Testing the Law of Aggregate Demand

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In fulfillment of the requirement for the
Tepper School of Business Senior Honors Thesis in Economics

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Abstract

We provide an algorithm to test the law of aggregate demand with runtime complexity polynomial to the length of the agent's preference relation over a set of contracts. Access to this preference relation is essential, as we show that any algorithms with only oracle access to the agent's choice function require a number of queries exponential to the number of contracts.

1 Introduction

A two-sided matching market is a two-sided market that operates in the absence of financial transactions. Normally, when we think of markets, we picture buyers, sellers, and money in exchange for goods and services. Prices play a crucial role, acting as an indicator for how willing someone is to buy or to sell a product. However, there also exists a class of naturally occurring markets which do not involve money or prices, where both sides of the market have to come to an agreement through some other mechanism, through a matching.

The study of matching markets has important practical applications. The matching procedure constructed by Roth and Peranson (1999) is used today in the National Resident Matching Program, which annually matches 20,000 medical school graduates to residency programs. Both New York City and Boston public schools have tackled the school choice problem by adopting a procedure advocated by Abdulkadiroglu and Sönmez (2003). In both cases, a centralized system is employed to construct an optimal and feasible allocation of economic resources, whether it is labor, services, or goods.

In order to construct such an allocation or matching, we must examine the preferences of each agent in the market. In 1962, Gale and Shapley published "College Admissions and the Stability of Marriage," creating the foundation of a theoretical framework for two-sided matching markets. This was improved upon by Hatfield and Milgrom in "Matching with Contracts," which refined the matching model by introducing a framework based on contracts. Contracts can be thought of as agreements which specify the matchings between two sides of a market. Subsequent analysis of contract based matching included two important properties of agents' preferences, the law of demand and substitutability.

The law of aggregate demand follows from its analogue in producer theory, where the law of demand states that as input prices falls, a firm demands more of an input good. In the contracts model, a lower price corresponds to an increase in the number of offered contracts, and higher demand corresponds to choosing a greater number of these contracts. The law of aggregate demand in matching theory states that when an agent chooses from an expanded set of contracts, the set of contracts it selects also weakly expands.

Substitutability derives inspiration from demand theory. Two substitutable goods are goods where an increase in the price of one results in an increase in demand for the other. In the contracts model, a price reduction corresponds to an expansion of the set of feasible contracts. Substitutability in matching theory states that when an agent chooses from an expanded set of contracts, the set of contracts it rejects also weakly expands.

Both the law of aggregate demand and substitutability impose restrictions on an agent's preferences. When they hold, other desirable conditions of preferences and properties of the market are satisfied as well, including group strategy-proofness, the rural hospitals theorem, and another condition known as the irrelevance of rejected contracts. Thus, testing for substitutability and the law of aggregate demand are problems of interest. Hatfield et al. provide a testing algorithm for substitutability in "Testing substitutability". Similarly, we seek an algorithm to test for the law of aggregate demand.

In this paper, we begin in section 2 with an overview of the formal model for a matching market. Section 3 follows by examining the necessary and sufficient conditions for an agent's preferences to satisfy the law of aggregate demand. From this, we obtain an algorithm, presented in section 4, which allows us to test if the law of aggregate demand is satisfied. However, the efficiency of testing depends on the form in which an agent's preferences are reported. In particular, the law of aggregate demand cannot be tested in better than exponential time given only oracle access to the agent's choice function, but when the underlying preference relation is visible.

2 Model

Throughout this paper, we consider a many-to-one matching market modeled after the school choice problem, where schools admit heterogeneous students. We denote the finite set of contracts as X. Each contract $x \in X$ is associated with one school and one student. A set of contracts $X' \subseteq X$ is an *allocation* if each student is associated with at most one contract under X'. Note that a school can admit multiple students, thus it can be associated with multiple contracts under an allocation.

The preferences of a school can be expressed as a preference relation $\succ$ of length N over possible contracts of the following form.

$$Y_1 \succ Y_2 \succ ... \succ Y_N \succ \emptyset,$$

where $Y_i \subseteq X$ and $\emptyset$ represents accepting no contract.

