Diversity Considerations in School Choice: Developing Fair and Effective Mechanisms

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Abstract

Controlled school choice policies aim to provide families a choice as to which school their child will attend while maintaining diversity in schools. These policies are often implemented in the form of minimum and maximum quotas on student types (that can be based on race, ethnicity or socioeconomic factors). When these quotas are interpreted as hard bounds, standard fairness and non-wastefulness properties are compromised, imposing a cost on student welfare. By interpreting the minimum quota as a flexible guideline (a "soft" bound), I employ a mixed interpretation of these quotas to formulate a modified version of the student-optimal deferred acceptance algorithm that guarantees fairness. Furthermore, I show that both this and a purely soft interpretation of the diversity bounds can be used to formulate effective mechanisms that ensure fairness, even in settings with notionally hard bounds.

1 Introduction

School choice policies aimed at providing families a choice with regards to which school their child will attend are becoming increasingly widespread in the United States. Controlled school choice aims to provide families with the same choice while maintaining diversity (racial, ethnic or otherwise) in schools. These policies often take the form of minimum and maximum quotas for different student types at schools. For example, affirmative action policies are a special case of controlled school choice, ones which give traditionally under-represented groups priority over others at the schools of their choice.

There are numerous other examples of controlled school choice in the United States. These include the policies of Kansas City and St. Louis, Missouri, where controlled school choice is implemented due to court orders enforcing desegregation. The city of White Plains (New York) voluntarily implements its controlled school choice program, with the White Plains Board of Education's racial and ethnic balance policy playing a part in elementary and middle school admissions. Historically, Boston Public Schools' controlled school choice program was employed due to laws preventing racial imbalance, though at present it is voluntarily used to achieve diversity across ethnic and socioeconomic lines.

Controlled school choice is not limited to racial or ethnic diversity. Chicago Public Schools promote diversity by categorizing students according to their socioeconomic status; students are divided into four tiers based on their family income and a certain number of seats is allocated to each tier. New York City's EdOpt programs control for student ability and are required to admit students of different ability ranges. The EdOpt schools use students' scores on the English Language Arts exam as a measure of student ability. This flexible interpretation of diversity emphasizes the relevance of controlled school choice and the importance of studying such policies.

The question then arises of how to implement controlled school choice while minimizing the adverse effects to students. An idea central to student welfare under school choice is that of *fairness*. A definition of fairness that is consistent with the literature on the subject is the removal of *justifiable envy*. Justifiable envy exists when there is a student and school that are not matched together such that the student prefers the school to her assignment and has priority over another student at that school. Another important notion of student welfare is that of *non-wastefulness*, which captures the idea that no student should be denied an empty seat at any school. From the perspective of a policy maker (such as a school district), fairness and non-wastefulness are highly desirable.

Abdulkadiroğlu and Sönmez (2003) consider a version of controlled schooled choice with type-specific quotas and propose the student-proposing deferred acceptance algorithm as a mechanism for controlled school choice. They go on to show that this algorithm removes justifiable envy, but only among students of the same type e.g. if gender is being controlled for, justifiable envy is removed between all male students, though a male student may still justifiably envy a female student. Their work exemplifies the welfare costs in terms of fairness that are introduced by these diversity quotas.

Ehlers et al. (2011) build on the approach introduced by Abdulkadiroğlu and Sönmez (2003) and show that when diversity quotas are "hard bounds" (that cannot be violated), then there may be no fair assignments. They go on to provide an alternative interpretation of diversity constraints as "soft bounds", which are to be used more as guidelines and not binding. They define a version of the student-proposing deferred acceptance algorithm, the DAASB, which results in a fair and non-wasteful assignment. However, the attractive fairness and non-wastefulness properties are achieved at the cost of weakening the diversity constraints; in fact, the soft bounds approach still allows schools to be segregated.

In cases when the diversity constraints are strictly enforced, guaranteeing that these constraints are met comes at the cost of fairness. As shown by Ehlers et al. (2011), fairness can be guaranteed by weakening the interpretation of the bounds on student types. Their DAASB does not ensure that the given diversity requirements are upheld, but can it still be a useful mechanism? If a view is taken in which fairness is indispensable, can useful mechanisms be constructed that respect the constraints in most cases? By using a laxer interpretation of the bounds, mechanisms can be designed to guarantee fairness while providing a high possibility of respecting constraints (without guaranteeing that they are not violated). The idea of fair mechanisms that are effective at meeting constraints encompasses the motivation behind this study.

In this paper, I adopt a mixed bounds approach (where minimum quotas are soft bounds and maximum quotas are hard bounds) as a compromise between diversity and student welfare. I define a version of the student-proposing deferred acceptance algorithm, the DAASFHC, under this framework and show that it always results in a fair assignment for students (Proposition 1). This approach eliminates the loss of fairness associated with hard bounds while imposing stronger diversity restrictions than those in a soft bounds approach. However, this comes at the cost of non-wastefulness and the possibility of some students being unassigned (Proposition 3). Since the DAASFHC and DAASB are closely related, I provide a comparison between the two and find that the DAASB assignment is always weakly preferred to the DAASFHC assignment by students (Proposition 4). I also provide conditions under which the outcomes of the DAASB and DAASFHC are the same (Proposition 5). Finally, I use computer simulations to test the efficacy of the DAASB and the DAASFHC at meeting the given diversity constraints, and quantify the costs to student welfare at which the DAASFHC provides improved effectiveness over the DAASB.

1.1 Review of Related Literature

Abdulkadiroğlu (2005) studies the college admissions problem with affirmative action. His results show that when college preferences are substitutable, there is no stable mechanism that makes truthful revelation of preferences a dominant strategy for every student. He goes on to show that truthful preference revelation is a dominant strategy for every student under student-proposing deferred acceptance algorithm when college preferences are responsive over sets of students that meet the type-specific quotas.

Kojima (2012) considers a school choice model with two types of students (majority and minority) in which affirmative action is implemented in the form of maximum quotas for majority students and proceeds to show that such policies might actually be detrimental to minority students - the supposed beneficiaries in this setting. Hafalir, Yenmez and Yildirim (2011) build on Kojima's results and propose the implementation of *minority reserves* (as opposed majority quotas) where minority students are given higher priority (than majority students) until they fill the minority reserves. They adapt both the student-proposing deferred acceptance algorithm and the top trading cycles algorithm to this setting and find that minorities, on average, are better off under minority reserves than under majority quotas.

Fragiadakis et al. (2012) revisit the problem of controlled school choice with both minimum and maximum quotas. They treat strategy-proofness as a necessary property of any assignment mechanism and propose variants of the student-proposing deferred acceptance algorithm that preserve strategy-proofness using relaxed definitions of fairness and non-wastefulness.

Kamada and Kojima (2010) study a related model of entry-level medical markets with regional caps (that restrict the number of doctors that can be assigned to hospitals in a region); their results are in a similar vein to those of Ehlers et al. (2011) where they show that *strongly stable* assignments may not exist and propose the *flexible deferred acceptance algorithm* which finds a *stable* and *non-wasteful* assignment[1].

This paper proceeds as follows. Section 2 introduces the controlled school choice model. Section 3 provides a general form of the student-proposing deferred acceptance algorithm, along with two results from literature that provide conditions on the choice functions under which the student-proposing deferred acceptance algorithm produces a fair assignment and is strategy-proof. Section 4 considers the mixed bounds approach to controlled school choice and presents the theoretical results as well as results from the simulations. Section 5 concludes.

[1]The notions of strong stability and stability correspond to "fairness across types" and "fairness for same types" (as proposed in Ehlers et al., 2011) respectively, and the flexible deferred acceptance algorithm possesses properties similar to those of the DAASB.

2 Controlled School Choice

In this paper, I consider the model studied by Ehlers et al. (2011).

1. A finite set of students $S = \{s_1, s_2, \dots, s_n\}$,
2. A finite set of schools $C = \{c_1, c_2, \dots, c_m\}$,
3. A capacity vector $q = (q_{c_1}, q_{c_2}, \dots, q_{c_m})$ where q_c is the capacity of (or number of seats in) school c,
4. A students' preference profile $\succ_S = (\succ_{s_1}, \succ_{s_2}, \dots, \succ_{s_n})$ where $\succ_s$ is a strict preference relation over $C \cup \{s\}$ for each student $s \in S$,
5. A schools' priority order profile $\succ_C = (\succ_{c_1}, \succ_{c_2}, \dots, \succ_{c_m})$ where $\succ_c$ is a strict priority ranking over S for each school $c \in C$,
6. A type space $T = \{t_1, t_2, \dots, t_k\}$,
7. A mapping $\tau : S \longmapsto T$ where $\tau(s)$ is the type of student s,
8. For each school c, the type-specific diversity constraint vectors $\underline{q}_c^T = (\underline{q}_c^{t_1}, \underline{q}_c^{t_2}, \dots, \underline{q}_c^{t_k})$ and $\overline{q}_c^T = (\overline{q}_c^{t_1}, \overline{q}_c^{t_2}, \dots, \overline{q}_c^{t_k})$ such that $\underline{q}_c^t \leq \overline{q}_c^t \leq q_c$ for all $t \in T$ and $\sum_{t \in T} \underline{q}_c^t \leq q_c \leq \sum_{t \in T} \overline{q}_c^t$; $\underline{q}_c^t$ and $\overline{q}_c^t$ are the minimum and maximum quota respectively for students of type t at school c.

The minimum quota $\underline{q}_c^t$ is the least number of seats that school c must allocate to students of type t, and is referred to as the **floor** or **lower bound** for type t at school c; similarly, the maximum quota $\overline{q}_c^t$ is the greatest number of seats that c can allocate to students of type t and is referred to as the **ceiling** or **upper bound** for type t at c. In this paper, the following two interpretations of the floors and ceilings are employed. A diversity constraint is **hard** if it cannot be violated under any circumstances. A diversity constraint is **soft** if it can be violated and is used as a guideline to modify school priorities. The manner of this modification is specified as follows: if a diversity constraint at c is met by students of type t but not of type t', then all students of type t' are given a higher priority ranking than all students of type t by c.

An assignment is a mapping $\mu : (C \cup S) \longmapsto (C \cup S)$ such that:

(i) $\mu(s) \in C \cup \{s\}$ for all $s \in S$

(ii) $|\mu(c)| \leq q_c$ and $\mu(c) \subseteq S$

(iii) $\mu(s) = c$ if and only if $s \in \mu(c)$

Less formally, an assignment matches each student to at most one school, and each school is matched to a subset of students such that the number of students it is matched to does not exceed its capacity. Moreover, if a student is matched to a school, then it must be in the subset that the same school is matched with and vice versa. If a student s is not matched with a school under μ, then $\mu(s) = s$ and the student is said to be **unassigned under** μ.

For the purposes of this study, I assume that every student prefers to be assigned to any school, rather than being unassigned. This assumption can be formalized as for any $s \in S$, $c \succ_s s$ for all $c \in C$. A consequence of this assumption is that **individual rationality** is never violated under this framework; that is, no student is assigned to a school that is unacceptable to her. This is crucial as fairness requires that individual rationality not be violated for any student.

Given two assignments μ_1 and μ_2, a student s **weakly prefers** μ_1 to μ_2 if either $\mu_1(s) \succ_s \mu_2(s)$ or $\mu_1(s) = \mu_2(s)$. Notationally, if s weakly prefers μ_1 to μ_2, then $\mu_1 \succeq_s \mu_2$. Given a subset $\hat{S} \subseteq S$, if $\mu_1 \succeq_s \mu_2$ for all $s \in \hat{S}$, then $\mu_1 \succeq_{\hat{S}} \mu_2$. Finally, μ_1 **Pareto dominates** μ_2 if $\mu_1 \succeq_S \mu_2$. An assignment μ is **Pareto optimal for students**, or simply **student-optimal**, in a set of assignments if it is not Pareto dominated by any assignment in that set.

3 General Deferred Acceptance in School Choice

In this section, I introduce a general form of the Deferred Acceptance Algorithm (DAA) in the context of school choice. I then provide two results from literature that provide conditions under which the DAA is fair and group strategy-proof.

A convenient way of adapting the classic DAA introduced by Gale and Shapley (1962) to many-to-one matching problems and, in particular, school choice problems is by specifying a choice function for schools. A choice function for a school c is a mapping $Ch_c : 2^S \longmapsto 2^S$ such that $Ch_c(S') \subseteq S'$ for all subsets $S' \subseteq S$. Given choice functions Ch_c for all schools $c \in C$, the general form of the student-proposing DAA is as follows:

General Deferred Acceptance Algorithm

Step 1: Start with the assignment μ_0 such that every student is unassigned under μ_0. Let $S_{c,1}$ be the set of students that have school c as their first choice; they apply to c at the first step of the algorithm. School c tentatively admits the students in $Ch_c(S_{c,1})$ and rejects the rest. Define the assignment μ_1 by $\mu_1(c) = Ch_c(S_{c,1})$.

