &\widehat{M}_{1,\mathcal{W},+}^n \circ \lambda_1 \overline{P}_1^n - \widehat{M}_{1,\mathcal{W},-}^n \circ \mu_0 \overline{P}_1^n + \widehat{M}_{2,\mathcal{W},+}^n \circ \lambda_1 \overline{P}_2^n + \widehat{M}_{5,\mathcal{W},+}^n \circ \lambda_1 \overline{P}_5^n \overset{J_1}{\Longrightarrow} \\ &\qquad B_{1,\mathcal{W},+} \circ \frac{\lambda_1^2}{\lambda_0 + \lambda_1} \overline{P}_{G^*}^+ - B_{1,\mathcal{W},-} \circ \frac{\mu_0 \lambda_1}{\lambda_0 + \lambda_1} \overline{P}_{G^*}^+ + B_{2,\mathcal{W},+} \circ \frac{\lambda_1(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} \overline{P}_{G^*}^+, \qquad (5.74) \end{aligned}$$

$$\begin{aligned} &\widehat{M}_{8,\mathcal{V},-}^n \circ \mu_1 \overline{P}_8^n - \widehat{M}_{8,\mathcal{V},+}^n \circ \lambda_0 \overline{P}_8^n + \widehat{M}_{7,\mathcal{V},-}^n \circ \mu_1 \overline{P}_7^n + \widehat{M}_{5,\mathcal{V},-}^n \circ \mu_1 \overline{P}_5^n \overset{J_1}{\Longrightarrow} \\ &\qquad B_{8,\mathcal{V},-} \circ \frac{\mu_1^2}{\mu_0 + \mu_1} \overline{P}_{G^*}^- - B_{8,\mathcal{V},+} \circ \frac{\lambda_0 \mu_1}{\mu_0 + \mu_1} \overline{P}_{G^*}^- + B_{7,\mathcal{V},-} \circ \frac{\mu_1(\mu_0 - \lambda_1)}{\mu_0 + \mu_1} \overline{P}_{G^*}^-. \qquad (5.75) \end{aligned}$$

Substituting into (5.72) and (5.73), we conclude that

$$\begin{aligned} &\sqrt{n}\big[(\lambda_1 - \mu_0)\overline{P}_1^n + \lambda_1 \overline{P}_2^n + \lambda_1 \overline{P}_5^n\big] \overset{J_1}{\Longrightarrow} \\ &\qquad -B_{1,\mathcal{W},+} \circ \frac{\lambda_1^2}{\lambda_0 + \lambda_1} \overline{P}_{G^*}^+ + B_{1,\mathcal{W},-} \circ \frac{\mu_0 \lambda_1}{\lambda_0 + \lambda_1} \overline{P}_{G^*}^+ - B_{2,\mathcal{W},+} \circ \frac{\lambda_1(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} \overline{P}_{G^*}^+, \qquad (5.76) \end{aligned}$$

$$\sqrt{n}\big[(\mu_1-\lambda_0)\overline{P}_8^n+\mu_1\overline{P}_7^n+\mu_1\overline{P}_5^n\big] \overset{J_1}{\Longrightarrow}$$
$$-B_{8,\mathcal{V},-}\circ\frac{\mu_1^2}{\mu_0+\mu_1}\overline{P}_{G^*}^- + B_{8,\mathcal{V},+}\circ\frac{\lambda_0\mu_1}{\mu_0+\mu_1}\overline{P}_{G^*}^- - B_{7,\mathcal{V},-}\circ\frac{\mu_1(\mu_0-\lambda_1)}{\mu_0+\mu_1}\overline{P}_{G^*}^-, \quad (5.77)$$

and, in particular, the processes on the left-hand sides of (5.76) and (5.77) have continuous limits.

To simplify notation, we define

$$\widehat{\Theta}_1^n := \widehat{M}_{1,\mathcal{W},+}^n\circ\lambda_1\overline{P}_1^n - \widehat{M}_{1,\mathcal{W},-}^n\circ\mu_0\overline{P}_1^n + \widehat{M}_{2,\mathcal{W},+}^n\circ\lambda_1\overline{P}_2^n, \quad (5.78)$$
$$\widehat{\Theta}_2^n := -\widehat{M}_{2,\mathcal{W},-}^n\circ\mu_1\overline{P}_2^n + \widehat{M}_{3,\mathcal{W},+}^n\circ\lambda_0\overline{P}_3^n - \widehat{M}_{3,\mathcal{W},-}^n\circ\mu_1\overline{P}_3^n, \quad (5.79)$$
$$\widehat{\Theta}_3^n := \widehat{M}_{1,\mathcal{V},+}^n\circ\lambda_2\overline{P}_1^n + \widehat{M}_{2,\mathcal{V},+}^n\circ\lambda_2\overline{P}_2^n - \widehat{M}_{2,\mathcal{V},-}^n\circ\mu_0\overline{P}_2^n + \widehat{M}_{3,\mathcal{V},+}^n\circ\lambda_1\overline{P}_3^n - \widehat{M}_{3,\mathcal{V},-}^n\circ\mu_0\overline{P}_3^n, \quad (5.80)$$
$$\widehat{\Theta}_4^n := -\widehat{M}_{8,\mathcal{W},-}^n\circ\mu_2\overline{P}_8^n - \widehat{M}_{7,\mathcal{W},-}^n\circ\mu_2\overline{P}_7^n + \widehat{M}_{7,\mathcal{W},+}^n\circ\lambda_0\overline{P}_7^n - \widehat{M}_{6,\mathcal{W},-}^n\circ\mu_1\overline{P}_6^n + \widehat{M}_{6,\mathcal{W},+}^n\circ\lambda_0\overline{P}_6^n, \quad (5.81)$$
$$\widehat{\Theta}_5^n := \widehat{M}_{7,\mathcal{V},+}^n\circ\lambda_1\overline{P}_7^n - \widehat{M}_{6,\mathcal{V},-}^n\circ\mu_0\overline{P}_6^n + \widehat{M}_{6,\mathcal{V},+}^n\circ\lambda_1\overline{P}_6^n, \quad (5.82)$$
$$\widehat{\Theta}_6^n := -\widehat{M}_{8,\mathcal{V},-}^n\circ\mu_1\overline{P}_8^n + \widehat{M}_{8,\mathcal{V},+}^n\circ\lambda_0\overline{P}_8^n - \widehat{M}_{7,\mathcal{V},-}^n\circ\mu_1\overline{P}_7^n, \quad (5.83)$$
$$\widehat{\Theta}_7^n := \widehat{M}_{4,\mathcal{W},+}^n\circ\lambda_0\overline{P}_4^n + \widehat{M}_{4,\mathcal{V},+}^n\circ\lambda_1\overline{P}_4^n + \widehat{M}_{4,\mathcal{W},-}^n\circ\mu_1\overline{P}_4^n + \widehat{M}_{4,\mathcal{V},-}^n\circ\mu_0\overline{P}_4^n + b\widehat{M}_{5,\mathcal{W},+}^n\circ\lambda_1\overline{P}_5^n + \widehat{M}_{5,\mathcal{V},+}^n\circ\lambda_2\overline{P}_5^n + \widehat{M}_{5,\mathcal{W},-}^n\circ\mu_2\overline{P}_5^n + a\widehat{M}_{5,\mathcal{V},-}^n\circ\mu_1\overline{P}_5^n, \quad (5.84)$$
$$\widehat{\Theta}_8^n := \widehat{M}_{4,\mathcal{W},+}^n\circ\lambda_0\overline{P}_4^n + \widehat{M}_{4,\mathcal{V},-}^n\circ\mu_0\overline{P}_4^n \quad (5.85)$$
$$\widehat{\Theta}_9^n := \widehat{M}_{5,\mathcal{W},+}^n\circ\lambda_1\overline{P}_5^n + \widehat{M}_{5,\mathcal{V},-}^n\circ\mu_1\overline{P}_5^n. \quad (5.86)$$

Then from (5.17), (5.18), and (5.19), we obtain

$$\big|\widehat{G}^n\big| = |\widehat{G}^n(0)| + b\widehat{\Theta}_1^n + \widehat{\Theta}_2^n + \widehat{\Theta}_3^n - \widehat{\Theta}_4^n - \widehat{\Theta}_5^n - a\widehat{\Theta}_6^n + \widehat{\Theta}_7^n + 2a\lambda_0\sqrt{n}(\overline{P}_4^n + \overline{P}_5^n), \quad (5.87)$$
$$\widehat{H}^n = \widehat{H}^n(0) + \widehat{\Theta}_1^n + \widehat{\Theta}_2^n - \widehat{\Theta}_5^n - \widehat{\Theta}_6^n + \widehat{\Theta}_8^n + \widehat{\Theta}_9^n + \sqrt{n}(\lambda_1+\mu_1)\overline{P}_5^n + \sqrt{n}(\lambda_0+\mu_0)\overline{P}_4^n + c\sqrt{n}(\overline{P}_3^n + \overline{P}_6^n - \overline{P}_1^n - \overline{P}_8^n) + \sqrt{n}(\mu_1-\lambda_1)(\overline{P}_7^n - \overline{P}_2^n), \quad (5.88)$$
$$\big|\widehat{H}^n\big| = \big|\widehat{H}^n(0)\big| + \widehat{\Theta}_1^n - \widehat{\Theta}_2^n + \widehat{\Theta}_5^n - \widehat{\Theta}_6^n - \widehat{\Theta}_8^n + \widehat{\Theta}_9^n + \sqrt{n}(\lambda_1+\mu_1)(\overline{P}_2^n + \overline{P}_7^n + \overline{P}_5^n) - c\sqrt{n}(\overline{P}_1^n + \overline{P}_3^n + \overline{P}_6^n + \overline{P}_8^n) - \sqrt{n}(\lambda_0+\mu_0)\overline{P}_4^n. \quad (5.89)$$

Note in (5.87) that $2a\lambda_0\sqrt{n}(\overline{P}_4^n+\overline{P}_5^n)$ is a nondecreasing process that increases only when the nonnegative process $|\widehat{G}^n|$ is at zero. Because of uniqueness of the solution to the Skorohod problem, we must have

$$2a\lambda_0\sqrt{n}(\overline{P}_4^n+\overline{P}_5^n) = \Gamma\big(|\widehat{G}^n(0)| + b\widehat{\Theta}_1^n + \widehat{\Theta}_2^n + \widehat{\Theta}_3^n - \widehat{\Theta}_4^n - \widehat{\Theta}_5^n - a\widehat{\Theta}_6^n + \widehat{\Theta}_7^n\big),$$

where $\Gamma : D[0,\infty) \to D[0,\infty)$ is given by

$$\Gamma(x)(t) = -\inf_{0\le s\le t}\Big(x(s)\vee 0\Big).$$

Therefore, we can rewrite (5.87) as

$$\begin{aligned}\left|\widehat{G}^n\right| &= |\widehat{G}^n(0)| + b\widehat{\Theta}_1^n + \widehat{\Theta}_2^n + \widehat{\Theta}_3^n - \widehat{\Theta}_4^n - \widehat{\Theta}_5^n - a\widehat{\Theta}_6^n + \widehat{\Theta}_7^n \\ &\quad + \Gamma\big(|\widehat{G}^n(0)| + b\widehat{\Theta}_1^n + \widehat{\Theta}_2^n + \widehat{\Theta}_3^n - \widehat{\Theta}_4^n - \widehat{\Theta}_5^n - a\widehat{\Theta}_6^n + \widehat{\Theta}_7^n\big). \qquad (5.90)\end{aligned}$$

On Page 87, we will define $\{\Theta_i^*\}_{i=1}^6$. From Proposition 5.1.6, we observe that

$$\begin{aligned}\widehat{\Theta}_1^n &\overset{J_1}{\Longrightarrow} \Theta_1^* \circ \overline{P}_G^+, \\ \widehat{\Theta}_2^n &\overset{J_1}{\Longrightarrow} \Theta_2^* \circ \overline{P}_G^+, \\ \widehat{\Theta}_3^n &\overset{J_1}{\Longrightarrow} \Theta_3^* \circ \overline{P}_G^+, \\ \widehat{\Theta}_4^n &\overset{J_1}{\Longrightarrow} \Theta_4^* \circ \overline{P}_G^-, \\ \widehat{\Theta}_5^n &\overset{J_1}{\Longrightarrow} \Theta_5^* \circ \overline{P}_G^-, \\ \widehat{\Theta}_6^n &\overset{J_1}{\Longrightarrow} \Theta_6^* \circ \overline{P}_G^-.\end{aligned}$$

Since $\widehat{G}^n(0) \overset{J_1}{\Longrightarrow} 0$, we can take the limit of (5.90) to get

$$\left|G^*\right| = b\tilde{\Theta}_1^* + \tilde{\Theta}_2^* + \tilde{\Theta}_3^* - \tilde{\Theta}_4^* - \tilde{\Theta}_5^* - a\tilde{\Theta}_6^* + \tilde{\Theta}_7^* + \Gamma\big(b\tilde{\Theta}_1^* + \tilde{\Theta}_2^* + \tilde{\Theta}_3^* - \tilde{\Theta}_4^* - \tilde{\Theta}_5^* - a\tilde{\Theta}_6^* + \tilde{\Theta}_7^*\big), \quad (5.91)$$

where $\tilde{\Theta}_i^*$ is the limit of $\widehat{\Theta}_i^n$ for $i = 1, \ldots, 9$. Also, from Theorem 5.1.1, (5.88), and (5.89), we have

$$\begin{aligned}&\sqrt{n}(\lambda_1 + \mu_1)\overline{P}_5^n + \sqrt{n}(\lambda_0 + \mu_0)\overline{P}_4^n + c\sqrt{n}(\overline{P}_3^n + \overline{P}_6^n - \overline{P}_1^n - \overline{P}_8^n) \\ &\quad + \sqrt{n}(\mu_1 - \lambda_1)(\overline{P}_7^n - \overline{P}_2^n) \overset{J_1}{\Longrightarrow} -(\tilde{\Theta}_1^* + \tilde{\Theta}_2^* - \tilde{\Theta}_5^* - \tilde{\Theta}_6^* + \tilde{\Theta}_8^* + \tilde{\Theta}_9^*), \qquad (5.92)\end{aligned}$$

$$\begin{aligned}&\sqrt{n}(\lambda_1 + \mu_1)(\overline{P}_2^n + \overline{P}_7^n + \overline{P}_5^n) - c\sqrt{n}(\overline{P}_1^n + \overline{P}_3^n + \overline{P}_6^n + \overline{P}_8^n) \\ &\quad - \sqrt{n}(\lambda_0 + \mu_0)\overline{P}_4^n \overset{J_1}{\Longrightarrow} -(\tilde{\Theta}_1^* - \tilde{\Theta}_2^* + \tilde{\Theta}_5^* - \tilde{\Theta}_6^* - \tilde{\Theta}_8^* + \tilde{\Theta}_9^*). \qquad (5.93)\end{aligned}$$

If we multiply (5.92) by $-\frac{1}{2}(b+1)$, multiply (5.93) by $\frac{1}{2}(b-1)$ and take the sum, we obtain

$$\begin{aligned}&\sqrt{n}\big[((\lambda_0 - \mu_1)\overline{P}_1^n + (b\mu_1 - \lambda_1)\overline{P}_2^n - \lambda_2\overline{P}_3^n - b(\lambda_0 + \mu_0)\overline{P}_4^n - (\mu_1 + \lambda_1)\overline{P}_5^n \\ &- \lambda_2\overline{P}_6^n + (\mu_0 - \lambda_2)\overline{P}_7^n + (\lambda_0 - \mu_1)\overline{P}_8^n\big] \Rightarrow \tilde{\Theta}_1^* + b\tilde{\Theta}_2^* - b\tilde{\Theta}_5^* - \tilde{\Theta}_6^* + b\tilde{\Theta}_8^* + \tilde{\Theta}_9^*, \quad (5.94)\end{aligned}$$

which will be used in the following proofs.

Theorem 5.2.2 *The sequence of càdlàg processes $\{\widehat{\mathcal{U}}^n\}_{n=1}^\infty$ and $\{\widehat{\mathcal{X}}^n\}_{n=1}^\infty$ are bounded in probability on compact time intervals.*

It suffices to prove $\{\widehat{\mathcal{U}}^n\}_{n=1}^\infty$ is bounded in probability on compact time intervals. The proof of this theorem is presented in Lemmas 5.2.3 and 5.2.4 below.

Lemma 5.2.3 *The sequence of processes $\{\widehat{\mathcal{U}}^n\}_{n=1}^{\infty}$ is bounded above in probability on compact time intervals.*

PROOF: To simplify notation, we rewrite (5.62) as

$$\widehat{\mathcal{U}}^n = \widehat{\mathcal{U}}^n(0) + Y_1^n + Y_2^n + Y_3^n + Y_4^n, \tag{5.95}$$

where

$$\begin{aligned}
Y_1^n(t) &= -\widehat{M}^n_{1,\mathcal{U},-}\left(\int_0^t \theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+ \, d\overline{P}^n_1(s)\right), && (5.96)\\
Y_2^n(t) &= \widehat{M}^n_{3,\mathcal{U},+}\big(\lambda_2\overline{P}^n_3(t)\big) + \widehat{M}^n_{4,\mathcal{U},+}\big(\lambda_2\overline{P}^n_4(t)\big) - \widehat{M}^n_{5,\mathcal{U},-}\big(\mu_0\overline{P}^n_5(t)\big)\\
&\quad + \widehat{M}^n_{6,\mathcal{U},+}\big(\lambda_2\overline{P}^n_6(t)\big) + \widehat{M}^n_{7,\mathcal{U},+}\big(\lambda_2\overline{P}^n_7(t)\big) - \widehat{M}^n_{7,\mathcal{U},-}\big(\mu_0\overline{P}^n_7(t)\big)\\
&\quad + \widehat{M}^n_{8,\mathcal{U},+}\big(\lambda_1\overline{P}^n_8(t)\big) - \widehat{M}^n_{8,\mathcal{U},-}\big(\mu_0\overline{P}^n_8(t)\big), && (5.97)\\
Y_3^n(t) &= -\sqrt{n}\int_0^t \theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+ \, d\overline{P}^n_1(s), && (5.98)\\
Y_4^n(t) &= \sqrt{n}\big[\lambda_2\overline{P}^n_3(t) + \lambda_2\overline{P}^n_4(t) - \mu_0\overline{P}^n_5(t) + \lambda_2\overline{P}^n_6(t)\\
&\quad - (\mu_0-\lambda_2)\overline{P}^n_7(t) - (\mu_0-\lambda_1)\overline{P}^n_8(t)\big]. && (5.99)
\end{aligned}$$

Then

$$Y_2^n \overset{J_1}{\Longrightarrow} Y_2^*, \tag{5.100}$$

where Y_2^* is a continuous process. We rewrite Y_4^n as

$$\begin{aligned}
Y_4^n &= \sqrt{n}(\lambda_0-\mu_1)\overline{P}^n_1 + \sqrt{n}(b\mu_1-\lambda_1)\overline{P}^n_2 - \sqrt{n}(b(\lambda_0+\mu_0)-\lambda_2)\overline{P}^n_4 - \sqrt{n}(\mu_1+\lambda_1+\mu_0)\,\overline{P}^n_5\\
&\quad + \sqrt{n}\big[-(\lambda_0-\mu_1)\overline{P}^n_1 - (b\mu_1-\lambda_1)\overline{P}^n_2 + \lambda_2\overline{P}^n_3 + b(\lambda_0+\mu_0)\overline{P}^n_4 + (\mu_1+\lambda_1)\overline{P}^n_5\\
&\quad + \lambda_2\overline{P}^n_6 - (\mu_0-\lambda_2)\overline{P}^n_7 - (\lambda_0-\mu_1)\overline{P}^n_8\big]\\
&= \sqrt{n}(\lambda_0-\mu_1)\overline{P}^n_1 + \sqrt{n}(b\mu_1-\lambda_1)\overline{P}^n_2 + Y_5^n, && (5.101)
\end{aligned}$$

where

$$\begin{aligned}
Y_5^n &= -\sqrt{n}(b(\lambda_0+\mu_0)-\lambda_2)\overline{P}^n_4 - \sqrt{n}(\mu_1+\lambda_1+\mu_0)\,\overline{P}^n_5\\
&\quad + \sqrt{n}\big[-(\lambda_0-\mu_1)\overline{P}^n_1 - (b\mu_1-\lambda_1)\overline{P}^n_2 + \lambda_2\overline{P}^n_3 + b(\lambda_0+\mu_0)\overline{P}^n_4 + (\mu_1+\lambda_1)\overline{P}^n_5\\
&\quad + \lambda_2\overline{P}^n_6 - (\mu_0-\lambda_2)\overline{P}^n_7 - (\lambda_0-\mu_1)\overline{P}^n_8\big]. && (5.102)
\end{aligned}$$

From (5.87), (5.90) and (5.91), we see that $2a\lambda_0\sqrt{n}(\overline{P}^n_4 + \overline{P}^n_5)$ has a continuous limit, and since both $\sqrt{n}\,\overline{P}^n_4$ and $\sqrt{n}\,\overline{P}^n_5$ are nondecreasing, they are bounded above in probability. Furthermore, the modulus of continuity of each of these processes is dominated by the modulus of continuity of their sum. Therefore, both sequences $\{\sqrt{n}\,\overline{P}^n_4\}_{n=1}^{\infty}$ and $\{\sqrt{n}\,\overline{P}^n_5\}_{n=1}^{\infty}$ are tight in $C[0,\infty)$, and we can choose a subsequence along which both have a continuous limit. According to (5.94), the last term in (5.102) converges to $-\tilde{\Theta}^*_1 - b\tilde{\Theta}^*_2 + b\tilde{\Theta}^*_5 + \tilde{\Theta}^*_6 - b\tilde{\Theta}^*_8 - \tilde{\Theta}^*_9$, a continuous process. Therefore taking limit along the subsequence we have

$$Y_5^n \overset{J_1}{\Longrightarrow} Y_5^*,$$

where Y_5^* is a continuous process.

From (5.76), (2.1), and the convergence of $\sqrt{n}\overline{P}_5^n$ to a continuous limit, we conclude that

$$\sqrt{n}[-(\lambda_0-\mu_1)\overline{P}_1^n+\lambda_1\overline{P}_2^n)]=\mathcal{O}_{cl}(1).$$

Thus,

$$\begin{aligned}\sqrt{n}[(\lambda_0-\mu_1)\overline{P}_1^n+(b\mu_1-\lambda_1)\overline{P}_2^n] &= (\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n\\ &\quad +\frac{b\mu_1-\lambda_1}{\lambda_1}\sqrt{n}[-(\lambda_0-\mu_1)\overline{P}_1^n+\lambda_1\overline{P}_2^n]\\ &= (\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n+\mathcal{O}_{cl}(1). \qquad (5.103)\end{aligned}$$

Substituting this into (5.101), we obtain

$$Y_4^n=(\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n+Y_5^n+\mathcal{O}_{cl}(1)=(\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n+\mathcal{O}_{cl}(1). \qquad (5.104)$$

If N is a unit-intensity Poisson process, then $-N(t)+\frac{1}{2}t$ is a supermartingale whose supremum S^* over $t\geq 0$ is finite almost surely. Therefore,

$$-\frac{1}{\sqrt{n}}\big(N(nt)-nt\big)-\sqrt{n}\,t=\frac{1}{\sqrt{n}}\left[-N(nt)+\frac{1}{2}nt\right]-\frac{1}{2}\sqrt{n}\,t\leq\frac{1}{\sqrt{n}}S^*-\frac{1}{2}\sqrt{n}\,t,$$

and hence

$$Y_1^n(t)+Y_3^n(t)\leq\frac{1}{2}Y_3^n(t)+\frac{1}{\sqrt{n}}\mathcal{O}_{cl}(1). \qquad (5.105)$$

Combining (5.100), (5.104), and (5.105) we obtain

$$\begin{aligned}\widehat{\mathcal{U}}^n(t) &\leq \frac{1}{2}Y_3^n(t)+(\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n+\mathcal{O}_{cl}(1)\\ &= \sqrt{n}\int_0^t\left((\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}-\frac{1}{2}\theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+\right)d\overline{P}_1^n(s)+\mathcal{O}_{cl}(1). \qquad (5.106)\end{aligned}$$

Let us fix $T>0$ and consider $t\in[0,T]$. Either

$$\int_0^t\left((\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}-\frac{1}{2}\theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+\right)d\overline{P}_1^n(s)\leq 0, \qquad (5.107)$$

or else

$$\int_0^t\theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+\,d\overline{P}_1^n(s)\leq 2(\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\overline{P}_1^n(t)\leq 2(\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}T. \qquad (5.108)$$

We define

$$\tau^n(t):=\begin{cases} t & \text{if (5.107) holds,}\\ \sup\left\{s\in[0,t]:\theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+\leq 2(\lambda_0-\mu_1)\frac{b\mu_1}{\lambda_1}\right\} & \text{if (5.108) holds.}\end{cases} \qquad (5.109)$$

If (5.107) holds and $\tau^n(t) = t$, $\widehat{\mathcal{U}}^n(t)$ is bounded by the $\mathcal{O}_{cl}(1)$ term in (5.106). If (5.108) holds, then

$$\widehat{\mathcal{U}}^n(t) \leq \widehat{\mathcal{U}}^n\big(\tau^n(t)\big) + \sum_{i=1}^{4} \Big[Y_i^n(t) - Y_i^n\big(\tau^n(t)\big)\Big]. \tag{5.110}$$

We consider each of the five terms on the right-hand side of (5.110). Since the jumps in $\widehat{\mathcal{U}}^n$ are of size $\frac{1}{\sqrt{n}}$, we must have

$$\widehat{\mathcal{U}}^n\big(\tau^n(t)\big) \leq 2(\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\frac{1}{\theta_b} + \frac{1}{\sqrt{n}}. \tag{5.111}$$

Because of the bound (5.108) on the argument of $\widehat{M}^n_{1,\mathcal{U},-}$, both $Y_1^n(t)$ and $Y_1^n\big(\tau^n(t)\big)$ are $\mathcal{O}_{cl}(1)$. Also, both $Y_2^n(t)$ and $Y_2^n\big(\tau^n(t)\big)$ are $\mathcal{O}_{cl}(1)$. It follows that

$$\begin{aligned}
\widehat{\mathcal{U}}^n(t) &\leq Y_3^n(t) - Y_3^n\big(\tau^n(t)\big) + \sqrt{n}(\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\Big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\tau^n(t)\big)\Big) + \mathcal{O}_{cl}(1) \\
&= \sqrt{n}\int_{\tau^n(t)}^{t} \left((\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1} - \theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+\right) d\overline{P}_1^n(s) + \mathcal{O}_{cl}(1) \\
&\leq -\sqrt{n}(\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\Big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\tau^n(t)\big)\Big) + \mathcal{O}_{cl}(1),
\end{aligned} \tag{5.112}$$

because $\theta(\widehat{\mathcal{U}}^n(s))^+ \geq 2(\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}$ for $s \in [\tau^n(t), t]$. Recall from (2.1) that $\lambda_0 - \mu_1 > 0$. Again we have an upper bound on $\widehat{\mathcal{U}}^n$. In conclusion, $\{\widehat{\mathcal{U}}^n\}_{n=1}^{\infty}$ is bounded above in probability on compact time intervals. □

Lemma 5.2.4 *The sequence of processes $\{\widehat{\mathcal{U}}^n\}_{n=1}^{\infty}$ is bounded below in probability on compact time intervals.*

Proof: We return to (5.62) and note that because $\widehat{\mathcal{U}}^n$ is bounded above in probability on compact time intervals and $d\overline{P}_1^n \leq dt$, the sequence of processes $\{\int_0^{\cdot} \theta_b(\widehat{\mathcal{U}}^n)^+ \, d\overline{P}_1^n\}_{n=1}^{\infty}$ is bounded in probability on compact time intervals. Consequently, the sequence of processes

$$\left\{\widehat{M}^n_{1,\mathcal{U},-} \circ \int_0^{\cdot} \theta_b(\widehat{\mathcal{U}}^n)^+ \, d\overline{P}_1^n\right\}_{n=1}^{\infty}$$

is bounded in probability on compact time intervals. In addition, the other processes on the right-hand side of (5.62) involving scaled, centered Poisson processes are bounded in probability on compact time intervals. This permits us to write

$$\begin{aligned}
\widehat{\mathcal{U}}^n(t) &= \sqrt{n}\Bigg[-\int_0^t \theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+ d\overline{P}_1^n(s) + \lambda_2\overline{P}_3^n(t) + \lambda_2\overline{P}_4^n(t) - \mu_0\overline{P}_5^n(t) + \lambda_2\overline{P}_6^n(t) \\
&\qquad - (\mu_0 - \lambda_2)\overline{P}_7^n(t) - (\mu_0 - \lambda_1)\overline{P}_8^n(t)\Bigg] + \mathcal{O}_{cl}(1).
\end{aligned} \tag{5.113}$$

Note that $\sqrt{n}\overline{P}_4^n = \mathcal{O}_{cl}(1)$, $\sqrt{n}\overline{P}_5^n = \mathcal{O}_{cl}(1)$, and the left-hand-side of (5.94) is also $\mathcal{O}_{cl}(1)$. We add the left-hand-side of (5.94) to (5.99) to get

$$Y_4^n = \sqrt{n}(\lambda_0 - \mu_1)\overline{P}_1^n + \sqrt{n}(b\mu_1 - \lambda_1)\overline{P}_2^n + \mathcal{O}_{cl}(1).$$