These preferences can also be expressed as a choice function C, which maps a set of contracts in X offered to the school to the subset of these contracts that is most preferred by the school. Formally, we define the choice function as the following.

$$C(S) = max_{\succ}\{Z : Z \subseteq S\}$$

Note that the choice function C is induced by the preference relation, though we suppress dependence on the preference relation when presenting C as a primitive. In the other direction, it is easy to see that construction of the preference relation from C is exponential to the total number of contracts. In this paper, we distinguish between using the preference relation and the choice function as primitives to represent the agent's preferences.

The law of aggregate demand[1] is defined formally as follows.

Definition 2.1. A choice function C satisfies the *law of aggregate demand* if

$$\forall S' \subset S \subseteq X, \;\; |C(S')| \leq |C(S)|$$

This tries to capture the idea that as the price falls, agents should demand more of a good. Here, a lower price corresponds to more contracts being available, and higher demand corresponds to choosing (weakly) more contracts. In the school choice problem, if the pool of applicants to a school expands, then the students admitted by the school either grows as well or stays the same. The corresponding property for student preferences is implied because each student chooses at most one contract.

We see that we can naïvely test for the law of aggregate demand in $\Omega(2^{|X|})$ by considering every subset of X for S' and S. Depending on the primitive of the preferences, there may be an additional cost of computing $C(S)$. Thus, the lower bound on a naïve test is exponential in $|X|$. The algorithm that we present, given a preference relation, will be polynomial in $|X|$ and N. We note that in some circumstances, it is possible that N is on the order of $O(2^{|X|})$, in which case the algorithm we present will run in time of the same order as that of the naïve algorithm. However, for practical purposes, preference relations tend to be polynomial, as opposed to an exhaustive ordering of all sets of contracts, so our algorithm provides a significant improvement from exponential to polynomial runtime.

Substitutability[2] is defined formally as follows.

Definition 2.2. A choice function C satisfies *substitutability* if

$$\forall x, z \in X, \; \forall Y \subseteq X, \; z \in C(\{x\} \cup Y \cup \{z\}) \Longrightarrow z \in C(Y \cup \{z\})$$

[1]For more analysis in the context of matching theory, see Alkan (2002), Alkan and Gale (2003), and Hatfield and Milgrom (2005).

[2]For more analysis in the context of matching theory, see Kelso and Crawford (1982) and Hatfield and Milgrom (2005).

This condition captures the idea that if a contract is chosen by an agent from some set of available contracts, then that contract will still be chosen from any smaller set that includes it. In the school choice problem, if a student is admitted by a school from a larger pool of applicants, then she would be admitted from a smaller pool as well.

If both the law of aggregate demand and substitutability hold, then a number of desirable results hold. One result is that truthful revelation of preferences is a dominant strategy (Hatfield and Milgrom, 2005). In the student-offering deferred acceptance algorithm, if school preferences satisfy the law of aggregate demand and substitutability, then it is a dominant strategy for students to truthfully reveal their preferences over schools. Otherwise, a group of students may find it advantageous for themselves to submit false preferences in order to obtain a better outcome for themselves while leaving other students worse off. Another result is known as the Rural Hospitals Theorem (Hatfield and Milgrom, 2005). In the matching problem of doctors to hospitals, rural hospitals typically had difficulty filling all of their positions. This lead to the question of whether there are other core matches in which rural hospitals can be better off in terms of filling these positions. It turns out that if hospital preferences satisfy the law of aggregate demand and substitutability, then every hospital hires the same number of doctors at every stable match.

The irrelevance of rejected contracts[3] is defined formally as follows.

Definition 2.3. A choice function C satisfies the *irrelevance of rejected contracts* if

$$\forall Y \subset X,\ \forall z \in X \setminus Y,\ z \notin C(Y \cup \{z\}) \Longrightarrow C(Y) = C(Y \cup \{z\})$$

This condition reflects the notion that the removal of rejected contracts from the set of possible contracts does not affect the choice set. In the school choice problem, if a student is rejected from a school, then the outcome of students admitted to that school would remain unchanged had that student not applied in the first place.

The irrelevance of rejected contracts always holds if choice sets are constructed from preference relations, but not necessarily when choice functions are primitives. However, it turns out that substitutability and the law of aggregate demand together imply the irrelevance of rejected contracts (Aygün and Sönmez, 2012).