Step k: Start with the assignment μ_{k-1} from step $k-1$. If no students were rejected at step $k-1$, then stop. Else, all students that were rejected in step $k-1$ apply to their next choice school. Let $S_{c,k}$ denote the set of students that were either tentatively admitted to c in step $k-1$ or that apply to school c at step k. School c tentatively admits those students in $Ch_c(S_{c,k})$ and rejects the rest. Define the assignment μ_k by $\mu_k(c) = Ch_c(S_{c,k})$.

If the algorithm terminates at step k, then the resulting DAA assignment is $\mu = \mu_{k-1}$.

If the choice functions are well-defined then this formulation is particularly useful due to results from literature that confirm the fairness and strategy-proof characteristics of the DAA, provided these choice functions satisfy certain conditions. Before these results are presented however, it is beneficial to introduce notions of fairness and strategy-proofness.

A student s and school c form a **blocking pair** (s, c) if $s \notin \mu(c)$, $c \succ_s \mu(s)$ and $s \in Ch_c(\mu(c) \cup \{s\})$. Less formally, (s, c) is a blocking pair if s prefers c to her assignment under μ and c prefers to give a seat to s (either by giving s an empty seat or by replacing a student in $\mu(c)$ with s). If (s, c) constitutes a blocking pair, then there is **justifiable envy**. An assignment μ is **fair** if there are no blocking pairs or equivalently, if it removes justifiable envy.

Roth and Sotomayor (1990) provide a result that guarantees not only the existence of a fair assignment, but also that the student-proposing DAA results in a student-optimal fair assignment. The idea of substitutability of schools' choice functions is central to this result. A choice function Ch_c satisfies **substitutability** if for any subset $\tilde{S} \subseteq S$ and $s, s' \in \tilde{S}$ with $s \neq s'$, if $s \in Ch_c(\tilde{S})$, then $s \in Ch_c(\tilde{S} \backslash \{s'\})$. Roth and Sotomayor's result, applied to the problem of school choice, can now be presented as below.

Theorem 1 (Roth and Sotomayor, 1990, Theorem 6.8)**.** *If every school's choice function satisfies substitutability (and if preferences are strict), then the student-proposing DAA results in a student-optimal fair assignment.*

The next result relates to the group strategy-proof property. Recall that a mechanism is a mapping from the space of student preference profiles to the set of assignments. A mechanism F is group strategy-proof if for any group of students $\hat{S} \subseteq S$ and for any student preference profile $\succ_S$, there is no $\tilde{\succ}_{\hat{S}}$ such that $F((\tilde{\succ}_{\hat{S}}, \succ_{S \backslash \hat{S}})) \succeq_{\hat{S}} F(\succ_S)$ i.e. no group of students can modify or misreport their preferences so as to make every student in the group better off. Hatfield and Kojima (2009) show that if the schools' choice functions satisfy substitutability and the law of aggregate demand, then the student-proposing DAA is group strategy-proof. A choice function Ch_c satisfies the **law of aggregate demand** if for any $S'' \subseteq S' \subseteq S$, $|Ch_c(S'')| \leq |Ch_c(S')|$. The result is formalized as follows.

Theorem 2 (Hatfield and Kojima, 2009, Theorem 1)**.** *If every school's choice function satisfies substitutability and the law of aggregate demand, then the student-proposing DAA is group strategy-proof.*

These preliminary results provide the basis for the approach adopted ahead in this paper, which is to construct choice functions that incorporate the diversity restrictions under consideration. In the next section, I consider controlled school choice with a mixed interpretation of the diversity constraints - ceilings are hard bounds and floors are soft bounds.

4 School Choice with Soft Floors and Hard Ceilings

Ehlers et al. (2011) show that there may be no fair assignments in a controlled school choice problem where minimum quotas are imposed as hard bounds. To circumvent this obstacle, I interpret the minimum quotas as soft bounds: constraints that are not binding and used as guidelines to modify school priorities. In the same paper, Ehlers et al. (2011) consider controlled school choice problems in which both minimum and maximum quotas are soft bounds to eliminate existence issues of fair assignments. However, such an interpretation allows schools to be segregated (since soft bounds can be violated). This is an unattractive proposition, since the idea behind the diversity quotas is to prevent segregation. Therefore, I interpret the ceilings as hard bounds. I adopt this mixed approach as a compromise between fairness and diversity.

In this section, I define the **Deferred Acceptance Algorithm with Soft Floors and Hard Ceilings (DAASFHC)** by defining choice functions consistent with the interpretation of the diversity constraints. I then establish the fairness and group strategy-proofness properties of the DAASFHC. I also provide a negative result which shows that the DAASFHC may leave some students unassigned. Section 4.1 reviews the DAASB introduced in Ehlers et al. (2011) and Section 4.2 compares the DAASFHC to the DAASB in terms of Pareto dominance relations between the two.

In context of this mixed interpretation of diversity constraints, relevant properties of assignments are defined as follows. Given a controlled school choice problem, an assignment

μ is **feasible** if, for all schools $c \in C$, $|\mu(c)| \leq q_c$ and $|\mu^t(c)| \leq \overline{q}_c^t$ for all $t \in T$ (that is, μ does not violate the capacity and the ceiling for any type at any school). An assignment μ **respects constraints** if it is feasible and if $|\mu^t(c)| \geq \underline{q}_c^t$ for all $t \in T$ and $c \in C$. In cases where schools are legally required to meet the imposed diversity restrictions, assignments that respect constraints become particularly relevant.

Two notions of non-wastefulness are considered: an assignment μ is **non-wasteful** if $c \succ_s \mu(s)$ for some student s and school c implies $|\mu(c)| = q_c$, and is **constrained non-wasteful** if for some student s and school c, $c \succ_s \mu(s)$ and $|\mu(c)| < q_c$ imply $|\mu^{\tau(s)}(c)| = \overline{q}_c^{\tau(s)}$. Less formally, if a student prefers some school over the one she is assigned to, non-wastefulness means that her preferred school must have filled its capacity while constrained non-wastefulness means that if her preferred school has empty seats, then it must have filled its ceiling for students of her type.

Under soft floors and hard ceilings, given an assignment μ, a student s **justifiably claims an empty seat** at school c if s can be enrolled at c without violating either the capacity constraint of c or the ceiling for students of type $\tau(s)$ at c. Formally, s justifiably claims an empty seat at c if $c \succ_s \mu(s)$, $|\mu(c)| < q_c$ and $|\mu^{\tau(s)}(c)| < \overline{q}^{\tau(s)}$. A student s **justifiably envies** another student s' if $c \succ_s \mu(s)$, $\mu(s') = c$, $\mu^{\tau(s)}(c) \leq \overline{q}_c^{\tau(s)}$ and either (1) or (2) where:

(1) $\tau(s) = \tau(s')$ and $s \succ_c s'$

(2) $\tau(s) \neq \tau(s')$ and either (a) or (b):

- (a) $|\mu^{\tau(s)}(c)| < \underline{q}_c^{\tau(s)}$ and $|\mu^{\tau(s')}(c)| \geq \underline{q}_c^{\tau(s')}$
- (b) Either $|\mu^t(c)| < \underline{q}_c^t$ or $|\mu^t(c)| \geq \underline{q}_c^t$ for all $t \in \{\tau(s), \tau(s')\}$, and $s \succ_c s'$

Therefore, an assignment μ **removes justifiable envy under soft floors and hard ceilings** if for any student s and school c such that $c \succ_s \mu(s)$ with $\tau(s) = t$ if both $|\mu^t(c)| \geq \underline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu^t(c)$ and either

(i) $|\mu^t(c)| = \overline{q}_c^t$, or

(ii) $\underline{q}_c^t \leq |\mu^t(c)| < \overline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu(c)$ such that $\underline{q}_c^{\tau(s')} < |\mu^{\tau(s')}(c)| \leq \overline{q}_c^{\tau(s')}$

An assignment is **fair** if it removes justifiable envy under soft floors and hard ceilings.

With these definitions in hand, the school's choice function can be formulated to account for the soft floors and hard ceilings. The choice function is defined in the following manner: given a subset of students $S' \subseteq S$ and a school $c \in C$, let $Ch_c(S', q_c, (q_c^t)_{t \in T})$ be the subset

$S'' \subseteq S'$ that includes the highest ranked students, according to $\succ_c$, in S' such that there are no more than q_c students in total and q_c^t students of type t in S''. Further, define:

$$Ch_c^{(1)}(S') = Ch_c(S', q_c, (\underline{q}_c^t)_{t \in T}), \text{ and}$$

$$Ch_c^{(2)}(S') = Ch_c(S' \backslash Ch_c^{(1)}(S'), q_c - |Ch_c^{(1)}(S')|, (\overline{q}_c^t - \underline{q}_c^t)_{t \in T}))$$

Less formally, $Ch_c^{(1)}(S')$ is the set of students with highest priorities in S' such that the floor is not exceeded for any student type. $Ch_c^{(2)}(S')$ is the subset of students remaining in S' (i.e. in $S' \backslash Ch_c^{(1)}(S')$) that have the highest priorities up to the ceiling of their types. Finally, define the **choice function under soft floors and hard ceilings** as:

$$Ch_c(S') = Ch_c^{(1)}(S') \cup Ch_c^{(2)}(S')$$

It is worth noting that the notion of fairness as the absence of blocking pairs is equivalent to the removal of justifiable envy as defined in this section.

The DAASFHC is simply defined by applying the choice function under soft floors and hard ceilings to the general DAA provided in Section 2. The next results confirm that the DAASFHC is fair, constrained non-wasteful and group strategy-proof.

Proposition 1. *The DAASFHC results in a fair assignment that is student-optimal among all such assignments.*

Proof. By Theorem 1, it suffices to show that Ch_c satisfies substitutability. Consider some $\tilde{S} \subseteq S$ and some $s \in Ch_c(\tilde{S})$. For any $s' \in \tilde{S} \backslash \{s\}$, note that $s \in Ch_c(\tilde{S})$ implies $s \in Ch_c(\tilde{S}) \backslash \{s'\}$. Consider some $s \in Ch_c(\tilde{S}) \backslash \{s'\}$; either $s' \notin Ch_c(\tilde{S})$ or $s' \in Ch_c(\tilde{S})$.

If $s' \notin Ch_c(\tilde{S})$, then $Ch_c(\tilde{S}) \backslash \{s'\} = Ch_c(\tilde{S} \backslash \{s'\})$ and $s \in Ch_c(\tilde{S} \backslash \{s'\})$. If $s' \in Ch_c(\tilde{S})$, then either $s' \succ_c s$ or $s \succ_c s'$. If $s' \succ_c s$, then the ranking of s according to $\succ_c$ in $S \backslash \{s'\}$ is either improved from or remains the same as in $\tilde{S}$ and hence, $s \in Ch_c(\tilde{S} \backslash \{s'\})$. If $s \succ_c s'$, then the ranking of s according to $\succ_c$ in $\tilde{S} \backslash \{s'\}$ is the same as in $\tilde{S}$ and $s \in Ch_c(\tilde{S} \backslash \{s'\})$.

Therefore, $Ch_c(\tilde{S}) \backslash \{s'\} \subseteq Ch_c(\tilde{S} \backslash \{s'\})$ and $s \in Ch_c(\tilde{S} \backslash \{s'\})$, which implies that Ch_c satisfies substitutability. Hence, the DAASFHC assignment is fair and student-optimal. □

Corollary 1. *The DAASFHC assignment is constrained non-wasteful.*

Proof. Let μ be the DAASFHC assignment for a controlled school choice problem. By way of contradiction, suppose μ is not constrained non-wasteful; then there exist a student s and school c such that $c \succ_s \mu(s)$, $|\mu(c)| < q_c$ and $|\mu^{\tau(s)}(c)| < \overline{q}_c^{\tau(s)}$. By definition of the choice function, $Ch_c(\mu(c) \cup \{s\}) = \mu(c) \cup \{s\} \Rightarrow s \in Ch_c(\mu(c) \cup \{s\})$. Hence, (s, c) is a blocking pair by definition. This contradicts the fairness of μ (Proposition 1). □

Proposition 2. *The DAASFHC is group-strategy proof.*

Proof. Since Ch_c satisfies substitutability (Proposition 1), by Theorem 2 it suffices to show that Ch_c satisfies the law of aggregate demand.

Let $Ch_c^t(\tilde{S}) = Ch_c(\tilde{S}) \cap S^t$ i.e. $Ch_c^t(\tilde{S})$ is the subset of all students with type t in $Ch_c(\tilde{S})$. For any $S'' \subseteq S' \subseteq S$, if $Ch_c(S'') = q_c$, then $Ch_c(S') = q_c$. If $Ch_c(S'') < q_c$, then either $|Ch_c(S')| = |Ch_c(S'')|$ (if $S' = S''$ or $|Ch_c^t(S'')| = \bar{q}_c^t \;\; \forall t \in \{\tau(s) : s \in S' \setminus S''\}$), or $|Ch_c(S')| > |Ch_c(S'')|$ (if $|S''| < |S'|$ and $|Ch_c^t(S'')| < \bar{q}_c^t$ for at least one $t \in \{\tau(s) : s \in S' \setminus S''\}$). Therefore, Ch_c satisfies the law of aggregate demand. □

Though the DAASFHC possesses the desired fairness property, there is a trade-off in terms of non-wastefulness. The next result shows, by means of an example, that some students may be unassigned under the DAASFHC.