From (5.104), we have

$$\sqrt{n}(\lambda_0 - \mu_1)\overline{P}_1^n + \sqrt{n}(b\mu_1 - \lambda_1)\overline{P}_2^n = (\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\overline{P}_1^n + \mathcal{O}_{cl}(1). \tag{5.114}$$

We define

$$\rho^n(t) := \sup\left\{s \in [0,t] : \widehat{\mathcal{U}}^n(s) \geq 0\right\}. \tag{5.115}$$

Then $\widehat{\mathcal{U}}^n(s) < 0$ for $\rho^n(t) < s \leq t$, and (5.113) and (5.114) imply

$$\begin{aligned}
\widehat{\mathcal{U}}^n(t) &= \widehat{\mathcal{U}}^n\big(\rho^n(t)\big) + \sqrt{n}\Big[\lambda_2\big(\overline{P}_3^n(t) - \overline{P}_3^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_4^n(t) - \overline{P}_4^n\big(\rho^n(t)\big)\big) \\
&\quad -\mu_0\big(\overline{P}_5^n(t) - \overline{P}_5^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_6^n(t) - \overline{P}_6^n\big(\rho^n(t)\big)\big) \\
&\quad -(\mu_0 - \lambda_2)\big(\overline{P}_7^n(t) - \overline{P}_7^n\big(\rho^n(t)\big)\big) - (\mu_0 - \lambda_1)\big(\overline{P}_8^n(t) - \overline{P}_8^n\big(\rho^n(t)\big)\big)\Big] + \mathcal{O}_{cl}(1) \\
&= \widehat{\mathcal{U}}^n\big(\rho^n(t)\big) + \sqrt{n}[(\lambda_0 - \mu_1)\big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\rho^n(t)\big)\big) + (b\mu_1 - \lambda_1)\big(\overline{P}_2^n(t) - \overline{P}_2^n\big(\rho^n(t)\big)\big)] \\
&\quad +\sqrt{n}\Big[- (\lambda_0 - \mu_1)\big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\rho^n(t)\big)\big) - (b\mu_1 - \lambda_1)\big(\overline{P}_2^n(t) - \overline{P}_2^n\big(\rho^n(t)\big)\big) \\
&\quad +\lambda_2\big(\overline{P}_3^n(t) - \overline{P}_3^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_4^n(t) - \overline{P}_4^n\big(\rho^n(t)\big)\big) \\
&\quad -\mu_0\big(\overline{P}_5^n(t) - \overline{P}_5^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_6^n(t) - \overline{P}_6^n\big(\rho^n(t)\big)\big) \\
&\quad -(\mu_0 - \lambda_2)\big(\overline{P}_7^n(t) - \overline{P}_7^n\big(\rho^n(t)\big)\big) - (\mu_0 - \lambda_1)\big(\overline{P}_8^n(t) - \overline{P}_8^n\big(\rho^n(t)\big)\big)\Big] + \mathcal{O}_{cl}(1) \\
&= \widehat{\mathcal{U}}^n\big(\rho^n(t)\big) + (\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\sqrt{n}\big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\rho^n(t)\big)\big) \\
&\quad +\sqrt{n}\Big[- (\lambda_0 - \mu_1)\big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\rho^n(t)\big)\big) - (b\mu_1 - \lambda_1)\big(\overline{P}_2^n(t) - \overline{P}_2^n\big(\rho^n(t)\big)\big) \\
&\quad +\lambda_2\big(\overline{P}_3^n(t) - \overline{P}_3^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_4^n(t) - \overline{P}_4^n\big(\rho^n(t)\big)\big) \\
&\quad -\mu_0\big(\overline{P}_5^n(t) - \overline{P}_5^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_6^n(t) - \overline{P}_6^n\big(\rho^n(t)\big)\big) \\
&\quad -(\mu_0 - \lambda_2)\big(\overline{P}_7^n(t) - \overline{P}_7^n\big(\rho^n(t)\big)\big) - (\mu_0 - \lambda_1)\big(\overline{P}_8^n(t) - \overline{P}_8^n\big(\rho^n(t)\big)\big)\Big] + \mathcal{O}_{cl}(1) \\
&\geq \widehat{\mathcal{U}}^n\big(\rho^n(t)\big) + \sqrt{n}\Big[- (\lambda_0 - \mu_1)\big(\overline{P}_1^n(t) - \overline{P}_1^n\big(\rho^n(t)\big)\big) - (b\mu_1 - \lambda_1)\big(\overline{P}_2^n(t) - \overline{P}_2^n\big(\rho^n(t)\big)\big) \\
&\quad \lambda_2\big(\overline{P}_3^n(t) - \overline{P}_3^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_4^n(t) - \overline{P}_4^n\big(\rho^n(t)\big)\big) \\
&\quad -\mu_0\big(\overline{P}_5^n(t) - \overline{P}_5^n\big(\rho^n(t)\big)\big) + \lambda_2\big(\overline{P}_6^n(t) - \overline{P}_6^n\big(\rho^n(t)\big)\big) \\
&\quad -(\mu_0 - \lambda_2)\big(\overline{P}_7^n(t) - \overline{P}_7^n\big(\rho^n(t)\big)\big) - (\mu_0 - \lambda_1)\big(\overline{P}_8^n(t) - \overline{P}_8^n\big(\rho^n(t)\big)\big)\Big] \\
&\quad +\mathcal{O}_{cl}(1).
\end{aligned} \tag{5.116}$$

From (5.94) and the convergence of $\sqrt{n}\overline{P}_4^n$ and $\sqrt{n}\overline{P}_5^n$, we conclude that the second term on the right-hand side of (5.116) is $\mathcal{O}_{cl}(1)$, and hence

$$\widehat{\mathcal{U}}^n(t) \geq \widehat{\mathcal{U}}^n\big(\rho^n(t)\big) + \mathcal{O}_{cl}(1).$$

Because $\widehat{\mathcal{U}}^n(\rho^n(t)) \geq -\frac{1}{\sqrt{n}}$, we conclude that $\widehat{\mathcal{U}}^n \geq \mathcal{O}_{cl}(1)$. $\square$

Remark 5.2.5 *We recall (5.62):*

$$\begin{aligned}
\widehat{\mathcal{U}}^n(t) &= \widehat{\mathcal{U}}^n(0) - \widehat{M}^n_{1,\mathcal{U},-}\Big(\int_0^t \theta_b\big(\widehat{\mathcal{U}}^n(s)\big)^+ \, d\overline{P}^n_1(s)\Big) + \widehat{M}^n_{3,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_3(t) + \widehat{M}^n_{4,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_4(t) \\
&\quad - \widehat{M}^n_{5,\mathcal{U},-} \circ \mu_0 \overline{P}^n_5(t) + \widehat{M}^n_{6,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_6(t) + \widehat{M}^n_{7,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_7(t) \\
&\quad - \widehat{M}^n_{7,,\mathcal{U},-} \circ \mu_0 \overline{P}^n_7(t) + \widehat{M}^n_{8,\mathcal{U},+} \circ \lambda_1 \overline{P}^n_8(t) - \widehat{M}^n_{8,\mathcal{U},-} \circ \mu_0 \overline{P}^n_8(t) \\
&\quad + \sqrt{n}\Big(- \int_0^t \big(\theta_b(\widehat{\mathcal{U}}^n(s)\big)^+ - (\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\big) \, d\overline{P}^n_1(s)\Big) \\
&\quad + \sqrt{n}\big[- (\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\overline{P}^n_1(t) + \lambda_2 \overline{P}^n_3(t) + \lambda_2 \overline{P}^n_4(t) - \mu_0 \overline{P}^n_5(t) + \lambda_2 \overline{P}^n_6(t) \\
&\quad - (\mu_0 - \lambda_2)\overline{P}^n_7(t) - (\mu_0 - \lambda_1)\overline{P}^n_8(t)\big]. \qquad (5.117)
\end{aligned}$$

From (5.104) and using $Y^n_5 \overset{J_1}{\Longrightarrow} Y^*_5$, we have

$$\begin{aligned}
&\sqrt{n}\big[- (\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\overline{P}^n_1(t) + \lambda_2 \overline{P}^n_3(t) + \lambda_2 \overline{P}^n_4(t) - \mu_0 \overline{P}^n_5(t) + \lambda_2 \overline{P}^n_6(t) \\
&\quad - (\mu_0 - \lambda_2)\overline{P}^n_7(t) - (\mu_0 - \lambda_1)\overline{P}^n_8(t)\big] \\
&= -\sqrt{n}(\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\overline{P}^n_1(t) + Y^n_4(t) = \mathcal{O}_{cl}(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widehat{\mathcal{U}}^n(t) &= \widehat{\mathcal{U}}^n(0) - \sqrt{n}\int_0^t \big(\theta_b(\widehat{\mathcal{U}}^n(s))^+ - (\lambda_0 - \mu_1)\frac{b\mu_1}{\lambda_1}\big) \, d\overline{P}^n_1(s) + C^n_{\mathcal{U}}(t) \\
&= \widehat{\mathcal{U}}^n(0) - \sqrt{n}\int_0^t \big(\theta_b(\widehat{\mathcal{U}}^n(s))^+ - \frac{\lambda_2\mu_1}{\lambda_1}\big) \, d\overline{P}^n_1(s) + C^n_{\mathcal{U}}(t), \qquad (5.118)
\end{aligned}$$

where $C^n_{\mathcal{U}} = \mathcal{O}_{cl}(1)$. In fact, $C^n_{\mathcal{U}}$ has a continuous limit along subsequences of sequences. Similarly, we can obtain

$$\widehat{\mathcal{X}}^n(t) = \widehat{\mathcal{X}}^n(0) + \sqrt{n}\int_0^t \big(\theta_s(\widehat{\mathcal{X}}^n(s))^- - \frac{\mu_2\lambda_1}{\mu_1}\big) \, d\overline{P}^n_8(s) + C^n_{\mathcal{X}}(t), \qquad (5.119)$$

where $C^n_{\mathcal{X}} = \mathcal{O}_{cl}(1)$. In fact, $C^n_{\mathcal{X}}$ has a continuous limit along subsequences of sequences.

5.2.2 $\mathcal{U}^*$ on negative excursions of G^* and $\mathcal{X}^*$ on positive excursions of G^*

In this section we identify the limit of $\widehat{\mathcal{U}}^n$ on negative excursions of $\widehat{G}^n$, and by an analogous argument, the limit of $\widehat{\mathcal{X}}^n$ on positive excursions of $\widehat{G}^n$. When $\widehat{G}^n$ is on a negative excursion, the terms $d\overline{P}^n_1$, $d\overline{P}^n_3$, $d\overline{P}^n_4$ and $d\overline{P}^n_5$ in formula (5.62) for $\widehat{\mathcal{U}}^n$ are zero. Thus, on such an excursion,

$$\begin{aligned} d\widehat{\mathcal{U}}^n &= d\big(\widehat{M}^n_{6,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_6\big) + d\big(\widehat{M}^n_{7,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_7\big) - d\big(\widehat{M}^n_{7,\mathcal{U},-} \circ \mu_0 \overline{P}^n_7\big) + d\big(\widehat{M}^n_{8,\mathcal{U},+} \circ \lambda_1 \overline{P}^n_8\big) \\ &\quad - d\big(\widehat{M}^n_{8,\mathcal{U},-} \circ \mu_0 d\overline{P}^n_8\big) + \sqrt{n}\big(\lambda_2\, d\overline{P}^n_6 - (\mu_0 - \lambda_2)\, d\overline{P}^n_7 - (\mu_0 - \lambda_1)\, d\overline{P}^n_8\big). \quad (5.120) \end{aligned}$$

The scaled centered Poisson processes $\widehat{M}^n_{i,\mathcal{U},*}$ are independent of $\widehat{G}^n$ and hence independent of the beginning and ending times of the excursion. These processes converge to independent Brownian motions. We have identified the limits of the scaled occupation times $\overline{P}^n_i$ in Proposition 5.1.6 and Theorem 5.1.8. Thus, we can determine the limits of the first five terms,

$$\begin{aligned} & d\big(\widehat{M}^n_{6,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_6\big) + d\big(\widehat{M}^n_{7,\mathcal{U},+} \circ \lambda_2 d\overline{P}^n_7\big) - d\big(\widehat{M}^n_{7,\mathcal{U},-} \circ \mu_0 d\overline{P}^n_7\big) \\ &\quad + d\big(\widehat{M}^n_{8,\mathcal{U},+} \circ \lambda_1\, d\overline{P}^n_8\big) - d\big(\widehat{M}^n_{8,\mathcal{U},-} \circ \mu_0\, d\overline{P}^n_8\big) \end{aligned} \quad (5.121)$$

on the right-hand side of (5.120) (see Proposition 5.2.12 below). The remaining term,

$$\sqrt{n}\big(\lambda_2\, d\overline{P}^n_6 - (\mu_0 - \lambda_2)\, d\overline{P}^n_7 - (\mu_0 - \lambda_1)\, d\overline{P}^n_8\big) \quad (5.122)$$

is more difficult. It is not independent of $\widehat{G}^n$, and hence depends on the fact that we are observing it during a negative excursion of $\widehat{G}^n$. We will see (Proposition 5.2.12 below) that it convergences to a constant times the excursion itself plus a one variance Brownian motion that is independent of the excursion and also independent of the limit of the first five terms on the right-hand side of (5.120).

To set the stage for this analysis, we introduce six independent one variance Brownian motions. To be consistent, we use the same notation that appeared in (5.91). Specifically, since $\widehat{M}^n_{i,\times,*} \overset{J_1}{\Longrightarrow} B_{i,\times,*}$, we define Θ^*_i, where $i = 1, \ldots, 6$, by

$$\begin{aligned} \Theta^*_1 &:= B_{1,\mathcal{W},+} \circ \frac{\lambda_1^2}{\lambda_0 + \lambda_1} e - B_{1,\mathcal{W},-} \circ \frac{\mu_0 \lambda_1}{\lambda_0 + \lambda_1} e + B_{2,\mathcal{W},+} \circ \frac{\lambda_1(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e, \\ \Theta^*_2 &:= -B_{2,\mathcal{W},-} \circ \frac{\mu_1(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e + B_{3,\mathcal{W},+} \circ \frac{\lambda_0 \mu_1}{\lambda_0 + \lambda_1} e - B_{3,\mathcal{W},-} \circ \frac{\mu_1^2}{\lambda_0 + \lambda_1} e, \\ \Theta^*_3 &:= B_{1,\mathcal{V},+} \circ \frac{\lambda_1 \lambda_2}{\lambda_0 + \lambda_1} e + B_{2,\mathcal{V},+} \circ \frac{\lambda_2(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e - B_{2,\mathcal{V},-} \circ \frac{\mu_0(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e \\ &\qquad + B_{3,\mathcal{V},+} \circ \frac{\lambda_1 \mu_1}{\lambda_0 + \lambda_1} e - B_{3,\mathcal{V},-} \circ \frac{\mu_0 \mu_1}{\lambda_0 + \lambda_1} e, \\ \Theta^*_4 &:= -B_{8,\mathcal{W},-} \circ \frac{\mu_1 \mu_2}{\lambda_0 + \lambda_1} e - B_{7,\mathcal{W},-} \circ \frac{\mu_2(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e + B_{7,\mathcal{W},+} \circ \frac{\lambda_0(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e \\ &\qquad - B_{6,\mathcal{W},-} \circ \frac{\mu_1 \lambda_1}{\lambda_0 + \lambda_1} e + B_{6,\mathcal{W},+} \circ \frac{\lambda_0 \lambda_1}{\lambda_0 + \lambda_1} e, \\ \Theta^*_5 &:= B_{7,\mathcal{V},+} \circ \frac{\lambda_1(\lambda_0 - \mu_1)}{\lambda_0 + \lambda_1} e - B_{6,\mathcal{V},-} \circ \frac{\mu_0 \lambda_1}{\lambda_0 + \lambda_1} e + B_{6,\mathcal{V},+} \circ \frac{\lambda_1^2}{\lambda_0 + \lambda_1} e, \\ \Theta^*_6 &:= -B_{8,\mathcal{V},-} \circ \frac{\mu_1^2}{\lambda_0 + \lambda_1} e + B_{8,\mathcal{V},+} \circ \frac{\lambda_0 \mu_1}{\lambda_0 + \lambda_1} e - B_{7,\mathcal{V},-} \circ \frac{\mu_1(\lambda_0 - \mu_1)}{\lambda_0 + \lambda 1} e, \end{aligned}$$

and these are six independent one variance Brownian motions. We observe that

$$\langle \Theta_1^*, \Theta_1^* \rangle = \frac{1}{\lambda_0+\lambda_1}(\lambda_1^2+\mu_0\lambda_1+\lambda_0\lambda_1-\lambda_1\mu_1)e = \frac{2\lambda_1}{b}e, \tag{5.123}$$

$$\begin{aligned} \langle \Theta_2^*, \Theta_2^* \rangle &= \frac{1}{\lambda_0+\lambda_1}(\mu_1\lambda_0-\mu_1^2+\lambda_0\mu_1+\mu_1^2)e \\ &= \frac{1}{\lambda_0+\lambda_1}2\mu_1\lambda_0 e = \frac{2\mu_1}{a}e, \end{aligned} \tag{5.124}$$

$$\begin{aligned} \langle \Theta_3^*, \Theta_3^* \rangle &= \frac{1}{\lambda_0+\lambda_1}(\lambda_1\lambda_2+\lambda_0\lambda_2-\mu_1\lambda_2+\mu_0\lambda_0-\mu_0\mu_1+\lambda_1\mu_1+\mu_0\mu_1)e \\ &= \frac{1}{\lambda_0+\lambda_1}(\lambda_1\lambda_2+\lambda_0\lambda_2-\mu_1\lambda_2+\mu_0\lambda_0+\lambda_1\mu_1)e = \frac{2\lambda_0}{b}e, \end{aligned} \tag{5.125}$$

$$\begin{aligned} \langle \Theta_4^*, \Theta_4^* \rangle &= \frac{1}{\lambda_0+\lambda_1}(\mu_1\mu_2+\lambda_0\mu_2-\mu_1\mu_2+\lambda_0^2-\lambda_0\mu_1+\lambda_1\mu_1+\lambda_0\lambda_1)e \\ &= \frac{1}{\lambda_0+\lambda_1}(\lambda_0\mu_2+\lambda_0^2-\lambda_0\mu_1+\lambda_1\mu_1+\lambda_0\lambda_1)e = \frac{2\lambda_0}{b}e, \end{aligned} \tag{5.126}$$

$$\langle \Theta_5^*, \Theta_5^* \rangle = \frac{1}{\lambda_0+\lambda_1}(\lambda_1\lambda_0-\mu_1\lambda_1+\mu_0\lambda_1+\lambda_1^2)e = \frac{2\lambda_1}{b}e, \tag{5.127}$$

$$\begin{aligned} \langle \Theta_6^*, \Theta_6^* \rangle &= \frac{1}{\lambda_0+\lambda_1}(\mu_1^2+\lambda_0\mu_1+\lambda_0\mu_1-\mu_1^2)e \\ &= \frac{1}{\lambda_0+\lambda_1}2\mu_1\lambda_0 e = \frac{2\mu_1}{a}e. \end{aligned} \tag{5.128}$$

Based the From (5.16), (5.17), (5.91), and Proposition 5.1.6, we have

$$G^* = (b\Theta_1^*+\Theta_2^*+\Theta_3^*)\circ \overline{P}^+_{G^*} + (\Theta_4^*+\Theta_5^*+a\Theta_6^*)\circ \overline{P}^-_{G^*}, \tag{5.129}$$

$$\begin{aligned} |G^*| &= (b\Theta_1^*+\Theta_2^*+\Theta_3^*)\circ \overline{P}^+_{G^*} - (\Theta_4^*+\Theta_5^*+a\Theta_6^*)\circ \overline{P}^-_{G^*} \\ &\quad +\Gamma\big((b\Theta_1+\Theta_2+\Theta_3)\circ \overline{P}^+_{G^*} - (\Theta_4+\Theta_5+a\Theta_6)\circ \overline{P}^-_{G^*}\big). \end{aligned} \tag{5.130}$$

For the analysis of this section we will also need several correlated one variance Brownian motions. For convenient reference, we collect their definitions and properties here.

Definition 5.2.6

$$
\begin{aligned}
\Phi_1 &:= b\Theta_1^* + \Theta_2^* + \Theta_3^*, \\
\Phi_2 &:= -\Theta_4^* - \Theta_5^* - a\Theta_6^*, \\
\Phi_3 &:= -\Theta_1^* - b\Theta_2^*, \\
\widetilde{\Phi}_3 &:= -\Theta_1^* - a\Theta_2^*, \\
\Phi_4 &:= b\Theta_5^* + \Theta_6^*, \\
\widetilde{\Phi}_4 &:= a\Theta_5^* + \Theta_6^*, \\
\alpha &:= \frac{-b\langle\Theta_1^*, \Theta_1^*\rangle - b\langle\Theta_2^*, \Theta_2^*\rangle}{b^2\langle\Theta_1^*, \Theta_1^*\rangle + \langle\Theta_2^*, \Theta_2^*\rangle + \langle\Theta_3^*, \Theta_3^*\rangle} = -\frac{(b-1)\lambda_0 + \lambda_1}{\lambda_0 + b\lambda_1}, \\
\beta &:= \frac{-b\langle\Theta_5^*, \Theta_5^*\rangle - a\langle\Theta_6^*, \Theta_6^*\rangle}{\langle\Theta_4^*, \Theta_4^*\rangle + \langle\Theta_5^*, \Theta_5^*\rangle + a^2\langle\Theta_6^*, \Theta_6^*\rangle} = -\frac{\lambda_1 + \mu_1}{\mu_0 + a\mu_1}, \\
\tilde{\alpha} &:= \frac{-b\langle\Theta_1^*, \Theta_1^*\rangle - a\langle\Theta_2^*, \Theta_2^*\rangle}{b^2\langle\Theta_1^*, \Theta_1^*\rangle + \langle\Theta_2^*, \Theta_2^*\rangle + \langle\Theta_3^*, \Theta_3^*\rangle} = -\frac{\lambda_1 + \mu_1}{\lambda_0 + b\lambda_1}, \\
\tilde{\beta} &:= \frac{-a\langle\Theta_5^*, \Theta_5^*\rangle - a\langle\Theta_6^*, \Theta_6^*\rangle}{\langle\Theta_4^*, \Theta_4^*\rangle + \langle\Theta_5^*, \Theta_5^*\rangle + a^2\langle\Theta_6^*, \Theta_6^*\rangle} = -\frac{(a-1)\mu_0 + \mu_1}{\mu_0 + a\mu_1}, \\
\Phi_5 &:= \Phi_3 - \alpha\Phi_1 = -(1 + b\alpha)\Theta_1^* - (b + \alpha)\Theta_2^* - \alpha\Theta_3^*, \\
\widetilde{\Phi}_5 &:= \widetilde{\Phi}_3 - \tilde{\alpha}\Phi_1 = -(1 + b\tilde{\alpha})\Theta_1^* - (a + \tilde{\alpha})\Theta_2^* - \tilde{\alpha}\Theta_3^*, \\
\Phi_6 &:= \Phi_4 - \beta\Phi_2 = \beta\Theta_4^* + (b + \beta)\Theta_5^* + (1 + a\beta)\Theta_6^*, \\
\widetilde{\Phi}_6 &:= \widetilde{\Phi}_4 - \tilde{\beta}\Phi_2 = \tilde{\beta}\Theta_4^* + (a + \tilde{\beta})\Theta_5^* + (1 + a\tilde{\beta})\Theta_6^*,
\end{aligned}
$$

so that $\{\Phi_i\}_{i=1,\dots,6}$ *and* $\{\widetilde{\Phi}_j\}_{j=3,4,5,6}$ *are one variance Brownian motions, and* α, $\tilde{\alpha}$, β, *and* $\tilde{\beta}$ *are constants.*

Lemma 5.2.7 *Following the definition above, we have*

$$
\begin{aligned}
\langle\Phi_5, \Phi_1\rangle &= 0, \quad \langle\Phi_6, \Phi_2\rangle = 0, && (5.131) \\
\langle\Phi_5, \Phi_2\rangle &= 0, \quad \langle\Phi_6, \Phi_1\rangle = 0, && (5.132) \\
\langle\widetilde{\Phi}_5, \Phi_1\rangle &= 0, \quad \langle\widetilde{\Phi}_6, \Phi_2\rangle = 0, && (5.133) \\
\langle\widetilde{\Phi}_5, \Phi_2\rangle &= 0, \quad \langle\widetilde{\Phi}_6, \Phi_1\rangle = 0, && (5.134)
\end{aligned}
$$

which implies $\left(\Phi_5, \widetilde{\Phi}_5, \Phi_6, \widetilde{\Phi}_6\right)$ *is independent of* $\left(\Phi_1, \Phi_2\right)$*, which follows from the Levy's Theorem on Page 157 of [19]. Moreover,*

$$
\begin{aligned}
\langle\Phi_1, \Phi_1\rangle &= c_+ e, \\
\langle\Phi_2, \Phi_2\rangle &= c_- e,
\end{aligned}
$$

where c_+ *and* c_- *are defined in Corollary 5.1.7.*

Proof: From Assumption 2.2.1 and (5.123)-(5.128), we can do a lengthy computation to show that

$$
\langle\Phi_1, \Phi_1\rangle = c_+ e, \quad \langle\Phi_2, \Phi_2\rangle = c_- e.
$$

Since Φ_1, Φ_2, Φ_5, Φ_6, $\widetilde{\Phi}_5$, and $\widetilde{\Phi}_6$ are one variance Brownian motions, it suffices to show (5.131)-(5.134). Following the definition of α, β and Φ_i where $i = 1, \ldots, 6$, we have

$$\begin{aligned}
\langle \Phi_5, \Phi_1 \rangle &= \langle \Phi_3 - \alpha\Phi_1, \Phi_1 \rangle = \langle \Phi_3, \Phi_1 \rangle - \alpha\langle \Phi_1, \Phi_1 \rangle \\
&= -b\langle \Theta_1^*, \Theta_1^* \rangle - b\langle \Theta_2^*, \Theta_2^* \rangle - \alpha(b^2\langle \Theta_1^*, \Theta_1^* \rangle + \langle \Theta_2^*, \Theta_2^* \rangle + \langle \Theta_3^*, \Theta_3^* \rangle) \\
&= 0, \\
\langle \Phi_6, \Phi_2 \rangle &= \langle \Phi_4 - \beta\Phi_2, \Phi_2 \rangle = \langle \Phi_4, \Phi_2 \rangle - \beta\langle \Phi_2, \Phi_2 \rangle \\
&= -b\langle \Theta_5^*, \Theta_5^* \rangle - a\langle \Theta_6^*, \Theta_6^* \rangle - \beta(\langle \Theta_4^*, \Theta_4^* \rangle + \langle \Theta_5^*, \Theta_5^* \rangle + a^2\langle \Theta_6^*, \Theta_6^* \rangle) \\
&= 0, \\
\langle \Phi_5, \Phi_2 \rangle &= \langle \Phi_3 - \alpha\Phi_1, \Phi_2 \rangle = \langle \Phi_3, \Phi_2 \rangle - \alpha\langle \Phi_1, \Phi_2 \rangle \\
&= 0, \\
\langle \Phi_6, \Phi_1 \rangle &= \langle \Phi_4 - \beta\Phi_2, \Phi_1 \rangle = \langle \Phi_4, \Phi_1 \rangle - \beta\langle \Phi_2, \Phi_1 \rangle \\
&= 0.
\end{aligned}$$

Similarly, following the definition of $\tilde{\alpha}$, $\tilde{\beta}$ and $\widetilde{\Phi}_i$ where $i = 1, \ldots, 6$, we have

$$\begin{aligned}
\langle \widetilde{\Phi}_5, \Phi_1 \rangle &= \langle \widetilde{\Phi}_3 - \tilde{\alpha}\Phi_1, \Phi_1 \rangle = \langle \widetilde{\Phi}_3, \Phi_1 \rangle - \tilde{\alpha}\langle \Phi_1, \Phi_1 \rangle \\
&= -b\langle \Theta_1^*, \Theta_1^* \rangle - a\langle \Theta_2^*, \Theta_2^* \rangle - \tilde{\alpha}(b^2\langle \Theta_1^*, \Theta_1^* \rangle + \langle \Theta_2^*, \Theta_2^* \rangle + \langle \Theta_3^*, \Theta_3^* \rangle) \\
&= 0, \\
\langle \widetilde{\Phi}_6, \Phi_2 \rangle &= \langle \widetilde{\Phi}_4 - \tilde{\beta}\Phi_2, \Phi_2 \rangle = \langle \widetilde{\Phi}_4, \Phi_2 \rangle - \tilde{\beta}\langle \Phi_2, \Phi_2 \rangle \\
&= -a\langle \Theta_5^*, \Theta_5^* \rangle - a\langle \Theta_6^*, \Theta_6^* \rangle - \tilde{\beta}(\langle \Theta_4^*, \Theta_4^* \rangle + \langle \Theta_5^*, \Theta_5^* \rangle + a^2\langle \Theta_6^*, \Theta_6^* \rangle) \\
&= 0, \\
\langle \widetilde{\Phi}_5, \Phi_2 \rangle &= \langle \widetilde{\Phi}_3 - \tilde{\alpha}\Phi_1, \Phi_2 \rangle = \langle \widetilde{\Phi}_3, \Phi_2 \rangle - \tilde{\alpha}\langle \Phi_1, \Phi_2 \rangle \\
&= 0, \\
\langle \widetilde{\Phi}_6, \Phi_1 \rangle &= \langle \widetilde{\Phi}_4 - \tilde{\beta}\Phi_2, \Phi_1 \rangle = \langle \widetilde{\Phi}_4, \Phi_1 \rangle - \tilde{\beta}\langle \Phi_2, \Phi_1 \rangle \\
&= 0,
\end{aligned}$$

which finishes the proof. □

Lemma 5.2.8 *According to Definition 5.2.6, we can write (5.129) and (5.130) as*

$$\begin{aligned}
G^* &= \Phi_1 \circ \overline{P}^+_{G^*} - \Phi_2 \circ \overline{P}^-_{G^*}, &(5.135)\\
|G^*| &= \Phi_1 \circ \overline{P}^+_{G^*} + \Phi_2 \circ \overline{P}^-_{G^*} + \Gamma(\Phi_1 \circ \overline{P}^+_{G^*} + \Phi_2 \circ \overline{P}^-_{G^*}). &(5.136)
\end{aligned}$$