3 Results

Here, we examine necessary and sufficient conditions for the law of aggregate demand to hold. This is utilized in constructing an algorithm to test for the law of aggregate demand. We begin with two definitions which prove crucial in checking if the law of aggregate demand is satisfied.

Definition 3.1. Given a choice function C, we say that a set of contracts A is *revealed preferred* to a set of contracts B if $C(B) = B$ and $C(A \cup B) = A$. We denote this as $A \succ^C_{RP} B$.

This captures the idea that even though both sets A and B are affordable to the school, the school prefers set A over B, thus A is reveled preferred over B. This definition is used in the following.

Definition 3.2. A choice function C satisfies *monotonicity* when $A \succ^C_{RP} B \Longrightarrow |A| \geq |B|$.

It turns out that monotonicity is necessary but not sufficient to satisfy the law of aggregate demand. This is key in formulating the subsequent two theorems regarding the law of aggregate demand.

Theorem 3.3. *If a choice function C satisfies the law of aggregate demand, then C satisfies monotonicity.*

Proof. Assume that the law of aggregate demand holds. We want to show that monotonicity is satisfied. Consider two sets of contracts $S' \subset S$ such that $C(S') = S'$ and $C(S \cup S') = S$. By definition, S is revealed preferred to S'. By the law of aggregate demand, we know $|C(S')| \leq |C(S \cup S')|$, thus $|S'| \leq |S|$ by substitution. Thus, we see that monotonicity holds. □

Theorem 3.4. *If a choice function C satisfies both monotonicity and the irrelevance of rejected contracts, then C satisfies the law of aggregate demand.*

[3]For more analysis in the context of matching theory, see Aygün and Sönmez (2012).

Proof. Assume that both monotonicity and irrelevance of rejected contracts hold, and consider two sets of contracts $S' \subseteq S$. We want to show that $|C(S)| \geq |C(S')|$. By irrelevance of rejected contracts, we know $C(C(S) \cup C(S')) = C(S)$. Furthermore, we know $C(C(S')) = C(S')$. Thus, $C(S)$ is revealed preferred to $C(S')$. By monotonicity, we obtain $|C(S)| \geq |C(S')|$. This shows that the law of aggregate demand holds. □

Again, we see that monotonicity is a necessary condition for the law of aggregate demand to hold, but it is not sufficient by itself. However, monotonicity with the irrelevance of rejected contracts are sufficient conditions for the law of aggregate demand.

4 Algorithm

4.1 Preference relation as primitive

Given access to the agent's preference relation, we provide an algorithm to test if the law of aggregate demand is satisfied. Theorem 3.5 provides the sufficient conditions for satisfiability of the law of aggregate demand. Because the preference relation is the primitive of the choice function, we know that irrelevance of rejected contracts automatically holds, so it suffices to test for monotonicity. This directly leads us to the following algorithm, which returns pairs that violate monotonicity, thus the law of aggregate demand as well.

Algorithm 1 Tests if C satisfies the law of aggregate demand

```
ℒ ← ∅
for i = 1 . . . N − 1 do
    for j = i + 1 . . . N do
        if Y_j = C(Y_j) and Y_i = C(Y_i ∪ Y_j) and |Y_i| < |Y_j| then
            ℒ ← ℒ ∪ (Y_i, Y_i ∪ Y_j)
        end if
    end for
end for
```

The algorithm tries every pair Y_i and Y_j in the preference relation such that $Y_i \succ Y_j$. We are only interested in pairs where Y_i is revealed preferred to Y_j, so we check that $Y_j = C(Y_j)$ (fixed point) and that $Y_i = C(Y_i \cup Y_j)$, which comes from the definition of revealed preferred. In order for $Y_i \succ^C_{RP} Y_j$ to violate monotonicity, it must be the case that $|Y_i| < |Y_j|$, so all such violating pairs are appended to $\mathcal{L}$. Therefore, by termination, $\mathcal{L}$ contains pairs of sets of contracts such that each pair (S', S) represents sets $S' \subseteq S \subseteq X$ which, when offered to the agent, the resulting chosen sets violate the law of aggregate demand.