Proposition 3. *The DAASFHC assignment may not be non-wasteful.*

Proof. This result can be shown by the following example. Consider the controlled school choice problem: $C = \{c_1, c_2\}$, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $T = \{t_1, t_2\}$. Each school has a capacity of 3, $q_{c_1} = q_{c_2} = 3$; the floors are given by $\underline{q}_{c_1}^T = \underline{q}_{c_2}^T = (0, 0)$ while the ceilings are $\overline{q}_{c_1}^T = (2, 3)$ and $\overline{q}_{c_2}^T = (3, 3)$. Student types are given by $\tau(s_1) = \tau(s_2) = \tau(s_3) = t_1$ and $\tau(s_4) = \tau(s_5) = \tau(s_6) = t_2$. Student preferences are: $c_1 \succ_{s_i} c_2$ for $i \in \{1, 2, 3\}$, and $c_2 \succ_{s_i} c_1$ for $i \in \{4, 5, 6\}$. School priorities are given by:

$$\succ_{c_1}: s_1 \succ_{c_1} s_2 \succ_{c_1} s_3 \succ_{c_1} s_4 \succ_{c_1} s_5 \succ_{c_1} s_6$$
$$\succ_{c_2}: s_6 \succ_{c_2} s_5 \succ_{c_2} s_4 \succ_{c_2} s_3 \succ_{c_2} s_2 \succ_{c_2} s_1$$

The DAASFHC assignment is:

$$\mu(c_1) = \{s_1, s_2\},\ \mu(c_2) = \{s_4, s_5, s_6\}$$

Here, student s_3 is unassigned under μ while school c_1 has an empty seat $\Rightarrow \mu$ is not non-wasteful. □

Corollary 2. *Students may be unassigned under the DAASFHC assignment.*

The observation that some students might be unassigned, even though schools have empty seats to accommodate them, is particularly concerning. From a policy perspective, it would be difficult to justify promoting diversity at the cost of leaving some students out of the public school system. While other options (such as private schools or homeschooling) are available, they may not be accessible to all students (for example, private schools might be too expensive and parents may be unable to facilitate homeschooling due to occupational commitments or tutor costs). Furthermore, since the DAASFHC assignment is the student-optimal fair assignment, any students unassigned under the DAASFHC will be unassigned under any fair assignment. This possibility of unassigned students is a major drawback of the soft floors and hard ceilings framework - a solution requires a relaxation in either the notion of fairness or the diversity constraints.

4.1 Review of Controlled School Choice with Soft Bounds

This approach, introduced by Ehlers et al. (2011), involves interpreting both the minimum and maximum quotas in a controlled school choice problem as soft bounds. Such an interpretation means that neither minimum nor maximum quotas are binding constraints, but both are used as guidelines that modify school priorities to accurately reflect the community's diversity in schools. To provide a basis for comparison between the DAASB and the DAASFHC, definitions of relevant properties under soft bounds are considered as they are defined in Ehlers et al. (2011).

Since both diversity constraints are soft, any assignment that respects the capacity of every school is **feasible under soft bounds**. Further note that non-wastefulness as defined in this paper is equivalent to **non-wastefulness under soft bounds**. The most significant difference between controlled school choice under soft bounds and under soft floors and hard ceilings is in the definition of fairness. An assignment is **fair under soft bounds** if it removes justifiable envy under soft bounds; an assignment **removes justifiable envy under soft bounds** if for any student s and school c such that $c \succ_s \mu(s)$ with $\tau(s) = t$ both $|\mu^t(c)| \geq \underline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu^t(c)$ and either

(1) $|\mu^t(c)| \geq \overline{q}_c^t$ and $s' \succ_c s$ for all $s' \in \mu(c)$ such that $|\mu^{\tau(s')}(c)| > \overline{q}_c^{\tau(s')}$, or

(2) $\underline{q}_c^t \leq |\mu^t(c)| < \overline{q}_c^t$, $|\mu^{t'}(c)| > \overline{q}_c^{t'}$ for all $t' \in T\backslash\{t\}$, and $s' \succ_c s$ for all $s' \in \mu(c)$ such that $\underline{q}_c^{\tau(s')} < |\mu^{\tau(s')}(c)| \leq \overline{q}_c^{\tau(s')}$

The choice function under soft bounds accounts for the difference in interpretation of the ceilings between the two contexts. Consider $Ch_c(S', q_c, (q_c^t)_{t\in T})$ as defined previously, and define:

$$Ch_c^{(3)}(S') = Ch_c(S' \backslash (Ch_c^{(1)}(S') \cup Ch_c^{(2)}(S')), q_c - |Ch_c^{(1)}(S') \cup Ch_c^{(2)}(S')|, (q_c - \overline{q}_c^t)_{t \in T}))$$

If $Ch_c^{(1)}(S')$ and $Ch_c^{(2)}(S')$ are defined as under soft floors and hard ceilings, then the **choice function under soft bounds** is defined as:

$$\begin{aligned} &Ch_c^{SB}(S') = Ch_c^{(1)}(S') \cup Ch_c^{(2)}(S') \cup Ch_c^{(3)}(S') \\ \Rightarrow &Ch_c^{SB}(S') = Ch_c(S') \cup Ch_c^{(3)}(S') \end{aligned}$$

where Ch_c is the choice function under soft floors and hard ceilings. The DAASB is defined by applying Ch_c^{SB} to the general DAA from Section 2. Ehlers et al. show that the DAASB results in an assignment that is fair under soft bounds, non-wasteful under soft bounds and student-optimal among such assignments (Theorem 4), and that the DAASB is group strategy-proof (Theorem 5).

The non-wastefulness of the DAASB serves to emphasize the trade-off between diversity and fairness/non-wastefulness properties. One would also expect the DAASB to improve student welfare vis-a-vis the DAASFHC; the similarities between the two algorithms allow for a natural comparison of them. The next section formalizes two results that relate the DAASB and DAASFHC in a Pareto dominance sense.

4.2 Theoretical Comparison of DAASFHC and DAASB

As noted previously, the DAASB supplies a solution to the non-wastefulness concerns associated with the DAASFHC. In particular, if the number of students is less than or equal to the total number of seats (across all schools), then non-wastefulness of the DAASB guarantees that every student will be assigned to a school. Intuitively, since the DAASB operates under weaker constraints than the DAASFHC, the DAASB would be expected to provide an improvement of student welfare over the DAASFHC - this is illustrated by the following proposition.

Proposition 4. *For any controlled school choice problem, the DAASB assignment Pareto dominates the DAASFHC assignment.*

Proof. This follows from Theorem 4 in Ehlers et al. (2011). Given a controlled school choice problem, let μ be the assignment resulting from the DAASFHC, and $\hat{\mu}$ the assignment resulting from the DAASB. Since μ is feasible as defined, μ is feasible under soft bounds.

Consider the following cases:

Case 1: μ is non-wasteful. Note that μ is non-wasteful under soft bounds. Further note that since μ is fair as defined and non-wasteful under soft bounds, μ is also fair under soft bounds. Theorem 4 from Ehlers et al. directly applies and hence, by the Pareto optimality of $\hat{\mu}$, $\hat{\mu}(s) \succeq_s \mu(s)$ for all $s \in S$.

Case 2: μ is not non-wasteful. There exist a student s and school c such that $c \succ_s \mu(s)$ and $|\mu(c)| < q_c$. Define the **improvement algorithm** as follows:

Step 1: Start with the assignment μ. Define $\tilde{C} = \{c \in C : |\mu(c)| < q_c\}$; choose a school $c \in \tilde{C}$ and define $\tilde{S}_c = \{s \in S : c \succ_s \mu(s)\}$. All students in $\tilde{S}_c$ apply to c. c first admits the highest ranked students in $\tilde{S}_c$, according to $\succ_c$, until the ceilings are filled and then up to capacity, or until $\tilde{S}_c$ is exhausted. Call the resulting assignment $\tilde{\mu}_1$.

Step k: Define $\tilde{C} = \{c \in C : |\tilde{\mu}_{k-1}(c)| < q_c\}$; choose a school $c \in \tilde{C}$ and define $\tilde{S}_c = \{s \in S : c \succ_s \tilde{\mu}_{k-1}(s)\}$. If either $|\tilde{C}| = 0$, or $|\tilde{S}_c| = 0$ for all $c \in \tilde{C}$, then stop. Else, all students in $\tilde{S}_c$ apply to c. c admits the highest ranked students in $\tilde{S}_c$, according to $\succ_c$, until first the floors, then the ceilings are filled and then up to capacity, or until $\tilde{S}_c$ is exhausted. Call the resulting assignment $\tilde{\mu}_k$.

Since the improvement algorithm improves the assignment of at least one student at each step, it ends in finite time.

Let $\tilde{\mu}$ be the assignment resulting from applying the improvement algorithm to μ. From the stopping condition, it is clear that $\tilde{\mu}$ is non-wasteful under soft bounds. Furthermore, since the algorithm improves the match of at least one student, we have $\tilde{\mu}(s) \succeq_s \mu(s)$ for all $s \in S$. The next step in the proof involves showing that $\tilde{\mu}$ is fair under soft bounds.

Consider a student s and a school c such that $c \succ_s \tilde{\mu}(s)$, and let $\tau(s) = t$. For any student $s' \in \tilde{\mu}^t(c)$, s' was matched with c under either the DA algorithm with soft floors and hard ceilings or the improvement algorithm, both of which imply that $s' \succ_c s$. Also note that the algorithm, by definition, implies that $|\tilde{\mu}^t(c)| \geq \underline{q}_c^t$. Now, consider the following subcases:

Subcase 2a: $|\tilde{\mu}^t(c)| \geq \overline{q}_c^t$. Consider $s' \in \tilde{\mu}(c)$ such that $|\tilde{\mu}^{\tau(s')}(c)| > \overline{q}_c^{\tau(s')}$; from the feasibility of μ, it must be that some students were admitted to c under the improvement algorithm.

The students admitted to c under the improvement algorithm must have decreasing priorities according to the order in which they were admitted. It must be that the last student of type $\tau(s')$ had higher priority than s since the ceiling for type $\tau(s')$ had already been filled. Therefore, we have $s' \succ_c s$ for all $s' \in \tilde{\mu}(c)$ such that $|\tilde{\mu}^{\tau(s')}(c)| > \overline{q}_c^{\tau(s'}$.

Subcase 2b: $\underline{q}_c^t \leq |\tilde{\mu}^t(c)| < \overline{q}_c^t$. In this case, we must have $|\tilde{\mu}^{t'}(c)| \leq \overline{q}_c^{t'}$ for all $t' \in T \backslash \{t\}$. By way of contradiction, suppose there is a type t' such that $|\tilde{\mu}^{t'}(c)| > \overline{q}_c^{t'}$. Since μ is feasible, there must be at least one student of type t' that was matched with c under the improvement algorithm. Consider the step (say step k) of the improvement algorithm at which the last student of type t' was matched with c. Since s was not matched to c at step k, it must have been the case that $|\tilde{\mu}_k^t(c)| = \overline{q}_c^t$. Since $|\tilde{\mu}^t(c)| < \overline{q}_c^t$, some students of type t in c must have been matched with some other school at some subsequent step. After this step, students of type t could have been admitted to c without violating the ceiling for type t. However, since s is not matched with c and type t students do not fill their ceiling at the end of the algorithm, we have a contradiction.

Now, consider a type t' such that $\underline{q}_c^{t'} < |\tilde{\mu}^{t'}(c)| \leq \overline{q}_c^{t'}$. Let s' be the student with lowest priority in $\tilde{\mu}^{t'}(c)$. If $s' \in \mu^{t'}(c)$ and $|\mu^{t'}(c)| > \underline{q}_c^{t'}$, then $s' \succ_c s$ since μ is fair.

If $s' \in \mu^{t'}(c)$ and $|\mu^{t'}(c)| = \underline{q}_c^{t'}$, then at least one type t' student must be matched to c under the improvement algorithm; this contradicts the fairness of μ since that student has higher priority than s' and prefers c to his or her match under μ. Finally, if $s' \notin \mu^{t'}(c)$, then s' must be matched with c under the improvement algorithm. If type t students do not fill their ceiling at the step when s' is admitted to c, then $s' \succ_c s$. If type t students do fill their ceiling at that step, then some of them must be subsequently matched away at some later step(s). Since s is not matched with c, and type t students do not fill their ceiling, we get a contradiction. Therefore, $s' \succ_c s$ for all $s' \in \tilde{\mu}(c)$ such that $\underline{q}_c^{\tau(s')} < |\tilde{\mu}^{\tau(s')}(c)| \leq \overline{q}_c^{\tau(s')}$.

Therefore, $\tilde{\mu}$ is non-wasteful and fair under soft bounds. Since $\hat{\mu}$ is Pareto efficient among all such assignments, we have $\hat{\mu}(s) \succeq_s \tilde{\mu}(s)$ for all $s \in S$, which implies $\hat{\mu}(s) \succeq_s \mu(s)$ for all $s \in S$ i.e. $\hat{\mu}$ is weakly preferred to μ by all $s \in S$. □

The next result provides a basis for comparison between the DAASB and DAASFHC in terms of how effective these algorithms are at respecting constraints.