Remark 5.2.9 *Note that $\overline{P}^+_{G^*}$ and $\overline{P}^-_{G^*}$ are determined by Φ_1 and Φ_2 and hence they are independent of any processes that are independent of Φ_1 and Φ_2. From Theorem 5.1.8, we know G^* is two-variance Brownian Motion, therefore, from Lemma 5.2.7 and Lemma 4.2.7, we obtain that Φ_5, $\widetilde{\Phi}_5$, Φ_6 and $\widetilde{\Phi}_6$ are independent of $\overline{P}^+_{G^*}$, $\overline{P}^-_{G^*}$ and G^*.*

From (5.18) and (5.19) and Theorem 5.1.1, we see that

$$
\begin{aligned}
\sqrt{n} \quad & \big[(\lambda_0-\mu_1)(\overline{P}_1^n+\overline{P}_8^n)-(\lambda_0-\mu_1)(\overline{P}_3^n+\overline{P}_6^n)+(\mu_1-\lambda_1)(\overline{P}_2^n-\overline{P}_7^n) \\
& \quad -(\lambda_0+\mu_0)\overline{P}_4^n-(\lambda_1+\mu_1)\overline{P}_5^n\big] \\
& \quad \overset{J_1}{\Longrightarrow} (\Theta_1^*+\Theta_2^*)\circ\overline{P}_{G^*}^+-(\Theta_5^*+\Theta_6^*)\circ\overline{P}_{G^*}^-, && (5.137)\\
\sqrt{n} \quad & \big[(\lambda_0-\mu_1)(\overline{P}_1^n+\overline{P}_8^n)+(\lambda_0-\mu_1)(\overline{P}_3^n+\overline{P}_6^n)-(\mu_1+\lambda_1)(\overline{P}_2^n+\overline{P}_7^n) \\
& \quad +(\lambda_0+\mu_0)\overline{P}_4^n-(\lambda_1+\mu_1)\overline{P}_5^n\big] \\
& \quad \overset{J_1}{\Longrightarrow} (\Theta_1^*-\Theta_2^*)\circ\overline{P}_{G^*}^++(\Theta_5^*-\Theta_6^*)\circ\overline{P}_{G^*}^-, && (5.138)
\end{aligned}
$$

According to Assumption 2.2.1, we can easily verify

$$b\lambda_1-\mu_1=\mu_0-\lambda_2, \quad a\mu_1-\lambda_1=\lambda_0-\mu_2.$$

We multiply (5.137) by $-\frac{1}{2}(b+1)$, multiply (5.138) by $\frac{1}{2}(b-1)$, and sum to obtain

$$\Pi^n \overset{J_1}{\Longrightarrow} (-\Theta_1^*-b\Theta_2^*)\circ\overline{P}_{G^*}^++(b\Theta_5^*+\Theta_6^*)\circ\overline{P}_{G^*}^-. \tag{5.139}$$

where

$$
\begin{aligned}
\Pi^n \quad := \quad & \sqrt{n}\big[-(\lambda_0-\mu_1)\overline{P}_1^n-(b\mu_1-\lambda_1)\overline{P}_2^n+\lambda_2\overline{P}_3^n+b(\lambda_0+\mu_0)\overline{P}_4^n+(\mu_1+\lambda_1)\overline{P}_5^n \\
& +\lambda_2\overline{P}_6^n-(\mu_0-\lambda_2)\overline{P}_7^n-(\lambda_0-\mu_1)\overline{P}_8^n\big].
\end{aligned}
$$

Similarly, we multiply (5.137) by $-\frac{1}{2}(a+1)$, multiply (5.138) by $\frac{1}{2}(a-1)$, and sum to obtain

$$\widetilde{\Pi}^n \overset{J_1}{\Longrightarrow} (-\Theta_1^*-a\Theta_2^*)\circ\overline{P}_{G^*}^++(a\Theta_5^*+\Theta_6^*)\circ\overline{P}_{G^*}^-. \tag{5.140}$$

where

$$
\begin{aligned}
\widetilde{\Pi}^n \quad := \quad & \sqrt{n}\big[-(\mu_0-\lambda_1)\overline{P}_1^n-(\lambda_0-\mu_2)\overline{P}_2^n+\mu_2\overline{P}_3^n+a(\lambda_0+\mu_0)\overline{P}_4^n+(\mu_1+\lambda_1)\overline{P}_5^n \\
& +\mu_2\overline{P}_6^n-(a\lambda_1-\mu_1)\overline{P}_7^n-(\lambda_0-\mu_1)\overline{P}_8^n\big].
\end{aligned}
$$

From Definition 5.2.6, we may rewrite (5.139) and (5.140) as

$$
\begin{aligned}
\Pi^n \quad & \overset{J_1}{\Longrightarrow} \quad \Phi_3\circ\overline{P}_{G^*}^++\Phi_4\circ\overline{P}_{G^*}^-, && (5.141)\\
\widetilde{\Pi}^n \quad & \overset{J_1}{\Longrightarrow} \quad \widetilde{\Phi}_3\circ\overline{P}_{G^*}^++\widetilde{\Phi}_4\circ\overline{P}_{G^*}^-, && (5.142)
\end{aligned}
$$

The following lemma will be crucial in the identification of the limit of the term (5.121).

Lemma 5.2.10 *Since*

$$
\begin{aligned}
\Phi_3 \quad &= \quad \alpha\Phi_1+\Phi_5, \quad \Phi_4=\beta\Phi_2+\Phi_6,\\
\widetilde{\Phi}_3 \quad &= \quad \tilde{\alpha}\Phi_1+\widetilde{\Phi}_5, \quad \widetilde{\Phi}_4=\tilde{\beta}\Phi_2+\widetilde{\Phi}_6,
\end{aligned}
$$

we can rewrite (5.141) and (5.145) as

$$
\begin{aligned}
\Pi^n \quad & \overset{J_1}{\Longrightarrow} \quad (\alpha\Phi_1\circ\overline{P}_{G^*}^++\beta\Phi_2\circ\overline{P}_{G^*}^-)+\Phi_5\circ\overline{P}_{G^*}^++\Phi_6\circ\overline{P}_{G^*}^-, && (5.143)\\
\widetilde{\Pi}^n \quad & \overset{J_1}{\Longrightarrow} \quad (\tilde{\alpha}\Phi_1\circ\overline{P}_{G^*}^++\tilde{\beta}\Phi_2\circ\overline{P}_{G^*}^-)+\widetilde{\Phi}_5\circ\overline{P}_{G^*}^++\widetilde{\Phi}_6\circ\overline{P}_{G^*}^-. && (5.144)
\end{aligned}
$$

After these preliminaries, we are ready to study the behavior of $\widehat{\mathcal{U}}^n$ on a negative excursion of $\widehat{G}^n$. To select such an excursion of $\widehat{G}^n$, we first select a negative excursion of G^* and appeal to Lemma 5.2.11 below. To do this, we use the Skorohod embedding theorem to place all processes on a common probability space so that all weak convergence results become almost sure convergence in the J_1 topology. In particular, convergence to a continuous limiting process is uniform convergence on compact time intervals.

Let $\varepsilon > 0$ and a positive integer k be given, and consider the k-th ε-long negative excursion of G^*. This excursion has a left endpoint Λ and a right endpoint R. The excursion itself is

$$E(t) = G^*\big((t+\Lambda) \wedge R\big), \quad t \geq 0.$$

In particular, $E(0) = 0$, $E(t) < 0$ for $0 < t < R - \Lambda$, and $E(t) = 0$ for $t \geq R - \Lambda > \varepsilon$. Consider also the k-th ε-long negative excursion of $\widehat{G}^n$. Denote its left endpoint by Λ^n, its right endpoint by R^n, and the excursion itself by

$$E^n(t) = \widehat{G}^n\big((t+\Lambda^n) \wedge R^n\big), \quad t \geq 0.$$

Then sufficiently large n, we have $-\frac{1}{\sqrt{n}} \leq E^n(0) \leq 0$, $E^n(t) < 0$ for $0 < t < R^n - \Lambda^n$, and $E^n(t) = 0$ for $t \geq R^n - \Lambda^n$.

Lemma 5.2.11 *The excursions and excursion endpoints defined above satisfy*

(i) $\Lambda^n \to \Lambda$ *a.s.,*

(ii) $R^n \to R$ *a.s.*

(iii) $E^n \to E$.

Proof: For any $x \in D[0,\infty)$, let $L_k : D[0,\infty) \to [0,\infty]$ and $R_k : D[0,\infty) \to [0,\infty]$ be defined by

$$\begin{aligned} L_k(x) &:= \text{ Left endpoint of } k^{th}\ \varepsilon\text{-long negative excursion of } x, \\ R_k(x) &:= \text{ Right endpoint of } k^{th}\ \varepsilon\text{-long negative excursion of } x. \end{aligned}$$

Since $\widehat{G}^n \to G^*$ almost surely, according to the Continuous Mapping Theorem, it suffices to show L_k and R_k are almost surely continuous under the two-variance Wiener measure. Given a path g of a two-variance Brownian motion and a sequence of paths $\{g_n\}_{n=1}^\infty$ converging to g, let $\{l_i, r_i\}_{i=1,\dots,k}$ denote the left endpoints and right endpoints of the first k ε-long negative excursions on g, and let $\{l_i^n, r_i^n\}_{i=1,\dots,k}$ denote the left endpoints and right endpoints of the first k ε-long negative excursion on g_n. Since $g_n \to g$ in J_1 and g is continuous, we have uniform convergence on compact intervals. Our goal is to show $l_k^n \to l_k$ and $r_k^n \to r_k$.

Note that g is a path of two-variance Brownian motion, so g crosses zero at l_i and r_i for $i = 1, \dots, k$. Because of the uniform convergence of $\{g_n\}_{n=1}^\infty$, we can find $\hat{l}_i^n \to l_i$ and $\hat{r}_i^n \to r_i$ such that $g_n(t) < 0$ for $t \in (\hat{l}_i^n, \hat{r}_i^n)$ and $g_n(\hat{l}_i^n) = g_n(\hat{r}_i^n) = 0$, for $i = 1, \dots, k$.

Therefore, for sufficiently large n, $\hat{l}_k^n$ and $\hat{r}_k^n$ are the left endpoint and right endpoint of at least the k^{th} ε-long negative excursion of g_n, which means that $\{l_k^n\}_{n=1}^\infty$ and $\{r_k^n\}_{n=1}^\infty$ are bounded and $l_k^n \le \hat{l}_k^n$, $r_k^n \le \hat{r}_k^n$.

It suffices to show every convergent subsequence of $\{l_k^n, r_k^n\}_{n=1}^\infty$ converges to $\{l_k, r_k\}$. Given any convergent subsequence (for convenience, we do not re-label it) $\{l_k^n, r_k^n\}_{n=1}^\infty$, let $l_k^n \to \tilde{l}_k$ and $r_k^n \to \tilde{r}_k$. Since $r_k^n - l_k^n > \varepsilon$, we must have $\tilde{r}_k - \tilde{l}_k \ge \varepsilon$. Moreover, since $g_n < 0$ on (l_k^n, r_k^n) and $g_n \to g$ uniformly on $[0, r_k]$, we have $g \le 0$ on $(\tilde{l}_k, \tilde{r}_k)$. Note that g is a path of two-variance Brownian motion, g must be on a negative excursion on $(\tilde{l}_k, \tilde{r}_k)$. Since the probability that g has a negative excursion with length exactly equal to ε is zero, $(\tilde{l}_k, \tilde{r}_k)$ must be an ε-long negative excursion of g. Note that $l_k^n \le \hat{l}_k^n$, $r_k^n \le \hat{r}_k^n$, $\hat{l}_k^n \to l_k$, and $\hat{r}_k^n \to r_k$, so we have $\tilde{l}_k \le l_k$ and $\tilde{r}_k \le r_k$.

In order to show $\tilde{l}_k \ge l_k$ and $\tilde{r}_k \ge r_k$, we can further choose a subsequence (for convenience, we still let n denote the index) $\{l_i^n, r_i^n\}_{n=1}^\infty$ such that $l_i^n \to \tilde{l}_i$ and $r_i^n \to \tilde{r}_i$ for $i = 1, \dots, k-1$. Similar to analysis above, for any $i = 1, \dots, k-1$, since $r_i^n - l_i^n > \varepsilon$, we must have $\tilde{r}_i - \tilde{l}_i \ge \varepsilon$. Moreover, since $g_n < 0$ on (l_i^n, r_i^n) and $g_n \to g$ uniformly on $[0, r_k]$, we have $g \le 0$ on $(\tilde{l}_i, \tilde{r}_i)$. Note that g is a path of two-variance Brownian motion, g must be on a negative excursion on $(\tilde{l}_i, \tilde{r}_i)$. Since the probability that g has a negative excursion with length exactly equal to ε is zero, $(\tilde{l}_i, \tilde{r}_i)$ must be a ε-long negative excursion of g. Therefore, $(\tilde{l}_i, \tilde{r}_i)_{i=1}^k$ are k ε-long negative excursions of g, which implies $\tilde{l}_k \ge l_k$ and $\tilde{r}_k \ge r_k$. □

Proposition 5.2.12 *Note that $\widehat{M}_{i,\times,*}^n \overset{J_1}{\Longrightarrow} B_{i,\times,*}$, let us define the one variance Brownian motion*

$$\begin{aligned}
\Phi_7 \quad := \quad & B_{6,\mathcal{U},+} \circ \frac{\lambda_2\lambda_1}{\lambda_0+\lambda_1} e + B_{7,\mathcal{U},+} \circ \frac{\lambda_2(\lambda_0-\mu_1)}{\lambda_0+\lambda_1} e - B_{7,\mathcal{U},-} \circ \frac{\mu_0(\lambda_0-\mu_1)}{\lambda_0+\lambda_1} e \\
& + B_{8,\mathcal{U},+} \circ \frac{\lambda_1\mu_1}{\lambda_0+\lambda_1} e - B_{8,\mathcal{U},-} \frac{\mu_0\mu_1}{\lambda_0+\lambda_1} e.
\end{aligned}$$

Then

$$\begin{aligned}
& \widehat{M}_{6,\mathcal{U},+}^n \circ \lambda_2 \overline{P}_6^n + \widehat{M}_{7,\mathcal{U},+}^n \circ \lambda_2 d\overline{P}_7^n - \widehat{M}_{7,\mathcal{U},-}^n \circ \mu_0 d\overline{P}_7^n \\
& \quad + \widehat{M}_{8,\mathcal{U},+}^n \circ \lambda_1 \, d\overline{P}_8^n - \widehat{M}_{8,\mathcal{U},-}^n \circ \mu_0 \, d\overline{P}_8^n \overset{J_1}{\Longrightarrow} \Phi_7 \circ \overline{P}_{G^*}^-,
\end{aligned}$$

and

$$\begin{aligned}
\langle \Phi_7, \Phi_7 \rangle \quad &= \quad \frac{\lambda_2\lambda_1 + \lambda_2\lambda_0 - \lambda_2\mu_1 + \mu_0\lambda_0 - \mu_0\mu_1 + \lambda_1\mu_1 + \mu_0\mu_1}{\lambda_0+\lambda_1} e \\
&= \quad \frac{2\lambda_0}{b} e. \qquad (5.145)
\end{aligned}$$

Moreover, because the scaled centered Poisson processes $\widehat{M}_{6,\mathcal{U},+}^n$, $\widehat{M}_{7,\mathcal{U},+}^n$, $\widehat{M}_{7,\mathcal{U},-}^n$, $\widehat{M}_{8,\mathcal{U},+}^n$, and $\widehat{M}_{8,\mathcal{U},-}^n$ are independent of $\widehat{G}^n$, Φ_7 is independent of G^*.

On the interval $[\Lambda^n, R^n]$, the processes $\overline{P}^n_1$, $\overline{P}^n_2$, $\overline{P}^n_3$, $\overline{P}^n_4$ and $\overline{P}^n_5$ are constant. Thus, for $0 \le t \le R^n - \Lambda^n$, (5.62) implies

$$\begin{aligned}
&\widehat{\mathcal{U}}^n(\Lambda^n + t) - \widehat{\mathcal{U}}^n(\Lambda^n)\\
&\quad = \big(\widehat{M}^n_{6,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_6(\Lambda^n + t) - \widehat{M}^n_{6,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_6(\Lambda^n)\big)\\
&\qquad +\big(\widehat{M}^n_{7,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_7(\Lambda^n + t) - \widehat{M}^n_{7,\mathcal{U},+} \circ \lambda_2 \overline{P}^n_7(\Lambda^n)\big)\\
&\qquad -\big(\widehat{M}^n_{7,\mathcal{U},-} \circ \mu_0 \overline{P}^n_7(\Lambda^n + t) - \widehat{M}^n_{7,\mathcal{U},-} \circ \mu_0 \overline{P}^n_7(\Lambda^n)\big) + \big(\widehat{M}^n_{8,\mathcal{U},+} \circ \lambda_1 \overline{P}^n_8(\Lambda^n + t)\\
&\qquad -\widehat{M}^n_{8,\mathcal{U},+} \circ \lambda_1 \overline{P}^n_8(\Lambda^n)\big) - \big(\widehat{M}^n_{8,\mathcal{U},-} \circ \mu_0 \overline{P}^n_8(\Lambda^n + t) - \widehat{M}^n_{8,\mathcal{U},-} \circ \mu_0 \overline{P}^n_8(\Lambda^n)\big)\\
&\qquad +\sqrt{n}\big[(\lambda_2 \overline{P}^n_6(\Lambda^n + t) - \lambda_2 \overline{P}^n_6(\Lambda^n)\big) - (\mu_0 - \lambda_2)\big(\overline{P}^n_7(\Lambda^n + t) - \overline{P}^n_7(\Lambda^n)\big)\\
&\qquad -(\mu_0 - \lambda_1)\big(\overline{P}^n_8(\Lambda^n + t) - \overline{P}^n_8(\Lambda^n)\big)\big].
\end{aligned}$$

Proposition 5.2.13 *We have*

$$\begin{aligned}
&\sqrt{n}\big[(\lambda_2\big(\overline{P}^n_6(\Lambda^n + t) - \overline{P}^n_6(\Lambda^n)\big) - (\mu_0 - \lambda_2)\big(\overline{P}^n_7(\Lambda^n + t) - \overline{P}^n_7(\Lambda^n)\big)\\
&-(\mu_0 - \lambda_1)\big(\overline{P}^n_8(\Lambda^n + t) - \overline{P}^n_8(\Lambda^n)\big)\big] \quad = \quad \Pi^n(\Lambda^n + t) - \Pi^n(\Lambda^n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \stackrel{J_1}{\Longrightarrow} \quad \beta\big(\Phi_2 \circ \overline{P}^-_{G^*}(\Lambda + t) - \Phi_2 \circ \overline{P}^-_{G^*}(\Lambda)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +\big(\Phi_6 \circ \overline{P}^-_{G^*}(\Lambda + t) - \Phi_6 \circ \overline{P}^-_{G^*}(\Lambda)\big),
\end{aligned}$$

and

$$\Big(\widehat{\mathcal{U}}^n\big((\Lambda^n + \cdot) \wedge R^n\big) - \widehat{\mathcal{U}}^n(\Lambda^n)\Big) \stackrel{J_1}{\Longrightarrow} (\Phi_7 + \Phi_6 - \beta E)\,\big(\,\cdot \wedge (R - \Lambda)\big). \tag{5.146}$$

Moreover, the increments $\Phi_7 \circ \overline{P}^-_{G^*}(\Lambda + t) - \Phi_7 \circ \overline{P}^-_{G^*}(\Lambda)$, $\Phi_2 \circ \overline{P}^-_{G^*}(\Lambda + t) - \Phi_2 \circ \overline{P}^-_{G^*}(\Lambda)$, *and* $\Phi_6 \circ \overline{P}^-_{G^*}(\Lambda + t) - \Phi_6 \circ \overline{P}^-_{G^*}(\Lambda)$ *are independent.*

Proof: The first convergence follows from (5.143) and Lemma 5.2.11. Proposition 5.2.12 then implies

$$\begin{aligned}
\Big(\widehat{\mathcal{U}}^n\big((\Lambda^n + \cdot) \wedge R^n\big) - \widehat{\mathcal{U}}^n(\Lambda^n)\Big) \quad \stackrel{J_1}{\Longrightarrow} \quad & \Big(\Phi_7 \circ \overline{P}^-_{G^*}\big((\Lambda + \cdot) \wedge R\big) - \Phi_7 \circ \overline{P}^-_{G^*}(\Lambda)\Big)\\
& +\beta\,\Big(\Phi_2 \circ \overline{P}^-_{G^*}\big((\Lambda + \cdot) \wedge R\big) - \Phi_2 \circ \overline{P}^-_{G^*}(\Lambda)\Big)\\
& +\,\Big(\Phi_6 \circ \overline{P}^-_{G^*}\big((\Lambda + \cdot) \wedge R\big) - \Phi_6 \circ \overline{P}^-_{G^*}(\Lambda)\Big)
\end{aligned} \tag{5.147}$$

Because Φ_2 and Φ_6 are defined in terms of $\{\Theta^*_i\}_{i=1,\ldots,6}$, and these processes are independent of the Brownian motions appearing in the definition of Φ_7, Φ_2 and Φ_6 are independent of Φ_7. From Lemma 5.2.7, we also have Φ_2 is independent of Φ_6.

Since Φ_7 is independent of G^*,

$$\Phi_7 \circ \overline{P}^-_{G^*}(\Lambda + t) - \Phi_7 \circ \overline{P}^-_{G^*}(\Lambda), \quad 0 \le t \le R - \Lambda,$$

has the same distribution as

$$\Phi_7(t), \quad 0 \le t < R - \Lambda.$$

According to Remark 5.2.9, we see that Φ_6 is independent of $\overline{P}^{-}_{G^*}$ and G^*, and hence independent of Λ and R, so the increment

$$\Phi_6 \circ \overline{P}^{-}_{G^*}(\Lambda + t) - \Phi_6 \circ \overline{P}^{-}_{G^*}(\Lambda), \quad 0 \le t \le R - \Lambda,$$

has the same distribution as

$$\Phi_6(t), \quad 0 \le t \le R - \Lambda.$$

The final component of the limit of $\widehat{\mathcal{U}}^n(\Lambda^n + t) - \widehat{\mathcal{U}}^n(\Lambda^n)$ during the k-th ε-long negative excursion of $\widehat{G}^n$ is

$$\beta\big(\Phi_2 \circ \overline{P}^{-}_{G^*}(\Lambda + t) - \Phi_2 \circ \overline{P}^{-}_{G^*}(\Lambda)\big).$$

From Lemma 5.2.8, we see that

$$\begin{aligned} (G^*)^-(t) &= \frac{1}{2}(|G^*| - G^*)(t) \\ &= \Phi_2 \circ \overline{P}^{-}_{G^*}(t) + \frac{1}{2}\Gamma\big(\Phi_1 \circ \overline{P}^{+}_{G^*} + \Phi_2 \circ \overline{P}^{-}_{G^*}\big)(t). \end{aligned}$$

Also, from Lemma 4.2.4 we have

$$\Gamma\big(\Phi_1 \circ \overline{P}^{+}_{G^*} + \Phi_2 \circ \overline{P}^{-}_{G^*}\big) = 2\Gamma\big(\Phi_2 \circ \overline{P}^{-}_{G^*}\big),$$

which implies

$$(G^*)^-(t) = \Phi_2 \circ \overline{P}^{-}_{G^*}(t) + \Gamma\big(\Phi_2 \circ \overline{P}^{-}_{G^*}\big)(t). \tag{5.148}$$

Therefore k-th ε-long negative excursion of G^* corresponds to the k-th ε-long excursion of the reflected Brownian motion

$$\widetilde{\Phi}_2(t) := \Phi_2(t) + \Gamma(\Phi_2)(t),$$

and this excursion has left endpoint$(\overline{P}^{-}_{G^*})^{-1}(\Lambda)$ and right endpoint $(\overline{P}^{-}_{G^*})^{-1}(R) = (\overline{P}^{-}_{G^*})^{-1}(\Lambda) + R - \Lambda$, where

$$(\overline{P}^{-}_{G^*})^{-1}(t) := \min\{s \ge 0 : \overline{P}^{-}_{G^*}(s) > t\}.$$

Since $E(t) = G^*\big((t + \Lambda) \wedge R\big)$, we have

$$E(t) = -\widetilde{\Phi}_2\Big(\big(t + (\overline{P}^{-}_{G^*})^{-1}(\Lambda)\big) \wedge (P^{-}_{G^*})^{-1}(R)\Big), \quad t \ge 0.$$

From Lemma 5.2.7, we have $\langle \Phi_2, \Phi_2 \rangle = c_-$, which implies the distribution of E is the same as the distribution of any ε-long negative excursion of a Brownian motion with variance c_- per unit time. In particular, $-\Phi_2 \circ \overline{P}^{-}_{G^*}(\Lambda + t) + \Phi_2 \circ \overline{P}^{-}_{G^*}(\Lambda), 0 \le t \le R - \Lambda$ is equal to $E(t)$ almost surely. Putting the three pieces together, we may rewrite the right hand side of (5.147), thereby obtaining (5.146). $\square$

From Definition 5.2.6 and Proposition 5.2.12, the total quadratic variation on the right-hand side of (5.148) is

$$\begin{aligned}
&\langle\Phi_6,\Phi_6\rangle+\langle\Phi_7,\Phi_7\rangle+\beta^2c_-e\\
&\quad= \beta^2\langle\Theta_4^*,\Theta_4^*\rangle+(b+\beta)^2\langle\Theta_5^*,\Theta_5^*\rangle+(1+a\beta)^2\langle\Theta_6^*,\Theta_6^*\rangle+\langle\Phi_7,\Phi_7\rangle+\beta^2c_-e\\
&\quad= (2ab-2b+2)\lambda_0 e\\
&\quad= c_+e,
\end{aligned}\tag{5.149}$$

which is exactly as expected.

Let $\varepsilon>0$ and a positive integer k be given, and consider the k-th ε-long positive excursion of G^*. This excursion has a left endpoint $\widetilde{\Lambda}$ and a right endpoint $\widetilde{R}$. The excursion itself is

$$\widetilde{E}(t)=G^*\big((t+\widetilde{\Lambda})\wedge\widetilde{R}\big),\quad t\geq 0.\tag{5.150}$$

Following the same argument, we have

$$\Big(\widehat{\mathcal{X}}^n\big((\widetilde{\Lambda}^n+\cdot)\wedge\widetilde{R}^n\big)-\widehat{\mathcal{X}}^n(\widetilde{\Lambda}^n)\Big)\overset{J_1}{\Longrightarrow}\Big(\widetilde{\Phi}_7-\widetilde{\Phi}_5+\tilde{\alpha}\widetilde{E}\Big)\big(\cdot\wedge(\widetilde{R}-\widetilde{\Lambda})\big),\tag{5.151}$$

where

$$\begin{aligned}
\widetilde{\Phi}_7 \;:=\;& -B_{3,\mathcal{X},-}\circ\frac{\mu_2\mu_1}{\lambda_0+\lambda_1}-B_{2,\mathcal{X},-}\circ\frac{\mu_2(\mu_0-\lambda_1)}{\lambda_0+\lambda_1}+B_{2,\mathcal{X},+}\circ\frac{\lambda_0(\mu_0-\lambda_1)}{\lambda_0+\lambda_1}\\
&-B_{1,\mathcal{X},-}\circ\frac{\mu_1\lambda_1}{\lambda_0+\lambda_1}+B_{1,\mathcal{X},+}\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}.
\end{aligned}\tag{5.152}$$

and

$$\langle\widetilde{\Phi}_7,\widetilde{\Phi}_7\rangle=\frac{2\lambda_0}{b}e.\tag{5.153}$$

From Definition 5.2.6 and (5.153), the total quadratic variation on the right-hand side of (5.151) is

$$\begin{aligned}
\langle\widehat{\mathcal{X}}^n,\widehat{\mathcal{X}}^n\rangle &= \langle\widetilde{\Phi}_5,\widetilde{\Phi}_5\rangle+\langle\widetilde{\Phi}_7,\widetilde{\Phi}_7\rangle+\tilde{\alpha}^2c_+e\\
&= \langle\Theta_1^*,\Theta_1^*\rangle+a^2\langle\Theta_2^*,\Theta_2^*\rangle+\langle\widetilde{\Phi}_7,\widetilde{\Phi}_7\rangle\\
&= \frac{(2a^2b-2a^2+2a)}{b}\lambda_0 e\\
&= c_-e,
\end{aligned}\tag{5.154}$$

which is also as expected.