Theorem 4.1. *Algorithm 1 tests if a choice function satisfies the law of aggregate demand in time $O(|X| \cdot 2^{|X|})$.*

Proof. The time bounds for the algorithm follow directly from the code. Clearly, the loops account for $O(N^2)$ total iterations. In each iteration, there are sequential operations involving sets, namely computing the union of two sets as well as the cardinality of a set. Each of these is bound by $O(|X|)$, though the latter could be computed in $O(1)$ if the value is explicitly stored in the data structure implementation instead. Furthermore, computing $C(Z)$ for any $Z \subseteq X$ takes $O(N \cdot |X|)$, since it requires checking if Z is a superset of Y_n for $n \in [1, N]$ and checking for the superset relation takes $O(|X|)$. Thus, in each iteration, the cost of all sequential operations is bound by $O(N \cdot |X|)$. Therefore, total time complexity is simply $O(N^3 \cdot |X|)$[4]. □

For an example of using the algorithm, consider a school with the following preference relation. Each contract is denoted by i, which represents the school admitting student i.

$$\{3\} \succ \{1, 2\} \succ \{1\} \succ \{2\} \succ \emptyset$$

[4]We note that the preference relation could have a length exponential to the number of contracts, e.g. an explicit ranking of all subsets of X. This would imply that total time complexity is $O(|X| \cdot 2^{|X|})$, which corresponds to our naïve algorithm time bound.

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By observation, we see that the law of aggregate demand fails to hold in this case. When the school chooses from the set $\{1,2\}$, it chooses $\{1,2\}$, but when the set is expanded to $\{1,2,3\}$, it prefers to just choose $\{3\}$.

With the algorithm, in the very first iteration, $i = 1$ and $j = 2$, so we consider $Y_i = \{3\}$ and $Y_j = \{1,2\}$. Now, $Y_j = \{1,2\} = C(Y_j)$ and $Y_i = C(Y_i \cup Y_j)$, which implies that Y_i is revealed preferred to Y_j. Then we see that $|Y_i| = 1$ and $|Y_j| = 2$, so $|Y_i| < |Y_j|$, meaning that monotonicity is now violated. Because monotonicity is a necessary condition, it cannot be the case that the law of aggregate demand holds for this preference relation. Thus, we add to the set $\mathcal{L}$ the pair $(\{1,2\},\{1,2,3\})$, since these are sets of contracts offered to the school that illustrate a violation of the law of aggregate demand.

4.2 Choice function as primitive

Suppose we only have oracle access to the agent's choice function C, which can be used to query for the chosen subset of a set of contracts in $O(1)$ time. We do not have access to the preference relation itself. We claim that there does not exist a better than exponential algorithm.[5]

Our proof uses Yao's minimax principle (1977), which states that the expected cost of the best randomized algorithm on a worst-case deterministic input is equal to the expected cost of the best deterministic algorithm on a worst-case probability distribution of inputs. Formally, for a problem with deterministic inputs $\mathcal{I}$ and deterministic algorithms $\mathcal{A}$, let $\Delta(\mathcal{I})$ denote the set of probability distributions over $\mathcal{I}$ and $\Delta(\mathcal{A})$ denote the set of probability distributions over $\mathcal{A}$. For $A \in \mathcal{A}$ and $I \in \mathcal{I}$, let $A(I)$ denote the run time of A on I.

Theorem 4.2 (Yao, 1977)**.** *For a problem with deterministic inputs $\mathcal{I}$ and deterministic algorithms $\mathcal{A}$,*

$$\min_{\delta\mathcal{A}\in\Delta(\mathcal{A})} \max_{I\in\mathcal{I}} E_{A\sim\delta\mathcal{A}}[A(I)] = \max_{\delta\mathcal{I}\in\Delta(\mathcal{I})} \min_{A\in\mathcal{A}} E_{I\sim\delta\mathcal{I}}[A(I)]$$

We want to show that no randomized algorithm on a worst-case deterministic input can do better than exponential run time. With the use of Theorem 4.1, we want to construct a worst-case probability distribution of inputs and examine the performance of the best deterministic algorithm. Consider the following choice function.

$$C_Y(Z) = \begin{cases} Y, & \text{if } Y \subseteq Z \\ Z, & \text{otherwise} \end{cases}$$

We note that C_Y does not satisfy the law of aggregate demand if $|Y| \leq |X| - 2$. For some $y \in Y$, we know that $|C_Y(X \setminus \{y\})| = |X| - 1$, but if we include y in the offered set, then $|C_Y(X)| = |Y| \leq |X| - 2$, thus $X \setminus \{y\} \subset X$ but $|C_Y(X \setminus \{y\})| > |C_Y(X)|$.