Proposition 5. *For a controlled school choice problem, the DAASB and the DAASFHC result in the same assignment if and only if the DAASB assignment respects the ceilings for each type in every school.*

Proof. Suppose the DAASB results in an assignment μ that respects ceilings (i.e. $\mu^t(c) \leq \overline{q}_c^t$ for all $t \in T$ and $c \in C$). Since μ respects ceilings and is fair under soft bounds, by definition μ is fair under soft floors and hard ceilings. Now, suppose $\hat{\mu}$ is the assignment resulting from the DAASFHC. By the student optimality of $\hat{\mu}$ (Proposition 1), $\hat{\mu}(s) \succeq_s \mu(s)$ for all $s \in S$. However, $\mu(s) \succeq_s \hat{\mu}(s)$ by Proposition 4. Hence, each student is indifferent between μ and $\hat{\mu}$. Since preferences are strict, $\mu(s) = \hat{\mu}(s)$ for all $s \in S$.

Conversely, suppose the DAASB results in an assignment that violates the ceiling for at least one type in at least one school. By definition, the DAASFHC cannot violate any ceilings $\Rightarrow$ the DAASB and DAASFHC assignments cannot be the same. □

Ehlers et al. (2011) provide a result to show that the DAASB assignment Pareto dominates assignments that respect constraints and are *strongly fair across types*. This result also holds for the DAASFHC and is included as a supplement in Appendix A.

4.3 Simulations

The advantages of the DAASB over the DAASFHC (Pareto dominance, non-wastefulness) are not wholly unexpected due to the stronger restrictions imposed in the DAASFHC. However, one would expect the DAASFHC to be better at respecting constraints for the very same reason. To reiterate, the motivation behind the formulation of the DAASFHC was to maintain fairness while maximizing the likelihood of meeting diversity requirements. An immediate consequence of Proposition 5 is that whenever the DAASB respects constraints, the DAASFHC too will respect constraints. Qualitatively, the DAASFHC is, at the very least, as good at respecting constraints as the DAASB - the question is of whether the DAASFHC is actually better for this purpose than the DAASB and if so, how much better it is.

I use computer simulations (in the spirit of Hafalir, Yenmez and Yildirim, 2011) to not only answer the question posed above, but also as an attempt to quantify the differences in student welfare between the DAASFHC and the DAASB. The methodology applied is to simulate controlled choice models, apply both the DAASFHC and DAASB, and then compare the resulting assignments by verifying whether they meet the diversity quotas along with quantifying the welfare costs to students.

For simplicity, the number of students is fixed at 2000 while the number of schools is fixed at 20 with each school having a maximum capacity of 100 seats. The diversity constraints are centered about the type-distribution of the student population and are set in the following manner: if p_t% of the student body is of type t, then for any school c the floor for type t

is $\max\{(p_t - 10), 0\}\%$ and the ceiling is $\min\{(p_t + 10), q_c\}\%$ of the school's capacity. The proportions by type of students are randomly determined. Student preferences as well as school priorities are strict and random; these are derived by defining utility functions for both students and schools. For a student s_i and school c_j, the utility functions are defined as follows:

$$U_{s_i}(c_j) = \alpha X(c_j) + (1 - \alpha) X_{s_i}(c_j)$$
$$U_{c_j}(s_i) = \beta X(s_i) + (1 - \beta) X_{c_j}(s_i)$$

where $\alpha, \beta \in [0, 1]$; α and β are the correlation parameters for student preferences and school priorities respectively. X denotes random variables that are drawn from a standard uniform distribution (i.e. $U[0, 1]$). I assume that these randomly generated utility functions are representative of student preferences as well as school priorities i.e. $c \succ_s c'$ if and only if $U_s(c) > U_s(c')$, and $s \succ_c s'$ if and only if $U_c(s) > U_c(s')$.

Models are simulated for five differing levels of the school priority correlation parameter β (0, 0.25, 0.5, 0.75 and 1), while the student preference correlation parameter α is varied from 0 to 1 in steps of 0.1. To capture the type-dependence of the two algorithms, I also vary the number of types from two to seven. The intention behind these parameter choices is to capture the behavior of these mechanisms over a wide variety of settings[2], which allows for a more insightful comparison between their relative effectiveness' at respecting constraints.

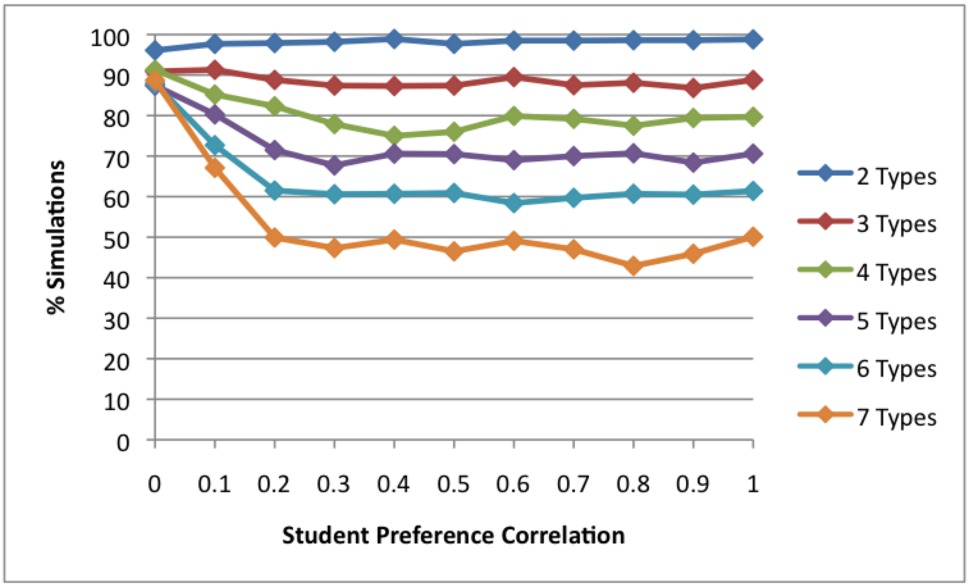
Both the mechanisms under consideration ensure fairness but do not guarantee an assignment that respects constraints. However, provided they are likely to do so, these algorithms can still be useful even when schools are required to respect constraints. For each algorithm, I use the percentage of simulations in which the resulting assignment respects constraints as an estimate of how likely that algorithm is to respect constraints. The results are graphically presented in Figure 1.

The results do not differ significantly for differing correlation levels of the school priorities and therefore the aggregated results for all school priority correlation levels are presented[3]. As expected, the DAASFHC performs better than the DAASB at respecting constraints - the improvement is marginal in models with two or three types but becomes increasingly stark for models with four or more types. For the DAASB, an almost stepwise decrease in this likelihood estimate is observed with an increase in the number of types (especially

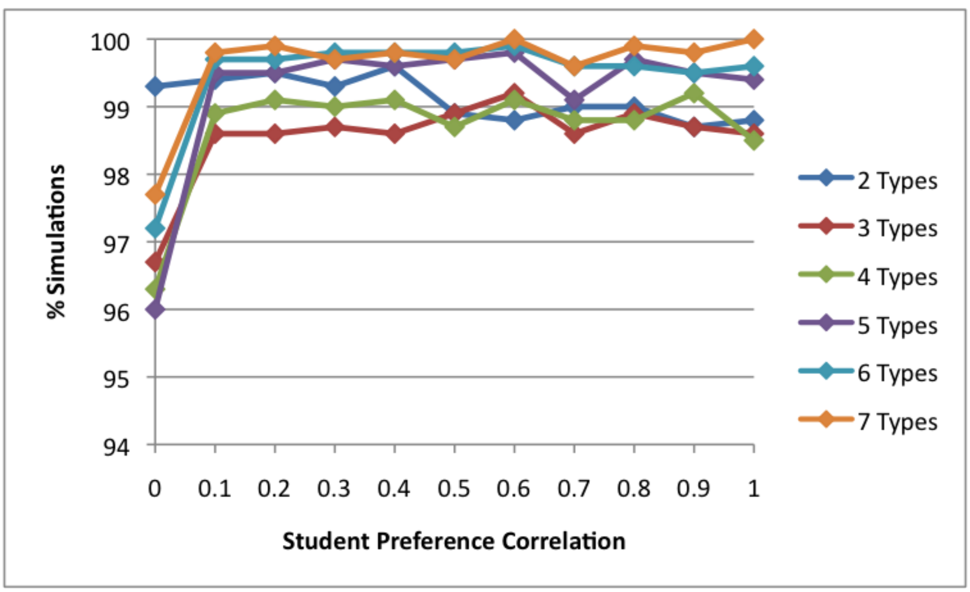
[2]The simulated results are robust to changes in the number of students and schools; variations in these parameters as well as the school capacities and the width of the bounds do not qualitatively affect the results.

[3]Figures for each level of school priority correlation are provided in Appendix B.

Figure 1: Percentage of simulations in which the DAASB and DAASFHC respect constraints.



(a) DAASB

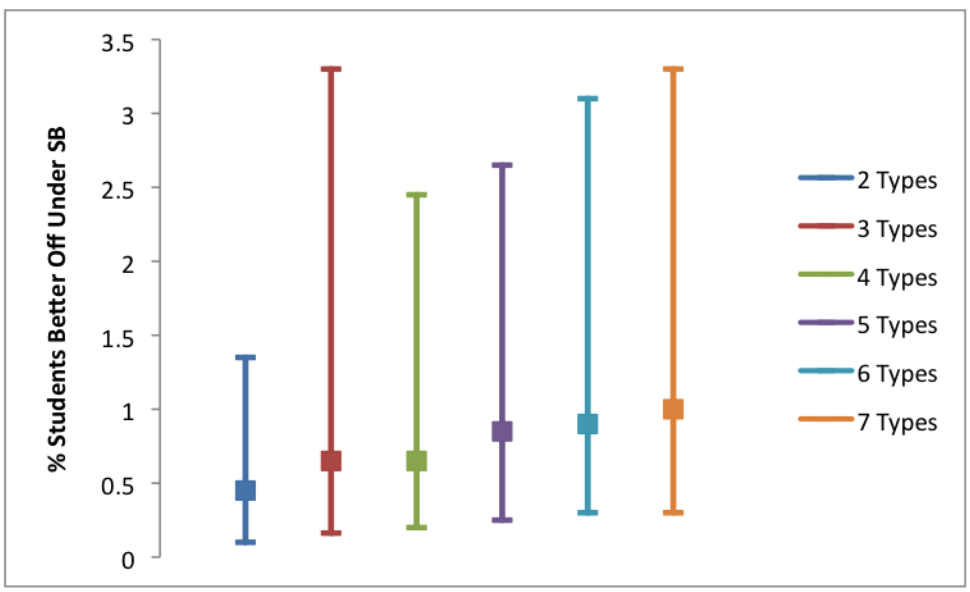


(b) DAASFHC

for higher levels of correlation between student preferences); it undergoes a considerable decrease in effectiveness from respecting constraints in over 90% of problems with two types to less than 50% of problems with seven types. In contrast, the DAASFHC does not appear to be sensitive to the number of student types and consistently produces assignments that respect constraints - for any class of the simulated school choice problems, the DAASFHC assignment respects constraints in at least 96% of them. Based solely on these results, the DAASFHC appears a much more powerful mechanism when it comes to meeting diversity constraints. However, any judgments without taking the welfare costs of the DAASFHC into account would be incomplete; the next results aim at quantifying the differences in student welfare between the DAASFHC and the DAASB.

Before proceeding further, the notion of **strict Pareto dominance** or simply, **strict dominance** of assignments is needed. An assignment μ_1 strictly dominates μ_2 if every student weakly prefers μ_1 (i.e. μ_1 Pareto dominates μ_2) and at least one student strictly prefers μ_1 to μ_2 ($\mu_1(s) \succ_s \mu_2(s)$ for at least one $s \in S$). From Propositions 4 and 5, the DAASB strictly dominates the DAASFHC whenever they result in different assignments. Therefore, it must be that some students are strictly better off under the DAASB in cases when the DAASB does not respect constraints, but the DAASFHC does. To provide some quantification of this welfare cost, Figure 2 shows the median (along with the first and third quartiles) percentage of students that strictly prefer the DAASB to the DAASFHC when the DAASB does not respect constraints, but the DAASFHC does.

Figure 2: Median percentage of students that strictly prefer the DAASB to the DAASFHC when they result in different assignments. The first and third quartiles are represented by the lower and upper bars, respectively.



The results are characterized by an interquartile range that appears to be increasing as well as shifting upwards in the number of types. A slight increase in the median percentage of students that are strictly better off under the DAASB is also observed with an increase in the number of types. However, these statistics are not particularly troublesome - ensuring that the diversity constraints are met may justify incurring these welfare costs, given their small magnitude. But what of the worst case scenario? This question is worth some consideration, and a cursory look at the extremal cases provides some telling results; over half of the students could strictly prefer the DAASB. Table 1 provides a compilation of the maximum number of students that strictly prefer the DAASB over the DAASFHC for each considered number of types.