5.2.3 Convergence of $(\widehat{\mathcal{U}}^n,\widehat{\mathcal{X}}^n)$

In this section, we want to enumerate the positive and negative excursions of G^* like the way they are defined in Section 4.2 of [2]. Following the same notation, we denote the left and right endpoints of k^{th} negative excursions by $\Lambda_{k,-}$ and $R_{k,-}$; we denote the left and right endpoints of k^{th} positive excursions by $\Lambda_{k,+}$ and $R_{k,+}$.

Proposition 5.2.14

$$\begin{aligned} \widehat{\mathcal{U}}^n &\Longrightarrow \mathcal{U}^*, \\ \widehat{\mathcal{X}}^n &\Longrightarrow \mathcal{X}^*, \end{aligned}$$

in $D[0-,\infty)$, *where* $\mathcal{U}^*(0-) = u_0$, $\mathcal{X}^*(0-) = x_0$, *and*

$$\mathcal{U}^*(t) = \begin{cases} \frac{\lambda_2\mu_1}{\theta_b\lambda_1}, & \text{if } t \in [\Lambda_{k,+}, R_{k,+}) \text{ for some } k = 1,2,\dots, \\ \frac{\lambda_2\mu_1}{\theta_b\lambda_1} + \mathcal{U}^*_{k,-}(t-\Lambda_{k,-}), & \text{if } t \in [\Lambda_{k,-}, R_{k,-}) \text{ for some } k = 1,2,\dots, \\ \frac{\lambda_2\mu_1}{\theta_b\lambda_1}, & \text{else.} \end{cases}$$

$$\mathcal{X}^*(t) = \begin{cases} -\frac{\mu_2\lambda_1}{\theta_s\mu_1}, & \text{if } t \in [\Lambda_{k,-}, R_{k,-}) \text{ for some } k = 1,2,\dots, \\ -\frac{\mu_2\lambda_1}{\theta_s\mu_1} + \mathcal{X}^*_{k,+}(t-\Lambda_{k,+}), & \text{if } t \in [\Lambda_{k,+}, R_{k,+}) \text{ for some } k = 1,2,\dots, \\ -\frac{\mu_2\lambda_1}{\theta_s\mu_1}, & \text{else.} \end{cases}$$

Here,

$$\begin{aligned} \mathcal{U}^*_{k,-} &:= C_{k,-}\big(\,\cdot \wedge (R_{k,-} - \Lambda_{k,-})\big) - \beta E_{k,-}, \\ \mathcal{X}^*_{k,-} &:= C_{k,+}\big(\,\cdot \wedge (R_{k,+} - \Lambda_{k,+})\big) + \tilde{\alpha} E_{k,+}, \end{aligned}$$

where $(C_{k,-}, k \geq 1)$ *is a sequence of independent Brownian motions that accumulate quadratic variation at rate* $c_+ - \beta^2 c_-$ *per unit time,* $(C_{k,+}, k \geq 1)$ *is a sequence of independent Brownian motions that accumulate quadratic variation at rate* $c_- - \tilde{\alpha}^2 c_+$ *per unit time, and*

$$\begin{aligned} E_{k,-} &:= G^*\big((\cdot + \Lambda_{k,-}) \wedge R_{k,-}\big), \\ E_{k,+} &:= G^*\big((\cdot + \Lambda_{k,+}) \wedge R_{k,+}\big), \end{aligned}$$

PROOF: Note that we assume $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{X}}^n)$ follow the same dynamic as those acting on $(\widehat{U}^n, \widehat{X}^n)$ when the bracketing processes are valid, so we can still use the result from [2]. From (5.118)-(5.119), (5.148)-(5.149), (5.151) and (5.154), the proof of Proposition 5.2.14 follows from Theorem 4.5.3 of [2]. □

5.2.4 Convergence of $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n, \widehat{\mathcal{X}}^n)$

We have already proved that,

$$(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n, \widehat{\mathcal{X}}^n) \Longrightarrow (\mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*, \mathcal{X}^*)$$

where convergence is weak convergence and the probability measure is on $\big(D[0-,\infty) \times D[0,\infty) \times D[0,\infty) \times D[0-,\infty)\big)$ equipped with topology $(M_1 \times J_1 \times J_1 \times M_1)$. Since M_1 and J_1 are separable, we can apply the Skorohod Representation Theorem to place all processes on a common probability space so that all weak convergences become almost sure convergences in the the topology mentioned above. Let us define,

$$\begin{aligned}
\tau_{\mathcal{U}}^n &= \inf\{t > 0 | \widehat{\mathcal{U}}^n(t) \le 0\}, \\
\tau_{\mathcal{X}}^n &= \inf\{t > 0 | \widehat{\mathcal{X}}^n(t) \ge 0\}, \\
\tau_{\mathcal{U}} &= \inf\{t > 0 | \mathcal{U}^*(t) \le 0\}, \\
\tau_{\mathcal{X}} &= \inf\{t > 0 | \mathcal{X}^*(t) \le 0\}.
\end{aligned}$$

Theorem 5.2.15

$$\begin{aligned}
\tau_{\mathcal{U}}^n &\to \tau_{\mathcal{U}}, \\
\tau_{\mathcal{X}}^n &\to \tau_{\mathcal{X}},
\end{aligned}$$

almost surely.

We will only prove $\tau_{\mathcal{U}}^n \to \tau_{\mathcal{U}}$, and the other convergence follows the same argument. Before we start the proof, we want to introduce a lemma.

Lemma 5.2.16 *For every $\delta > 0$,*

$$\inf_{0<t<0\vee(\tau_{\mathcal{U}}-\delta)} \{\mathcal{U}^*(t)\} > 0, \tag{5.155}$$

almost surely.

Proof: Note that before $\tau_{\mathcal{U}} - \delta$, $\mathcal{U}^*$ has infinitely many attempts to hit zero while $\mathcal{W}^*$ is on a negative excursion. Obviously, there are countably many such attempts, and let us call these negative excursion intervals (a_i, b_i), and their lengths $\ell_i = b_i - a_i$ for $i \ge 1$. Then we can order these intervals by decreasing lengths. For simplicity, let $\{\ell_i\}_{i\ge1}$ be ordered sequence. Fix $\epsilon > 0$. We can find sufficiently large $N_0 \in \mathbb{N}$ such that $\sum_{i=N_0}^{\infty} \ell_i < \epsilon$. Moreover, $\mathcal{U}^*$ only jumps when $\mathcal{W}^*$ reaches the right end point of a negative excursion, and during a negative excursion of $\mathcal{W}^*$, $\mathcal{U}^*$ behaves like a Brownian motion correlated with $\mathcal{W}^*$. Among the finitely many excursion intervals indexed by $i = 1, \cdots, N_0 - 1$, there is none on which the $\inf_{t\in(a_i,b_i)}\{\mathcal{U}^*(t)\}$ is zero because that would require an excursion interval to end just as $\mathcal{U}^*$ was reaching zero, which is a probability zero event. Therefore $\mathcal{U}^*$ is always strictly positive on negative excursions of $\mathcal{W}^*$ that terminate before $\tau_{\mathcal{U}} - \delta$. In particular,

$$\min_{i\le N_0} \left\{ \inf_{t\in(a_i,b_i)} \{\mathcal{U}^*(t)\} \right\} > 0. \tag{5.156}$$

For any $i > N_0$, $\mathcal{U}^*$ will start from $\frac{\lambda_2\mu_1}{\theta_b\lambda_1} > 0$ at the beginning of the i^{th} negative excursion of $\mathcal{W}^*$. Let T^i be the first passage time to $\frac{\lambda_2\mu_1}{2\theta_b\lambda_1}$ of $\mathcal{U}^*$ starting from beginning of the i^{th} excursion. Let $\theta = \frac{\theta_b\lambda_1}{\lambda_2\mu_1}$. Then, by reflection principle,

$$\mathbb{P}\{T^i > \ell_i\} = 2N(\frac{1}{2\theta\sqrt{c_+\ell_i}}) - 1.$$

Hence,

$$\begin{aligned}\mathbb{P}\Big\{\inf_{0<t<0\vee(\tau_{\mathcal{U}}-\delta)}\{\mathcal{U}^*(t)\}=0\Big\} &\le \mathbb{P}\{\exists i\ge N_0, T^i\le \ell_i\}\\ &= 1-\mathbb{P}\{\forall i\ge N_0, T^i>\ell_i\}\\ &= 1-\Pi_{i\ge N_0}^{\infty}(2N(\frac{1}{2\theta\sqrt{c_+\ell_i}})-1) \qquad (5.157)\end{aligned}$$

Let $x_i=\frac{1}{2\theta\sqrt{c_+\ell_i}}$. Since $\sum_{i=N}^{\infty}\ell_i<\epsilon$, we have

$$\sum_{i=N}^{\infty}\frac{1}{x_i^2}<4\theta^2c_+\epsilon. \qquad (5.158)$$

From Page 112, Problem 9.22 of [19], we have $1-\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}\le N(x)$, and this implies

$$1-\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{x_i^2}{2}}}{x_i}\le 2N(x_i)-1. \qquad (5.159)$$

Therefore, by possibly increasing N_0, we have

$$\begin{aligned}\Pi_{i\ge N_0}^{\infty}(2N(\frac{1}{2\theta\sqrt{c_+\ell_i}})-1) &= \Pi_{i\ge N_0}^{\infty}(2N(x_i)-1)\ge \Pi_{i\ge N_0}^{\infty}(1-\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{x_i^2}{2}}}{x_i})\\ &= \exp\{\sum_{i\ge N_0}^{\infty}\log(1-\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{x_i^2}{2}}}{x_i})\}\\ &\ge \exp\{\sum_{i\ge N_0}^{\infty}(-2\sqrt{\frac{2}{\pi}}\frac{e^{\frac{-x_i^2}{2}}}{x_i})\}=\exp\{-2\sqrt{\frac{2}{\pi}}\sum_{i\ge N_0}^{\infty}(\frac{e^{-\frac{x_i^2}{2}}}{x_i})\}\\ &= \exp\{-2\sqrt{\frac{2}{\pi}}\sum_{i\ge N_0}^{\infty}(\frac{x_ie^{-\frac{x_i^2}{2}}}{x_i^2})\}\ge \exp\{-2\sqrt{\frac{2}{\pi}}\sum_{i\ge N_0}^{\infty}(\frac{1}{x_i^2})\}\\ &> \exp\{-2\sqrt{\frac{2}{\pi}}4\theta^2\epsilon\},\end{aligned}$$

where the second inequality comes from $-2x\le\log(1-x)$ for sufficiently small x, and the third inequality comes from $xe^{-\frac{x^2}{2}}\le 1$ for all x, and the last inequality follows from (5.158). Now from (5.157), we have,

$$\mathbb{P}\{\inf_{0<t<0\vee(\tau_{\mathcal{U}}-\delta)}\{\mathcal{U}^*(t)\}=0\}<1-\exp\{-2\sqrt{\frac{2}{\pi}}4\theta^2\epsilon\}.$$

By sending $\epsilon\to 0$, we have proved that

$$\inf_{0<t<0\vee(\tau_{\mathcal{U}}-\delta)}\{\mathcal{U}^*(t)\}>0,$$

almost surely. $\square$

PROOF OF THEOREM 5.2.15: Let us fix $\omega \in \Omega$, and for any $t \geq 0$, let

$$\begin{aligned} f_n(t) &:= \widehat{\mathcal{U}}^n(\omega, t), \\ f(t) &:= \mathcal{U}^*(\omega, t), \end{aligned}$$

and

$$\begin{aligned} T_n &:= \tau^n_{\mathcal{U}}(\omega) = \inf\{t > 0 | f_n(t) \leq 0\}, \\ T &:= \tau_{\mathcal{U}}(\omega) = \inf\{t > 0 | f(t) \leq 0\}. \end{aligned}$$

Our goal is to show $|T_n - T| \to 0$. We can divide $\{T_n\}_{n\geq 1}$ into two subsequence $\{T_{n_k}\}_{k\geq 1}$ and $\{T_{n_p}\}_{p\geq 1}$ such that

$$T_{n_k} \geq T, \text{ for all } k \geq 1, \quad T_{n_p} < T, \text{ for all } p \geq 1.$$

Of course, one of these sequences may be empty.

Case 1: Fix $\epsilon > 0$, we want to show there exists $N \in \mathbb{N}$ such that $T_{n_k} - T < \epsilon$ for all $k \geq N$.

Note that f is continuous at T because $\mathcal{U}^*$ only jumps when $\mathcal{W}^*$ reaches the right end of a negative excursion, and the probability that these two events happen at the same time is zero. In fact, we can find $0 < \delta < \epsilon$ such that f is continuous $\forall t \in [T-\delta, T+\delta]$. Also, we know that $\mathcal{U}^*$ behaves like a Brownian motion between two consecutive jumps, so we can define

$$\begin{aligned} 0 > m &:= \min_{t \in [T-\frac{\delta}{2}, T+\frac{\delta}{2}]} \{f(t)\}, \\ t_m &:= \min\{t \in [T - \frac{\delta}{2}, T + \frac{\delta}{2}] | f(t) = m\}. \end{aligned}$$

Since $f_n \to f$ in the M_1 topology, from Theorem 12.5.1 of [26], we have

$$\lim_{r\to 0} \overline{\lim_{k\to\infty}} v(f_{n_k}, f, t_m, r) = 0,$$

where v is the uniform-distance function defined by

$$v(x_1, x_2, t, r) := \sup_{0\vee(t-r)\leq t_1, t_2 \leq (t+r)} \{\|x_1(t_1) - x_2(t_2)\|\}.$$

Therefore, there exists $r_0 > 0$ such that

$$\overline{\lim_{k\to\infty}} v(f_{n_k}, f, t_m, r_0) < \frac{|m|}{2}.$$

Note that

$$|f_{n_k}(t_m) - f(t_m)| \leq v(f_{n_k}, f, t_m, r_0).$$

Therefore, we have

$$\overline{\lim_{k\to\infty}} |f_{n_k}(t_m) - f(t_m)| \leq \overline{\lim_{k\to\infty}} v(f_{n_k}, f, t_m, r_0) \leq \frac{|m|}{2},$$

which implies that for sufficiently large k, we have

$$f_{n_k}(t_m) \leq \frac{m}{4} < 0,$$

which implies $T_{n_k} < t_m < T + \epsilon$.

Case 2: Fix $\epsilon > 0$, we want to show there exists $N \in \mathbb{N}$ such that $T_{n_p} > T - \epsilon$ for all $p \geq N$.

For convenience, we just label this subsequence as $(T_n)_{n\geq 1}$. From Lemma 5.2.16, we have

$$0 < \tilde{m} := \inf_{0\leq t\leq T-\frac{\epsilon}{2}} \{f(t)\}.$$

Also, let $\delta = \tilde{m} \wedge \frac{\epsilon}{2}$. Since $f_n \to f$ in the M_1 topology. Given $x \in D[0,\infty)$, let Γ_x denote the graph of x. Specifically,

$$\Gamma_x := \{(z,t) \in \mathbb{R} \times [0,\infty) : z \in [x(t-), x(t)]\},$$

where $x(0-) = x(0)$. A parametric representation of x is a continuous nondecreasing function (u,r) mapping $[0,\infty)$ onto Γ_x, and $\Pi(x)$ denotes the set of parametric representations of x. Applying Theorem 12.9.3 of [26], we have for all sufficiently large n, there exists $(u,r) \in \Pi(f)$, $(u_n, r_n) \in \Pi(f_n)$ such that

$$\|u_n - u\|_t \vee \|r_n - r\|_t < \delta, \tag{5.160}$$

for each $t > 0$ where $\|\cdot\|_t$ denotes the supremum norm over $[0,t]$. By definition of (u_n, r_n) we can find $s_n^* \in [0,\infty)$ satisfies

$$\begin{aligned} r_n(s_n^*) &= T_n, \\ u_n(s_n^*) &= 0. \end{aligned}$$

From (5.160), we have

$$r(s_n^*) < T_n + \delta \leq T_n + \frac{\epsilon}{2}. \tag{5.161}$$

Meanwhile,

$$|u(s_n^*)| = |u_n(s_n^*) - u(s_n^*)| < \delta \leq \tilde{m},$$

which implies that $r(s_n^*) > T - \frac{\epsilon}{2}$. Together with (5.161), we get $T_n > T - \epsilon$. □

Corollary 5.2.17 *Let*

$$\begin{aligned} \tau_{min}^n &= \tau_{\mathcal{U}}^n \wedge \tau_{\mathcal{X}}^n, \\ \tau_{min} &= \tau_{\mathcal{U}} \wedge \tau_{\mathcal{X}}. \end{aligned}$$

Then from Theorem 5.2.15, we have

$$\tau_{min}^n \to \tau_{min},$$

almost surely.

Corollary 5.2.18

$$(\widehat{\mathcal{U}}^n_{\cdot\wedge\tau^n_{min}}, \widehat{\mathcal{V}}^n_{\cdot\wedge\tau^n_{min}}, \widehat{\mathcal{W}}^n_{\cdot\wedge\tau^n_{min}}, \widehat{\mathcal{X}}^n_{\cdot\wedge\tau^n_{min}}) \longrightarrow (\mathcal{U}^*_{\cdot\wedge\tau_{min}}, \mathcal{V}^*_{\cdot\wedge\tau_{min}}, \mathcal{W}^*_{\cdot\wedge\tau_{min}}, \mathcal{X}^*_{\cdot\wedge\tau_{min}})$$

almost surely under the topology $(M_1 \times J_1 \times J_1 \times M_1)$.

Proof Since $\mathcal{V}^*$ and $\mathcal{W}^*$ are continuous and $(\widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n) \xrightarrow{J_1} (\mathcal{V}^*, \mathcal{W}^*)$, following the same argument as in Corollary 3.5.3, we can prove $(\widehat{\mathcal{V}}^n_{\cdot\wedge\tau^n_{min}}, \widehat{\mathcal{W}}^n_{\cdot\wedge\tau^n_{min}}) \xrightarrow{J_1} (\mathcal{V}^*_{\cdot\wedge\tau_{min}}, \mathcal{W}^*_{\cdot\wedge\tau_{min}})$. It suffices to show $(\widehat{\mathcal{U}}^n_{\cdot\wedge\tau^n_{min}}, \widehat{\mathcal{X}}^n_{\cdot\wedge\tau^n_{min}}) \xrightarrow{M_1} (\mathcal{U}^*_{\cdot\wedge\tau_{min}}, \mathcal{X}^*_{\cdot\wedge\tau_{min}})$. Note that $\mathcal{U}^*$ only jumps when $\mathcal{W}^*$ reaches the right end of a negative excursion, and the probability that $\mathcal{U}^*$ reaches zero when $\mathcal{W}^*$ reaches the right end of a negative excursion is zero. Similarly $\mathcal{X}^*$ only jumps when $\mathcal{V}^*$ reaches the right end of a positive excursion, and the probability that $\mathcal{X}^*$ reaches zero when $\mathcal{V}^*$ reaches the right end of a positive excursion is zero. Therefore $\mathcal{U}^*$ and $\mathcal{X}^*$ are continuous at τ_{min} almost surely. It suffices to show $\widehat{\mathcal{U}}^n_{\cdot\wedge\tau^n_{min}} \xrightarrow{M_1} \mathcal{U}^*_{\cdot\wedge\tau_{min}}$, and the other convergence follows from the exact same argument.

Let us fix $\omega \in \Omega$, and for any $t \geq 0$, let

$$\begin{aligned} f_n(t) &:= \widehat{\mathcal{U}}^n(\omega, t), \\ f(t) &:= \mathcal{U}^*(\omega, t), \\ T_n &:= \tau^n_{min}(\omega), \\ T &:= \tau_{min}(\omega), \\ g_n(t) &:= \widehat{\mathcal{U}}^n(\omega, t \wedge T_n), \\ g(t) &:= \mathcal{U}^*(\omega, t \wedge T). \end{aligned}$$

It suffices to show $g_n \xrightarrow{M_1} g$. From Theorem 12.5.1 of [26], it is equivalent to show for each $t \notin Disc(g)$ where $Disc(x)$ denotes the set of discontinuity points of x, we have

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, v(g_n, g, t, \delta) = 0,$$

where v is the uniform-distance function defined by

$$v(x_1, x_2, t, \delta) := \sup_{0\vee(t-\delta)\leq t_1, t_2 \leq (t+\delta)} \{\|x_1(t_1) - x_2(t_2)\|\},$$

and for each $t \in Disc(g)$,

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, w(g_n, t, \delta) = 0,$$

where w is defined by

$$w(g_n, t, \delta) := \sup_{0\vee(t-\delta)\leq t_1 < t_2 < t_3 \leq (t+\delta)} \Big\{\|g_n(t_2) - [g_n(t_1), g_n(t_3)]\|\Big\},$$

where

$$\|x_1 - [x_2, x_3]\| := \min_{x\in[x_2,x_3]} \{|x_1 - x|\}.$$

Since $f_n \xrightarrow{M_1} f$, we have for each $t \notin Disc(f)$,

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} v(f_n, f, t, \delta) = 0,$$

and for each $t \in Disc(f)$,

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} w(f_n, t, \delta) = 0.$$

Case 1: $t = T$. Since g is continuous at T, we need to show $\lim_{\delta\to 0} \overline{\lim}_{n\to\infty} v(g_n, g, t, \delta) = 0$. Since $T_n \to T$, fix any $\delta > 0$. For sufficiently large n, we have $|T - T_n| < \delta$, and this implies

$$v(g_n, g, t, \delta) \le v(f_n, f, t, \delta),$$

which implies

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} v(g_n, g, t, \delta) = 0.$$

Case 2: $t \notin Disc(g)$ and $t < T$. Let $\delta_0 = \frac{1}{2}(T - t)$. Then for sufficiently large n, we have $|T - T_n| < \delta_0$, which implies

$$v(g_n, g, t, \delta) = v(f_n, f, t, \delta),$$

for any $\delta < \delta_0$. Hence, we have

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} v(g_n, g, t, \delta) = 0.$$

Case 3: $t \notin Disc(g)$ and $t > T$. Let $\delta_0 = \frac{1}{2}(t - T)$. Then for sufficiently large n, we have $|T - T_n| < \delta_0$, which implies

$$v(g_n, g, t, \delta) = |g_n(T_n) - g(T)|,$$

for any $\delta < \delta_0$. It suffices to show $|g_n(T_n) - g(T)| \to 0$. Fix $\epsilon > 0$. Since

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} v(g_n, g, T, \delta) = 0,$$

there exists $\delta_1 > 0$ such that

$$\overline{\lim_{n\to\infty}} v(g_n, g, T, \delta_1) < \epsilon.$$

For sufficiently large n, we have $|T - T_n| < \delta_1$, and hence

$$|g_n(T_n) - g(T)| \le v(g_n, g, T, \delta_1) < 2\epsilon.$$

Case 4: $t \in Disc(g)$. Since g is continuous after T, then $t < T$. Let $\delta_0 = \frac{1}{2}(T - t)$. For sufficiently large n, we have $|T - T_n| < \delta_0$, which implies

$$w(g_n, t, \delta) = w(f_n, t, \delta),$$

for any $\delta < \delta_0$. Therefore,

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} w(g_n, t, \delta) = 0.$$

$\square$

Remark 5.2.19 *Note that the dynamic acting on* $(\widehat{U}^n, \widehat{V}^n, \widehat{W}^n, \widehat{X}^n)$ *is the same as the one acting on* $(\widehat{\mathcal{U}}^n, \widehat{\mathcal{V}}^n, \widehat{\mathcal{W}}^n, \widehat{\mathcal{X}}^n)$ *till* τ^n_{min}. *Hence, from Corollary 5.2.17, we can find a probability measure on* $\big(D[0,\infty) \times D[0,\infty) \times D[0,\infty) \times D[0,\infty)\big)$ *equipped with topology* $(M_1 \times J_1 \times J_1 \times M_1)$, *and under this probability measure we have*

$$(\widehat{U}^n_{\cdot\wedge\tau^n_{min}}, \widehat{V}^n_{\cdot\wedge\tau^n_{min}}, \widehat{W}^n_{\cdot\wedge\tau^n_{min}}, \widehat{X}^n_{\cdot\wedge\tau^n_{min}}) \longrightarrow (\mathcal{U}^*_{\cdot\wedge\tau_{min}}, \mathcal{V}^*_{\cdot\wedge\tau_{min}}, \mathcal{W}^*_{\cdot\wedge\tau_{min}}, \mathcal{X}^*_{\cdot\wedge\tau_{min}}),$$

almost surely, where (U^*, W^*) *is the split two-variance Brownian motion defined in Corollary 5.1.9.*

Chapter 6

Waiting time between two different renewal states

In Chapter 3 we began with five queues at adjacent price ticks, labeled T^n, U^n, V^n, W^n and X^n. We assumed the initial conditions

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}T^n(0)=t_0>0,\quad \lim_{n\to\infty}\frac{1}{\sqrt{n}}U^n(0)=u_0>0,$$
$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}V^n(0)=0,$$
$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}W^n(0)=w_0<0,\quad \lim_{n\to\infty}\frac{1}{\sqrt{n}}X^n(0)=x_0<0.$$

Therefore, the initial condition for the diffusion-scaled limit processes is

$$T^*(0)=t_0,\quad U^*(0)=u_0,\quad V^*(0)=0,\quad W^*(0)=w_0,\quad X^*(0)=x_0.$$

For the analysis, we replaced T^*, U^*, V^*, W^* and X^* by processes $\mathcal{T}^*,\mathcal{U}^*,\mathcal{V}^*,\mathcal{W}^*$ and $\mathcal{X}^*$ that agree with T^*, U^*, V^*, W^* and X^* until the first time U^* or W^* reaches zero. For the moment, we discuss these processes only until the first time U^* or W^* reaches zero, and hence we dispense with the notation $\mathcal{T}^*,\mathcal{U}^*,\mathcal{V}^*,\mathcal{W}^*$ and $\mathcal{X}^*$.

The results of Chapter 3 show that T^* is snapped to the value κ_L given by (3.5) immediately after time zero, X^* is snapped to κ_R given by (3.6) immediately after time zero (Proposition 3.7.4), and both these processes thereafter remain constant. The process V^* remains at zero (Corollary 3.5.3). We further saw in Corollaries 3.5.1 and 5.1.7 that (U^*, W^*) is a two-dimensional correlated Brownian motion, both components having zero drift. The variance per unit time of U^* is c_+ and the variance per unit time of W^* is c_-.

Eventually, one of the processes U^* or W^* reaches zero. We assumed without loss of generality at the beginning of Chapter 5 that W^* reaches zero before U^*. Resetting the clock, we further assumed at the beginning of Chapter 5 that

$$U^*(0)=u_0>0,\quad V^*(0)=0,\quad W^*(0)=0,\quad X^*(0)=\kappa_R. \tag{6.1}$$

Proceeding from this configuration, we designated U^* and X^* the *bracketing processes* and V^* and W^* the *interior processes*. In Chapter 5, as in Chapter 3, the analysis was aided by considering processes $\mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*$ and $\mathcal{X}^*$ that agree with U^*, V^*, W^* and X^* until the first time one of the bracketing processes reaches zero. We shall do that in this chapter as well, but for the present discussion about the behavior of these processes prior to the time that one of the bracketing processes vanishes, we continue with the notation U^*, V^*, W^* and X^*.

We showed in Chapter 5 that until one of the bracketing processes reaches zero, the pair (V^*, W^*) is a split two-variance Brownian motion (Corollary 5.1.9). In particular, Theorem 5.1.8 provides the existence of a two-variance Brownian motion (Definition 4.1.1)

$$G^* = B^* \circ \left(\frac{1}{c_+} P^+_{B^*} + \frac{1}{c_-} P^-_{B^*} \right)^{-1}, \tag{6.2}$$

where B^* is a standard Brownian motion and

$$P^{\pm}_{B^*}(t) = \int_0^t \mathbb{I}_{\{\pm B^*(s) > 0\}}\, ds,$$

such that

$$(V^*, W^*) = \big(\max\{G^*, 0\}, \min\{G^*, 0\} \big). \tag{6.3}$$

In every interval of time after the initial time and initial state given by (6.1), G^* has both positive and negative excursions away from zero. When G^* is on a negative excursion, we are in the situation studied in Chapter 3, where U^* takes positive values, V^* is at zero, and W^* is negative. In this situation, U^* is a Brownian motion with variance c_+ per unit time. On the other hand, when G^* is on a positive excursion, V^* takes positive values while W^* is zero and X^* is negative. This is the situation studied in Chapter 3 translated right by one price tick, and rather than T^*, now U^* is snapped to κ_L and frozen there. In conclusion, when G^* takes a negative excursion away from zero, the bracketing process U^* has a chance to fall to zero, but if U^* fails to reach zero before the negative excursion of G^* ends, U^* is snapped back to κ_L. Analogously, when G^* takes a positive excursion, the bracketing process X^* has a chance to rise to zero, but if X^* fails to reach zero before the positive excursion of G^* ends, X^* is snapped back to κ_R. On the negative excursions of G^*, the bracketing process X^* is frozen at κ_R, and on the positive excursions of G^*, the bracketing process U^* is frozen at κ_L

We refer to the configuration in (6.1) in which two adjacent queues are at zero as a *renewal state*. In light of the discussion above, the positive value of $U^*(0)$ and the negative value of $X^*(0)$ at the moment when $V^*(0) = W^*(0) = 0$ are irrelevant. Whatever their values at this moment, a moment later $U^*(0)$ will be snapped to κ_L and $X^*(0)$ will be snapped to κ_R.