Now we consider some deterministic algorithm A which tests for the law of aggregate demand on some choice function C, defined as follows. With $\frac{1}{2}$ probability, let C be the identity choice function C_{id}, where $C_{id}(Z) = Z$. Note that $C_{id}(Z)$ satisfies the law of aggregate demand. With the remaining $\frac{1}{2}$ probability, select a set Y of cardinality $\lfloor \frac{|X|}{2} \rfloor$ from X and let C be C_Y as defined earlier.[6]

We observe that C_{id} and C_Y only differ on input sets Z where $Y \subset Z$. Thus, to verify if the law of aggregate demand holds for C, i.e. C is C_{id}, it suffices for A to query all supersets Z of Y to check that $C(Z) = Z$ as opposed to $C(Z) = Y$. All supersets of Y can be expressed as the following.

$$\{Y \cup W : W \in X \setminus Y\}$$

Intuitively, we can construct all supersets of Y by taking the powerset of $X \setminus Y$ and taking the union of each individual element set with Y. We know that $|X \setminus Y| = \lceil \frac{X}{2} \rceil$, so there are $2^{\lceil \frac{X}{2} \rceil}$ elements in the powerset. Thus, we see that A must query $O(2^{|X|})$ elements to test C. Because C is the C_{id} with probability $\frac{1}{2}$, the expected running time must be at least $\frac{1}{2}$ the previous bound, thus the run time of A still lies in $O(2^{|X|})$. This leads to the following theorem.

Theorem 4.3. *There exists a distribution of choice functions δ such that for any deterministic algorithm A that tests the law of aggregate demand given only oracle access to the choice function, the expected running time of A given δ is at least exponential in $|X|$, the number of contracts.*

[5] Our proof mimics the analogous proof for substitutability found in Hatfield et al. (2012).

[6] We remark that the choice function does not have a preference relation encoding of length polynomial in $|X|$, which also implies that Algorithm 1 cannot test if a choice function satisfies the law of aggregate demand in polynomial time.

Combining Theorems 4.1 and 4.2, we obtain the following key corollary.

Corollary 4.1. *The worst-case expected running time of any randomized algorithm that tests the law of aggregate demand given only oracle access to the choice function is at least exponential in* $|X|$*, the number of contracts.*

5 Conclusion

Given the mathematical framework behind matching theory, we have shown necessary and sufficient conditions for the law of aggregate demand. In particular, monotonicity is a necessary condition for the law of aggregate demand. Monotonicity along with the irrelevance of rejected contracts are sufficient for the law of aggregate demand to hold.

From these results, we directly obtained an algorithm to test if the law of demand is satisfied given a preference relation. This algorithm has a total time complexity of $O(N^3 \cdot |X|)$, where $|X|$ is the total number of contracts and N is the length of the preference relation. We also show that given access to only the agent's choice function, any algorithm is essential a brute force algorithm requiring an exponential number of queries. We remark that the preference relation could have a length exponential to the number of contracts, which in fact implies that total time complexity would be exponential even with the preference relation. However, we assume that agents are, in general, only able to construct short, i.e. polynomial length, preference lists. Thus, the algorithm is an effective method for market designers to test whether the law of aggregate demand holds for submitted preferences.

It has already been showed that substitutability can be tested for in a similar manner. Thus, given access to preference relations, a market designer can test for both the law of aggregate demand as well as substitutable, both of which reflect practical restrictions on preferences. Knowing that both the law of aggregate demand and substitutability hold guarantees other desirable properties in the matching market.

6 Acknowledgments

I thank Dr. Bumin Yenmez for being my Honors Thesis advisor. I also thank Dr. Carol Goldburg for her guidance throughout my undergraduate experience at Carnegie Mellon University.

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