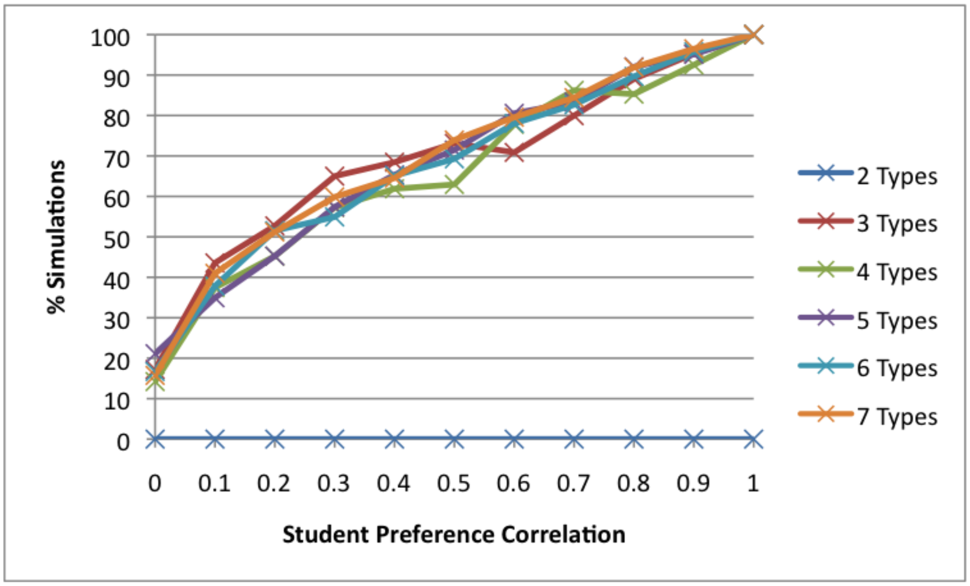
Table 1: Maximum percentage of students that are strictly better off under the DAASB.

Number of Types	Percentage of Students
2	8.95
3	45.20
4	39.90
5	52.40
6	47.30
7	35.55

The DAASFHC performs strongly when it comes to respecting constraints, at some cost to student welfare (with the DAASB as the reference point). These observations are a consequence of imposing a hard ceiling rather than a soft one, but how exactly does the hard ceiling provide this diversity improvement? A possible explanation of how the DAASFHC enforces diversity bounds is that it could simply be leaving unmatched some of those students that would cause ceiling violations in the DAASB. Figure 3 shows the percentage of simulations in which at least one student is unassigned under the DAASFHC assignment when it respects constraints and the DAASB does not.

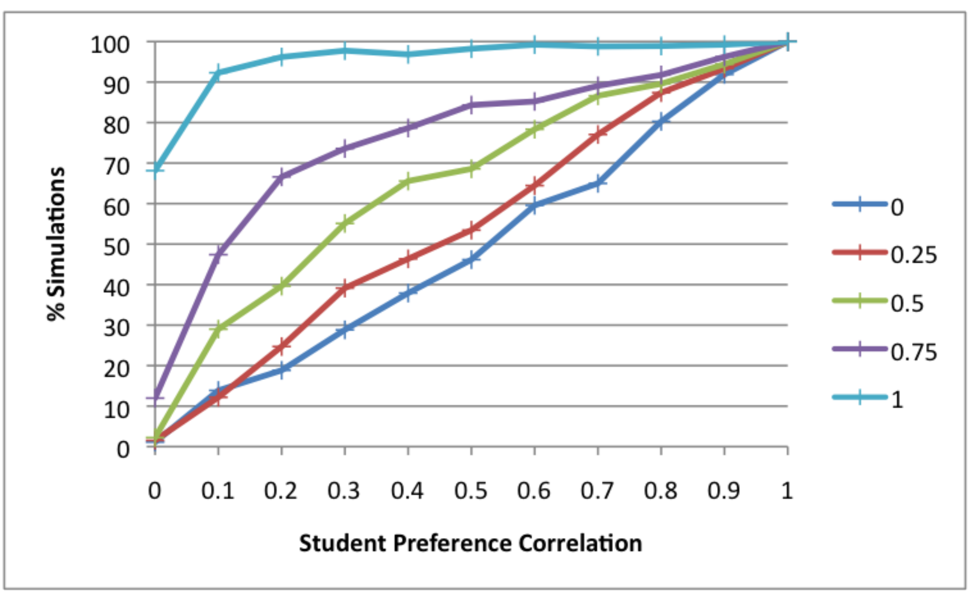
For models with two types, the DAASFHC never leaves students unassigned, even when the DAASB fails to respect constraints. However, the fraction of simulations in which students are left unassigned for models with more than two types is considerable, especially given the increasing trend with respect to the correlation of student preferences. Again, the DAASFHC outcomes display a large degree of independence from the number of student types, with results for only models with two types deviating significantly. It can be inferred that the DAASFHC does indeed leave students unassigned to enforce a high likelihood of respecting constraints. Exceptionally, this result is also the only one that displays significant variation with changes in the school priority correlation. Figure 4 depicts the changes in this statistic for the five levels of school priority correlation; these results also do not vary

Figure 3: Percentage of simulations in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not.



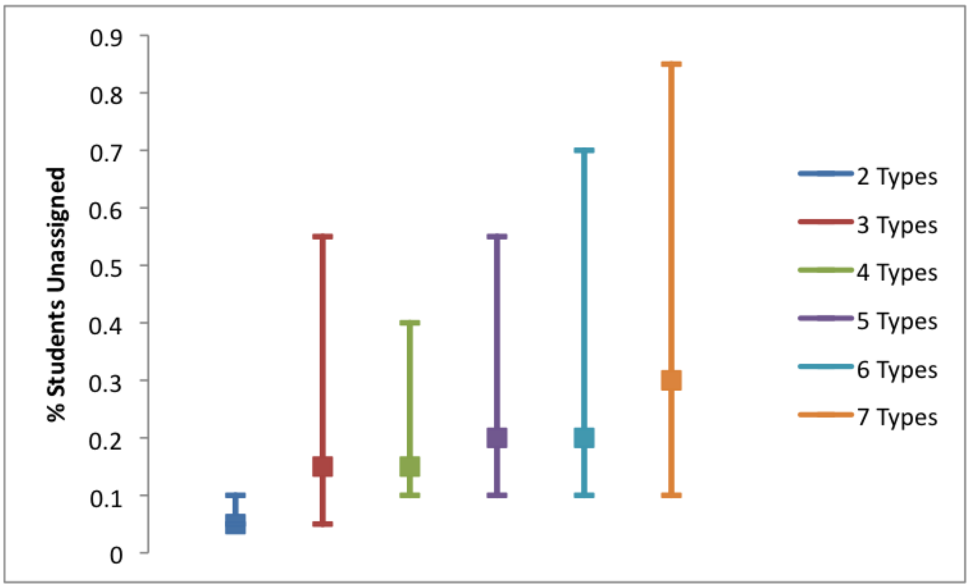
qualitatively with a change in the number of types, emphasizing the apparent lack of sensitivity of the DAASFHC outcomes to changes in the same. The same results for models with different numbers of types can be found in Appendix B.

Figure 4: Percentage of simulations of models in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. Different series represent different levels of school priority correlation.



While even one unassigned student is undesirable, it is nonetheless important to gauge the magnitude of this particular problem. Figure 5 provides the median along with the first and third quartiles of the percentage of students that are unassigned under the DAASFHC, conditional on at least one student being unassigned.

Figure 5: Median percentage of students that are unassigned under the DAASFHC (given that at least one student is unassigned under the same). The first and third quartiles are represented by the lower and upper bars, respectively.



As was the case with the statistics for the fraction of students that were strictly better off under the DAASB, each of the first quartile, median and third quartile shows a slight increase with an increase in the number of types. Though the magnitudes are not very large, even one unassigned student poses a problem; in fact, this wastefulness is the greatest drawback of the DAASFHC, which means an understanding of the worst case outcome is imperative. Table 2 presents the maximum percentage of unassigned students for each considered number of types.

In the maximum, 6.55% of the students were unassigned under the DAASFHC, emphasizing its wastefulness in being unable to allocate students to schools and consequentially leaving seats empty. As mentioned previously, the implications of having unassigned students are serious - on one hand, leaving students unassigned is unacceptable. On the other, accommodating these students by allocating empty seats at schools to them would require sacrificing the fairness of the DAASFHC assignment as well as violating the diversity constraints, essentially defeating the purpose of the DAASFHC as an assignment mechanism.

Table 2: Maximum percentage of students that are unassigned under the DAASFHC.

Number of Types	Percentage of Students
2	0.25
3	3.00
4	4.00
5	5.00
6	6.00
7	6.55

5 Conclusion

This study serves as an illustration of the trade-offs between diversity, fairness and non-wastefulness that are inherent in controlled school choice. By imposing controlled choice constraints as hard bounds, one can ensure that these diversity requirements are met at the cost of losing fairness and adversely affecting student welfare. Since school choice programs are implemented to improve the well-being of students, I take a view that fairness is a necessity and should not be compromised for the sake of diversity (rather than the other way around). My aim is to develop solutions that possess the desired fairness property while providing a high likelihood of meeting the diversity constraints.

Preserving fairness requires a weaker interpretation of the bounds on student types, and I examine two mechanisms that make use of soft bounds to guarantee fairness. The DAASB proposed by Ehlers et al. (2011) ensures both non-wastefulness and fairness by interpreting both upper and lower bounds as soft bounds. In addition, the DAASB is effective at respecting diversity constraints when the number of student types is small, but as this number increases, the DAASB exhibits a considerable decrease in effectiveness. Nevertheless, it is important to note that the DAASB still respects constraints more often than not, and can be a practical solution to controlled school choice problems of this mold.

As an alternative, I consider a mixed bounds approach with soft floors and hard ceilings. The mechanism obtained under this framework, the DAASFHC, also guarantees the fairness property. Furthermore, my results suggest that this approach is more powerful than the DAASB when it comes to meeting diversity requirements and it maintains its effectiveness independent of the number of student types. However, these positives come at the expense of non-wastefulness. While the weaker notion of constrained non-wastefulness is satisfied, the loss of non-wastefulness still has detrimental effects on student welfare. In particular, the possibility of unassigned students is an unattractive trait, especially from a policy perspective. However, if unassigned students can be absorbed into other parts of the school choice system, then the DAASFHC is feasible as an assignment mechanism.

To conclude, the purpose of this work was to show that sacrificing fairness is not necessary in the implementation of this variant of controlled school choice. Both the DAASB and the DAASFHC are easily implementable mechanisms, and can be useful even in settings with notionally hard bounds due to their effectiveness at respecting constraints. In these mechanisms, school districts have viable alternatives that can preserve student welfare while enabling them to achieve their diversity objectives.

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Appendix A

Ehlers et al. (2011) define a stronger notion of fairness under hard bounds by defining **weakly justifiable envy**. Under an assignment μ, a student s weakly justifiably envies (or simply **weakly envies**) a student s' at school c if:

(a) $c \succ_s \mu(s)$ i.e. s prefers the school c that s' is assigned to over her own.

(b) $s \succ_{\mu(s')} s'$ i.e. s has higher priority than s' at the school c

(c) s can be admitted to c without violating any of the diversity constraints, by removing s' from c.

An assignment is **strongly fair across types** if no student weakly envies another. Ehlers et al. show that the DAASB assignment Pareto dominates any assignment that respects constraints and is strongly fair across types. The same result holds for the DAASFHC.

Proposition 6. *The DAASFHC assignment Pareto dominates any assignment that respects constraints and is strongly fair across types.*

Proof. Given a controlled school choice problem, let $\hat{\mu}$ be the assignment resulting from the DA algorithm with soft floors and hard ceilings. Consider any assignment μ that respects constraints and is strongly fair across types. We have the following cases:

Case 1: μ is constrained non-wasteful (under soft floors and hard ceilings). Along with strong fairness across types, constrained non-wastefulness implies that μ is fair under soft floors and hard ceilings (by definition). By the Pareto efficiency of $\hat{\mu}$, $\hat{\mu}$ Pareto dominates μ.

Case 2: μ is not constrained non-wasteful. In this case, apply the following **constrained improvement algorithm** to μ:

Step 1: Start with the assignment μ. Define $\tilde{C} = \{c \in C : |\mu(c)| < q_c\}$. For each school $c \in \tilde{C}$, define the set of students $\tilde{S}_c = \{s \in S : c \succ_s \mu(s), |\mu^{\tau(s)}(c)| < \overline{q}_c^{\tau(s)}\}$. Choose a school c such that $|\tilde{S}_c| > 0$. All students in $\tilde{S}_c$ apply to c. c admits students the students of highest ranking, according to $\succ_c$, until either the ceilings are filled, capacity is reached, or until $\tilde{S}_c$ is exhausted. Call the resulting assignment μ_1.