In this chapter we compute the probability, given the initial condition (6.1), that the next different renewal state reached is when $U^* = V^* = 0$. This probability is provided by (6.75). Let us call this a *leftward renewal state transition*. Of course, one minus the probability of a leftward renewal state transition is the probability that the next renewal state reached is $W^* = X^* = 0$. We call moving to this state a *rightward renewal state transition*. A *renewal state transition* is a leftward or a rightward renewal state transition. In this chapter we compute the characteristic function of the time to see a renewal state transition, conditioned on the renewal state transition being leftward, conditioned on it being rightward, and unconditionally.

At the moment of a renewal state transition, we must change our designation of interior processes and bracketing processes, shifting all of them one tick to the left in the case of a leftward transition or one tick to the right in the case of a rightward transition. The processes then proceed as described above, but with these new designations. However, for the analysis in this chapter, we need processes that agree with U^*, V^*, W^* and X^* up to the time of a renewal state transition, but then continue on without reference to the renewal state transition. For this purpose, we recall that the standard Brownian motion B^* in (6.2) is defined for all time, and that G^* is defined by (6.2) likewise for all time. We define $\mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*$ and $\mathcal{X}^*$ as follows. First, we set

$$(\mathcal{V}^*, \mathcal{W}^*) = \big(\max\{G^*, 0\}, \min\{G^*, 0\}\big). \tag{6.4}$$

Unlike (6.3), which is valid only up to the time of a renewal state transition, (6.4) is valid for all time. Next, we take $\mathcal{U}^*$ and $\mathcal{X}^*$ to be the càdlàg processes constructed in Proposition 5.2.14. In particular $\mathcal{U}^*$ is a Brownian motion with variance c_+ per unit time when G^* is on a negative excursion and is equal to κ_L when G^* is at zero or on a positive excursion. Similarly, $\mathcal{X}^*$ is a Brownian motion with variance c_- per unit time when G^* is on a positive excursion and is equal to κ_R when G^* is at zero or on a negative excursion. These descriptions are valid for all time, not just until the time of a renewal state transition. We define

$$\begin{aligned}
\tau_{\mathcal{U}^*} &:= \inf\{t \geq 0 : \mathcal{U}^*(t) \leq 0\}, \\
\tau_{\mathcal{X}^*} &:= \inf\{t \geq 0 : \mathcal{X}^*(t) \geq 0\}, \\
\tau_{min} &:= \tau_{\mathcal{U}^*} \wedge \tau_{\mathcal{X}^*}.
\end{aligned} \tag{6.5}$$

Then τ_{min} is the time of a renewal state transition, and

$$(U^*, V^*, W^*, X^*) = (\mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*, \mathcal{X}^*)$$

on $[0, \tau_{min}]$. We now undertake the computation of $\mathbb{P}\{\tau_{\mathcal{U}^*} < \tau_{\mathcal{X}^*}\}$, $\mathbb{P}\{\tau_{\mathcal{U}^*} > \tau_{\mathcal{X}^*}\}$, and the joint probability density function of $(\tau_{\mathcal{U}^*}, \tau_{\mathcal{X}^*})$.

6.1 Conditional on the length of the excursion

In this section we compute the probability that $\mathcal{U}^*$ hits zero when G^* is on a negative excursion of length ℓ. Following the same strategy, we can compute the probability that $\mathcal{X}^*$ hits zero when G^* is on a positive excursion of length ℓ, a formula we present without proof. Following the derivation of these probabilities, in Section 6.2 we use P. Lévy's theory of Brownian excursions to remove the conditioning on the length of the excursion of G^* to obtain the desired distribution of $(\tau_{\mathcal{U}^*}, \tau_{X^*})$.

For the computation of this section, we follow the notation of Proposition 5.2.14 in which the negative excursions of G^* are enumerated. The k-th negative excursion is denoted $E_{k,-}$, and its left and right endpoints are denoted $\Lambda_{k,-}$ and $R_{k,-}$, respectively. During the time interval $[\Lambda_{k,-}, R_{k,-}]$, the process $\mathcal{U}^*$ is given by

$$\mathcal{U}^*(t + \Lambda_{k,-}) = \kappa_L + C_{k,-}\big(t\big) - \beta E_{k,-}(t), \quad 0 \le t < R_k,$$

where (see Definition 5.2.6)

$$\beta = -\frac{\lambda_1 + \mu_1}{\mu_0 + a\mu_1}$$

and where $C_{k,-}$ is a Brownian motion independent of G^* with variance $c_+ - \beta^2 c_-$ per unit time. We set

$$D = \kappa_L + C_{k,-} - \beta E_{k,-} \text{ on } [0, R_{k,-} - \Lambda_{k,-}].$$

In this section we fix ℓ, condition on $R_{k,-} - \Lambda_{k,-} = \ell$, and compute the conditional probability that D reaches zero on $[0, \ell]$. The calculation proceeds in steps. In order to avoid consideration of the entrance law for the excursion $E_{k,-}$, we let $\varepsilon \in (0, \ell)$ be given and restrict attention to paths of D that do not hit zero before time ε. On these paths, we condition on $(D(\varepsilon), E(\varepsilon)) = (x, y)$ and show that (D, E) is a correlated two-dimensional Brownian motion conditioned on E first hitting zero at time ℓ. For this analysis, it is helpful to characterize E on $[\varepsilon, \ell]$ as a Brownian motion absorbed at zero at time ℓ. Under these conditions, we can compute the probability that D reaches zero before time ℓ. By this device we not only obtain the probability that D reaches zero before the excursion $E_{k,-}$ ends at time ℓ, but we also obtain the distribution of the time that D reaches zero conditional on doing so before the excursion $E_{k,-}$ ends at time ℓ. Finally, we take ε down to zero in the formulas thus obtained.

Note that on $[0, R_{k,-} - \Lambda_{k,-}]$, we can rewrite D as

$$D(t) = \kappa_L + \kappa C(t) - \beta\sqrt{c_-}E(t),$$

where

$$\kappa := \sqrt{c_+ - \beta^2 c_-},$$

and C is a standard Brownian motion independent of E and E is an excursion of length $R_{k,-} - \Lambda_{k,-}$ of a standard Brownian motion. Let

$$\tau_d := \inf\big\{t \ge 0 \,|\, D(t) = 0\big\}, \tag{6.6}$$

$$\tau_e := \inf\big\{t > 0 \,|\, E(t) = 0\big\}. \tag{6.7}$$

The conditional probability we want to compute is,

$$\mathbb{P}\big\{\tau_d < \tau_e \big| \tau_e = \ell\big\} = \mathbb{P}\big\{\tau_d < \ell \big| \tau_e = \ell\big\}.$$

6.1.1 The computation of conditional probability

We fix ℓ to be the length of the excursion, and let $\mathbb{P}^\ell$ be a probability measure that is restricted to the space of positive excursions of E that have length ℓ. In particular, $\mathbb{P}^\ell$ is defined on $\mathcal{W}^+ \cap \{w : \beta(w) = \ell\}$, where $\mathcal{W}^+$ is the space of positive excursions of E and $\beta(w)$ denotes the length of excursion w.

We take $\mathbb{W}$ to Wiener measure on $C[0,\infty)$, and define $\mathbb{Q}^\ell = \mathbb{P}^\ell \times \mathbb{W}$, a probability measure on $(\mathcal{W}^+ \cap \{w \,|\, \beta(w) = \ell\}) \times C[0,\infty)$. Since C is independent of E, we redefine $D\colon (\mathcal{W}^+ \cap \{w \,|\, \beta(w) = \ell\}) \times C[0,\infty) \to C[0,\infty)$ and $E\colon (\mathcal{W}^+ \cap \{w \,|\, \beta(w) = \ell\}) \times C[0,\infty) \to C[0,\infty)$ by

$$
\begin{aligned}
D(w_1, w_2) &:= \frac{\lambda_2 \mu_1}{\theta_b \lambda_1} + \kappa w_2 - \beta\sqrt{c_-} w_1, && (6.8)\\
E(w_1, w_2) &:= w_1. && (6.9)
\end{aligned}
$$

Then we rewrite (6.6) as

$$\tau_d(w_1, w_2) := \inf\big\{t \ge 0 : D(w_1, w_2)(t) = 0\big\},$$

and the probability we wish to compute is

$$\mathbb{Q}^\ell\{\tau_d < \ell\}. \tag{6.10}$$

Instead of computing this probability directly, we consider the probability that D hits zero before the excursion ends under the condition that we are already in the middle of the excursion and D has not hit zero so far. In particular, fix $\varepsilon \in (0, \ell)$, and define

$$
\begin{aligned}
A_\varepsilon &:= \{\tau_d > \varepsilon\},\\
\tau_{d,\varepsilon}(w_1, w_2) &:= \begin{cases} \inf\big\{t > \varepsilon \,|\, D(w_1, w_2)(t) \le 0\big\} & \text{if}(w_1, w_2) \in A_\varepsilon,\\ +\infty & \text{if}(w_1, w_2) \in A_\varepsilon^c. \end{cases}
\end{aligned}
$$

Since $\mathbb{Q}^\ell\{A_\varepsilon\} \uparrow 1$ as $\varepsilon \downarrow 0$, and

$$
\begin{aligned}
\mathbb{Q}^\ell\{\tau_d < \ell\} &= \mathbb{Q}^\ell\{(\tau_d < \ell) \cap A_\varepsilon\} + \mathbb{Q}^\ell\{(\tau_d < \ell) \cap A_\varepsilon^c\}\\
&= \mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell\} + \mathbb{Q}^\ell\{A_\varepsilon^c\},
\end{aligned}
$$

we have

$$\mathbb{Q}^\ell\{\tau_d < \ell\} = \lim_{\varepsilon \downarrow 0} \mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell\}. \tag{6.11}$$

Thus, it suffices to compute $\mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell\}$. Let us consider the pair of processes (E, D). At time ε, from (6.8) and (6.9), we have

$$
\begin{aligned}
E(w_1, w_2)(\varepsilon) &= w_1(\varepsilon) > 0,\\
D(w_1, w_2)(\varepsilon) &= \frac{\lambda_2 \mu_1}{\theta_b \lambda_1} + \kappa w_2(\varepsilon) - \beta\sqrt{c_-} w_1(\varepsilon).
\end{aligned}
$$

Therefore, we can compute $\mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell\}$ through

$$\mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell\big\} = \int_{y=-\infty}^{\infty}\int_{x=0}^{\infty} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell \big| E(\varepsilon) = x, D(\varepsilon) = y\big\}\mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\}. \tag{6.12}$$

Note that $D(\varepsilon) \le 0$ implies D already hits zero before time ε, which means $\tau_{d,\varepsilon} = +\infty$. We can upgrade (6.12) to

$$\mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell\big\} = \int_{y=0}^{\infty}\int_{x=0}^{\infty} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell \big| E(\varepsilon) = x, D(\varepsilon) = y\big\}\mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\}. \tag{6.13}$$

So our job is to compute

$$\mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\}, \tag{6.14}$$

and

$$\mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell \big| E(\varepsilon) = x, D(\varepsilon) = y\big\}, \tag{6.15}$$

for $x > 0$ and $y > 0$. In the remainder of this subsection, we compute (6.14). We compute (6.15) in Section 6.1.4 (see (6.37), (6.38) and (6.43)).

According to the formula at the bottom page 124 of Ikeda & Watanabe, we know

$$\mathbb{P}^\ell\big\{E(\varepsilon) \in dx\big\} = \sqrt{\frac{\pi}{2}\ell^3}\, K^+(\varepsilon, x)K^+(\ell - \varepsilon, x)\, dx, \tag{6.16}$$

where

$$K^+(t, x) = \sqrt{\frac{2}{\pi t^3}}\, x e^{-\frac{x^2}{2t}}, \quad t > 0, x \ge 0.$$

We also have

$$\mathbb{W}\big\{w_2(\varepsilon) \in dz\big\} = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{z^2}{2\varepsilon}}\, dz. \tag{6.17}$$

Note that

$$\begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} w_1 \\ \frac{\lambda_2\mu_1}{\theta_b\lambda_1} + \kappa w_2 - \beta\sqrt{c_-}w_1 \end{pmatrix}, \tag{6.18}$$

which implies,

$$D \le y \Leftrightarrow w_2 \le \frac{1}{\kappa}(y - \frac{\lambda_2\mu_1}{\theta_b\lambda_1} + \beta\sqrt{c_-}w_1).$$

So we have,

$$\begin{aligned}
&\mathbb{Q}^\ell\big\{E(\varepsilon) \le x, D(\varepsilon) \le y\big\} \\
&= \mathbb{Q}^\ell\big\{w_1(\varepsilon) \le x, w_2(\varepsilon) \le \frac{1}{\kappa}(y - \frac{\lambda_2\mu_1}{\theta_b\lambda_1} + \beta\sqrt{c_-}w_1(\varepsilon))\big\} \\
&= \int_0^x \int_{-\infty}^{\frac{1}{\kappa}(y - \frac{\lambda_2\mu_1}{\theta_b\lambda_1} + \beta\sqrt{c_-}u)} \mathbb{P}^\ell\big\{w_1(\varepsilon) \in du\big\}\mathbb{W}\big\{w_2(\varepsilon) \in dv\big\} \\
&= \int_0^x \sqrt{\frac{\pi}{2}\ell^3} K^+(\varepsilon, u)K^+(\ell - \varepsilon, u) \left(\int_{-\infty}^{\frac{1}{\kappa}(y - \frac{\lambda_2\mu_1}{\theta_b\lambda_1} + \beta\sqrt{c_-}u)} \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{v^2}{2\varepsilon}}\, dv\right) du
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{Q}^\ell\big\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\big\}\\
&= \left(\frac{d}{dy}\frac{d}{dx}(\mathbb{Q}^\ell\big\{E(\varepsilon)\le x, D(\varepsilon)\le y\big\})\right)dxdy\\
&= \frac{1}{\kappa}\sqrt{\frac{\pi}{2}}\ell^3 K^+(\varepsilon,x)K^+(\ell-\varepsilon,x)\frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{(\frac{1}{\kappa}(y-\frac{\lambda_2\mu_1}{\theta_b\lambda_1}+\beta\sqrt{c_-}x))^2}{2\varepsilon}}dxdy\\
&= \frac{1}{\kappa\pi\varepsilon^2}\sqrt{(\frac{\ell}{\ell-\varepsilon})^3}x^2e^{-\frac{x^2}{2\varepsilon}-\frac{x^2}{2(\ell-\varepsilon)}-\frac{(y-\frac{\lambda_2\mu_1}{\theta_b\lambda_1}+\beta\sqrt{c_-}x)^2}{2\kappa^2\varepsilon}}dxdy. \qquad (6.19)
\end{aligned}$$

Remark 6.1.1 *Let B_δ be the ball centered at $(0,\frac{\lambda_2\mu_1}{\theta_b\lambda_1})$ with radius $\delta>0$, then*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{(x>0,y>0)\cap B_\delta^c}\mathbb{Q}^\ell\big\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\big\}\\
&= \int_{(x>0,y>0)\cap B_\delta^c}\lim_{\varepsilon\to 0}\mathbb{Q}^\ell\big\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\big\}\\
&= 0, \qquad (6.20)
\end{aligned}$$

where we have used the Monotone Convergence Theorem. Moreover, we claim that

$$\lim_{\varepsilon\to 0}\int_{(x>0,y>0)}\mathbb{Q}^\ell\big\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\big\}=1. \qquad (6.21)$$

This is true because

$$\int_{x=0}^{\infty}\int_{y=-\infty}^{\infty}\mathbb{Q}^\ell\big\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\big\}=1,$$

and

$$\begin{aligned}
&\mathbb{P}\{(E,D) \text{ reaches lower half of the plane at time } \varepsilon\}\\
&\quad= \int_{x=0}^{\infty}\int_{y=-\infty}^{0}\mathbb{Q}^\ell\big\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\big\}.
\end{aligned}$$

Obviously,

$$\lim_{\varepsilon\to 0}\mathbb{P}\{(E,D) \text{ reaches lower half of the plane at time } \varepsilon\}=0.$$

6.1.2 Absorbed Brownian motion

We will show that the excursion E after time ε, assuming the length of the excursion is greater than ε, is a Brownian motion absorbed at the origin. In this section, we derive the properties of absorbed Brownian motion.

Brownian motion absorbed at origin is defined to be

$$B^0(t) = B(t \wedge \tau_0),$$

where B is a standard Brownian motion starting at $B(0) = x_0 \geq 0$ and

$$\tau_0 = \inf \{ t \geq 0 : B(t) = 0 \}.$$

This a Markov process. Strictly speaking it does not have a transition density because mass accumulates at zero. However, it has a *defective* transition density, were *defective* refers to the fact that it does not integrate to 1. This defective transition density is

$$p^0(t,x,y) = \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{(y-x)^2}{2t} \right) - \exp\left(-\frac{(y+x)^2}{2t} \right) \right], \quad x > 0, y > 0, t > 0.$$

In fact, from Appendix B, we have

$$\int_0^\infty p^0(t,x,y)\,dy = 1 - \frac{2}{\sqrt{2\pi t}} \int_x^\infty \exp\left(-\frac{z^2}{2t} \right) dz. \tag{6.22}$$

According to the reflection principle, for $x > 0$,

$$\begin{aligned} \mathbb{P}\{ B^0(t+s) = 0 \big| B^0(s) = x \} &= \mathbb{P}\{ \tau_0 \leq t+s \big| B^0(s) = x \} \\ &= \frac{2}{\sqrt{2\pi t}} \int_x^\infty \exp\left(-\frac{z^2}{2t} \right) dz \\ &= 1 - \int_0^\infty p^0(t,x,y)\,dy. \end{aligned} \tag{6.23}$$

For a Borel subset A of $(0, \infty)$ and $x > 0$,

$$\mathbb{P}\{ B^0(t+s) \in A \big| B^0(s) = x \} = \int_A p^0(t,x,y)\,dy. \tag{6.24}$$

Of course,

$$\mathbb{P}\{ B^0(t+s) = 0 \big| B^0(s) = 0 \} = 1. \tag{6.25}$$

Equations (6.23)–(6.25) provide the transition probabilities for B^0.

The defective transition density p^0 has the semigroup property

$$\int_0^\infty p^0(s,x,y) p^0(t,y,z)\,dy = p^0(s+t,x,z), \quad s > 0, t > 0, x > 0, z > 0, \tag{6.26}$$

which we establish by direct calculation in Appendix B.

6.1.3 Excursion starting at time ε

Now let $t_1 < t_2 < \cdots < t_n < T_1 < T_2$ be given and let $A_2, \ldots, A_n$ be Borel subsets of $(0, \infty)$. For $x_1 > 0$, we have

$$\begin{aligned}
&\mathbb{P}\big\{B^0(t_2) \in A_2, \ldots, B^0(t_n) \in A_n, T_1 < \tau_0 \le T_2 \big| B^0(t_1) = x_1\big\} \\
&\quad = \mathbb{P}\big\{B^0(t_2) \in A_2, \ldots, B^0(t_n) \in A_n, B^0(T_1) > 0, B^0(T_2) = 0 \big| B^0(t_1) = x_1\big\} \\
&\quad = \int_{x_2 \in A_2} \cdots \int_{x_n \in A_n} \int_{x=0}^{\infty} p^0(t_2 - t_1, x_1, x_2) \cdots p^0(t_n - t_{n-1}, x_{n-1}, x_n) p^0(T_1 - t_n, x_n, x) \\
&\qquad\qquad \times \left(1 - \int_0^\infty p^0(T_2 - T_1, x, y)\, dy\right) dx\, dx_n \cdots dx_2. \qquad (6.27)
\end{aligned}$$

We consider the inner-most integral in (6.27),

$$\begin{aligned}
&\int_{x=0}^{\infty} p^0(T_1 - t_n, x_n, x) \left(1 - \int_0^\infty p^0(T_2 - T_1, x, y)\, dy\right) dx \\
&\quad = \int_0^\infty p^0(T_1 - t_n, x_n, x)\, dx - \int_0^\infty \int_0^\infty p^0(T_1 - t_n, x_n, x) p^0(T_2 - T_1, x, y)\, dx\, dy \\
&\quad = \int_0^\infty p^0(T_1 - t_n, x_n, x)\, dx - \int_0^\infty p^0(T_2 - t_n, x_n, y)\, dy \\
&\quad = \frac{2}{\sqrt{2\pi(T_2 - t_n)}} \int_{x_n}^\infty \exp\left(-\frac{z^2}{2(T_2 - t_n)}\right) dz \\
&\qquad\qquad - \frac{2}{\sqrt{2\pi(T_1 - t_n)}} \int_{x_n}^\infty \exp\left(-\frac{z^2}{2(T_1 - t_n)}\right) dz \\
&\quad = h(T_2 - t_n, x_n) - f(T_1 - t_n, x_n), \qquad (6.28)
\end{aligned}$$

where we have used the semigroup property (6.26), equation (6.22), and where

$$h(t, x) = \frac{2}{\sqrt{2\pi t}} \int_x^\infty \exp\left(-\frac{z^2}{2t}\right) dz = \frac{2}{\sqrt{2\pi}} \int_{x/\sqrt{t}}^\infty \exp\left(-\frac{w^2}{2}\right) dw. \qquad (6.29)$$

We substitute (6.28) into (6.27) to obtain

$$\begin{aligned}
&\mathbb{P}\big\{B^0(t_2) \in A_2, \ldots, B^0(t_n) \in A_n, T_1 < \tau_0 \le T_2 \big| B^0(t_1) = x_1\big\} \\
&\quad = \int_{x_2 \in A_2} \cdots \int_{x_n \in A_n} p^0(t_2 - t_1, x_1, x_2) \cdots p^0(t_n - t_{n-1}, x_{n-1}, x_n) \\
&\qquad\qquad \times \big(h(T_2 - t_n, x_n) - f(T_1 - t_n, x_n)\big)\, dx_n \cdots dx_2. \qquad (6.30)
\end{aligned}$$

Because

$$\frac{\partial}{\partial t} h(t, x) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right), \qquad (6.31)$$

we can also write (6.30) as

$$\begin{aligned}
&\mathbb{P}\big\{B^0(t_2)\in A_2,\ldots,B^0(t_n)\in A_n, T_1<\tau_0\le T_2\big|B^0(t_1)=x_1\big\}\\
&\quad= \int_{x_2\in A_2}\cdots\int_{x_n\in A_n} p^0(t_2-t_1,x_1,x_2)\cdots p^0(t_n-t_{n-1},x_{n-1},x_n)\\
&\qquad\qquad\times\int_{T_1}^{T_2}\frac{x_n}{\sqrt{2\pi(\ell-t_n)^3}}\exp\left(-\frac{x_n^2}{2(\ell-t_n)^3}\right)d\ell\,dx_n\cdots dx_2\\
&\quad= \int_{T_1}^{T_2}\int_{x_2\in A_2}\cdots\int_{x_n\in A_n} p^0(t_2-t_1,x_1,x_2)\cdots p^0(t_n-t_{n-1},x_{n-1},x_n)\\
&\qquad\quad\times\sqrt{\frac{(\ell-t_1)^3}{(\ell-t_n)^3}}\cdot\frac{x_n}{x_1}\exp\left(-\frac{x_n^2}{2(\ell-t_n)}+\frac{x_1^2}{2(\ell-t_1)}\right)dx_n\cdots dx_2\\
&\qquad\quad\times\frac{x_1}{\sqrt{2\pi(\ell-t_1)^3}}\exp\left(-\frac{x_1^2}{2(\ell-t_1)}\right)d\ell\\
&\quad= \int_{T_1}^{T_2}\int_{x_2\in A_2}\cdots\int_{x_n\in A_n} p^0(t_2-t_1,x_1,x_2)\cdots p^0(t_n-t_{n-1},x_{n-1},x_n)\\
&\qquad\quad\times\sqrt{\frac{(\ell-t_1)^3}{(\ell-t_n)^3}}\cdot\frac{x_n}{x_1}\exp\left(-\frac{x_n^2}{2(\ell-t_n)}+\frac{x_1^2}{2(\ell-t_1)}\right)dx_n\cdots dx_2\\
&\qquad\quad\times\mathbb{P}\big\{\tau_0\in d\ell\big|B^0(t_1)=x_1\big\}.
\end{aligned}\tag{6.32}$$

We also have

$$\begin{aligned}
&\mathbb{P}\big\{B^0(t_2)\in A_2,\ldots,B^0(t_n)\in A_n, T_1<\tau_0\le T_2\big|B^0(t_1)=x_1\big\}\\
&\quad= \int_{T_1}^{T_2}\mathbb{P}\big\{B^0(t_2)\in A_2,\ldots,B^0(t_n)\in A_n\big|\tau_0=\ell,B^0(t_1)=x_1\big\}\mathbb{P}\big\{\tau_0\in d\ell\big|B^0(t_1)=x_1\big\}.
\end{aligned}$$

Since $\{A_i\}_{i=1}^n$ are arbitrary Borel sets, with (6.32), we conclude that

$$\begin{aligned}
&\mathbb{P}\big\{B^0(t_2)\in A_2,\ldots,B^0(t_n)\in A_n\big|\tau_0=\ell,B^0(t_1)=x_1\big\}\\
&\quad= \int_{x_2\in A_2}\cdots\int_{x_n\in A_n} p^0(t_2-t_1,x_1,x_2)\cdots p^0(t_n-t_{n-1},x_{n-1},x_n)\\
&\qquad\qquad\times\sqrt{\frac{(\ell-t_1)^3}{(\ell-t_n)^3}}\cdot\frac{x_n}{x_1}\exp\left(-\frac{x_n^2}{2(\ell-t_n)}+\frac{x_1^2}{2(\ell-t_1)}\right)dx_n\cdots dx_2.
\end{aligned}\tag{6.33}$$

We can write this more intuitively as

$$\begin{aligned}
&\mathbb{P}\big\{B^0(t_2)\in dx_2,\ldots,B^0(t_n)\in dx_n\big|\tau_0=\ell,B^0(t_1)=x_1\big\}\\
&\quad= p^0(t_2-t_1,x_1,x_2)\cdots p^0(t_n-t_{n-1},x_{n-1},x_n)\\
&\qquad\qquad\times\sqrt{\frac{(\ell-t_1)^3}{(\ell-t_n)^3}}\cdot\frac{x_n}{x_1}\exp\left(-\frac{x_n^2}{2(\ell-t_n)}+\frac{x_1^2}{2(\ell-t_1)}\right)dx_2\cdots dx_n.
\end{aligned}\tag{6.34}$$

From page 124 of Ikeda & Watanabe, we see that $\mathbb{P}^\ell$ appearing in Section 6.1.1 satisfies

$$\begin{aligned}
&\mathbb{P}^\ell\big\{w(t_2) \in dx_2, \ldots, w(t_n) \in dx_n \big| w(t_1) = x_1\big\} \\
&= \frac{h(0,0;t_1,x_1)h(t_1,x_1;t_2,x_2)\cdots h(t_{n-1},x_{n-1};t_n,x_n)\,dx_1\,dx_2\cdots dx_n}{h(0,0;t_1,x_1)\,dx_1} \\
&= \frac{K^+(\ell-t_2,x_2)}{K^+(\ell-t_1,x_1)}p^0(t_2-t_1,x_1,x_2)\frac{K^+(\ell-t_3,x_3)}{K^+(\ell-t_2,x_2)}p^0(t_3-t_2,x_2,x_3)\cdots \\
&\qquad \cdots\frac{K^+(\ell-t_n,x_n)}{K^+(\ell-t_{n-1},x_{n-1})}p^0(t_n-t_{n-1},x_{n-1},x_n)\,dx_2\cdots dx_n \\
&= p^0(t_2-t_1,x_1,x_2)p^0(t_3-t_2,x_2,x_3)\cdots p^0(t_n-t_{n-1},x_{n-1}x_n) \\
&\qquad \times\frac{K^+(\ell-t_n,x_n)}{K^+(\ell-t_1,x_1)}\,dx_2\cdots dx_3 \\
&= \mathbb{P}\big\{B^0(t_2)\in dx_2,\ldots,B^0(t_n)\in dx_n\big|\tau_0=\ell, B^0(t_1)=x_1\big\} \qquad (6.35)
\end{aligned}$$

as in (6.34). We conclude that under the measure $\mathbb{P}^\ell$, once we condition on $E(\varepsilon) = x$, the distribution of $E(t), \varepsilon \le t \le \ell$, is the same as a Brownian motion absorbed at zero conditioned on taking the value x at time ε and first reaching zero at time ℓ.