Step k: Start with the assignment μ_{k-1}. Define $\tilde{C} = \{c \in C : |\mu_{k-1}(c)| < q_c\}$. For each school $c \in \tilde{C}$, define the set of students $\tilde{S}_c = \{s \in S : c \succ_s \mu_{k-1}(s), |\mu_{k-1}^{\tau(s)}(c)| < \overline{q}_c^{\tau(s)}\}$. If either $|\tilde{C}| = 0$ or $|\tilde{S}_c| = 0$ for all $c \in C$, then stop. Otherwise, choose a school c such that $|\tilde{S}_c| > 0$. All students in $\tilde{S}_c$ apply to c. c admits students the students of highest ranking, according to $\succ_c$, until either the ceilings are filled, capacity is reached, or until $\tilde{S}_c$ is exhausted. Call the resulting assignment μ_k.

Since the algorithm improves the match of at least one student at every step, it ends in finite time.

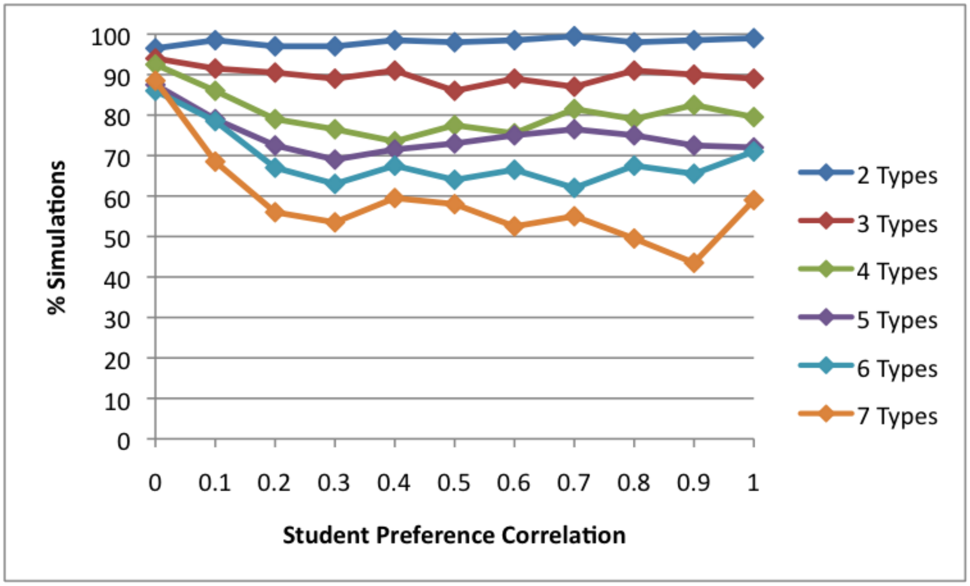
Let $\tilde{\mu}$ be the assignment resulting from the application of the constrained improvement algorithm to μ. Note that $\tilde{\mu}$ is constrained non-wasteful. Furthermore, since μ respects constraints, for each school c we must have $\underline{q}_c^t \leq |\tilde{\mu}^t(c)| \leq \overline{q}_c^t$ for all $t \in T$. Consider a student s with $\tau(s) = t$ and school c such that $c \succ_s \tilde{\mu}(s)$. If $|\tilde{\mu}^t(c)| = \overline{q}_c^t$ then note that any student s' of type t in c was admitted to c either under μ, in which case $s' \succ_c s$, or under the constrained improvement algorithm, which again means $s' \succ_c s$. Otherwise, if $\tilde{\mu}^t < \overline{q}_c^t$, consider any student $s' \in \tilde{\mu}(c)$. If s' was admitted to c under μ, then by the strong fairness across types of μ we have $s' \succ_c s$. Else, if s' was admitted to c under the constrained improvement algorithm, by definition of the algorithm it must be that $s' \succ_c s$. Therefore, by definition, $\tilde{\mu}$ is fair under soft floors and hard ceilings.

Since the constrained improvement algorithm only improves students' assignments, $\tilde{\mu}$ Pareto dominates μ. Furthermore, since $\tilde{\mu}$ is constrained non-wasteful and fair under soft floors and hard ceilings, by theorem 1 $\hat{\mu}$ Pareto dominates $\tilde{\mu}$. Therefore, $\hat{\mu}$ Pareto dominates μ. □

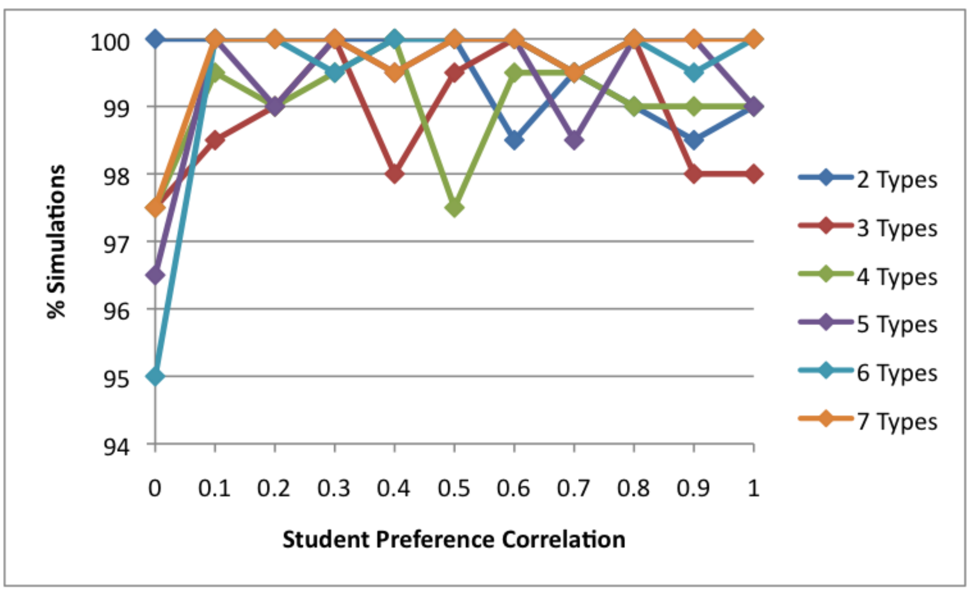
Appendix B

Supplementary Figures:

Figure 6: Percentage of simulations in which the DAASB and DAASFHC respect constraints for school priority correlation of 0.

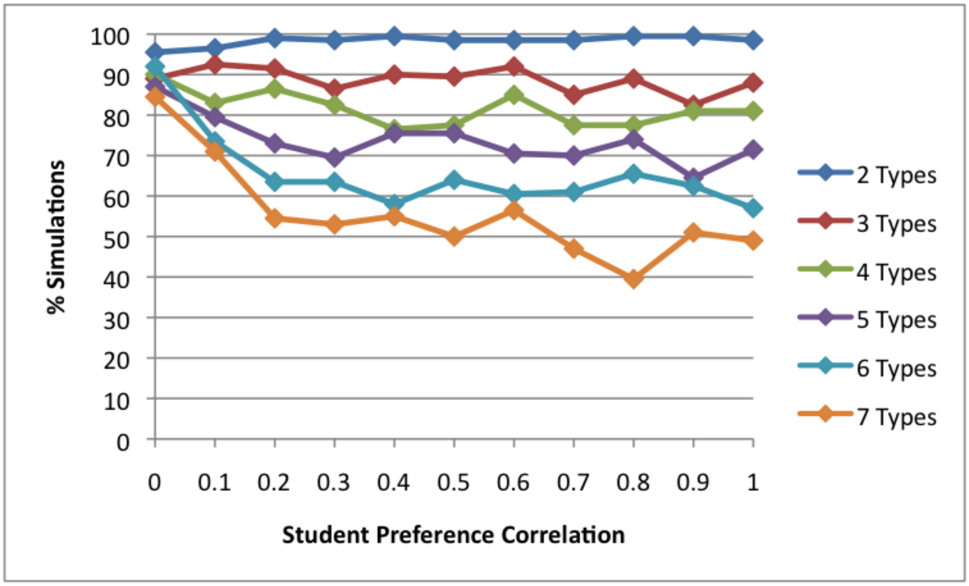


(a) DAASB

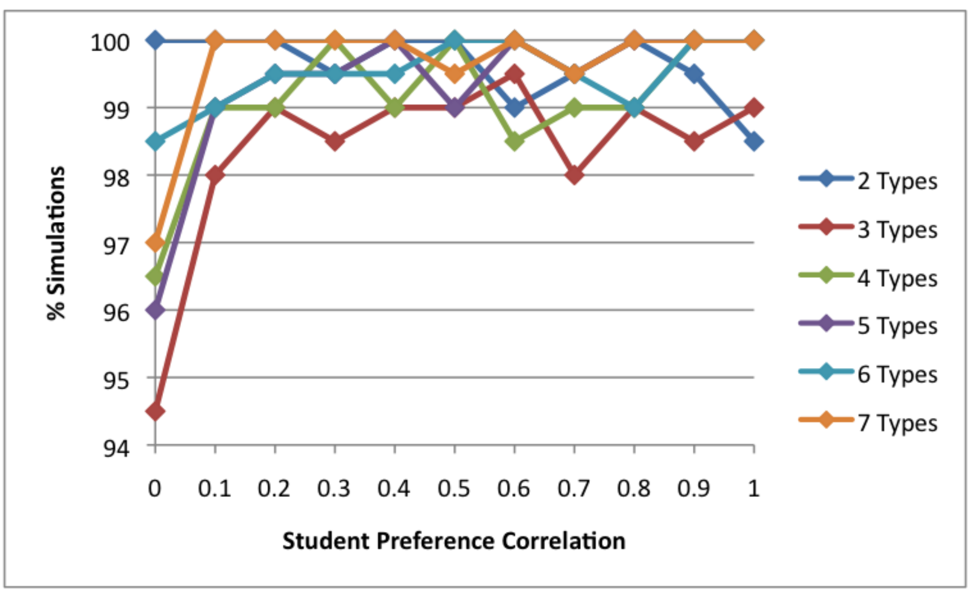


(b) DAASFHC

Figure 7: Percentage of simulations in which the DAASB and DAASFHC respect constraints for school priority correlation of 0.25.

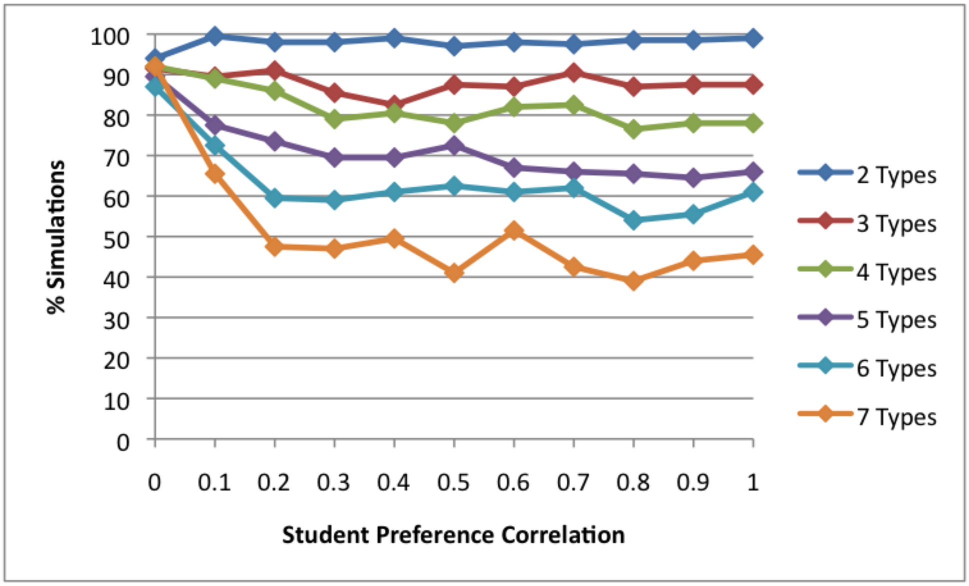


(a) DAASB

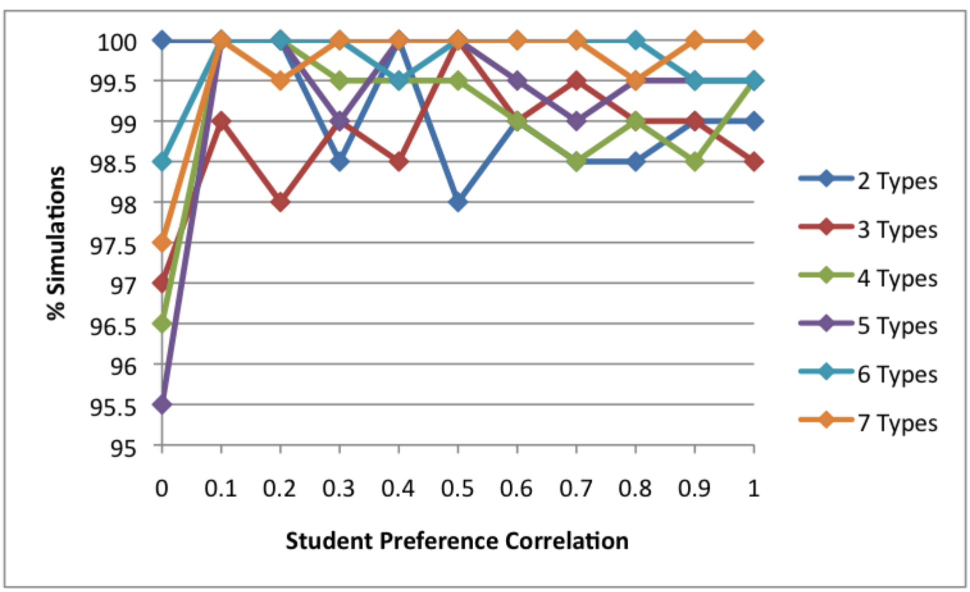


(b) DAASFHC

Figure 8: Percentage of simulations in which the DAASB and DAASFHC respect constraints for school priority correlation of 0.5.

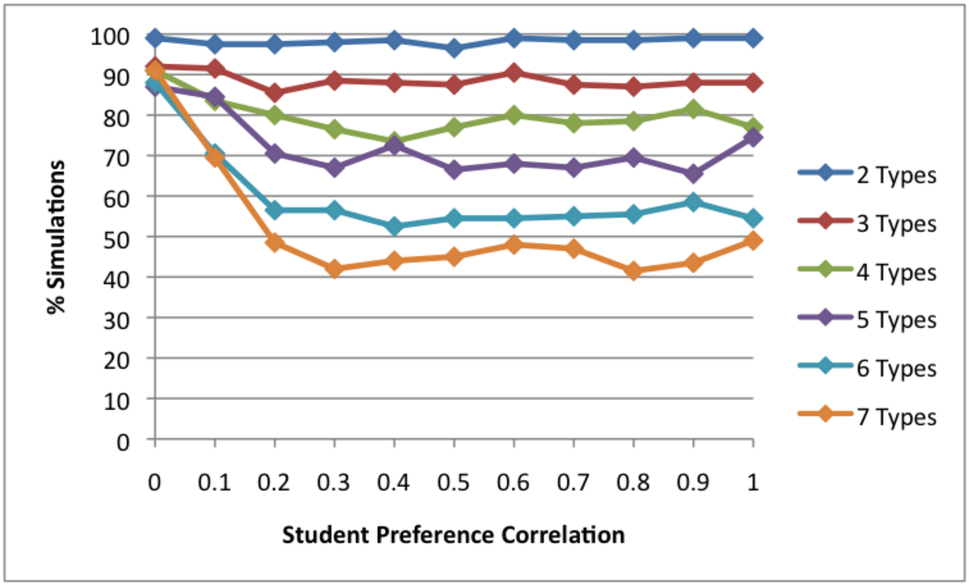


(a) DAASB

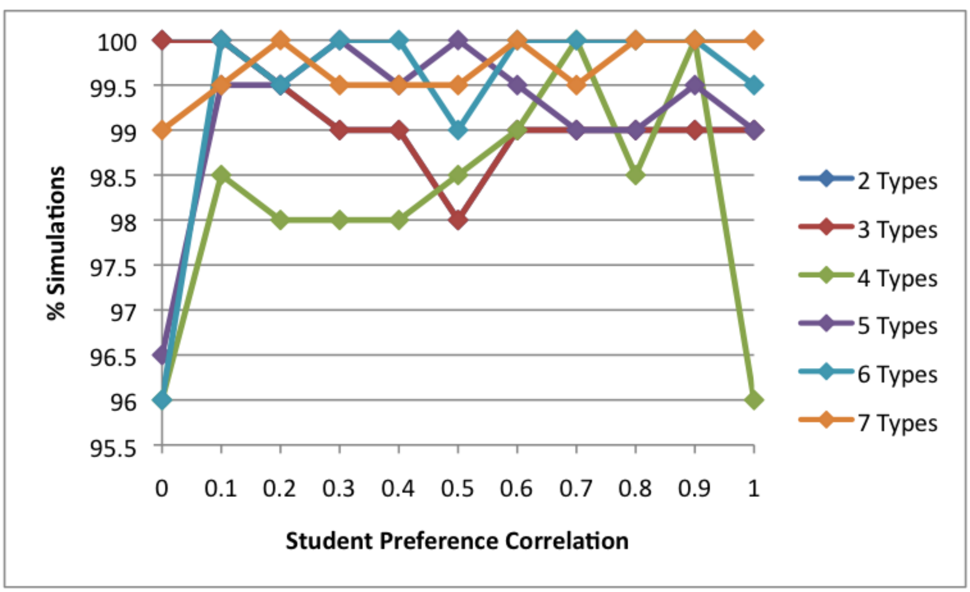


(b) DAASFHC

Figure 9: Percentage of simulations in which the DAASB and DAASFHC respect constraints for school priority correlation of 0.75.

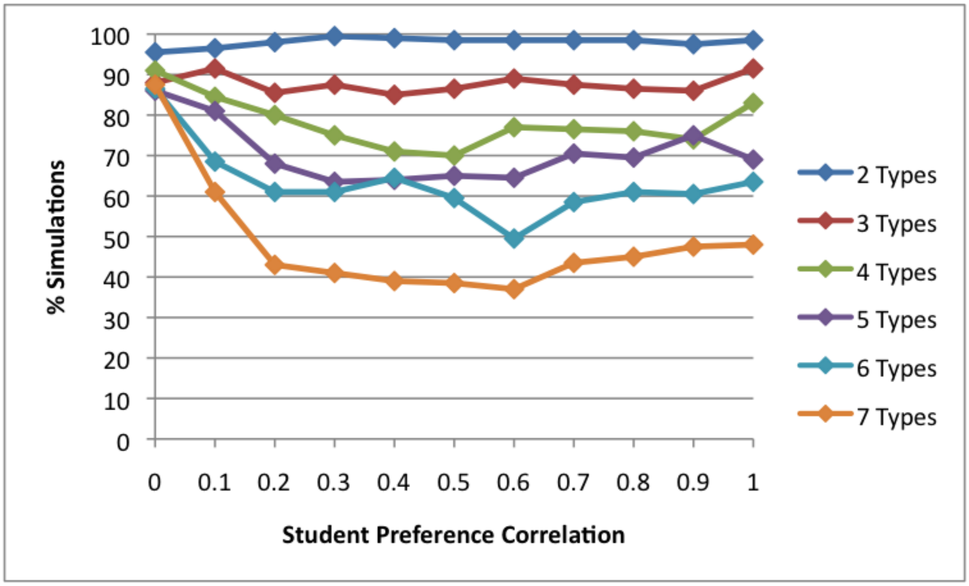


(a) DAASB

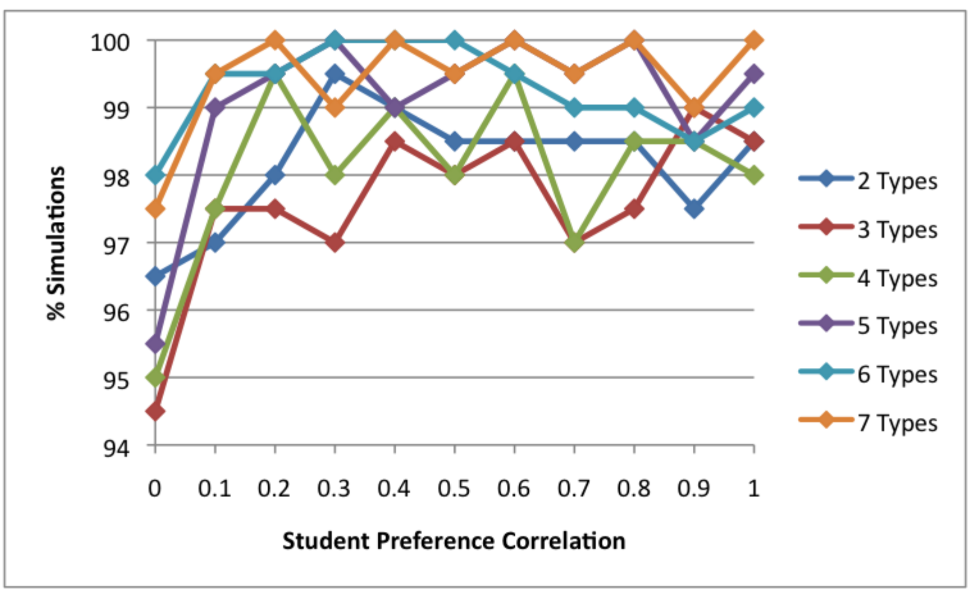


(b) DAASFHC

Figure 10: Percentage of simulations in which the DAASB and DAASFHC respect constraints for school priority correlation of 1.



(a) DAASB



(b) DAASFHC

Figure 11: Percentage of simulations in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. School priority correlation is 0.

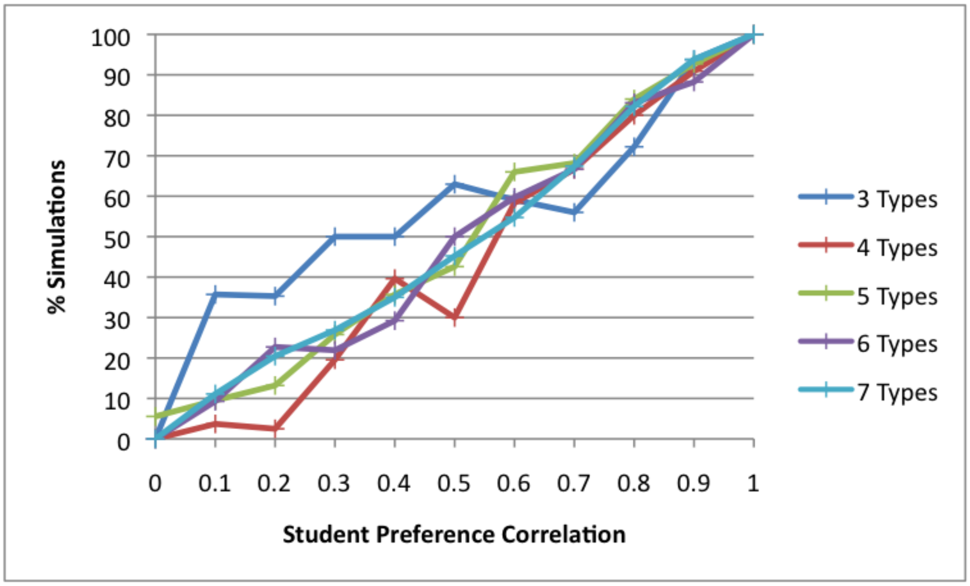


Figure 12: Percentage of simulations in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. School priority correlation is 0.25.

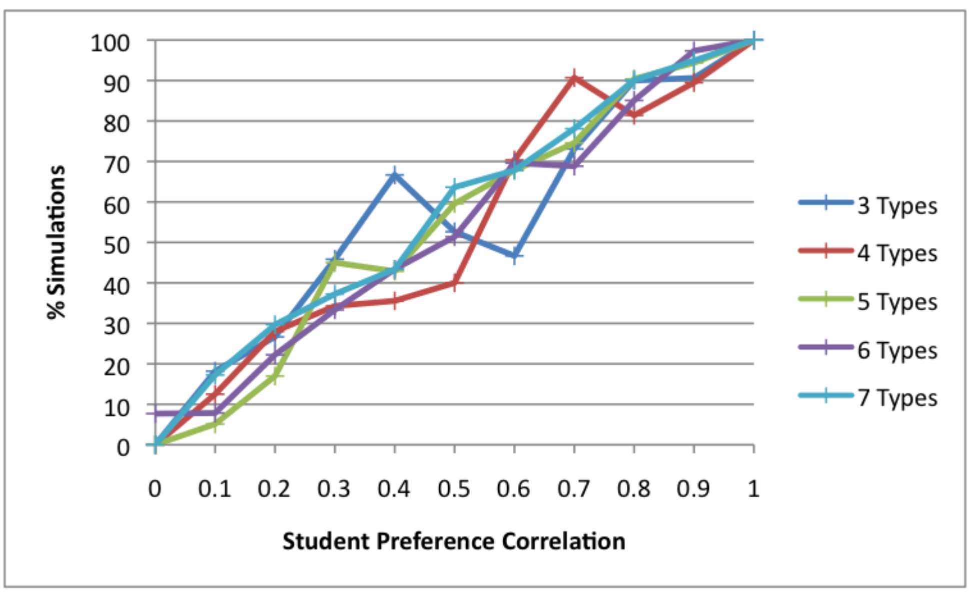


Figure 13: Percentage of simulations in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. School priority correlation is 0.5.