6.1.4 Replacing (E, D) by correlated Brownian motions

Our next goal is to compute $\mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell | E(\varepsilon) = x, D(\varepsilon) = y\}$. According to our observation from the previous two sections, we can replace (E, D) by two correlated Brownian Motions starting from (x, y). In particular, we can consider a Brownian motion B_1 with initial condition

$$B_1(0) = x,$$

and variance 1 per unit time. We have just seen that conditional on $E(\varepsilon) = x$, the law of $E(\varepsilon+t)$, $0 \le t \le \ell-\varepsilon$, under $\mathbb{P}^\ell$ is the same as the law of $B_1(t)$, $0 \le t \le \ell-\varepsilon$, conditioned on B_1 first reaching zero at time $\ell - \varepsilon$. We consider a second Brownian motion B_2 with initial condition

$$B_2(0) = y,$$

with variance c_+ per unit time, and such that

$$d\langle B_1, B_2\rangle(t) = -\beta\sqrt{c_-}\,dt,$$

or equivalently,

$$\mathrm{Corr}(B_1, B_2) = \frac{-\beta\sqrt{c_-}}{\sqrt{c_+}} := \rho < 0. \qquad (6.36)$$

From subsection 6.1.2 and 6.1.3, conditional on $E(\varepsilon) = x$ and $D(\varepsilon) = y$, the law of

$$\big(E(\varepsilon+t), D(\varepsilon+t)\big), \quad 0 \le t \le \ell-\varepsilon,$$

under $\mathbb{Q}^\ell$ is the same as the law of

$$\big(B_1(t), B_2(t)\big), \quad 0 \le t \le \ell-\varepsilon,$$

conditional on B_1 first reaching zero at time $\ell-\varepsilon$. For convenience, let (B_1, B_2) to be defined for all nonnegative times, and set

$$\begin{aligned}\tau_1 &:= \inf\{t \geq 0 : B_1(t) = 0\},\\ \tau_2 &:= \inf\{t \geq 0 : B_2(t) = 0\}.\end{aligned}$$

Then

$$\mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell \big| E(\varepsilon) = x, D(\varepsilon) = y\} = \mathbb{P}^{x,y}\{\tau_2 < \ell-\varepsilon \big| \tau_1 = \ell-\varepsilon, \tau_d > \varepsilon\}, \tag{6.37}$$

where $\mathbb{P}^{x,y}$ is the probability measure on the two-dimensional Brownian motions (B_1, B_2) starting at (x, y). We have

$$\mathbb{P}^{x,y}\{\tau_2 < \ell-\varepsilon \big| \tau_1 = \ell-\varepsilon, \tau_d > \varepsilon\} = \frac{\int_0^{\ell-\epsilon} f(\ell-\varepsilon, t_2)dt_2}{p(x;\ell-\varepsilon)}, \tag{6.38}$$

where $f(s,t)$ is the joint density of (τ_1, τ_2) and $p(x;\ell-\varepsilon)$ is the density of the first passage time of a Brownian Motion starting from x to zero, so we have

$$p(x;\ell-\epsilon) = \frac{x}{\sqrt{2\pi(\ell-\varepsilon)^3}} e^{-\frac{x^2}{2(\ell-\varepsilon)}} \tag{6.39}$$

Then it suffices to evaluate $\int_0^{\ell-\epsilon} f(\ell-\epsilon, t_2)dt_2$.

The first step is to find a transformation that transforms (B_1, B_2) into a pair of independent Brownian Motions. To this end, we define,

$$\Xi := \frac{1}{\sqrt{c_+}\sqrt{1-\rho^2}} \begin{pmatrix} \sqrt{c_+} & -\rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix}. \tag{6.40}$$

Then we define linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(B_1, B_2) = \Xi \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} := Z.$$

So

$$\begin{aligned} Z_1 &= \frac{1}{\sqrt{1-\rho^2}} B_1 - \frac{\rho}{\sqrt{c_+}\sqrt{1-\rho^2}} B_2, \\ Z_2 &= \frac{1}{\sqrt{c_+}} B_2. \end{aligned} \tag{6.41}$$

We can easily verify (Z_1, Z_2) are independent Brownian Motions. Moreover, the horizontal axis-B_1 is invariant under transformation, while the vertical axis-B_2 is mapped to the line $z_2 = \tan(\alpha)z_1$, where $\alpha = \tan^{-1}(-\frac{\sqrt{1-\rho^2}}{\rho})$. Also, the initial point (x, y) is mapped to $z_0 := (z_0^1, z_0^2)$ whose polar coordinates are given by

$$\begin{aligned} r_0 &:= \sqrt{\frac{a_1^2 + a_2^2 - 2\rho a_1 a_2}{1-\rho^2}}, \\ \theta_0 &:= \tan^{-1}(\frac{a_2\sqrt{1-\rho^2}}{a_1 - \rho a_2}), \end{aligned} \tag{6.42}$$

where $a_1 = x$, and $a_2 = \frac{y}{\sqrt{c_+}}$. As such we define,

$$\begin{aligned} \eta_2 &:= \text{The first passage time of } Z(t) \text{ to the horizontal axis.} \\ \eta_1 &:= \text{The first passage time of } Z(t) \text{ to the line } z_2 = \tan(\alpha) z_1. \end{aligned}$$

Obviously, after linear transformation T, we have

$$\begin{aligned} \tau_1 &= \eta_1, \\ \tau_2 &= \eta_2. \end{aligned}$$

According to Page 282 of [23], the joint density of (η_1, η_2) in the region where $\eta_2 < \eta_1$ is given by,

$$\begin{aligned} f(s,t) &= \frac{\pi \sin\alpha}{2\alpha^2 \sqrt{t(s - t\cos^2\alpha)}(s-t)} \exp\left(-\frac{r_0^2}{2t} \frac{s - t\cos(2\alpha)}{(s-t) + (s - t\cos(2\alpha))}\right) \\ & \quad \sum_{n=1}^{\infty} n \sin(\frac{n\pi\theta_0}{\alpha}) I_{\frac{n\pi}{2\alpha}} \left(\frac{r_0^2}{2t} \frac{s-t}{(s-t) + (s - t\cos(2\alpha))}\right), \quad 0 < t < s, \end{aligned} \tag{6.43}$$

where I_ν denotes the modified Bessel function of the first kind of order ν. Now we have all ingredients we need to compute (6.13). Before the computation, we observe that

$$\begin{aligned} \mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell\} &= \int_{y=0}^{\infty} \int_{x=0}^{\infty} \mathbb{P}^{x,y}\{\tau_2 < \ell - \varepsilon | \tau_1 = \ell - \varepsilon, \tau_d > \varepsilon\} \mathbb{Q}^\ell\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\} \\ &= \int_{y=0}^{\infty} \int_{x=0}^{\infty} \int_0^{\ell - \varepsilon} \frac{f(\ell - \varepsilon, t)}{p(x; \ell - \varepsilon)} dt \mathbb{Q}^\ell\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\}, \end{aligned} \tag{6.44}$$

Since $f(\ell - \varepsilon, t) \to +\infty$ as $t \to \ell - \varepsilon$, in order to get away from the singularity, we want to compute $\mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell - \delta\}$ for some $\delta > 0$, and then get $\mathbb{Q}^\ell\{\tau_d < \ell\}$ by sending δ to zero. In particular,

$$\mathbb{Q}^\ell\{\tau_d < \ell\} = \lim_{\delta \to 0} \mathbb{Q}^\ell\{\tau_d < \ell - \delta\} = \lim_{\delta \to 0} \left(\lim_{\varepsilon \to 0} \mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell - \delta\}\right). \tag{6.45}$$

Then our goal is to compute

$$\lim_{\varepsilon \to 0} \mathbb{Q}^\ell\{\tau_{d,\varepsilon} < \ell - \delta\} = \lim_{\varepsilon \to 0} \int_{y=0}^{\infty} \int_{x=0}^{\infty} \int_0^{\ell - \varepsilon - \delta} \frac{f(\ell - \varepsilon, t)}{p(x; \ell - \varepsilon)} dt \mathbb{Q}^\ell\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\}. \tag{6.46}$$

Note that

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \Xi^{-1} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2} Z_1 + \rho Z_2 \\ \sqrt{c_+} Z_2 \end{pmatrix}.$$

Then under (Z_1, Z_2)-coordinates, we can rewrite (6.39) as

$$p(x; \ell - \varepsilon) = \frac{(\sqrt{1-\rho^2} z_0^1 + \rho z_0^2)}{\sqrt{2\pi(\ell - \varepsilon)^3}} e^{-\frac{(\sqrt{1-\rho^2} z_0^1 + \rho z_0^2)^2}{2(\ell - \varepsilon)}}. \tag{6.47}$$

Then we switch to polar coordinates, where $z_0^1 = r_0 \cos\theta_0$ and $z_0^2 = r_0 \sin\theta_0$, and (6.47) becomes

$$\begin{aligned} p(x; \ell - \varepsilon) &= \frac{(\sqrt{1-\rho^2} r_0 \cos\theta_0 + \rho r_0 \sin\theta_0)}{\sqrt{2\pi(\ell-\varepsilon)^3}} e^{-\frac{(\sqrt{1-\rho^2} r_0 \cos\theta_0 + \rho r_0 \sin\theta_0)^2}{2(\ell-\varepsilon)}} \\ &=: p(r_0, \theta_0; \ell - \varepsilon). \end{aligned} \tag{6.48}$$

Lemma 6.1.2 *Suppose* $\lim_{\varepsilon\to 0}\left(\lim_{x\to 0, y\to \frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\}\right)$ *exists. Then we have following,*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{y=0}^{\infty}\int_{x=0}^{\infty} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} \mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\} \\ &= \lim_{\varepsilon\to 0}\left(\lim_{x\to 0, y\to \frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\}\right). \end{aligned} \tag{6.49}$$

Proof: Let B_ζ be the ball centered at $(0, \frac{\lambda_2\mu_1}{\theta_b\lambda_1})$ with radius $\zeta > 0$. According to (6.20) and (6.21), we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{(x>0,y>0)\cap B_\zeta^c} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \zeta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} \mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\} \\ &\le \lim_{\varepsilon\to 0} \int_{(x>0,y>0)\cap B_\zeta^c} \mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\} = 0, \end{aligned} \tag{6.50}$$

and

$$\lim_{\varepsilon\to 0} \int_{(x>0,y>0)\cap B_\zeta} \mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\} = 1. \tag{6.51}$$

Thus from (6.50)

$$\begin{aligned} A &:= \lim_{\varepsilon\to 0} \int_{y=0}^{\infty}\int_{x=0}^{\infty} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} \mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\} \\ &= \lim_{\varepsilon\to 0} \int_{(x>0,y>0)\cap B_\zeta} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} \mathbb{Q}^\ell\big\{E(\varepsilon) \in dx, D(\varepsilon) \in dy\big\} \end{aligned} \tag{6.52}$$

One can check that $\mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\}$ is continuous with respect to (x, y). Then $\forall \gamma > 0$, $\exists \xi > 0$ such that $\forall (x, y) \in B_\xi$

$$\begin{aligned} &\lim_{x\to 0, y\to \frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} - \gamma \\ &\le \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} \\ &\le \lim_{x\to 0, y\to \frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\big\{\tau_{d,\varepsilon} < \ell - \delta \big| E(\varepsilon) = x, D(\varepsilon) = y\big\} + \gamma, \end{aligned} \tag{6.53}$$

which implies

$$\begin{aligned}
&\left(\lim_{x\to 0, y\to\frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\} - \gamma\right) \int_{(x>0,y>0)\cap B_\zeta} \mathbb{Q}^\ell\left\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\right\} \\
\leq\ & \int_{(x>0,y>0)\cap B_\zeta} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\} \mathbb{Q}^\ell\left\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\right\} \\
\leq\ & \left(\lim_{x\to 0, y\to\frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\} + \gamma\right) \int_{(x>0,y>0)\cap B_\zeta} \mathbb{Q}^\ell\left\{E(\varepsilon)\in dx, D(\varepsilon)\in dy\right\}.
\end{aligned} \tag{6.54}$$

Taking the limit as $\varepsilon \to 0$, using (6.51), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \left(\lim_{x\to 0, y\to\frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\} - \gamma\right) \\
\leq\ & A \leq \lim_{\varepsilon\to 0} \left(\lim_{x\to 0, y\to\frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\} + \gamma\right).
\end{aligned} \tag{6.55}$$

Finally, send $\gamma \to 0$, we get

$$A = \lim_{\varepsilon\to 0} \left(\lim_{x\to 0, y\to\frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\}\right). \tag{6.56}$$

$\square$

Lemma 6.1.3 *The limit*

$$\lim_{\varepsilon\to 0} \left(\lim_{x\to 0, y\to\frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^\ell\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\}\right)$$

exists.

PROOF: The starting point $(0, \frac{\lambda_2\mu_1}{\theta_b\lambda_1})$ is mapped to

$$\begin{pmatrix} z_s^1 \\ z_s^2 \end{pmatrix} := \Xi \begin{pmatrix} 0 \\ \frac{\lambda_2\mu_1}{\theta_b\lambda_1} \end{pmatrix} = \begin{pmatrix} -\frac{\rho}{\sqrt{c_+}\sqrt{1-\rho^2}}\frac{\lambda_2\mu_1}{\theta_b\lambda_1} \\ \frac{1}{\sqrt{c_+}}\frac{\lambda_2\mu_1}{\theta_b\lambda_1} \end{pmatrix}, \tag{6.57}$$

whose polar coordinates representation is (r_s, α), where

$$r_s = \frac{\lambda_2\mu_1}{\theta_b\lambda_1\sqrt{c_+}\sqrt{1-\rho^2}}.$$

Therefore we have following,

$$\lim_{x\to 0, y\to \frac{\lambda_2\mu_1}{\theta_b\lambda_1}} \mathbb{Q}^{\ell}\left\{\tau_{d,\varepsilon} < \ell-\delta \middle| E(\varepsilon)=x, D(\varepsilon)=y\right\}$$
$$= \lim_{r_0\to r_s,\theta_0\to\alpha} \int_0^{\ell-\varepsilon-\delta} \frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)}dt \qquad (6.58)$$

Substituting (6.43) and (6.48) into (6.58), we have

$$\lim_{r_0\to r_s,\theta_0\to\alpha} \frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)} = \lim_{r_0\to r_s,\theta_0\to\alpha} \frac{\pi\sin\alpha}{2\alpha^2\sqrt{t((\ell-\varepsilon)-t\cos^2\alpha)}(\ell-\varepsilon-t)}$$
$$\exp\left(-\frac{r_0^2}{2t}\frac{(\ell-\varepsilon)-t\cos(2\alpha)}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right)$$
$$\frac{\sum_{n=1}^{\infty} n\sin(\frac{n\pi\theta_0}{\alpha}) I_{\frac{n\pi}{2\alpha}}\left(\frac{r_0^2}{2t}\frac{(\ell-\varepsilon)-t}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right)}{\frac{(\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)}{\sqrt{2\pi(\ell-\varepsilon)^3}}e^{-\frac{(\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)^2}{2(\ell-\varepsilon)}}} \qquad (6.59)$$

When $\theta_0\to\alpha$, $\sin(\frac{n\pi\theta_0}{\alpha})\to 0$. We also see that $\tan\alpha = -\frac{\sqrt{1-\rho^2}}{\rho}$, which implies

$$(\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)\to 0.$$

In order to apply Dominated Convergence Theorem, we are going to show $\frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)}$ is bounded for θ_0 in a neighborhood of α and r_0 in a neighborhood of r_s on $t\in(0,\ell-\varepsilon-\delta)$. First, we show

$$\frac{\pi\sin\alpha}{2\alpha^2\sqrt{t((\ell-\varepsilon)-t\cos^2\alpha)}(\ell-\varepsilon-t)}\exp\left(-\frac{r_0^2}{2t}\frac{(\ell-\varepsilon)-t\cos(2\alpha)}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right)$$
$$\times\sqrt{2\pi(\ell-\varepsilon)^3}\exp\left(\frac{(\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)^2}{2(\ell-\varepsilon)}\right)\le M_1<\infty \qquad (6.60)$$

for (r_0,θ_0) in a neighborhood of (r_s,α) and $t\in(0,\ell-\varepsilon-\delta)$ where M_1 is a finite constant. Let us fix $\eta_1>0$ and $0<\eta_2<\frac{r_s}{2}$, obviously

$$\sqrt{2\pi(\ell-\varepsilon)^3}\exp\left(\frac{(\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)^2}{2(\ell-\varepsilon)}\right)\le N_1<\infty$$

for the neighborhood $|\theta_0-\alpha|<\eta_1$ and $|r_0-r_s|<\eta_2$ where N_1 is a finite constant. On the other hand, for $t\in(0,\ell-\varepsilon-\delta)$, we have

$$\frac{\pi\sin\alpha}{2\alpha^2\sqrt{t((\ell-\varepsilon)-t\cos^2\alpha)}(\ell-\varepsilon-t)}\exp\left(-\frac{r_0^2}{2t}\frac{(\ell-\varepsilon)-t\cos(2\alpha)}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right)$$
$$<\frac{\pi\sin\alpha}{2\alpha^2\sqrt{t\delta}\delta}\exp\left(-\frac{(\frac{r_s}{2})^2}{2t}\frac{\delta}{2(\ell-\varepsilon)}\right).$$

Thus it suffices to show

$$\frac{\pi \sin\alpha}{2\alpha^2\sqrt{t\delta}\delta}\exp\left(-\frac{(\frac{r_s}{2})^2}{2t}\frac{\delta}{2(\ell-\varepsilon)}\right) \le N_2 < \infty$$

for $t \in (0, \ell-\varepsilon-\delta)$ where N_2 is a finite constant. Since $\frac{\pi \sin\alpha}{2\alpha^2\sqrt{t\delta}\delta}\exp\left(-\frac{(\frac{r_s}{2})^2}{2t}\frac{\delta}{2(\ell-\varepsilon)}\right)$ is continuous on t, it is sufficient to show

$$\lim_{t\to 0}\frac{\pi \sin\alpha}{2\alpha^2\sqrt{t\delta}\delta}\exp\left(-\frac{(\frac{r_s}{2})^2}{2t}\frac{\delta}{2(\ell-\varepsilon)}\right) = N_3 < \infty,$$

where N_3 is a finite constant. From L'Hospital's rule, we have

$$\begin{aligned}
&\lim_{t\to 0}\frac{\pi \sin\alpha}{2\alpha^2\sqrt{t\delta}\delta}\exp\left(-\frac{(\frac{r_s}{2})^2}{2t}\frac{\delta}{2(\ell-\varepsilon)}\right)\\
&= \lim_{y\to\infty}\frac{y\,\pi\sin\alpha}{2\alpha^2\delta^{\frac{3}{2}}}\frac{1}{\exp\left((\frac{r_s}{2})^2y^2\frac{\delta}{2(\ell-\varepsilon)}\right)}\\
&= \lim_{y\to\infty}\frac{\pi\sin\alpha}{2\alpha^2\delta^{\frac{3}{2}}}\frac{1}{\exp\left((\frac{r_s}{2})^2y^2\frac{\delta}{2(\ell-\varepsilon)}\right)\left(2(\frac{r_s}{2})^2y\frac{\delta}{2(\ell-\varepsilon)}\right)} = 0,
\end{aligned}$$

which finishes the proof of (6.60).

Next, we are going to show

$$\sum_{n=1}^{\infty}\frac{n\sin(\frac{n\pi\theta_0}{\alpha})}{\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)}I_{\frac{n\pi}{2\alpha}}\left(\frac{r_0^2}{2t}\frac{(\ell-\varepsilon)-t}{(\ell-\varepsilon-t)+(\ell-\varepsilon-t\cos(2\alpha))}\right) \le M_2 < \infty$$

for θ_0 in a neighborhood of α and r_0 in a neighborhood of r_s where M_2 is a finite constant.

Let us fix $\eta_1 = \frac{\pi}{4}$, and define

$$z_0 := \frac{r_0^2}{2t}\frac{(\ell-\varepsilon)-t}{(\ell-\varepsilon-t)+(\ell-\varepsilon-t\cos(2\alpha))}, z_s := \frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t}{(\ell-\varepsilon-t)+(\ell-\varepsilon-t\cos(2\alpha))}.$$

Then, there exists $0 < \eta_2 \le \frac{r_s}{2}$ such that for any r_0 such that $|r_0 - r_s| \le \eta_2$, we have $|z_0 - z_s| \le \frac{z_s}{2}$. We want to show

$$\begin{aligned}
\sup_{0<|\theta_0-\alpha|\le\eta_1, |r_0-r_s|\le\eta_2}\Bigg\{\sum_{n=1}^{\infty}\quad&\left|\frac{n\sin(\frac{n\pi\theta_0}{\alpha})}{\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0)}I_{\frac{n\pi}{2\alpha}}(z_0)\right|\Bigg\}\\
&= N_4 < \infty,
\end{aligned} \tag{6.61}$$

where N_4 is a finite constant. We observe that $\sqrt{1-\rho^2}r_0\cos\theta_0+\rho r_0\sin\theta_0 = \sin\alpha\cos\theta_0 - \cos\alpha\sin\theta_0 = \sin(\alpha-\theta_0)$. By the Mean Value Theorem, there exists ζ between $n\pi\theta_0/\alpha$

and $n\pi\alpha/\alpha$ such that

$$
\begin{aligned}
|\frac{n\sin(\frac{n\pi\theta_0}{\alpha})}{\sqrt{1-\rho^2}\cos\theta_0+\rho\sin\theta_0)}| &= \frac{n^2\pi}{\alpha}|\frac{\sin(\frac{n\pi\theta_0}{\alpha})-\sin(\frac{n\pi\alpha}{\alpha})}{\frac{n\pi\theta_0}{\alpha}-\frac{n\pi\alpha}{\alpha}}||\frac{\theta_0-\alpha}{\sin(\theta_0-\alpha)}| \\
&= \frac{n^2\pi}{\alpha}|\cos\zeta||\frac{\sin(\theta_0-\alpha)}{\theta_0-\alpha}|^{-1} \\
&\leq \frac{n^2\pi}{\alpha}|\frac{\sin(\theta_0-\alpha)}{\theta_0-\alpha}|^{-1} \\
&\leq \frac{n^2\pi^2}{2\sqrt{2}\alpha},
\end{aligned}
$$

for all $0<|\theta_0-\alpha|\leq\eta_1=\frac{\pi}{4}$. It suffices to show that

$$\sup_{|r_0-r_s|\leq\eta_2}\sum_{n=1}^{\infty}\frac{n^2}{r_0}I_{\frac{n\pi}{2\alpha}}(z_0)=N_5<\infty, \tag{6.62}$$

where N_5 is a finite constant. We recall that

$$I_\nu(z)=(\frac{z}{2})^\nu\sum_{k=0}^{\infty}(\frac{z^2}{4})^k\frac{1}{k!\Gamma(\nu+k+1)} \tag{6.63}$$

is increasing for $z>0$. Since $|z_0-z_s|\leq\frac{z_s}{2}$ for $|r_0-r_s|\leq\eta_2$, so the supremum in (6.62) is attained at $z:=z_s+\frac{z_s}{2}=\frac{3z_s}{2}$. Also because $\eta_2\leq\frac{r_s}{2}$, we have $r_0\geq\frac{r_s}{2}>0$. Therefore, it suffices to show

$$\sum_{n=1}^{\infty}n^2 I_{\frac{n\pi}{2\alpha}}(z)=N_6<\infty, \tag{6.64}$$

where N_6 is a finite constant. It is sufficient to show the successive terms in the sum have a ratio less than $\frac{1}{2}$. We use the fact that for any $\nu>0$, $\lim_{x\to\infty}\frac{\Gamma(x)}{\Gamma(x+\nu)}=0$. Thus, let us fix $\nu=\frac{\pi}{2\alpha}$, there exists sufficiently large $M\in\mathbb{N}$ such that $\frac{\Gamma(x)}{\Gamma(x+\nu)}<\frac{1}{2}\frac{1}{(z/2)^\nu}$ for every $x\geq M$. Let $N=\lceil M/\nu\rceil+1$, then for every $n\geq N$ we have $n\nu>M$. From (6.63), we have

$$
\begin{aligned}
I_{\frac{(n+1)\pi}{2\alpha}}(z) &= I_{(n+1)\nu}(z) = (\frac{z}{2})^{(n+1)\nu}\sum_{k=0}^{\infty}(\frac{z^2}{4})^k\frac{1}{k!\Gamma((n+1)\nu+k+1)} \\
&= (\frac{z}{2})^{n\nu}\sum_{k=0}^{\infty}(\frac{z^2}{4})^k\frac{(\frac{z}{2})^{\nu}}{k!\Gamma((n+1)\nu+k+1)} \\
&= (\frac{z}{2})^{n\nu}\sum_{k=0}^{\infty}(\frac{z^2}{4})^k\frac{(\frac{z}{2})^{\nu}}{k!\Gamma(n\nu+k+1)}\frac{\Gamma(n\nu+k+1)}{\Gamma((n+1)\nu+k+1)} \\
&< \frac{1}{2}(\frac{z}{2})^{n\nu}\sum_{k=0}^{\infty}(\frac{z^2}{4})^k\frac{(\frac{z}{2})^{\nu}}{k!(\frac{z}{2})^{\nu}\Gamma(n\nu+k+1)} \\
&= \frac{1}{2}(\frac{z}{2})^{n\nu}\sum_{k=0}^{\infty}(\frac{z^2}{4})^k\frac{1}{k!\Gamma(n\nu+k+1)} \\
&= \frac{1}{2}I_{n\nu}(z) = \frac{1}{2}I_{\frac{n\pi}{2\alpha}}(z), \qquad (6.65)
\end{aligned}
$$

which proves (6.64). This implies (6.61).

By Dominated Convergence Theorem and L'Hospital's rule, we get

$$
\begin{aligned}
&\lim_{r_0\to r_s,\theta_0\to\alpha}\frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)} \\
&= \frac{\pi\sin\alpha\sqrt{2\pi(\ell-\varepsilon)^3}}{2\alpha^2\sqrt{t((\ell-\varepsilon)-t\cos^2\alpha)}(\ell-\varepsilon-t)}\exp\left(-\frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t\cos(2\alpha)}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right) \\
&\quad \sum_{n=1}^{\infty}\left(\lim_{r_0\to r_s,\theta_0\to\alpha}\frac{\frac{n^2\pi}{\alpha}\cos(\frac{n\pi\theta_0}{\alpha})}{-\sqrt{1-\rho^2}r_0\sin\theta_0+\rho r_0\cos\theta_0}\right)I_{\frac{n\pi}{2\alpha}}\left(\frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right) \\
&= \frac{\pi\sin\alpha\sqrt{2\pi(\ell-\varepsilon)^3}}{2\alpha^2\sqrt{t((\ell-\varepsilon)-t\cos^2\alpha)}(\ell-\varepsilon-t)}\exp\left(-\frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t\cos(2\alpha)}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right) \\
&\quad \sum_{n=1}^{\infty}\frac{n^2\pi}{\alpha r_s}(-1)^{n-1}I_{\frac{n\pi}{2\alpha}}\left(\frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right). \qquad (6.66)
\end{aligned}
$$

Since for $t<\ell-\varepsilon-\delta$, $\lim_{r_0\to r_s,\theta_0\to\alpha}\frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)}$ is well-defined, then we can use the Dominated Convergence Theorem to get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\lim_{r_0\to r_s,\theta_0\to\alpha}\int_0^{\ell-\varepsilon-\delta}\frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)}dt\right)\\
&= \lim_{\varepsilon\to 0}\left(\int_0^{\ell-\varepsilon-\delta}\lim_{r_0\to r_s,\theta_0\to\alpha}\frac{f(\ell-\varepsilon,t)}{p(r_0,\theta_0;\ell-\varepsilon)}dt\right)\\
&= \lim_{\varepsilon\to 0}\int_0^{\ell-\varepsilon-\delta}\frac{\pi\sin\alpha\sqrt{2\pi(\ell-\varepsilon)^3}}{2\alpha^2\sqrt{t((\ell-\varepsilon)-t\cos^2\alpha)}(\ell-\varepsilon-t)}\exp\left(-\frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t\cos(2\alpha)}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right)\\
&\qquad\sum_{n=1}^{\infty}\frac{n^2\pi}{\alpha r_s}(-1)^{n-1}I_{\frac{n\pi}{2\alpha}}\left(\frac{r_s^2}{2t}\frac{(\ell-\varepsilon)-t}{((\ell-\varepsilon)-t)+((\ell-\varepsilon)-t\cos(2\alpha))}\right)dt\\
&= \int_0^{\ell-\delta}\frac{\pi\sin\alpha\sqrt{2\pi\ell^3}}{2\alpha^2\sqrt{t(\ell-t\cos^2\alpha)}(\ell-t)}\exp\left(-\frac{r_s^2}{2t}\frac{\ell-t\cos(2\alpha)}{(\ell-t)+(\ell-t\cos(2\alpha))}\right)\\
&\qquad\sum_{n=1}^{\infty}\frac{n^2\pi}{\alpha r_s}(-1)^{n-1}I_{\frac{n\pi}{2\alpha}}\left(\frac{r_s^2}{2t}\frac{\ell-t}{(\ell-t)+(\ell-t\cos(2\alpha))}\right)dt.
\end{aligned}
\tag{6.67}
$$

□

Therefore, by Lemma 6.1.2, 6.1.3 and (6.45), we have

$$
\begin{aligned}
&\mathbb{Q}^{\ell}\{\tau_d<\ell\} = \lim_{\delta\to 0}\mathbb{Q}^{\ell}\{\tau_d<\ell-\delta\}\\
&= \lim_{\delta\to 0}\int_0^{\ell-\delta}\frac{\pi\sin\alpha\sqrt{2\pi\ell^3}}{2\alpha^2\sqrt{t(\ell-t\cos^2\alpha)}(\ell-t)}\exp\left(-\frac{r_s^2}{2t}\frac{\ell-t\cos(2\alpha)}{(\ell-t)+(\ell-t\cos(2\alpha))}\right)\\
&\qquad\sum_{n=1}^{\infty}\frac{n^2\pi}{\alpha r_s}(-1)^{n-1}I_{\frac{n\pi}{2\alpha}}\left(\frac{r_s^2}{2t}\frac{\ell-t}{(\ell-t)+(\ell-t\cos(2\alpha))}\right)dt\\
&= \int_0^{\ell}\frac{\pi\sin\alpha\sqrt{2\pi\ell^3}}{2\alpha^2\sqrt{t(\ell-t\cos^2\alpha)}(\ell-t)}\exp\left(-\frac{r_s^2}{2t}\frac{\ell-t\cos(2\alpha)}{(\ell-t)+(\ell-t\cos(2\alpha))}\right)\\
&\qquad\sum_{n=1}^{\infty}\frac{n^2\pi}{\alpha r_s}(-1)^{n-1}I_{\frac{n\pi}{2\alpha}}\left(\frac{r_s^2}{2t}\frac{\ell-t}{(\ell-t)+(\ell-t\cos(2\alpha))}\right)dt\\
&= \int_0^{\ell}p_{\mathcal{U}}(t,l)dt =: p_{\mathcal{U}}(\ell).
\end{aligned}
\tag{6.68}
$$

This finishes the computation of the conditional probability that $\mathcal{U}^*$ hits zero before the negative excursion of $\mathcal{W}^*$ ends given that the length of the excursion equals ℓ. Following the same logic, we can show that the conditional probability that $\mathcal{X}^*$ hits zero before the positive excursion of $\mathcal{V}^*$ ends given that the length of the excursion equals ℓ is

$$\begin{aligned}
p_{\mathcal{X}}(\ell) &:= \int_0^{\ell} p_{\mathcal{X}}(t,l)dt \\
&= \int_0^{\ell} \frac{\pi \sin\alpha\sqrt{2\pi\ell^3}}{2\alpha^2\sqrt{t(\ell - t\cos^2\alpha)}(\ell - t)} \exp\left(-\frac{\hat{r}_s^2}{2t}\frac{\ell - t\cos(2\alpha)}{(\ell - t) + (\ell - t\cos(2\alpha))}\right) \\
&\quad \sum_{n=1}^{\infty} \frac{n^2\pi}{\alpha\hat{r}_s}(-1)^{n-1} I_{\frac{n\pi}{2\alpha}}\left(\frac{\hat{r}_s^2}{2t}\frac{\ell - t}{(\ell - t) + (\ell - t\cos(2\alpha))}\right) dt, \qquad (6.69)
\end{aligned}$$

where $\hat{r} = \frac{\mu_2\lambda_1}{\theta_s\mu_1\sqrt{c_-}\sqrt{1-\rho^2}}$.

6.2 Waiting time between two consecutive renewal states

6.2.1 P.Levy's theory of Brownian local time

Now we want to relate our problem to Levy's theory of Brownian local time. Let B be a standard Brownian motion, let $L_B(t)$ denote the local time of B at zero up to time t, and let L_B^{-1} be its right-continuous inverse. It is well known that the Levy measure of L_B^{-1} is

$$\mu(d\ell) = \frac{d\ell}{\sqrt{2\pi\ell^3}},$$

and it tells us the frequency of the excursions of length ℓ. In this section, we want to distinguish positive and negative excursions, and since an excursion of a standard Brownian motion has half chance to be a positive, and half chance to be a negative, we can define Levy measures for positive excursions and negative excursions

$$\begin{aligned}
\mu^+(d\ell) &= \frac{d\ell}{2\sqrt{2\pi\ell^3}}, \\
\mu^-(d\ell) &= \frac{d\ell}{2\sqrt{2\pi\ell^3}},
\end{aligned}$$

and their corresponding Poisson random measures ν^+ and ν^-.

Now consider the standard Brownian motion B^* appearing in (6.2). From (6.2), we see that once B^* is on a negative excursion with length ℓ, G^* is actually on a negative excursion with length ℓ/c_-. Similarly, once B^* is on a positive excursion with length ℓ, G^* is actually on a positive excursion with length ℓ/c_+. In (6.74) and (6.75), we proved that the probability of $\mathcal{U}^*$ reaching zero before the negative excursion of $\mathcal{W}^*$ ends given the length of the excursion equals ℓ/c_- is $p_{\mathcal{U}}(\frac{\ell}{c_-})$, and the probability of $\mathcal{X}^*$ reaching zero before the positive excursion of $\mathcal{V}^*$ ends given the length of the excursion equals ℓ/c_+ is $p_{\mathcal{X}}(\frac{\ell}{c_+})$. This suggests construction of *thinned* measures over μ^- that distinguish the negative excursions of $\mathcal{W}^*$ on which $\mathcal{U}^*$ hits zero from the negative excursions of $\mathcal{W}^*$ on which $\mathcal{U}^*$ does not hit zero. We do the same thing for positive excursions of $\mathcal{V}^*$, distinguishing those on which $\mathcal{X}^*$ hits zero from those on which $\mathcal{X}^*$ does not reach zero. In particular, we define

$$\begin{aligned}
\mu_\circ^-(d\ell) &= p_{\mathcal{U}}(\frac{\ell}{c_-})\frac{d\ell}{2\sqrt{2\pi\ell^3}}, \\
\mu_\times^-(d\ell) &= (1 - p_{\mathcal{U}}(\frac{\ell}{c_-}))\frac{d\ell}{2\sqrt{2\pi\ell^3}}, \\
\mu_\circ^+(d\ell) &= p_{\mathcal{X}}(\frac{\ell}{c_+})\frac{d\ell}{2\sqrt{2\pi\ell^3}}, \\
\mu_\times^+(d\ell) &= (1 - p_{\mathcal{X}}(\frac{\ell}{c_+}))\frac{d\ell}{2\sqrt{2\pi\ell^3}},
\end{aligned}$$

where $\mu_\circ^-$ is the Levy measure for negative excursions on which $\mathcal{U}^*$ hits zero before the negative excursion of B^*(and also G^* and $\mathcal{W}^*$) ends given the length of the excursion equals ℓ, $\mu_\times^-$ is the Levy measure for negative excursions on which $\mathcal{U}^*$ does not hit zero before the excursion ends given the length of the excursion equals ℓ. Similarly, $\mu_\circ^+$ and $\mu_\times^+$ represent the corresponding measures for positive excursions of B^*. Furthermore, we denote the corresponding Poisson random measures as $\nu_\circ^-$, $\nu_\times^-$, $\nu_\circ^+$, and $\nu_\times^+$. Because they are thinned independent Poisson random measures, $\nu_\circ^-$, $\nu_\times^-$, $\nu_\circ^+$, and $\nu_\times^+$ are independent.

Lemma 6.2.1

$$A_{\mathcal{U}} := \mu_\circ^-\big([0,\infty)\big) = \int_0^\infty p_{\mathcal{U}}(\frac{\ell}{c_-})\frac{d\ell}{2\sqrt{2\pi\ell^3}} < \infty$$

and

$$A_{\mathcal{X}} := \mu_\circ^+\big([0,\infty)\big) = \int_0^\infty p_{\mathcal{X}}(\frac{\ell}{c_+})\frac{d\ell}{2\sqrt{2\pi\ell^3}} < \infty.$$

Proof: We prove the first equation. The proof of the second is analogous. By a change of variable, we see that

$$\int_0^\infty p_{\mathcal{U}}(\frac{\ell}{c_-})\frac{d\ell}{2\sqrt{2\pi\ell^3}} = \frac{1}{\sqrt{c_-}}\int_0^\infty p_{\mathcal{U}}(\ell)\frac{d\ell}{2\sqrt{2\pi\ell^3}}.$$

We prove finiteness of the second integral. Note

$$\begin{aligned}
p_{\mathcal{U}}(\ell) &= \mathbb{P}\{\tau_d < \ell | \tau_e = \ell\} \\
&= \mathbb{P}\{\max_{0\le t\le\ell}(\beta\sqrt{c_-}E(t) - \kappa C(t)) \ge \frac{\lambda_2\mu_1}{\theta_b\lambda_1} | \tau_e = \ell\} \\
&\le \mathbb{P}\{\beta\sqrt{c_-}\max_{0\le t\le\ell}E(t) + \kappa\max_{0\le t\le\ell}(-C(t)) \ge \frac{\lambda_2\mu_1}{\theta_b\lambda_1} | \tau_e = \ell\} \\
&\le \mathbb{P}\{\beta\sqrt{c_-}\max_{0\le t\le\ell}E(t) \ge \frac{\lambda_2\mu_1}{2\theta_b\lambda_1} \text{ or } \kappa\max_{0\le t\le\ell}(-C(t)) \ge \frac{\lambda_2\mu_1}{2\theta_b\lambda_1} | \tau_e = \ell\} \\
&\le \mathbb{P}\{\beta\sqrt{c_-}\max_{0\le t\le\ell}E(t) \ge \frac{\lambda_2\mu_1}{2\theta_b\lambda_1} | \tau_x = \ell\} + \mathbb{P}\{\max_{0\le t\le\ell}(-C(t)) \ge \frac{\lambda_2\mu_1}{2\kappa\theta_b\lambda_1}\}.
\end{aligned}$$

Since

$$\begin{aligned}\mathbb{P}\{\max_{0\le t\le \ell}(-C(t)) \ge \frac{\lambda_2\mu_1}{2\kappa\theta_b\lambda_1}\} &\le 2\mathbb{P}\{C(\ell) \ge \frac{\lambda_2\mu_1}{2\kappa\theta_b\lambda_1}\} \\ &\le \Big(2\frac{4\kappa^2\theta_b^2\lambda_1^2}{\lambda_2^2\mu_1^2}\mathbb{E}C^2(\ell)\Big) \wedge 1 \\ &= \Big(\frac{8\kappa^2\theta_b^2\lambda_1^2}{\lambda_2^2\mu_1^2}\ell\Big) \wedge 1,\end{aligned}$$

we have

$$\begin{aligned}\int_0^\infty \mathbb{P}\{\max_{0\le t\le \ell}(-C(t)) \ge \frac{\lambda_2\mu_1}{2\kappa\theta_b\lambda_1}\}\frac{d\ell}{2\sqrt{2\pi\ell^3}} &\le \int_0^\infty \Big((\frac{8\kappa^2\theta_b^2\lambda_1^2}{\lambda_2^2\mu_1^2}\ell) \wedge 1\Big)\frac{d\ell}{2\sqrt{2\pi\ell^3}} \\ \le \int_0^1 \frac{8\kappa^2\theta_b^2\lambda_1^2}{\lambda_2^2\mu_1^2}\ell\frac{d\ell}{2\sqrt{2\pi\ell^3}} + \int_1^\infty \frac{d\ell}{2\sqrt{2\pi\ell^3}} < \infty.\end{aligned}$$

Now it remains to show

$$\int_0^\infty \mathbb{P}\{\max_{0\le t\le \ell} E(t) \ge \frac{\lambda_2\mu_1}{2\beta\sqrt{c_-}\theta_b\lambda_1}|\tau_e = \ell\}\frac{d\ell}{2\sqrt{2\pi\ell^3}} < \infty. \tag{6.70}$$

According to [6], Theorem 7, for $\xi > 0$, we have

$$\mathbb{P}\{\max_{0\le t\le \ell} E(t) \ge \xi|\tau_e = \ell\} = 2\sum_{n=1}^\infty(\frac{4n^2\xi^2}{\ell} - 1)\exp\{-\frac{2n^2\xi^2}{\ell}\}.$$

Thus,

$$\begin{aligned}&\int_0^\infty \mathbb{P}\{\max_{0\le t\le \ell} E(t) \ge \frac{\lambda_2\mu_1}{2\beta\sqrt{c_-}\theta_b\lambda_1}|\tau_e = \ell\}\frac{d\ell}{2\sqrt{2\pi\ell^3}} \\ \le& \int_0^1 \sum_{n=1}^\infty(\frac{4n^2\xi^2}{\ell})\exp\{-\frac{2n^2\xi^2}{\ell}\}\frac{d\ell}{\sqrt{2\pi\ell^3}} + \int_1^\infty \frac{d\ell}{2\sqrt{2\pi\ell^3}},\end{aligned}$$

where $\xi = \frac{\lambda_2\mu_1}{2\beta\sqrt{c_-}\theta_b\lambda_1}$. It suffices to show $\int_0^1 \sum_{n=1}^\infty(\frac{4n^2\xi^2}{\ell})\exp\{-\frac{2n^2\xi^2}{\ell}\}\frac{d\ell}{\sqrt{2\pi\ell^3}} < \infty$. From the Monotone Convergence Theorem, we have

$$\int_0^1 \sum_{n=1}^\infty(\frac{4n^2\xi^2}{\ell})\exp\{-\frac{2n^2\xi^2}{\ell}\}\frac{d\ell}{\sqrt{2\pi\ell^3}} = \sum_{n=1}^\infty \int_0^1 (\frac{4n^2\xi^2}{\ell})\exp\{-\frac{2n^2\xi^2}{\ell}\}\frac{d\ell}{\sqrt{2\pi\ell^3}}.$$

Obviously, $\int_0^1 (\frac{4n^2\xi^2}{\ell})\exp\{-\frac{2n^2\xi^2}{\ell}\}\frac{d\ell}{\sqrt{2\pi\ell^3}} < \infty$ for any $n \ge 1$. Let

$$f(\ell) := (\frac{4n^2\xi^2}{\ell})\exp\{-\frac{2n^2\xi^2}{\ell}\}\frac{1}{\sqrt{2\pi\ell^3}}.$$

Then its first derivative is

$$f'(\ell) = \frac{4n^2\xi^2}{\sqrt{2\pi}}\exp\{-\frac{2n^2\xi^2}{\ell}\}\ell^{-\frac{7}{2}}(-\frac{5}{2} + \frac{2n^2\xi^2}{\ell}).$$

This implies $f(\ell)$ is increasing over $(0, \frac{4n^2\xi^2}{5})$. We can find sufficiently large $M \in \mathbb{N}$ such that $\frac{4n^2\xi^2}{5} > 1$ for all $n \geq M$. Then

$$\begin{aligned}&\int_0^1 (\frac{4n^2\xi^2}{\ell}) \exp\{-\frac{2n^2\xi^2}{\ell}\} \frac{d\ell}{\sqrt{2\pi\ell^3}} \\ = & \int_0^1 f(\ell) d\ell \leq f(1) = (\frac{4n^2\xi^2}{\sqrt{2\pi}}) \exp\{-2n^2\xi^2\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{n=M}^{\infty} \int_0^1 (\frac{4n^2\xi^2}{\ell}) \exp\{-\frac{2n^2\xi^2}{\ell}\} \frac{d\ell}{\sqrt{2\pi\ell^3}} \\ \leq & \sum_{n=M}^{\infty} (\frac{4n^2\xi^2}{\sqrt{2\pi}}) \exp\{-2n^2\xi^2\} < \infty,\end{aligned}$$

which finishes our proof. □

6.2.2 Computation of the waiting time between two renewal states

According to Lemma 6.2.1, we see that $\mu_\circ^-$ and $\mu_\circ^+$ are a finite measures. Let us define subordinators

$$\begin{aligned}H_\circ^+(s) &:= \int_{u=0}^{s} \int_{\ell=0}^{\infty} \frac{\ell}{c_+} \nu_\circ^+(du \times d\ell), \\ H_\times^+(s) &:= \int_{u=0}^{s} \int_{\ell=0}^{\infty} \frac{\ell}{c_+} \nu_\times^+(du \times d\ell), \\ H_\circ^-(s) &:= \int_{u=0}^{s} \int_{\ell=0}^{\infty} \frac{\ell}{c_-} \nu_\circ^-(du \times d\ell), \\ H_\times^-(s) &:= \int_{u=0}^{s} \int_{\ell=0}^{\infty} \frac{\ell}{c_-} \nu_\times^-(du \times d\ell).\end{aligned}$$

$H_\circ^+(s)$ keeps track of the chronological time spent on positive excursions where $\mathcal{X}^*$ vanishes accumulated up to local time s of B; $H_\times^+(s)$ keeps track of the chronological time spent on positive excursions where $\mathcal{X}^*$ does not vanish accumulated up to local time s of B; $H_\circ^-(s)$ keeps track of the chronological time spent on negative excursions where $\mathcal{U}^*$ vanishes accumulated up to local time s; $H_\times^-(s)$ keeps track of the chronological time spent on negative excursions where $\mathcal{U}^*$ does not vanish accumulated up to local time s.

Let us define

$$\zeta_{\mathcal{U}} := \min\{t > 0 : \int_0^t \int_0^{\infty} \nu_\circ^-(du \times d\ell) > 0\}, \tag{6.71}$$

$$\zeta_{\mathcal{X}} := \min\{t > 0 : \int_0^t \int_0^{\infty} \nu_\circ^+(du \times d\ell) > 0\}, \tag{6.72}$$

so that $\zeta_{\mathcal{U}}$ is the first time (in the local time clock) when $\mathcal{U}^*$ vanishes on a negative excursion, and $\zeta_{\mathcal{X}}$ is the first time (in the local time clock) when $\mathcal{X}^*$ vanishes on a positive excursion. We can compute $\zeta_{\mathcal{U}}$'s cumulative distribution function by

$$\begin{aligned}\mathbb{P}\{\zeta_{\mathcal{U}} < t\} &= \mathbb{P}\{\nu_\circ^-([0,t]\times[0,\infty)) \geq 1\} \\ &= 1 - \mathbb{P}\{\nu_\circ^-([0,t]\times[0,\infty)) = 0\} \\ &= 1 - e^{-\lambda([0,t]\times[0,\infty))},\end{aligned}$$

where

$$\lambda([0,t]\times[0,\infty)) = \int_0^\infty \mu_\circ^-(d\ell)t = A_{\mathcal{U}}t.$$

From Lemma 6.2.1, we have

$$A_{\mathcal{U}} < \infty,$$

which implies

$$\mathbb{P}\{\zeta_{\mathcal{U}} \in dt\} = A_{\mathcal{U}}e^{-A_{\mathcal{U}}t}dt. \tag{6.73}$$

Similarly, we have

$$\mathbb{P}\{\zeta_{\mathcal{X}} \in dt\} = A_{\mathcal{X}}e^{-A_{\mathcal{X}}t}dt. \tag{6.74}$$

Obviously, $\zeta_{\mathcal{U}}$ and $\zeta_{\mathcal{X}}$ have the exponential distributions with parameter $A_{\mathcal{U}}$ and $A_{\mathcal{X}}$. Since $\zeta_{\mathcal{U}}$ is determined by $\nu_\circ^-$, and $\zeta_{\mathcal{X}}$ is determined by $\nu_\circ^+$, we see that $\zeta_{\mathcal{U}}$ and $\zeta_{\mathcal{X}}$ are independent because of the independence between $\nu_\circ^-$ and $\nu_\circ^+$. This implies that the minimum of these two random variables has an exponential distribution, too. In particular,

$$\zeta_{min} := \min\{\zeta_{\mathcal{U}}, \zeta_{\mathcal{X}}\} \sim \exp\{A_{\mathcal{U}} + A_{\mathcal{X}}\}.$$

Also, we can compute the probability that $\mathcal{U}^*$ hits zero before $\mathcal{X}^*$ and vice versa. In particular,

$$\begin{aligned}\mathbb{P}(\zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}) &= \int_0^\infty A_{\mathcal{X}}e^{-A_{\mathcal{X}}x}\int_0^x A_{\mathcal{U}}e^{-A_{\mathcal{U}}u}du dx \\ &= \int_0^\infty A_{\mathcal{X}}e^{-A_{\mathcal{X}}x}(1-e^{-A_{\mathcal{U}}x})dx \\ &= 1 - \int_0^\infty A_{\mathcal{X}}e^{-(A_{\mathcal{X}}+A_{\mathcal{U}})x}dx \\ &= 1 - \frac{A_{\mathcal{X}}}{A_{\mathcal{U}}+A_{\mathcal{X}}} \\ &= \frac{A_{\mathcal{U}}}{A_{\mathcal{U}}+A_{\mathcal{X}}}.\end{aligned} \tag{6.75}$$

Recall definition in (6.5). Now we compute τ_{min}, the chronological time until $\mathcal{U}^*$ or $\mathcal{X}^*$ hits zero. We can define τ_{min} conditional on either $\tau_{\mathcal{U}} < \tau_{\mathcal{X}}$ (equivalently, $\zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}$) or $\tau_{\mathcal{X}} < \tau_{\mathcal{U}}$ (equivalently, $\zeta_{\mathcal{X}} < \zeta_{\mathcal{U}}$). We have

$$\tau_{min} := \begin{cases} \int_{u=0}^{\zeta_{min}}\int_{\ell=0}^\infty \frac{\ell}{c_+}(\nu_\circ^+ + \nu_\times^+)(du\times d\ell) + \int_{u=0}^{\zeta_{min}}\int_{\ell=0}^\infty \frac{\ell}{c_-}\nu_\times^-(du\times d\ell) + R_{\mathcal{U}} & \text{if } \zeta_{min} = \zeta_{\mathcal{U}} \\ \\ \int_{u=0}^{\zeta_{min}}\int_{\ell=0}^\infty \frac{\ell}{c_-}(\nu_\circ^- + \nu_\times^-)(du\times d\ell) + \int_{u=0}^{\zeta_{min}}\int_{\ell=0}^\infty \frac{\ell}{c_+}\nu_\times^+(du\times d\ell) + R_{\mathcal{X}} & \text{if } \zeta_{min} = \zeta_{\mathcal{X}}. \end{cases}$$

When $\zeta_{min} = \zeta_{\mathcal{U}}$, $\int_{u=0}^{\zeta_{min}} \int_{\ell=0}^{\infty} \frac{\ell}{c_+}(\nu_\circ^+ + \nu_\times^+)(du \times d\ell)$ represents the time spent when $\mathcal{W}^*$ is on positive excursions, $\int_{u=0}^{\zeta_{min}} \int_{\ell=0}^{\infty} \frac{\ell}{c_-}\nu_\times^-(du \times d\ell)$ represents the time spent when $\mathcal{W}^*$ is on negative excursions before the last excursion on which $\mathcal{U}^*$ vanishes, and $R_{\mathcal{U}}$ is the time spent on the last excursion on which $\mathcal{U}^*$ vanishes. Note that by (6.71) and (6.72), we have

$$\int_0^{\zeta_{min}} \int_0^{\infty} \nu_\circ^+(du \times d\ell) = 0, \text{ if } \zeta_{min} = \zeta_{\mathcal{U}},$$
$$\int_0^{\zeta_{min}} \int_0^{\infty} \nu_\circ^-(du \times d\ell) = 0, \text{ if } \zeta_{min} = \zeta_{\mathcal{X}},$$

so we can rewrite τ_{min} as

$$\tau_{min} := \begin{cases} \int_{u=0}^{\zeta_{min}} \int_{\ell=0}^{\infty} \frac{\ell}{c_+}\nu_\times^+(du \times d\ell) + \int_{u=0}^{\zeta_{min}} \int_{\ell=0}^{\infty} \frac{\ell}{c_-}\nu_\times^-(du \times d\ell) + R_{\mathcal{U}} & \text{if } \zeta_{min} = \zeta_{\mathcal{U}} \\ \\ \int_{u=0}^{\zeta_{min}} \int_{\ell=0}^{\infty} \frac{\ell}{c_-}\nu_\times^-(du \times d\ell) + \int_{u=0}^{\zeta_{min}} \int_{\ell=0}^{\infty} \frac{\ell}{c_+}\nu_\times^+(du \times d\ell) + R_{\mathcal{X}} & \text{if } \zeta_{min} = \zeta_{\mathcal{X}}. \end{cases}$$

In order to simplify the notation, we define,

$$\begin{aligned} H_\times^+(s) &:= \int_{u=0}^{s} \int_{\ell=0}^{\infty} \frac{\ell}{c_+}\nu_\times^+(du \times d\ell), \\ H_\times^-(s) &:= \int_{u=0}^{s} \int_{\ell=0}^{\infty} \frac{\ell}{c_-}\nu_\times^-(du \times d\ell). \end{aligned}$$

Our goal is to compute the characteristic function of τ_{min} conditional on $\zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}$. From the definition of τ_{min}, we have

$$\mathbb{P}\{\tau_{min} = dt | \zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}\} = \mathbb{P}\{H_\times^+(\zeta_{\mathcal{U}}) + H_\times^-(\zeta_{\mathcal{U}}) + R_{\mathcal{U}} = dt | \zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}\}.$$

Note that $R_{\mathcal{U}}$ is independent of the excursions before time $\zeta_{\mathcal{U}}$. Thus it is independent of $H_\times^+(\zeta_{\mathcal{U}})$, $H_\times^-(\zeta_{\mathcal{U}})$. Also because $\zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}$, the characteristic function of τ_{min} satisfies

$$\begin{aligned} \mathbb{E}[e^{i\alpha\tau_{min}} | \zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}] &= \mathbb{E}[e^{i\alpha(H_\times^+(\zeta_{\mathcal{U}}) + H_\times^-(\zeta_{\mathcal{U}}) + R_{\mathcal{U}})} | \zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}] \\ &= \mathbb{E}[e^{i\alpha R_{\mathcal{U}}}] E[e^{i\alpha(H_\times^+(\zeta_{\mathcal{U}}) + H_\times^-(\zeta_{\mathcal{U}}))} | \zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}]. \end{aligned} \tag{6.76}$$

We compute the two factors on the right-hand side of (6.76).