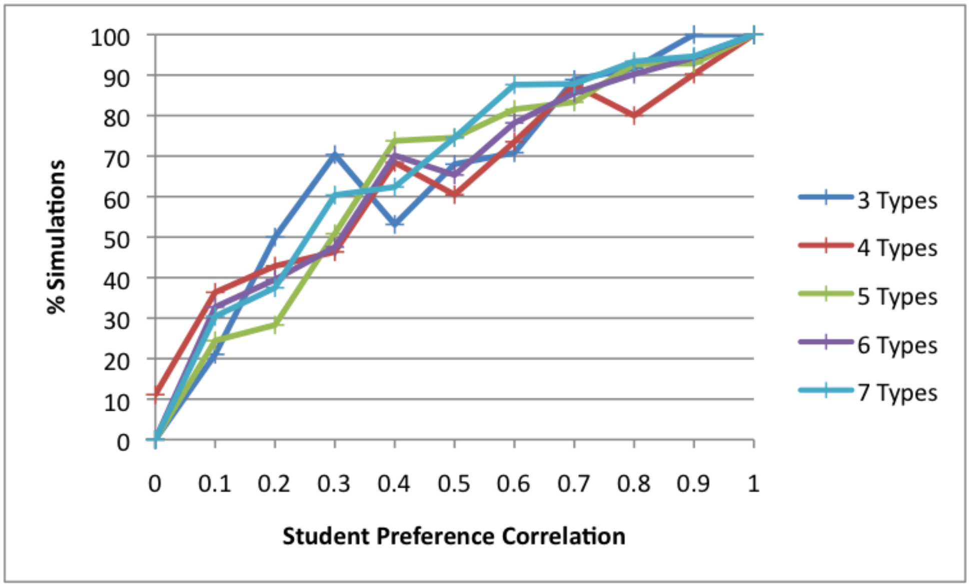


Figure 14: Percentage of simulations in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. School priority correlation is 0.75.

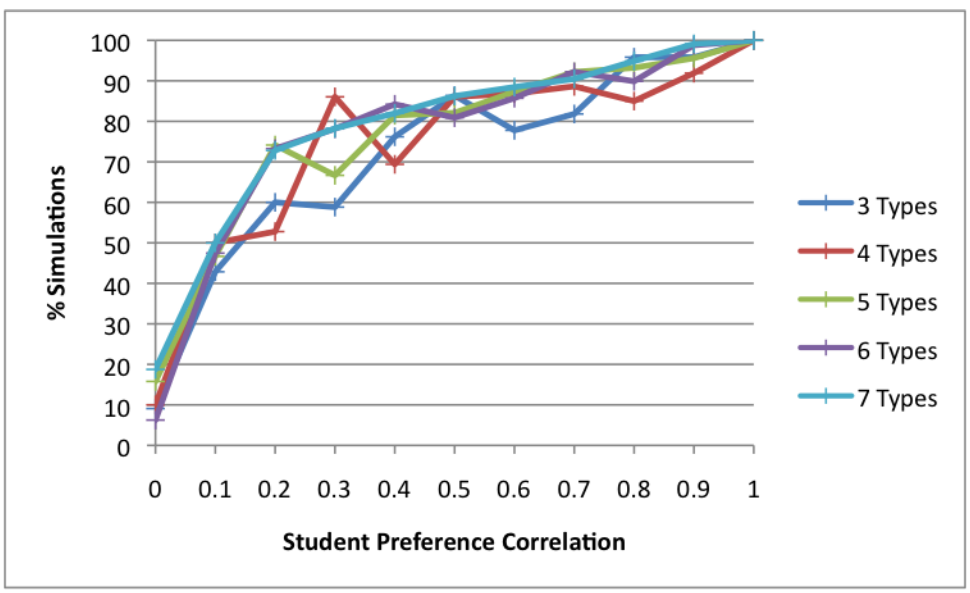


Figure 15: Percentage of simulations in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. School priority correlation is 1.

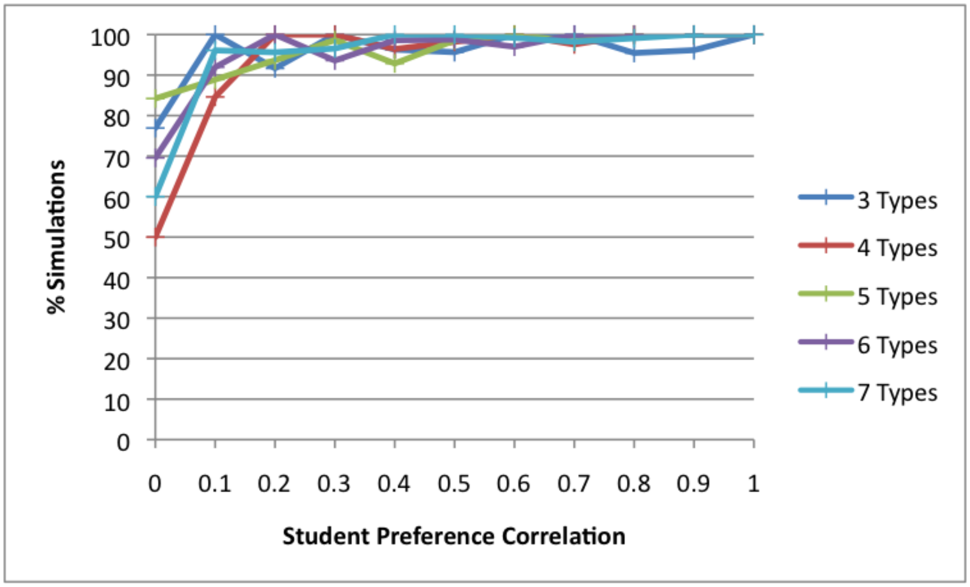


Figure 16: Percentage of simulations of models with 3 types in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. Different series represent different levels of school priority correlation.

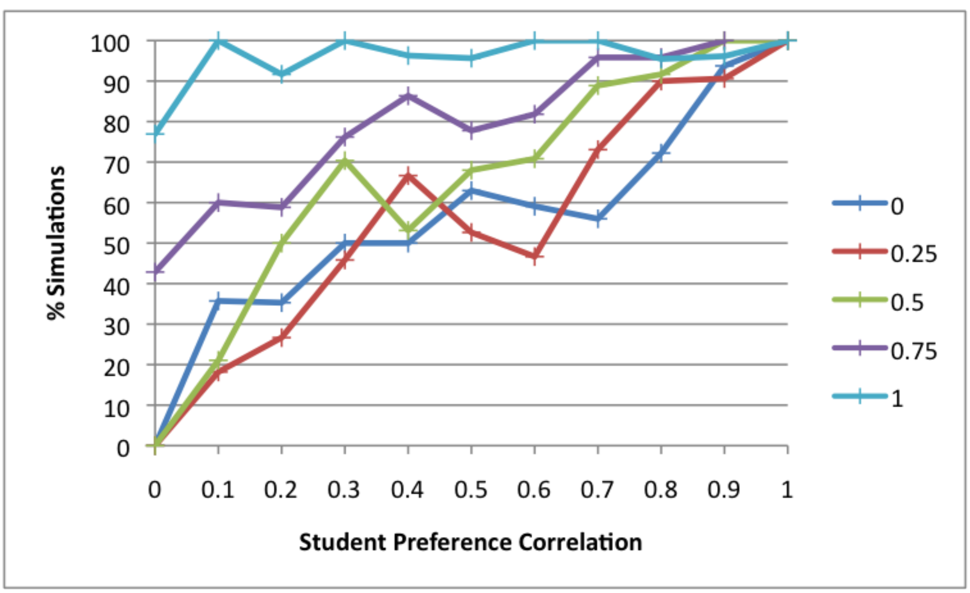


Figure 17: Percentage of simulations of models with 4 types in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. Different series represent different levels of school priority correlation.

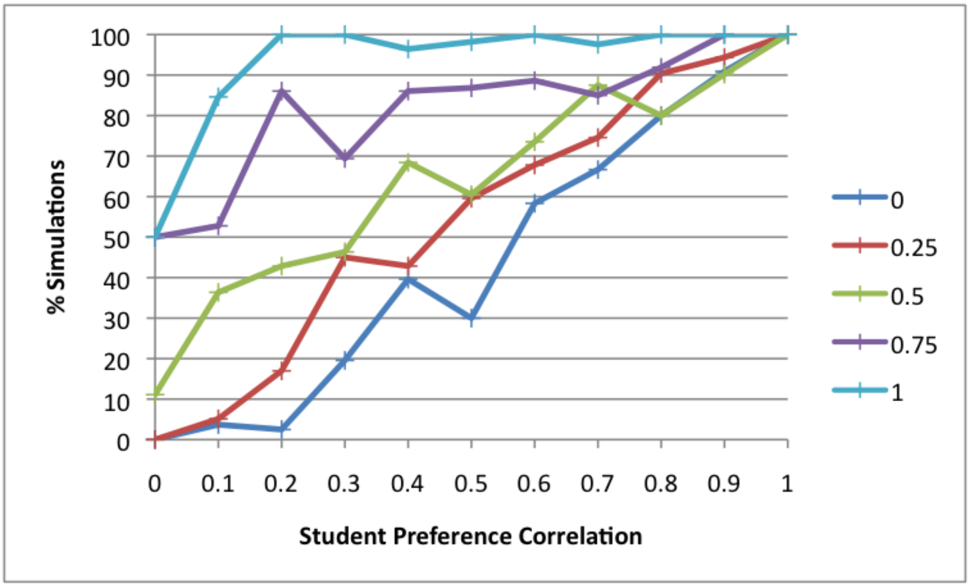


Figure 18: Percentage of simulations of models with 5 types in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. Different series represent different levels of school priority correlation.

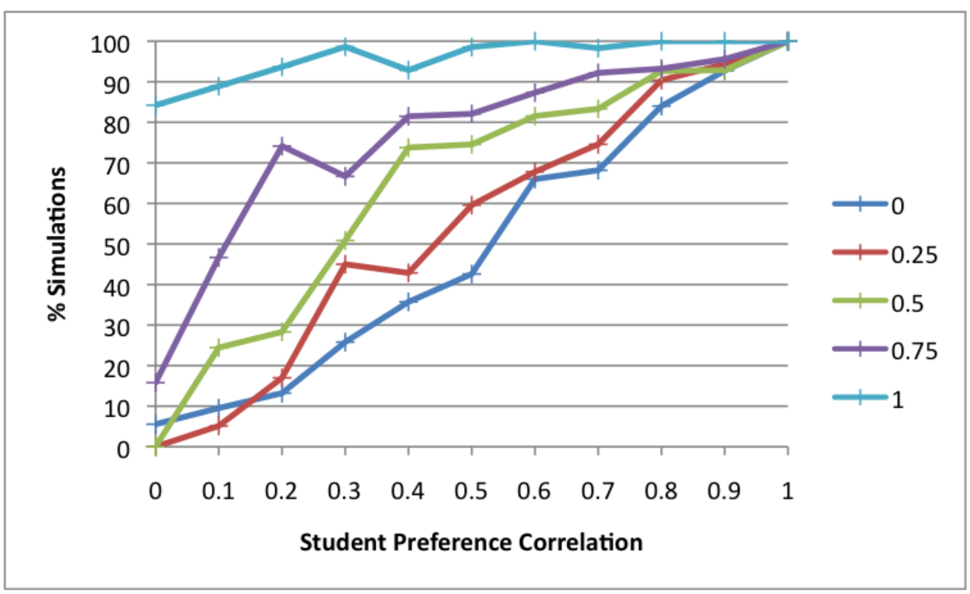


Figure 19: Percentage of simulations of models with 6 types in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. Different series represent different levels of school priority correlation.

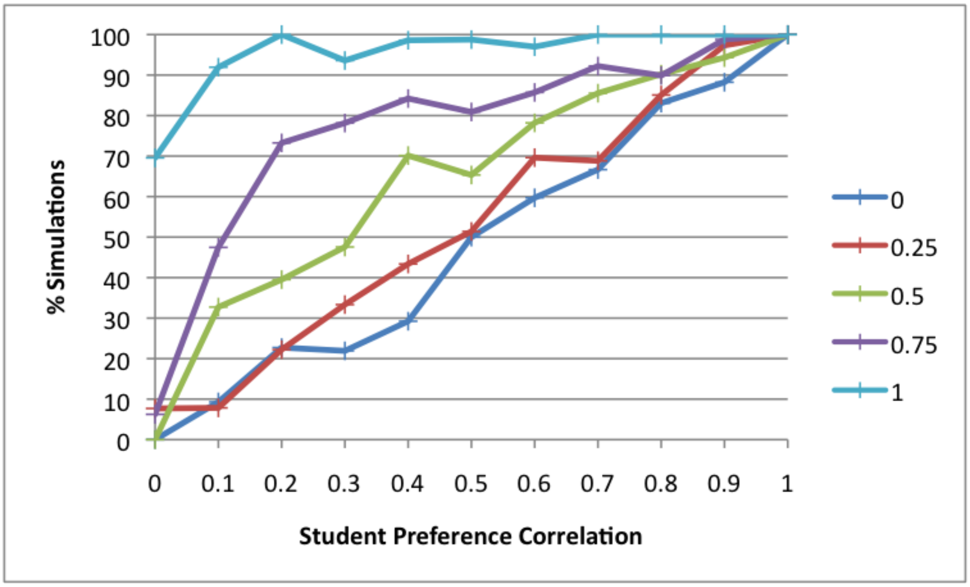


Figure 20: Percentage of simulations of models with 7 types in which students are unassigned under the DAASFHC when it respects constraints and the DAASB does not. Different series represent different levels of school priority correlation.