Since $\zeta_{\mathcal{U}}$ and $\zeta_{\mathcal{X}}$ are independent exponentially distributed random variables, we have

$$\begin{aligned} &\mathbb{E}[e^{i\alpha(H_\times^+(\zeta_{\mathcal{U}}) + H_\times^-(\zeta_{\mathcal{U}}))} | \zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}] \\ &= \frac{1}{\mathbb{P}\{\zeta_{\mathcal{U}} < \zeta_{\mathcal{X}}\}} \int_0^{\infty} \int_u^{\infty} E[e^{i\alpha(H_\times^+(u) + H_\times^-(u))}] A_{\mathcal{U}} e^{-A_{\mathcal{U}} u} A_{\mathcal{X}} e^{-A_{\mathcal{X}} x} dx du \\ &= \frac{A_{\mathcal{U}} + A_{\mathcal{X}}}{A_{\mathcal{U}}} \int_0^{\infty} \mathbb{E}[e^{i\alpha(H_\times^+(u) + H_\times^-(u))}] A_{\mathcal{U}} e^{-(A_{\mathcal{U}} + A_{\mathcal{X}})u} du \\ &= \int_0^{\infty} \mathbb{E}[e^{i\alpha H_\times^+(u)}] \mathbb{E}[e^{i\alpha H_\times^-(u))}] (A_{\mathcal{U}} + A_{\mathcal{X}}) e^{-(A_{\mathcal{U}} + A_{\mathcal{X}})u} du. \end{aligned}$$

Now we use the Lévy-Hinčin formula to write

$$\begin{aligned}
&\int_0^\infty \mathbb{E}[e^{i\alpha H_\times^+(u)}]\mathbb{E}[e^{i\alpha H_\times^-(u))}](A_\mathcal{U}+A_\mathcal{X})e^{-(A_\mathcal{U}+A_\mathcal{X})u}du \\
&= \int_0^\infty \exp\left(-u\int_0^\infty (1-e^{i\frac{\alpha}{c_+}\ell})\mu_\times^+(d\ell)\right) \\
&\qquad \exp\left(-u\int_0^\infty (1-e^{i\frac{\alpha}{c_-}\ell})\mu_\times^-(d\ell)\right)(A_\mathcal{U}+A_\mathcal{X})e^{-(A_\mathcal{U}+A_\mathcal{X})u}du \\
&= \frac{A_\mathcal{U}+A_\mathcal{X}}{A_\mathcal{U}+A_\mathcal{X}+\int_0^\infty(1-e^{i\frac{\alpha}{c_+}\ell})\mu_\times^+(d\ell)+\int_0^\infty(1-e^{i\frac{\alpha}{c_-}\ell})\mu_\times^-(d\ell)} \\
&= \frac{A_\mathcal{U}+A_\mathcal{X}}{A_\mathcal{U}+A_\mathcal{X}+\int_0^\infty(1-e^{i\frac{\alpha}{c_+}\ell})(1-p_\mathcal{X}(\frac{\ell}{c_+}))\frac{d\ell}{2\sqrt{2\pi\ell^3}}+\int_0^\infty(1-e^{i\frac{\alpha}{c_-}\ell})(1-p_\mathcal{U}(\frac{\ell}{c_-}))\frac{d\ell}{2\sqrt{2\pi\ell^3}}}.
\end{aligned} \tag{6.77}$$

We can simplify the integral by the following calculation. Let $\theta = \frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)$, then $\mathrm{Re}(\theta) > 0$. Let W be a standard Brownian motion with $W_0 = 0$. Define the martingale

$$M_t = e^{2\theta W_t - 2\theta^2 t} = e^{2\theta W_t + i\alpha t}.$$

Let T_b be the first passage time of W to $b > 0$. Then

$$1 = \mathbb{E}[M_{t\wedge T_b}] = \mathbb{E}[e^{2\theta W_{t\wedge T_b} + i\alpha(t\wedge T_b)}]. \tag{6.78}$$

Because $\mathrm{Re}(\theta) > 0$ and $W_{t\wedge T_b}$ is bounded above by b, $|M_{t\wedge T_b}| \le b$. We can thus let $t \to \infty$ in (6.78) and use the Dominated Convergence Theorem to conclude that

$$1 = \mathbb{E}[e^{2\theta b + i\alpha T_b}],$$

i.e.

$$\mathbb{E}[e^{i\alpha T_b}] = e^{-2\theta b}. \tag{6.79}$$

Now

$$T_b = \int_0^\infty \ell\, \nu\big([0,b]\times d\ell\big),$$

where $\nu(db\, d\ell)$ is the Poisson random measure with Lévy measure $\mu(d\ell) = \frac{d\ell}{\sqrt{2\pi\ell^3}}$. (See, e.g., [19], Page 411). The Lévy-Hinčin formula implies

$$\mathbb{E}[e^{i\alpha T_b}] = \exp\big\{-b\int_0^\infty (1-e^{i\alpha\ell})\mu(d\ell)\big\}. \tag{6.80}$$

Comparing this with (6.79), we see that

$$\int_0^\infty (1-e^{i\alpha\ell})\frac{d\ell}{\sqrt{2\pi\ell^3}} = 2\theta. \tag{6.81}$$

In (6.77) we have

$$\begin{aligned}
&\int_0^\infty (1-e^{\frac{i\alpha\ell}{c\pm}})\frac{d\ell}{2\sqrt{2\pi\ell^3}} \\
&= \frac{1}{\sqrt{c_\pm}}\int_0^\infty (1-e^{i\alpha y})\frac{y}{2\sqrt{2\pi y^3}} \\
&= \frac{1}{\sqrt{c_\pm}}\theta, \qquad (6.82)
\end{aligned}$$

where $y=\frac{\ell}{c_\pm}$. We can write (6.77) as

$$\frac{A_{\mathcal{U}}+A_{\mathcal{X}}}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}. \qquad (6.83)$$

Note for future reference that

$$\begin{aligned}
&\mathbb{E}[e^{i\alpha(H^+_\times(\zeta_{\mathcal{X}})+H^-_\times(\zeta_{\mathcal{X}}))}|\zeta_{\mathcal{X}}<\zeta_{\mathcal{U}}] \\
&= \frac{A_{\mathcal{U}}+A_{\mathcal{X}}}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)} \\
&= \mathbb{E}[e^{i\alpha(H^+_\times(\zeta_{\mathcal{U}})+H^-_\times(\zeta_{\mathcal{U}}))}|\zeta_{\mathcal{U}}<\zeta_{\mathcal{X}}]. \qquad (6.84)
\end{aligned}$$

We now turn to the other factor on the right-hand side of (6.76). Let ϑ be the length of last excursion on which $\mathcal{U}^*$ vanishes. Recall from (6.68), for $t<\frac{\ell}{c_-}$, we have

$$\mathbb{P}\{R_{\mathcal{U}}\in dt\big|\vartheta=\ell\}=\mathbb{P}\{R_{\mathcal{U}}\in dt\big|\tau_e=\frac{\ell}{c_-}\}=\frac{p_{\mathcal{U}}(t,\frac{\ell}{c_-})}{p_{\mathcal{U}}(\frac{\ell}{c_-})}dt,$$

and

$$\mathbb{P}\{\vartheta=d\ell\}=\frac{\mu_\circ^-(d\ell)}{\int_0^\infty\mu_\circ^-(d\ell)}=\frac{p_{\mathcal{U}}(\frac{\ell}{c_-})}{2A_{\mathcal{U}}\sqrt{2\pi\ell^3}}d\ell.$$

The characteristic function of $R_{\mathcal{U}}$ is

$$\begin{aligned}
\mathbb{E}[e^{i\alpha R_{\mathcal{U}}}] &= \mathbb{E}\big[\mathbb{E}[e^{i\alpha R_{\mathcal{U}}}\big|\vartheta]\big] \\
&= \int_0^\infty\int_0^{\frac{\ell}{c_-}} e^{i\alpha t}\mathbb{P}\{R_{\mathcal{U}}\in dt\big|\tau_B=\ell\}\mathbb{P}\{\tau_B=d\ell\} \\
&= \int_0^\infty\int_0^{\frac{\ell}{c_-}} e^{i\alpha t}\frac{p_{\mathcal{U}}(t,\frac{\ell}{c_-})}{2A_{\mathcal{U}}\sqrt{2\pi\ell^3}}dtd\ell. \qquad (6.85)
\end{aligned}$$

Thus from (6.76), (6.77), and (6.85), we have

$$
\begin{aligned}
&\mathbb{E}[e^{i\alpha\tau_{min}}|\zeta_{\mathcal{U}}<\zeta_{\mathcal{X}}]\\
&= \frac{(A_{\mathcal{U}}+A_{\mathcal{X}})\int_0^\infty\int_0^{\frac{\ell}{c_-}} e^{i\alpha t}\frac{p_{\mathcal{U}}(t,\frac{\ell}{c_-})}{2A_{\mathcal{U}}\sqrt{2\pi\ell^3}}dtd\ell}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}\\
&= \frac{\frac{A_{\mathcal{U}}+A_{\mathcal{X}}}{\sqrt{c_-}}\int_0^\infty\int_0^{\ell} e^{i\alpha t}\frac{p_{\mathcal{U}}(t,\ell)}{2A_{\mathcal{U}}\sqrt{2\pi\ell^3}}dtd\ell}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\mathbb{E}[e^{i\alpha\tau_{min}}|\zeta_{\mathcal{U}}>\zeta_{\mathcal{X}}]\\
&= \frac{\frac{A_{\mathcal{U}}+A_{\mathcal{X}}}{\sqrt{c_+}}\int_0^\infty\int_0^{\ell} e^{i\alpha t}\frac{p_{\mathcal{X}}(t,\ell)}{2A_{\mathcal{X}}\sqrt{2\pi\ell^3}}dtd\ell}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathbb{E}[e^{i\alpha\tau_{min}}]\\
&= \mathbb{E}[e^{i\alpha\tau_{min}}\mathbb{1}_{\zeta_{\mathcal{U}}<\zeta_{\mathcal{X}}}]+\mathbb{E}[e^{i\alpha\tau_{min}}\mathbb{1}_{\zeta_{\mathcal{U}}>\zeta_{\mathcal{X}}}]\\
&= \mathbb{P}\{\zeta_{\mathcal{U}}<\zeta_{\mathcal{X}}\}\mathbb{E}[e^{i\alpha\tau_{min}}|\zeta_{\mathcal{U}}<\zeta_{\mathcal{X}}]+\mathbb{P}\{\zeta_{\mathcal{U}}>\zeta_{\mathcal{X}}\}\mathbb{E}[e^{i\alpha\tau_{min}}|\zeta_{\mathcal{U}}>\zeta_{\mathcal{X}}]\\
&= \frac{\frac{A_{\mathcal{U}}}{\sqrt{c_-}}\int_0^\infty\int_0^{\ell} e^{i\alpha t}\frac{p_{\mathcal{U}}(t,\ell)}{2A_{\mathcal{U}}\sqrt{2\pi\ell^3}}dtd\ell}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}\\
&\quad+\frac{\frac{A_{\mathcal{X}}}{\sqrt{c_+}}\int_0^\infty\int_0^{\ell} e^{i\alpha t}\frac{p_{\mathcal{X}}(t,\ell)}{2A_{\mathcal{X}}\sqrt{2\pi\ell^3}}dtd\ell}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}\\
&= \frac{\frac{1}{\sqrt{c_-}}\int_0^\infty\int_0^{\ell} e^{i\alpha t}\frac{p_{\mathcal{U}}(t,\ell)}{2\sqrt{2\pi\ell^3}}dtd\ell+\frac{1}{\sqrt{c_+}}\int_0^\infty\int_0^{\ell} e^{i\alpha t}\frac{p_{\mathcal{X}}(t,\ell)}{2\sqrt{2\pi\ell^3}}dtd\ell}{\frac{1}{\sqrt{c_-}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{U}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+\frac{1}{\sqrt{c_+}}\int_0^\infty e^{i\alpha\ell}\frac{p_{\mathcal{X}}(\ell)d\ell}{2\sqrt{2\pi\ell^3}}+(\frac{1}{\sqrt{c_-}}+\frac{1}{\sqrt{c_+}})\big(\frac{1}{2}\sqrt{|\alpha|}(1-\mathrm{sign}(\alpha)i)\big)}.
\end{aligned} \tag{6.86}
$$

6.3 Future work

In this thesis we followed a stylized stochastic LOB model proposed in [8], where the occurrences of market orders, limit orders, and cancellations are governed by independent Poisson processes. We assume that arrival rates of such market events are constant parameters that depend on the relative distance between price of arriving and opposite best price. The formulation of the model can be viewed using queueing theory and by applying the idea of heavy-traffic scaling and the "crushing" argument from Peterson [24], we derive the diffusion limit of scaled sequence of LOB models. In particular, the limiting model has a "two-tick" wide bid-ask spread, and the processes of volumes on the best bid and best ask follow a pair of correlated Brownian motions. We then discuss the evolution of the limiting LOB model and calculate the probability of upward/downward price movement via P.Lévy's theory of Brownian local time. We finish the thesis with the derivation of the characteristic function of the waiting time between two different renewal states, i.e., a price movement.

Taking into account a set of empirically observed properties of real LOB, desirable extensions of this model are the following:

- The model assumes limit orders arrive within two-tick distance from the opposite best price. It could be extended by allowing limit orders arrive at deeper price levels.
- Given the fact that cancellations actually happen at any price level in a real LOB, we can add cancellation to any price levels with non-zero volumes.
- The model uses six constant parameters for arrival rates of market and limit orders, which have three degrees of freedom. We could consider more general arrival rates, i.e., arrival rates with full freedom, time-dependent arrival rates or arrival rates depending on the volumes at the arriving price levels.
- Since we calculate the probability the next mid-price move is up/down conditional on the state of the LOB, and we compute the distribution of the waiting time between two adjacent price move events, it is natural to consider the optimal execution strategy for a large chunk of orders. One can study how to divide the large orders into small pieces and when to submit them in LOB.

We hope the results derived in this thesis can provide a good starting point for the development of more versatile LOB models.

Appendices

Appendix A

Measurability

A.1 Spaces

We denote by $C[0,\infty)$ the space of continuous functions from $[0,\infty)$ to $\mathbb{R}$ equipped with the metric

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n}\big(1 \wedge \max_{0\le t\le n} |x(t)-y(t)|\big), \quad x,y \in C[0,\infty). \tag{1.1}$$

Convergence in this metric is uniform convergence on compact subsets of $C[0,\infty)$.

Lemma 1.1 *Under the metric (1.1), $C[0,\infty)$ is a complete separable metric space.*

Proof: We first show that $C[0,\infty)$ is separable. For each finite T, the space $C[0,T]$ under uniform convergence is a complete separable metric space. For each n, let Q_n be a countable dense subset of $C[0,n]$. Given ε, there exists N such that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon/2$. Given $y \in C[0,\infty)$, there exists $x_N \in Q_N$ such that $\max_{0\le t\le N} |x_N(t)-y(t)| \le \varepsilon/2$. Let x be the extension of x_N to $[0,\infty)$ obtained by setting $x(t)=x_N(N)$ for $t \ge N$. Then $d(x,y) < \varepsilon$. It follows that the countable collection of continuous constant extensions of functions in $\cup_{n=1}^{\infty} Q_n$ is dense in $C[0,\infty)$.

To see that $C[0,\infty)$ is complete, let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $C[0,\infty)$. Then for each n, $\{x_k|_{[0,n]}\}_{k=1}^{\infty}$, the sequence obtained by restriction to $[0,n]$, is Cauchy in $C[0,n]$, and consequently this sequence has a limit x_n^*. It is obvious that $x_n^*(t) = x_m^*(t)$ for $0 \le t \le n \wedge m$, and this permits us to define $x^*(t) = x^*_{\lceil t\rceil}(t)$ for $0 \le t < \infty$, where $\lceil t\rceil$ is the smallest integer greater than or equal to t. The sequence $\{x_k\}_{k=1}^{\infty}$ converges to x^* uniformly on compact subsets of $C[0,\infty)$. $\square$

For any topological space X, we denote by $\mathcal{B}(X)$ the Borel σ-algebra generated by the open subsets of X. In particular, $\mathcal{B}(C[0,\infty))$ is the Borel σ-algebra in $C[0,\infty)$. This σ-algebra is generated by the countable collection of open balls

$$B_{x,q} := \big\{y \in C[0,\infty) : d(x,y) < q\big\}, \tag{1.2}$$

where x ranges over a countable dense subset of $C[0,\infty)$ and q ranges over the set of positive rational numbers. For $r \in \mathbb{R}$, we define

$$C_r[0,\infty) := \big\{x \in C[0,\infty) : x(0) = r\big\}, \tag{1.3}$$

so that $\mathcal{B}(C_r[0,\infty))$ is the trace σ-algebra of $\mathcal{B}(C[0,\infty))$ on $C_r[0,\infty)$. Finally, we define

$$\mathcal{D}^r := \big\{(z_+, z_-) \in C_{r^+}[0,\infty) \times C_{r^-}[0,\infty) : \liminf_{t\to\infty} z_+(t) = \liminf_{t\to\infty} z_-(t) = -\infty\big\} \tag{1.4}$$

and denote by $\mathcal{B}(\mathcal{D}^r)$ the trace σ-algebra of $\mathcal{B}(C_{r^+}[0,\infty)) \otimes \mathcal{B}(C_{r^-}[0,\infty))$ on $\mathcal{D}^r$.

A.2 Mappings

For each $t \geq 0$, we define the *evaluation map* $E_t : C[0,\infty) \to \mathbb{R}$ by

$$E_t(x) = x(t), \quad x \in C[0,\infty). \tag{2.1}$$

This map is continuous and hence $\mathcal{B}(C[0,\infty))/\mathcal{B}(\mathbb{R})$-measurable.

Lemma 2.1 *For each $r \in \mathbb{R}$, the set $C_r[0,\infty)$ belongs to the σ-algebra $\mathcal{B}(C[0,\infty))$, and the set $\mathcal{D}^r$ belongs to the product σ-algebra $\mathcal{B}(C_{r^+}[0,\infty)) \otimes \mathcal{B}(C_{r^-}[0,\infty))$.*

PROOF: The set $C_r[0,\infty)$ is the pre-image of $\{r\}$ under the measurable mapping E_0, and hence belongs to $\mathcal{B}(C[0,\infty))$. Let $\mathbb{Q}$ denote the set of rational numbers, We have

$$\big\{z \in C[0,\infty) : \liminf_{t\to\infty} z(t) = -\infty\big\} = \bigcap_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{q\in\mathbb{Q}\cap[n,\infty)} \{z \in C[0,\infty) : E_q(z) < -k\},$$

which is thus a set in $\mathcal{B}(C[0,\infty))$. It follows that $\mathcal{D}^r$ belongs to $\mathcal{B}(C_{r^+}[0,\infty)) \otimes \mathcal{B}(C_{r^-}[0,\infty))$. $\square$

We define the *Skorkhod map* $\Gamma : C[0,\infty) \to C[0,\infty)$ by

$$\Gamma(x)(t) := -\min_{0\leq u\leq t}\big(x(u) \wedge 0\big), \quad x \in C[0,\infty),\ 0 \leq t < \infty. \tag{2.2}$$

This map is continuous and hence $\mathcal{B}(C[0,\infty))/\mathcal{B}(C[0,\infty))$-measurable.

Given $(z_+, z_-) \in \mathcal{D}^r$, we construct a function $z \in C_r[0,\infty)$ as follows. We first set $\ell_\pm = \Gamma(z_\pm)$. By the definition of $\mathcal{D}^r$, we have

$$\lim_{t\to\infty} \ell_+(t) = \lim_{t\to\infty} \ell_-(t) = \infty. \tag{2.3}$$

We define the mappings $\Phi_\pm$ from $\mathcal{D}^r$ to $C[0,\infty)$ (continuity is established in Lemma 4.2.3) by the formulas

$$\Phi_+(z_+, z_-)(t) := \sup\big\{u \in [0,t] : \ell_+(u) = \ell_-(t-u)\big\}, \tag{2.4}$$

$$\Phi_-(z_+, z_-)(t) := \inf\big\{u \in [0,t]; \ell_-(u) = \ell_+(t-u)\big\}, \tag{2.5}$$

for all $t \geq 0$. We then set $p_\pm = \Phi_\pm(z_+, z_-)$ and define

$$z = \Psi(z_+, z_-) := z_+ \circ p_+ - z_- \circ p_-. \tag{2.6}$$

In the next subsection we show that $\Psi\colon \mathcal{D}^r \to C_r[0,\infty)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(C_r[0,\infty))$-measurable.

A.3 Measurability

Lemma 3.1 *The mapping $\Lambda : [0,\infty) \times C[0,\infty) \to \mathbb{R}$ defined by*

$$\Lambda(t,x) = x(t), \quad t \in [0,\infty),\ x \in C[0,\infty)$$

is continuous, and hence $\mathcal{B}[0,\infty) \otimes \mathcal{B}(C[0,\infty))/\mathcal{B}(\mathbb{R})$-measurable.

Proof: Let (t_n, x_n) converge to $(t,x) \in [0,\infty) \times C[0,\infty)$. For large enough n, we have $t_n \le t+1 =: T$, and

$$\begin{aligned}\big|\Lambda(t_n,x_n) - \Lambda(t,x)\big| &= \big|x_n(t_n) - x(t)\big| \\ &\le \big|x_n(t_n) - x(t_n)\big| + \big|x(t_n) - x(t)\big| \\ &\le \max_{0\le t\le T} \big|x_n(t) - x(t)\big| + \big|x(t_n) - x(t)\big|.\end{aligned}$$

This last expression has limit zero as $n \to \infty$. $\square$

Lemma 3.2 *Let $\Xi : \mathcal{D}^r \to C[0,\infty)$ have the property that for every $t \ge 0$, the mapping $(z_+, z_-) \to \Xi(z_+,z_-)(t)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R})$-measurable. Then Ξ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(C[0,\infty))$-measurable.*

Proof: It suffices to show that $\Xi^{-1}(B_{x,q}) \in \mathcal{B}(\mathcal{D}^r)$ for each of the open balls in (1.2). But

$$\begin{aligned}\Xi^{-1}(B_{x,q}) &= \big\{(z_+,z_-) \in \mathcal{D}^r : d\big(x, \Xi(z_+,z_-)\big) < q\big\} \\ &= \left\{(z_+,z_-) \in \mathcal{D}^r : \sum_{n=1}^{\infty} \frac{1}{2^n}\Big(1 \wedge \sup_{t\in\mathbb{Q}\cap[0,n]} \big|x - \Xi(z_+,z_-)(t)\big|\Big) < q\right\},\end{aligned}$$

which is in $\mathcal{B}(\mathcal{D}^r)$ because of the assumption that the mapping $(z_+,z_-) \to \Xi(z_+,z_-)(t)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R})$-measurable for each t. $\square$

Lemma 3.3 *For every $t \ge 0$, the mapping $(z_+,z_-) \to \Phi_\pm(z_+,z_-)(t)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R})$-measurable, where Φ_+ and Φ_- are defined by (2.4) and (2.5).*

Proof: For $t \ge 0$ and $u \in \mathbb{R}$, we have

$$\begin{aligned}\big\{(z_+,z_-) \in \mathcal{D}^r : \Phi_+(z_+,z_-)(t) \ge u\big\} &= \big\{(z_+,z_-) \in \mathcal{D}^r : \ell_+(u) \le \ell_-(t-u)\big\} \\ &= \big\{(z_+,z_-) \in \mathcal{D}^r : \Gamma(z_+)(u) \le \Gamma(z_-)(t-u)\big\}.\end{aligned} \tag{3.1}$$

Because Γ_+ and Γ_- are continuous and hence $\mathcal{B}(C[0,\infty))/\mathcal{B}(C[0,\infty))$-measurable, the mapping

$$(z_+,z_-) \to \big(\Gamma(z_+)(u), \Gamma(z_-)(t-u)\big) = \big(E_u \circ \Gamma(z_+), E_{t-u} \circ \Gamma(z_-)\big)$$

is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R}^2)$-measurable. Consequently, the set in (3.1) is in $\mathcal{B}(\mathcal{D}^r)$. Because this is the case for every u, for each fixed t the function $(z_+, z_-) \to \Phi_+(z_+, z_-)(t)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R})$-measurable. The proof that $(z_+, z_-) \to \Phi_-(z_+, z_-)(t)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R})$-measurable, which uses the equivalence

$$\Phi_-(z_+, z_-) \leq u \Longleftrightarrow \ell_-(u) \geq \ell_+(t-u),$$

is similar. □

Theorem 3.4 *The mapping $\Psi : \mathcal{D}^r \to C_r[0,\infty)$ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(C_r[0,\infty))$-measurable.*

Proof: We fix $t \geq 0$ and decompose Ψ into the concatenation of mappings

$$\begin{aligned}
(z_+, z_-) &\to \big(\Phi_+(z_+, z_-)(t), z_+, z_-, \Phi_-(z_+, z_-)(t)\big) && (3.2)\\
&\to \Big(\Lambda\big(\Phi_+(z_+, z_-)(t), z_+\big), \Lambda\big(\Phi_-(z_+, z_-)(t), z_-\big)\Big) && (3.3)\\
&\to \Lambda\big(\Phi_+(z_+, z_-)(t), z_+\big) - \Lambda\big(\Phi_-(z_+, z_-)(t), z_-\big) && (3.4)\\
&= \Psi(z_+, z_-)(t).
\end{aligned}$$

The mapping in (3.2) is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{D}^r) \otimes \mathcal{B}(\mathbb{R})$-measurable by Lemma 3.3. The mapping in (3.3) is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{D}^r) \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R}^2)$ measurable by Lemma 3.1. The mapping in (3.4), which is subtraction in $\mathbb{R}^2$, is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$-measurable. Therefore, for every $t \geq 0$, the mapping

$$(z_+, z_-) \to \Psi(z_+, z_-)(t)$$

is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(\mathbb{R})$-measurable. It follows from Lemma 3.2 that Ψ is $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(C[0,\infty))$-measurable. Since Ψ maps into $C_r[0,\infty)$, it is also $\mathcal{B}(\mathcal{D}^r)/\mathcal{B}(C_r[0,\infty))$-measurable. □

Appendix B

Absorbed Brownian Motion

B.1 Proof of (6.22)

$$
\begin{aligned}
\int_0^\infty p^0(t,x,y)\,dy &= \frac{1}{\sqrt{2\pi t}}\int_0^\infty \exp\left(-\frac{(y-x)^2}{2t}\right)\,dy - \frac{1}{\sqrt{2\pi t}}\int_0^\infty \exp\left(-\frac{(y+x)^2}{2t}\right)\,dy \\
&= \frac{1}{\sqrt{2\pi t}}\int_{-x}^\infty \exp\left(-\frac{z^2}{2t}\right)\,dz - \frac{1}{\sqrt{2\pi t}}\int_x^\infty \exp\left(-\frac{z^2}{2t}\right)\,dz \\
&= \frac{2}{\sqrt{2\pi t}}\int_0^x \exp\left(-\frac{z^2}{2t}\right)\,dz \\
&= 2\left(\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^x \exp\left(-\frac{z^2}{2t}\right)\,dz - \frac{1}{2}\right) \\
&= 2\left(\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^x \exp\left(-\frac{z^2}{2t}\right)\,dz - 1\right) + 1 \\
&= 1 - \frac{2}{\sqrt{2\pi t}}\int_x^\infty \exp\left(-\frac{z^2}{2t}\right)\,dz.
\end{aligned}
$$

B.2 Proof of (6.26)

We begin with the observation

$$
\begin{aligned}
&p^0(s,x,y)p^0(t,y,z) \\
&= \frac{1}{2\pi\sqrt{st}}\left[\exp\left(-\frac{(y-x)^2}{2s}\right)-\exp\left(-\frac{(y+x)^2}{2s}\right)\right] \\
&\qquad\times\left[\exp\left(-\frac{(z-y)^2}{2t}\right)-\exp\left(-\frac{(z+y)^2}{2t}\right)\right] \\
&= \frac{1}{2\pi\sqrt{st}}\left[\exp\left(-\frac{(y-x)^2}{2s}-\frac{(z-y)^2}{2t}\right)-\exp\left(-\frac{(y+x)^2}{2s}-\frac{(z-y)^2}{2t}\right)\right. \\
&\qquad\left.-\exp\left(-\frac{(y-x)^2}{2s}-\frac{(z+y)^2}{2t}\right)+\exp\left(-\frac{(y+x)^2}{2s}-\frac{(z+y)^2}{2t}\right)\right] \\
&= \frac{1}{2\pi\sqrt{st}}\left[\exp\left(-\frac{s+t}{2st}\left(y-\frac{tx+sz}{s+t}\right)^2-\frac{1}{2(s+t)}(x-z)^2\right)\right. \\
&\qquad-\exp\left(-\frac{s+t}{2st}\left(y+\frac{tx-sz}{s+t}\right)^2-\frac{1}{2(s+t)}(x+z)^2\right) \\
&\qquad-\exp\left(-\frac{s+t}{2st}\left(y-\frac{tx-sz}{s+t}\right)^2-\frac{1}{2(s+t)}(x+z)^2\right) \\
&\qquad\left.+\exp\left(-\frac{s+t}{2st}\left(y+\frac{tx+sz}{s+t}\right)^2-\frac{1}{2(s+t)}(x-z)^2\right)\right] \\
&= \frac{1}{\sqrt{2\pi(s+t)}}\exp\left(-\frac{1}{2(s+t)}(x-z)^2\right)\cdot\sqrt{\frac{s+t}{2\pi st}} \\
&\qquad\times\left[\exp\left(-\frac{s+t}{2st}\left(y-\frac{tx+sz}{s+t}\right)^2\right)+\exp\left(-\frac{s+t}{2st}\left(-y-\frac{tx+sz}{s+t}\right)^2\right)\right] \\
&\quad-\frac{1}{\sqrt{2\pi(s+t)}}\exp\left(-\frac{1}{2(s+t)}(x+z)^2\right)\cdot\sqrt{\frac{s+t}{2\pi st}} \\
&\qquad\times\left[\exp\left(-\frac{s+t}{2st}\left(y+\frac{tx-sz}{s+t}\right)^2\right)+\exp\left(-\frac{s+t}{2st}\left(-y+\frac{tx-sz}{s+t}\right)^2\right)\right]. \quad (2.1)
\end{aligned}
$$

Because

$$
\begin{aligned}
&\sqrt{\frac{s+t}{2\pi st}}\int_0^\infty\left[\exp\left(-\frac{s+t}{2st}\left(y-\frac{tx+sz}{s+t}\right)^2\right)+\exp\left(-\frac{s+t}{2st}\left(-y-\frac{tx+sz}{s+t}\right)^2\right)\right]dy \\
&\qquad= \sqrt{\frac{s+t}{2\pi st}}\int_{-\infty}^\infty\exp\left(-\frac{s+t}{2st}\left(y-\frac{tx+sz}{s+t}\right)^2\right)dy \\
&\qquad= 1
\end{aligned}
$$

and

$$\sqrt{\frac{s+t}{2\pi st}}\int_0^\infty \left[\exp\left(-\frac{s+t}{2st}\left(y+\frac{tx-sz}{s+t}\right)^2\right)+\exp\left(-\frac{s+t}{2st}\left(-y+\frac{tx-sz}{s+t}\right)^2\right)\right]\,dy$$
$$\begin{aligned} &= \sqrt{\frac{s+t}{2\pi st}}\int_{-\infty}^\infty \exp\left(-\frac{s+t}{2st}\left(y+\frac{tx-sz}{s+t}\right)^2\right)\,dy \\ &= 1, \end{aligned}$$

we can integrate (2.1) to obtain

$$\begin{aligned} &\int_0^\infty p^0(s,x,y)p^0(t,z,z)\,dy \\ &\quad= \frac{1}{\sqrt{2\pi(s+t)}}\exp\left(-\frac{1}{2(s+t)}(x-z)^2\right)-\frac{1}{\sqrt{2\pi(s+t)}}\exp\left(-\frac{1}{2(s+t)}(x+z)^2\right) \\ &\quad= p^0(s+t,x,z). \end{aligned}$$

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